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Discrete Morse Theory and Simplicial Sets

Master's thesis in Applied Physics and Mathematics
Supervisor: Marius Thaule
Co-supervisor: Melvin Vaupel
June 2023

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ABSTRACT. In this thesis, we use concepts and methods from the theory of simplicial sets to study discrete Morse theory. We focus on the discrete flow category introduced by Vidit Nanda, and investigate its properties in the case where it is defined from a discrete Morse function on a regular CW complex. We construct an algorithm to efficiently compute the Hom posets of the discrete flow category in this case. Furthermore, we show that in the special case where the discrete Morse function is defined on a simplicial complex, then each Hom poset has the structure of a face poset of a regular CW complex. Finally, we prove that the spectral sequence associated to the double nerve of the discrete flow category collapses on page 2.

CONTENTS

1. INTRODUCTION

Differentiable functions on smooth manifolds allow us to deduce properties of the manifolds' topology, through the use of Morse theory. For Morse theory to apply, the differential function has to satisfy the *Morse condition*. Such functions are called *Morse functions*, and they are in many ways abundant. A Morse function can be viewed as a "height function" on a manifold that allows you to decompose the manifold into smaller, more manageable parts. Morse theory has had vast applications, not only in topology, but also in other areas, such as in the study of dynamical systems [21].

The combinatorial counterpart of Morse theory, discrete Morse theory, was introduced by Robin Forman [6], and studies *discrete Morse functions* on *simplicial complexes*. Like Morse theory, discrete Morse theory allows us to deduce information about the topology of a simplicial complex. As the name suggests, the data of a discrete Morse function is discrete, making discrete Morse theory suitable for computer applications. Discrete Morse theory has seen much use in computer applications in later years, particularly in topological data analysis (TDA). As an example, in [20, pp. 13–14], Scoville explains how discrete Morse theory can be used to reduce the computational complexity of homology computations in TDA.

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Simplicial sets are a generalization of simplicial complexes with several useful theoretical properties. They are, however, not used much in computer applications. One of the focus areas of this thesis is to use concepts and methods from the theory of simplicial sets to study discrete Morse theory.

One of the main foundations for this thesis is a paper by Nanda [17] that introduces the *discrete flow category*, an analogue in discrete Morse theory for the *flow category* of Cohen, Jones and Segal [4]. The flow category is a category consisting of critical points and flow lines of a Morse function, and whose classifying space captures the homotopy type of the manifold on which the Morse function is defined. Likewise, the discrete flow category consists of critical *cells* and *gradient paths* of a discrete Morse function, and has a classifying space with the homotopy type of the simplicial complex on which the discrete Morse function is defined. While the flow category is a *topological category*, meaning that the Hom sets are topological spaces, the discrete flow category is a *p-category*, meaning that the Hom sets are posets. Constructing the classifying space of either of these categories utilizes concepts from simplicial sets, inspiring the further study of discrete Morse theory from the perspective of simplicial sets.

Another inspiration for this thesis is a paper by Vaupel, Hermansen and Trygsland on section complexes of simplicial height functions [23]. In the paper, a bisimplicial set is constructed from a *height function* on a simplicial set. This bisimplicial set gives a spectral sequence that can be used to compute the homology of the simplicial set, and it is shown that under certain conditions the spectral sequence collapses on the second page.

In this thesis, we give a perspective on the connections between smooth Morse theory and discrete Morse theory. For this, we use concepts from both simplicial sets and from category theory. In particular, we study the gradient paths out of both regular and critical points of a Morse function, and their discrete analogues, and show how these paths form spaces that behave the same way in the smooth and discrete cases. This idea is made formal by comparing both of these path concepts to the under-fiber of an object in an appropriate category.

We further explore Nanda's discrete flow category (in the special case where it is defined from a discrete Morse function), and prove several properties of it. In particular, we show that in the case where the discrete Morse function is defined on a simplicial complex (not a general CW complex), then the Hom posets in the discrete flow category are CW posets, meaning that they have the structure of a face poset of a regular CW complex. We will refer to this as Theorem A.

Theorem A. Let C be the discrete flow category of a discrete Morse function on a simplicial complex. Then the for all objects w and z in C, the poset $\mathrm{Hom}_C(w, z)^{\mathrm{op}}$ is the face poset of some regular CW complex.

This result gives a simpler way of realizing the Hom posets as topological spaces: instead of taking the geometric realization of the nerve, one can take the corresponding regular CW complex.

Furthermore, we construct an algorithm, Algorithm 1, to compute the Hom posets of the discrete flow category, where the input is a discrete Morse function defined on a regular CW complex. For the algorithm, we use several of the results needed for proving Theorem A. We also provide a Python implementation for the algorithm, which can be found at `https://github.com/bjornarhem/discrete_flow` .

Finally, we investigate the spectral sequence associated to the *double nerve* of the discrete flow category, a bisimplicial set whose realization is the classifying space of the category. As in the paper by Vaupel et al. [23], this spectral sequence computes the homology of the regular CW complex from which we define the discrete flow category. Classifying spaces of p-categories are often complicated and hard to compute, and this is also the case for the discrete flow category. However, as we prove in this thesis, the spectral sequence associated to its double nerve has a particularly nice structure that causes it to collapse on page 2. We will refer to this as Theorem B.

Theorem B. The spectral sequence associated to the double nerve of a discrete flow category collapses on page 2.

To prove this theorem, we generalize the concept of *simplicial collapse* to simplicial sets. The definition of the mentioned spectral sequence will be made formal in the text (in particular, there are two possible choices). We also show how you can ignore degeneracies when computing the spectral sequence, which makes the computation relatively simple. Finally, we provide several examples of computing the spectral sequence, in which we make use of Algorithm 1 to compute the Hom sets.

Outline. Section 2 and 3 are dedicated to reviewing the theory on simplicial sets and discrete Morse theory, respectively, that will be needed for the following sections. In Section 4, we present our work on connections between smooth and discrete Morse theory. Section 5 contains the necessary preliminaries on p-categories, and Section 6 is a review of the discrete flow category of a discrete Morse function. In Section 7, we prove several results on the Hom posets of the discrete flow category, and conclude with constructing Algorithm 1 and proving Theorem A. In Section 8, we generalize simplicial collapse to simplicial sets, and use this to prove a result which is later used in the proof of Theorem B. Finally, in section 9, we explain how the discrete flow category gives rise to a spectral sequence, and prove Theorem B.

Acknowledgments. I would like to thank my supervisors, Marius Thaule and Melvin Vaupel, who have both been incredibly helpful and supportive. They have gone above and beyond, both in encouraging me to explore my ideas and in helping me seek answers to the countless questions I have asked them. Their feedback, which has consistently been both thorough and constructive, has been invaluable in writing this thesis.

2. Preliminaries on simplicial sets

In this section we state some facts about simplicial sets that will be used throughout the thesis. We will not give an in-depth explanation of simplicial sets, but instead just define the notation we will use. For a thorough exposition of simplicial sets, see e.g., [8] or [7].

A *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to \mathsf{Set},$$

where Δ is the simplex category and Set is the category of sets. A morphism of simplicial sets is a natural transformation between two such functors, and we denote the category of simplicial sets by sSet. The ith face map is denoted by d_i, and the ith degeneracy map is denoted by s_i. For a simplicial set X, we denote its geometric realization by $|X|$. For a small category C, we denote its nerve by $\mathcal{N}(C)$.

2.1. Simplicial sets, simplicial complexes and regular CW complexes.

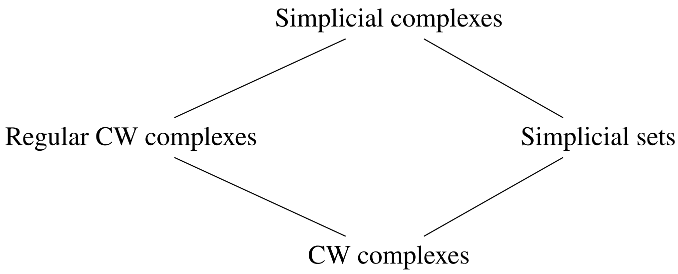
We comment on the relations between simplicial sets, simplicial complexes and regular CW complexes. We assume familiarity with CW complexes and simplicial complexes.

The geometric realization of a simplicial set is a CW complex with one n–cell for each nondegenerate n–simplex [14, Theorem 14.1].

A *regular* CW complex is a CW complex where each attaching map is a homeomorphism onto its image. A simplicial complex can be viewed as a regular CW complex with an n–cell for each n–simplex.

A simplicial complex can be viewed as a simplicial set by giving the vertices an ordering and adding degeneracies. Different choices of orderings may lead to different simplicial sets, but their geometric realization will all be homeomorphic to the simplicial complex.

All these observations can be summed up in the following diagram, where the edges mean that the top is a subclass of the bottom.



2.2. **Skeletons and coskeletons.** Let $\Delta_{\leq n}$ be the full subcategory of Δ on the objects $[0], \dots, [n]$. From the inclusion $\Delta_{\leq n} \hookrightarrow \Delta$, we get a truncation functor

$$\mathrm{tr}_n \colon \mathsf{sSet} = [\Delta^{\mathrm{op}}, \mathsf{Set}] \to [\Delta^{\mathrm{op}}_{\leq n}, \mathsf{Set}] = \mathsf{sSet}_{\leq n},$$

given by precomposition with the inclusion. Here, we denote by $[\mathscr{A}, \mathscr{B}]$ the category of functors $\mathscr{A} \to \mathscr{B}$ and natural transformations between them.

The functor tr_n has a left adjoint, $\mathrm{sk}_n \colon \mathsf{sSet}_{\leq n} \to \mathsf{sSet}$, given by producing degenerate simplices for degrees $> n$. Furthermore, tr_n has a right adjoint, $\mathrm{cosk}_n \colon \mathsf{sSet}_{\leq n} \to \mathsf{sSet}$, which adds a single simplex in degree $m > n$ for each compatible set of faces [10, pp. 7–8].

Now, define the two functors

$$\mathbf{cosk}_n = \mathrm{cosk}_n \circ \mathrm{tr}_n \colon \mathsf{sSet} \to \mathsf{sSet},$$
$$\mathbf{sk}_n = \mathrm{sk}_n \circ \mathrm{tr}_n \colon \mathsf{sSet} \to \mathsf{sSet}.$$

These are in fact adjoint, so that for a simplicial set X,

$$(\mathbf{cosk}_n\, X)_k = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^k, \mathbf{cosk}_n\, X) = \mathrm{Hom}_{\mathsf{sSet}}(\mathbf{sk}_n\, \Delta^k, X).$$

Simplicial sets isomorphic to objects in the image of cosk_n are called *n–coskeletal*. In other words, a simplicial set is n–coskeletal if for each compatible set of $(m-1)$–faces, with $m > n$, there is a unique m–simplex with those faces.

Theorem 2.1. *Let C be a small category. Then the nerve of C is 2–coskeletal.*

Proof. Let $m > 2$, and assume we have a compatible set of $(m-1)$–faces. This is equivalent to having vertices $x_0, \dots, x_m$ and maps $f_{ij} \colon x_i \to x_j$ for $0 \leq i \leq j \leq m$ such that for any $(m-1)$–subset I of $\{0, \dots, m\}$ the maps form a simplex, meaning that $f_{ik} = f_{jk} \circ f_{ij}$ for $i, j, k \in I$. Since $m > 2$ this means that $f_{ik} = f_{jk} \circ f_{ij}$ for all i, j and k, so there is a unique m–simplex with these $(m-1)$–simplices as faces. □

2.3. **Barycentric subdivision.** We here define what we mean by the barycentric subdivision of a regular CW complex.

Definition 2.2. Let X be a regular CW complex. The barycentric subdivision of X, denoted $T(X)$, is a simplicial complex whose vertices are cells of X and whose n–simplices are sequences of distinct cells $(\sigma_0, \dots, \sigma_n)$ such that $\sigma_0 \subseteq \cdots \subseteq \sigma_n$.

An example of a barycentric subdivision is given in Figure 1.

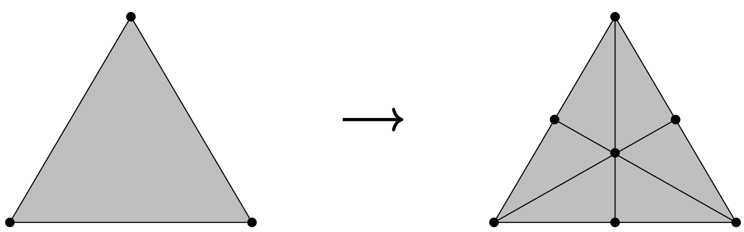


FIGURE 1. The barycentric subdivision of the simplicial complex Δ^2.

Theorem 2.3. *Let X be a regular CW complex. Then the geometric realization of the barycentric subdivision, $|T(x)|$, is homeomorphic to X.*

For a proof, see [12, pp. 80–81].

The barycentric subdivision is a *flag complex*, that is, a simplicial complex such that for each set of $n > 2$ vertices, if each pair forms an edge, then the set defines a n–simplex. Viewing the simplicial complex as a simplicial set, this is the same as being 1–coskeletal.

2.4. **Simplicial homology.** Given a set S, let $\mathbb{Z}S$ denote the free abelian group on S. For a simplicial set X, we can define a chain complex C with $C_n = \mathbb{Z}X_n$ and the differential $\partial_n : C_n \to C_{n-1}$ given on basis elements $\sigma \in X_n$ as

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma.$$

The *simplicial homology* of a simplicial set X is defined as the homology of the complex C defined above.

A proof of the following result can be found in [14, p. 63].

Theorem 2.4. *Let X be a simplicial set. Then for all n,*

$$H_n(|X|) \cong H_n(X),$$

where the right hand side is the simplicial homology of X and the left hand side is the singular homology (with integer coefficients) of the realization of X.

For a complex C associated to a simplicial set, let D_n denote the subgroup of C_n generated by the degenerate simplices in X_n. One can show that $\partial_n(D_n) \subseteq D_{n-1}$, so that ∂_n induces a well-defined map $C_n/D_n \to C_{n-1}/D_{n-1}$. We thus get a new complex with C_n/D_n in the nth degree, which we shall denote by C/D.

We get the following result by combining Theorem 2.1 and Theorem 2.4 in [8, Chapter III].

Theorem 2.5. *Let C be a complex associated to a simplicial set, and D its subcomplex of degeneracies. Then, for all n there is an isomorphism*

$$H_n(C) \cong H_n(C/D),$$

induced by the projection map $p : C \to C/D$.

The following corollary follows immediately from applying the long exact sequence of homology to the short exact sequence of chain complexes $0 \to D \to C \to C/D \to 0$.

Corollary 2.6. *Let C be a complex associated to a simplicial set, and D its subcomplex of degeneracies. Then,*

$$H_n(D) \cong 0,$$

for all n.

2.5. **Weak Kan complexes.** The *Kan condition* is a condition on simplicial sets that allows you to relate the notions of homotopies between simplicial maps and homotopies between continuous maps. The condition says that all *horns* in the simplicial set has unique fillers, and is satisfied, for example, by the singular simplicial set of a topological space. We here present a weaker condition, called the *weak Kan condition*, where only the *inner* horns are required to have unique fillers.

Definition 2.7. A simplicial set X satisfies the *weak Kan condition* if for all n and $0 < k < n$, any simplicial map $\Lambda^n_k \to X$ can be extended to a simplicial map $\Delta^n_k \to X$.

Horns of the form Λ^n_k with $0 < k < n$ are called *inner horns*. Simplicial sets satisfying the weak Kan condition are called *weak Kan complexes*.

Now, let a *weak Kan complex with unique fillers* be a weak Kan complex such that the extension in Definition 2.7 is always unique. Note that in this case, you can define composition of 1–simplices: Let σ and τ be 1–simplices with $d_1\sigma = d_0\tau$. Then extending the map from Λ^2_1 into these two 1–simplices gives a unique 1–simplex υ with $d_0\upsilon = d_0\sigma$ and $d_1\upsilon = d_1\sigma$ (see illustration Figure 2). We define $\tau\circ\sigma$ to be this υ.

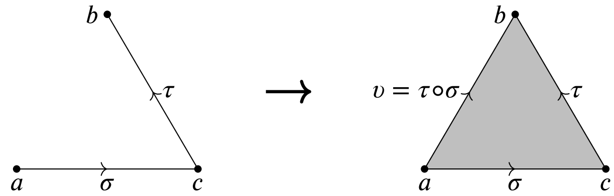


FIGURE 2. Applying the unique inner horn filling to define the composition of σ and τ.

Observe further that σ, τ and $\tau\circ\sigma$ form the edges of a 2–simplex, similar to the nerve of a category. In fact, taking the nerve of a category with these composition rules gives our original simplicial set.

Theorem 2.8. [13, Proposition 0031] *A simplicial set is the nerve of some category if and only if it is a weak Kan complex with unique fillers.*

Given a weak Kan complex with unique fillers X, its corresponding category has as objects 0–simplices in X. The morphisms are 1–simplices in X and the composition rule is as described above.

2.6. **Bisimplicial sets.** A *bisimplicial set* is a functor

$$\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set},$$

or equivalently,

$$\Delta^{\mathrm{op}} \to \mathsf{sSet}.$$

For a bisimplicial set X, we define the *diagonal* of X, denoted $\operatorname{diag} X$, as the simplicial set given by:

- $(\operatorname{diag} X)([n]) = X([n],[n])$,
- For $\theta : [m] \to [n]$, $(\operatorname{diag} X)(\theta) = X(\theta, \theta)$.

We define the *geometric realization* of a bisimplicial set X, denoted $|X|$, as the realization of the diagonal, i.e., $|X| \coloneqq |\operatorname{diag} X|$.

3. DISCRETE MORSE THEORY

In this section, we briefly summarize the core ideas and definitions from discrete Morse theory.

In Morse theory, one studies certain differentiable functions, called *Morse functions*, on smooth manifolds. These Morse functions allow us to deduce information about the topology of the manifold. For more information about Morse theory, see e.g. [16]. Discrete Morse theory is a discrete analogue of this, where one studies real-valued functions on CW complexes. The functions assign a single real value to each cell of the CW complex, and hence we get a discrete set of values.

We will only define discrete Morse functions for regular CW complexes. For the general definition, see [6]. Note also that one often consider only discrete Morse functions on simplicial complexes, which can be considered a special class of regular CW complexes.

Definition 3.1. Let X be a regular CW complex. A discrete Morse function on X is a function $f : X \to \mathbb{R}$ that assigns a real value to each cell of X such that for each k–cell x, the following holds:

(1) There is at most one $(k+1)$–cell y such that $x \subseteq y$ and $f(x) \geq f(y)$.
(2) There is at most one $(k-1)$–cell y such that $y \subseteq x$ and $f(y) \geq f(x)$.

Definition 3.2. An k–cell x is called *critical* if

(1) There are no $(k+1)$–cells y such that $x \subseteq y$ and $f(x) \geq f(y)$, and
(2) There are no $(k-1)$–cells y such that $y \subseteq x$ and $f(y) \geq f(x)$.

Cells that are not critical are called *regular*. Note that for a discrete Morse function, it is impossible for both of these conditions to fail to hold for a single cell (this is often called the exclusion lemma) [6, Lemma 2.5]. This also implies that if $x \subseteq y$ and $\dim y > \dim x + 1$, then $f(y) > f(x)$. An example of a discrete Morse function is given in Figure 3.

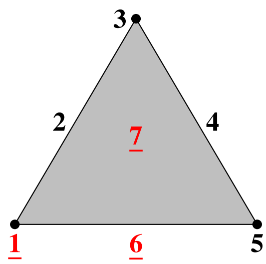


FIGURE 3. An example of a discrete Morse function. The critical simplices are underlined and colored red.

3.1. **Gradient vector fields.** In discrete Morse theory, the essential information in a discrete Morse function is for what pairs one of the conditions in Definition 3.2 fails to hold, its *regular pairs*. Therefore, instead of talking about a discrete Morse function f, we often talk about its induced *gradient vector field* V_f.

Definition 3.3. Let f be a discrete Morse function, let x be a k–cell and let y be a $(k+1)$–cell. Then $\{x, y\}$ is called a *regular pair* if $x \subseteq y$ and $f(x) \geq f(y)$.

Definition 3.4. Let $f : X \to \mathbb{R}$ be a discrete Morse function. The *induced gradient vector field* of f, denoted V_f, is defined as the set of *regular pairs*, i.e.,

$$V_f = \{\{x, y\} : x \subsetneq y, f(x) \geq f(y)\}.$$

An example of a gradient vector field is given in Figure 4. Two discrete Morse functions that induce the same gradient vector field are said to be *Forman equivalent*.

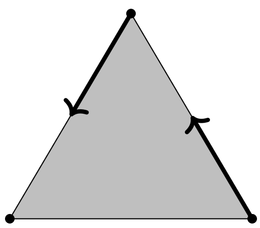


FIGURE 4. The gradient vector field for the discrete Morse function in Figure 3.

Now, given a regular CW complex X, a *discrete vector field* on X is a set V of mutually disjoint pairs $\{x, y\}$ such that $\dim y = \dim x + 1$. Given a discrete vector field V, a V–path is a sequence of simplices $(x_0, y_0, x_1, \dots, y_m, x_{m+1})$ such that $x_i \in X_k, y_i \in X_{k+1}$, $x_{i+1} \subseteq y_i$, $\{x_i, y_i\} \in V$ and $x_{i+1} \neq x_i$. Discrete vector fields and gradient vector fields are related in the following way.

Theorem 3.5. [6, Theorem 9.3] *A discrete vector field V is a gradient vector field of some discrete Morse function if and only if there are no nontrivial V–paths that start and end on the same cell.*

3.2. **Simplicial collapse.** In this section, we restrict our attention to simplicial complexes, and define what is known as a *simplicial collapse*. A *free face* in a simplicial complex is a k–simplex that is the face of exactly one $(k + 1)$–simplex. If $x \in X^k$ is a free face, and y is its $(k + 1)$–dimensional coface, then $\{x, y\}$ is called a *free pair*. Now, removing a free pair from a simplicial complex is called an *elementary collapse*. A series of elementary collapses is called a *compound collapse* or *simplicial collapse*, and if the simplicial complex Y can be produced from a simplicial complex X through a compound collapse, we say that X *collapses* to Y and write $X \searrow Y$. Of particular importance is the fact that elementary collapses, and as a consequence compound collapses, yields a deformation retract. That is, if $X \searrow Y$, then X deformation retracts to Y.

An example of an elementary collapse is given in Figure 5.

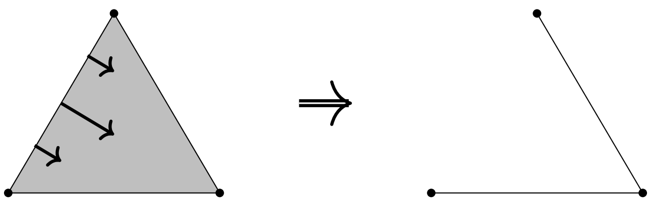


FIGURE 5. An elementary collapse.

For a simplicial complex X, let $X(a)$ denote the subcomplex consisting of the simplices in $f^{-1}((-\infty, a])$ and all their faces. The following result shows why simplicial collapses are relevant to discrete Morse theory.

Lemma 3.6. [6, Theorem 3.3] *Let X be a simplicial complex and $f : X \to \mathbb{R}$ be a discrete Morse function. Suppose the interval $(a, b]$ is such that $f^{-1}((a, b])$ contains no critical simplices. Then $X(b) \searrow X(a)$.*

This result is one example of how discrete Morse functions contains information about the topology of their domain. We end this section with another powerful result, which is considered one of the main results in discrete Morse theory.

Theorem 3.7. [6, Theorem 10.2] *Let X be a regular CW complex and $f : X \to \mathbb{R}$ a discrete Morse function. Then X is homotopy equivalent to a CW complex with number of n–cells equal to the number of critical n–cells of f.*

4. CONNECTIONS BETWEEN SMOOTH AND DISCRETE MORSE THEORY

In this section, we investigate some connections between Morse theory and discrete Morse theory. We also compare these settings to height functions on simplicial sets, and define an analogue of the flow category for height functions on Weak Kan complexes with unique fillers.

4.1. **Height functions on simplicial sets.**

Definition 4.1. Let $(\mathbb{R}, \leq)$ be the poset of real numbers (equipped with its usual ordering) viewed as a category. We define the *simplicial real line*, $\mathbf{R}$, as the nerve of $(\mathbb{R}, \leq)$, i.e., $\mathbf{R} = \mathcal{N}(\mathbb{R}, \leq)$.

An n–simplex in $\mathbf{R}$ can be represented by a non-decreasing sequence of real numbers $(a_0, \dots, a_n)$. Observe also that $\mathbf{R}$ is 1–coskeletal.

Definition 4.2. Let X be a simplicial set. A *height function* on X is a simplicial map $h\colon X \to \mathbf{R}$.

A height function $h\colon X \to \mathbf{R}$ is uniquely determined by its restriction to X_0. A map $h_0\colon X_0 \to \mathbb{R}$ extends to a height function $h\colon X \to \mathbf{R}$ as long as the orientation of the 1–simplices in X is respected (i.e., for $v \in X_1$, $h_0(d_1 v) \leq h_0(d_0 v)$). We can therefore characterize height functions on X as maps $h_0\colon X_0 \to \mathbb{R}$ that respect orientation of 1–simplices.

Now, recall that picking an ordering of the vertices of a simplicial complex defines a simplicial set. Therefore, given a simplicial complex X and a real-valued function on its vertices $h_0\colon X^0 \to \mathbb{R}$, we get a height function $\tilde{h}\colon \tilde{X} \to \mathbf{R}$ by picking and ordering on X that's respected by h_0. Furthermore, if h_0 is injective, then $\tilde{X}$ and $\tilde{h}$ is uniquely determined.

Example 4.3. Let X be a regular CW complex and $f\colon X \to \mathbb{R}$ a discrete Morse function. Then f defines a function on the vertices of the barycentric subdivision of X, $h_0\colon T(X)^0 \to \mathbb{R}$ (given by $h_0(\sigma) = f(\sigma)$). This in turn gives a height function on $T(X)$ viewed as a simplicial set, and by an abuse of notation we write $h\colon T(X) \to \mathbf{R}$.

An illustration of this is given in Figure 6. Here, the height function is represented by marking the 1–simplices with the direction the height function is increasing.

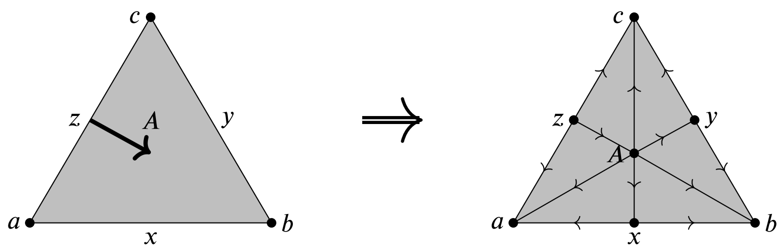


FIGURE 6. Left: A discrete Morse function. Right: The induced simplicial height function on the barycentric subdivision.

4.2. **Regular vertices of height functions, formalized.** Firstly, to make everything more similar to the smooth case, we define a height function as a function $h\colon X \to \mathbf{R}^{\mathrm{op}} = \mathcal{N}((\mathbb{R}, \leq)^{\mathrm{op}})$. In other words for $v \in X_1$, we have that $h(d_1 v) \geq h(d_0 v)$, opposite to our previous definition. Furthermore, we say that a height function is *strictly decreasing* if $h(d_1 v) > h(d_0 v)$ for all non-degenerate $v \in X_1$.

Now, given a strictly decreasing height function $h\colon X \to \mathbf{R}^{\mathrm{op}}$, define the lower link of $v \in X_0$ as follows.

Definition 4.4. Let $h\colon X \to \mathbf{R}^{\mathrm{op}}$ be a strictly decreasing height function, and let $v \in X_1$. Then the lower link of v, denoted $\operatorname{llink}_h v$, is defined as follows.

- For $n \geq 0$, the n–simplices in $\operatorname{llink}_h v$ are $(n+1)$–simplices $w \in X_{n+1}$ such that $d_1 d_2 \cdots d_{n+1} w = v$, and such that for all other 0–simplices $u < w$, we have $h(u) < h(v)$.

- For $w \in (\operatorname{llink}_h v)_n$ and $0 \le i \le n$, the degeneracy maps s'_i are $s_{i+1} : X_{n+1} \to X_{n+2}$. Likewise, for $w \in (\operatorname{llink}_h v)_n$, the face maps d'_i are $d_{i+1} : X_{n+1} \to X_n$.

For this to be a well-defined simplicial set, the face and degeneracy maps need to be well-defined and satisfy the following equations.

$$\begin{aligned} d'_i d'_j &= d'_{j-1} d'_i && \text{if } i < j \\ d'_i s'_j &= s'_{j-1} d'_i && \text{if } i < j \\ d'_j s'_j &= d'_{j+1} s'_j = \mathrm{id} && \\ d'_i s'_j &= s'_j d'_{i-1} && \text{if } i > j+1 \\ s'_i s'_j &= s'_{j+1} s'_i && \text{if } i \le j. \end{aligned} \tag{1}$$

It is easily verified that the equations in (1) follow from the fact that the same equations hold for d_i and s_i. Hence, it is enough to verify that the face and degeneracy maps are well-defined. To show this, we must show that for $w \in (\operatorname{llink}_h v)_n$ and $0 \le i \le n$, $d'_i w$ is in $(\operatorname{llink}_h v)_{n-1}$. This is equivalent to showing that $d_1 \cdots d_n d_{i+1} w = v$. Applying the first equation in [9, Equation 1] repeatedly, we get

$$\begin{aligned} d_1 \cdots d_n d_{i+1} w &= d_1 \cdots d_i d_{i+1} d_{i+2} \cdots d_{n+1} w \\ &= d_1 \cdots d_{n+1} w = v. \end{aligned}$$

Similarly, for the degeneracy maps, we must show that $d_1 \cdots d_{n+2} s_{i+1} w = v$ for $w \in (\operatorname{llink}_h v)_n$ and $0 \le i \le n$. Using the third and fourth equation in (1), we get

$$\begin{aligned} d_1 \cdots d_{n+2} s_i w &= d_1 \cdots d_{i+2} s_{i+1} d_{i+2} \cdots d_{n+1} w \\ &= d_1 \cdots d_{i+1} \,\mathrm{id}\, d_{i+2} \cdots d_{n+1} w \\ &= d_1 \cdots d_{n+1} w = v. \end{aligned}$$

Hence, $\operatorname{llink}_h v$ is a well-defined simplicial set.

Note that h doesn't really play a role in this definition, only X does, but it can be useful when generalizing to not-strictly decreasing height functions.

Finally, we define regular vertices as follows.

Definition 4.5. Let $h : X \to \mathbf{R}^{\mathrm{op}}$ be a strictly decreasing height function, and let $v \in X_0$. Then v is *regular* if $|\operatorname{llink}_h v|$ is contractible.

As usual, vertices that are not regular will be called *critical*.

This definition is closely related to regular simplices in discrete Morse theory.

Theorem 4.6. *Let $f : X \to \mathbb{R}$ be an injective discrete Morse function on a regular CW complex. Then the induced height function $h : T(X) \to \mathbf{R}^{\mathrm{op}}$ on the barycentric subdivision is strictly decreasing, and the regular vertices are precisely the regular simplices in X.*

Proof. First, observe that as $T(X)$ is 1–coskeletal, for $\sigma \in T(X)_0$, $\operatorname{llink}_h \sigma$ is isomorphic to the subcomplex spanned by all $\tau \in T(X)_0$ that shares an edge with σ and has lower h–value. This is the subcomplex spanned by all faces and cofaces of σ that has lower f–value.

Now, we show that if σ is a critical simplex of f, then it's a critical vertex of h. In this case, all its faces have lower f–value and all its cofaces have higher f–value. Thus, the lower link of a critical n–simplex σ is the barycentric subdivision of $\partial\Delta^n$, which does not have contractible realization, and hence, σ is a critical vertex of h.

Next, we show that if σ is a regular simplex of f, then it's a regular vertex of h. Let σ be regular n–simplex. There are two cases: either σ has a single $(n+1)$–dimensional coface with lower f–value than σ, or σ a single $(n-1)$–dimensional face with higher f–value than σ. In the first case, $\operatorname{llink}_h \sigma$ is isomorphic to the barycentric subdivision of Δ^n. In the second case, it's isomorphic to the barycentric subdivision of a horn $|\Lambda^n_k|$. In both cases, $|\operatorname{llink}_h v|$ is contractible (an illustration of these cases is given in Figure 7). This concludes the proof. □

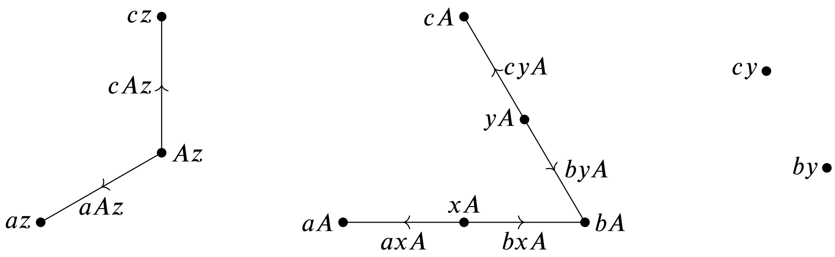


FIGURE 7. An illustration of the lower links of three vertices with respect to the height function in Figure 6. The realization of the lower link is contractible exactly for those vertices corresponding to regular simplices. Left: The lower link of the regular vertex z. Middle: The lower link of the regular vertex A. Right: The lower link of the critical vertex y.

4.3. **Morse theory.** Let M be an n–dimensional smooth manifold, and $f : M \to \mathbb{R}$ a Morse function. The Morse lemma tells us that the local topological behavior of f at a critical point is entirely decided by the index of the critical point. Given a critical point p of index k, let us consider the "space of flow directions" out of the points. More formally, for a sufficiently small neighbourhood around p, consider the quotient space of points on flow lines starting at p where points on the same flow line are identified. Using the Morse lemma, f can be written locally around p as

$$\hat{f}(x) = f(p) - \sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{n} x_n^2.$$

The flow lines out of this point is those on the form

$$t \mapsto (a_1 e^{-2t}, \dots, a_k e^{-2t}, 0, \dots, 0),$$

for $-\infty < t < t_0$, where the a_i are arbitrary real constants and t_0 is a constant decided by the a_i. Thus, the points on flow lines starting at p are of the form $(x_1, \dots, x_k, 0, \dots, 0)$, and two such points are identified in the quotient space if they differ by a positive scaling factor. Hence, the quotient space of "flow directions" is precisely S^{k-1} (taking S^{-1} to be the empty set). As the index k completely describes the topological information around p, we can equivalently categorize critical points by what their "space of flow directions" is. This will be useful when passing to the discrete world.

To illustrate, consider the Morse function in Figure 8. The top critical point has index 2, and its space of flow directions is S^1. The two middle critical points has index 1, and their respective spaces of flow directions are both S^0. The bottom critical point has index 0, and its space of flow directions is $\emptyset$.

One could also try to look at the space of flow lines passing through a regular point. Given a regular point, there is only a single flow line passing through it, so such a space would always be the one-point space. In discrete Morse theory, however, there can be several gradient paths passing through a regular simplex. Nevertheless, we will see that the corresponding analogue of the "space of flow directions" will always be contractible, capturing the intuition that the flow "flows in one direction."

4.4. **The discrete case.** A simplicial height function can be viewed as a piecewise linear function on a CW complex, as described on page 7 in [23]. From this perspective, we can ask what the critical and regular points of a simplicial height function should be. It's clear that the critical points should all be at the 0–simplices. Attempting to use our "space of flow directions" characterization, a natural replacement for "flow directions" in the previous section is 1–simplices going out of a vertex. However, one should also include

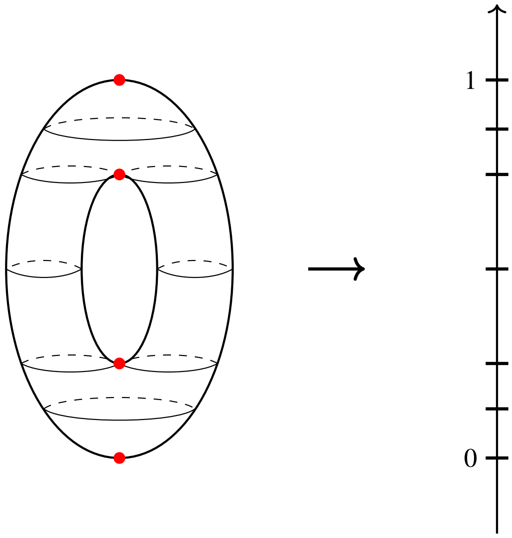


FIGURE 8. An example of a Morse function, the height function on a torus. The critical points are denoted by red dots.

2–simplices as a "line segment of direction" between two one segments, a 3–simplex as an "area of directions" between these, and so on. This leads to the definition of lower link from Definition 4.4, and we defined a 0–simplex as *critical* if it's lower link is not contractible.

We now move on to discrete Morse theory, where we have discrete Morse functions on simplicial complexes. We can get a discrete analogue by taking the barycentric subdivision, and looking at the lower link of a simplex. In the proof of Theorem 4.6 we showed how the geometric realization of the lower link of a:

- Critical simplex of dimension k is $|\partial\Delta^k| \cong S^{k-1}$ (taking S^{-1} to be the empty set),
- Regular simplex is contractible.

Some examples of this is given in Figure 7.

In conclusion, the "flow direction space" of the previous section naturally translates to the lower link in the discrete setting, and regular simplices are characterized by having contractible lower link, just as in the smooth setting. Furthermore, the dimension of a critical simplex is given by the lower link, just as the index of a critical point is in the smooth case.

4.5. **The simplicial flow category of a weak Kan complex with unique fillers.** In this section, we will define the simplicial flow category of a weak Kan complex with unique fillers, and show that its classifying space is homotopy equivalent to the realization of the weak Kan complex. Recall from Section 2.5 that a weak Kan complex with unique fillers is the nerve of some category. Given a height function h on a weak Kan complex X with unique fillers, we denote by $\mathscr{S}_h$ the corresponding category of the weak Kan complex.

Definition 4.7. Let X be a weak Kan complex with unique fillers, and let $h\colon X \to \mathbf{R}^{\mathrm{op}}$ be a strictly decreasing height function. The *simplicial flow category* of h, denoted $\mathscr{S}_h^c$, is the subcategory of $\mathscr{S}_h$ generated by the critical vertices of h.

Let X and h be as above. To show that the classifying space of $\mathscr{S}_h^c$ is homotopy equivalent to $|X|$, we will apply Quillen's theorem A to the inclusion functor

$$J\colon \mathscr{S}_h^c \to \mathscr{S}_h.$$

This will show that J induces a homotopy equivalence $|\mathcal{N}(\mathscr{S}_h^c)| \to |\mathcal{N}(\mathscr{S}_h)|$, and, as $X = \mathcal{N}(\mathscr{S}_h)$, this will give our desired result.

To state Quillen's theorem A, we must first define the *under-fiber* of a functor.

Definition 4.8. Let C and D be categories, $F : C \to D$ a functor, and $z \in D$ an object. The *fiber of F under z*, denoted $z \downarrow F$, is a category whose:

- objects are pairs (w, f), where $w \in C$ and $f \in \mathrm{Hom}_D(z, F(w))$, and
- morphisms from (w, f) to (w', f') are morphisms $h \in \mathrm{Hom}_C(w, w')$ such that $F(h) \circ f = f'$, i.e., such that the following diagram commutes.

$$\begin{array}{ccc} z & \xrightarrow{\quad f' \quad} \searrow & \\ {\scriptstyle f}\downarrow & & \\ F(w) & \xrightarrow{F(h)} & F(w') \end{array}$$

Theorem 4.9 (Quillen's theorem A)**.** [18, Theorem A] *Let C and D be categories, $F : C \to D$ a functor. If the classifying space $|\mathcal{N}(z \downarrow F)|$ of the fiber under z is contractible for each $z \in D$, then F induces a homotopy equivalence on classifying spaces $|\mathcal{N}(C)| \to |\mathcal{N}(D)|$.*

Theorem 4.10. *If $\mathscr{S}_h$ has a finite amount of objects, the inclusion functor $J : \mathscr{S}_h^c \to \mathscr{S}_h$ induces a homotopy equivalence on the classifying spaces $|\mathcal{N}(\mathscr{S}_h^c)| \to |\mathcal{N}(\mathscr{S}_h)|$.*

Proof. Denote by $(\mathscr{S}_h^c)^i$ the subcategory of $\mathscr{S}_h$ where the i highest-valued regular vertices have been removed. We will show that $J_i : (\mathscr{S}_h^c)^{i+1} \to (\mathscr{S}_h^c)^i$ induces a homotopy equivalence on classifying spaces, and the result will follow from induction.

We now show that $|\mathcal{N}(z \downarrow J_i)|$ is contractible for all $z \in (\mathscr{S}_h^c)^i$. Let $r \in X_0$ be the unique object in $(\mathscr{S}_h^c)^i$ not in $(\mathscr{S}_h^c)^{i+1}$.

First, suppose $z \neq r$. Then (z, id_z) is an initial object in $z \downarrow J_i$, so $|\mathcal{N}(z \downarrow J_i)|$ is contractible.

Now, suppose $z = r$. Then $\mathcal{N}(z \downarrow J_i)$ is precisely $\mathrm{llink}_h\, v$. To see this, observe that the objects in $z \downarrow J_i$ are 1–simplices v with $d_0 v = z$ and $d_1 v \neq z$, which are exactly the 0–simplices in $\mathrm{llink}_h\, v$. Morphisms in $z \downarrow J_i$ are commutative triangles

$$\begin{array}{ccc} z & \xrightarrow{\quad f' \quad} \searrow & \\ {\scriptstyle f}\downarrow & & \\ w & \xrightarrow{h} & w' \end{array}$$

which are precisely the 1–simplices in $\mathrm{llink}_h\, v$. Finally, a set of morphisms $h_i j : (w_i, f_i) \to (w_j, f_j)$, for $0 \leq i \leq j \leq n$, correspond to exactly one n–simplex in $\mathrm{llink}_h\, v$ due to the unique filling property of X. Thus, as r is regular, $|N(z \downarrow J_i)| = |\,\mathrm{llink}_h\, v|$ is contractible.

Finally, by Quillen's theorem A, J_i induces a homotopy equivalence on classifying spaces, which concludes the proof. □

The following corollary follows from the fact that $X = \mathcal{N}(\mathscr{S}_h)$.

Corollary 4.11. *Let X be a weak Kan complex with unique fillers, and let $h : X \to \mathbf{R}^{\mathrm{op}}$ be a strictly decreasing height function. If X_0 is a finite set, the classifying space of the simplicial flow category of h, $|\mathcal{N}(\mathscr{S}_h^c)|$, is homotopy equivalent to $|X|$.*

Example 4.12. Let X be a weak Kan complex with unique fillers and $h : X \to \mathbf{R}^{\mathrm{op}}$ a strictly decreasing height function as illustrated in Figure 9.

The critical vertices are the four vertices with value 0 and 2. Thus, $\mathscr{S}_h^c$ is as illustrated in Figure 10. Observe that $|\mathcal{N}(\mathscr{S}_h^c)| \simeq |X| \simeq S^1$.

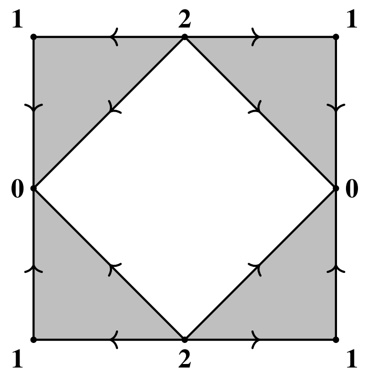


FIGURE 9. A strictly decreasing height function on a weak Kan complex with unique fillers.

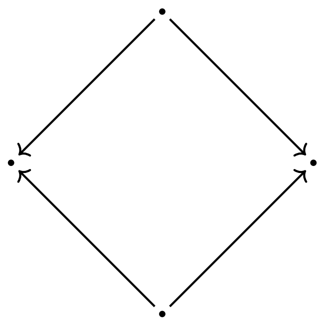


FIGURE 10. The simplicial flow category of the height function in Figure 9.

4.6. **Connections to category theory.** The lower link and the space of flow directions are closely connected to the under-fiber, which is used in the statement of Quillen's theorem A. The objects in the under-fiber of an object are morphisms out of that object. In the flow category, morphisms are compositions of flow lines, and in the discrete flow category and my simplicial flow category, they are gradient paths. In both my simplicial flow category and in Nanda's discrete flow category, Quillen's theorem A is used to prove that regular simplices or regular vertices can be removed from the category without changing the homotopy type of the classifying space.

5. P-CATEGORIES

A p-category is a category where the Hom sets are enriched with a poset structure (poset meaning partially ordered set). The partial orders have to be compatible with the composition rule, i.e., if $f \Rightarrow f'$ and $g \Rightarrow g'$ then $f \circ g \Rightarrow f' \circ g'$. Here we have used the symbol $\Rightarrow$ to denote a partial order relation between two morphisms, which we often will.

A *strict p-functor* $F : \boldsymbol{C} \to \boldsymbol{D}$ maps objects in $\boldsymbol{C}$ to objects in $\boldsymbol{D}$, and maps each Hom poset $\mathrm{Hom}_{\boldsymbol{C}}(x, y)$ monotonically to $\mathrm{Hom}_{\boldsymbol{D}}(F(x), F(y))$, such that

(1) $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for all objects x in $\boldsymbol{C}$, and
(2) $F(f \circ g) = F(f) \circ F(g)$ for all composable f and g.

A p-category can be viewed as a special case of a strict 2–category, where the 2–morphisms are partial order relations between morphisms. One can easily verify that this satisfies all the axioms of a strict 2–category. In particular, for a p-category viewed as a

2–category, there can only be zero or one 2–morphisms between two 1–morphism (either there is a partial order relation, or there is not).

5.1. **The geometric nerve of a p-category.** Nerves of p-categories, and more generally 2-categories, can be defined in many ways. One of them is the *geometric nerve*, denoted Δ, and is defined as follows.

- $(\Delta C)_0$ is the objects in C.
- An n–simplex, for $n \geq 1$, is a set of morphisms $\{f_{ij} : x_i \to x_{i+1} : 0 \leq i < j \leq n\}$, satisfying $f_{ik} \Rightarrow f_{jk} \circ f_{ij}$ for all i, j, k.

The face and degeneracy maps are defined just as for the nerve of an ordinary category.

The geometric nerve of a p-category can be very large, even though its realization has a simple homotopy type. As an example, consider a p-category $\mathscr{P}$ with two objects, a and b, two non-identity morphisms, $f : a \to b$ and $g : a \to b$, and a single non-identity partial order relation $f \Rightarrow g$ (see also Figure 11 for an illustration). We will compute the simplices of $\Delta\mathscr{P}$ in the first few dimensions.

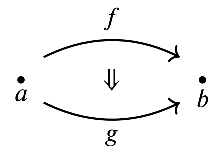
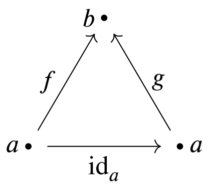
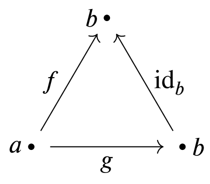


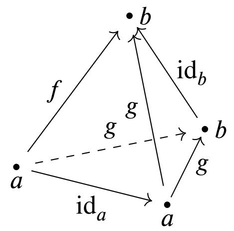
FIGURE 11. The p-category $\mathscr{P}$.

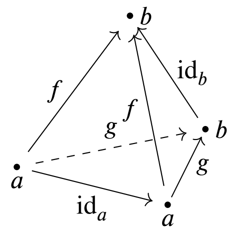
- The 0–simplices $(\Delta\mathscr{P})_0$ are the objects a and b.
- The non-degenerate 1–simplices are f and g.
- There are two non-degenerate 2–simplices, given by the following triangles.





- There are two non-degenerate 3–simplices, given by the following tetrahedrons.



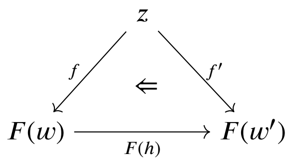


Thus, the nerve of $\mathscr{P}$ has many simplices in higher dimensions, despite $\mathscr{P}$ only having two morphisms and a single partial order relation. The homotopy type of the classifying space, however, is that of a point, as we will now show. For this, we will use Quillen's theorem A for p-categories.

First, we define the under-fiber of a strict p-functor.

Definition 5.1. Let C and D be p-categories, let $F : C \to D$ be a strict p-functor, and let z be an object in D. The *fiber of F under z* denoted $z \downarrow F$ is a p-category whose:

- objects are pairs (w, f), where $w \in C$ and $f \in \mathrm{Hom}_D(z, F(w))$, and
- morphisms from (w, f) to (w', f') are morphisms $h \in \mathrm{Hom}_C(w, w')$ satisfying $f' \Rightarrow F(h) \circ f$:



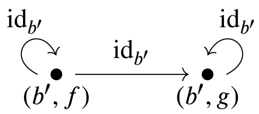
- The poset structure is inherited from C, i.e., $h \Rightarrow h'$ in $z \downarrow F$ iff $h \Rightarrow h'$ in C.

The following result is a generalization of Quillen's theorem A to p-categories, and follows directly from [2, Theorem 2].

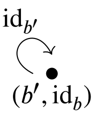
Theorem 5.2 (Quillen's theorem A for p-categories)**.** *Let C and D be p-categories, and $F : C \to D$ a strict p-functor. If the classifying space $|\Delta(z \downarrow F)|$ of the fiber under z is contractible for each $z \in D$, then F induces a homotopy equivalence on classifying spaces $|\Delta(C)| \to |\Delta(D)|$.*

Now let C' be a p-category consisting of a single object b' and a single morphism $\mathrm{id}_{b'}$. Let $J : C' \to C$ be the strict p-functor sending b' to b. Then the under-fibers of J are:

- $a \downarrow J =$



- $b \downarrow J =$



It's clear that $|\Delta(C')|$ is contractible. Then, as both $|\Delta(a \downarrow J)|$ and $|\Delta(b \downarrow J)|$ are contractible, the classifying space $|\Delta(C)|$ is contractible by Theorem 5.2.

5.2. **Simplicial categories.** A simplicial category is a simplicial object in Cat, i.e., a functor

$$\Delta^{\mathrm{op}} \to \mathsf{Cat}.$$

You can compose such a functor $F : \Delta^{\mathrm{op}} \to \mathsf{Cat}$ with the nerve functor $\mathcal{N} : \mathsf{Cat} \to \mathsf{sSet}$ to get a bisimplicial set $(\mathcal{N} \circ F)$. You can further take the diagonal of this bisimplicial set to get the simplicial set $\mathrm{diag}(\mathcal{N} \circ F)$.

Example 5.3. Consider the following simplicial category $\mathscr{S}$.

- $\mathscr{S}[0]$ is a category with two objects, a and b, and two non-identity morphisms, $f : a \to b$ and $g : a \to b$.
- $\mathscr{S}[1]$ consists of degeneracies of $\mathscr{S}[0]$ and a single non-degenerate morphism $F : s_0 a \to s_0 b$, with $d_1 F = f$ and $d_0 F = g$.
- For $n \geq 2$, all objects and morphisms in $\mathscr{S}[n]$ are degenerate.

This category can also be thought of as a *simplicially enriched* category with two objects a, b, such that $\mathrm{Hom}(a, b) \cong \Delta^1$.

Now, let's compute the simplicial set $\mathrm{diag}(\mathcal{N} \circ \mathscr{S})$.

- The 0–simplices are elements of $(\mathcal{N} \circ \mathscr{S})([0], [0])$, which are the objects a and b.
- The 1–simplices are elements of $(\mathcal{N} \circ \mathscr{S})([1], [1])$, which are morphisms in $\mathscr{S}[1]$. These are $F, s_0 f, s_0 g, s_0\,\mathrm{id}_a$ and $s_0\,\mathrm{id}_b$. Of these, the last two are degenerate as $s_0\,\mathrm{id}_a = (s_0, s_0)a$ and $s_0\,\mathrm{id}_b = (s_0, s_0)b$ (here (s_0, s_0) is a degeneracy map in the bisimplicial set $\mathcal{N} \circ \mathscr{S}$, while $s_0\,\mathrm{id}_a$ is the degeneracy of id_a in the simplicial category $\mathscr{S}$).

- The 2–simplices are 2–simplices in $\mathcal{N}(\mathscr{S}[2])$, i.e., pairs of composable morphisms in $\mathscr{S}[2]$. The morphisms in $\mathscr{S}[2]$ are all the degeneracies of the morphisms in $\mathscr{S}[1]$. There are thus many composable pairs, but only two are nondegenerate in $\operatorname{diag}(\mathcal{N}\circ\mathscr{S})$:
  - $\{s_0F, s_0s_0\,\mathrm{id}_b\} = (s_0, s_1)F$, and
  - $\{s_0s_0\,\mathrm{id}_a, s_1F\} = (s_1, s_0)F$.

  To see that these are nondegenerate, let's compute the face maps. First, for $(s_1, s_0)F$:
  - $(d_0, d_0)(s_1, s_0)F = (d_0s_1, d_0s_0)F = (s_0d_0, \mathrm{id})F = s_0d_0F = s_0g$.
  - $(d_1, d_1)(s_1, s_0)F = (d_1s_1, d_1s_0)F = (\mathrm{id}, \mathrm{id})F = F$.
  - $(d_2, d_2)(s_1, s_0)F = (d_2s_1, d_2s_0)F = (\mathrm{id}, s_0d_1)F = \mathrm{id}_{s_0a} = s_0\,\mathrm{id}_a$.

  Similar computations for $(s_0, s_1)F$ gives:
  - $(d_0, d_0)(s_0, s_1)F = s_0\,\mathrm{id}_b$.
  - $(d_1, d_1)(s_0, s_1)F = F$.
  - $(d_2, d_2)(s_0, s_1)F = s_0f$.
- All higher simplices are degenerate.

Finally, Figure 12 illustrates the simplicial set $\operatorname{diag}(\mathcal{N}\circ\mathscr{S})$ (omitting unimportant degenerate simplices). Observe that the geometric realization is a suspension of $|\Delta^1|$, i.e., homeomorphic to a disk D^2.

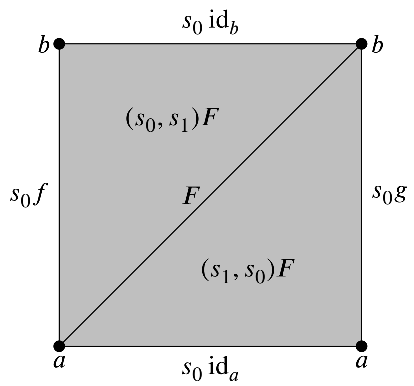


FIGURE 12. The simplicial set $\operatorname{diag}(\mathcal{N}\circ\mathscr{S})$. Note that the top and bottom edges are degenerate, as $s_0\,\mathrm{id}_a = (s_0, s_0)a$, and similar for $s_0\,\mathrm{id}_b$. The left and right edges are, however, not degenerate as 1–simplices in $\operatorname{diag}(\mathcal{N}\circ\mathscr{S})$.

5.3. **The double nerve of a p-category.** Given a p-category C, one can construct a simplicial category $\overline{\mathrm{N}}\,C$ as follows:

- The objects in $(\overline{\mathrm{N}}\,C)[0]$ are the objects in C.
- For $n \geq 1$, the objects in $(\overline{\mathrm{N}}\,C)[n]$ are degeneracies of the objects in $(\overline{\mathrm{N}}\,C)[0]$.
- The morphisms $a \to b$ in $(\overline{\mathrm{N}}\,C)[0]$ are the morphisms $a \to b$ in C.
- For $n \geq 1$, the morphisms $(s_0)^n a \to (s_0)^n b$ in $(\overline{\mathrm{N}}\,C)[n]$ are n–simplices in $\mathcal{N}\left(\mathrm{Hom}_C(a, b)\right)$ (where we consider the poset $\mathrm{Hom}_C(a, b)$ as an ordinary category and take its nerve).

  In other words, morphisms in $\mathrm{Hom}_{(\overline{\mathrm{N}}\,C)[n]}\left((s_0)^n a, (s_0)^n b\right)$ are sets of morphisms $\{f_0, \dots, f_n\}$ in $\mathrm{Hom}_C(a, b)$ such that

$$f_0 \Rightarrow f_1 \Rightarrow \cdots \Rightarrow f_n$$

  (here $(s_0)^n$ means s_0 applied n times).

  The face and degeneracy maps are as in $\mathcal{N}\left(\mathrm{Hom}_C(a, b)\right)$.

- The composition of $\{f_0, \dots, f_n\} : (s_0)^n a \to (s_0)^n b$ with $\{g_0, \dots, g_n\} : (s_0)^n b \to (s_0)^n c$ is $\{g_0 \circ f_0, \dots, g_n \circ f_n\}$.

It's easy to verify that $g_0 \circ f_0 \Rightarrow g_1 \circ f_1 \Rightarrow \cdots \Rightarrow g_n \circ f_n$, so that the composition rule is well-defined. It's also easily verified that this composition rule is associative, and that $\{\mathrm{id}_x, \dots, \mathrm{id}_x\}$ acts as the identity on $(s_0)^n x$. To show that $\overline{\mathrm{N}}\, C$ is a well-defined simplicial category, it remains to show that the face maps and degeneracy maps are functors between small categories. It's clear that the face map $d_i : (\overline{\mathrm{N}}\, C)[n] \to (\overline{\mathrm{N}}\, C)[n-1]$ sends a morphisms $\{f_0, \dots, f_n\} \in \mathrm{Hom}_{(\overline{\mathrm{N}}\, C)([n])}\left((s_0)^n a, (s_0)^n b\right)$ to $\mathrm{Hom}_{(\overline{\mathrm{N}}\, C)([n-1])}\left(d_i (s_0)^n a, d_i (s_0)^n b\right)$, as $d_i (s_0)^n = (s_0)^{n-1}$. Furthermore, d_i commutes with composition:

$$
\begin{aligned}
& d_i \left(\{g_0, \dots, g_n\} \circ \{f_0, \dots, f_n\}\right) \\
&\quad = d_i \{g_0 \circ f_0, \dots, g_n \circ f_n\} \\
&\quad = \{g_0 \circ f_0, \dots, g_{i-1} \circ f_{i-1}, g_{i+1} \circ f_{i+1}, \dots, g_n \circ f_n\} \\
&\quad = \{g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n\} \circ \{f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n\} \\
&\quad = \left(d_i \{g_0, \dots, g_n\}\right) \circ \left(d_i \{f_0, \dots, f_n\}\right)
\end{aligned}
$$

Hence, the face maps are functors. Similar computations for s_i gives that the degeneracy maps are functors. In conclusion, $\overline{\mathrm{N}}\, C$ is a well-defined simplicial category.

Example 5.4. Let's consider the p-category $\mathscr{P}$ from Section 5.1. Let $\overline{\mathrm{N}}\, \mathscr{P}$ be the corresponding simplicial category as described above. Then the objects in $(\overline{\mathrm{N}}\, \mathscr{P})[0]$ are a and b, while the morphisms in $(\overline{\mathrm{N}}\, \mathscr{P})[0]$ are f and g, together with identities. There is a single non-degenerate morphism in $(\overline{\mathrm{N}}\, \mathscr{P})[1]$, namely $F := \{f, g\}$, which has the faces $d_1 F = f$ and $d_0 F = g$. All other objects and morphisms are degenerate. Thus, $\overline{\mathrm{N}}\, \mathscr{P}$ is precisely the simplicial category $\mathscr{S}$ from Example 5.3.

Now, as mentioned in the previous section, we can concatenate the simplicial category $\overline{\mathrm{N}}\, C$ with the nerve operation $\mathcal{N}$ to get a bisimplicial set. We will call this composite operation the *double nerve*.

Definition 5.5. Let C be a p-category. The *double nerve* of C, denoted $\mathrm{N}\overline{\mathrm{N}}\, C$, is the bisimplicial set $\mathcal{N} \circ (\overline{\mathrm{N}}\, C)$.

The double nerve is also defined in [3] and [2]. Note that the intermediate simplicial category $\overline{\mathrm{N}}\, C$ ($\underline{\mathrm{N}}C$ in their papers) is defined differently there; there, the n–simplices are horizontal compositions, not vertical compositions. The double nerve, however, is the same in both definitions, except that the indices in the bisimplicial set are swapped.

In both [3] and [2], the following is shown.

Theorem 5.6. *Let C be a p-category. The (realizations of the) geometric nerve and the double nerve of C are homotopy equivalent, i.e.,*

$$|\Delta C| \simeq |\,\mathrm{N}\overline{\mathrm{N}}\, C|.$$

6. The discrete flow category

In this section we present the definition of the discrete flow category, as defined by Nanda in [17]. Note that we only define the discrete flow category of a discrete Morse function on a regular CW complex. For the general discrete flow category, see [17]. The following three subsections (6.1–6.3) was written as part of the project thesis preceding this thesis [9, Section 8.2–8.4], and is included here to provide the reader with the background needed for the following sections in this thesis.

6.1. **The entrance path category.** We now describe an example of a p-category, the *entrance path category* of a regular CW complex. It is similar to the face poset, but has more structure: it includes data on *how* a cell is a face of another cell.

Definition 6.1. Let X be a regular CW complex. The *entrance path category* $\mathrm{Ent}[X]$ of X is a p-category given by the following.

(1) The objects are the cells in X.
(2) The morphisms $x \to y$ are strictly descending sequences of cells ($x = x_0 > x_1 > \cdots > x_k = y$), with the understanding that id_x is the sequence (x) with a single element.
(3) The partial order is defined so that $f \Rightarrow f'$ if and only if f is a (not necessarily contiguous) subsequence of f'.
(4) Composition of morphisms are given by concatenating sequences as follows.

$$\big(z > x_1 > \cdots > x_k\big) \circ \big(y_0 > \cdots > y_{l-1} > z\big) = \big(y_0 > \cdots > y_{l-1} > z > x_1 > \ldots x_k\big)$$

A theorem, proved in [17, p. 11], states the following.

Theorem 6.2. *Let X be a finite regular CW complex. Then X is homotopy equivalent to $|\mathcal{N}(\mathrm{Ent}[X])|$.*

6.2. **Localization.** Given a category C and a collection of morphisms Q that contains all identities and is closed under composition, one can construct a new category $C[Q^{-1}]$, called the *localization* of C about Q. This category is the minimal category containing C where all morphisms in Q has inverses. The localization comes with a functor $L\colon C \to C[Q^{-1}]$ that sends objects in C to their copies in $C[Q^{-1}]$ and morphisms in C to their equivalence classes in $C[Q^{-1}]$.

We will now describe how to localize a p-category about a special class of morphisms.

Definition 6.3. A morphism $f\colon x \to y$ in a p-category C is called an *atom* if

(1) $f \Rightarrow f'$ holds for any $f' \in C(x, y)$,
(2) $x = y$ implies $f = \mathrm{id}_x$, and
(3) Solutions to $h \circ g \Rightarrow f$ for morphisms $g\colon x \to z$ and $h\colon z \to y$ only exist when
   - $z = x$, in which case $(g, h) = (\mathrm{id}_x, f)$, or
   - $z = y$, in which case $(g, h) = (f, \mathrm{id}_y)$.

As an example, in the entrance path category of a regular CW complex, the atoms are precisely the sequences of length 2, i.e., those on the form $(x > y)$, together with the identities.

Definition 6.4. A collection Σ of morphisms in a p-category is called *directed* if

(1) all morphisms in Σ are atoms,
(2) if $f\colon x \to y$ is in Σ, then $x \neq y$, and
(3) if $f\colon x \to y$ is in Σ, then there are no morphisms $y \to x$ in Σ.

We again give an example in the entrance path category. Let $f\colon X \to \mathbb{R}$ be a discrete Morse function. The gradient vector field $V_f = \{(x_i, y_i)\}$ gives a collection $\Sigma = \{(y_i > x_i)\}$ of morphisms in $\mathrm{Ent}[X]$. It's easily verified that Σ is directed.

Now, given a p-category C and a directed collection Σ of morphisms in C so that the union Σ^+ with all identities is closed under composition, we can define the *localization of C about* Σ. We write this as $\mathrm{Loc}_\Sigma C$, and define it as follows.

Definition 6.5. Let C, Σ and Σ^+ be as above. Then $\mathrm{Loc}_\Sigma C$ is a p-category given by the following.

- The objects are the same as the objects in C.

- The morphisms $w \to z$ are equivalence classes of zigzags of the form
$$w \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_0 \xrightarrow{g_1} y_1 \xleftarrow{f_1} \cdots \xleftarrow{f_k} x_k \xrightarrow{g_{k+1}} z$$
where the f_i's are in Σ^+, and the g_i's are arbitrary, and the equivalence relation is generated by the following relations. Two zigzags are related *horizontally* if they differ by intermediate identity maps, and *vertically* if they form the rows of a commutative diagram of the form
$$\begin{array}{ccccccccccccc}
w & \xrightarrow{g_0} & y_0 & \xleftarrow{f_0} & x_0 & \xrightarrow{g_1} & y_1 & \xleftarrow{f_1} & \cdots & \xleftarrow{f_k} & x_k & \xrightarrow{g_{k+1}} & z \\
{\scriptstyle \mathrm{id}_w}\downarrow & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle v_k} & & \downarrow{\scriptstyle \mathrm{id}_z} \\
w & \xrightarrow{g'_0} & y'_0 & \xleftarrow{f'_0} & x'_0 & \xrightarrow{g'_1} & y'_1 & \xleftarrow{f'_1} & \cdots & \xleftarrow{f'_k} & x'_k & \xrightarrow{g'_{k+1}} & z
\end{array}$$
where the u_i's and v_i's are in Σ^+.
- The partial order is defined so that $\gamma' \Rightarrow \gamma$ if and only if there exist representatives of γ and γ' that fit into the top and bottom row, respectively, of a (not necessarily commutative) diagram of the form
$$\begin{array}{ccccccccccccc}
w & \xrightarrow{g_0} & y_0 & \xleftarrow{f_0} & x_0 & \xrightarrow{g_1} & y_1 & \xleftarrow{f_1} & \cdots & \xleftarrow{f_k} & x_k & \xrightarrow{g_{k+1}} & z \\
{\scriptstyle \mathrm{id}_w}\downarrow & \Nearrow & \downarrow{\scriptstyle u_0} & \Nwarrow & \downarrow{\scriptstyle v_0} & \Nearrow & \downarrow{\scriptstyle u_1} & \Nwarrow & & \Nwarrow & \downarrow{\scriptstyle v_k} & \Nearrow & \downarrow{\scriptstyle \mathrm{id}_z} \\
w & \xrightarrow{g'_0} & y'_0 & \xleftarrow{f'_0} & x'_0 & \xrightarrow{g'_1} & y'_1 & \xleftarrow{f'_1} & \cdots & \xleftarrow{f'_k} & x'_k & \xrightarrow{g'_{k+1}} & z
\end{array}$$
where, again, u_i and v_i are in Σ^+.
- Composition of morphisms are given by concatenating representatives as follows
$$\left[w \xrightarrow{g_0} y_0 \xleftarrow{f_0} \cdots \xleftarrow{f_k} x_k \xrightarrow{g_{k+1}} z \right] \circ \left[v \xrightarrow{g'_0} y'_0 \xleftarrow{f'_0} \cdots \xleftarrow{f'_k} x'_k \xrightarrow{g'_{k+1}} w \right]$$
$$= \left[v \xrightarrow{g'_0} y'_0 \xleftarrow{f'_0} \cdots \xleftarrow{f'_k} x'_k \xrightarrow{g_0 \circ g'_{k+1}} y_0 \xleftarrow{f_0} \cdots \xleftarrow{f_k} x_k \xrightarrow{g_{k+1}} z \right].$$

One also gets a functor (or equivalently, a strict p-functor) in this case, which we write $L_\Sigma : C \to \mathrm{Loc}_\Sigma C$. The functor sends objects to themselves and morphisms to their respective equivalence classes.

As mentioned above, a gradient vector field $V_f = \{(x_i, y_i)\}$ gives a directed collection $\Sigma = \{(y_i > x_i)\}$ on the entrance path category. Thus, given a discrete Morse function $f : X \to \mathbb{R}$, we can localize $\mathrm{Ent}[X]$ about Σ, which will give us a category where the morphisms corresponding to regular pairs have inverses. We will call this Σ the *Morse system* induced by f.

6.3. **The discrete flow category.**

We are now ready to define the discrete flow category of a discrete Morse function.

Definition 6.6. Let $f : X \to \mathbb{R}$ be a discrete Morse function, and let Σ be its induced Morse system. Let $\mathrm{Loc}_\Sigma[X]$ be the p-category localization of $\mathrm{Ent}[X]$ about Σ. The *discrete flow category* of f, denoted $\mathrm{Flo}_\Sigma[X]$, is the full subcategory of $\mathrm{Loc}_\Sigma[X]$ generated by the critical cells of f.

A special case of the main result of [17] states the following.

Theorem 6.7. *Let $f : X \to \mathbb{R}$ be a discrete Morse function, and let Σ be its induced Morse system. Then the classifying space $|\mathcal{N}(\mathrm{Flo}_\Sigma[X])|$ of the discrete flow category of f is homotopy equivalent to X.*

Example 6.8. As an example, consider the discrete Morse function on $X \cong S^1$ illustrated in Figure 13.

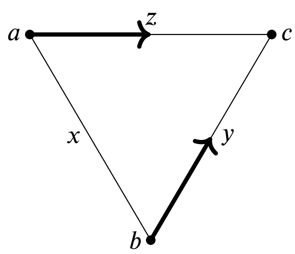


FIGURE 13. A gradient vector field of a discrete Morse function $f : X \to \mathbb{R}$.

The objects of $\mathrm{Flo}_\Sigma[X]$ are the critical cells of f, which are x and c. There are two morphisms from x to c, corresponding to the two zigzag paths $x > b < y > c$ and $x > a < z > c$. There are no partial order relation between these morphisms. The other Hom posets are $\mathrm{Hom}(x, x) = \{\mathrm{id}_x\}$, $\mathrm{Hom}(c, c) = \{\mathrm{id}_c\}$ and $\mathrm{Hom}(c, x) = \emptyset$. Thus, the classifying space $|\mathcal{N}(\mathrm{Flo}_\Sigma[X])|$ becomes S^1, as illustrated in Figure 14.

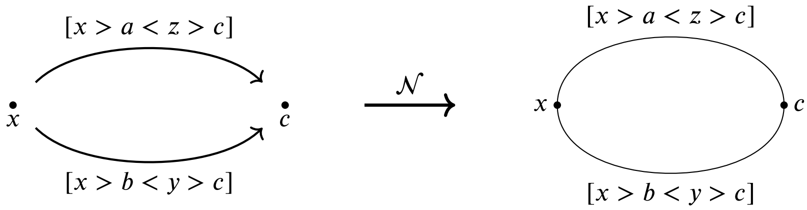


FIGURE 14. The discrete flow category and its nerve (the identity morphisms are omitted).

7. HOM POSETS OF DISCRETE FLOW CATEGORIES

In this section we study the structure of the Hom posets of the discrete flow category of a discrete Morse function. We first develop some technical results on the Hom posets, and in particular show that the posets are *graded*. We then use these results to construct an algorithm to compute said posets. Finally, we prove Theorem A: for a discrete Morse function on a simplicial complex X, the opposite poset of each Hom poset in its discrete flow category is a *CW poset*, i.e., it has the structure of a face poset of a regular CW complex.

7.1. **CW posets.** In [1], Björner describes sufficient conditions for a poset to be the face poset of a regular CW complex. Note that he defines face posets to have an added least element $\hat{0}$, as opposed to our definition of $\mathrm{Fac}[X]$. Björner's definition of CW posets is as follows.

Definition 7.1. A poset P is said to be a *CW poset* if

(1) P has a least element $\hat{0}$,
(2) P is nontrivial, i.e., has more than one element,
(3) For all $x \in P \setminus \{\hat{0}\}$ the realization of the open interval $(\hat{0}, x)$ is homeomorphic to a sphere (i.e., to some S^k, where k depends on x).

We now formalize the statement that CW posets are the face posets of regular CW complexes (augmented with a least element).

Definition 7.2. Let X be a regular CW complex. Let the *face poset* of X, denoted $\mathrm{Fac}[X]$, be the poset where the elements are the cells in X and $x \geq y$ whenever y is contained in x.

Denote by $\mathrm{Fac}^+[X]$ the face poset $\mathrm{Fac}[X]$ augmented with a least element $\hat{0}$. In other words, $\mathrm{Fac}^+[X] = \mathrm{Fac}[X] \cup \{\hat{0}\}$, where $\hat{0}$ is defined to be smaller than all other elements.

The following statement is proved in [1, p. 11].

Theorem 7.3. *A poset P is a CW poset if and only if it is isomorphic to* $\mathrm{Fac}^+[X]$ *for some regular CW complex X.*

7.2. **Graded posets.** In this section we define what it means for a poset to be graded. We will say that a poset is *bounded* if it has a least and greatest element. Furthermore, we say that x *covers* y if $x > y$ and there exists no element z such that $x > z > y$.

Definition 7.4. A graded poset is a poset P equipped with a *rank function*, which is a function $\rho : P \to \mathbb{Z}$ satisfying the following two properties.

(1) The function ρ is compatible with the partial order, meaning that $x > y$ implies $\rho(x) > \rho(y)$.
(2) The function ρ is compatible with the covering relation, meaning that if x covers y then $\rho(x) = \rho(y) + 1$.

For an element x, we will call $\rho(x)$ the *rank* of x.

7.3. **Hom posets of a discrete flow category.** In this section, we let X be a finite regular CW complex, $f : X \to \mathbb{R}$ a discrete Morse function, and Σ the Morse system on $\mathrm{Ent}[X]$ consisting of the regular pairs of f. Furthermore, we let w and z be arbitrary objects in $\mathrm{Flo}_\Sigma[X]$ and consider the Hom poset $\mathrm{Hom}(w, z)$ (note that whenever we write $\mathrm{Hom}(w, z)$, we will mean $\mathrm{Hom}_{\mathrm{Flo}_\Sigma[X]}(w, z)$).

First, we construct an algebraic invariant for morphisms in the discrete flow category, which we will use when proving that two different morphisms are not equal.

Let G be the free Abelian group on all morphisms in $\mathrm{Ent}[X]$ that are atoms, modulo all identities (i.e., we define identities to be 0). The algebraic invariant will be an element of this group. Given a representative

$$\gamma = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right)$$

of some $[\gamma] \in \mathrm{Hom}(w, z)$, we let

$$\alpha(g_i) = \left(g_i^0 + g_i^1 + \cdots + g_i^{N_i} \right) \in G,$$

where $g_i = g_i^{N_i} \circ \ldots \circ g_i^0$ is the atom decomposition of g_i. We now define the algebraic invariant I as follows.

Definition 7.5. Let $[\gamma] \in \mathrm{Hom}(w, z)$. Define $I([\gamma])$ as

$$I([\gamma]) = \sum_{i=0}^{k} \alpha(g_i) - \sum_{i=0}^{k-1} f_i. \tag{2}$$

Lemma 7.6. *The function I, as defined in Definition 7.5, is well-defined.*

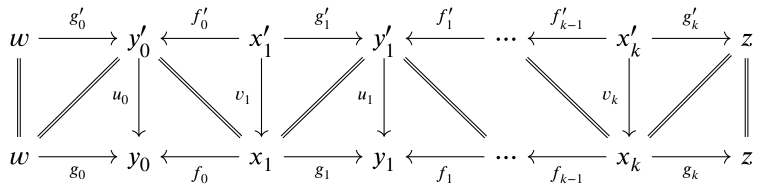
Proof. To show that I is well-defined, we must show that it is preserved under horizontal and vertical relations.

(H) To see that it is preserved under horizontal relations, it is enough to check that $I([\gamma \circ \gamma']) = I([\gamma]) + I([\gamma'])$, and that

$$I([x_i \xrightarrow{g_i} y_i \xleftarrow{\mathrm{id}} x_{i+1} \xrightarrow{g_{i+1}} y_{i+1}]) = I([x_i \xrightarrow{g_{i+1} \circ g_i} y_{i+1}]),$$

which follows from the fact that identities are defined to be 0 in G, and that $\alpha(g \circ h) = \alpha(g) + \alpha(h)$.

(V) To see that I is preserved under vertical relations, consider a diagram:



We have $\alpha(g_i') = \alpha(g_i) + v_i - u_i$, and $f_i' = f_i + u_i - v_{i+1}$ (as elements of G). Putting this together gives

$$\begin{aligned} I([\gamma]) &= \sum_{i=0}^{k} \alpha(g_i') - \sum_{i=0}^{k-1} f_i' \\ &= \sum_{i=0}^{k} \alpha(g_i) + \sum_{i=0}^{k} v_i - \sum_{i=0}^{k} u_i - \left(\sum_{i=0}^{k-1} f_i + \sum_{i=0}^{k-1} u_i - \sum_{i=0}^{k-1} v_{i+1} \right) \\ &= \sum_{i=0}^{k} \alpha(g_i) - \sum_{i=0}^{k-1} f_i, \end{aligned}$$

where we use that $v_0 = \mathrm{id}_w = 0$ and $u_k = \mathrm{id}_z = 0$. Hence, I is preserved under vertical relations. □

Theorem 7.7. *Let $[\gamma]$ and $[\gamma']$ be morphisms in* $\mathrm{Hom}(w, z)$ *such that $[\gamma] \Rightarrow [\gamma']$. Let*

$$\tau = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right)$$

be a representative of $[\gamma]$. Then it is possible to choose a representative τ' of $[\gamma']$, such that

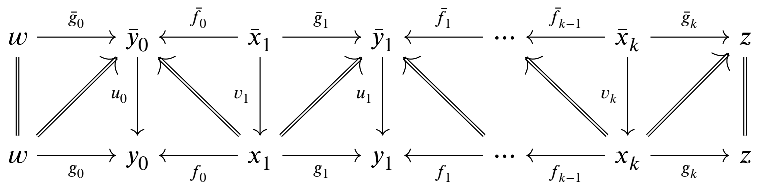
$$\tau' = \left(w = x_0 \xrightarrow{g_0'} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1'} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k'} y_k = z \right),$$

for some g_i', such that $g_i \Rightarrow g_i'$ holds for all i.

Furthermore, the g_i' are unique.

Note that a partial order $g_i \Rightarrow g_i'$ means that $g_i = (z_0 > z_1 > \cdots > z_m)$ is a subsequence of $g_i' = (z_0' > z_1' > \cdots > z_n')$. Therefore, this theorem tells us that a partial order $[\gamma] \Rightarrow [\gamma']$ corresponds to picking a representative of $[\gamma]$ and adding elements to the sequences that constitutes the right-pointing arrows (the g_i).

Proof. We prove this in two parts. First we show that there exists a diagram



(3)

where the bottom row is τ and the top row is a representative of $[\gamma']$. Then we show that we can modify this diagram so that the top row is of the form

$$\tau' = \left(w = x_0 \xrightarrow{g_0'} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1'} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k'} y_k = z \right),$$

where τ' is still a representative of $[\gamma']$.

To show the first part, we show that given a diagram representing a partial order relation (as the one in (3)), we can replace the bottom row by either

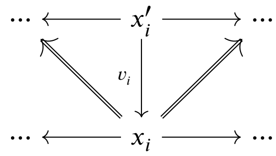
(V) a vertically related representative, or
(H) a horizontally related representative,

and modify the top row appropriately to get a new diagram.

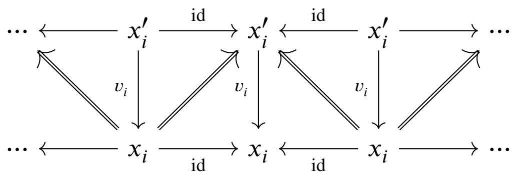
(V) The case with vertically related representatives is simple; you simply compose the partial order diagram with the vertical relation diagram.

(H) For horizontal relations, we must show that you can both add intermediate identity maps, and remove them.

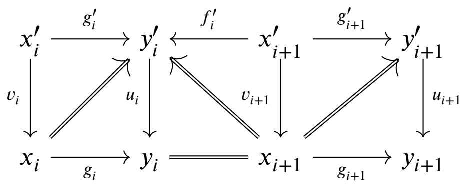
The case with adding intermediate identity maps is also simple, you simply add identity maps to both the top and bottom rows and copy the vertical map (clearly, the top rows is also horizontally related). For example, for adding an identity map at an x_i, you replace



with



Removing intermediate identity maps is more complicated (in fact, it only works on the bottom row, not the top row). Now, given a diagram

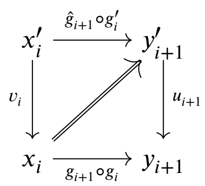


If $f'_i = \mathrm{id}$, we can clearly remove the identity maps from both rows. Suppose not. Then u_i must be id and $v_{i+1} = f'_i$. As $g_{i+1} \circ v_{i+1} \Rightarrow u_{i+1} \circ g'_{i+1}$, and v_{i+1} is an atom on the form $(x'_{i+1} > x_{i+1})$ with $\dim x'_{i+1} = \dim x_{i+1} + 1$, we must have that $v_{i+1} \circ g'_{i+1}$ is sequence that starts with $(x'_1 > x_1)$. There are now two cases: either $g'_{i+1} = \hat{g}_{i+1} \circ v_{i+1}$ for some $\hat{g}_{i+1}$, or $g'_{i+1} = \mathrm{id}$ and $u_{i+1} = v_{i+1}$.

In the first case, the top row is equivalent to

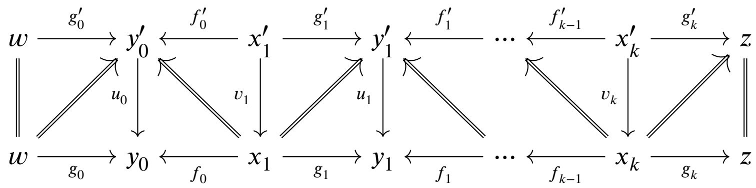
$$x'_i \xrightarrow{\hat{g}_{i+1} \circ g'_i} y_{i+1},$$

as $g'_{i+1} = \hat{g}_i \circ v_{i+1} = \hat{g}_i \circ f'_i$, and we can replace the diagram with



The reader can verify that in the case $g'_{i+1} = \mathrm{id}$, f'_{i+1} will be forced to equal id, which allows us to make a similar rewrite to remove the identity map from the bottom row.

Now, we have a diagram



such that the bottom row is a $[\tau]$ and the top row is a representative of $[\gamma']$.

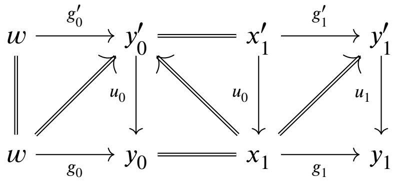
Assume either $x_1 \neq x'_1$ or $y_0 \neq y'_0$. There are three cases to consider:

(I) $x_1 \neq x'_1$ and $y_0 \neq y'_0$

(II) $x_1 = x'_1$ and $y_0 \neq y'_0$

(III) $x_1 \neq x'_1$ and $y_0 = y'_0$.

Case I In this case, the exhaustion axiom gives that $y_0 = x_1$ and $y'_0 = x'_1$, which implies that $v_1 = u_0$. The first three squares in the diagram is then:

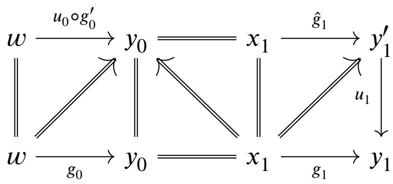


As $g_1 \circ u_0 \Rightarrow u_1 \circ g'_1$, and u_0 is an atom on the form $(x'_1 > x_1)$ with $\dim x'_1 = \dim x_1 + 1$, we must have that $u_1 \circ g'_1$ is sequence that starts with $(x'_1 > x_1)$. There are now two cases: either $g'_1 = \hat{g}_1 \circ u_0$ for some $\hat{g}_1$, or $g'_1 = \mathrm{id}_{x'_1}$ and $u_1 = u_0 = g_1$.

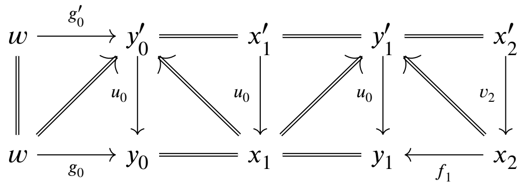
In the first case, the top row is vertically related to

$$w \xrightarrow{u_0 \circ g'_0} y_0 \xleftarrow{\mathrm{id}} x_1 \xrightarrow{\hat{g}_1} y'_1,$$

so we can replace the diagram with:



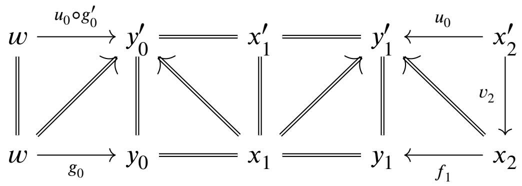
In the second case, we must have $g_1 = \mathrm{id}$ and $u_1 = \mathrm{id}$. Hence, the first four squares in the diagram:



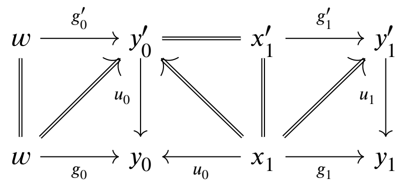
Now, the top row is vertically related to

$$w \xrightarrow{u_0 \circ g'_0} y_0 \xleftarrow{\mathrm{id}} x_1 \xrightarrow{\mathrm{id}} y'_1 \xleftarrow{u_0} x'_2,$$

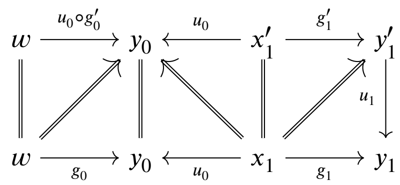
so we can replace the diagram with:



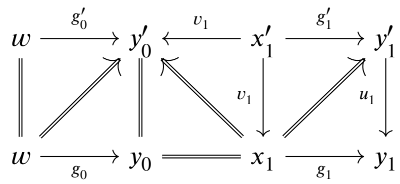
Case II In this case the exhaustion axiom gives $y'_0 = x'_1$ and $f_0 = u_0$, so the first three squares of the diagram is:



The top row is vertically related to $x_0 \xrightarrow{u_0 \circ g'_0} y_0 \xleftarrow{u_0} x_1 \xrightarrow{g_1} y'_1$ (see [17, Remark 2.9]). We can thus replace this part of the diagram with:



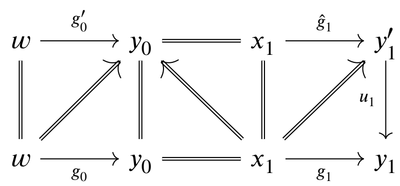
Case III In this case the exhaustion axiom gives $y_0 = x_1$ and $f'_0 = v_1$, so the first three squares of the diagram is:



As in the first case, either $g'_1 = \mathrm{id}$ or $g'_1 = \hat{g}_1 \circ v_1$ for some $\hat{g}_1$. In the latter case, the top row is vertically related to

$$w \xrightarrow{g'_0} y'_0 \xleftarrow{\mathrm{id}} x_1 \xrightarrow{\hat{g}_1} y'_1,$$

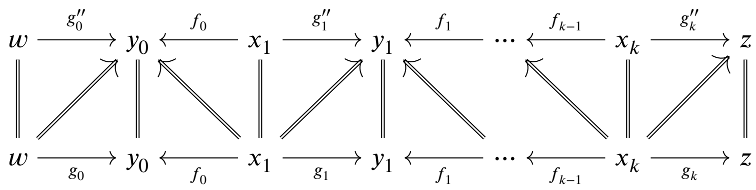
and we can replace the diagram with



If, instead, $g'_1 = \mathrm{id}$, we must have $g_1 = \mathrm{id}$ and $u_1 = v_1$. As in case 1, we look at the first four squares of the diagram, and see that we can replace the top row, in this case with

$$w \xrightarrow{g'_0} y'_0 \xleftarrow{\mathrm{id}} x_1 \xrightarrow{\mathrm{id}} y_1 \xleftarrow{v_1} x'_2.$$

Continuing this process on the right until you reach the end of the diagram, you end up with a new diagram on the form



where the and bottom row is τ and the top row is a representative of $[\gamma']$. It follows from the diagram that $g_i \Rightarrow g_i''$ for all i, and hence the top row is our desired representative τ'.

Finally, we prove that the g_i' are unique. We do this by using the algebraic invariant I, as defined in Definition 7.5, and showing that given I, the g_i' are decided by the x_i, y_i and f_i.

We know that I is an algebraic invariant for morphisms in $\mathrm{Hom}(w, z)$. Thus, given a representation τ' as in the theorem statement, where the x_i, y_i and f_i are given, we know what all the atoms in the g_i' are. It remains to show that these atoms can only appear in one specific order, which will mean that all the g_i' are uniquely defined.

For this, recall that $f : X \to \mathbb{R}$ is the Morse function from which the Morse system is induced. We can assume without loss of generality that f is injective on the cells of X. Now, for an atom $(x > y)$, let $\beta((x > y)) = (\dim x, f(x), f(y)) \in \mathbb{R}^3$. Then $\beta((x > y)) = \beta((x' > y'))$ if and only if $(x > y) = (x' > y')$ (as f is injective). Furthermore, using the lexicographical ordering on $\mathbb{R}^3$, β is decreasing on the atoms in the right-pointing arrows of a representation, in the order they appear:

- If $(x' > y')$ directly succeeds $(x > y)$ (possibly with an intermediate identity maps), then $x' = y$, and $\dim x' > \dim y' = \dim x > \dim y$.
- If there is a non-identity left-pointing map f_i directly between two atoms $(x > y)$ and $(x' > y')$, then $f_i = (x' > y)$, and $\dim x = \dim x'$. Then, we have $f(y) > f(x')$, and either $x = x'$ or $f(x) > f(y)$, so $f(x) \geq f(x')$. Likewise, either $y = y'$ or $f(x') > f(y')$ so $f(y) \geq f(y')$. In conclusion, this gives $\beta((x > y)) \geq \beta((x' > y'))$.

Thus, given the x_i, y_i and f_i in a representation τ' as in the theorem statement, the atoms in the g_i', and their order, is uniquely determined, and hence the g_i' themselves are uniquely determined, which proves the last part of the theorem. □

Now, let $\mathrm{Hom}(w, z)^{\mathrm{op}}$ be the *opposite* of the poset $\mathrm{Hom}(w, z)$, i.e., the poset with elements equal to the elements of $\mathrm{Hom}(w, z)$ and with the partial order reversed. Define further $P_{w,z}$ as $\mathrm{Hom}(w, z)^{\mathrm{op}}$ augmented with a least element $\hat{0}$. In other words,

$$P_{w,z} = \mathrm{Hom}(w, z)^{\mathrm{op}} \cup \{\hat{0}\}, \tag{4}$$

where $\hat{0}$ is defined to be smaller than all other elements.

Next, we show that $P_{w,z}$ is graded by defining a rank function on it. To do this, we first define the following rank function r on $\mathrm{Hom}_{\mathrm{Ent}[X]}(a, b)^{\mathrm{op}}$, for arbitrary objects $a, b \in \mathrm{Ent}[X]$.

$$r\left(a = x_0 > x_1 > \cdots > x_k = b\right) = k \tag{5}$$

Here, we take $r(\mathrm{id}_x)$ to be 0.

It's easy to verify that r is, in fact, a rank function. Also, observe that r satisfies the following identity.

$$r(\gamma \circ \gamma') = r(\gamma) + r(\gamma') \tag{6}$$

We now define a rank function on $P_{w,z}$.

Definition 7.8. Define the function $\rho\colon P_{w,z} \to \mathbb{Z}$ as follows.

Let $x \in P_{w,z}$. If $x = \hat{0}$, then $\rho(x) = -1$. Else, let x be represented by

$$\tau = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right).$$

Then,

$$\rho(x) = N + \sum_{i=0}^{k-1} r(f_i) - \sum_{i=0}^{k} r(g_i),$$

where $N = \dim w - \dim z$.

The rank function ρ can be interpreted as the number of cells that are "skipped" in the g_i's in the representative.

Theorem 7.9. *The function ρ, as defined in Definition 7.8, is well-defined and a rank function.*

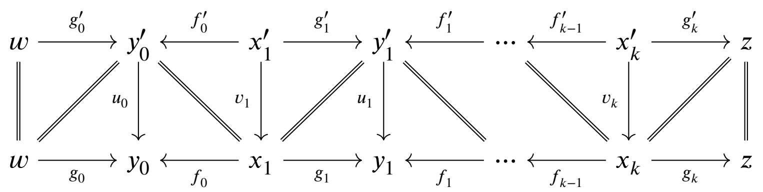
Proof. First, to prove well-definedness of ρ, we need to show that equivalent representations give the same value of ρ. To do this, we first show that horizontally related representations give the same value of ρ, and then we show that vertically related representations give the same value of ρ.

(1) Let τ and τ' be horizontally related representations. Then τ can be transformed to τ' by making substitutions of the form

$$\left(x_i \xrightarrow{g_i} y_i \xleftarrow{\mathrm{id}_{y_i}} y_i \xrightarrow{g_{i+1}} y_{i+1} \right) \leftrightarrow \left(x_i \xrightarrow{g_{i+1} \circ g_i} y_{i+1} \right).$$

As $r(g_{i+1} \circ g_i) = r(g_{i+1}) + r(g_i)$ and $r(\mathrm{id}_{y_i}) = 0$, these substitutions preserve the value of ρ, so both τ and τ' give the same value of ρ.

(2) Let τ and τ' be vertically related representations. Then they form the bottom and top rows of a commutative diagram:



We adopt the notation that $v_0 = \mathrm{id}_w$ and $u_k = \mathrm{id}_z$. Then $r(v_0) = r(u_k) = 0$. Now, commutativity of the diagram gives that $r(u_i) + r(f_i') = r(u_i \circ f_i') = r(f_i \circ v_{i+1}) = r(f_i) + r(v_{i+1})$. Similarly, $r(u_i) + r(g_i') = r(g_i) + r(v_i)$. Applying this, we get

$$\begin{aligned} &N + \sum_{i=0}^{k-1} r(f_i') - \sum_{i=0}^{k} r(g_i') \\ &= N + \sum_{i=0}^{k-1} \big(r(f_i) + r(u_i) - r(v_{i+1}) \big) - \sum_{i=0}^{k} \big(r(g_i) + r(u_i) - r(v_i) \big) \\ &= N + r(u_k) - r(v_0) + \sum_{i=0}^{k-1} r(f_i) - \sum_{i=0}^{k} r(g_i) \\ &= N + \sum_{i=0}^{k-1} r(f_i) - \sum_{i=0}^{k} r(g_i). \end{aligned}$$

Thus, τ and τ' give the same value of ρ.

Now, we prove that ρ is a rank function. To prove this, we must prove that it satisfies the two conditions in Definition 7.4.

(1) We prove that ρ is compatible with the partial order.

First, let x and y be elements in $P_{w,z}$ different from $\hat{0}$ such that $x > y$. Then $x \Rightarrow y$ when considered as elements of $\mathrm{Hom}(w, z)$. We can thus pick representatives τ and τ' as in Theorem 7.7, and we get that

$$\rho(x) - \rho(y) = \sum_{i=0}^{k} r(g_i') - \sum_{i=0}^{k} r(g_i) = \sum_{i=0}^{k} \left(r(g_i') - r(g_i)\right) > 0,$$

where the last inequality follows from the fact that $g_i \Rightarrow g_i'$ and that $g_i \neq g_i'$ for at least one i (otherwise, x would equal y).

Now, we must show that $\rho(x) > \rho(\hat{0})$ for $x \neq 0$. For this, let

$$\tau = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right)$$

be a representative of x. Observe that $r(g_i) \leq \dim x_i - \dim y_i$. Furthermore, $r(f_i) = \dim x_{i+1} - \dim y_i$ (as $\dim x_{i+1} - \dim y_i$ is either 0 or 1, due to the fact that the Morse system Σ consists of regular pairs of a discrete Morse function). Hence,

$$\begin{aligned} \rho(x) =& N + \sum_{i=0}^{k-1} r(f_i) - \sum_{i=0}^{k} r(g_i) \\ \geq& N + \sum_{i=0}^{k-1} \left(\dim x_{i+1} - \dim y_i\right) - \sum_{i=0}^{k} \left(\dim x_i - \dim y_i\right) \\ =& N + \dim y_k - \dim x_0 = N + \dim z - \dim w = 0. \end{aligned} \tag{7}$$

Thus, $\rho(x) \geq 0 > -1 = \rho(\hat{0})$, as desired.

(2) We prove that ρ is compatible with the covering relation.

First, let x and y be elements in $P_{w,z}$ different from $\hat{0}$ such that x covers y. We again choose representatives τ and τ' as in Theorem 7.7. It follows that $g_i = g_i'$ for all but one i, or else we could choose a representative γ so that $x > [\gamma] > y$. Similarly, it follows that for this i, g_i' covers g_i, or else we could find a $\hat{g}$ such that $g_i' > \hat{g} > g_i$, which again would let us choose a representative γ so that $x > [\gamma] > y$. As g_i' covers g_i, we have that $r(g_i') = r(g_i) + 1$, and thus

$$\rho(x) - \rho(y) = \sum_{i=0}^{k} r(g_i') - \sum_{i=0}^{k} r(g_i) = 1,$$

as desired.

Now, we must show that if x covers $\hat{0}$, then $\rho(x) = \rho(\hat{0}) + 1 = 0$. An element x covers $\hat{0}$ if and only if there are no $z \in \mathrm{Hom}(w, z)$ such that $x \Rightarrow z$. Let the following be a representative of such a x.

$$\tau = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right)$$

Then, all g_i must be maximal in $\mathrm{Hom}_{\mathrm{Ent}[X]}(x_i, y_i)$, or else we could replace this g_i to get a representative τ' with $[\tau] \Rightarrow [\tau']$. It is easy maximal elements in $\mathrm{Hom}_{\mathrm{Ent}[X]}(x_i, y_i)$ are sequences on the form $(x_i = a_0 > a_1 > \cdots > a_m = y_i)$ with $m = \dim x_i - \dim y_i$, and hence $r(g_i) = \dim x_i - \dim y_i$. We get that the inequality in (7) is an equality in this case, and $\rho(x) = 0$, as desired. □

7.4. **Computing the Hom posets.** In this section, we describe an algorithm for computing the Hom posets in the discrete flow category. We still operate in the case of a Morse system induced by a discrete Morse function on a regular CW complex, and use the same notation as the previous section. We first state some results that are needed for the algorithm.

First, we prove a useful lemma. Recall that the regular pairs of a discrete Morse function are the pairs of simplices (x, y), where $x \in X^k$ and $y \in X^{k+1}$, such that $x < y$ and $f(x) \geq f(y)$ (the set of these is the induced gradient vector field, V_f). The following lemma essentially tells us that we can always assume, without loss of generality, that this inequality is an equality for all regular pairs.

Lemma 7.10. *Let $f : X \to \mathbb{R}$ be a discrete Morse function. There exists a Forman equivalent discrete Morse function $\tilde{f}$ such that $\tilde{f}(x) = \tilde{f}(y)$ for each regular pair (x, y) of f.*

Proof. Define $\tilde{f} : X \to \mathbb{R}$ as follows.

$$\tilde{f}(x) = \begin{cases} f(y), & \text{if } (x, y) \in V_f \text{ for some } y, \\ f(x), & \text{otherwise.} \end{cases}$$

It's clear that $\tilde{f}(x) = \tilde{f}(y)$ for all regular pairs of f, so it remains to show that $\tilde{f}$ is a discrete Morse function, and that $\tilde{f}$ and f are Forman equivalent. We prove both of these things by showing that their induced gradient vector fields, $V_{\tilde{f}}$ and V_f are the same.

Clearly, $V_f \subseteq V_{\tilde{f}}$. We need to show that $V_{\tilde{f}} \subseteq V_f$. First, observe that $\tilde{f}(x) \leq f(x)$ for all x.

Now, suppose $(x, y) \in V_{\tilde{f}}$. Then $x < y$ and $\tilde{f}(x) \geq \tilde{f}(y)$. If $\tilde{f}(y) = f(y)$, then $f(y) = \tilde{f}(y) \leq \tilde{f}(x) \leq f(x)$, so $(x, y) \in V_f$. Suppose $\tilde{f}(y) \neq f(y)$. Then $(y, z) \in V_f$ for some z, and $\tilde{f}(y) = f(z)$. Let $w \neq y$ be such that $x < w < z$. If (x, w) is a regular pair, then $\tilde{f}(x) = f(w) < f(z) = \tilde{f}(y)$, which is a contradiction. If (x, w) is not a regular pair, then $\tilde{f}(x) \leq f(x) < f(w) < f(z) = \tilde{f}(y)$; also a contradiction. Hence, we cannot have that $f(y) \neq \tilde{f}(y)$, which completes the proof. ☐

Theorem 7.11. *A morphism $[\gamma] \in \mathrm{Hom}(w, z)$ is represented by a unique sequence $(w = x_0, x_1, \dots, x_k = z)$ of cells, such that*

- *no two elements in the sequence are equal, and*
- *for all $i < k$, either $x_{i+1} < x_i$ or $\{x_i, x_{i+1}\}$ is a regular pair.*

It is clear that such a sequence always defines a morphism in $\mathrm{Hom}(w, z)$, so this tells us that we can count the morphisms in $\mathrm{Hom}(w, z)$ by counting the simple paths from w to z in a directed graph with face relations and regular pairs as edges.

Proof. We get a sequence representation from a representation

$$\gamma = \left(x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k \right)$$

by concatenating all the g_i's. For example,

$$\left(x_0 \xrightarrow{g_0=(x_0>x_0^1>y_0)} y_0 \xleftarrow{f_0=(x_1>y_0)} x_1 \xrightarrow{g_1=(x_1>x_1^1>x_1^2>y_1)} y_1 \right)$$

becomes

$$(x_0, x_0^1, y_0, x_1, x_1^1, x_1^2, y_1).$$

To get a sequence without repeating elements, we choose an appropriate representation. To see that this is possible, first observe that we can eliminate successive repeating elements by removing intermediate identity maps (horizontal relation). Furthermore, by Lemma 7.10, given a sequence $(x_0, \dots, x_k)$, there is a discrete Morse function $\tilde{f}$ such that $\tilde{f}(x_0) \geq \tilde{f}(x_1) \geq \cdots \geq \tilde{f}(x_k)$. Thus, if a cell x' repeats in the sequence, the only other cell that can be between the two x' is y' where $\{x', y'\}$ form a regular pair. Now, we can simplify the representation, as per [17, Remark 2.9], to turn $(\dots, x_i, x', y', x', x_{i+4}, \dots)$ into $(\dots, x_i, x', x_{i+4}, \dots)$.

It remains to show that for a given morphism, this sequence representation is unique. For this, we show that the sequence representation is decided by the algebraic invariant I, as defined in Definition 7.5.

Let $(w = x_0, x_1, \dots, x_k = z)$ and $(w = y_0, y_1, \dots, y_l = z)$ be sequence representations of $[\tau]$ and $[\sigma]$, such that $I([\tau]) = I([\sigma])$. Now, as x_0 appears only once in the first sequence, the coefficient before the atom $(x_0 > x_1)$ (or $(x_1 > x_0)$) in $I([\tau])$ is nonzero. Hence, as $y_0 = x_0$ appears only once in the second sequence, we must have $y_1 = x_1$ for $I([\sigma])$ to equal $I([\tau])$. Similarly, we must have $y_2 = x_2$, and so on, so the sequences must be equal in all places. This completes the proof of uniqueness. □

For the following theorem, recall the definitions of r (5) and ρ (Definition 7.8) from the previous section.

Theorem 7.12. *Let a morphism* $[\gamma] \in \mathrm{Hom}(w, z)$ *be represented by the sequence* $(x_0, \dots, x_k)$*. Let* I *be the set of indices* $i \in \{0, \dots, k-1\}$ *such that* $x_i > x_{i+1}$*. Then,*

$$\rho([\gamma]) = \sum_{i \in I} \left(\dim x_i - \dim x_{i+1} - 1\right)$$

Proof. The morphism $[\gamma]$ is represented by

$$\gamma = \left(x_{j_0} \xrightarrow{g_0} x_{j_1 - 1} \xleftarrow{f_0} x_{j_1} \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_{j_m} \xrightarrow{g_k} x_{j_{m+1}-1} \right)$$

for some sequence $(j_0, \dots, j_{m+1})$ (with $j_0 = 0$ and $j_{m+1} = k + 1$). Then,

$$\begin{aligned}
\sum_{i \in I} \left(\dim x_i - \dim x_{i+1} - 1\right) &= \sum_{i=0}^{m} \sum_{l=j_i}^{j_{i+1}-2} \left(\dim x_l - \dim x_{l+1} - 1\right) \\
&= \sum_{i=0}^{m} \Big(\dim x_{j_i} - \dim x_{j_{i+1}-1} - (j_{i+1} - j_i - 1)\Big) \\
&= \dim x_{j_0} - \dim x_{j_{m+1}-1} + \sum_{i=1}^{m} \left(\dim x_{j_i} - \dim x_{j_i - 1}\right) - \sum_{i=0}^{m} \left(j_{i+1} - j_i - 1\right) \\
&= \dim x_0 - \dim x_k + \sum_{i=0}^{m-1} r(f_i) - \sum_{i=0}^{m} r(g_i) = \rho([\gamma]).
\end{aligned}$$

□

Theorem 7.13. *Let* $[\gamma]$ *and* $[\tau]$ *be morphisms in* $\mathrm{Hom}(w, z)$*, such that* $\rho([\tau]) = \rho([\gamma]) + 1$*. Then,* $[\tau]$ *covers* $[\gamma]$ *if and only if the sequence representation of* $[\gamma]$ *equals the sequence representation of* $[\tau]$ *with exactly one element added or exactly one element removed.*

Proof. First, suppose $[\tau]$ covers $[\gamma]$. By Theorem 7.7, you can get $[\gamma]$ from $[\tau]$ by adding an element x' to the sequence representation of $[\tau]$ and possibly reducing to a non-repeating sequence. There are two cases:

(I) The element x' was not in the sequence representation of $[\tau]$.

(II) The element x' was in the sequence representation of $[\tau]$.

Case I In this case, the sequence representation of $[\gamma]$ is simply the sequence representation of $[\tau]$ with x' added.

Case II In this case, adding x' to the sequence representation of $[\tau]$ gives something of the form $(x_0, \dots, x_i, x', x_{i+1}, x', x_{i+2}, \dots, x_k)$, which is equivalent to to $(x_0, \dots, x_i, x', x_{i+2}, \dots, x_k)$. Hence, the sequence representation of $[\gamma]$ is the sequence representation of $[\tau]$ with x_{i+1} removed.

Now, for the other direction, there are again two cases to consider.

(I) The sequence representation of $[\gamma]$ equals that of $[\tau]$ with one element added.

(II) The sequence representation of $[\gamma]$ equals that of $[\tau]$ with one element removed.

Case I Let the sequence representations of $[\tau]$ and $[\gamma]$ be $(x_0, \dots, x_k)$ and $(x_0, \dots, x_i, x', x_{i+1}, \dots, x_k)$, respectively. We use Theorem 7.12 and the fact that $\rho([\tau]) = \rho([\gamma]) + 1$ to conclude that $\dim x_i > \dim x' > \dim x_{i+1}$. Hence, $[\gamma]$ equals $[\tau]$ with the element x' added to a right-pointing arrow, so $[\gamma] \Rightarrow [\tau]$.

Case II Let the sequence representations of $[\tau]$ and $[\gamma]$ be $(x_0, \dots, x_k)$ and $(x_0, \dots, x_i, x_{i+2}, \dots, x_k)$, respectively. We use Theorem 7.12 and the fact that $\rho([\tau]) = \rho([\gamma]) + 1$ to conclude that either $\dim x_{i+1} > \dim x_i$ or $\dim x_{i+2} > \dim x_{i+1}$. In the first case, $[x_i > x_{i+2}] = [x_i < x_{i+1} > x_i > x_{i+2}]$. In the second case, $[x_i > x_{i+2}] = [x_i > x_{i+2} > x_{i+1} < x_{i+2}]$. In any case, $[\gamma]$ can be constructed by adding a single element (either x_i or x_{i+2}) to a right-pointing arrow of $[\tau]$, so $[\gamma] \Rightarrow [\tau]$.

When $[\gamma] \Rightarrow [\tau]$, $[\gamma]$ covers $[\tau]$, as their rank differ by one, and this concludes the proof. □

Algorithm 1 Compute Hom poset

Input: A regular CW complex X, a gradient vector field V_f, a source w, and a target z.
Output: The Hom poset $\mathrm{Hom}(w, z)$ of the discrete flow category.

1: Construct a directed graph G with cells as vertices and an edge $u \to v$ if $v < u$ or if $\{u, v\}$ is a regular pair
2: In G, find all paths with non-repeating vertices from w to z
3: For all paths, compute the rank with Theorem 7.12.
4: **for** $i = 1..(\dim w - \dim z)$ **do**
5: Compute the covering relations between paths of rank $i-1$ and paths of rank i, using Theorem 7.13.
6: **end for**

Note that the only data needed for the CW complex X is the graph of its covering relations. Only the covering relations of $\mathrm{Hom}(w, z)$ are outputted with this algorithm, but the full set of partial order relations is easily computed from this.

Regarding complexity of this algorithm, the main computational cost is comparing the covering relations. Comparing two paths to determine if one covers the other takes $\mathcal{O}(l)$ time, where l is the length of the longest path. For each path γ, the algorithm makes one comparison to each path of rank $\rho(\gamma) + 1$. Hence, the algorithm makes $\mathcal{O}(n \cdot W)$ comparisons, where n is the number of morphisms in $\mathrm{Hom}(w, z)$, and W is the maximum number of morphisms of a given rank (the maximal "width" of a rank layer in the Hasse diagram). In total, the complexity of the algorithm is $\mathcal{O}(l \cdot n \cdot W)$ (it's easy to verify that steps 1.-3. have smaller complexity than this). As W is clearly smaller than n, we could write this more simply as $\mathcal{O}(l \cdot n^2)$. For comparison, listing the sequence representations of all morphisms in $\mathrm{Hom}(w, z)$ takes $\Theta(l \cdot n)$ time (and this is thus a lower bound of how fast an algorithm to compute $\mathrm{Hom}(w, z)$ can be).

7.5. **The Hom posets are CW posets.** In this section, we prove Theorem A: that in the case where X is a simplicial complex, then for any $w, z \in \mathrm{Flo}_\Sigma[X]$ with $\mathrm{Hom}(w, z)$ nonempty, the poset $P_{w,z}$, as defined in (4), is a CW poset. (Note that when $\mathrm{Hom}(w, z)$ is empty, then $P_{w,z}$ is just the poset with one element.)

Lemma 7.14. *Let P and Q be posets with greatest elements $\hat{1}_P$ and $\hat{1}_Q$, respectively. Suppose that $|P \setminus \{\hat{1}_P\}| \cong S^m$ and $|Q \setminus \{\hat{1}_Q\}| \cong S^n$. Then the realization of the poset $(P \times Q) \setminus \{(\hat{1}_P, \hat{1}_Q)\}$ is homeomorphic to S^{m+n+1}.*

Proof. From [19, Proposition 1.9], the realization of $(P \times Q) \setminus \{(\hat{1}_P, \hat{1}_Q)\}$ is the join of S^m and S^n, which is S^{m+n+1}. □

Definition 7.15. Let X be a simplicial complex, and let $\sigma \in X$. We define the *coface complex* as the set

$$\operatorname{cof} \sigma = \{\tau \setminus \sigma : \sigma \subsetneq \tau\}$$

In words, the coface complex of σ contains all the proper cofaces of σ, with σ subtracted as a set from the complexes.

Lemma 7.16. *Let X be a simplicial complex, and let $\sigma \in X$. The coface complex* $\operatorname{cof} \sigma$ *is a simplicial complex.*

Proof. We need to prove that for each $\gamma \in \operatorname{cof} \sigma$, each nonempty subset $\gamma' \subseteq \gamma$ is also an element of $\operatorname{cof} \sigma$. Suppose $\gamma \in \operatorname{cof} \sigma$. Then $\gamma = \tau \setminus \sigma$ for some proper coface τ of σ. Then, $(\gamma' \cup \sigma) \subseteq \tau$, so $\gamma' \cup \sigma$ is a simplex in X, and $(\gamma' \cup \sigma)$ is a proper coface of σ, so $\gamma' = (\gamma' \cup \sigma) \setminus \sigma$ is in $\operatorname{cof} \sigma$. □

We illustrate the above definition and lemma with an example.

Example 7.17. Consider the simplicial complex consisting of a single tetrahedron, as illustrated in Figure 15. Let a be one of the vertices. The coface complex of $\{a\}$ consists of all the proper cofaces of $\{a\}$, with a removed, i.e.:

$$\begin{aligned}
\{a,b\} \setminus \{a\} &= \{b\},\\
\{a,c\} \setminus \{a\} &= \{c\},\\
\{a,d\} \setminus \{a\} &= \{d\},\\
\{a,b,c\} \setminus \{a\} &= \{b,c\},\\
\{a,b,d\} \setminus \{a\} &= \{b,d\},\\
\{a,c,d\} \setminus \{a\} &= \{c,d\},\\
\{a,b,c,d\} \setminus \{a\} &= \{b,c,d\}.
\end{aligned}$$

This simplicial complex is illustrated in Figure 15. We see that $\operatorname{cof}\{a\}$ is a simplicial complex consisting of a single 2–simplex.

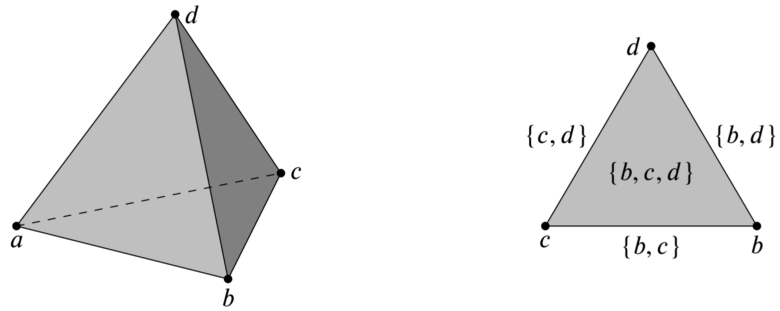


FIGURE 15. Left: a tetrahedron as a simplicial complex. Right: the coface complex of $\{a\}$, $\operatorname{cof}\{a\}$.

Similar computations give that $\operatorname{cof}\{a,b\}$ is the 1–simplex $\{c,d\}$ and $\operatorname{cof}\{a,b,c\}$ is simply $\{d\}$.

Lemma 7.18. *Let X be a simplicial complex, and let $\sigma \in X^n$ and $\tau \in X^m$ with $\tau \subseteq \sigma$. Then the realization of* $\operatorname{Hom}_{\operatorname{Ent}[X]}(\sigma,\tau) \setminus \{(\sigma > \tau)\}$ *is homeomorphic to S^{n-m-2}.*

Proof. The poset $\operatorname{Hom}_{\operatorname{Ent}[X]}(\sigma,\tau) \setminus \{(\sigma > \tau)\}$ contains all sequences $(\sigma > v_1 > \cdots > v_k > \tau)$, with $k \geq 1$, and the partial order is given by inclusion. This poset is isomorphic to the poset of descending sequences $(v_1 > \cdots > v_k)$ of simplices in $\partial(\sigma \setminus \tau) \subseteq \operatorname{cof} \tau$, i.e., the boundary of $\sigma \setminus \tau$ in the coface complex of τ (where the ordering is still given by inclusion). This is again isomorphic to the barycentric subdivision $T(\partial(\sigma \setminus \tau))$.

Now,

$$|T(\partial(\sigma \setminus \tau))| \cong |\partial(\sigma \setminus \tau)| \cong |\partial\Delta^{n-m-1}| \cong |S^{n-m-2}|,$$

as desired (here we use that $\sigma \setminus \tau$ is a set of cardinality $(n+1)-(m+1) = n-m$, and hence an $(n-m-1)$–simplex). □

In the proof of the following lemma, we will use the fact that for a poset P, $|P^{\text{op}}| \cong |P|$, which is a special case of the fact that $|\mathcal{N}(C^{\text{op}})| \cong |\mathcal{N}(C)|$ for a general category C.

Lemma 7.19. *Let X be a simplicial complex. Let $P = \text{Hom}_{\text{Ent}[X]}(w,z)^{\text{op}} \cup \{\hat{0}\}$, where $\hat{0}$ is a least element. Then, for any $g \in P \setminus \{\hat{0}\}$, the realization of the open interval $(\hat{0}, g)$ is homeomorphic to a sphere.*

Proof. Let $g = (x_0, \dots, x_k)$. Then,

$$(\hat{0}, g] \quad \cong \quad \text{Hom}(x_0, x_1)^{\text{op}} \times \text{Hom}(x_1, x_2)^{\text{op}} \times \cdots \times \text{Hom}(x_{k-1}, x_k)^{\text{op}},$$

where the $\text{Hom}(x_i, x_{i+1})$ are Hom posets in $\text{Ent}[X]$. Now, observe that the atom $(x_i > x_{i+1})$ is the greatest element in $\text{Hom}(x_i, x_{i+1})^{\text{op}}$. Furthermore, by Lemma 7.18, the realization of $\text{Hom}(x_i, x_{i+1})^{\text{op}} \setminus \{(x_i > x_{i+1})\}$ is homeomorphic to a sphere. Thus, by applying Lemma 7.14 to the above equation, we get that the realization $|(\hat{0}, g)| = |(\hat{0}, g] \setminus \{g\}|$ is homeomorphic to a sphere. □

Theorem 7.20 (Theorem A)**.** *In the case that X is a simplicial complex, the poset $P_{w,z}$, as defined in* (4)*, is a CW poset.*

Proof. We use Definition 7.1. Condition (1.) and (2.) is clearly fulfilled, so we only need to show that for all $x \in P_{w,z} \setminus \{\hat{0}\}$, the realization of the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.

Let x be represented by

$$\tau = \left(w = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = z \right).$$

Applying Theorem 7.7, we see that

$$(\hat{0}, x] \quad \cong \quad (\hat{0}, g_0] \times (\hat{0}, g_1] \times \cdots \times (\hat{0}, g_k].$$

By Lemma 7.19, the realization of $(0, g_i)$ is a sphere. Now, as in the proof for Lemma 7.19, we apply Lemma 7.14 to the above equation and get that $|(\hat{0}, x)|$ is homeomorphic to a sphere. □

In conclusion, what this result tells us is that in the discrete flow category, the opposite poset of a Hom set is the face poset of a regular CW complex. The nerve of the face poset of a regular CW complex is its barycentric subdivision, and hence has a geometric realization which is homeomorphic to the CW complex. Furthermore, the nerve of the opposite of a poset is isomorphic to the nerve of the poset. Thus, we get a simpler way to describe the geometric realization of the nerve of a Hom set in the discrete flow category: instead of taking the nerve and realizing, we can construct the CW complex of which it is a face poset of, which results in a description with fewer cells. To construct this CW complex, we need only compute the rank values and the covering relations, which we can do with Algorithm 1.

Example 7.21. We illustrate the observations in the last paragraph with an example.

Consider a regular CW decomposition of the filled sphere D^3, as illustrated in Figure 16. We define the gradient vector field $V_f = \{(w > y), (b > z)\}$, and let Σ be its corresponding Morse system. We compute $\text{Hom}(f, x)$ in the discrete flow category.

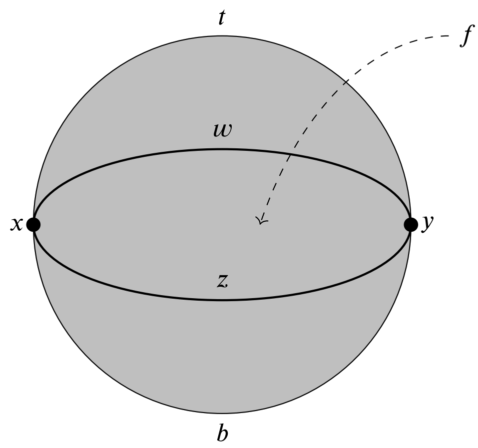


FIGURE 16. A CW decomposition of D^3. The 0–cells are x and y, the 1–cells are w and z, the 2–cells are t and b and the single 3–cell is f.

As we have shown that $\mathrm{Hom}(f, x)$ is a graded poset, we can represent it by a Hasse diagram. The highest rank is $\dim f - \dim x - 1 = 2$. All rank 2 elements are gradient paths on the form

$$f = x_0 \xrightarrow{g_0} y_0 \xleftarrow{f_0} x_1 \xrightarrow{g_1} \cdots \xleftarrow{f_{k-1}} x_k \xrightarrow{g_k} y_k = x,$$

where all g_i are atoms. There are four of these:

- $f > x$
- $f > z < b > x$
- $f > y < w > x$
- $f > z < b > y < w > x$

All elements of lower rank can be constructed from these by inserting intermediate cells, and all Hasse diagram edges correspond to inserting an intermediate cell in a sequence. For example, inserting b between f and x in $f > x$ gives $f > b > x$, and inserting b between f and z in $f > z < b > x$ gives $f > b > z < b > x = f > b > x$. We now draw the Hasse diagram, which is illustrated in Figure 17.

The corresponding regular CW complex is illustrated in Figure 18, and it's easy to see that this is a contractible space.

Example 7.22. We illustrate that Theorem 7.20 may fail to hold when X is not a simplicial complex. In fact, Lemma 7.19, which may be viewed as a special case of Theorem 7.20 where the Morse system is empty (so that $\mathrm{Flo}_\Sigma[X] = \mathrm{Ent}[X]$), will fail to hold in our example.

First, we describe how the suspension of a simplicial complex is also a simplicial complex. Let X be a simplicial complex. Its cone CX can be viewed as a simplicial complex as follows:

$$CX = X \cup \{\{0\}\} \cup \{\sigma \cup \{0\} : \sigma \in X\}.$$

The suspension ΣX, which is just the union of two cones CX along X, can then be viewed as the simplicial complex:

$$\Sigma X = X \cup \{\{0\}\} \cup \{\sigma \cup \{0\} : \sigma \in X\} \cup \{\{1\}\} \cup \{\sigma \cup \{1\} : \sigma \in X\}. \quad (8)$$

Our counterexample will use the following fact: there exists a simplicial complex Y such that the suspension ΣY is homeomorphic to a sphere, but Y is not homeomorphic to S^k for any k. For this, we will consider the Poincaré sphere, which is a simplicial complex and a

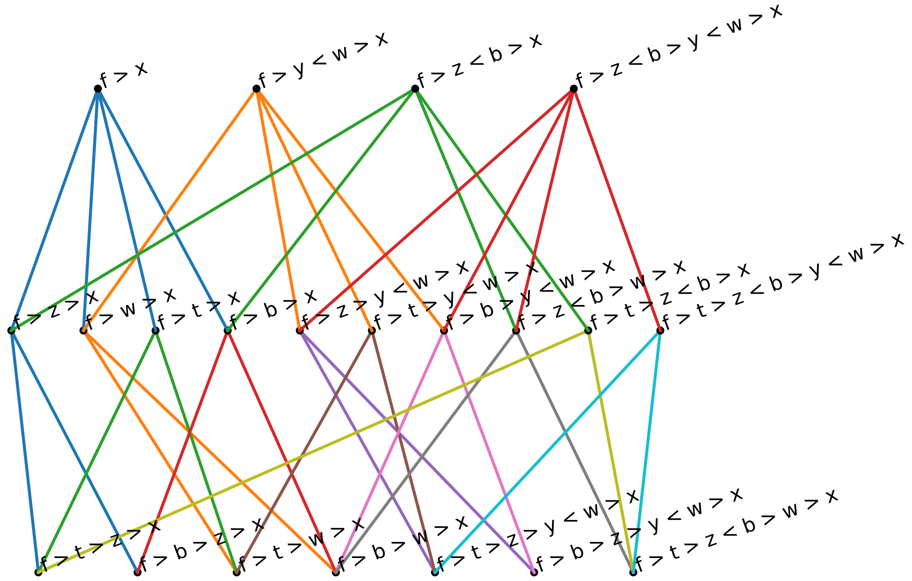


FIGURE 17. The lattice for the Hom poset $\mathrm{Hom}(f, x)$ in the discrete flow category.

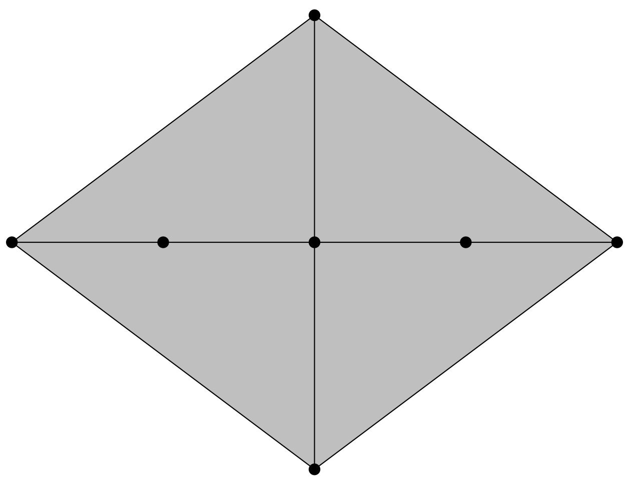


FIGURE 18. The face poset of this regular CW complex is $\mathrm{Hom}(f, x)$.

homology 3–sphere, but not homeomorphic to a sphere. Its suspension is not homeomorphic to a sphere, but its double suspension is (for more details, see [22, Example 3.2.11]). Hence, letting Y be the suspension of the Poincaré sphere, we get our desired properties.

Now, let Y be the suspension of the Poincaré sphere. As ΣY is a simplicial complex and homeomorphic to S^5, it is also a regular CW complex of dimension 4. Hence we can construct a new regular CW complex Z by attaching a 5–cell e along ΣY. The CW complex Z is illustrated in Figure 19.

Now, consider the Hom poset

$$\mathrm{Hom}_{\mathrm{Ent}[Z]}(e, 0) \setminus \{(e > 0)\}, \tag{9}$$

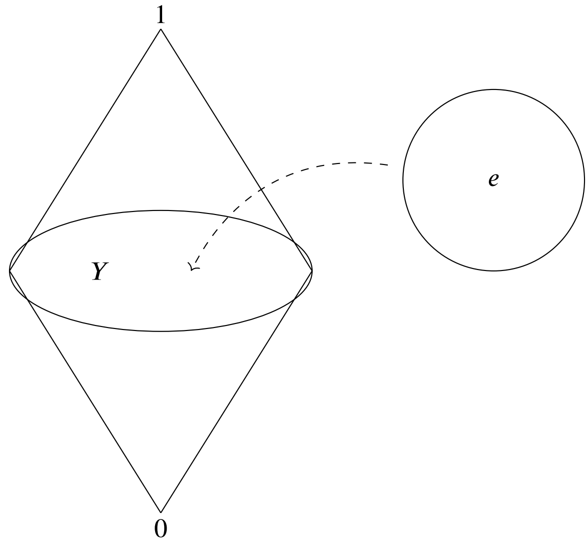


FIGURE 19. An illustration of the CW complex Z.

where 0 is a 0–simplex in ΣY as per the notation in (8). This poset consists of all descending chains $(e > \tau_1 > \cdots > \tau_k > 0)$, with $k \geq 1$, ordered by inclusion. The set of τ_i satisfying $e > \tau_i > 0$ is precisely $\{\{0\} \cup \sigma : \sigma \in Y\}$. Hence, the poset in (9) is isomorphic to the barycentric subdivision of Y. Thus,

$$|\operatorname{Hom}_{\mathrm{Ent}[Z]}(e, 0) \setminus \{(e > 0)\}| \cong |T(Y)| \cong |Y|,$$

which is not homeomorphic to a sphere. Thus, the interval $(\hat{0}, (e > 0)) \in P_{e,0}$ (which is just the opposite of the poset in (9)) is not a sphere, and hence $P_{e,0}$ is not a sphere.

8. Simplicial collapse on simplicial sets

In this section we define *regular* simplicial sets, and generalize the concept of *simplicial collapse* to these simplicial sets. We further prove how the nerves of certain categories are regular, and apply simplicial collapse to a subclass of these categories.

8.1. Collapses on simplicial complexes.

In Section 3.2, we defined *free faces* and *collapses*.

We now provide a more general definition of free faces, that allows for codimension greater than 1.

Definition 8.1. Let X be a simplicial complex and let τ and σ be simplices in X such that

(1) $\tau \subsetneq \sigma$, and
(2) all cofaces of τ are faces of σ.

Then τ is called a *free face* and $\{\tau, \sigma\}$ is a *free pair*.

Recall that in our old definition of free pairs, removing a free pair of X from a simplicial complex constitutes an elementary collapse. Recall that we write $X \searrow Y$ if Y can be reached through a series of elementary collapses on X. The following theorem justifies our generalization of the definition of free pairs.

Theorem 8.2. [11, Proposition 9.18] *Let X be a simplicial complex and let $\{\tau, \sigma\}$ be a free pair according to Definition 8.1. Let $Y \setminus [\tau, \sigma]$ be the simplicial complex where all simplices γ satisfying $\tau \subseteq \gamma \subseteq \sigma$ have been removed from X. Then Y is a simplicial complex, and $X \searrow Y$.*

Note that in the previous theorem, we might as well have written $\tau \subseteq \gamma$ instead of $\tau \subseteq \gamma \subseteq \sigma$, as all cofaces of τ are faces of σ.

An example of a simplicial collapse of the kind described in Theorem 8.2 is given in Figure 20.

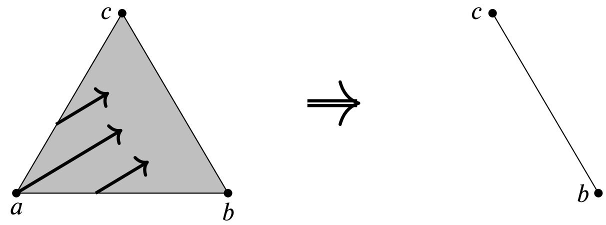


FIGURE 20. A simplicial collapse given by the free pair $\{\{a\}, \{a, b, c\}\}$.

8.2. **Collapses on simplicial sets.** We generalize free pairs and collapses to simplicial sets. We will restrict ourselves to a certain class of simplicial sets, which we will name *regular* simplicial sets.

Definition 8.3. Let X be a simplicial set. Then X is called *regular* if for all $n \in \mathbb{N}$ (including 0) and all nondegenerate $x \in X_n$, the map

$$\begin{aligned} \mathrm{Hom}_\Delta([0],[n]) &\to X_0 \\ \theta &\mapsto X(\theta)(x) \end{aligned}$$

is injective.

This definition says that for a nondegenerate n–simplex, all of the $(n+1)$ 0–dimensional faces are different. This also implies that for each nondegenerate n–simplex x and all $m < n$, all of the m–dimensional faces of x are different[1], because no two m–dimensional faces have the same 0–dimensional faces. In other words, for $x \in X_n$ nondegenerate, the simplicial set consisting of x, all the faces of x, and all their degeneracies, is isomorphic to Δ^n. This last part is the motivation behind the definition. Just as for simplicial complexes, the sub-simplicial set generated by a nondegenerate n–simplex x looks like Δ^n, which will allow us to define free pairs and collapses, just as for simplicial complexes.

Example 8.4. An example of a regular simplicial set is given in Figure 21. Observe how there are two 1–simplices with b and c as faces, something that is not possible for simplicial complexes. However, just as for simplicial complexes, all 1–simplices lie between two different 0–simplices, and the 2–simplex A looks like a 2–simplex in a simplicial complex. To make the last point more precise, the simplicial set consisting of A and all its faces and degeneracies, is isomorphic to Δ^2.

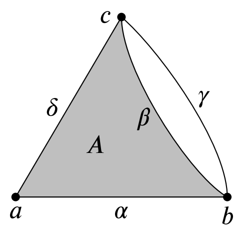


FIGURE 21. A regular simplicial set (degeneracies omitted).

[1]More precisely, the map $\mathrm{Hom}_\Delta([m],[n]) \to X_m$ that maps θ to $X(\theta)(x)$ is injective.

Lemma 8.5. *The realization of a regular simplicial set is a regular CW complex.*

Proof. Let X be a regular simplicial set. Recall that a CW complex is regular if all attaching maps are homeomorphisms onto their image. From [14, Theorem 14.1], we know that $|X|$ is a CW complex with one n–cell for each nondegenerate simplex. Furthermore, for an n–cell corresponding to a nondegenerate n–simplex x, the attaching map is given by sending $|\partial\Delta^n|$ to $|\partial x|$ (in the canonical way). Now, from the observations in the above paragraph, we know that this map is a homeomorphism, so we are finished. □

We introduce some new notation. We say that $\tau \in X_m$ is a *face* of $\sigma \in X_n$ if $\tau = X(\theta)(\sigma)$ for some $\theta \in \mathrm{Hom}_\Delta([m],[n])$. Note that in this definition, degeneracies are also considered faces, but the nondegenerate faces are just as for simplicial complexes. We say that σ is a coface of τ when τ is a face of σ. We write $\tau \subseteq \sigma$ when τ is a face of σ, and we write $\tau \subsetneq \sigma$ when τ is a *proper* face of σ, i.e., when $\tau \subseteq \sigma$ and $\tau \neq \sigma$. It's clear that this relation is transitive, as if $\tau \subseteq \tau'$ and $\tau' \subseteq \tau''$, then $\tau = X(\theta)(\tau')$ and $\tau' = X(\theta')(\tau'')$ for some θ, θ', so $\tau = X(\theta \circ \theta')(\tau'')$. It's not, however, antisymmetric, as all simplices are faces of all their degeneracies.

Example 8.6. In the simplicial set in Figure 21, the faces of A are

- A itself,
- the degeneracies of A,
- the nondegenerate simplices a, b, c, α, β and δ,
- the degeneracies of the above mentioned simplices.

The simplex γ, on the other hand, is *not* a face of A.

Definition 8.7. Let X be a regular simplicial set, and let τ and σ be nondegenerate simplices such that

(1) $\tau \subsetneq \sigma$, and
(2) all cofaces of τ are faces of σ.

Then τ is called a *free face* and $\{\tau, \sigma\}$ is a *free pair*.

Note that the fact that τ and σ are nondegenerate and that $\tau \subsetneq \sigma$ implies that $\dim \tau < \dim \sigma$.

Theorem 8.8. *Let X be a simplicial set and let $\{\tau, \sigma\}$ be a free pair according to Definition 8.1. Let $Y \setminus [\tau, \sigma]$ be the simplicial set where all simplices γ satisfying $\tau \subseteq \gamma \subseteq \sigma$ have been removed from X. Then Y is a simplicial set, and the realization $|Y|$ is a deformation retract of $|X|$.*

Proof. We first prove that Y is a simplicial set. We must show that for all $x \in Y$, all faces (including degeneracies) of x are contained in Y. Let x be an n–simplex in Y, and suppose y is a face of x not in Y. Then y is a coface of τ. But then x is also a coface of τ (and hence also a face of σ), so $x \notin Y$, which is a contradiction. This proves that Y is a well-defined simplicial set.

We now show that $|Y|$ is a deformation retract of $|X|$. From Lemma 8.5, we know that $|X|$ is a regular CW complex, and so is $|Y|$. Let e_τ and e_σ denote the cells in $|X|$ corresponding to τ and σ, respectively. Then $\overline{e_\sigma}$ (i.e., the closure of e_σ) is homeomorphic to $|\Delta^n|$ (for some n), and under this homeomorphism, e_τ maps to some face of $|\Delta^n|$. Thus, $\overline{e_\sigma}$ deformation retracts to $\left(\overline{e_\sigma} \setminus \{e' : e_\tau \subseteq e'\}\right)$ (i.e., $\overline{e_\sigma}$ with all the cofaces of e_τ removed), just as for simplicial complexes. This deformation retract then extends to a deformation retract from $|X|$ to $|Y|$ by setting it to be identity everywhere else. □

When X deformation retracts to Y in this way, we call it a *collapse*, and say that X *collapses* to Y.

Example 8.9. In Figure 22, $\{a, A\}$ is a free pair in the regular simplicial set on the left hand side. The right hand side shows the simplicial set after the corresponding collapse. Observe how the collapse gives a deformation retract on the closure of the 2–cell corresponding to A, which extends to a deformation retract on the entire realization by defining it to be identity everywhere else.

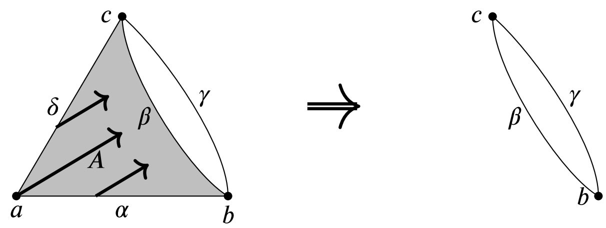


FIGURE 22. A collapse of a regular simplicial set.

8.3. **Unique factorization categories.** We now define a certain class of categories, which we will call *unique factorization categories*. In these categories, all morphism can be written as a composition of one or more *indecomposable* morphisms in a unique way. We shall see that these categories have nerves that are regular simplicial sets, and with many free faces.

Definition 8.10. Let C be a category and $f : A \to B$ a morphism in C. Then f is called *indecomposable* if it cannot be written as a composite of non-identity morphisms, i.e., if there exist no morphisms $h \neq \mathrm{id}_B$ and $g \neq \mathrm{id}_A$ such that $f = h \circ g$.

We say that a category is *finite* if both its set of objects and set of morphisms is finite.

Definition 8.11. A *finite directed category* is a finite category such that:

- For all objects a, $\mathrm{Hom}(a, a)$ contains only the identity id_a.
- For all objects $a \neq b$, if $\mathrm{Hom}(a, b)$ is nonempty, then $\mathrm{Hom}(b, a) = \emptyset$.

Any finite poset (viewed as a category) is an example of a finite directed category.

It's clear that for a finite directed category, each morphism f can be written as a composition $f = f_k \circ \cdots \circ f_1$ of indecomposable morphisms. However, its possible that there are several different ways to do this. For example, consider the face poset of the simplicial complex Δ^2. Here, the morphism $(\{0\} \leq \{0, 1, 2\})$ has two decompositions: $(\{0, 1\} \leq \{0, 1, 2\}) \circ (\{0\} \leq \{0, 1\})$ and $(\{0, 2\} \leq \{0, 1, 2\}) \circ (\{0\} \leq \{0, 2\})$

Theorem 8.12. *Let C be a finite directed category. Then the nerve of C is a regular simplicial set.*

Proof. A nondegenerate n–simplex in $\mathcal{N}(C)$ is a set of composable non-identity morphisms $\{f_i : x_i \to x_{i+1} : 0 \leq i < n\}$. Suppose $x_i = x_j$ for some $i < j$. If $j = i+1$, then f_i is a non-identity morphism in $\mathrm{Hom}(x_i, x_i)$, which contradicts Definition 8.11. If $j > i+1$, then both $\mathrm{Hom}(x_i, x_{i+1})$ and $\mathrm{Hom}(x_{i+1}, x_i)$ are nonempty, which also contradicts Definition 8.11.

Hence, any nondegenerate n–simplex in $\mathcal{N}(C)$ has $n + 1$ different 0–dimensional faces, and so $\mathcal{N}(C)$ is regular. □

Definition 8.13. A *unique factorization category* is a finite directed category C such that all non-identity morphisms f in C can be written as a composition of non-identity indecomposable morphisms in a unique way.

Example 8.14. Let C be the entrance path category of a finite CW complex X considered as an ordinary category, i.e., with the poset structure of the Hom sets removed. Then C is a unique factorization category. To see this, observe that each non-identity morphism goes to a cell of strictly lower dimension, so C is finite directed. Furthermore, an entrance path $(x_0 > \cdots > x_k)$ factors uniquely to $(x_k > x_{k-1})\circ \ldots \circ(x_0 > x_1)$.

Similarly, we can show that a discrete flow category of a finite CW complex, with the poset structure of the Hom sets removed, is a unique factorization category. The factorization of a gradient path is given by subsequences between critical cells. As each non-identity morphism goes from a critical cell to a critical cell of lower dimension, this gives a well-defined factorization into indecomposable morphisms. Now, to see that this factorization is unique, suppose we have $f, f' : a \to b$ and $g, g' : b \to c$ with $g\circ f = g'\circ f'$. Then the sequence representations of $g\circ f$ and $g'\circ f'$ are equal, and as b is critical, both sequence representations contain b. Now, the subsequences starting at a and ending at b have to be equal, and these are precisely the sequence representations of f and f', so $f = f'$. Similarly, $g = g'$. This shows that factorizations are unique.

To give an example, in the discrete flow category in Example 7.21, the morphism $[f > t > z < b > x]$ has the unique factorization $[t > z < b > x]\circ[f > t]$.

Theorem 8.15. *Let C be a unique factorization category. Then the nerve of C deformation retracts to a simplicial set whose n–simplices are all degenerate for $n > 1$.*

Proof. The main idea of the proof is to collapse $\mathcal{N}(C)$ by removing 1–simplices represented by higher compositions.

We collapse $\mathcal{N}(C)$ inductively. Let k be the largest integer such that there is a 1–simplex in $\mathcal{N}(C)$ represented by a morphism f that decomposes into k indecomposable morphisms, i.e., $f = f_k\circ \ldots \circ f_1$. We claim that f is a free face, and f is in a free pair with the k–simplex $\{f_1, \ldots, f_k\}$. To see why this is true, let $\{g_1, \ldots, g_m\}$ be a coface of f. Then $g = g_m\circ \ldots \circ g_1$ is a 1–simplex that factors through f, and thus $g = f$, as otherwise, g would factor into more than k indecomposable morphisms. It follows that each g_i is a composite $g_i = f_{j_{i+1}-1}\circ \ldots \circ f_{j_i}$, and hence, $\{g_1, \ldots, g_m\}$ is a face of $\{f_1, \ldots, f_k\}$.

Now, we can remove any k–composable 1–simplex f from the simplicial set through a series of collapses. Now, all remaining 1–simplices decomposes into $k - 1$ or fewer indecomposable factors. We can repeat the process with $k-1$, and so on, until all remaining 1–simplices are indecomposable morphisms. Now, all remaining nondegenerate simplices are of dimension 0 or 1, as any higher simplex $\{f_1, \ldots, f_k\}$ has the decomposable morphism $f_k\circ \ldots \circ f_1$ as a face. □

By combining the previous theorem with Theorem 2.5, we get the following result.

Corollary 8.16. *Let C be a unique factorization category. Then*

$$H_n(\mathcal{N}(C)) \cong 0$$

for $n \geq 2$.

9. Spectral sequences

9.1. **Spectral sequences and double complexes.** In this section, we briefly summarize the topic of spectral sequences, so that a reader unfamiliar with them can follow the computations in the following sections. For more details on spectral sequences, we refer the reader to, e.g., [15, Chapter 2].

A spectral sequence can be viewed as a computational tool to compute homology. Formally, it can be defined as a collection of bigraded differential abelian groups[2] $\{E^r_{*,*}, \partial^r : r = 0, 1, \ldots\}$, where the bidegree of ∂^r is $(-r, r - 1)$[3]. Spectral sequences arise in many ways, one of the most common being from filtered differential graded Abelian groups,

$$\{0\} \subseteq F^n A \subseteq F^{n-1} A \subseteq \cdots \subseteq F^0 A \subseteq F^{-1} A = A.$$

[2]Or more generally, bigraded differential R–modules

[3]There is also a cohomological version, where the bidegree of the differentials are $(r, 1 - r)$.

We say that a spectral sequence $E^r_{p,q}$ *converges* to H_* if H_n is the sum of the nth diagonal of the limit page $E^\infty_{p,q}$.

One example of such a filtered differential graded Abelian group arises from a double complex. A double complex is a bigraded Abelian group C with two differentials $\partial_h : C_{*,*} \to C_{*-1,*}$ and $\partial_v : C_{*,*} \to C_{*,*-1}$, such that

$$\begin{aligned}\partial_h\partial_h &= 0,\\ \partial_v\partial_v &= 0, \text{ and}\\ \partial_h\partial_v &= \partial_v\partial_h.\end{aligned}$$

From a double complex C, we can define a total complex $\operatorname{Tot} C$, with $(\operatorname{Tot} C)_n = \bigoplus_{p+q=n} C_{p,q}$, and whose differentials are given by $\partial^{\mathrm{Tot}}_n = \partial_v + (-1)^{\text{vertical degree}}\partial_h$. Given a double complex, we can define a filtration on $\operatorname{Tot} C$ in two ways:

$$\begin{aligned}F^p_I(\operatorname{Tot} C)_t &= \bigoplus_{r\geq p} C_{r,t-r}\\ F^p_{II}(\operatorname{Tot} C)_t &= \bigoplus_{r\geq p} C_{t-r,r}\end{aligned} \tag{10}$$

Each of these filtrations gives a spectral sequence, and it turns out that they both converge to the homology of $\operatorname{Tot} C$. We explain how to compute the first pages of the spectral sequences, and we refer the reader to [15, pp. 47–49] for details. We only do this for F^*_I; the other case is exactly the same, except with the indices and differentials of C swapped.

On the 0th page, we have $E^0_{p,q} = C_{p,q}$ and $\partial^0 = \partial_v$. On the 1st page, we then have $E^1_{p,q} = \ker(\partial_v)_{p,q}/\operatorname{im}(\partial_v)_{p,q+1}$, which we shall denote as $H_{II}(C)_{p,q}$ (the homology of ∂_v). The differential on E^1 is the induced map $\partial_h : H_{II}(C)_{p,q} \to H_{II}(C)_{p-1,q}$ (one can show that this is well-defined, due to the fact that $\partial_v\partial_h = \partial_h\partial_v$). The 2nd page is now the homology of this induced map ∂_h, which we shall denote with $H_I H_{II}(C)$. The differential on E^2 goes from $E^2_{p,q}$ to $E^2_{p-2,q+1}$, and is slightly more complicated to define. The process continues with E^3, and so on. The first three pages are illustrated in Figure 23.

The previous paragraphs can be summarized in the following result, which is the homological version of Theorem 2.15 in [15] (see also [24, pp. 141–143] for the homological formulation).

Theorem 9.1. *Let C be a double complex such that $C_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Then there are two spectral sequences ${}_IE$ and ${}_{II}E$ with*

$${}_IE^2_{p,q} \cong H_I H_{II}(C)_{p,q}, \quad \text{and} \quad {}_{II}E^2_{p,q} \cong H_{II} H_I(C)_{p,q},$$

that both converge to $H_*(\operatorname{Tot} C)$.

9.2. **The spectral sequence of a bisimplicial set.** To a bisimplicial set X, we can associate a double complex FX with $(FX)_{p,q} = \mathbb{Z}X_{p,q}$. The horizontal and vertical differentials are induced by the horizontal and vertical face maps as for the chain complex of a simplicial set, i.e.,

$$\begin{aligned}\partial_h &= \sum_i (-1)^i (d_i, \mathrm{id}),\\ \partial_v &= \sum_i (-1)^i (\mathrm{id}, d_i).\end{aligned}$$

A simple calculation shows that $\partial_h^2 = 0$, $\partial_v^2 = 0$ and $\partial_h\partial_v = \partial_v\partial_h$.

The following theorem of Dold and Puppe tells us that the homology of $\operatorname{Tot}(FX)$ equals the homology $\operatorname{diag} X$ [5, Theorem 2.9] (see also [8, Theorem 2.5]). Hence, we can compute the homology of $\operatorname{diag} X$ with the spectral sequence in Theorem 9.1.

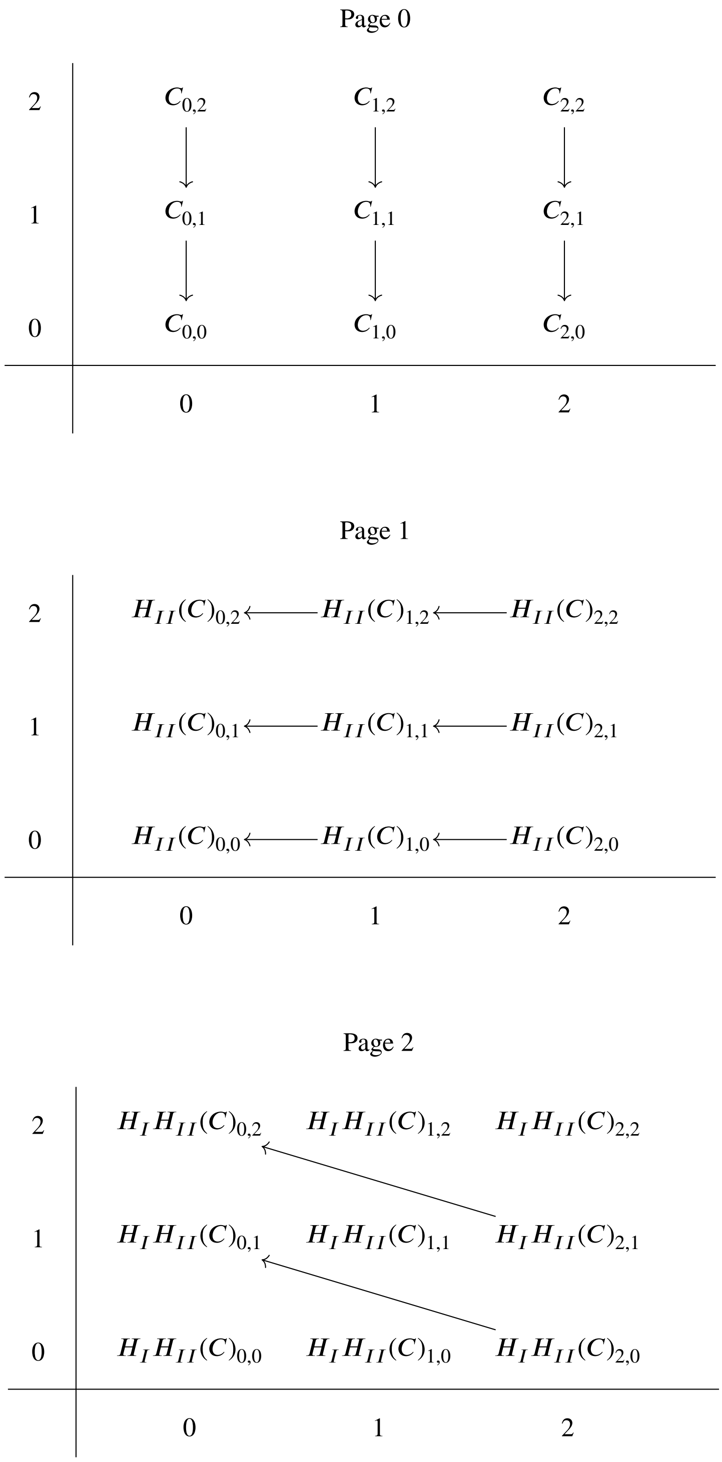


FIGURE 23. The first three pages of the spectral sequence ${}_I E$ associated to the double complex C.

Theorem 9.2 (Dold–Puppe)**.** *Let X be a bisimplicial set and FX its associated double complex. Then*

$$H_* \operatorname{Tot}(FX) \cong H_*(\operatorname{diag} X)$$

As a bisimplicial set has an infinite amount of degenerate simplices, computing the spectral sequence associated to $\operatorname{Tot}(FX)$ can be impractical. However, we will show that, just as for the chain complex of a simplicial set, we can in fact ignore the degenerate simplices (both the vertical and horizontal ones). For the proof, we will need the following lemma.

Lemma 9.3. *Let C_* be a chain complex, and let D_* and D'_* be subcomplexes of C_*. Suppose*

$$\begin{aligned} H_n(D) &\cong 0, \\ H_n(D') &\cong 0, \text{ and} \\ H_n(D \cap D') &\cong 0 \end{aligned}$$

for all $n \in \mathbb{Z}$. Then,

$$H_n(D + D') \cong 0$$

for all $n \in \mathbb{Z}$.

Here, $D_* \cap D'_*$ and $D_* + D'_*$ are the subcomplexes of C_* defined by [4]

$$\begin{aligned} (D \cap D')_n &= D_n \cap D'_n, \text{ and} \\ (D + D')_n &= D_n + D'_n = \{x + y : x \in D_n, y \in D'_n\}. \end{aligned}$$

Proof. There is a short exact sequence of chain complexes

$$0 \to D_* \cap D'_* \to D_* \oplus D'_* \to D_* + D'_* \to 0,$$

where the first map is $x \mapsto (x, -x)$, and the second map is $(x, y) \mapsto x + y$. The induced long exact sequence in homology then proves the lemma. □

Now, let $D^h_{p,q}$ be the set of horizontal degeneracies in $X_{p,q}$, i.e., the simplices on the form $(s_i, \mathrm{id})x$ for some $x \in X_{p-1,q}$ and some i. Likewise, let $D^v_{p,q}$ be the set of vertical degeneracies in $X_{p,q}$. Then $\mathbb{Z}D^h_{p,q}$ and $\mathbb{Z}D^v_{p,q}$ are subgroups of $(FX)_{p,q}$. Now, as $D^h_{*,q}$ are just the degeneracies in the simplicial set $X_{*,q}$, we have that $\partial_h(D^h_{p,q}) \subseteq D^h_{p-1,q}$ (as stated in Section 2.4). Furthermore, for $(s_i, \mathrm{id})x \in D^h_{p,q}$,

$$\begin{aligned} \partial_v(s_i, \mathrm{id})x &= \left(\sum_i (-1)^i (\mathrm{id}, d_i) \right) (s_i, \mathrm{id})x \\ &= \sum_i (-1)^i (s_i, d_i)x = (s_i, \mathrm{id}) \left(\sum_i (-1)^i (\mathrm{id}, d_i) \right) x, \end{aligned} \tag{11}$$

so $\partial_v(D^h_{p,q}) \subseteq D^h_{p,q-1}$. Hence, FD^h given by $(FD^h)_{p,q} = \mathbb{Z}D^h_{p,q}$ is a well defined sub-double complex of FX, and we can form the quotient double complex FX/FD^h. Likewise, we have well-defined double complexes FD^v and FX/FD^v. We then also have a sub-double complex $F(D^h \cup D^v)$, again given by $F(D^h \cup D^v)_{p,q} = \mathbb{Z}(D^h_{p,q} \cup D^v_{p,q})$, and we can form the quotient $FX/F(D^h \cup D^v)$.

Theorem 9.4. *Let X be a bisimplicial set. Then*

$$H_* \operatorname{Tot}(FX) \cong H_* \operatorname{Tot}\left(FX/F(D^h \cup D^v)\right).$$

[4] Note the difference between $D_* + D'_*$ and the direct sum $D_* \oplus D'_*$.

Proof. From Corollary 2.6, we get that $H_I(FD^h) \cong 0$ everywhere. Theorem 9.1 then gives us that $H_*(\text{Tot}\, FD^h) \cong 0$. By the same argument for D^v, we get that $H_*(\text{Tot}\, FD^v) \cong 0$.

Now, the degenerate simplices in the diagonal diag X are precisely the elements of $D^h \cap D^v$. Applying Corollary 2.6 again, we get that $H_*(\text{diag}(D^h \cap D^v)) = 0$, and applying Theorem 9.2, we get that $H_*(\text{Tot}\, F(D^h \cap D^v)) \cong 0$.

Note now that $\text{Tot}\, F(D^h \cup D^v) = \text{Tot}(FD^h + FD^v) = (\text{Tot}\, FD^h) + (\text{Tot}\, FD^v)$ and that $\text{Tot}\, F(D^h \cap D^v) = (\text{Tot}\, FD^h) \cap (\text{Tot}\, FD^v)$. It now follows from Lemma 9.3 that $H_*(\text{Tot}\, F(D^h \cup D^v)) \cong 0$. Hence, $H_*(\text{Tot}\, FX/F(D^h \cup D^v)) \cong H_*\big((\text{Tot}\, FX)/(\text{Tot}\, F(D^h \cup D^v))\big) \cong H_*(\text{Tot}\, FX)$. □

9.3. **The spectral sequence of the discrete flow category.** From the discrete flow category $\text{Flo}_\Sigma[X]$, we get a bisimplicial set $S := \text{N}\overline{\text{N}}\,\text{Flo}_\Sigma[X]$, as described in Section 5.3. As the realization of the diagonal, $|\,\text{diag}\, S|$, is (homotopy equivalent to) the classifying space of $\text{Flo}_\Sigma[X]$, we can compute the homology of this classifying space through simplicial homology on diag S. This is again equal to the homology of the total complex, Tot FS, by Theorem 9.2. Finally, we can compute the homology of Tot FS with one of two the spectral sequences in Theorem 9.1. We will use the spectral sequence ${}_IE$, i.e., we will take vertical homology first. As we shall show, this spectral sequence collapses on page 2 (Theorem B in the introduction).

Recall that an element of $S_{p,q}$ consists of q horizontally composable sets of p vertically composable 2–morphisms. For p-categories, this is a set $\{f_{0,j} \Rightarrow \cdots \Rightarrow f_{p,j} : f_{i,j} : x_j \to x_{j+1}, 0 \le j < q\}$. The face and degeneracy maps, both in the vertical and in the horizontal direction, are as for the ordinary nerve.

First, we compute some examples.

Example 9.5. We consider the discrete flow category of a discrete Morse function on S^2, as described in [17, Section 6.1]. Let S be the double nerve of the discrete flow category. By Theorem 9.4, we need only consider the elements of S that are not in the image of any degeneracy map (either vertical or horizontal) when constructing the spectral sequence.

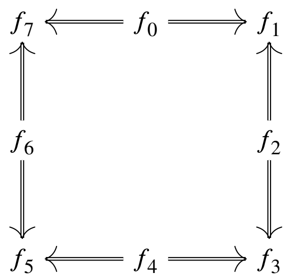


FIGURE 24. The Hom poset Hom(t, w).

As the discrete flow category has only two objects, t and w, there are no nondegenerate horizontal compositions, and thus $E^0_{0,q} = 0$ for $q \ge 2$. In the bottom row, we have $E^0_{0,0} = \mathbb{Z}^2$, corresponding to the two objects, and $E^0_{p,0} = 0$ for $p \ge 1$. Furthermore, Hom(t, w) (illustrated in Figure 24) has 8 nondegenerate 1-morphisms, 8 nondegenerate 2-morphisms (i.e., partial orders) and no nondegenerate compositions of morphisms, so $E^0_{0,1} = E^0_{1,1} = \mathbb{Z}^8$ and $E^0_{p,1} = 0$ for $p \ge 2$.

The 0th page, E^0, is illustrated in Figure 25, together with the two succeeding pages. To compute E^1, we need only find the differential $\partial^0_{0,1} : E^0_{0,1} \to E^0_{0,0}$. This image takes $f_i : t \to w$ to $(w - t)$, and hence its image is $\mathbb{Z}(1, -1) \cong \mathbb{Z}$. This gives us E^1, as illustrated.

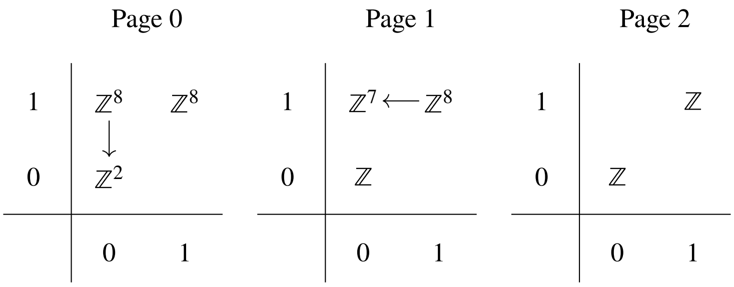


FIGURE 25. The first three pages of the spectral sequence for the discrete flow category of a discrete Morse function on S^2.

To compute E^2, we need the differential $\partial^1_{1,1} : E^1_{1,1} \to E^1_{0,1}$. Observe that that $E^1_{0,1}$ is generated by $[f_i - f_0], i \in 1, \dots, 7$. From this, it's clear that $\partial^1_{1,1}$ is surjective (for example, $[f_3 - f_0]$ is $\partial^1 \left[(f_2 \Rightarrow f_3) - (f_2 \Rightarrow f_1) + (f_0 \Rightarrow f_1)\right]$). Hence, we get E^2 as illustrated. The resulting homology of the total complex, and hence S^2, is then as follows.

$$H_n(S^2) = \begin{cases} \mathbb{Z}, & n = 0, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Example 9.6. We compute a slightly more complicated example, the discrete flow category of the discrete Morse function on D^3 from Example 7.21. There are three critical cells: f, t and x. We computed the Hom poset $\mathrm{Hom}(f, x)$ in Example 7.21. The Hom poset $\mathrm{Hom}(t, x)$ is as for the S^2 example, i.e., the one illustrated in Figure 24. Finally, $\mathrm{Hom}(f, t)$ consists of just one morphism, $(f > t)$.

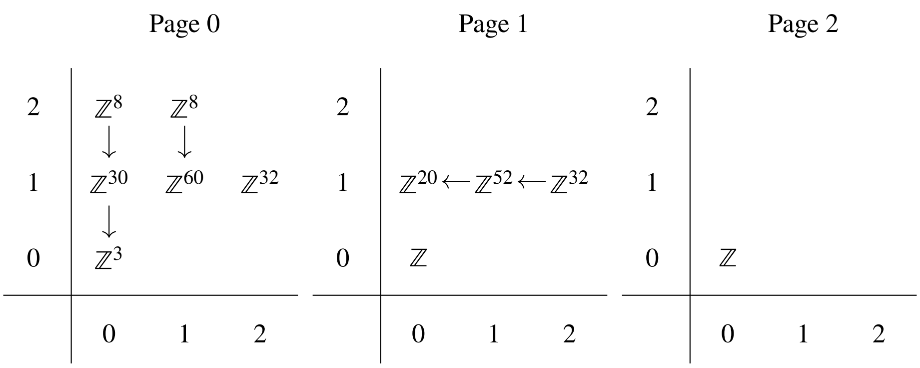


FIGURE 26. The first three pages of the spectral sequence for the discrete flow category of a discrete Morse function on D^3.

We count the nondegenerate simplices to get E^0. In $E_{p,0}$, as in the previous example, we get the $E^0_{p,0} = 0$ for $p \geq 1$, and $E^0_{0,0} = \mathbb{Z}^3$, one copy of $\mathbb{Z}$ for each of the critical cells. There are 30 non-identity morphisms, so $E^0_{0,1} = \mathbb{Z}^{30}$. There are 60 non-identity partial order relations, so $E^0_{1,1} = \mathbb{Z}^{60}$. The only Hom poset with composable partial orders is $\mathrm{Hom}(f, x)$, and there are 32 possible nondegenerate pairs of composable partial orders, so $E^0_{2,1} = \mathbb{Z}^{32}$. Finally, there are no nondegenerate triples of composable partial orders, so $E^0_{p,1} = 0$ for $p \geq 3$.

The only non-identity (horizontally) composable morphisms are $(f > t)$ composed with one of the eight morphisms in $\mathrm{Hom}(t, x)$, so $E^0_{0,2} = \mathbb{Z}^8$. Similarly, the only nondegenerate horizontally composable partial orders are $((f > t) \Rightarrow (f > t))$ composed with one of the eight non-identity partial orders in $\mathrm{Hom}(t, x)$[5]. Thus, $E^0_{1,2} = \mathbb{Z}^8$. Now, all elements in $S_{2,2}$ includes either a vertical identity (on $(f > t)$) or a horizontal identity (on f, t or x), and is thus degenerate, so $E^0_{2,2} = 0$, and the same holds for $E^0_{p,2}$ for $p \geq 3$.

Finally, any horizontal composition of 3 or more must include an identity (as we only have 3 critical cells), so $E^0_{p,q} = 0$ for $q \geq 3$. We can now draw the 0th page of the spectral sequence. The first three pages are shown in Figure 26.

To get E^1 from E^0, me must inspect the three differentials

$$\partial^0_{0,1} : E^0_{0,1} \to E^0_{0,0},$$
$$\partial^0_{0,2} : E^0_{0,2} \to E^0_{0,1}, \text{ and}$$
$$\partial^0_{1,2} : E^0_{1,2} \to E^0_{1,1}.$$

The image of $\partial^0_{0,1}$ is the span of $\{f - t, f - x, t - x\}$, so its cokernel is $\mathbb{Z}$. Now, $\partial^0_{0,2}$ takes a composable pair $\{f, g\}$ to $(f + g - f \circ g)$. Now, as $f \circ g$ is different for each of the composable pairs $\{f, g\}$ in $E^0_{0,2}$, the kernel of $\partial^0_{0,2}$ is 0. Furthermore, we see that computing the cokernel of $\partial^0_{0,2}$ amounts to identifying $f \circ g$ with $f + g$ for each of these 8 compositions, so the cokernel is torsion-free and $E^1_{0,1} = \mathbb{Z}^{30-8-2} = \mathbb{Z}^{20}$. By a similar argument, the kernel of $\partial^0_{1,2}$ is 0 and its cokernel is $\mathbb{Z}^{52}$.

Computing E^2 from E^1 can be done through tedious computation. Working out this computation gives that $\partial^1_{1,1}$ is surjective and that $\operatorname{im} \partial^1_{2,1} = \ker \partial^1_{1,1}$, so $E^2_{0,1}$, $E^2_{1,1}$ and $E^2_{2,1}$. The spectral sequence now collapses, and we get that the computed homology $\mathbb{Z}$ in degree 0 and 0 everywhere else, as expected from D^3.

The key takeaway from this example is that all rows above $q = 1$ became 0 everywhere on the E^1–page. As we shall now show, this always happens, which causes the spectral sequence to collapse on the second page. For the rest of the section, when we write a *discrete flow category*, we shall mean a discrete flow category induced from a discrete Morse function on a regular CW complex, as defined in Section 6.

For the following result, recall that for a p-category C, $\overline{\mathrm{N}}\, C$ is the simplicial category described in Section 5.3.

Theorem 9.7. *Let X be a regular CW complex and let Σ be a Morse system induced from a discrete Morse function. Then $(\overline{\mathrm{N}}\,\mathrm{Flo}_\Sigma[X])([n])$ is a unique factorization category for all $n \geq 0$.*

Proof. As observed in Example 8.14, $(\overline{\mathrm{N}}\,\mathrm{Flo}_\Sigma[X])([0])$, which is just $\mathrm{Flo}_\Sigma[X]$ with the poset structure removed, is a unique factorization category.

Now, let $n \geq 1$. The morphisms in $(\overline{\mathrm{N}}\,\mathrm{Flo}_\Sigma[X])([n])$ are represented by $f_0 \Rightarrow \cdots \Rightarrow f_n$ with f_i morphisms in $\mathrm{Flo}_\Sigma[X]$. It is clear that $(\overline{\mathrm{N}}\,\mathrm{Flo}_\Sigma[X])([n])$ is a finite directed category. Furthermore, given a representative $f_0 \Rightarrow \cdots \Rightarrow f_n$, $f_0 : x \to y$ has a unique factorization through critical cells $x = c_0, c_1, \ldots, c_k = y$. By Theorem 7.7, then all f_i factors through these critical cells. Let $f_i = f_i^1 \circ \ldots \circ f_i^k$ be the the decomposition of f_i through $c_0, \ldots, c_k$. We then have a decomposition $\{f_0, \ldots, f_n\} = \{f_0^1, \ldots, f_n^1\} \circ \ldots \circ \{f_0^k, \ldots, f_n^k\}$. The sets on the right hand side are indecomposable, as f_0^i is indecomposable for all i. Furthermore, the decomposition is unique, as $f_0^1 \circ \ldots f_0^k$ is the unique decomposition of f_0, and all other f_i^j are decided uniquely by this decomposition. □

Corollary 9.8. *Let $\{E^r_{*,*}\}$ be the spectral sequence associated to a discrete flow category. Then $E^1_{p,q} = 0$ for $q \geq 2$*

[5]Even though the partial order $((f > t) \Rightarrow (f > t))$ is degenerate, its horizontal composition with a nondegenerate partial order is not.

Proof. Let C be the discrete flow category. By definition, $E^1_{p,q}$ is the qth homology of the simplicial set $E^0_{p,*} = (\mathrm{N}\overline{\mathrm{N}}\,C)_{p,*} = \mathcal{N}\left((\overline{\mathrm{N}}\,C)([p])\right)$. By combining Theorem 9.7 and Corollary 8.16, we get that this is 0 when $q \geq 2$. □

Lemma 9.9. *Let* $\{E^r_{*,*}\}$ *be the spectral sequence associated to a discrete flow category. Then* $E^2_{p,0} = 0$ *for* $p \geq 1$.

Proof. Let C be the discrete flow category. We have that $E^1_{p,0}$ is the free Abelian group on the connected components of $\mathcal{N}\left((\overline{\mathrm{N}}\,C)([p])\right)$. Now, if a and b are connected in $\mathcal{N}\left((\overline{\mathrm{N}}\,C)([p])\right)$, then they are connected in $\mathcal{N}\left((\overline{\mathrm{N}}\,C)([i])\right)$ for all i. Hence, $E^1_{p,0} = \{(s_0)^p[x] : [x] \in E^1_{0,0}\}$. Now, as $\partial_h(s_0)^p[x]$ is $[x]$ when p is odd and 0 when p is even, taking the horizontal homology gives us 0 everywehere except in $E^1_{0,0}$. □

With the previous two results, we are finally ready to prove Theorem B.

Theorem 9.10 (Theorem B)**.** *The spectral sequence associated to a discrete flow category collapses on the second page.*

Proof. As $E^1_{p,q} = 0$ for $q \geq 2$, the same applies to E^2. Hence, the only potential nonzero entries on page 2 are $E^2_{0,0}$ and $E^2_{p,1}, p \geq 0$. Hence, there can be no nonzero differentials from page 2 and onward. □

Note that the above results applies to the ordinary spectral sequence associated to a bisimplicial set, *not* the one where degenerate simplices have been removed (as in Theorem 9.4). However, for computational purposes it is useful to remove degenerate simplices from the spectral sequence, as the amount of degenerate simplices is always infinite. Therefore, we now show that the spectral sequence also collapses on page 2 when removing degeneracies.

Theorem 9.11. *Let X be the double nerve of a discrete flow category. Then the spectral sequence associated to the double complex*

$$FX/F(D^h \cup D^v)$$

collapses on the second page.

Proof. Let $\{E^r_{*,*}\}$ be the associated spectral sequence. Firstly, there are no nondegenerate simplices in $X_{p,0}$ for $p \geq 1$, so $E^0_{p,0} \cong 0$ for $p \geq 1$ (and the same holds for the succeeding pages).

We now show that $E^1_{p,q} \cong 0$ for $q \geq 2$. Recall that $E^1_{p,q} = H_{II}(FX/F(D^h \cup D^v))_{p,q}$ First, we show that $H_{II}(FX/FD^h)_{p,q} \cong 0$ for $q \geq 2$. Suppose that $[x] \in (FX/FD^h)_{p,q}$, with $q \geq 2$, such that $\partial_v[x] = 0$. Then $x \in FX_{p,q}$ is such that $\partial_v x \in FD^h$, so $\partial_v x = s^h_0 y_0 + \cdots + s^h_{p-1} y_{p-1}$ for some $y_0, \ldots, y_{p-1} \in FX_{p-1,q}$, where $s^h_i = (s_i, \mathrm{id})$ is the ith horizontal degeneracy map. Now, let $x' = x - s^h_{p-1} d^h_p x$. Then $[x'] = [x]$ in $(FX/FD^h)_{p,q}$, and

$$\begin{aligned}\partial_v x' &= \partial_v x - \partial_v(s^h_{p-1} d^h_p)x = \partial_v x - (s^h_{p-1} d^h_p)\partial_v x \\ &= \left(s^h_0 y_0 + \cdots + s^h_{p-1} y_{p-1}\right) - \left(s^h_{p-1} d^h_p s^h_0 y_0 + \cdots + s^h_{p-1} d^h_p s^h_{p-1} y_{p-1}\right).\end{aligned}$$

We now utilize the fact that $d_p s_{p-1} = \mathrm{id}$ and that when $i < p-1$, $s_{p-1} d_p s_i = s_{p-1} s_i d_{p-1} = s_i s_{p-2} d_{p-1}$. Thus, the sum above rewrites to

$$\begin{aligned}& s^h_0(y_0 - s^h_{p-2} d^h_{p-1} y_0) + \cdots + s^h_{p-2}(y_{p-2} - s^h_{p-2} d^h_{p-1} y_{p-2}) + s^h_{p-1}(y_{p-1} - y_{p-1}) \\ &= s^h_0 y'_0 + \ldots s^h_{p-2} y'_{p-2}.\end{aligned}$$

Now, continuing this process by letting $x'' = x' - s^h_{p-2} d^h_{p-1} x'$, and so on, we end up with a $\hat{x} \in FX_{p,q}$ such that $[\hat{x}] = [x] \in (FX/FD^h)_{p,q}$ and $\partial_v \hat{x} = 0$. But as $H_{II}(FX)_{p,q}$ is 0 when $q \geq 2$, then $\hat{x}$ is in the image of ∂_v, and thus $[x]$ is in the image of ∂_v. Hence, $H_{II}(FX/FD^h)_{p,q} \cong 0$ when $q \geq 2$.

It remains to show that $H_{II}(FX/F(D^h \cup D^v))_{p,q} \cong 0$ when $q \geq 2$. For this, we use that $G/(AB) = (G/A)/(B/(A \cap B))$ for groups $A, B < G$, so we have a short exact sequence of chain complexes

$$0 \to FD^v/(FD^h \cap FD^v) \to FX/FD^h \to FX/F(D^h \cup D^v) \to 0. \tag{12}$$

We know from Corollary 2.6 that $H_{II}(FD^v) \cong 0$. Furthermore, one can show that $D^h_{p,*}$ together with the vertical face and degeneracy maps is a simplicial set[6]. The degeneracies in this simplicial set are $D^h \cap D^v$, so, again by Corollary 2.6, $H_{II}(F(D^h \cap D^v)) = H_{II}(FD^h \cap FD^v)) \cong 0$. In then follows (from the long exact sequence in homology of short exact sequences of chain complexes) that $H_{II}(F^v/(FD^h \cap FD^v)) = 0$.

Finally, we can use the long exact sequence in homology from the short exact sequence in (12) to conclude that $H_{II}(FX/F(D^h \cup D^v))_{p,q} \cong H_{II}(FX/FD^h)_{p,q}$ (for all p and q). In particular, $H_{II}(FX/F(D^h \cup D^v))_{p,q} \cong 0$ for $q \geq 2$.

Hence, $E^1_{p,q}$ is 0 when $q \geq 2$, and this, together with the fact that $E^1_{p,0} \cong 0$ when $p \geq 1$, shows that the spectral sequence collapses on the second page. □

Observe that due to this theorem, when computing the spectral sequence, we never need to know $E^0_{p,q}$ for $q \geq 3$. In fact, the only things we need to compute at page 1 is the row $E^1_{*,1}$ and the single square $E^1_{0,0}$. The squares in $E^1_{*,1}$ also have a relatively simple description; the differential $\partial_v E^0_{p,2} \to E^0_{p,1}$ sends a composable pair $\{f, g\}$ to $f + g - g \circ f$. Hence, $E^1_{p,1}$ is the same as $E^0_{p,1}$ where all morphisms have been identified with the sum of the morphisms in their (unique) decomposition of indecomposable morphisms.

We now show another example of a spectral sequence computation, using Theorem 9.11 and the observations in the last example.

Example 9.12. We compute the discrete flow category of a discrete Morse function on the 2–torus, illustrated in Figure 27. We compute the associated spectral sequence (with degeneracies removed), with rational coefficients (to make the computations easier).

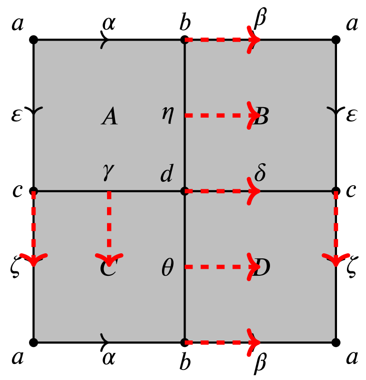


FIGURE 27. A discrete Morse function (represented by its gradient vector field) on a regular CW decomposition of the 2–torus. The regular pairs are given by red, dashed arrows.

[6]This follows from the fact that if x is a horizontal degeneracy, then so is $d^v_i x$ and $s^v_i x$, as $d^v_i s^h_j = s^h_j d^v_i$ and $s^v_i s^h_j = s^h_j s^v_i$.

The critical cells are A, α, ϵ and a. We compute the Hom posets with Algorithm 1, and get:

$$\begin{aligned}
\mathrm{Hom}(A,\alpha) &= \left\{f_1 = [A > \alpha],\ f_2 = [A > \gamma < C > \alpha]\right\},\\
\mathrm{Hom}(A,\varepsilon) &= \left\{f_1' = [A > \varepsilon],\ f_2' = [A > \eta < B > \varepsilon]\right\},\\
\mathrm{Hom}(\alpha,a) &= \left\{g_1 = [\alpha > a],\ g_2 = [\alpha > b < \beta > a]\right\},\\
\mathrm{Hom}(\varepsilon,a) &= \left\{g_1' = [\varepsilon > a],\ g_2' = [\varepsilon > c < \zeta > a]\right\}.
\end{aligned}$$

The only Hom poset with nontrivial partial orders is $\mathrm{Hom}(A,a)$, which consists of 36 morphisms in four connected components, as illustrated in Figure 28.

$$\begin{array}{l}
g_1 \circ f_1 \Rightarrow \bullet \Leftarrow g_1' \circ f_1' \\[1em]
g_2 \circ f_1 \Rightarrow \bullet \Leftarrow \quad \dots \quad \Rightarrow \bullet \Leftarrow g_1' \circ f_2' \\[1em]
g_1 \circ f_2 \Rightarrow \bullet \Leftarrow \quad \dots \quad \Rightarrow \bullet \Leftarrow g_2' \circ f_1' \\[1em]
g_2 \circ f_2 \Rightarrow \bullet \Leftarrow \quad \dots \quad \Rightarrow \bullet \Leftarrow g_2' \circ f_2'
\end{array}$$

FIGURE 28. The four components in the Hom poset $\mathrm{Hom}(A,a)$. The unnamed morphisms (represented by dots) are all indecomposable.

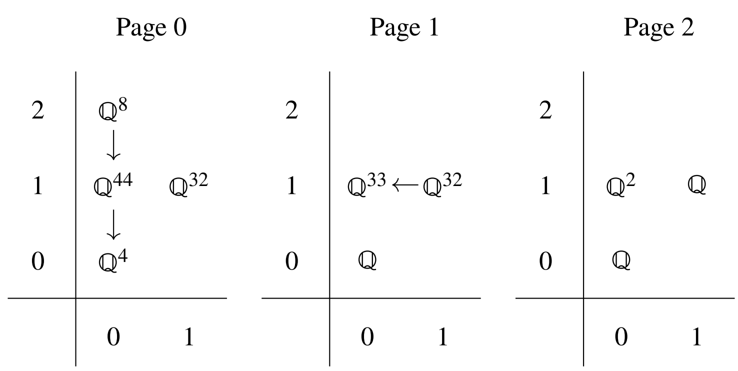


FIGURE 29. The first three pages of the spectral sequence for a discrete flow category of the 2–torus.

The discrete flow category has 4 objects, 44 morphisms, 8 nondegenerate horizontal compositions of morphisms, 32 non-identity partial orders, and no other nondegenerate compositions. Hence, the E^0 page is as illustrated in Figure 29.

On page 1, $E^1_{0,0}$ is $\mathbb{Q}$, as the nerve of the category has one connected component. Hence, $E^1_{0,1} = \mathbb{Q}^{44-8-3} = \mathbb{Q}^{33}$. Furthermore, $E^1_{0,1}$ can be described as the cycles in $E^0_{0,1}$, where all compositions $f \circ g$ has been identified with $f + g$.

Now, to compute E^2, we need only determine the map $\partial^2_{1,1} : E^1_{1,1} \to E^1_{0,1}$. For this, observe that for each of the four components in Figure 28, we can assign an alternating sum of partial orders x_i, so that

$$\begin{aligned}\partial^2 x_1 &= g_1'\circ f_1' - g_1\circ f_1 = g_1' + f_1' - g_1 - f_1,\\ \partial^2 x_2 &= g_1'\circ f_2' - g_2\circ f_1 = g_1' + f_2' - g_2 - f_1,\\ \partial^2 x_3 &= g_2'\circ f_1' - g_1\circ f_2 = g_2' + f_1' - g_1 - f_2,\\ \partial^2 x_4 &= g_2'\circ f_2' - g_2\circ f_2 = g_2' + f_2' - g_2 - f_2.\end{aligned}$$

Now, $\partial^2(x_1 - x_2 - x_3 + x_4) = 0$. One can verify (for example through writing ∂^2 as a matrix and row reducing) that $(x_1 - x_2 - x_3 + x_4)$ generates the kernel of ∂^2, so that $E^2_{1,1} = \mathbb{Q}$ and $E^2_{0,1} = \mathbb{Q}^2$. The spectral sequence now collapses, as predicted by Theorem 9.11, and the computed homology of the 2–torus is:

$$H_n(T^2;\mathbb{Q}) = \begin{cases}\mathbb{Q}, & n = 0, 2,\\ \mathbb{Q}^2, & n = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Observe that in this case we got a nonzero group in $E^2_{1,1}$ even though the homology of (the nerve of) the Hom poset $\mathrm{Hom}(A, a)$ is 0 in degree 1. This happens because on page 1, the compositions $f\circ g$ becomes identified with the sums $f + g$, so that the differential $\partial^2 : E^1_{1,1} \to E^1_{0,1}$ has a nonzero element in the kernel, even though the differential $\partial_h : E^0_{1,1} \to E^0_{0,1}$ does not.

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