Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Metaplectic Transformations for Gabor Frames and Equivalence Bimodules

Master's thesis in Mathematical Sciences
Supervisor: Franz Luef
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Abstract

The aim of this thesis is to present and explore a symplectic approach to Gabor analysis – Gabor analysis is a subject in time-frequency analysis with close ties to information and communication technologies as well as to theoretical and applied physics. Whereas the structure of Gabor frames over the two-dimensional time-frequency plane is well-understood, the higher-dimensional cases remain much more elusive. We present and develop a classification scheme for lattices in arbitrary dimensions; we show that the structure of Gabor frames over a given lattice is uniquely determined by a set of symplectic forms determined by the lattice. This framework is built on the metaplectic representation, which relates Gabor systems over lattices determining the same symplectic forms via unitary transformations.

It is becoming increasingly clear that equivalence bimodules over noncommutative tori provide a powerful framework for Gabor analysis. We also lift this classification scheme to the setting of such bimodules by constructing isomorphisms between equivalence bimodules associated to lattices which determine the same symplectic forms. We do this by extending metaplectic transformations. In addition to this, we explore how the notion of Morita equivalence of noncommutative tori relates to duality in Gabor analysis from a novel point of view. This leads us to consider the notion of *partially adjoint lattices*.

The thesis also contains a detailed introduction to the theory of C*-algebras, Hilbert C*-modules and the equivalence bimodules that feature in Gabor analysis. We have attempted to build the theory in a systematic manner while assuming minimal prerequisite knowledge.

Sammendrag

Målet med denne avhandlingen er å presentere og utforske en symplektisk tilnærming til gaboranalyse – gaboranalyse er et emne innen tid-frekvens-analyse som er nært knyttet både til informasjon- og kommunikasjonsteknologier og til teoretisk og anvendt fysikk. Teorien om gaborrammer over det todimensjonale tid-frekvens-planet er godt utviklet, men tilfellet med et vilkårlig antall dimensjoner har vist seg å være mer komplisert. Vi gjør rede for og videreutvikler et system for klassifisering av gitter i et vilkårlig antall dimensjoner; vi viser at strukturen til gaborrammer over et gitt gitter kun er avhengig av en mengde med symplektiske former bestemt av gitteret. Dette rammeverket bygger på den metaplektiske representasjonen, som relaterer gaborsystemer over gitter som bestemmer de samme symplektiske formene via unitære transformasjoner.

Det blir stadig mer tydelig at ekvivalens-bimoduler over ikkekommutative toruser utgjør et nyttig rammeverk for gaboranalyse. Vi løfter også dette klassifiseringssystemet til disse ekvivalens-bimodulene ved å konstruere isomorfier mellom ekvivalens-bimoduler assosiert med gitter som bestemmer de samme symplektiske formene. Dette gjør vi ved å utvide metaplektiske transformasjoner. I tillegg til dette utforsker vi hvordan moritaekvivalens mellom ikkekommutative toruser er relatert til dualiteten i gaboranalyse fra et nytt perspektiv. Dette fører oss til å betrakte konseptet *delvis adjungerte gitter*.

Utover dette inneholder avhandlingen en detaljert introduksjon til teorien om C*-algebraer, Hilbert C*-moduler og ekvivalens-bimodulene som inngår i gaboranalyse. Vi har forsøkt å bygge teorien på en systematisk måte som antar færrest mulig forkunnskaper.

Contents

Introduction

Gabor analysis is a subject in time-frequency analysis. As an independent branch of mathematics, time-frequency analysis is quite modern – emerging only around the year 1980. Its historical roots, however, stretch across the 20th century and are deeply entangled with the origins of quantum mechanics as well as the development of information and communication technologies. It is a rapidly growing and already vast interdisciplinary subject, of interest to mathematicians, physicist and engineers alike.

The aim of time-frequency analysis, broadly construed, is to provide a framework for analysing the frequency spectrum of a signal *locally*. While the Fourier transform lets us study the frequency spectrum of a signal, it provides only global information – it cannot answer the question of *when* a given frequency is prominent. In this manner, time-frequency analysis is a refinement of Fourier theory: the two domains of Fourier theory, the time and frequency domains, unify to form the *time-frequency plane*. A function no longer has one representation in the time domain and another in the frequency domain, but a single *time-frequency representation*: a function on the time-frequency plane.

The subject of Gabor analysis is where the historical and contemporary ties to engineering and signal processing are most apparent. Indeed, the person after which the subject is named, Dennis Gabor, was an electrical engineer (as well as a physicist). The foundational idea is to replace the time-frequency plane with a discrete set, typically a lattice. This is of course a prerequisite for numerical time-frequency analysis. In mathematical terms, we are studying classes of highly structured and countable bases for spaces of functions on the time-frequency plane. To be more precise, we are not studying bases, but *frames*, which generalize the notion of bases. The frames that feature in Gabor analysis are known as *Gabor frames*. For every choice of lattice, there is an associated set of Gabor frames *supported* by that lattice.

There is a notion of duality that shapes the subject of Gabor analysis: for every lattice on the time-frequency plane, there is an associated lattice, referred to as its *adjoint lattice*. The properties of a lattice and its adjoint, from the vantage of Gabor analysis, are in many ways dual to each other. Over the last two decades, my advisor, Franz Luef, has uncovered an intriguing connection between duality in Gabor analysis and the notion of *equivalence bimodules*. These are quite abstract constructions motivated by the representation theory of *C*-algebras*, which are particular kinds of algebras of operators. Just as Fourier theory benefits greatly from the abstract theory of Hilbert spaces, Gabor analysis is enriched by the theory of C*-algebras. This connection between duality and equivalence bimodules is a fascinating and promising aspect of this relationship.

Much is know about Gabor frames in the case of one-dimensional signals, corresponding to a two-dimensional time-frequency plane. The general case, with signals of arbitrary dimension, has proven to be much more unwieldy. The main argument of this thesis is that the structure of Gabor frames supported by a given lattice is uniquely determined by a certain *symplectic* structure determined by the lattice. This leads to a classification scheme for lattices in Gabor analysis. This is not really relevant in the case of one-dimensional signals, for there is essentially a unique symplectic structure in that case. Our goal is to show how precise and powerful this classification scheme is in the general case. In particular, we will show how this method of classification can be lifted to the setting of equivalence bimodules. This approach is based on ideas by Luef.

Symplectic geometry is a deep mathematical subject with close ties to both classical and quantum mechanics. We will spend the first chapter introducing those parts of the subject that will be of relevance to us. Because the theory of C*-algebras features so prominently in our constructions, we have devoted the second chapter to a detailed introduction to the subject. The third chapter serves as an introduction to time-frequency analysis and Gabor analysis. In particular, we wish to highlight Subsection 3.2.2: this is where we provide a more precise outline of our classification scheme. The reader who is already familiar with Gabor analysis may want to peek at this subsection in order to get a better understanding of what we aim to achieve.

The equivalence bimodules introduced into Gabor analysis by Luef are particular kinds of Hilbert C*-modules over noncommutative tori. Chapters 4 and 5 are spent introducing these notions and constructing the relevant bimodules in great detail. In Chapter 5, we also spend some time further exploring the connection between duality and equivalence bimodules.

Finally, in Chapter 6, we introduce *metaplectic transformations*. These are the transformations that facilitate our classification scheme – hence the title. The final subsection, Subsection 6.1.3, contains the precise formulation of this scheme. In the end, the proofs are fairly simple; it is the development of all the required background material that fills the pages.

Notation and conventions

Throughout this thesis, d will denote an arbitrary positive integer and $\{e_j\}_{j=1}^{2d}$ will denote the standard basis for $\mathbb{R}^{2d}$. The set of all $2d \times 2d$ matrices over $\mathbb{R}$ will be denoted by $M_{2d}(\mathbb{R})$ and we will write $\mathrm{GL}(2d, \mathbb{R})$ for its group of invertible elements. We will frequently identify elements of $M_{2d}(\mathbb{R})$ with the linear transformations they represent with respect to the standard basis. We will write $\mathbb{N}_0$ and $\mathbb{N}_1$ to denote the natural numbers including and excluding zero, respectively – we will simply write $\mathbb{N}$ when the distinct doesn't matter. For sequences $\mathbb{N} \to X$ (for any set X), we may write either $(x_n)_{n\in\mathbb{N}}, (x_n)_n$ or simply (x_n). When it comes to linear maps, we may or may not use parentheses for evaluations and we may or may not use the symbol $\circ$ when considering compositions; the particulars of each situation dictate our choices. Finally, overlines will be used for complex conjugation and for the closures of sets in topological spaces (and sometimes for completions).

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Michael Gjertsen

Chapter 1

Linear Symplectic Algebra

This chapter is intended to serve as an introduction to the subject of linear symplectic algebra. Our presentation is largely shaped by our needs; the reader may consult our main source, de Gosson [15], for a much more complete treatment.

1.1 | Symplectic Vector Spaces

1.1.1 Symplectic Forms

The structure studied in linear symplectic algebra consists of a real vector space equipped with a symplectic form, a notion we now introduce.

1.1.1 Definition (Symplectic forms)**.** Let V be a vector space over $\mathbb{R}$. A bilinear map $\Omega\colon V \times V \to \mathbb{R}$ is called a *symplectic form* on V if it is

(i) antisymmetric: $\Omega(w, v) = -\Omega(v, w)$ for all $v, w \in V$.

(ii) nondegenerate: $\Omega(v, w) = 0$ for all $v \in V$ implies that $w = 0$.

The pair (V, Ω) is referred to as a *symplectic vector space*. For $w \in V$, we will write $\Omega(-, w)$ to denote the linear map $V \to \mathbb{R}$ defined by $\Omega(-, w)(v) = \Omega(v, w)$.

We will almost exclusively restrict our attention to the case $V = \mathbb{R}^{2d}$. A choice of basis lets us represent symplectic forms on $\mathbb{R}^{2d}$ in terms of matrices. We will use the standard basis $\{e_j\}_{j=1}^{2d}$ for $\mathbb{R}^{2d}$. Fix a symplectic form Ω on $\mathbb{R}^{2d}$, define

$$\theta_{ij} \coloneqq \Omega(e_j, e_i) \quad \text{for } 1 \leq i, j \leq 2d \tag{1.1}$$

and consider the $2d \times 2d$-matrix $\theta \coloneqq (\theta_{ij})$. For $z = (z_1, \ldots, z_{2d})$ and $w = (w_1, \ldots, w_{2d})$ in $\mathbb{R}^{2d}$, we find that

$$\Omega(z,w) = \Omega\Big(\sum_{j=1}^{2d} z_j e_j, \sum_{i=1}^{2d} w_i e_i\Big) = \sum_{j=1}^{2d}\sum_{i=1}^{2d} w_i \theta_{ij} z_j = w^T \theta z.$$

The conditions for Ω to define a symplectic form translate into conditions on the matrix θ. We capture this relationship between symplectic forms and matrices in the following lemma and the subsequent definition.

1.1.2 Lemma. *Let $\theta \in M_{2d}(\mathbb{R})$. The bilinear map $\Omega_\theta \colon \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}$ defined by*

$$\Omega_\theta(z,w) = w^T \theta z \quad \text{for } z, w \in \mathbb{R}^{2d} \tag{1.2}$$

is a symplectic form if and only if θ is antisymmetric and invertible, i.e. if and only if $\theta^T = -\theta$ and $\det \theta \neq 0$.

Proof. Assume that Ω_θ is a symplectic form. Inserting standard basis elements into Equation (1.2) gives $\theta_{ij} = \Omega_\theta(e_j, e_i)$, so

$$\theta_{ji} = \Omega_\theta(e_i, e_j) = -\Omega_\theta(e_j, e_i) = -\theta_{ij} \quad \text{for } 1 \leq i, j \leq 2d.$$

Thus, θ is antisymmetric. If $\theta z = 0$ for some $z \in \mathbb{R}^{2d}$, then $\Omega_\theta(z,w) = 0$ for all $w \in \mathbb{R}^{2d}$, so $z = 0$ by nondegeneracy of Ω_θ. This shows that θ is invertible.

Assume now that θ is antisymmetric and invertible. Then,

$$\Omega_\theta(w,z) = z^T \theta w = (w^T \theta^T z)^T = w^T \theta^T z = -w^T \theta z = -\Omega_\theta(z,w)$$

for all $z, w \in \mathbb{R}^{2d}$, so Ω_θ is antisymmetric. Nondegeneracy follows from the fact that $\Omega_\theta(z, \theta^{-T} z) = z^T \theta^{-1} \theta z = z^T z$ for all $z \in \mathbb{R}^{2d}$. $\square$

1.1.3 Definition. Consider the set $M_{2d}(\mathbb{R})$ of all $2d \times 2d$-matrices over $\mathbb{R}$ and define the subsets

$$\mathcal{T}_{2d} \coloneqq \big\{A \in M_{2d}(\mathbb{R}) : A^T = -A\big\} \quad \text{and} \quad \mathcal{S}_{2d} \coloneqq \mathcal{T}_{2d} \cap \mathrm{GL}(2d, \mathbb{R}).$$

For $\theta \in \mathcal{S}_{2d}$, we refer to the symplectic form Ω_θ defined by Equation (1.2) as the symplectic form *represented* by θ.

Lemma 1.1.2, along with the discussion prior to it, shows that symplectic forms on $\mathbb{R}^{2d}$ are in bijective correspondence with the elements of $\mathcal{S}_{2d}$.

We now turn our attention to a particularly simple symplectic form. Let I_d denote the $d \times d$ identity matrix. The matrix

$$J := \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \tag{1.3}$$

is known as the *standard symplectic matrix*. It satisfies

$$J^2 = I \quad \text{and} \quad J^{-1} = J^T = -J,$$

where I is the $2d \times 2d$ identity matrix. The symplectic form Ω_J represented by J is called the *standard symplectic form* on $\mathbb{R}^{2d}$. If we write $z = (x, \omega)$ and $w = (y, \eta)$ with $x, \omega, y, \eta \in \mathbb{R}^d$, it is given by

$$\Omega_J(z, w) = w^T J z = y \cdot \omega - \eta \cdot x, \tag{1.4}$$

where the dot denotes the standard inner product on $\mathbb{R}^d$.

It turns out that any symplectic form is related to Ω_J by a change of basis. This is the content of Proposition 1.1.8. Before we show this, we collect some basic facts from linear algebra regarding dual spaces and introduce some convenient terminology.

Let V be a finite dimensional vector space over $\mathbb{R}$. Linear maps $\tau \colon V \to \mathbb{R}$ are referred to as *linear functionals* on V, and the vector space of all linear functionals on V (with addition and scalar multiplication defined pointwise) is called the *dual space* of V and denoted by V^*.

1.1.4 Lemma. *Let V be a finite dimensional vector space over $\mathbb{R}$ and assume that $\tau_1, \dots, \tau_k \in V^*$ are linearly independent (for some integer $k \geq 1$). Then, the following statements are true.*

(i) $\tau \in \operatorname{span}\{\tau_1, \dots, \tau_k\} \quad \iff \quad \bigcap_{j=1}^k \operatorname{Ker} \tau_j \subset \operatorname{Ker} \tau$.

(ii) There exist vectors $v_1, \dots, v_k$ in V such that $\tau_i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq k$.

(iii) We have $\dim\left(\bigcap_{j=1}^k \operatorname{Ker} \tau_j\right) = \dim V - k$.

Moreover, we have $\dim V^* = \dim V$.

Proof. The forward direction ($\Longrightarrow$) of the first point is immediate. For the converse, we use induction on $k \geq 1$. For $k = 1$, the claim is that if $\operatorname{Ker} \tau_1 \subset \operatorname{Ker} \tau$, then there is some $r \in \mathbb{R}$ such that $\tau = r\tau_1$. Let $w \in V$ be such that $\tau_1(w) = 1$ (linearly independent vectors are nonzero). Then,

$$\tau(v) = \tau\Big(\tau_1(v)w + \big(v - \tau_1(v)w\big)\Big) = \tau_1(v)\tau(w) \quad \text{for all } v \in V$$

since $v - \tau_1(v)w \in \operatorname{Ker}\tau_1 \subset \operatorname{Ker}\tau$. Thus, $\tau = \tau(w)\tau_1$, which proves the base case.

Let now $k > 1$. Our induction hypothesis is the following: for any collection of $k-1$ linearly independent functionals $\tau'_1, \ldots, \tau'_{k-1}$, we have that

$$\bigcap_{j=1}^{k-1} \operatorname{Ker}\tau'_j \subset \operatorname{Ker}\tau' \qquad \Longrightarrow \qquad \tau' \in \operatorname{span}\{\tau'_1, \ldots, \tau'_{k-1}\} \tag{1.5}$$

for all $\tau' \in V^*$. We need to show that for any collection of k linearly independent functional $\tau_1, \ldots, \tau_k$, we have that

$$\bigcap_{j=1}^{k} \operatorname{Ker}\tau_j \subset \operatorname{Ker}\tau \qquad \Longrightarrow \qquad \tau \in \operatorname{span}\{\tau_1, \ldots, \tau_k\}$$

for all $\tau \in V^*$.

If $\tau_1, \ldots, \tau_k$ are linearly independent, then τ_i (for $1 \le i \le k$) is not in the span of $\{\tau_1, \ldots, \tau_k\}\backslash\{\tau_i\}$. The contrapositive of our induction hypothesis (1.5) (with $\tau' = \tau_i$ and $\{\tau'_1, \ldots, \tau'_{k-1}\} = \{\tau_1, \ldots, \tau_k\}\backslash\{\tau_i\}$) therefore implies that

$$\bigcap_{j\in\{1,\ldots,k\}\backslash\{i\}} \operatorname{Ker}\tau_j \not\subset \operatorname{Ker}\tau_i \quad (\text{for } 1 \le i \le k).$$

Thus, we can find $v_1, \ldots, v_k$ such that $\tau_i(v_j) = \delta_{ij}$ for $1 \le i, j \le k$. This also shows how (i) implies (ii) in general. We then have that

$$v - \sum_{j=1}^{k} v_j \tau_j(v) \in \bigcap_{j=1}^{k} \operatorname{Ker}\tau_j \quad \text{for all } v \in V.$$

Now, if $\tau \in V^*$ is such that $\bigcap_{j=1}^{k} \operatorname{Ker}\tau_j \subset \operatorname{Ker}\tau$, then

$$\tau(v) = \tau\left(\sum_{j=1}^{k} v_j\tau_j(v) + \Big(v - \sum_{j=1}^{k} v_j\tau_j(v)\Big)\right) = \sum_{j=1}^{k} \tau(v_j)\tau_j(v),$$

for all $v \in V$. Thus, $\tau = \sum_{j=1}^{k} \tau(v_j)\tau_j \in \operatorname{span}\{\tau_1, \ldots, \tau_k\}$, which completes the proof of the first point.

We have already seen that (ii) follows from (i). To prove (iii), consider the linear map $T\colon V \to \mathbb{R}^k$ defined by

$$T(v) = \big(\tau_1(v), \ldots, \tau_k(v)\big) \quad \text{for } v \in V.$$

Since $\bigcap_{j=1}^k \operatorname{Ker}\tau_j = \operatorname{Ker} T$, the rank-nullity theorem applied to T implies that

$$\dim\Big(\bigcap_{j=1}^{k} \operatorname{Ker}\tau_j\Big) + \dim T(V) = \dim V.$$

The result now follows if we can show that $\dim T(V) = k$, i.e. that T is surjective. By (ii), we can find $v_1, \dots, v_k$ such that $\tau_i(v_j) = \delta_{ij}$, and then $T(v_1), \dots, T(v_k)$ is the standard basis for $\mathbb{R}^k$, so T is surjective. This concludes the proof of (iii) and also shows that $\dim V^* \leq \dim V$ (if we could take $k > \dim V$, the rank nullity theorem would imply that $\dim(\operatorname{Ker} T) < 0$).

In order to see that $\dim V^* \geq \dim V$, let $n := \dim V$ and let $(v_1, \dots, v_n)$ be a basis for V. We can define linear functionals $\tau_1, \dots, \tau_n$ by $\tau_j(v_i) = \delta_{ij}$ for $1 \leq i, j \leq n$. This gives $\bigcap_{j\in\{1,\dots,n\}\setminus\{i\}} \operatorname{Ker}\tau_j \not\subset \operatorname{Ker}\tau_i$ (consider v_i), so $\{\tau_1, \dots, \tau_n\}$ is a linearly independent set by (i), hence $\dim V^* \geq n$. This concludes the proof. $\square$

We will frequently appeal to the first part of the following lemma. We will have no use for the second part, but it gives us an alternate view of symplectic forms which seems worth mentioning.

1.1.5 Lemma. *Let Ω be a symplectic form on $\mathbb{R}^{2d}$. The linear map*

$$\begin{aligned} \mathbb{R}^{2d} &\to (\mathbb{R}^{2d})^* \\ w &\mapsto \Omega(-, w) \end{aligned}$$

is then an isomorphism.

Conversely, if $S\colon \mathbb{R}^{2d} \to (\mathbb{R}^{2d})^*$ is a linear isomorphism such that $w \in \operatorname{Ker} S(w)$ for all $w \in \mathbb{R}^{2d}$, then the bilinear map $(z, w) \mapsto S(w)(z)$ is a symplectic form on $\mathbb{R}^{2d}$.

Proof. By nondegeneracy of Ω, we have $\Omega(-, w) = 0$ if and only if $w = 0$. This proves that the map in question is injective, so it is an isomorphism since $\dim \mathbb{R}^{2d} = \dim(\mathbb{R}^{2d})^*$ by Lemma 1.1.4.

If $S\colon \mathbb{R}^{2d} \to (\mathbb{R}^{2d})^*$ is as in the statement of the lemma, then $(z, w) \mapsto S(w)(z)$ is nondegenerate since $S(w)(z) = 0$ for all $z \in \mathbb{R}^{2d}$ implies that $S(w) = 0$ and S is injective. Finally, antisymmetry follows from

$$\begin{aligned} 0 = S(w+z)(w+z) &= S(w)(w) + S(w)(z) + S(z)(w) + S(z)(z) \\ &= S(w)(z) + S(z)(w) \end{aligned}$$

for all $z, w \in \mathbb{R}^{2d}$. $\square$

The following terminology will be useful. It is borrowed from differential geometry.

1.1.6 Definition (Pullbacks of bilinear maps)**.** Let V and W be vector spaces over $\mathbb{R}$, let $T\colon V \to W$ be a linear map and let $\Gamma\colon W \times W \to \mathbb{R}$ be a bilinear map. The bilinear map

$$\begin{aligned} T^*\Gamma\colon V \times V &\to \mathbb{R} \\ (v,w) &\mapsto T^*\Gamma(v,w) := \Gamma(Tv, Tw) \end{aligned}$$

is called the *pullback* of Γ by T.

In the case that $V = W = \mathbb{R}^{2d}$ and $A \in M_{2d}(\mathbb{R})$ represents a linear map, we will write $A^*\Gamma$ for the pullback of Γ by the linear map represented by A, i.e. $A^*\Gamma(z,w) = \Gamma(Az, Aw)$.

Pullbacks are a very natural and simple notion, for $T^*\Gamma$ is essentially just the composition "Γ after T", but accounting for the fact that Γ takes two arguments. This is made precise by introducing the map $T \times T : V \times V \to W \times W$ defined by $(v,w) \mapsto (Tv, Tw)$, for then $T^*\Gamma = \Gamma \circ (T \times T)$. The ordering-reversing property shown in the following lemma is a reflection of this fact.

1.1.7 Lemma (Basic properties of pullbacks)**.** *Let $A, B \in M_{2d}(\mathbb{R})$ and let Γ be a bilinear map on $\mathbb{R}^{2d}$. Then, $(AB)^*\Gamma = B^*(A^*\Gamma)$. Moreover, if $A \in \mathrm{GL}(2d, \mathbb{R})$, then $(A^{-1})^*(A^*\Gamma) = \Gamma$.*

Proof. Let $z, w \in \mathbb{R}^{2d}$. Then,

$$B^*(A^*\Gamma)(z,w) = A^*\Gamma(Bz, Bw) = \Gamma(ABz, ABw) = (AB)^*\Gamma(z,w),$$

so that $B^*(A^*\Gamma) = (AB)^*\Gamma$. If $A \in \mathrm{GL}(2d, \mathbb{R})$, then $(A^{-1})^*(A^*\Gamma) = \Gamma$ follows from $B^*(A^*\Gamma) = (AB)^*\Gamma$ by setting $B = A^{-1}$. $\square$

We have now gathered the tools we need to see how all symplectic forms relate to the standard symplectic form.

1.1.8 Proposition (Ω-symplectic bases)**.** *Let Ω be a symplectic form on $\mathbb{R}^{2d}$. Then, there exists a basis $\{v_1, \dots, v_d, w_1, \dots, w_d\}$ for $\mathbb{R}^{2d}$ such that*

$$\Omega(v_i, v_j) = \Omega(w_i, w_j) = 0 \quad \textit{and} \quad \Omega(w_i, v_j) = \delta_{ij}$$

for $1 \le i, j \le d$. A basis of this form is referred to as an Ω-symplectic basis.

Moreover, the matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ defined by

$$Av_j = e_j \quad \textit{and} \quad Aw_j = e_{d+j} \quad \textit{for } 1 \le j \le d$$

satisfies $\Omega = A^*\Omega_J$. If Ω is represented by $\theta \in \mathcal{S}_{2d}$, then $\Omega = A^*\Omega_J$ is equivalent to $A^TJA = \theta$.

Proof. We will prove the existence of the Ω-symplectic basis by induction on the integer $d \geq 1$. We begin with the base case $d = 1$. Pick any non-zero $w \in \mathbb{R}^2$. Then there exists some $v \in \mathbb{R}^2$ such that $\Omega(w, v) \neq 0$, for otherwise Ω would be degenerate. Scaling v and using bilinearity, we can assume that $\Omega(w, v) = 1$. By antisymmetry, $\Omega(v, v) = 0 = \Omega(w, w)$. Since $\Omega(w, rw) = r\Omega(w, w) = 0$ for all $r \in \mathbb{R}$, v and w must be linearly independent. Thus, $\{v, w\}$ is a basis of the desired form.

Assume now that $d > 1$ and that the result has been shown for $\mathbb{R}^{2(d-1)}$. Using the same argument as for the base case, we can find two (necessarily linearly independent) vectors v_1 and w_1 such that $\Omega(w_1, v_1) = 1$. We will show that

$$\mathbb{R}^{2d} = \mathbb{R}v_1 \oplus \mathbb{R}w_1 \oplus \big(\operatorname{Ker}\Omega(-, v_1) \cap \operatorname{Ker}\Omega(-, w_1)\big). \tag{1.6}$$

If $t, s \in \mathbb{R}$ were such that $tw_1 + sv_1 \in \operatorname{Ker}\Omega(-, v_1) \cap \operatorname{Ker}\Omega(-, w_1)$, then

$$0 = \Omega(-, v_1)\big(tw_1 + sv_1\big) = t\Omega(w_1, v_1) + s\Omega(v_1, v_1) = t,$$

and similarly we find that $s = 0$. This shows that the sum on the right hand side of Equation (1.6) is direct. To see that it adds up to $\mathbb{R}^{2d}$, we count dimensions. The linear functionals $\Omega(-, v_1)$ and $\Omega(-, w_1)$ are linearly independent (by Lemma 1.1.5 and linear independence of v_1 and w_1). Thus, by point (iii) of Lemma 1.1.4, we can conclude that

$$\dim\big(\operatorname{Ker}\Omega(-, v_1) \cap \operatorname{Ker}\Omega(-, w_1)\big) = 2d - 2,$$

which gives Equation (1.6).

By the induction hypothesis, we can choose a basis $\{v_2, \ldots, v_d, w_2, \ldots, w_d\}$ for $\operatorname{Ker}\Omega(-, v_1) \cap \operatorname{Ker}\Omega(-, w_1) \cong \mathbb{R}^{2(d-1)}$ such that

$$\Omega(v_i, v_j) = \Omega(w_i, w_j) = 0 \quad \text{and} \quad \Omega(w_i, v_j) = \delta_{ij} \quad \text{for } 2 \leq i, j \leq d.$$

Since we know that

$$\{v_2, \ldots, v_d, w_2, \ldots, w_d\} \subset \operatorname{Ker}\Omega(-, v_1) \cap \operatorname{Ker}\Omega(-, w_1)$$

and that $\Omega(w_1, v_1) = 1$, the basis $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ has the desired form. This concludes the proof that there exists an Ω-symplectic basis.

We now turn to the matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ defined by $Av_j = e_j$ and $Aw_j = e_{d+j}$ for $1 \leq j \leq d$. We have

$$\Omega_J(Aw_i, Av_j) = \Omega_J(e_{d+i}, e_j) = \delta_{ij} = \Omega(w_i, v_j) \quad \text{for } 1 \leq i, j \leq d,$$

and similarly

$$\Omega_J(Av_i, Av_j) = 0 = \Omega(v_i, v_j) \quad \text{and} \quad \Omega_J(Aw_i, Aw_j) = 0 = \Omega(w_i, w_j)$$

for $1 \leq i, j \leq d$. This shows that the two bilinear forms $A^*\Omega_J$ and Ω agree on all pairs of the basis $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$, so they must be equal.

Finally, if Ω is represented by $\theta \in \mathcal{S}_{2d}$, i.e. $\Omega = \Omega_\theta$, then $A^*\Omega_J = \Omega_\theta$ means that

$$w^T\theta z = \Omega_\theta(z, w) = A^*\Omega_J(z, w) = (Aw)^T J(Az) = w^T(A^T JA)z$$

for all $z, w \in \mathbb{R}^{2d}$. Taking z and w to be standard basis elements, we see that this holds if and only if $\theta_{ij} = (A^T JA)_{ij}$ for $1 \leq i, j \leq 2d$, i.e. if and only if $\theta = A^T JA$. $\square$

1.1.2 Lagrangian Subspaces and Polarizations

In Lemma 1.1.5 we saw how a symplectic form on $\mathbb{R}^{2d}$ determines an isomorphism between $\mathbb{R}^{2d}$ and its dual. We will now investigate a more subtle relation between symplectic forms and duality. We will show that a symplectic form Ω on $\mathbb{R}^{2d}$ *decomposes* $\mathbb{R}^{2d}$ as the direct sum of $\mathbb{R}^d$ and its dual $(\mathbb{R}^d)^*$ in such a manner that Ω takes a particularly nice form. We should note that this decomposition is far from unique. In the next section we will obtain a precise characterization of the redundancy (see the paragraph preceding Lemma 1.2.7).

To introduce the topic, let's momentarily widen our perspective and consider an arbitrary finite-dimensional vector space L over $\mathbb{R}$. There is a canonical symplectic form Ω^c on the (external) direct sum $L \oplus L^*$ given by

$$\begin{aligned} \Omega^c\colon (L \oplus L^*) \times (L \oplus L^*) &\to \mathbb{R} \\ \big((v_1, \tau_1), (v_2, \tau_2)\big) &\mapsto \tau_2(v_1) - \tau_1(v_2) \end{aligned} \tag{1.7}$$

Symplectic forms of this kind appear naturally and play a central role in the Hamiltonian formulation of classical mechanics.

Note that

$$\begin{aligned} \Omega^c\big((v_1, 0), (v_2, 0)\big) &= 0 \quad \text{for all } v_1, v_2 \in L \\ \text{and} \quad \Omega^c\big((0, \tau_1), (0, \tau_2)\big) &= 0 \quad \text{for all } \tau_1, \tau_2 \in L^*. \end{aligned}$$

In the terminology of the upcoming definition, the subspaces $L \oplus \{0\} \cong L$ and $\{0\} \oplus L^* \cong L^*$ would be referred to as Ω^c-*Lagrangian subspaces* of $L \oplus L^*$ and the pair $(L \oplus \{0\}, \{0\} \oplus L^*)$ as an Ω^c-*polarization* of the space.

1.1.9 Definition (Lagrangian subspaces)**.** Let Ω be a symplectic form on $\mathbb{R}^{2d}$.

(i) A d-dimensional subspace $L \subset \mathbb{R}^{2d}$ is called Ω-*Lagrangian* if $\Omega|_{L\times L} = 0$ (we say that Ω *vanishes* on L).

(ii) Two Ω-Lagrangian subspaces L and L' are said to be *transversal* if $V = L \oplus L'$. In this case, we refer to (L, L') an Ω-*polarization* of $\mathbb{R}^{2d}$.

When $\Omega = \Omega_J$, we will omit the Ω-prefix and speak of *Lagrangian subspaces* and *polarizations* of $\mathbb{R}^{2d}$.

We note that our main reference, de Gosson [15], does not use the terminology of polarizations, but speaks of pairs of transversal Lagrangian planes instead.

We will now see that all Ω-polarizations (L, L') of $\mathbb{R}^{2d}$ determine a decomposition of $\mathbb{R}^{2d}$ as a direct sum of L and its dual L^*, with the symplectic form Ω taking the canonical form. Note that Proposition 1.1.8 on symplectic bases shows that there exists an Ω-polarization for any symplectic form Ω on $\mathbb{R}^{2d}$ (take $L = \operatorname{span}\{v_1, \dots, v_d\}$ and $L' = \operatorname{span}\{w_1, \dots, w_d\}$).

1.1.10 Proposition. *Let Ω be a symplectic form on $\mathbb{R}^{2d}$ and let (L, L') be an Ω-polarization of $\mathbb{R}^{2d}$. The linear map*

$$\begin{aligned} T\colon L' &\to L^* \\ w &\mapsto \Omega(-, w)|_L \end{aligned}$$

is then an isomorphism. Moreover, introducing the isomorphism

$$\begin{aligned} I \oplus T\colon L \oplus L' &\to L \oplus L^* \\ v + w &\mapsto (v, Tw) \quad (\textit{where } v \in L \textit{ and } w \in L') \end{aligned}$$

and letting Ω^c denote the canonical symplectic form on $L \oplus L^*$ defined by Equation (1.7)*, we have that $\Omega = (I \oplus T)^*\Omega^c$.*

Proof. To verify that T is an isomorphism, all we need to show is that it is injective, since $\dim L' = d = \dim L$ and $\dim L = \dim L^*$ by Lemma 1.1.4.

Let $w \in L$ be such that $\Omega(-, w)|_L = Tw = 0$. For any $z \in \mathbb{R}^{2d} = L \oplus L'$, we can write $z = z_L + z_{L'}$ with $z_L \in L$ and $z_{L'} \in L'$. We now find that

$$\Omega(z, w) = \Omega(z_L, w) + \Omega(z_{L'}, w) = \Omega(-, w)|_L(z_L) + \Omega(z_{L'}, w) = 0,$$

where $\Omega(z_{L'}, w) = 0$ since $z_{L'}, w \in L'$ and L' is Ω-Lagrangian. Since $z \in \mathbb{R}^{2d}$ was arbitrary and Ω is nondegenerate, we must have $w = 0$, so T is injective.

For the identity $\Omega = (I \oplus T)^*\Omega^c$, let $v_1, v_2 \in L$ and $w_1, w_2 \in L'$. Then,

$$\begin{aligned}(I \oplus T)^*\Omega^c(v_1 + w_1, v_2 + w_2) &= \Omega^c\big((v_1, Tw_1), (v_2, Tw_2)\big)\\ &= Tw_2(v_1) - Tw_1(v_2)\\ &= \Omega(v_1, w_2) - \Omega(v_2, w_1)\\ &= \Omega(v_1 + w_1, v_2 + w_2),\end{aligned}$$

where the last equality relies on the fact that L and L' are Ω-Lagrangian. This concludes the proof. □

Consider now the canonical symplectic form Ω^c on $\mathbb{R}^d \oplus (\mathbb{R}^d)^*$. If we identify $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$ using the standard inner product, we obtain the standard symplectic form Ω_J on $\mathbb{R}^{2d}$. This is one reason as to why the standard symplectic form is particularly natural.

Before embarking on the study of linear maps preserving symplectic forms, which will lead to the notion of symplectic groups, we introduce the symplectic analogue of orthogonal complements.

1.1.11 Lemma (Ω-complements)**.** *Let Ω be a symplectic form on $\mathbb{R}^{2d}$ and let $V \subset \mathbb{R}^{2d}$ be a subspace. Define the Ω-complement of V to be the subspace*

$$V^\Omega := \{z \in \mathbb{R}^{2d} : \Omega(z, v) = 0 \text{ for all } v \in V\}.$$

Let $V, W \subset \mathbb{R}^{2d}$ be subspaces. Then,

(i) $\{0\}^\Omega = \mathbb{R}^{2d}$, $(\mathbb{R}^{2d})^\Omega = \{0\}$ *and* $(V^\Omega)^\Omega = V$.

(ii) $V^\Omega \cap W^\Omega = (V + W)^\Omega$.

(iii) $(V \cap W)^\Omega = V^\Omega + W^\Omega$.

(iv) $\dim V + \dim V^\Omega = 2d$.

(v) V *is Ω-Lagrangian if and only if* $V^\Omega = V$.

Proof. We begin with (i). The equalities $\{0\}^\Omega = \mathbb{R}^{2d}$ and $(\mathbb{R}^{2d})^\Omega = \{0\}$ follow from bilinearity and nondegeneracy of Ω, respectively. If $v \in V$, then $\Omega(v', v) = -\Omega(v, v') = 0$ for all $v' \in V^\Omega$ (by definition of V^Ω), so $v \in (V^\Omega)^\Omega$. Thus, $V \subset (V^\Omega)^\Omega$. The opposite inclusion is more subtle, but the results we have developed makes the proof fairly quick.

If $V = \{0\}$, then $(V^\Omega)^\Omega = (\mathbb{R}^{2d})^\Omega = \{0\}$, so we assume that $V \neq \{0\}$. Let $\{v_1, \ldots, v_k\}$ be a basis for V. If $v \in (V^\Omega)^\Omega$, then $V^\Omega \subset \operatorname{Ker}\Omega(-, v)$, and so

$$\bigcap_{j=1}^{k} \operatorname{Ker}\Omega(-, v_j) = V^\Omega \subset \operatorname{Ker}\Omega(-, v).$$

By Lemma 1.1.5 (i.e. the fact that $w \mapsto \Omega(-,w)$ is an isomorphism), the linear functionals $\Omega(-,v_1),\ldots,\Omega(-,v_k)$ are linearly independent, and so

$$\Omega(-,v) \in \operatorname{span}\big\{\Omega(-,v_1),\ldots,\Omega(-,v_k)\big\}$$

by point (i) of Lemma 1.1.4. Appealing again to Lemma 1.1.5, this gives $v \in \operatorname{span}\{v_1,\ldots,v_k\} = V$.

For (ii), assume first that $u \in V^\Omega \cap W^\Omega$. If $v \in V$ and $w \in W$, then $\Omega(u, v+w) = \Omega(u,v) + \Omega(u,w) = 0$, so $u \in (V+W)^\Omega$. For the converse inclusion, let $u \in (V+W)^\Omega$. Since $V \subset V+W$ and $W \subset V+W$, we find that $u \in V^\Omega \cap W^\Omega$.

Point (iii) follows from (ii) and (i) by noting that

$$V \cap W = (V^\Omega)^\Omega \cap (W^\Omega)^\Omega = (V^\Omega + W^\Omega)^\Omega$$

and taking Ω-complements (note also that $V^\Omega + W^\Omega$ is a subspace).

We now turn to (iv). If $V = \{0\}$, the result follows from (i). If V is nonzero, let $\{v_1,\ldots,v_k\}$ be a basis for V. Then, $V^\Omega = \bigcap_{j=1}^k \operatorname{Ker}\Omega(-,v_j)$. As in the proof of (i), Lemma 1.1.5 implies that $\Omega(-,v_1),\ldots,\Omega(-,v_k)$ are linearly independent, and point (iii) of Lemma 1.1.4 implies that

$$\dim V^\Omega = \dim\left(\bigcap_{j=1}^k \operatorname{Ker}\Omega(-,v_j)\right) = 2d - k = 2d - \dim V,$$

as desired.

We are now left only with (v). If V is Ω-Lagrangian, then $V \subset V^\Omega$ since $\Omega|_{V\times V} = 0$. Since V^Ω has dimension $2d - d = d$, we must have $V = V^\Omega$. Conversely, if $V = V^\Omega$, we must have $\dim V = d$ and $\Omega|_{V\times V} = 0$, so V is Ω-Lagrangian. □

1.2 | The Symplectic Group

1.2.1 Definition and Characterizations

We now introduce the notion of structure preserving maps in linear symplectic algebra.

1.2.1 Definition (Ω-symplectic matrices)**.** Let Ω be a symplectic form on $\mathbb{R}^{2d}$. A matrix $S \in M_{2d}(\mathbb{R})$ is called Ω-*symplectic* if

$$\Omega(Sz, Sw) = \Omega(z,w) \quad \text{for all } z, w \in \mathbb{R}^{2d},$$

i.e. if $S^*\Omega = \Omega$. We write $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ for the set of all Ω-symplectic matrices. When $\Omega = \Omega_J$, we will omit the Ω-prefix and speak of *symplectic* matrices and write $\mathrm{Sp}(2d,\mathbb{R})$ for the set of all symplectic matrices.

1.2.2 Proposition. *Let Ω be a symplectic form on $\mathbb{R}^{2d}$. The set $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ of all Ω-symplectic matrices is a subgroup of $\mathrm{GL}(2d,\mathbb{R})$.*

Proof. It is clear that $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ is closed under multiplication of matrices and that it contains the identity, so we only show that $\mathrm{Sp}_\Omega(2d,\mathbb{R}) \subset \mathrm{GL}(2d,\mathbb{R})$ and that the inverse of an Ω-symplectic matrix is Ω-symplectic.

Let $S \in \mathrm{Sp}_\Omega(2d,\mathbb{R})$ and assume that $z \in \mathbb{R}^{2d}$ is such that $Sz = 0$. Then $\Omega(z,w) = \Omega(Sz,Sw) = 0$ for all $w \in \mathbb{R}^{2d}$, so $z = 0$ by nondegeneracy of Ω. This shows that S is injective, hence invertible. Since S is Ω-symplectic, we find that

$$\Omega(S^{-1}z, S^{-1}w) = \Omega\big(S(S^{-1}z), S(S^{-1}w)\big) = \Omega(z,w) \quad \text{for all } z,w \in \mathbb{R}^{2d},$$

so S^{-1} is Ω-symplectic as well. □

We will refer to $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ as the Ω-*symplectic group* and $\mathrm{Sp}(2d,\mathbb{R})$ simply as the *symplectic group*. The following lemma shows that there is no loss of generality in restricting our attention to $\mathrm{Sp}(2d,\mathbb{R})$, as we often will.

1.2.3 Lemma. *Let Ω be a symplectic form on $\mathbb{R}^{2d}$ and let $A \in \mathrm{GL}(2d,\mathbb{R})$ be such that $A^*\Omega_J = \Omega$ (see Prop. 1.1.8). Then*

$$\mathrm{Sp}_\Omega(2d,\mathbb{R}) = A^{-1}\big(\mathrm{Sp}(2d,\mathbb{R})\big)A$$

and $A\mathrm{Sp}_\Omega(2d,\mathbb{R}) = \mathrm{Sp}(2d,\mathbb{R})A$.

Proof. Let $S \in \mathrm{Sp}(2d,\mathbb{R})$. By Lemma 1.1.7 on pullbacks, we have $(A^{-1})^*\Omega = \Omega_J$ and

$$(A^{-1}SA)^*\Omega = (SA)^*\big((A^{-1})^*\Omega\big) = (SA)^*\Omega_J = A^*(S^*\Omega_J) = A^*\Omega_J = \Omega,$$

so $A^{-1}SA \in \mathrm{Sp}_\Omega(2d,\mathbb{R})$. Thus, $A^{-1}(\mathrm{Sp}(2d,\mathbb{R}))A \subset \mathrm{Sp}_\Omega(2d,\mathbb{R})$.

If $S_\Omega \in \mathrm{Sp}_\Omega(2d,\mathbb{R})$, a similar calculation gives $AS_\Omega A^{-1} \in \mathrm{Sp}(2d,\mathbb{R})$, so

$$S_\Omega = A^{-1}(AS_\Omega A^{-1})A \in A^{-1}\big(\mathrm{Sp}(2d,\mathbb{R})\big)A,$$

which shows that $\mathrm{Sp}_\Omega(2d,\mathbb{R}) = A^{-1}(\mathrm{Sp}(2d,\mathbb{R}))A$.

Since $\mathrm{GL}(2d,\mathbb{R})$ is a group, left multiplication by A is a bijection of $\mathrm{GL}(2d,\mathbb{R})$. Since $\mathrm{Sp}(2d,\mathbb{R})$ and $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ are subsets of $\mathrm{GL}(2d,\mathbb{R})$, we find that $A\mathrm{Sp}_\Omega(2d,\mathbb{R}) = \mathrm{Sp}(2d,\mathbb{R})A$ as well. □

Recall that symplectic forms on $\mathbb{R}^{2d}$ correspond precisely to antisymmetric matrices in $\mathrm{GL}(2d,\mathbb{R})$, the set of which is denoted by $\mathcal{S}_{2d}$ (Definition 1.1.3), and that we write Ω_θ for the symplectic form represented by $\theta \in \mathcal{S}_{2d}$. To avoid towers of subscripts, we will write $\mathrm{Sp}_\theta(2d,\mathbb{R}) \coloneqq \mathrm{Sp}_{\Omega_\theta}(2d,\mathbb{R})$. This is in line with our convention of identifying matrices with the linear transformations they represent; here, we are identifying matrices with the bilinear forms they represent.

We now characterize membership in $\mathrm{Sp}_\theta(2d,\mathbb{R})$ in terms of a simple matrix equation.

1.2.4 Lemma. *Let $\theta \in \mathcal{S}_{2d}$ and $S \in M_{2d}(\mathbb{R})$. Then, $S \in \mathrm{Sp}_\theta(2d,\mathbb{R})$ if and only if $S^T\theta S = \theta$. In particular (when $\theta = J$), we have $S \in \mathrm{Sp}(2d,\mathbb{R})$ if and only if $S^TJS = J$.*

Proof. We have $S \in \mathrm{Sp}_\theta(2d,\mathbb{R})$ if and only if $\Omega_\theta(Sz,Sw) = \Omega_\theta(z,w)$ for all $z,w \in \mathbb{R}^{2d}$. In other words, $S \in \mathrm{Sp}_\theta(2d,\mathbb{R})$ if and only if

$$w^T(S^T\theta S)z = (Sw)^T\theta(Sz) = \Omega_\theta(Sz,Sw) = \Omega_\theta(z,w) = w^T\theta z$$

for all $z,w \in \mathbb{R}^{2d}$. Taking z and w to be standard basis elements, we see that this holds if and only if $(S^T\theta S)_{ij} = \theta_{ij}$ for $1 \le i,j \le 2d$, i.e. if and only if $S^T\theta S = \theta$. $\square$

Since $J^T = -J = J^{-1}$, we have $J^TJJ = J$, so $J \in \mathrm{Sp}(2d,\mathbb{R})$. Calling J the standard symplectic matrix is therefore consistent with our definition of symplectic matrices. We emphasize that J appears in our theory in two distinct ways: it is both the matrix in $\mathcal{S}_{2d}$ *representing* the standard symplectic form, and it represents a linear transformation *preserving* the standard symplectic form. It is important to understand the distinction between matrices *representing* symplectic forms, i.e. $\mathcal{S}_{2d}$, and matrices *preserving* the standard symplectic form, i.e. $\mathrm{Sp}(2d,\mathbb{R})$. *Symplectic matrices* refers to the latter kind.

Since J has a simple block-structure, Lemma 1.2.4 suggests that it may be advantageous to block-decompose symplectic matrices. The following lemma shows that this indeed is this case.

1.2.5 Lemma. *Let $S \in \mathrm{GL}(2d,\mathbb{R})$ and write*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{where } A,B,C,D \in M_d(\mathbb{R}).$$

We then have

$$S \in \mathrm{Sp}(2d,\mathbb{R}) \qquad \Longleftrightarrow \qquad S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

Moreover, if $S \in \mathrm{Sp}(2d,\mathbb{R})$, then $S^T \in \mathrm{Sp}(2d,\mathbb{R})$.

Proof. Using the fact that $J^{-1} = -J$, we have

$$S^T J S = J \iff (-JS^T J)S = I \iff -JS^T J = S^{-1}.$$

The claimed equivalence now follows from the calculation

$$\begin{aligned} -JS^T J &= \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix} \begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \end{aligned}$$

If $S^{-1} = -JS^T J$, then (using that $J^T = -J$)

$$(S^T)^{-1} = (S^{-1})^T = (-JS^T J)^T = J^T (S^T)^T (-J)^T = -J(S^T)^T J$$

shows that $S^T \in \mathrm{Sp}(2d, \mathbb{R})$. $\square$

1.2.2 A Convenient Set of Generators

Our goal for this subsection is to prove the following proposition, which gives us a very convenient set of generators for $\mathrm{Sp}(2d, \mathbb{R})$.

1.2.6 Proposition (Generators of the symplectic group)**.** *For each symmetric matrix $P^T = P \in M_d(\mathbb{R})$ and invertible matrix $L \in \mathrm{GL}(d, \mathbb{R})$, define*

$$V_P = \begin{pmatrix} I_d & 0 \\ -P & I_d \end{pmatrix} \quad \textit{and} \quad M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}.$$

Then,

$$G := \{V_P : P^T = P \in M_d(\mathbb{R})\} \cup \{M_L : L \in \mathrm{GL}(d, \mathbb{R})\} \cup \{J\}$$

generates $\mathrm{Sp}(2d, \mathbb{R})$, which is to say that $G \subset \mathrm{Sp}(2d, \mathbb{R})$ and that every $S \in \mathrm{Sp}(2d, \mathbb{R})$ can be written as a finite product of elements of G.

The proof of Proposition 1.2.6 turns out to be somewhat involved, for it relies on another factorization of symplectic matrices which seems difficult to describe explicitly. This factorization is the content of Lemma 1.2.10, after which we will prove Proposition 1.2.6. There is another proof of Proposition 1.2.6 which relies on topology, but which would be be no less involved at our stage of development. An instance of this proof can be found in Folland [14, Proposition 4.10 on p. 174]. We have opted for the algebraic proof given by de

Gosson [15, Corollary 63 on p. 40], which has the added benefit of deepening our understanding of Ω-polarizations of $\mathbb{R}^{2d}$.

It is not difficult to show that if $S \in \mathrm{Sp}_\Omega(2d, \mathbb{R})$ and if (L, L') is an Ω-polarization of $\mathbb{R}^{2d}$, then $(S(L), S(L'))$ is also an Ω-polarization of $\mathbb{R}^{2d}$. The following lemma shows that any two Ω-polarizations of $\mathbb{R}^{2d}$ are related by an Ω-symplectic transformation. Thus, if we are given one Ω-polarization, we obtain all of them by acting with the Ω-symplectic group.

1.2.7 Lemma. *Let Ω be a symplectic form on $\mathbb{R}^{2d}$ and let (L, L') and (K, K') be two Ω-polarizations of $\mathbb{R}^{2d}$. Then, there exists an $S \in \mathrm{Sp}_\Omega(2d, \mathbb{R})$ such that $S(L) = K$ and $S(L') = K'$.*

Proof. Let $\{w_1, \dots, w_d\}$ be a basis for L'. By Lemma 1.1.10, the linear functionals $\{\Omega(-, w_1)|_L, \dots, \Omega(-, w_d)|_L\}$ form a basis for L^*. By point (ii) of Lemma 1.1.4, we can now find $v_1, \dots, v_d \in L$ such that $\Omega(v_i, w_j) = \delta_{ij}$ for $1 \leq i, j \leq d$. These vectors must be linearly independent, for if some v_j was in the span of the other v's, we would have $\Omega(v_j, w_j) = 0$. Thus, $\{v_1, \dots, v_d\}$ is a basis for L. Since L and L' are Lagrangian and $\mathbb{R}^{2d} = L \oplus L'$, $\{v_1, \dots, v_d, w_1, \dots, w_d\}$ is an Ω-symplectic basis for $\mathbb{R}^{2d}$, i.e.

$$\Omega(v_i, v_j) = \Omega(w_i, w_j) = 0 \quad \text{and} \quad \Omega(w_i, v_j) = \delta_{ij}$$

for $1 \leq i, j \leq d$. By Proposition 1.1.8, the matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ mapping $\{v_1, \dots, v_d, w_1, \dots, w_d\}$ to $\{e_j\}_{j=1}^{2d}$ satisfies $\Omega = A^*\Omega_J$.

Repeating the argument for K and K', we similarly obtain an Ω-symplectic basis $\{v'_1, \dots, v'_d, w'_1, \dots, w'_d\}$ such that

$$K = \mathrm{span}\{v'_1, \dots, v'_d\} \quad \text{and} \quad K' = \mathrm{span}\{w'_1, \dots, w'_d\}$$

and the matrix $B \in \mathrm{GL}(2d, \mathbb{R})$ mapping this basis to the standard one also satisfies $\Omega = B^*\Omega_J$.

If we set $S = B^{-1}A$, then S maps the basis $\{v_1, \dots, v_d, w_1, \dots, w_d\}$ to the basis $\{v'_1, \dots, v'_d, w'_1, \dots, w'_d\}$, so $S(L) = K$ and $S(L') = K'$. Moreover, appealing to Lemma 1.1.7 on pullbacks, we find that $(B^{-1})^*\Omega = \Omega_J$ and that

$$S^*\Omega = (B^{-1}A)^*\Omega = A^*\big((B^{-1})^*\Omega\big) = A^*\Omega_J = \Omega,$$

so $S \in \mathrm{Sp}_\Omega(2d, \mathbb{R})$, which concludes the proof. $\square$

We will need the following simple lemma from linear algebra.

1.2.8 Lemma. *Let V, V_1 and V_2 be subspaces of $\mathbb{R}^{2d}$. If $V \backslash (V_1 \cup V_2) = \emptyset$, then $V \subset V_1$ or $V \subset V_2$.*

Proof. First of all, it suffices to show that

$$V\backslash(W_1 \cup W_2) = \emptyset \quad \Longrightarrow \quad V \subset W_1 \text{ or } V \subset W_2$$

in the case that W_1 and W_2 are subspaces of V. To see this, set $W_1 = V \cap V_1$ and $W_2 = V \cap V_2$. Then,

$$V\backslash(V_1 \cup V_2) = V\backslash\big((V \cap V_1) \cup (V \cap V_2)\big) = V\backslash(W_1 \cup W_2),$$

and if $V \subset W_1 = V \cap V_1$ or $V \subset W_2 = V \cap V_2$, then $V \subset V_1$ or $V \subset V_2$. We will show the contrapositive statement, namely that for subspaces W_1 and W_2 of V, we have

$$V \not\subset W_1 \text{ and } V \not\subset W_2 \quad \Longrightarrow \quad V\backslash(W_1 \cup W_2) \neq \emptyset.$$

If $W_1 \subset W_2$ or $W_2 \subset W_1$, then $V\backslash(W_1 \cup W_2)$ is either $V\backslash W_1$ or $V\backslash W_2$, both of which are nonempty by assumption ($V\backslash W = \emptyset \iff V \subset W$), so the claim follows in this case. Assume therefore that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. We can then choose $w_1 \in W_1\backslash W_2$ and $w_2 \in W_2\backslash W_1$, which implies that $w_1 + w_2 \in V\backslash(W_1 \cup W_2)$ since

$$\begin{aligned} & w_1 + w_2 \in W_1 \quad \Longrightarrow \quad w_2 = (w_1 + w_2) - w_1 \in W_1 \\ \text{and} \quad & w_1 + w_2 \in W_2 \quad \Longrightarrow \quad w_1 = (w_1 + w_2) - w_2 \in W_2, \end{aligned}$$

and either case contradicts our choice of v_1 and v_2. □

The following lemma is the last ingredient we need to prove the factorization result needed for the proof of Proposition 1.2.6. In particular (with $L_1 = L_2$), this lemma shows that any Ω-Lagrangian subspace $L \subset \mathbb{R}^{2d}$ is part of an Ω-polarization of $\mathbb{R}^{2d}$ (there are much simpler proofs of this fact).

1.2.9 Lemma. *Let L_1 and L_2 be two Ω-Lagrangian subspaces of $\mathbb{R}^{2d}$. Then, there exists an Ω-Lagrangian subspace L which is transversal to both L_1 and L_2, i.e. such that $L \oplus L_1 = \mathbb{R}^{2d} = L \oplus L_2$.*

Proof. By points (iv) and (v) of Lemma 1.1.11, a d-dimensional subspace $L \subset \mathbb{R}^{2d}$ satisfying $L \subset L^\Omega$ must be Ω-Lagrangian. We are therefore looking for a d-dimensional subspace $L \subset \mathbb{R}^{2d}$ satisfying

$$L \subset L^\Omega \quad \text{and} \quad L \cap L_1 = \{0\} = L \cap L_2 \tag{$*$}$$

(the second condition, along with the dimensions involved, implies that $L \oplus L_1 = \mathbb{R}^{2d} = L \oplus L_2$). We will say that a subspace $M \subset \mathbb{R}^{2d}$ (of any dimension) satisfies condition ($*$) if $M \subset M^\Omega$ and $M \cap L_1 = \{0\} = M \cap L_2$.

Thus, our goal is to show that there exists some d-dimensional subspace satisfying $(*)$.

It is simple to find a one-dimension subspace satisfying $(*)$, for if we choose any $m \in \mathbb{R}^{2d} \backslash (L_1 \cup L_2)$ (which is nonempty by Lemma 1.2.8), then $M = \mathbb{R}m$ does the job. We will now show that we can inductively increase the dimension of M until we obtain a d-dimensional subspace satisfying $(*)$. To be precise, we will show that the following statement is true.

> If $M \subset \mathbb{R}^{2d}$ is a subspace satisfying $(*)$ and $\dim M < d$, then
> $$M^\Omega \backslash \big((L_1 + M) \cup (L_2 + M)\big) \neq \emptyset,$$
> and for any $m \in M^\Omega \backslash ((L_1 + M) \cup (L_2 + M))$, $M + \mathbb{R}m$ is a $(\dim M + 1)$-dimensional subspace satisfying $(*)$.

We will show the contrapositive of the first part, namely that for a subspace $M \subset \mathbb{R}^{2d}$ satisfying $(*)$, we have

$$M^\Omega \backslash \big((L_1 + M) \cup (L_2 + M)\big) = \emptyset \quad \Longrightarrow \quad \dim M \geq d.$$

By Lemma 1.2.8, emptiness of $M^\Omega \backslash ((L_1 + M) \cup (L_2 + M))$ implies that M^Ω is contained in either $L_1 + M$ or $L_2 + M$. Let's say $M^\Omega \subset L_1 + M$ for concreteness. Since $M \cap L_1 = \{0\}$, Lemma 1.1.11 on Ω-complements gives

$$\mathbb{R}^{2d} = \{0\}^\Omega = (M \cap L_1)^\Omega = M^\Omega + L_1^\Omega = M^\Omega + L_1,$$

so $\mathbb{R}^{2d} \subset (L_1 + M) + L_1 = M + L_1$, which implies $\dim M \geq d$, as desired.

Assume now that $m \in M^\Omega \backslash ((L_1 + M) \cup (L_2 + M))$. First of all, $m \notin M$, for $m \in M$ implies that $m \in L_1 + M$. Thus, $\dim(M + \mathbb{R}m) = \dim M + 1$. Let now

$$m_1 + r_1 m, m_2 + r_2 m \in M + \mathbb{R}m \quad \text{with } m_1, m_2 \in M \text{ and } r_1, r_2 \in \mathbb{R}.$$

Then,

$$\Omega(m_1 + r_1 m, m_2 + r_2 m) = \Omega(m_1, m_2) + r_2 \Omega(m_1, m) + r_1 \Omega(m, m_2) = 0$$

because $M \subset M^\Omega$ and $m \in M^\Omega$. Thus, $(M + \mathbb{R}m) \subset (M + \mathbb{R}m)^\Omega$. All that remains is to show that $(M + \mathbb{R}m) \cap L_1 = \{0\} = (M + \mathbb{R}m) \cap L_2$.

Let $v \in (M + \mathbb{R}m) \cap L_1$, so that $v = m_v + r_v m$ for some $m_v \in M$ and $r_v \in \mathbb{R}$. If $r_v \neq 0$, then $m = (r_v)^{-1}(v - m_v) \in L_1 + M$, which contradicts our choice of m. We must therefore have $v = m_v$, but then $v \in M \cap L_1 = \{0\}$, so $v = 0$. This shows that $(M + \mathbb{R}m) \cap L_1 = \{0\}$. An identical argument shows that $(M + \mathbb{R}m) \cap L_2 = \{0\}$, which concludes the proof. $\square$

We are now ready for the factorization result which we will use to prove Proposition 1.2.6. Free symplectic matrices are important in their own right and they deserve a much more detailed treatment. Again, we refer the reader to de Gosson [15]. Let

$$L_x \coloneqq \{(x,0) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d\} \quad \text{and} \quad L_\omega \coloneqq \{(0,\omega) \in \mathbb{R}^{2d} : \omega \in \mathbb{R}^d\},$$

so that (L_x, L_ω) is a polarization of $\mathbb{R}^{2d}$.

1.2.10 Lemma (Free symplectic matrices)**.** *We say that a symplectic matrix* $S \in \mathrm{Sp}(2d,\mathbb{R})$ *is free if* $S(L_\omega) \cap L_\omega = \{0\}$*.*

(i) Let $S \in \mathrm{Sp}(2d,\mathbb{R})$ *and write*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \textit{with } A, B, C, D \in M_d(\mathbb{R}).$$

Then S *is free if and only if* $\det B \neq 0$*.*

(ii) Any $S \in \mathrm{Sp}(2d,\mathbb{R})$ *can be written as the product of two free symplectic matrices.*

Proof. We begin with (i). With S written in block form, the condition $S(L_\omega) \cap L_\omega = \{0\}$ becomes

$$\{(B\omega, D\omega) \in \mathbb{R}^{2d} : \omega \in \mathbb{R}^d\} \cap \{(0,\omega') \in \mathbb{R}^{2d} : \omega' \in \mathbb{R}^d\} = \{0\}.$$

If $\det B = 0$, then we can find $\omega \neq 0$ such that $B\omega = 0$. Since S is invertible, we cannot have $D\omega = 0$ (for then S maps $(0,\omega)$ to zero), so $(B\omega, D\omega) = (0, D\omega) \neq 0$ shows that S is not free. If $\det B \neq 0$, then $B\omega = 0$ implies that $\omega = 0$ (and $D\omega = 0$), so S is free. This proves the claimed equivalence.

For (ii), use Lemma 1.2.9 to fix a Lagrangian plane L that is transversal to both L_ω and $S(L_\omega)$, so that (L_ω, L) and $(L, S(L_\omega))$ are polarizations of $\mathbb{R}^{2d}$. By Lemma 1.2.7, there exists $S_0 \in \mathrm{Sp}(2d,\mathbb{R})$ such that

$$S_0(L_\omega) = L \quad \text{and} \quad S_0(L) = S(L_\omega).$$

This gives $S_0(L_\omega) \cap L_\omega = L \cap L_\omega = \{0\}$ and

$$(S_0^{-1}S)(L_\omega) \cap L_\omega = S_0^{-1}(S(L_\omega)) \cap L_\omega = L \cap L_\omega = \{0\},$$

so both S_0 and $S_0^{-1}S$ are free. Since $S = S_0(S_0^{-1}S)$, we are done. $\square$

We are now finally ready to prove Proposition 1.2.6.

Proof of Proposition 1.2.6. Recall that

$$G := \{V_P : P^T = P \in M_d(\mathbb{R})\} \cup \{M_L : L \in \mathrm{GL}(d,\mathbb{R})\} \cup \{J\}.$$

We first show that $G \subset \mathrm{Sp}(2d,\mathbb{R})$. We have already remarked that $J \in \mathrm{Sp}(2d,\mathbb{R})$ (see the paragraph following the proof of Lemma 1.2.4). Using the characterization of $\mathrm{Sp}(2d,\mathbb{R})$ in terms of inverses written in block-form (Lemma 1.2.5), it is straightforward to check that $V_P \in \mathrm{Sp}(2d,\mathbb{R})$ and $M_L \in \mathrm{Sp}(2d,\mathbb{R})$ for all $P^T = P \in M_d(\mathbb{R})$ and $L \in \mathrm{GL}(d,\mathbb{R})$.

We now show that G generates all of $\mathrm{Sp}(2d,\mathbb{R})$. Because of Lemma 1.2.10, it suffices to show that any free symplectic matrix can be written as a finite product of elements of G. Let therefore $S \in \mathrm{Sp}(2d,\mathbb{R})$ be free. Lemma 1.2.10 shows that we can write

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with } B \in \mathrm{GL}(d,\mathbb{R}).$$

Now, multiplying out shows that

$$S = \begin{pmatrix} I_d & 0 \\ DB^{-1} & I_d \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & DB^{-1}A - C \end{pmatrix} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ B^{-1}A & I_d \end{pmatrix}. \tag{1.8}$$

By Lemma 1.2.5, the equation $SS^{-1} = I = S^{-1}S$ becomes

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Multiply out the products above, we find that

$$BA^T - AB^T = 0 = D^TB - B^TD \quad \text{and} \quad DA^T - CB^T = I_d,$$

which implies that

$$(B^{-1}A)^T = B^{-1}(BA^T)B^{-T} = B^{-1}(AB^T)B^{-T} = B^{-1}A,$$
$$(DB^{-1})^T = B^{-T}(D^TB)B^{-1} = B^{-T}(B^TD)B^{-1} = DB^{-1},$$

and that

$$B^{-T} = (DA^T - CB^T)B^{-T} = D(B^{-1}A)^T - C = D(B^{-1}A) - C.$$

Combining the last three equations with Equation (1.8), we have shown that

$$S = \begin{pmatrix} I_d & 0 \\ DB^{-1} & I_d \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-T} \end{pmatrix} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ B^{-1}A & I_d \end{pmatrix}$$

with $(DB^{-1})^T = DB^{-1}$ and $(B^{-1}A)^T = B^{-1}A$. In other words, we have $S = V_{-(DB^{-1})} M_{B^{-1}} J V_{-(B^{-1}A)}$. This shows that any free symplectic matrix is a finite product of elements of G, which concludes the proof. □

Chapter 2

Banach Algebras and C*-Algebras

The goal of this chapter is to give the reader an introduction to the theory of Banach algebras and C*-algebras. We will assume that the reader is familiar with the basics of complex analysis, Banach spaces and functional analysis. Statements of those results we need from functional analysis, along with references to proofs, can be found in Appendix A. For a general and approachable introduction to functional analysis, we recommend Bowers and Kalton [7]. Our development of Banach algebras and C*-algebras is largely based on Murphy [23].

The Banach algebras and C*-algebras we will encounter in Gabor analysis will all be algebras of bounded operators on complex normed spaces. In the development of the theory, it may be useful to think of Banach algebras as an abstraction of algebras of bounded operators on Banach spaces and to think of C*-algebras as an abstraction of algebras of bounded operators on Hilbert spaces. The simplicity of Hilbert spaces compared to general Banach spaces is reflected in the theories of C*-algebras and Banach algebras; C*-algebras are much better behaved (or more constrained) than general Banach algebras.

2.1 | Involutions and Banach Algebras

This section is mainly concerned with the development of Banach algebras. Involutions and $\star$-algebras will not be of relevance until we consider C*-algebras in the next section. Nevertheless, they are introduced alongside algebras and spectra in the following subsection. This grouping of topics is meant to emphasize that algebras, spectra and involutions are purely algebraic constructions, whereas the theories of Banach algebras and C*-algebras are shaped by the interplay between algebra and topology.

2.1.1 Algebras, Spectra and Involutions

Some of the proofs in this subsection are quite scant; we assume that the reader is already somewhat familiar with algebras.

2.1.1 Definition (Algebras)**.** An *algebra* A is a nonzero vector space A over $\mathbb{C}$ with a bilinear and associative product $A \times A \to A$ (typically denoted by juxtaposition). We also require that there be an element $1_A \in A$ such that $1_A a = a = a 1_A$ for all $a \in A$. The element 1_A is called the *unit* of A.

Let A and B be two algebras. A map $\Phi\colon A \to B$ is an *algebra homomorphism* if it is $\mathbb{C}$-linear,

- *preserves the product:* $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in A$,
- and *preserves the unit:* $\Phi(1_A) = 1_B$.

An algebra homomorphism is an *isomorphism of algebras* if it is a bijection. We refer to the existence of an isomorphism $\Phi\colon A \to B$ of algebras by saying that $A \cong B$ *as algebras*.

There are a couple of remarks we wish to make regarding this definition.

- We are conforming to quite restrictive conventions and packing a lot of content into the word *algebra*. What we are calling an algebra would more generally be called an *associative nontrivial unital algebra over* $\mathbb{C}$. The requirement that our algebras be unital is *not* typical in the theory of operator algebras.[1]
- An algebra is, in particular, a ring. All the basic algebraic results for rings therefore hold for algebras as well. For example: the unit is unique, $0a = 0 = a0$ (where $0 \in A$ is the additive identity), $(-1_A)a = -a = a(-1_A)$ for all $a \in A$, etc.
- We are not considering the "trivial algebra" $A = \{0\}$ as an algebra. This implies that $1_A \neq 0$ in any algebra A.

[1] A foundational feature of nonunital C*-algebras is that they contain *approximate units*. These are nets of elements which, when acting on the algebra via the product, behave like a unit in the limit. The existence of approximate units ensures that the theory of nonunital C*-algebras greatly resembles the theory of unital C*-algebras.

- The inverse of an isomorphism of algebras is necessarily an algebra homomorphism (and indeed an isomorphism of algebras), and a composition of algebra homomorphisms is an algebra homomorphism as well.

We now briefly discuss the notion of inverses in an algebra. Let A be an algebra and fix any $a \in A$. If there exists an element $b \in A$ such that $ab = 1_A$, we say that a is *right invertible* and call b a *right inverse* of a. Similarly, a is *left invertible* with a *left inverse* $c \in A$ if $ca = 1_A$.

An element may in general have multiple right inverses or multiple left inverses. However, assume now that $a \in A$ has both a right inverse b and a left inverse c. Then,

$$b = 1_A b = (ca)b = c(ab) = c1_A = c, \quad \text{so } b = c.$$

There cannot be any other left or right inverses, for they must all equal b. In this situation, we say that a is *invertible*, write $a^{-1} := b$ and call a^{-1} *the inverse* of a. In summary, a^{-1} (if it exists) is the unique element such that $a^{-1}a = 1_A = aa^{-1}$, and whenever $a, b, c \in A$ satisfy $ab = 1_A$ and $ca = 1_A$, it follows that $b = c = a^{-1}$.

We will denote the set of all invertible elements in A by $\operatorname{Inv}(A)$. It is straightforward to verify that

$$1_A^{-1} = 1_A, \quad (a^{-1})^{-1} = a \quad \text{and} \quad (ba)^{-1} = a^{-1}b^{-1},$$

and that $\operatorname{Inv}(A)$ is a group with respect to the product.

2.1.2 Example (Shift operators)**.** Consider the Banach space $\ell^1(\mathbb{N})$ and the *shift operators* $L, R \in \mathcal{B}(\ell^1(\mathbb{N}))$ defined by

$$Lv = (v_2, v_3, \ldots) \quad \text{and} \quad Rv = (0, v_1, v_2, \ldots) \quad \text{for } v = (v_j)_{j=1}^{\infty} \in \ell^1(\mathbb{N}).$$

In the algebra $\mathcal{B}(\ell^1(\mathbb{N}))$ (with composition as the product), these satisfy $LR = \mathrm{Id}_{\ell^1(\mathbb{N})} = 1_{\mathcal{B}(\ell^1(\mathbb{N}))}$, while $RL : v \mapsto (0, v_2, v_3, \ldots)$ is not injective and hence not invertible.

We say that two elements $a, b \in A$ *commute* if $ab = ba$. If all elements of A commute with each other, then A is called a *commutative* algebra. The factors of an invertible product need *not* be invertible in general, as illustrated by the example above.

2.1.3 Lemma (Inverses and products)**.** *Let A be an algebra, let $N \geq 2$ be an integer and suppose that $a_1, \ldots, a_N \in A$ all commute with each other. Then, the product $a_1 \cdots a_N$ is invertible if and only if the factors $a_1, \ldots, a_N$ are all invertible.*

Proof. Assume first that $N = 2$. If a_1 and a_2 are invertible, it is an immediate consequence of associativity that $a_2^{-1}a_1^{-1}$ is the inverse of a_1a_2. For the converse, assume that a_1a_2 is invertible. Then,

$$a_1\big(a_2(a_1a_2)^{-1}\big) = (a_1a_2)(a_1a_2)^{-1} = 1_A = (a_1a_2)^{-1}(a_1a_2) = \big((a_1a_2)^{-1}a_2\big)a_1.$$

By our discussion of inverses prior to the lemma, this means that a_1 is invertible and that $a_1^{-1} = a_2(a_1a_2)^{-1} = (a_1a_2)^{-1}a_2$. A similar calculation shows that a_2 is invertible. This proves the lemma for $N = 2$.

For $N > 2$, we may write $a_1 \cdots a_N = (a_1 \cdots a_{N-1})a_N$. Thus, the $N = 2$ case shows that $a_1 \ldots a_N$ is invertible if and only if $a_1 \cdots a_{N-1}$ and a_N are both invertible. The general case now follows by a simple inductive argument. □

We now consider substructures and quotients of algebras. For a subset S of an algebra A and an element $a \in A$, we write $aS \coloneqq \{as : s \in S\}$.

2.1.4 Definition (Subalgebras and ideals)**.** Let A be an algebra.

(i) A *subalgebra* of A is a vector-subspace $B \subset A$ such that $1_A \in B$ and such that $ab \in B$ whenever $a, b \in B$.

(ii) An *ideal* of A is a vector-subspace $I \subset A$ satisfying $aI \subset I$ and $Ia \subset I$ for every $a \in A$. An ideal $I \subset A$ is *proper* if $I \neq A$.

Let A be an algebra. It is straightforward to verify that a vector-subspace of A is a subalgebra of A if and only if it is an algebra in its own right (with the product restricted from A) with the same unit as A.

Ideals are precisely those vector-subspaces we can quotient out if we want the product on A to descend to the quotient vector space. In order to avoid $A/I = \{0\}$, we have to restrict attention to proper ideals.

2.1.5 Proposition (Quotients of algebras)**.** *Let A be an algebra and let $I \subset A$ be a proper ideal of A. Then, the quotient vector space $A/I \coloneqq \{a + I : a \in A\}$ is an algebra with respect to the product defined by*

$$(a + I)(b + I) \coloneqq ab + I \quad \textit{for all } a, b \in I.$$

The unit of A/I is $1_A + I$ and the quotient map $q\colon A \to A/I$ is an algebra homomorphism.

Proof. We only show that the product is well-defined, for then associativity, bilinearity and the claimed unit are simple verifications. The definition of the product on A/I is equivalent to the quotient map preserving the product, so it is immediate that the quotient map is an algebra homomorphism.

If $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$ for some $a_1, a_2, b_1, b_2 \in A$, then

$$a_2(b_1 - b_2) \in I \quad \text{and} \quad (a_1 - a_2)b_1 \in I$$

by the defining properties of ideals. Thus,

$$a_2b_2 + I = a_2b_2 + a_2(b_1 - b_2) + (a_1 - a_2)b_1 + I = a_1b_1 + I.$$

This shows that the product is well-defined. $\square$

It is straightforward to show that the kernel $\operatorname{Ker}\Phi$ of any algebra homomorphism $\Phi\colon A \to B$ is an ideal in A. Moreover, it is necessarily a *proper* ideal because our algebra homomorphism preserve the unit and we are excluding the possibility that $1_B = 0$.

2.1.6 Theorem (The first isomorphism theorem for algebras)**.** *Let* $\Phi\colon A \to B$ *be an algebra homomorphism. Then,* $\operatorname{Ker}\Phi$ *is a proper ideal of* A*,* $\Phi(B)$ *is a subalgebra of* B *and the map*

$$\begin{aligned}\widetilde{\Phi}\colon A/\operatorname{Ker}\Phi &\to \Phi(A)\\ a + \operatorname{Ker}\Phi &\mapsto \Phi(a)\end{aligned}$$

is an isomorphism of algebras.

Proof. From the corresponding theorem for vector spaces, we already know (by $\mathbb{C}$-linearity of Φ) that $\operatorname{Ker}\Phi$ is a vector-subspace of A, that $\Phi(B)$ is a vector-subspace of B and that $\widetilde{\Phi}$ is a well-defined linear bijection of vector spaces. The remaining verifications are quite immediate consequences of Φ preserving the product and the unit, so we omit them. $\square$

2.1.7 Example. (The evaluation map) Let X be a nonempty set and let $\mathbb{C}^X$ denote the vector space of all functions from X to $\mathbb{C}$ (with the vector space operations defined pointwise). Equipping $\mathbb{C}^X$ with the pointwise product,

$$\begin{aligned}\mathbb{C}^X \times \mathbb{C}^X &\to \mathbb{C}^X\\ (f, g) &\mapsto \big(fg : x \mapsto fg(x) \coloneqq f(x)g(x)\big),\end{aligned}$$

turns $\mathbb{C}^X$ into a commutative algebra with the constant function $1_{\mathbb{C}^X} : x \mapsto 1$ as the unit.

Fix now some $x \in X$ and consider $\mathbb{C}$ as an algebra with the usual vector space structure and product. The *evaluation map* at x, defined by

$$\begin{aligned}\mathrm{ev}_x\colon \mathbb{C}^X &\to \mathbb{C}\\ f &\mapsto \mathrm{ev}_x(f) \coloneqq f(x)\end{aligned}$$

is an algebra homomorphism: we find that $\mathrm{ev}_x(1_{\mathbb{C}^X}) = 1$ and that

$$\mathrm{ev}_x(fg) = fg(x) = f(x)g(x) = \mathrm{ev}_x(f)\mathrm{ev}_x(g) \quad \text{for all } f, g \in \mathbb{C}^X.$$

Clearly $\mathrm{ev}_x(\mathbb{C}^X) = \mathbb{C}$, so the first isomorphism theorem tells us that the map

$$\begin{aligned} \mathbb{C}^X / \operatorname{Ker}(\mathrm{ev}_x) &\to \mathbb{C} \\ g + \operatorname{Ker}(\mathrm{ev}_x) &\mapsto g(x) \end{aligned}$$

is an isomorphism of algebras. There is no mystery here;

$$\operatorname{Ker}(\mathrm{ev}_x) = \big\{ f \in \mathbb{C}^X : f(x) = 0 \big\},$$

so the only information remaining in a coset $g + \operatorname{Ker}(\mathrm{ev}_x)$ is the common value of all its elements at the point x.

If X has additional structure, there may be interesting subalgebras of $\mathbb{C}^X$. For example, if X has a topology, we can consider the subalgebra $C(X)$ of all continuous functions on X. This will be our main example of commutative algebras going forward.

We now introduce the notion of the spectrum of an element in an algebra. This notion, as we will see, lies at the heart of the theory of Banach algebras and C*-algebras.

2.1.8 Definition (The spectra of elements)**.** Let A be an algebra and let $a \in A$. We define the *spectrum* of a to be the set

$$\sigma(a) \coloneqq \big\{ \lambda \in \mathbb{C} : \lambda 1_A - a \notin \operatorname{Inv}(A) \big\}.$$

We also define the *spectral radius* of a to be the (possibly infinite) number

$$r(a) \coloneqq \sup\big\{ |\lambda| : \lambda \in \sigma(a) \big\} \in [0, \infty].$$

We will give multiple examples of spectra when we discuss Banach algebras. In Example 2.1.7, the spectrum of a function $f \in \mathbb{C}^X$ is given by its image, i.e. $\sigma(f) = f(X)$. This is because an inverse in $\mathbb{C}^X$ corresponds to a pointwise inverse, and $\lambda 1_{\mathbb{C}^X} - f$ has a pointwise inverse if and only if it never vanishes, which happens if and only if $\lambda \notin f(X)$.

We now prove some basic properties of spectra.

2.1.9 Lemma (Basic properties of spectra)**.** *Let A be an algebra. Then, the following statements are true.*

(i) We have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ *for all* $a, b \in A$.

(ii) If $a \in \mathrm{Inv}(A)$, *then* $\lambda \in \sigma(a)$ *if and only if* $\lambda^{-1} \in \sigma(a^{-1})$.

(iii) If $\Phi\colon A \to B$ *is an algebra homomorphism, then* $\sigma(\Phi(a)) \subset \sigma(a)$ *for all* $a \in A$.

Proof. We will write λ instead of $\lambda 1_A$ (for any $\lambda \in \mathbb{C}$) for parts (i) and (ii) of this proof.

We begin with (i). Suppose that $\lambda \in \mathbb{C}\backslash\{0\}$ and $\lambda - ab \in \mathrm{Inv}(A)$. Then,

$$\begin{aligned}(\lambda - ba)\big(1 + b(\lambda - ab)^{-1}a\big) &= \lambda + \lambda b(\lambda - ab)^{-1}a - ba - bab(\lambda - ab)^{-1}a \\ &= \lambda + b\Big(\lambda(\lambda - ab)^{-1} - 1 - ab(\lambda - ab)^{-1}\Big)a \\ &= \lambda + b\Big((\lambda - ab)(\lambda - ab)^{-1} - 1\Big)a = \lambda.\end{aligned}$$

A very similar calculation yields $(1 + b(\lambda - ab)^{-1}a)(\lambda - ba) = \lambda$. Thus, since $\lambda \neq 0$, we see that $\lambda - ba \in \mathrm{Inv}(A)$. The same result clearly holds with a and b swapped, so we find that $\lambda - ab \in \mathrm{Inv}(A)$ if and only if $\lambda - ba \in \mathrm{Inv}(A)$, as long as $\lambda \neq 0$. Thus, $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, which proves (i).

For (ii), suppose that $a \in \mathrm{Inv}(A)$ and let $\lambda \in \sigma(a)$. Then $\lambda \neq 0$ (since $0 - a = -a$ is invertible) and

$$\lambda^{-1} - a^{-1} = -\lambda^{-1}a^{-1}(\lambda - a) \in \mathrm{Inv}(A), \quad \text{so } \lambda^{-1} \in \sigma(a^{-1}).$$

The fact that $(a^{-1})^{-1} = a$ gives the converse inclusion and concludes the proof of (ii).

For (iii), fix any $a \in A$. We will show that $\sigma(\Phi(a)) \subset \sigma(a)$ by contraposition. If $\lambda \notin \sigma(a)$, then $\lambda 1_A - a$ is invertible in A, and so

$$1_B = \Phi(1_A) = \Phi\big((\lambda 1_A - a)^{-1}(\lambda 1_A - a)\big) = \Phi\big((\lambda 1_A - a)^{-1}\big)\Phi(\lambda 1_A - a).$$

This shows that $\Phi(\lambda 1_A - a) = \lambda 1_B - \Phi(a)$ is left-invertible in B. A similar calculation shows that it is right-invertible as well. Thus, $\lambda 1_B - \Phi(a)$ is invertible in B, so $\lambda \notin \sigma(\Phi(a))$. This gives $\sigma(\Phi(a)) \subset \sigma(a)$, which concludes the proof. $\square$

We now introduce the notion of an involution on an algebra. The star-superscript in "C*-algebra" refers to the presence of an involution. Thinking of a C*-algebra as an algebra of bounded operators on a Hilbert space, the involution corresponds to the function mapping each operator to its adjoint.

2.1.10 Definition (Involutions and $\star$-algebras)**.** An *involution* on a algebra A is a map $\star\colon A \to A$, denoted $x^* \coloneqq \star(x)$, that satisfies

(i) $a^{**} := (a^*)^* = a$

(ii) $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$ (conjugate-linearity)

(iii) $(ab)^* = b^* a^*$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. If an algebra A is equipped with an involution, we call it a *$\star$-algebra*.

If A and B are two $\star$-algebras, a map $\Phi\colon A \to B$ is a *$\star$-algebra homomorphism* if it is an algebra homomorphism which also preserves the involution, meaning that

$$\Phi(a^*) = \Phi(a)^* \quad \text{for all } a \in A.$$

A $\star$-algebra homomorphism is an *isomorphism of $\star$-algebras* if it is a bijection. We refer to the existence of an isomorphism $\Phi\colon A \to B$ of $\star$-algebras by saying that $A \cong B$ *as $\star$-algebras*.

The inverse of an isomorphism of $\star$-algebras automatically preserves the involution, so it is a $\star$-algebra homomorphism as well (and indeed an isomorphism of $\star$-algebras). A composition of $\star$-algebra homomorphisms is also a $\star$-algebra homomorphism.

Note that $1_A^* = 1_A$ in any $\star$-algebra A:

$$1_A^* a = (a^* 1_A)^* = a = (1_A a^*)^* = a 1_A^* \quad \text{for all } a \in A,$$

from which uniqueness of the unit gives the result.

2.1.11 Definition ($\star$-subalgebras and $\star$-ideals)**.** Let A be a $\star$-algebra.

(i) A *$\star$-subalgebra* of A is a subalgebra $B \subset A$ such that $b^* \in B$ whenever $b \in B$.

(ii) A *$\star$-ideal* of A is an ideal $I \subset A$ such that $b^* \in I$ whenever $b \in I$.

Let A be a $\star$-algebra. It is straightforward to verify that a subalgebra of A is a $\star$-subalgebra of A if and only if it is a $\star$-algebra in its own right (with the involution restricted from A).

If $I \subset A$ is an ideal of A, the property that $b^* \in I$ for every $b \in I$ is precisely the condition we need in order for the involution to descend to the quotient algebra. The first isomorphism theorem for $\star$-algebras is precisely what one would expect. We will meet its incarnation for C*-algebras later.

2.1.2 The Very Basics of Banach Algebras

We begin by recalling some basic facts about linear maps between normed spaces. We will implicitly assume that all normed spaces in this chapter are nonzero and over the complex numbers (these are the conditions required for their algebras of operators to be algebras in our sense of the word).

For a linear map $T\colon V \to W$ between normed spaces, the *operator norm* $\|T\|$ of T is defined by

$$\|T\| \coloneqq \sup\big\{\|Tv\| : v \in V \text{ with } \|v\| \leq 1\big\} = \sup_{\|v\|\leq 1} \|Tv\| = \sup_{\|v\|=1} \|Tv\|.$$

Equivalently, $\|T\|$ is the least number $M \in [0, \infty]$ such that $\|Tv\| \leq M\|v\|$ for all $v \in V$. The map T is called *norm-decreasing* if $\|T\| \leq 1$ and it is called *isometric* (or said to be an *isometry*) if $\|Tv\| = \|v\|$ for all $v \in V$.

If we let $\mathcal{B}(V, W)$ denote the vector space of all linear maps $T\colon V \to W$ such that $\|T\|$ is finite, then the operator norm is indeed a norm on $\mathcal{B}(V, W)$. If W is a Banach space, then $\mathcal{B}(V, W)$ is a Banach space as well.

Regardless of whether V is Banach or not, $\mathcal{B}(V) \coloneqq \mathcal{B}(V, V)$ is an algebra with the product given by composition, and we have that

$$\|ST\| = \sup_{\|v\|\leq 1} \|STv\| \leq \sup_{\|v\|\leq 1} \|S\|\|T\|\|v\| = \|S\|\|T\| \quad \text{for all } S, T \in \mathcal{B}(V).$$

This is a highly desirable property for normed algebras to possess.

2.1.12 Definition (Submultiplicativity of the norm)**.** Let A be an algebra equipped with a norm $\|\cdot\|$. We say that the norm on A is *submultiplicative* if

$$\|ab\| \leq \|a\|\|b\| \quad \text{for all } a, b \in A.$$

The term *normed algebra* is often reserved for an algebra with a submultiplicative norm. We will spell out "an algebra with a submultiplicative norm" so as not to further overload our language just yet.

Submultiplicativity allows us to write products of limits as limits of products, as the following proposition shows.

2.1.13 Proposition (Continuity of multiplication)**.** *Let A be an algebra with a submultiplicative norm and let $a, b \in A$. If $(a_n), (b_n) \subset A$ are sequences such that $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.*

Proof. By submultiplicativity of the norm, we find that

$$\|a_n b_n - ab\| \leq \|a_n(b_n - b)\| + \|(a_n - a)b\| \leq \|a_n\|\|b_n - b\| + \|a_n - a\|\|b\|,$$

so $\|a_n b_n - ab\| \to 0$ as $n \to \infty$ (where we have also used continuity of the norm: $\|a_n\| \to \|a\|$). $\square$

This property is referred to as *continuity of multiplication* because it is equivalent to the statement that the algebra product $A \times A \to A$ is continuous when $A \times A$ is equipped with the product topology. This property will be referenced by name and not by number if referenced at all.

We have seen that if V is a Banach space, then $\mathcal{B}(V)$ is a Banach space which is also an algebra with a submultiplicative norm.

2.1.14 Definition (Banach algebras)**.** An algebra $\mathcal{B}$ with a submultiplicative norm $\|\cdot\|$ is called a *Banach algebra* if $(\mathcal{B}, \|\cdot\|)$ is a Banach space and $\|1_{\mathcal{B}}\| = 1$.

An *isomorphism of Banach algebras* is a bounded algebra isomorphism (between two Banach algebras) with a bounded inverse.[2] We will refer to the existence of an isomorphism $\Phi\colon \mathcal{B} \to \mathcal{B}'$ of Banach algebras by saying that $\mathcal{B} \cong \mathcal{B}'$ *as Banach algebras.*

We emphasize that a Banach algebra is *not* simply a Banach space which is also an algebra, for we are also requiring the norm to be submultiplicative and the unit to be normalized. Note also that any closed subalgebra of $\mathcal{B}(V)$ (where V is a Banach space) is a Banach algebra as well. More generally, any closed subalgebra of a Banach algebra is a Banach algebra.

Any Banach algebra $\mathcal{B}$ can be realized as a closed subalgebra of operators acting on a Banach space V, for we may take $\mathcal{B} = V$ (which leads to the slightly unfortunate notation $\mathcal{B}(V) = \mathcal{B}(\mathcal{B})$) and define the algebra homomorphism

$$\begin{aligned} \Phi\colon \mathcal{B} &\to \mathcal{B}(\mathcal{B}) \\ a &\mapsto \big(L_a : b \mapsto L_a(b) := ab\big), \end{aligned}$$

where $\mathcal{B}$ acts on itself by “multiplication from the left”. Submultiplicativity of the norm and the existence of a normalized unit straightforwardly lead to the conclusion that $\|L_a\| = \|a\|$ for all $a \in \mathcal{B}$, so that Φ is an isometric algebra homomorphism “embedding” $\mathcal{B}$ as a closed subalgebra of $\mathcal{B}(\mathcal{B})$.

Nevertheless, Banach algebras occur naturally outside the context of operator algebras. Here are some examples of relevance to us:

[2] As our notion of isomorphism reveals, we are considering bounded algebra homomorphism to be homomorphisms of Banach algebras, but we will simply call them bounded algebra homomorphisms. There is a good argument to be made that homomorphisms of Banach algebras should be algebra homomorphism which are not only bounded, but norm-decreasing. This would make isomorphisms of Banach algebras *isometric* algebra homomorphism. The choice depends on whether one wants to consider equivalent norms as isomorphic or not (equivalently: whether one wants to emphasize the particular norm or the set of Cauchy sequences it determines).

2.1.15 Example (Banach algebras)**.**

(i) The complex numbers $\mathbb{C}$ form a commutative Banach algebra with respect to the usual vector space structure, product and norm.

(ii) Let X be a topological space that is compact and Hausdorff.[3] Then, the algebra $C(X)$ of continuous $\mathbb{C}$-valued functions on X with pointwise operations (see Example 2.1.7) and the supremum norm

$$\|f\|_\infty := \sup\big\{|f(x)| : x \in X\big\} \quad \text{for } f \in C(X)$$

is a commutative Banach algebra. The supremum norm determines the topology of uniform convergence on X, so completeness is a consequence of the fact that a uniform limit of continuous functions is continuous.

(iii) The Banach space $\ell^1(\mathbb{Z})$ with its usual ℓ^1-norm and the product given by convolution,

$$(a * b)(n) = \sum_{j\in\mathbb{Z}} a(j)b(n-j) \quad \text{for all } n \in \mathbb{Z} \text{ and } a, b \in \ell^1(\mathbb{Z}),$$

is a commutative Banach algebra.

Whenever we refer to $\mathbb{C}$ or $C(X)$ (for some compact Hausdorff space X) as Banach algebras, these will be the structures we have in mind.

Let $\mathcal{B}$ be a Banach algebra. We call a series $\sum_{n=0}^\infty a_n$ with $(a_n) \subset \mathcal{B}$ *absolutely convergent* if $\sum_{n=0}^\infty \|a_n\| < \infty$. The standard proofs that absolute convergence implies convergence for $\mathbb{C}$-valued series can be adapted to $\mathcal{B}$-valued series simply by replacing absolute values with norms: for any $N, M \in \mathbb{N}$ with $N > M$ we see that

$$\left\| \sum_{n=0}^{N} a_n - \sum_{m=0}^{M} a_m \right\| \le \sum_{n=M+1}^{N} \|a_n\|,$$

so the fact that the partial sums of $\sum_{n=0}^\infty \|a_n\|$ are Cauchy implies that the partial sums of $\sum_{n=0}^\infty a_n$ are Cauchy, and so $\sum_{n=0}^\infty a_n$ converges by completeness.

Similarly, by using submultiplicativity of the norm, we can mimic the standard proofs for $\mathbb{C}$-valued series to show that we have

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{m=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$$

[3]The Hausdorff requirement is not necessary, but it makes the algebra $C(X)$ more pleasant, so it is often included as an assumption is examples.

in $\mathcal{B}$ whenever both series on the left converge absolutely. See Rudin [28, Theorem 3.50 on p. 74] for a particular proof that straightforwardly generalizes to the setting of a general Banach algebra.

In fact, equipped with submultiplicativity of the norm, we can lift a surprising amount of complex analysis to the setting of a general Banach algebra. We will barely scrape the surface; we recommend Rudin [30, Chapter 10] for further exploration.

2.1.16 Lemma (Power series and the exponential function)**.** *Let $\mathcal{B}$ be a Banach algebra and let $\sum_{n=0}^{\infty} \lambda_n z^n$ be a power series with complex coefficients and radius of convergence $R \in [0, \infty]$. If $a \in \mathcal{B}$ satisfies $\|a\| < R$, then $\sum_{n=0}^{\infty} \lambda_n a^n$ is an absolutely convergent series in $\mathcal{B}$.*

In particular, the exponential function

$$e^a := \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

is absolutely convergent for all $a \in \mathcal{B}$. Moreover, if $a, b \in \mathcal{B}$ commute, then $e^a e^b = e^{a+b}$.

Proof. Suppose $a \in \mathcal{B}$ satisfies $\|a\| < R$. Submultiplicativity of the norm gives $\|a^n\| \leq \|a\|^n$, so the fact that $\sum_{n=0}^{\infty} \lambda_n z^n$ converges absolutely for $|z| < R$ gives $\sum_{n=0}^{\infty} \|\lambda_n a^n\| \leq \sum_{n=0}^{\infty} |\lambda_n| \|a\|^n < \infty$. That is, we have absolute convergence of $\sum_{n=0}^{\infty} \lambda_n a^n$.

The radius of convergence of $\sum_{n=0}^{\infty} z^n/(n!)$ is infinite, so e^a is absolutely convergent for all $a \in \mathcal{B}$. By the discussion preceding the lemma, we have

$$\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{b^m}{m!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)$$

regardless of whether a and b commute or not. If they do commute, then the binomial expansion $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ holds. Inserting this above gives $e^a e^b = e^{a+b}$. $\square$

We can now say a fair bit about the structure of the group $\mathrm{Inv}(\mathcal{B})$ of invertible elements in a Banach algebra $\mathcal{B}$.

2.1.17 Proposition (Invertible elements in Banach algebras)**.** *Let $\mathcal{B}$ be a Banach algebra. Then, the following statements are true.*

(i) If $a \in \mathcal{B}$ satisfies $\|a\| < 1$, then

$$1_{\mathcal{B}} - a \in \mathrm{Inv}(\mathcal{B}) \quad \textit{and} \quad (1_{\mathcal{B}} - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

In particular, $\mathrm{Inv}(\mathcal{B})$ contains the open unit ball centered at $1_{\mathcal{B}}$.

(ii) The set $\mathrm{Inv}(\mathcal{B})$ *is open in* $\mathcal{B}$.

(iii) If $(a_n) \subset \mathrm{Inv}(\mathcal{B})$ *converges to* $a \in \mathrm{Inv}(\mathcal{B})$*, then* $a_n^{-1} \to a^{-1}$.

We will refer to (iii) as continuity of the inverse.

Proof. We begin with (i). By Lemma 2.1.16, $\sum_{n=0}^{\infty} a^n$ is convergent whenever $\|a\| < 1$. Now, for any $N \in \mathbb{N}$, we have

$$(1_{\mathcal{B}} - a)\left(\sum_{n=0}^{N} a^n\right) = 1_{\mathcal{B}} - a^{N+1} = \left(\sum_{n=0}^{N} a^n\right)(1_{\mathcal{B}} - a)$$

because of a pattern of cancellations that is easily observed by expanding out the products for small values of N (or proved by induction, if one wants to be precise). Taking the limit of this equation as $N \to \infty$, using continuity of the product and the fact that $a^{N+1} \to 0$ (since $\|a\| < 1$ and $\|a^{N+1}\| \le \|a\|^{N+1}$), we obtain $(1_{\mathcal{B}} - a)^{-1} = \sum_{n=0}^{\infty} a^n$, as desired.

For (ii), let $a \in \mathrm{Inv}(\mathcal{B})$. We will show that $a - b$ is invertible whenever $b \in \mathcal{B}$ satisfies $\|b\| < \|a^{-1}\|^{-1}$. Then, $\mathrm{Inv}(\mathcal{B})$ must be open, for it contains the open ball of radius $\|a^{-1}\|^{-1}$ centered at a.

If $b \in \mathcal{B}$ is such that $\|b\| < \|a^{-1}\|^{-1}$, then $\|ba^{-1}\| \le \|b\|\|a^{-1}\| < 1$, so

$$1_{\mathcal{B}} - ba^{-1} \in \mathrm{Inv}(\mathcal{B})$$

by point (i). Since $a \in \mathrm{Inv}(\mathcal{B})$, we have $a - b = (1_{\mathcal{B}} - ba^{-1})a \in \mathrm{Inv}(\mathcal{B})$ as well. This concludes the proof of (ii).

For (iii), we define $b_n := a - a_n$, so that $b_n \to 0$ and

$$a_n a^{-1} = (a - b_n)a^{-1} = 1_{\mathcal{B}} - b_n a^{-1}.$$

For large enough $n \in \mathbb{N}$, we will have $\|b_n a^{-1}\| < 1$ ($b_n a^{-1} \to 0$ by continuity of the product), and so

$$a_n^{-1} = a^{-1}(a_n a^{-1})^{-1} = a^{-1}(1_{\mathcal{B}} - b_n a^{-1})^{-1} = a^{-1}\sum_{k=0}^{\infty}(b_n a^{-1})^k$$

by point (i). Thus, for such n, we find that

$$\|a^{-1} - a_n^{-1}\| = \left\| a^{-1}\left(1_{\mathcal{B}} - \sum_{k=0}^{\infty}(b_n a^{-1})^k\right)\right\| \le \|a^{-1}\|\left\|\sum_{k=1}^{\infty}(b_n a^{-1})^k\right\|. \tag{2.1}$$

Now, since $b_n a^{-1} \to 0$, we have

$$\left\|\sum_{k=1}^{\infty}(b_n a^{-1})^k\right\| \le \sum_{k=1}^{\infty}\|b_n a^{-1}\|^k = \frac{\|b_n a^{-1}\|}{1 - \|b_n a^{-1}\|} \to 0$$

as well, from which Equation (2.1) gives $a_n^{-1} \to a^{-1}$ and concludes the proof. $\square$

We now begin our exploration of spectra in Banach algebras. We begin with a couple of examples

2.1.18 Example (Examples of spectra)**.** The following examples illustrate the notion of spectra in various Banach algebras.

(i) When $\mathcal{B} = \mathcal{B}(\mathbb{C}^n) \cong M_n(\mathbb{C})$ (for some $n \in \mathbb{N}_1$), the spectrum of an operator $T \in \mathcal{B}(\mathbb{C}^n)$ is the set of its eigenvalues. This is because a linear operator on $\mathbb{C}^n$ fails to be invertible precisely when it is not injective, and any non-zero vector in the kernel of $\lambda I_{\mathbb{C}^n} - T$ (for some $\lambda \in \mathbb{C}$) is an eigenvector for T with eigenvalue λ. The spectral radius of T is therefore the absolute value of its largest eigenvalue.

(ii) When $\mathcal{B} = \mathcal{B}(V)$ for an infinite dimensional Banach space V, the situation is more complicated, for an operator may fail to be invertible while still being injective (consider e.g. the shift operator R in Example 2.1.2). Bowers and Kalton [7, Example 8.16 on p. 187] show that the spectrum $\sigma(T)$ of $T \in \mathcal{B}(V)$ consists of the eigenvalues of T, the eigenvalues of its transpose $T^* : V^* \to V^*$ and the *approximate eigenvalues* of T, which are $\lambda \in \mathbb{C}$ for which there exists a sequence $(v_n) \subset V$ of unit vectors such that $(T - \lambda \mathrm{Id}_V)v_n \to 0$.

(iii) When $\mathcal{B} = C(X)$ for a compact Hausdorff space X, the spectrum of a function $f \in C(X)$ is simply its image: $\sigma(f) = f(X)$. This is because a function in $C(X)$ has an inverse in $C(X)$ if and only if it has a pointwise inverse,[4] from which our discussion of spectra in $\mathbb{C}^X$ gives the result (see the paragraph preceding Lemma 2.1.9). It now follows from our definitions that $r(f) = \|f\|_\infty$.

On a finite-dimensional complex vector space, any linear operator has an eigenvalue. This is a foundational result in linear algebra, but it is *not* true for bounded linear operators on infinite dimensional Banach spaces (consider e.g. the shift operator R in Example 2.1.2). The following theorem, whose importance can hardly be overstated, shows that spectra are capable of filling the void left by eigenvalues in the infinite-dimensional case. Murphy [23] suggest that we think of this as *the fundamental theorem of Banach algebras.*

2.1.19 Theorem. *Let $\mathcal{B}$ be a Banach algebra and let $a \in \mathcal{B}$. Then, $\sigma(a)$ is nonempty and compact. Moreover,*

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}$$

[4] The pointwise inverse of a continuous function is continuous by continuity of the map $\mathbb{C}\backslash\{0\} \to \mathbb{C}$ defined by $\lambda \mapsto \lambda^{-1}$. The absolute value of a continuous function attains its minimum on a compact space, so the inverse of a nonzero continuous function is bounded.

and, in particular, we have $r(a) \leq \|a\|$.

Proof. This is a rather long proof, so we have divided it into three parts. In this proof, and throughout this thesis, $\mathcal{B}^*$ will denote the continuous dual of $\mathcal{B}$ as a Banach space.

Part 1: The spectrum is nonempty

We will prove this by contradiction, so assume that $\sigma(a) = \emptyset$. If this is the case, then $\lambda 1_{\mathcal{B}} - a$ is invertible for every $\lambda \in \mathbb{C}$, so we have a well-defined map

$$\begin{aligned} g\colon \mathbb{C} &\to \mathcal{B} \\ \lambda &\mapsto (\lambda 1_{\mathcal{B}} - a)^{-1}. \end{aligned}$$

We claim that for any $\tau \in \mathcal{B}^*$, the map $\tau \circ g\colon \mathbb{C} \to \mathbb{C}$ is an entire function that is also bounded. Before we show this, let's see how this results in a contradiction.

Liouville's theorem states that the only entire and bounded functions are constants. Thus, we get

$$\tau(-a^{-1}) = \tau \circ g(0) = \tau \circ g(1) = \tau\big((1_{\mathcal{B}} - a)^{-1}\big) \quad \text{for all } \tau \in \mathcal{B}^*.$$

By the Hahn-Banach theorem (see Corollary A.2.2), this implies that $-a^{-1} = (1_{\mathcal{B}} - a)^{-1}$. Taking the inverse of both sides gives $1_{\mathcal{B}} = 0$, which is a possibility we have excluded by our definition of algebras. Thus, if we can show that $\tau \circ g$ is entire and bounded for each $\tau \in \mathcal{B}^*$, then we will have shown that $\sigma(a) \neq \emptyset$.

We first show that the $\tau \circ g$'s are entire. For the remainder of this part of the proof, we will write λ instead of $\lambda 1_{\mathcal{B}}$ (for any $\lambda \in \mathbb{C}$). Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be arbitrary but distinct. We find that

$$\begin{aligned} g(\lambda_2) - g(\lambda_1) &= (\lambda_2 - a)^{-1} - (\lambda_1 - a)^{-1} \\ &= (\lambda_2 - a)^{-1}\big(1_{\mathcal{B}} - (\lambda_2 - a)(\lambda_1 - a)^{-1}\big) \\ &= (\lambda_2 - a)^{-1}\big((\lambda_1 - a) - (\lambda_2 - a)\big)(\lambda_1 - a)^{-1} \\ &= (\lambda_2 - a)^{-1}(\lambda_1 - \lambda_2)(\lambda_1 - a)^{-1}, \end{aligned} \tag{2.2}$$

and so

$$\frac{g(\lambda_2) - g(\lambda_1)}{\lambda_2 - \lambda_1} = -(\lambda_2 - a)^{-1}(\lambda_1 - a)^{-1} \to -(\lambda_1 - a)^{-2}$$

as $\lambda_2 \to \lambda_1$. Fix any $\tau \in \mathcal{B}^*$. Since τ is linear and continuous, we can conclude that

$$\frac{\tau \circ g(\lambda_2) - \tau \circ g(\lambda_1)}{\lambda_2 - \lambda_1} = \tau\left(\frac{g(\lambda_2) - g(\lambda_1)}{\lambda_2 - \lambda_1}\right) \to \tau\big(-(\lambda_1 - a)^{-2}\big)$$

as $\lambda_2 \to \lambda_1$ (by continuity of the product and the inverse). This shows that $\tau \circ g$ has a complex derivative at the arbitrary point $\lambda_1 \in \mathbb{C}$, so it is entire.

We now show that $\tau \circ g$ is bounded. As an analytic function, it is certainly continuous, so it is bounded on any compact subset of $\mathbb{C}$, e.g. the closed disk of radius $2\|a\|$ centered at the origin. It suffices therefore to show that $\tau \circ g$ is bounded outside of this disk: let $\lambda \in \mathbb{C}$ be such that $|\lambda| > 2\|a\|$. Setting $\lambda_2 = \lambda$ and $\lambda_1 = 0$ in Equation (2.2), we find that

$$g(\lambda) = g(0) + (\lambda - a)^{-1}(-\lambda)(-a)^{-1} = g(0) + \left(1 - \frac{a}{\lambda}\right)^{-1} a^{-1}.$$

Since $|\lambda| > 2\|a\|$, Proposition 2.1.17 gives $(1 - a/\lambda)^{-1} = \sum_{n=0}^{\infty}(a/\lambda)^n$ and hence

$$\left\|\left(1 - \frac{a}{\lambda}\right)^{-1}\right\| \leq \sum_{n=0}^{\infty} \|a/\lambda\|^n = \frac{1}{1 - \|a/\lambda\|} < 2.$$

Thus, we see that

$$\|g(\lambda)\| \leq \|g(0)\| + \left\|\left(1 - \frac{a}{\lambda}\right)^{-1}\right\| \|a^{-1}\| < \|g(0)\| + 2\|a^{-1}\|.$$

Since $|\tau \circ g(\lambda)| = |\tau(g(\lambda))| \leq \|\tau\|\|g(\lambda)\|$, this concludes the proof that $\tau \circ g$ is bounded, and moreover the proof that $\sigma(a) \neq \emptyset$.

Part 2: The spectrum is compact

We will show that $\sigma(a) \subset \mathbb{C}$ is compact by showing that it is closed and bounded. If $\lambda_1 \notin \sigma(a)$, then $\lambda_1 1_{\mathcal{B}} - a \in \mathrm{Inv}(\mathcal{B})$. Since

$$\|(\lambda 1_{\mathcal{B}} - a) - (\lambda_1 1_{\mathcal{B}} - a)\| = |\lambda - \lambda_1| \quad \text{for all } \lambda \in \mathbb{C},$$

the fact that $\mathrm{Inv}(\mathcal{B})$ is open (point (ii) of Proposition 2.1.17) implies that for λ sufficiently close to λ_1, we will have $\lambda 1_{\mathcal{B}} - a \in \mathrm{Inv}(\mathcal{B})$ and so $\lambda \notin \sigma(a)$. This shows that the complement of $\sigma(a)$ in $\mathbb{C}$ is open, so $\sigma(a)$ is closed.

We now show that $\sigma(a)$ is bounded. If $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|a\|$, then point (i) of Proposition 2.1.17 tells us that $1_{\mathcal{B}} - \lambda^{-1}a$ is invertible, so $\lambda 1_{\mathcal{B}} - a$ is as well, hence $\lambda \notin \sigma(a)$. Thus, we have shown that $\sigma(a)$ is bounded and that $r(a) \leq \|a\|$.

Part 3: The formula for the spectral radius

Our goal is to show that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n}, \tag{2.3}$$

from which the general fact that

$$\inf_{n\geq 1} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n}$$

(which holds by virtue of the definitions) gives the result. The first inequality in (2.3) will require much more work than the second, so we show the second and simplest first.

The well-known formula for the difference of two nth powers gives

$$\lambda^n 1_{\mathcal{B}} - a^n = (\lambda 1_{\mathcal{B}} - a) \sum_{k=1}^{n} \lambda^{n-k} a^{k-1} \quad \text{for any } n \in \mathbb{N}_1 \text{ and } \lambda \in \mathbb{C}.$$

By Lemma 2.1.3 on products and inverses (everything commutes here), this implies that if $\lambda 1_{\mathcal{B}} - a$ is not invertible, then $\lambda^n 1_{\mathcal{B}} - a^n$ cannot be invertible either. In other words, $\lambda \in \sigma(a)$ implies that $\lambda^n \in \sigma(a^n)$. We have shown that $r(a^n) \leq \|a^n\|$ (see the last sentence of Part 2 of this proof), so this gives

$$\lambda \in \sigma(a) \quad \Longrightarrow \quad |\lambda|^n = |\lambda^n| \leq r(a^n) \leq \|a^n\| \text{ for all } n \in \mathbb{N}_1.$$

Taking nth roots and then the supremum over all $\lambda \in \sigma(a)$, we find that $r(a) \leq \|a^n\|^{1/n}$ for all $n \in \mathbb{N}_1$. Thus, $r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n}$.

All that remains is the first inequality in Equation (2.3). With slightly sloppy notation, let $\{|\lambda| < r(a)^{-1}\} \subset \mathbb{C}$ denote the open disk of radius $r(a)^{-1}$ centered at the origin, with the agreement that $r(a)^{-1} = \infty$ if $r(a) = 0$. We claim that the function

$$\begin{aligned} h\colon \left\{|\lambda| < r(a)^{-1}\right\} &\to \mathcal{B} \\ \lambda &\mapsto (1_{\mathcal{B}} - \lambda a)^{-1} \end{aligned}$$

is well-defined. At $\lambda = 0$ there is no problem, so consider $0 \neq |\lambda| < r(a)^{-1}$. Since $|\lambda^{-1}| > r(a)$, we must have $\lambda^{-1} \notin \sigma(a)$. This means that $\lambda^{-1} 1_{\mathcal{B}} - a$ is invertible and so $1_{\mathcal{B}} - \lambda a$ is as well. Thus, h is well-defined.

The exact same factorization trick that we used in Equation (2.2) gives

$$h(\lambda_2) - h(\lambda_1) = (1_{\mathcal{B}} - \lambda_2 a)^{-1} (\lambda_2 - \lambda_1) a (1_{\mathcal{B}} - \lambda_1 a)^{-1}$$

for all $\lambda_2, \lambda_1 \in \{|\lambda| < r(a)^{-1}\}$. The same reasoning as in Part 1 of this proof shows that $\tau \circ h$ is holomorphic on $\{|\lambda| < r(a)^{-1}\} \subset \mathbb{C}$ for all $\tau \in \mathcal{B}^*$.

By point (i) of Proposition 2.1.17, we have $(1_{\mathcal{B}} - \lambda a)^{-1} = \sum_{n=0}^{\infty} (\lambda a)^n$ whenever $|\lambda| < \|a\|^{-1}$. Fix any $\tau \in \mathcal{B}^*$. Linearity and continuity of τ gives

$$\tau \circ h(\lambda) = \sum_{n=0}^{\infty} \tau(a^n) \lambda^n \quad \text{whenever } |\lambda| < \|a\|^{-1}.$$

This is a power series expansion for $\tau \circ h$ centered at the origin. A power series representation of a holomorphic function converges in the largest possible disk where the function is holomorphic, and we know that $\tau \circ h$ is holomorphic on $\{|\lambda| < r(a)^{-1}\}$, so $\sum_{n=0}^{\infty} \tau(a^n)\lambda^n$ must converge whenever $|\lambda| < r(a)^{-1}$. This requires $|\tau(a^n)\lambda^n| \to 0$ as $n \to \infty$. In particular, there must be some constant M_τ such that $|\tau(a^n)\lambda^n| \leq M_\tau$ for all $n \in \mathbb{N}$.

We have now shown that the set $\{\lambda^n a^n\}_{n\in\mathbb{N}}$ is weakly bounded whenever $|\lambda| < r(a)^{-1}$. By the uniform boundedness principle (see Corollary A.2.5), $\{\lambda^n a^n\}_{n\in\mathbb{N}}$ must then also be bounded in norm. That is, there must exist some constant $M > 0$ such that $\|\lambda^n a^n\| \leq M$ for all $n \in \mathbb{N}$. Since $\limsup_{n\to\infty} M^{1/n} = 1$ for any $M > 0$, we have shown that

$$|\lambda| \limsup_{n\to\infty} \|a^n\|^{1/n} = \limsup_{n\to\infty} \|\lambda^n a^n\|^{1/n} \leq 1 \quad \text{whenever } |\lambda| < r(a)^{-1}.$$

This gives

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq \inf\left\{\frac{1}{|\lambda|} : |\lambda| < r(a)^{-1}\right\} = r(a),$$

which completes the proof of Equation (2.3) and the theorem as a whole. □

Here is an immediate and important corollary of Theorem 2.1.19.

2.1.20 Corollary. *If $\mathcal{B}$ is a Banach algebra such that every nonzero element is invertible, then $\mathcal{B} = \mathbb{C}1_{\mathcal{B}}$.*

Proof. Let $a \in \mathcal{B}$. By Theorem 2.1.19, we can find some $\lambda \in \sigma(a)$. This means that $\lambda 1_{\mathcal{B}} - a$ is not invertible, but then $\lambda 1_{\mathcal{B}} - a = 0$ by our assumption. Thus, $a = \lambda 1_{\mathcal{B}}$. Since a was arbitrary, we have $\mathcal{B} = \mathbb{C}1_{\mathcal{B}}$. □

We now consider quotients of Banach algebras; let $\mathcal{B}$ be a Banach algebra and let $I \subset \mathcal{B}$ be a proper ideal. We know from Proposition 2.1.5 that the quotient space $\mathcal{B}/I$ has the structure of an algebra. If I is closed in $\mathcal{B}$, then $\mathcal{B}/I$ is a Banach space when equipped with the quotient norm (see e.g. Bowers and Kalton [7, Proposition 3.47 on p. 54]). The following proposition shows that these structures combine to yield a Banach algebra.

2.1.21 Proposition (Quotients of Banach algebras)**.** *Let $\mathcal{B}$ be a Banach algebra and let $I \subset \mathcal{B}$ be a proper and closed ideal. Then, the quotient algebra $\mathcal{B}/I$ equipped with the quotient norm*

$$\|a + I\|_q := \inf\big\{\|a + b\| : b \in I\big\} \quad \text{for } a \in \mathcal{B}$$

is a Banach algebra.

Proof. By the paragraph preceding the proposition, we only need to verify that the quotient norm is submultiplicative and that the unit is normalized.

Fix any $\epsilon > 0$. Let $a, b \in \mathcal{B}$ and choose $c_a, c_b \in I$ such that

$$\|a + c_a\| < \|a + I\|_q + \epsilon \quad \text{and} \quad \|b + c_b\| < \|b + I\|_q + \epsilon.$$

Submultiplicativity in $\mathcal{B}$ gives

$$\|(a + c_a)(b + c_b)\| \leq \|a + c_a\|\|b + c_b\| < \big(\|a + I\|_q + \epsilon\big)\big(\|b + I\|_q + \epsilon\big).$$

Since $(a + c_a)(b + c_b) = ab + c$ with $c := (ac_b + c_a b + c_a c_b) \in I$, we find that

$$\|ab + I\|_q \leq \|ab + c\| = \|(a + c_a)(b + c_b)\| < \big(\|a + I\|_q + \epsilon\big)\big(\|b + I\|_q + \epsilon\big).$$

Taking the limit $\epsilon \to 0$ gives submultiplicativity of the quotient norm.

As for the normalization of the unit, submultiplicativity in $\mathcal{B}/I$ gives

$$\|1_{\mathcal{B}} + I\|_q = \|(1_{\mathcal{B}} + I)^2\|_q \leq \|1_{\mathcal{B}} + I\|_q^2,$$

which implies that either $\|1_{\mathcal{B}} + I\|_q \geq 1$ or $\|1_{\mathcal{B}} + I\|_q = 0$. However, we already know that $1_{\mathcal{B}} + I \neq I$ (since I is proper, for if $1_{\mathcal{B}} \in I$, then $\mathcal{B}1_{\mathcal{B}} \subset I$) and that the quotient norm is nondegenerate. Thus, we must have $\|1_{\mathcal{B}} + I\|_q \geq 1$. The fact that $0 \in I$ gives $\|1_{\mathcal{B}} + I\|_q \leq \|1_{\mathcal{B}}\| = 1$, so we find that $\|1_{\mathcal{B}} + I\|_q = 1$, which concludes the proof. □

2.1.3 The Gelfand Representation

In this subsection we will focus exclusively on commutative Banach algebras. We have seen that if X is a compact Hausdorff space, then $C(X)$ is a commutative Banach algebra (see Example 2.1.15). These are algebras that we understand particularly well. We have e.g. seen that the spectrum of a function $f \in C(X)$ is its image: $\sigma(f) = f(X)$ (Example 2.1.18).

In this subsection, we will ask the question: given a commutative Banach algebra $\mathcal{B}$, can we construct a compact Hausdorff space X such that $\mathcal{B} \cong C(X)$ as Banach algebras? The answer to this question in general is *no*, as is revealed by the existence of a commutative Banach algebra where $\sigma(b) = \{0\}$ does *not* imply that $b = 0$, as it would have to in a Banach space of the form $C(X)$.[5]

Nevertheless, we will comes as close to achieving this as is possible: given any commutative Banach algebra $\mathcal{B}$, we will construct a compact

[5] The matrix $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ generates a commutative Banach algebra in $M_2(\mathbb{C})$, namely the closed span of $\{N^m : m = \mathbb{N}_0\}$. One may verify that $(\lambda I - N)^{-1} = \lambda^{-2}(\lambda I + N)$ for all $\lambda \neq 0$, so that $\sigma(N) = \{0\}$ although $N \neq 0$.

Hausdorff space X and a norm-decreasing algebra homomorphism $\mathcal{B} \to C(X)$ whose kernel is precisely those $b \in \mathcal{B}$ for which $\sigma(b) = \{0\}$. *The Gelfand representation* will refer to this algebra homomorphism $\mathcal{B} \to C(X)$.

Given a compact Hausdorff space X, each $x \in X$ determines an algebra homomorphism $\mathrm{ev}_x \colon C(X) \to \mathbb{C}$ given by evaluation at x (see Example 2.1.7). Thus, given a commutative Banach algebra $\mathcal{B}$, it is natural to build the corresponding topological space out of algebra homomorphisms $\mathcal{B} \to \mathbb{C}$. The kernels of such algebra homomorphism are maximal ideals, which is where we will begin our development of the Gelfand representation in earnest.

An ideal I of an algebra A is called *maximal* if it is a proper ideal that is not properly contained in any other proper ideal (thus, it is a maximal element among proper ideals ordered by inclusion). Maximal ideals are typically denoted by the letter M. We emphasize that all maximal ideals are proper by definition.

2.1.22 Lemma (Maximal ideals)**.** *Let $\mathcal{B}$ be a commutative Banach algebra. Then, the following statements are true.*

(i) If $I \subset \mathcal{B}$ is a proper ideal, then there exists a maximal ideal $M \subset \mathcal{B}$ such that $I \subset M$.

(ii) Any maximal ideal $M \subset \mathcal{B}$ is closed and $\mathcal{B}/M \cong \mathbb{C}$ as Banach algebras.

Proof. Point (i) is a consequence of Zorn's lemma. We assume that the reader is familiar with the required terminology.

> **Zorn's lemma.** Let $\mathcal{P}$ be a nonempty partially ordered set (i.e. a set with a reflexive, antisymmetric and transitive relation). If every nonempty chain (i.e. totally ordered subset) of $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.

Let $\mathcal{P}$ be the set of all proper ideals of $\mathcal{B}$ which contain I and order $\mathcal{P}$ by inclusion (i.e. let the partial order $\leq$ be given by the subset relation $\subset$). Then, $I \in \mathcal{P}$, so $\mathcal{P}$ is nonempty. If $\mathcal{J} \subset \mathcal{P}$ is a nonempty chain, we claim that $\bigcup \mathcal{J} \in \mathcal{P}$ and that $\bigcup \mathcal{J}$ is an upper bound for $\mathcal{J}$ (where $\bigcup \mathcal{J}$ denotes the union of all elements of $\mathcal{J}$). We first show that $\bigcup \mathcal{J} \in \mathcal{P}$.

If $a, b \in \bigcup \mathcal{J}$ then there are $J_1, J_2 \in \mathcal{J}$ such that $a \in J_1$ and $b \in J_2$. Since $\mathcal{J}$ is a chain, either $J_1 \subset J_2$ or $J_2 \subset J_1$. Since both J_1 and J_2 are subspaces, we have $\lambda a + b \in J_1 \cup J_2 \subset \bigcup \mathcal{J}$ for all $\lambda \in \mathbb{C}$. This shows that $\bigcup \mathcal{J}$ is a vector-subspace of $\mathcal{B}$. Since all elements of $\mathcal{J}$ are ideals, we have $a(\bigcup \mathcal{J}), (\bigcup \mathcal{J})a \subset \bigcup \mathcal{J}$ for all $a \in \mathcal{B}$, so $\bigcup \mathcal{J}$ is an ideal. To see that it is proper, note that if $1_{\mathcal{B}} \in \bigcup \mathcal{J}$, then $1_{\mathcal{B}} \in J$ for some $J \in \mathcal{J}$, but then $\mathcal{B} = J \in \mathcal{J} \subset \mathcal{P}$, which contradicts our definition of $\mathcal{P}$ as a collection of proper ideals. This shows that $\bigcup \mathcal{J}$ is a proper ideal, and hence an element of $\mathcal{P}$.

It is now immediate that $\bigcup \mathcal{J}$ is an upper bound for $\mathcal{J}$: $J \subset \bigcup \mathcal{J}$ for all $J \in \mathcal{J}$. Thus, we have verified that $\mathcal{P}$ satisfies the hypothesis of Zorn's lemma, so there is a maximal element $M \in \mathcal{P}$. This means that M is a proper ideal of $\mathcal{B}$ that contains I and is not contained in any other proper ideal containing I. However, since M contains I, it cannot be contained in any proper ideal not containing I either. Thus, M is a maximal ideal and we have proved (i).

For (ii), let $M \subset \mathcal{B}$ be a maximal ideal. We will show that $\overline{M}$ (the closure of M in $\mathcal{B}$) is a proper ideal. Since $M \subset \overline{M}$, maximality of M will then imply that $M = \overline{M}$.

The closure of any subspace is a subspace by continuity of addition. Similarly, the closure of an ideal is an ideal by continuity of the product. To see this, let $m \in \overline{M}$ and $b \in \mathcal{B}$. Choose a sequence $(m_n) \subset M$ such that $m_n \to m$. Continuity of the product gives $bm_n \to bm$ and $m_n b \to mb$. Since M is an ideal, we have $(bm_n), (m_n b) \subset M$, so their limits, mb and bm, are in $\overline{M}$. This shows that $\overline{M}$ is an ideal.

We now show that $\overline{M}$ is proper, i.e. $\overline{M} \neq \mathcal{B}$. We first note that no element of a proper ideal can be invertible: if $m \in M$ is invertible, then $b = (bm^{-1})m \in M$ for all $b \in \mathcal{B}$, so $M = \mathcal{B}$. If we had $\overline{M} = \mathcal{B}$, then we could find a sequence $(m_n) \subset M$ such that $m_n \to 1_{\mathcal{B}}$. But any element of the open unit ball centered at $1_{\mathcal{B}}$ is invertible (Proposition 2.1.17), so then M would contain invertible elements. The fact that M is proper therefore prevents $\overline{M} = \mathcal{B}$.

We have now shown that $\overline{M}$ is a proper ideal, so $M = \overline{M}$ by maximality of M, as remarked at the beginning of our argument. It remains to show that $\mathcal{B}/M \cong \mathbb{C}$ as Banach algebras.

We claim that every nonzero element of $\mathcal{B}/M$ is invertible. For now, let $a \in \mathcal{B}$ be arbitrary. Using the fact that $\mathcal{B}$ is commutative, it is simple to verify that $\mathcal{B}a := \{ba : b \in \mathcal{B}\}$ is an ideal in $\mathcal{B}$. Moreover, the sum (as vector spaces) of two ideals is clearly an ideal, so $M + \mathcal{B}a$ is an ideal as well.

An element $a + M \in \mathcal{B}/M$ is nonzero precisely when $a \notin M$, so assume now that $a \notin M$. Then, $\mathcal{B}a \not\subset M$, because $a = 1_{\mathcal{B}}a \in \mathcal{B}a$. Thus, $M + \mathcal{B}a$ is an ideal of $\mathcal{B}$ properly containing M. By maximality of M, we must have $M + \mathcal{B}a = \mathcal{B}$. This means that there exists some $b \in \mathcal{B}$ and $m \in M$ such that $m + ba = 1_{\mathcal{B}}$. But then $(b + M)(a + M) = 1_{\mathcal{B}} + M = 1_{\mathcal{B}/M}$. By commutativity of $\mathcal{B}$ (and hence $\mathcal{B}/M$), this shows that the arbitary nonzero element $a + M \in \mathcal{B}/M$ is invertible, which is what we wanted to show.

By Corollary 2.1.20, it now follows that $\mathcal{B}/M = \mathbb{C}1_{\mathcal{B}/M}$, so clearly $\mathcal{B}/M \cong \mathbb{C}$ as Banach algebras. □

2.1.23 Definition (Characters and the spectrum of an algebra)**.** Let $\mathcal{B}$ be a commutative Banach algebra. An algebra homomorphism $\mu\colon \mathcal{B} \to \mathbb{C}$ is called a *character* on $\mathcal{B}$. We write $\mathcal{M}_{\mathcal{B}}$ for the set of all characters on $\mathcal{B}$ and call it the *spectrum* of $\mathcal{B}$.

The following lemma clarifies the relation between the spectrum of an algebra and the spectra of its elements. Once it has been proved, the Gelfand representation and its properties will follow without much effort.

2.1.24 Lemma. *Let $\mathcal{B}$ be a commutative Banach algebra. Then, the following statements are true.*

(i) For any $a \in \mathcal{B}$, we have that

$$\sigma(a) = \big\{\mu(a) : \mu \in \mathcal{M}_{\mathcal{B}}\big\} =: \mathcal{M}_{\mathcal{B}}(a).$$

In particular, $\mathcal{M}_{\mathcal{B}}$ is nonempty.

(ii) Characters are continuous and

$$\mathcal{M}_{\mathcal{B}} = \big\{\tau \in \mathcal{B}^*\backslash\{0\} : \tau(ab) = \tau(a)\tau(b) \text{ for all } a, b \in \mathcal{B}\big\}.$$

In fact, $\|\mu\| = 1$ for all $\mu \in \mathcal{M}_{\mathcal{B}}$.

Proof. We begin with (i). Let $\mu \in \mathcal{M}_{\mathcal{B}}$. If $b \in \operatorname{Inv}(\mathcal{B})$, then $\mu(b) \neq 0$ because

$$1 = \mu(1_{\mathcal{B}}) = \mu(bb^{-1}) = \mu(b)\mu(b^{-1}).$$

Let now $a \in \mathcal{B}$. Since $\mu(\mu(a)1_{\mathcal{B}} - a) = 0$, the element $\mu(a)1_{\mathcal{B}} - a$ cannot be invertible, so $\mathcal{M}_{\mathcal{B}}(a) \subset \sigma(a)$. The opposite inclusion, which we now wish to show, is less obvious.

Let $a \in \mathcal{B}$ be nonzero ($\sigma(0) = \{0\} = \mathcal{M}_{\mathcal{B}}(0)$ is trivial) and fix $\lambda \in \sigma(a)$. We will construct a character μ such that $\mu(a) = \lambda$.

Since $\lambda 1_{\mathcal{B}} - a$ is not invertible, the ideal $\mathcal{B}(\lambda 1_{\mathcal{B}} - a)$ is proper, because none of its elements are invertible (by Lemma 2.1.3 on inverses and products). By Lemma 2.1.22, $\mathcal{B}(\lambda 1_{\mathcal{B}} - a)$ is contained in a maximal ideal M and there is an algebra isomorphism $\Phi\colon \mathcal{B}/M \to \mathbb{C}$. The quotient map $q\colon \mathcal{B} \to \mathcal{B}/M$ is an algebra homomorphism, so the composition $\mu := \Phi \circ q\colon \mathcal{B} \to \mathbb{C}$ is a character on $\mathcal{B}$.

Since $\operatorname{Ker} q = M$ and Φ is an isomorphism, $\operatorname{Ker}\mu = M$. Finally, since $(\lambda 1_{\mathcal{B}} - a) \in \mathcal{B}(\lambda 1_{\mathcal{B}} - a) \subset M = \operatorname{Ker}\mu$, we have $\mu(\lambda 1_{\mathcal{B}} - a) = 0$, so $\lambda = \mu(a) \in \mathcal{M}_{\mathcal{B}}(a)$. Since $\lambda \in \sigma(a)$ was arbitrary, we have shown that $\sigma(a) \subset \mathcal{M}_{\mathcal{B}}(a)$, which concludes the proof of (i).

For (ii), we first note that if $\tau \in \mathcal{B}^*\backslash\{0\}$ and τ preserves the product, then it preserves the unit. This happens because

$$\tau(1_{\mathcal{B}}) = \tau(1_{\mathcal{B}}^2) = \tau(1_{\mathcal{B}})^2$$

and the only complex numbers which square to themselves are 0 and 1, so we must have $\tau(1_{\mathcal{B}}) = 1$, for otherwise $\tau = 0$. With this in mind, elements of $\mathcal{B}^*\backslash\{0\}$ are characters if and only if they preserve the product, so the equality

$$\mathcal{M}_{\mathcal{B}} = \big\{\tau \in \mathcal{B}^*\backslash\{0\} : \tau(ab) = \tau(a)\tau(b) \text{ for all } a, b \in \mathcal{B}\big\}$$

follows if we can show that characters are bounded, i.e. that $\mathcal{M}_{\mathcal{B}} \subset \mathcal{B}^*$.

Fix any $\mu \in \mathcal{M}_{\mathcal{B}}$. By point (i) we have $\mu(a) \in \sigma(a)$, so $|\mu(a)| \leq r(a) \leq \|a\|$ by Theorem 2.1.19. This shows that $\|\mu\| \leq 1$. Since $\mu(1_{\mathcal{B}}) = 1$, we must have $\|\mu\| = 1$, which concludes the proof. □

Let $\mathcal{B}$ be a Banach algebra and consider $\mathcal{B}^*$ with the weak* topology. The weak* topology is just the topology of pointwise convergence, so we have that

$$\tau_n \to \tau \text{ in } \mathcal{B}^* \qquad \Longleftrightarrow \qquad \tau_n(a) \to \tau(a) \text{ in } \mathbb{C} \text{ for all } a \in \mathcal{B}$$

for all sequences (and nets) $(\tau_n) \subset \mathcal{B}$ and $\tau \in \mathcal{B}^*$. If we fix some $\tau \in \mathcal{B}^*$, the sets

$$W_\tau(a_1, \ldots, a_N; \epsilon) \coloneqq \big\{\tau' \in \mathcal{B}^* : |\tau(a_j) - \tau'(a_j)| < \epsilon \text{ for } 1 \leq j \leq N\big\} \tag{2.4}$$

(for all $N \in \mathbb{N}_1$, $a_1, \ldots, a_N \in \mathcal{B}^*$ and $\epsilon > 0$) form a local base at τ.

Having identified the spectrum $\mathcal{M}_{\mathcal{B}}$ as a subset of $\mathcal{B}^*$, we may equip it with the subspace topology (w.r.t. the weak* topology). From now on, we will always assume that $\mathcal{M}_{\mathcal{B}}$ is equipped with this topology. In other words, convergence in $\mathcal{M}_{\mathcal{B}}$ is precisely pointwise convergence.

2.1.25 Proposition (The spectrum is compact Hausdorff)**.** *Let $\mathcal{B}$ be a commutative Banach algebra. Then, its spectrum $\mathcal{M}_{\mathcal{B}}$ is compact and Hausdorff.*

Proof. The weak* topology on $\mathcal{B}^*$ is Hausdorff, so $\mathcal{M}_{\mathcal{B}}$ is Hausdorff (this claim follows quite easily from consideration of the sets in Equation (2.4)).

The Banach-Alaoglu theorem (Theorem A.2.3) tells us that the closed unit ball in $\mathcal{B}^*$ is compact in the weak* topology. Closed subsets of compact sets are compact, so if we can show that $\mathcal{M}_{\mathcal{B}}$ is closed in $\mathcal{B}^*$ (and hence in the closed unit ball of $\mathcal{B}^*$), then we are done. The fact that $\mathcal{M}_{\mathcal{B}}$ is closed in $\mathcal{B}^*$ is very simple to show if one is familiar with nets. For the benefit of a potential reader who is not, we give an alternate proof.

Suppose $\tau \in \mathcal{B}^*$ is in the closure of $\mathcal{M}_{\mathcal{B}}$. Let $a, b \in \mathcal{B}$ and fix any $\epsilon > 0$. Since $\tau \in \overline{\mathcal{M}_{\mathcal{B}}}$, its neighborhood $W_\tau(a, b, ab; \epsilon)$ must intersect $\mathcal{M}_{\mathcal{B}}$. Thus, there is some character $\mu \in \mathcal{M}_{\mathcal{B}} \cap W_\tau(a, b, ab; \epsilon)$. But then

$$\begin{aligned} |\tau(ab) - \tau(a)\tau(b)| &\leq |\tau(ab) - \mu(ab)| + |\mu(a)\mu(b) - \tau(a)\tau(b)| \\ &< \epsilon + |\mu(a)\mu(b) - \tau(a)\mu(b)| + |\tau(a)\mu(b) - \tau(a)\tau(b)| \\ &= \epsilon + |\mu(a) - \tau(a)||\mu(b)| + |\tau(a)||\mu(b) - \tau(b)| \\ &< \epsilon\big(1 + |\mu(b)| + |\tau(a)|\big) \\ &\leq \epsilon\big(1 + \|b\| + |\tau(a)|\big), \end{aligned}$$

where we used $\|\mu\| = 1$ in the last step (see Lemma 2.1.24). Since $\epsilon > 0$ was arbitrary, we must have $\tau(ab) = \tau(a)\tau(b)$. Since $W_\tau(1_{\mathcal{B}}; \epsilon)$ is a neighborhood of τ for any $\epsilon > 0$, we similarly find that $\tau(1_{\mathcal{B}})$ is arbitrarily close to 1, so that $\tau(1_{\mathcal{B}}) = 1$. Thus, $\tau \in \mathcal{M}_{\mathcal{B}}$.

We have now shown that $\mathcal{M}_{\mathcal{B}}$ contains it closure, so it must be closed. By the second paragraph of the proof, we are done. $\square$

We motivated our investigation of characters with the following observation. If X is a compact Hausdorff space, then $\mathrm{ev}_x \colon C(X) \to \mathbb{C}$ is an algebra homomorphisms for each $x \in X$. Thus, in order to realize a commutative Banach algebra $\mathcal{B}$ as an algebra of the form $C(X)$, it would be wise to investigate algebra homomorphisms $\mu \colon \mathcal{B} \to \mathbb{C}$. We have now built a compact Hausdorff space out of such algebra homomorphisms, namely its spectrum $\mathcal{M}_{\mathcal{B}}$.

In the example of $C(X)$, we can think of a function $f \in C(X)$ as an *evaluation on evaluations*. That is, for every $f \in C(X)$, we have a map

$$\begin{aligned} \mathrm{ev}_f \colon \{\mathrm{ev}_x : x \in X\} &\to \mathbb{C} \\ \mathrm{ev}_x &\mapsto \mathrm{ev}_f(\mathrm{ev}_x) \coloneqq \mathrm{ev}_{\mathrm{x}}(f) = f(x). \end{aligned}$$

This is the idea behind the Gelfand representation.

2.1.26 Theorem (The Gelfand representation). *Let $\mathcal{B}$ be a commutative Banach algebra. Then, the evaluation map*

$$\begin{aligned} \mathrm{ev} \colon \mathcal{B} &\to C(\mathcal{M}_{\mathcal{B}}) \\ a &\mapsto \big(\mathrm{ev}_a \colon \mu \mapsto \mathrm{ev}_a(\mu) \coloneqq \mu(a)\big) \end{aligned}$$

is a norm-decreasing algebra homomorphism. Moreover,

$$\sigma(a) = \mathrm{ev}_a(\mathcal{M}_{\mathcal{B}}) \quad \textit{and} \quad r(a) = \|\mathrm{ev}_a\|_\infty$$

for all $a \in \mathcal{B}$.

Proof. We first show that the evaluation map is well-defined, which amounts to showing that evaluation at $a \in \mathcal{B}$ is a continuous map on $\mathcal{M}_{\mathcal{B}}$. This is immediate using nets, but we will give an alternate argument.

Let $\mu \in \mathcal{M}_{\mathcal{B}}$, let $\epsilon > 0$ be arbitary and let $B(\mu(a), \epsilon) \subset \mathbb{C}$ denote the open ball of radius ϵ centered at $\mu(a) = \mathrm{ev}_a(\mu)$. We will show that the preimage of this ball under ev_a is a neighborhood of μ, which means that $\mathrm{ev}_a \colon \mathcal{M}_{\mathcal{B}} \to \mathbb{C}$ is continuous. This is immediate, for

$$\begin{aligned}\mathrm{ev}_a^{-1}\big(B(\mathrm{ev}_a(\mu), \epsilon)\big) &= \big\{\mu' \in \mathcal{M}_{\mathcal{B}} : \mathrm{ev}_a(\mu') = \mu'(a) \in B\big(\mu(a), \epsilon\big)\big\} \\ &= \big\{\mu' \in \mathcal{M}_{\mathcal{B}} : |\mu(a) - \mu'(a)| < \epsilon\big\} = W_\mu(a; \epsilon) \cap \mathcal{M}_{\mathcal{B}}\end{aligned}$$

is a neighborhood of μ in $\mathcal{M}_{\mathcal{B}}$.

For any $a \in \mathcal{B}$, we find that

$$\|\mathrm{ev}_a\|_\infty = \sup\big\{|\mathrm{ev}_a(\mu)| : \mu \in \mathcal{M}_{\mathcal{B}}\big\} \leq \sup\big\{\|\mu\|\|a\| : \mu \in \mathcal{M}_{\mathcal{B}}\big\} = \|a\|$$

since $\|\mu\| = 1$ for all $\mu \in \mathcal{M}_{\mathcal{B}}$ (Lemma 2.1.24). This shows that the evaluation map $\mathrm{ev} \colon \mathcal{B} \to C(\mathcal{M}_{\mathcal{B}})$ is norm-decreasing. The fact that $\mathrm{ev}(1_{\mathcal{B}}) = \mathrm{ev}_{1_{\mathcal{B}}} = 1_{C(\mathcal{M}_{\mathcal{B}})}$ and that

$$\mathrm{ev}_{ab}(\mu) = \mu(ab) = \mu(a)\mu(b) = \mathrm{ev}_a(\mu)\mathrm{ev}_b(\mu) = (\mathrm{ev}_a \mathrm{ev}_b)(\mu)$$

for all $\mu \in \mathcal{M}_{\mathcal{B}}$ and all $a, b \in \mathcal{M}$ shows that it is an algebra homomorphism as well (linearity should also be shown, but this is even simpler).

In Lemma 2.1.24, we proved that $\sigma(a) = \{\mu(a) : \mu \in \mathcal{M}_{\mathcal{B}}\}$. The claim that $\sigma(a) = \mathrm{ev}_a(\mathcal{M}_{\mathcal{B}})$ is just different notation for this fact, since $\mu(a) = \mathrm{ev}_a(\mu)$. The claim that $r(a) = \|\mathrm{ev}_a\|_\infty$ is now just a matter of recalling definitions:

$$r(a) = \sup\big\{|\lambda| : \lambda \in \sigma(a)\big\} = \sup\big\{|\lambda| : \lambda \in \mathrm{ev}_a(\mathcal{M}_{\mathcal{B}})\big\} = \|\mathrm{ev}_a\|_\infty.$$

This concludes the proof. □

As we have remarked, the Gelfand representation need not be injective; its kernel clearly consists of those $b \in \mathcal{B}$ for which $\sigma(b) = \{0\}$. Another shortcoming is that $\mathrm{ev}(\mathcal{B})$ need not be closed in $C(\mathcal{M}_{\mathcal{B}})$, so our representation of $\mathcal{B}$ need not be a Banach algebra. We will eventually see that both of these problems are remedied if the commutative Banach algebra in question is a C*-algebra.

This is not that to say that the Gelfand representation in the general setting of Banach algebras is not important or profound. Indeed, there are deep ties to Fourier theory. We recommend Deitmar and Echterhoff [11] for further exploration.

2.2 | C*-Algebras

We are now fully prepared to explore the wonderful world of C*-algebras. As in the previous section, we are mainly following Murphy [23].

2.2.1 The Very Basics of C*-Algebras

Let H be a Hilbert space and consider the Banach algebra $\mathcal{B}(H)$ equipped with the involution given by the adjoint. For any $T \in \mathcal{B}(H)$, we have that

$$\|T\|^2 = \sup_{\|v\|\leq 1} \|Tv\|^2 = \sup_{\|v\|\leq 1} \langle T^*Tv, v\rangle \leq \|T^*T\| \leq \|T^*\|\|T\|,$$

and so $\|T\| \leq \|T^*\|$. Applying this to T^* gives $\|T^*\| \leq \|T^{**}\| \leq \|T\|$, so we must have $\|T\| = \|T^*\|$. We now find that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2,$$

and so $\|T^*T\| = \|T\|^2$.

2.2.1 Definition (C*-algebras)**.** A Banach algebra $\mathcal{A}$ equipped with an involution $\star\colon a \mapsto a^*$ satisfying

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in \mathcal{A} \tag{2.5}$$

is said to be a *C*-algebra*. Condition (2.5) is called *the C*-equality.*

There is some redundancy in our definitions: normalization of the unit is superfluous, since $0 \neq 1_{\mathcal{A}}$ and $\|1_{\mathcal{A}}\| = \|1_{\mathcal{A}}^* 1_{\mathcal{A}}\| = \|1_{\mathcal{A}}\|^2$ by the C*-equality.

The C*-equality is extremely powerful, for it not only *relates* the algebraic structure of a C*-algbera to its norm (as submultiplicativity does), but it implies that the norm is *uniquely determined* by the algebraic structure. A fantastic consequence of this, as we will see, is that injective $\star$-algebra homomorphisms between C*-algebras *must* be isometric.

The manner in which the algebraic structure determines the norm is part of Lemma 2.2.3. Before we state it, we introduce some basic terminology for distinguished elements of $\star$-algebras and C*-algebras. The inspiration from Hilbert space theory should be clear.

2.2.2 Definition. Let A be a $\star$-algebra. An element $a \in A$ is called

- *normal* if $aa^* = a^*a$.
- *unitary* if $aa^* = 1_A = a^*a$, i.e. if $a^{-1} = a^*$.

- *self-adjoint* if $a^* = a$.

Let A be a $\star$-algebra. Then, any $a \in A$ can be written as $a = a_1 + ia_2$, where a_1 and a_2 are self-adjoint:

$$a = \left(\frac{a + a^*}{2}\right) + i\left(\frac{a - a^*}{2i}\right).$$

We will use this on a couple of occasions.

2.2.3 Lemma (Basic properties of C*-norms). *Let $\mathcal{A}$ be a C*-algebra. Then, the following statements are true.*

(i) If $a \in \mathcal{A}$ is self-adjoint, then $\|a\| = r(a)$.

(ii) There is no other norm on the $\star$-algebra $\mathcal{A}$ with respect to which it is a C-algebra.*

(iii) The involution is an isometry: we have $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.

Proof. For (i), let $a^* = a \in \mathcal{A}$. Then, $\|a^2\| = \|a\|^2$ by the C*-equality. Since any power of a is self-adjoint, we can use induction to show that $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$. By our formula for the spectral radius (Theorem 2.1.19), we obtain

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|a\| = \|a\|,$$

as desired.

For (ii), assume that $\|\cdot\|'$ is a (possibly different) norm on $\mathcal{A}$ which turns it into a C*-algebra. For any $a \in \mathcal{A}$, the element $a^*a \in \mathcal{A}$ is self-adjoint, so (i) implies that

$$(\|a\|')^2 = \|a^*a\|' = r(a^*a) = \|a^*a\| = \|a\|^2$$

(recall that the spectral radius is defined purely in terms of the algebraic structure on $\mathcal{A}$ and makes no reference to the norm). Thus, $\|\cdot\|' = \|\cdot\|$.

For (iii), let $a \in \mathcal{A}$ be any nonzero element. The C*-equality and submultiplicativity of the norm gives

$$\|a\|^2 = \|a^*a\| \le \|a^*\|\|a\|, \quad \text{so } \|a\| \le \|a^*\|.$$

The same result applied to a^* instead of a gives $\|a^*\| \le \|a^{**}\| = \|a\|$, so we must have $\|a\| = \|a^*\|$, which concludes the proof. □

We now define a class of Banach algebras with involutions which are not necessarily C*-algebras, but for which the involution is an isometry.

2.2.4 Definition (Banach $\star$-algebras)**.** A Banach algebra $\mathcal{B}$ equipped with an involution $\star\colon a \mapsto a^*$ is a *Banach $\star$-algebra* if $\|a^*\| = \|a\|$ for all $a \in \mathcal{B}$.

By point (iii) of the lemma we just proved, all C*-algebras are Banach $\star$-algebras. We have now defined three kinds of algebras with deceptively simple names. We summarize our definitions in Table 2.2.1 and then provide some examples.

Table 2.1: Central structures, listed in decreasing order of generality.

	Structure	Compatibility condition
Banach algebra	Banach space and algebra	Submultiplicative norm and normalized unit
Banach $\star$-algebra	Banach algebra with involution	Isometric involution
C*-algebra	Banach algebra with involution	C*-equality ($\implies$ isometric involution)

2.2.5 Example (C*-algebras and Banach $\star$-algebras)**.** See Example 2.1.15 for the underlying Banach algebra structures.

- The Banach algebra $\mathbb{C}$ with the involution given by complex conjugation is a C*-algebra.
- Let H be a complex nonzero Hilbert space. Any closed subalgebra of $\mathcal{B}(H)$ that is also closed under the taking of adjoints is a C*-algebra (where the involution is given by the adjoint).
- Consider the Banach algebra $C(X)$ of continuous functions on a compact Hausdorff space X. This is a C*-algebra with the involution given by pointwise complex conjugation. For $f \in C(X)$, we will denote the pointwise complex conjugate of f by either $\overline{f}$ or f^*.
- The convolution algebra $\ell^1(\mathbb{Z})$ with the involution defined by

  $$a^*(n) := \overline{a(-n)} \quad \text{for all } n \in \mathbb{Z} \text{ and } a \in \ell^1(\mathbb{Z})$$

  is *not* a C*-algebra (the C*-equality does not hold), but it is a commutative Banach $\star$-algebra.

Whenever we refer to $\mathbb{C}$, $\mathcal{B}(H)$ or $C(X)$ as C*-algebras, these are always the structures we will have in mind.

The following proposition begins to reveal the convenience of having a norm determined by algebraic structure.

2.2.6 Proposition. *Let $\mathcal{B}$ be a Banach $\star$-algebra, let $\mathcal{A}$ be a C*-algebra and let $\Phi\colon \mathcal{B} \to \mathcal{A}$ be a $\star$-algebra homomorphism. Then, Φ is norm-decreasing.*

In particular, $\star$-algebra homomorphisms between C-algebras are norm-decreasing and $\star$-algebra isomorphisms between C*-algebras are isometric.*

Proof. Let $b \in \mathcal{B}$. By point (iii) of Lemma 2.1.9, we have $\sigma(\Phi(b^*b)) \subset \sigma(b^*b)$ and hence $r(\Phi(b^*b)) \leq r(b^*b)$. Thus,

$$\|\Phi(b)\|^2 = \|\Phi(b)^*\Phi(b)\| = r\big(\Phi(b^*b)\big) \leq r(b^*b) \leq \|b^*b\| \leq \|b^*\|\|b\| = \|b\|^2,$$

where the second equality follows from point (i) of Lemma 2.2.3 and the second inequality follows from Theorem 2.1.19 on spectra in Banach algebras. This proves that Φ is norm-decreasing.

By what we have already shown, a $\star$-algebra isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ between C*-algebras (which are Banach $\star$-algebras) must both be norm-decreasing and have a norm-decreasing inverse, so $\|a\| = \|\Phi^{-1}\Phi(a)\| \leq \|\Phi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$. Thus, such algebra isomorphisms are isometric. □

This result makes it clear that $\star$-algebra isomorphisms are the correct notion of isomorphisms for C*-algebras – we obtain isometry for free! We may refer to the presence of a $\star$-algebra isomorphism $\Phi\colon \mathcal{A} \to \mathcal{A}'$ between C*-algebras by saying that $\mathcal{A} \cong \mathcal{A}'$ *as C*-algebras.*

We now investigate the spectra of distinguished elements in a C*-algebra. We will use the symbol $\mathbb{T}$ to denote the unit circle in $\mathbb{C}$.

2.2.7 Proposition. *Let $\mathcal{A}$ be a C*-algebra.*

(i) If $a \in \mathcal{A}$ is unitary, then $\sigma(a) \subset \mathbb{T}$.

(ii) If $a \in \mathcal{A}$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.

Proof. We begin with (i). Let $a \in \mathcal{A}$ be unitary, i.e. $a^* = a^{-1}$. By point (ii) of Lemma 2.1.9, we have $\lambda \in \sigma(a)$ if and only if $\lambda^{-1} \in \sigma(a^{-1})$. Now, the C*-equality gives $\|a\|^2 = \|a^*a\| = \|1_{\mathcal{A}}\| = 1$, so we have

$$r(a) \leq \|a\| = 1 \quad \text{and} \quad r(a^{-1}) = r(a^*) \leq \|a^*\| = \|a\| = 1$$

by Theorem 2.1.19 and the fact that the involution is isometric.

Let now $\lambda \in \sigma(a)$. Since $\lambda^{-1} \in \sigma(a^{-1})$, we obtain both $|\lambda| \leq r(a) = 1$ and $|\lambda^{-1}| \leq r(a^{-1}) = 1$, which gives $|\lambda| = 1$. This shows that $\sigma(a) \subset \mathbb{T}$.

For (ii), let $a \in \mathcal{A}$ be self-adjoint, i.e. $a^* = a$. Since ia and $-ia$ trivially commute, we have $e^{ia}e^{-ia} = e^0 = 1_{\mathcal{A}}$ and $e^{-ia}e^{ia} = 1_{\mathcal{A}}$ (see Lemma 2.1.16). That is, $(e^{ia})^{-1} = e^{-ia}$. Since the involution is an isometry, it is also continuous, so

$$(e^{ia})^* = \left(\sum_{n=0}^{\infty} \frac{(ia)^n}{n!}\right)^* = \sum_{n=0}^{\infty} \left(\frac{(ia)^n}{n!}\right)^* = \sum_{n=0}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia} = (e^{ia})^{-1}.$$

This shows that e^{ia} is unitary. By (i), we now know that $\sigma(e^{ia}) \subset \mathbb{T}$.

For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} e^{i\lambda}1_{\mathcal{A}} - e^{ia} &= e^{i\lambda}\big(1_{\mathcal{A}} - e^{i(a-\lambda 1_{\mathcal{A}})}\big) = -e^{i\lambda} \sum_{n=1}^{\infty} \frac{\big(i(a-\lambda 1_{\mathcal{A}})\big)^n}{n!} \\ &= \left(e^{i\lambda} \sum_{n=1}^{\infty} \frac{i^n (a-\lambda 1_{\mathcal{A}})^{n-1}}{n!}\right)(\lambda 1_{\mathcal{A}} - a). \end{aligned}$$

In the last step, we could just as well have written $\lambda 1_{\mathcal{A}} - a$ to the left of the sum, so $\lambda 1_{\mathcal{A}} - a$ commutes with the sum. By Lemma 2.1.3 on inverses and products, we see that if $\lambda 1_{\mathcal{A}} - a$ is not invertible, then $e^{i\lambda}1_{\mathcal{A}} - e^{ia}$ cannot be invertible either. In other words: if $\lambda \in \sigma(a)$, then $e^{i\lambda} \in \sigma(e^{ia}) \subset \mathbb{T}$. Thus, we must have $\sigma(a) \subset \mathbb{R}$, which is what we wanted to show. □

We now consider C*-subalgebras.

2.2.8 Definition (C*-subalgebras)**.** Let $\mathcal{A}$ be a C*-algebra. A *C*-subalgebra* $\mathcal{C}$ of $\mathcal{A}$ is a $\star$-subalgebra $\mathcal{C} \subset \mathcal{A}$ that is closed.

C*-subalgebras of a C*-algebra $\mathcal{A}$ are of course precisely those subalgebras of $\mathcal{A}$ which are C*-algebras in their own right (with the norm and involution restricted from $\mathcal{A}$).

Let now $\mathcal{C}$ be a C*-subalgebra of $\mathcal{A}$. Since the spectrum of an element is defined with respect to invertibility in the C*-algebra containing it, an element $c \in \mathcal{C} \subset \mathcal{A}$ will have one spectrum with respect to $\mathcal{C}$ and a potentially different spectrum with respect to $\mathcal{A}$. We will denote these spectra by $\sigma_{\mathcal{C}}(c)$ and $\sigma_{\mathcal{A}}(c)$, respectively.

Clearly $\mathrm{Inv}(\mathcal{C}) \subset \mathrm{Inv}(\mathcal{A}) \cap \mathcal{C}$, for an inverse of $c \in \mathcal{C}$ in $\mathcal{C}$ is an inverse of c in $\mathcal{A}$ as well. Thus, for any $\lambda \in \mathbb{C}$ and $c \in \mathcal{C}$, we have that

$$\lambda 1_{\mathcal{A}} - c \notin \mathrm{Inv}(\mathcal{A}) \implies \lambda 1_{\mathcal{A}} - c \notin \mathrm{Inv}(\mathcal{C}).$$

This shows that $\sigma_{\mathcal{A}}(c) \subset \sigma_{\mathcal{C}}(c)$ (recall that $1_{\mathcal{C}} = 1_{\mathcal{A}}$ by our definitions). We will now see that these spectra in fact must be equal. This means that we are free to drop the subscripts going forward.

2.2.9 Proposition (Spectral permanence)**.** *Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{C} \subset \mathcal{A}$ be a C*-subalgebra of $\mathcal{A}$. Then,*

$$\operatorname{Inv}(\mathcal{C}) = \operatorname{Inv}(\mathcal{A}) \cap \mathcal{C} \quad \textit{and} \quad \sigma_{\mathcal{C}}(c) = \sigma_{\mathcal{A}}(c) \textit{ for all } c \in \mathcal{C}.$$

Proof. Assume first that $c \in \mathcal{C}$ is self-adjoint. We will show that $\sigma_{\mathcal{C}}(c) = \sigma_{\mathcal{A}}(c)$. As noted before the proposition, $\sigma_{\mathcal{A}}(c) \subset \sigma_{\mathcal{C}}(c)$ is immediate from the observation that $\operatorname{Inv}(\mathcal{C}) \subset \operatorname{Inv}(\mathcal{A}) \cap \mathcal{C}$. We will show that $\sigma_{\mathcal{C}}(c) \subset \sigma_{\mathcal{A}}(c)$ by contraposition.

For any $t \in \mathbb{R}$, self-adjointness of c implies that

$$\Big(t + \frac{i}{n}\Big)1_{\mathcal{A}} - c \in \operatorname{Inv}(\mathcal{C}) \quad \text{for all } n \in \mathbb{N}_1, \tag{2.6}$$

for otherwise the non-real number $t + i/n$ would be in $\sigma_{\mathcal{C}}(c)$, which cannot happen since $\sigma_{\mathcal{C}}(c) \subset \mathbb{R}$ (Proposition 2.2.7). If $t \notin \sigma_{\mathcal{A}}(c)$, then $t1_{\mathcal{A}} - c$ is invertible in $\mathcal{A}$, and by continuity of the inverse we see that

$$\bigg(\Big(t + \frac{i}{n}\Big)1_{\mathcal{A}} - c\bigg)^{-1} \to \big(t1_{\mathcal{A}} - c\big)^{-1} \quad \text{in } \mathcal{A} \text{ as } n \to \infty.$$

The sequence on the left is in $\mathcal{C}$ by Equation (2.6). Since $\mathcal{C}$ is closed, we can conclude that $(t1_{\mathcal{A}} - c)^{-1} \in \mathcal{C}$. This shows that $t \notin \sigma_{\mathcal{C}}(c)$ whenever $t \notin \sigma_{\mathcal{A}}(c)$, so $\sigma_{\mathcal{C}}(c) \subset \sigma_{\mathcal{A}}(c)$.

We have now shown that $\sigma_{\mathcal{C}}(c) = \sigma_{\mathcal{A}}(c)$ for all self-adjoint $c \in \mathcal{C}$. The next step is to show that $\operatorname{Inv}(\mathcal{A}) \cap \mathcal{C} \subset \operatorname{Inv}(\mathcal{C})$, so that these sets must be equal.

Let $c \in \operatorname{Inv}(\mathcal{A}) \cap \mathcal{C}$. Then,

$$(c^*c)\big(c^{-1}(c^{-1})^*\big) = 1_{\mathcal{A}} = \big(c^{-1}(c^{-1})^*\big)(c^*c), \quad \text{so } c^*c \in \operatorname{Inv}(\mathcal{A}).$$

Now, c^*c is a self-adjoint element of $\mathcal{C}$, so we know that $\sigma_{\mathcal{C}}(c^*c) = \sigma_{\mathcal{A}}(c^*c)$ by the previous part of the proof. Since

$$c^*c \in \operatorname{Inv}(\mathcal{A}) \iff 0 \notin \sigma_{\mathcal{A}}(c^*c) \quad \text{and} \quad c^*c \in \operatorname{Inv}(\mathcal{C}) \iff 0 \notin \sigma_{\mathcal{C}}(c^*c),$$

we obtain $c^*c \in \operatorname{Inv}(\mathcal{C})$ from $c^*c \in \operatorname{Inv}(\mathcal{A})$. Finally, we have

$$c\big((c^*c)^{-1}c^*\big) = c\big(c^{-1}(c^{-1})^*c^*\big) = 1_{\mathcal{A}} = \big((c^*c)^{-1}c^*\big)c,$$

so $c^{-1} = (c^*c)^{-1}c^* \in \mathcal{C}$ (since $(c^*c)^{-1} \in \mathcal{C}$ and $c^* \in \mathcal{C}$). This concludes the proof that $\operatorname{Inv}(\mathcal{A}) \cap \mathcal{C} = \operatorname{Inv}(\mathcal{C})$.

For any $c \in \mathcal{C}$ and $\lambda \in \mathbb{C}$, the fact that $\operatorname{Inv}(\mathcal{A}) \cap \mathcal{C} = \operatorname{Inv}(\mathcal{C})$ gives

$$\lambda 1_{\mathcal{A}} - c \notin \operatorname{Inv}(\mathcal{C}) \quad \iff \quad \lambda 1_{\mathcal{A}} - c \notin \operatorname{Inv}(\mathcal{A}),$$

so $\sigma_{\mathcal{C}}(c) = \sigma_{\mathcal{A}}(c)$, which concludes the proof. $\square$

2.2.2 The Continuous Functional Calculus

In this subsection, commutative C*-algebras will be denoted by $\mathcal{C}$ and general C*-algebras by $\mathcal{A}$. We will always specify our assumptions, but this should make the text more transparent. We have already seen that $\star$-algebra isomorphisms between C*-algebras are isometric (Proposition 2.2.6), but we will nevertheless speak of "isometric isomorphisms of $\star$-algebras" for emphasis.

We begin by revisiting the Gelfand representation in the case that our commutative Banach algebra is a C*-algebra $\mathcal{C}$. Recall that we have defined characters on $\mathcal{C}$ to be algebra homomorphism $\mu\colon \mathcal{C} \to \mathbb{C}$. The following lemma shows that characters on $\mathcal{C}$ must be $\star$-algebra homomorphisms as well.

2.2.10 Lemma. *Let $\mathcal{C}$ be a commutative C*-algebra and let $\mu \in \mathcal{M}_{\mathcal{C}}$ be a character on $\mathcal{C}$. Then, μ preserves the involution:*

$$\mu(c^*) = \overline{\mu(c)} \quad \text{for all } c \in \mathcal{C}.$$

Thus, characters on (commutative) C-algebras are $\star$-algebra homomorphisms.*

Proof. Let $c \in \mathcal{C}$ and write $c = c_1 + ic_2$ with c_1 and c_2 self-adjoint (see the paragraph following Definition 2.2.2). We know that $\mu(c_1) \in \sigma(c_1) \subset \mathbb{R}$ and $\mu(c_2) \in \sigma(c_2) \subset \mathbb{R}$ (Lemma 2.1.24 and Proposition 2.2.7). Since $c^* = c_1 - ic_2$, we find that

$$\mu(c^*) = \mu(c_1) - i\mu(c_2) = \overline{\mu(c_1) + i\mu(c_2)} = \overline{\mu(c_1 + ic_2)} = \overline{\mu(c)},$$

which is what we wanted to show. □

2.2.11 Theorem (The Gelfand representation)**.** *Let $\mathcal{C}$ be a commutative C*-algebra. Then, the Gelfand representation*

$$\begin{aligned} \mathrm{ev}\colon \mathcal{C} &\to C(\mathcal{M}_{\mathcal{C}}) \\ c &\mapsto \big(\mathrm{ev}_c\colon \mu \mapsto \mathrm{ev}_c(\mu) := \mu(c)\big) \end{aligned}$$

is an isometric isomorphism of $\star$-algebras.

Proof. We already know that $\mathrm{ev}\colon \mathcal{C} \to C(\mathcal{M}_{\mathcal{C}})$ is an algebra homomorphism by Theorem 2.1.26. We need to show that it preserves the involution and that is isometric and bijective.

Let $c \in \mathcal{C}$. By Lemma 2.2.10, we have

$$\mathrm{ev}_{c^*}(\mu) = \mu(c^*) = \overline{\mu(c)} = \overline{\mathrm{ev}_c}(\mu) = (\mathrm{ev}_c)^*(\mu) \quad \text{for all } \mu \in \mathcal{M}_{\mathcal{C}},$$

so $\mathrm{ev}_{c^*} = (\mathrm{ev}_c)^*$, which shows that ev preserves the involution. To see that it is isometric, we use the C*-equality and the fact that $r(c^*c) = \|\mathrm{ev}_{c^*c}\|_\infty$ (Theorem 2.1.26):

$$\|c\|^2 = \|c^*c\| = r(c^*c) = \|\mathrm{ev}_{c^*c}\|_\infty = \|(\mathrm{ev}_c)^*\mathrm{ev}_c\|_\infty = \|\mathrm{ev}_c\|_\infty^2.$$

As an isometry between complete spaces, ev is injective and its image is closed. If we can show that its image $\mathrm{ev}(\mathcal{C})$ is dense in $C(\mathcal{M}_\mathcal{C})$, then it must be surjective, which will conclude the proof that is is an isometric isomorphism of $\star$-algebras.

To conclude that $\mathrm{ev}(\mathcal{C})$ is dense in $C(\mathcal{M}_\mathcal{C})$, we appeal to the Stone-Weierstrass theorem (Theorem A.1.1). The Stone-Weierstrass theorem applies because $\mathrm{ev}(\mathcal{C})$ is a $\star$-subalgebra of $C(\mathcal{M}_\mathcal{C})$ such that

- $\mathrm{ev}(\mathcal{C})$ separates points of $\mathcal{M}_\mathcal{C}$: as functions on $\mathcal{C}$, μ and μ' are distinct if and only if they differ on an element of $\mathcal{C}$.
- $\mathrm{ev}(\mathcal{C})$ vanishes at no point of $\mathcal{C}$: $\mathrm{ev}_{1_\mathcal{C}}(\mu) = \mu(1_\mathcal{C}) = 1 \neq 0$ for all $\mu \in \mathcal{M}_\mathcal{C}$.

This concludes the proof. □

We now turn to the continuous functional calculus.

2.2.12 Definition (The C*-algebra generated by a set)**.** Let $\mathcal{A}$ be a C*-algebra and let $S \subset \mathcal{A}$ be a subset. The C*-algebra *generated* by S is the smallest (w.r.t. inclusion) C*-subalgebra of $\mathcal{A}$ containing S. It is denoted by $C^*(S)$. We will write $C^*(a) := C^*(\{a\})$ for all $a \in \mathcal{A}$.[6]

The C*-algebra generated by $S \subset \mathcal{A}$ equals the intersection of all C*-subalgebras of $\mathcal{A}$ containing S. It always exists, as $\mathcal{A}$ is a C*-subalgebra of itself containing any of its subsets, and an arbitrary intersection of C*-subalgebras is a C*-subalgebra. These claims are straightforward to verify.

Let $\mathcal{A}$ be a C*-algebra and let $a \in A$ be a normal element. Then,

$$C^*(a) = \overline{\mathrm{span}}_\mathbb{C}\big\{a^n(a^*)^m \in \mathcal{A} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\big\}, \tag{2.7}$$

because the closed span on the right hand side is a C*-subalgebra of $\mathcal{A}$ containing a, and any C*-algebra containing a must contain this closed span. Note that this is a commutative C*-algebra (by normality of a).

[6]As our C*-algebras are unital, $1_\mathcal{A} \in C^*(a)$ by definition. When working with nonunital C*-algebras, what we are calling $C^*(a)$ would be denoted by $C^*(1_\mathcal{A}, a)$ and $C^*(a)$ would carry a different meaning.

The continuous functional calculus refers to an identification of $C^*(a)$ with the C*-algebra $C(\sigma(a))$ of continuous functions on the spectrum of a. The Gelfand representation identifies $C^*(a)$ with $C(\mathcal{M}_{C^*(a)})$. To obtain the continuous functional calculus, we will identify $\mathcal{M}_{C^*(a)}$ and $\sigma(a)$ as topological spaces. The following lemma provides the link.

2.2.13 Lemma. *Let $\mathcal{C}$ be a commutative C*-algebra that is generated by a single element $a \in \mathcal{C}$. Then, (the corestriction of)*[7] *evaluation at a,*

$$\begin{aligned} \mathrm{ev}_a|^{\sigma(a)} \colon \mathcal{M}_{\mathcal{C}} &\to \sigma(a) \\ \mu &\mapsto \mu(a), \end{aligned}$$

is a homeomorphism.

Proof. We will write ev_a instead of $\mathrm{ev}_a|^{\sigma(a)}$ for the duration of this proof.

From the construction of the Gelfand representation for commutative Banach algebras (see Proposition 2.1.25 and Theorem 2.1.26), we know that $\mathcal{M}_{\mathcal{C}}$ is compact, that ev_a is continuous and that $\mathrm{ev}_a(\mathcal{M}_{\mathcal{C}}) = \sigma(a)$.

We now show that ev_a is injective. Suppose that $\mu, \mu' \in \mathcal{C}$ are such that $\mu(a) = \mathrm{ev}_a(\mu) = \mathrm{ev}_a(\mu') = \mu'(a)$. This implies that $\mu(a^*) = \mu'(a^*)$ (Lemma 2.2.10) and we know that $\mu(1_{\mathcal{C}}) = 1 = \mu'(1_{\mathcal{C}})$. Now, since

$$\mathcal{C} = C^*(a) = \overline{\mathrm{span}}_{\mathbb{C}}\big\{a^n(a^*)^m \in \mathcal{C} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\big\},$$

continuity of μ and μ' implies that $\mu = \mu'$.

We have now shown that ev_a is a continuous bijection of the compact space $\mathcal{M}_{\mathcal{C}}$ onto $\sigma(a)$. This is enough to conclude that ev_a^{-1} is continuous, as the following argument shows.

Let $E \subset \mathcal{M}_{\mathcal{C}}$ be any closed set. As a closed subset of a compact space, E is compact, so by continuity of ev_a, the set

$$\big((\mathrm{ev}_a)^{-1}\big)^{-1}(E) = \mathrm{ev}_a(E)$$

is compact and hence closed in $\sigma(a)$ (compact subsets of Hausdorff spaces are closed). Thus, the preimage by ev_a^{-1} of any closed set is closed, so ev_a^{-1} is continuous. This show that ev_a is a homeomorphism and concludes the proof. □

[7] We have defined evaluation at a as the map $\mathrm{ev}_a \colon \mathcal{M}_{\mathcal{C}} \to \mathbb{C}$. We have seen that $\mathrm{ev}_a(\mathcal{M}_{\mathcal{C}}) = \sigma(a)$, so we are free to restrict the range of ev_a to $\sigma(a)$. This is called *corestriction* (restriction of range) and is what is meant by the notation $\mathrm{ev}_a|^{\sigma(a)}$.

For a continuous map $F\colon X \to Y$ between topological spaces, we define the *transpose* of F to be the map

$$\begin{aligned} F^t\colon C(Y) &\to C(X) \\ f &\mapsto F^t(f) := f \circ F. \end{aligned}$$

In words, F^t is simply precomposition with F.

2.2.14 Lemma. *Let X and Y be compact Hausdorff spaces. If $F\colon X \to Y$ is a homeomorphism, then $F^t\colon C(Y) \to C(X)$ is an isometric isomorphism of $\star$-algebras.*

Proof. The fact that F^t is a $\star$-algebra homomorphism requires only straightforward verifications like

$$\begin{aligned} F^t(fg)(x) &= (fg)\big(F(x)\big) = f\big(F(x)\big)g\big(F(x)\big) \\ &= F^t f(x) F^t g(x) = \big((F^t f)(F^t g)\big)(x) \quad \text{for all } x \in X; \end{aligned}$$

we omit the rest. Since

$$\big((F^{-1})^t \circ F^t\big)(f) = f \circ F \circ F^{-1} = f \quad \text{for all } f \in C(Y),$$

we have $(F^{-1})^t \circ F^t = \mathrm{Id}_{C(Y)}$. Similarly, $F^t \circ (F^{-1})^t = \mathrm{Id}_{C(X)}$. Thus, F^t is bijective and hence an isomorphism of $\star$-algebras. Since $C(X)$ and $C(Y)$ are C*-algebras, F^t must be isometric (Proposition 2.2.6). $\square$

We now obtain the continuous functional calculus simply by putting the pieces together.

2.2.15 Theorem. *Let $\mathcal{C}$ be a commutative C*-algebra that is generated by a single element $a \in \mathcal{C}$, and let $\mathrm{ev}\colon \mathcal{C} \to C(\mathcal{M}_{\mathcal{C}})$ be the Gelfand representation. Then, the map*

$$\begin{aligned} \Gamma_a := \mathrm{ev}^{-1} \circ (\mathrm{ev}_a|^{\sigma(a)})^t : C\big(\sigma(a)\big) &\to \mathcal{C} \\ f &\mapsto \mathrm{ev}^{-1}(f \circ \mathrm{ev}_a|^{\sigma(a)}) \end{aligned}$$

is an isometric isomorphism of $\star$-algebras. Moreover, if $z\colon \sigma(a) \to \mathbb{C}$ denotes the inclusion of $\sigma(a)$ into $\mathbb{C}$, then $\Gamma_a(z) = a$.

Proof. The evaluation map $\mathrm{ev}\colon \mathcal{C} \to C(\mathcal{M}_{\mathcal{C}})$ is an isometric isomorphism of $\star$-algebras by Theorem 2.2.11. Combining Lemma 2.2.13 with Lemma 2.2.14 implies that $(\mathrm{ev}|^{\sigma(a)})^t\colon C(\sigma(a)) \to C(\mathcal{M}_{\mathcal{A}})$ is an isometric isomorphism of $\star$-algebras. The composition $\Gamma_a := \mathrm{ev}^{-1} \circ (\mathrm{ev}_a|^{\sigma(a)})^t$ is then an isometric isomorphism of $\star$-algebras as well.

Finally, if $z\colon \sigma(a) \to \mathbb{C}$ denotes the inclusion, then $z \circ \mathrm{ev}_a|^{\sigma(a)} = \mathrm{ev}_a$, so

$$\Gamma_a(z) = \mathrm{ev}^{-1}(z \circ \mathrm{ev}_a|^{\sigma(a)}) = \mathrm{ev}^{-1}(\mathrm{ev}_a) = a,$$

which concludes the proof. □

2.2.16 Corollary (The continuous functional calculus)**.** *Let $\mathcal{A}$ be a C*-algebra and let $a \in \mathcal{A}$ be a normal element. Then, there is an isometric $\star$-algebra isomorphism*

$$\Gamma_a\colon C\big(\sigma(a)\big) \to C^*(a)$$

such that $\Gamma_a(z) = a$.

Proof. If a is normal, then $C^*(a)$ is commutative (see Equation (2.7) and the surrounding discussion). By spectral permanence (Proposition 2.2.9) we have $\sigma_{C^*(a)}(a) = \sigma_{\mathcal{A}}(a)$, so Theorem 2.2.15 with $\mathcal{C} = C^*(a)$ gives the result. □

In the situation of the corollary, we will write

$$f(a) \coloneqq \Gamma_a(f) \quad \text{for any } f \in C\big(\sigma(a)\big).$$

This is sensible, because we can think of Γ_a as the identification of

$$C\big(\sigma(a)\big) = \overline{\operatorname{span}}_{\mathbb{C}}\big\{z^n \overline{z}^m \in C\big(\sigma(a)\big) : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\big\}$$

(see Corollary A.1.2) with

$$C^*(a) = \overline{\operatorname{span}}_{\mathbb{C}}\big\{a^n (a^*)^m \in \mathcal{A} : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\big\}$$

obtained by mapping $a \mapsto z$ (and consequently $a^* \mapsto \overline{z}$). It is the map $\Gamma_a\colon C(\sigma(a)) \to C^*(a)$ (potentially followed by the inclusion $C^*(a) \to \mathcal{A}$) that is referred to as the *continuous functional calculus* at a.

We now give a characterization of $f(a)$ which will be useful in proving the next result.

2.2.17 Lemma. *Let $\mathcal{A}$ be a C*-algebra, let $a \in \mathcal{A}$ be a normal element and let $f \in C(\sigma(a))$. Then, $f(a)$ is the unique element in $C^*(a)$ such that*

$$\mu\big(f(a)\big) = f\big(\mu(a)\big) \quad \text{for all } \mu \in \mathcal{M}_{C^*(a)}.$$

Proof. This is just a matter of unwrapping our definitions. Recall that $\Gamma_a\colon C(\sigma(a)) \to C^*(a)$ is defined by the composition

$$\begin{array}{ccccc} C\big(\sigma(a)\big) & \longrightarrow & C(\mathcal{M}_{C^*(a)}) & \longrightarrow & C^*(a) \\ f & \longmapsto & f \circ \mathrm{ev}_a|^{\sigma(a)} & & \\ & & g & \longmapsto & \mathrm{ev}^{-1}(g). \end{array}$$

This means that Γ_a maps $f \in C(\sigma(a))$ to the unique (by bijectivity) element $f(a) := \Gamma_a(f) \in C^*(a)$ such that

$$\mathrm{ev}_{f(a)} = f \circ \mathrm{ev}_a|^{\sigma(a)} \in C\big(\mathcal{M}_{C^*(a)}\big).$$

This is precisely the statement that $\mu(f(a)) = f(\mu(a))$ for all $\mu \in \mathcal{M}_{C^*(a)}$. $\square$

The following result shows that the continuous functional calculus interacts nicely with compositions and spectra.

2.2.18 Theorem (The spectral mapping theorem)**.** *Let $\mathcal{A}$ be a C*-algebra, let $a \in \mathcal{A}$ be a normal element and let $f \in C(\sigma(a))$. Then,*

$$\sigma\big(f(a)\big) = f\big(\sigma(a)\big), \quad \text{and} \quad (g \circ f)(a) = g\big(f(a)\big)$$

for all $g \in C\big(\sigma(f(a))\big)$.

Proof. By Lemma 2.1.24 and spectral permanence (Proposition 2.2.9), we have $\sigma(f(a)) = \{\mu(f(a)) : \mu \in \mathcal{M}_{C^*(a)}\}$ and $\sigma(a) = \{\mu(a) : \mu \in \mathcal{M}_{C^*(a)}\}$. By Lemma 2.2.17, we now find that

$$\sigma\big(f(a)\big) = \big\{f\big(\mu(a)\big) : \mu \in \mathcal{M}_{C^*(a)}\big\} = f\big(\sigma(a)\big).$$

For the other assertion of the theorem, note that $C^*(f(a)) \subset C^*(a)$ because $f(a) \in C^*(a)$. Moreover, $f(a)$ is normal, for both $f(a)$ and $f(a)^* = \overline{f}(a)$ are contained in the commutative C*-algebra $C^*(a)$. By Lemma 2.2.17, $g(f(a))$ is the unique element in $C^*(f(a))$ such that

$$\mu\big(g(f(a))\big) = g\big(\mu(f(a))\big) \quad \text{for all } \mu \in \mathcal{M}_{C^*(f(a))}.$$

Since every character on $C^*(a)$ restricts to a character on $C^*(f(a))$, this means that $g(f(a)) \in C^*(f(a)) \subset C^*(a)$ satisfies

$$\mu\big(g(f(a))\big) = g\big(\mu(f(a))\big) = g\big(f(\mu(a))\big) = (g \circ f)\big(\mu(a)\big) \text{ for all } \mu \in \mathcal{M}_{C^*(a)}$$

(where we have used Lemma 2.2.17 applied to f for the second equality). But $(g \circ f)(a)$ is the unique element in $C^*(a)$ such that $\mu((g \circ f)(a)) = (g \circ f)(\mu(a))$ for all $\mu \in \mathcal{M}_{C^*(a)}$, so we must have $g(f(a)) = (g \circ f)(a)$. $\square$

Finally, we show that the continuous functional calculus interacts nicely with $\star$-algebra homomorphisms. Note that our notation is somewhat sloppy with regard to restrictions.[8]

[8] This will often be the case when working with the continuous functional calculus. Restrictions of continuous functions are continuous, so this leads to no harm.

2.2.19 Proposition. *Let $\Phi\colon \mathcal{A} \to \mathcal{A}'$ be a $\star$-algebra homomorphism between C*-algebras and let $a \in \mathcal{A}$ be a normal element. Then, $\Phi(a)$ is normal, any $f \in C(\sigma(a))$ restricts to $f \in C(\sigma(\Phi(a)))$, and we have*

$$\Phi(f(a)) = f(\Phi(a))$$

for all $f \in C(\sigma(a))$.

Proof. The calculation $\Phi(a)^*\Phi(a) = \Phi(a^*a) = \Phi(aa^*) = \Phi(a)\Phi(a)^*$ shows that $\Phi(a)$ is normal. We know that $\sigma(\Phi(a)) \subset \sigma(a)$ (point (iii) of Lemma 2.1.9), so the claimed restriction poses no problems.

By the Stone-Weierstrass theorem, we can find a sequence (p_n) of polynomials in z and $\overline{z}$ such that $p_n \to f$ uniformly on $\sigma(a)$ (see Corollary A.1.2). The continuous functional calculus at a is continuous (even isometric), so we have $p_n(a) \to f(a)$ in $\mathcal{A}$.

Since Φ is an algebra homomorphism, we have $\Phi(p_n(a)) = p_n(\Phi(a))$ for all $n \in \mathbb{N}$. We know that Φ is norm-decreasing and hence continuous (Proposition 2.2.6), so

$$\Phi(f(a)) = \Phi(\lim_{n\to\infty} p_n(a)) = \lim_{n\to\infty} \Phi(p_n(a)) = \lim_{n\to\infty} p_n(\Phi(a)).$$

Finally, if $p_n \to f$ uniformly on $\sigma(a)$, then $p_n \to f$ uniformly on $\sigma(\Phi(a)) \subset \sigma(a)$ as well. Thus, we find that $p_n(\Phi(a)) \to f(\Phi(a))$ in $\mathcal{A}'$ by continuity of the continuous functional calculus at $\Phi(a)$. This gives $\Phi(f(a)) = f(\Phi(a))$ and concludes the proof. $\square$

2.2.3 Positive Elements

In Proposition 2.2.7, we showed that any self-adjoint element in a C*-algebra has a spectrum consisting entirely of real numbers. We now consider those self-adjoint elements whose spectra consists entirely of nonnegative numbers.

2.2.20 Definition (Positive elements)**.** Let $\mathcal{A}$ be a C*-algebra. An element $a \in \mathcal{A}$ is called *positive* if a is self-adjoint and $\sigma(a) \subset [0, \infty)$. We will write $a \geq 0$ to signify that a is positive and we will denote the set of all positive elements in $\mathcal{A}$ by $\mathcal{A}^+$.

If $\mathcal{C}$ is any (not necessarily commutative) C*-subalgebra of $\mathcal{A}$, then $c \in \mathcal{C}$ is positive in $\mathcal{C}$ if and only if it is positive in $\mathcal{A}$, because of spectral permanence (Proposition 2.2.9).

In analogy with the complex numbers (where $\sigma(\lambda) = \{\lambda\}$ for all $\lambda \in \mathbb{C}$, so that $\lambda \geq 0$ carries the usual meaning), positive elements have unique positive square roots.

2.2.21 Proposition (Square roots of positive elements)**.** *Let $\mathcal{A}$ be a C^*-algebra. Then, for any $a \in \mathcal{A}^+$, there exists a unique element $b \in \mathcal{A}^+$ such that $b^2 = a$. We write $a^{1/2} := b$ and call b the positive square root of a. Moreover, $a^{1/2} \in C^*(a)$.*

Proof. Fix $a \in \mathcal{A}^+$. We first show that the condition that $b \in \mathcal{A}^+$ satisfies $b^2 = a$ determines b uniquely. Define $\mathrm{sq}\colon [0,\infty) \to \mathbb{C}$ and $\mathrm{sqrt}\colon [0,\infty) \to \mathbb{C}$ by $\mathrm{sq}(\lambda) = \lambda^2$ and $\mathrm{sqrt}(\lambda) = \sqrt{\lambda}$. Then, $\mathrm{sq}(b) = b^2 = a$. Since $\sigma(\mathrm{sq}(b)) = \sigma(a) \subset [0,\infty)$, we have $\mathrm{sqrt} \in C(\sigma(\mathrm{sq}(b)))$. The spectral mapping theorem (Theorem 2.2.18) now gives

$$b = (\mathrm{sqrt} \circ \mathrm{sq})(b) = \mathrm{sqrt}(\mathrm{sq}(b)) = \mathrm{sqrt}(a) = \Gamma_a(\mathrm{sqrt}).$$

This means that any $b \geq 0$ such that $b^2 = a$ must be the result of applying the continuous functional calculus at a to the square root function, which proves uniqueness.

We now show existence. We have $\mathrm{sqrt} \in C(\sigma(a))$, and the continuous functional calculus at a gives

$$(\mathrm{sqrt}(a))^2 = (\Gamma_a(\mathrm{sqrt}))^2 = \Gamma_a(\mathrm{sqrt}^2) = \Gamma_a(z) = a$$

as well as $\mathrm{sqrt}(a)^* = \overline{\mathrm{sqrt}}(a) = \mathrm{sqrt}(a)$. Finally, the spectral mapping theorem gives $\sigma(\mathrm{sqrt}(a)) = \mathrm{sqrt}(\sigma(a)) \subset [0,\infty)$, so $b := \mathrm{sqrt}(a)$ is a positive element which squares to a, as desired, and we clearly have $b \in C^*(a)$. □

We now collect a handful of simple but useful results regarding positive elements.

2.2.22 Lemma. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$. Then, the following statements are true.*

(i) If $a^* = a$ and $\|t1_{\mathcal{A}} - a\| \leq t$ for some $t \in \mathbb{R}$, then $a \geq 0$.

(ii) If $a \geq 0$, then, for all $t \in \mathbb{R}$: $t \geq \|a\| \implies \|t1_{\mathcal{A}} - a\| \leq t$.

(iii) The sum of two positive elements is positive.

(iv) If a is normal and $f \in C(\sigma(a))$, then:

$$f(a) \geq 0 \text{ in } \mathcal{A} \iff f \geq 0 \text{ in } C\big(\sigma(a)\big) \iff f\big(\sigma(a)\big) \subset [0,\infty).$$

(v) If $a \geq 0$, then $\|a\| \in \sigma(a)$.

(vi) If $a^* = a$, then we can write $a = a_+ - a_-$, where $a_+, a_- \in \mathcal{A}^+$ and $a_+a_- = 0$.

Proof. In both (i) and (ii), we are assuming that a is self-adjoint, so we are free to apply the continuous functional calculus at a, and we know that $\sigma(a) \subset \mathbb{R}$ by Proposition 2.2.7.

If there exists some $t \in \mathbb{R}$ such that $\|t1_{\mathcal{A}} - a\| \leq t$, then (letting t denote the constant function at t) we have $\|t - z\|_\infty \leq t$ in $C(\sigma(a))$, since the continuous functional calculus is an isometry. For this to be true, the inclusion $z\colon \sigma(a) \to \mathbb{C}$ cannot take negative values, so $\sigma(a) \subset [0, \infty)$, which proves (i).

For (ii), suppose that $a \geq 0$ and let $t \in \mathbb{R}$. We have $\sigma(a) \subset [0, \|a\|]$ by Lemma 2.2.3, so if $t \geq \|a\|$, then $\|t - z\|_\infty \leq t$ in $C(\sigma(a))$. Thus, $t \geq \|a\|$ implies that $\|t1_{\mathcal{A}} - a\| \leq t$ by isometry of the continuous functional calculus.

For (iii), let $a, b \in \mathcal{A}^+$. Then,

$$\big\|\|a\|1_{\mathcal{A}} - a\big\| \leq \|a\| \quad \text{and} \quad \big\|\|b\|1_{\mathcal{A}} - b\big\| \leq \|b\|$$

by (ii). We now have $(a+b)^* = a + b$ and

$$\big\|\big(\|a\| + \|b\|\big)1_{\mathcal{A}} - (a+b)\big\| \leq \big\|\|a\|1_{\mathcal{A}} - a\big\| + \big\|\|b\|1_{\mathcal{A}} - b\big\| \leq \|a\| + \|b\|,$$

so $a + b \geq 0$ by (i), which proves (iii).

For (iv), let a be normal and $f \in C(\sigma(a))$. It is quite immediate that

$$f(\sigma(a)) \subset [0, \infty) \qquad \Longleftrightarrow \qquad f \geq 0 \text{ in the C*-algebra } C(\sigma(a)). \tag{2.8}$$

To see this, note that $f(\sigma(a)) = \sigma(f)$ (see point (iii) of Example 2.1.18) and that $f^* = \overline{f} = f$ if $f(\sigma(a)) \subset [0, \infty)$.

By the spectral mapping theorem (Theorem 2.2.18), we have $\sigma(\Gamma_a(f)) = \sigma(f(a)) = f(\sigma(a)) = \sigma(f)$, so the continuous functional calculus Γ_a preserves spectra. As a $\star$-algebra homomorphism, it also preserves the involution, so it is now immediate that it preserves the positivity condition (in both directions). Thus, $\Gamma_a(f) = f(a) \geq 0$ if and only if $f \geq 0$ in $C(\sigma(a))$, which concludes the proof of (iv).

For (v), the facts that

$$\sigma(a) \subset [0, \infty), \quad \sigma(a) \text{ is compact} \quad \text{and} \quad r(a) = \|a\|$$

(Theorem 2.1.19 and Lemma 2.2.3) together imply that $\|a\| \in \sigma(a)$. To see this, choose a sequence $(\lambda_n) \subset \sigma(a)$ such that $|\lambda_n| \to r(a) = \|a\|$. By compactness, there is a convergent subsequence $(\lambda'_n) \subset \sigma(a)$ such that $\lim_{n\to\infty} \lambda'_n \in \sigma(a)$, and we must have $\lambda'_n \to \|a\|$ since $\sigma(a) \subset [0, \infty)$. This gives (v).

For (vi), the fact that $a^* = a$ implies that $\sigma(a) \subset \mathbb{R}$ (Proposition 2.2.7). We can write the inclusion $z\colon \mathbb{R} \to \mathbb{C}$ as $z = z_+ - z_-$, where

$$z_+ := \max\{0, z\} \quad \text{and} \quad z_- := \max\{0, -z\}$$

are continuous functions on $\mathbb{R}$. We then have $z_+(\sigma(a)) \subset [0, \infty)$ and $z_-(\sigma(a)) \subset [0, \infty)$ as well as $z_+z_- = 0$. Define

$$a_+ := \Gamma_a(z_+) = z_+(a) \quad \text{and} \quad a_- := \Gamma_a(z_-) = z_-(a).$$

By (iv), we have $a_+ \geq 0$ and $a_- \geq 0$. Finally, since Γ_a is an algebra homomorphism, we have $a = a_+ - a_-$ and $a_+a_- = 0$, which concludes the proof. $\square$

The following theorem is of great importance.

2.2.23 Theorem. *Let $\mathcal{A}$ be a C^*-algebra. Then, $a^*a \geq 0$ for all $a \in \mathcal{A}$.*

Proof. Fix any $a \in \mathcal{A}$. By point (vi) of Lemma 2.2.22, we can write $a^*a = (a^*a)_+ - (a^*a)_-$ with $(a^*a)_+, (a^*a)_- \geq 0$ and $(a^*a)_+(a^*a)_- = 0$. We wish to show that $(a^*a)_- = 0$, for then $a^*a = (a^*a)_+ \geq 0$.

Set $b := a(a^*a)_-$. We will show that $b^*b = 0$, and then explain why this implies that $(a^*a)_- = 0$. We find that

$$b^*b = (a^*a)_-a^*a(a^*a)_- = (a^*a)_-\big((a^*a)_+ - (a^*a)_-\big)(a^*a)_- = -(a^*a)_-^3. \tag{2.9}$$

Since $\sigma((a^*a)_-) \subset [0, \infty)$, we have $-b^*b = (a^*a)_-^3 \geq 0$ by point (iv) of Lemma 2.2.22, and hence $\sigma(b^*b) = -\sigma(-b^*b) \subset (-\infty, 0]$.

Write $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint (see the paragraph following Definition 2.2.2). Then, $b^* = b_1 - ib_2$, and we find $bb^* + b^*b = 2b_1^2 + 2b_2^2$ by multiplying out. Now, $b_1^2 = \Gamma_{b_1}(z^2) \geq 0$ and $b_2^2 = \Gamma_{b_2}(z^2) \geq 0$ by point (iv) of Lemma 2.2.22. Since sums of positive elements are positive (point (iii) of Lemma 2.2.22), this gives

$$bb^* = 2b_1^2 + 2b_2^2 + (-b^*b) \geq 0.$$

By point (i) of Lemma 2.1.9, we have $\sigma(bb^*) \cup \{0\} = \sigma(b^*b) \cup \{0\}$, so $bb^* \geq 0$ gives $\sigma(b^*b) \subset [0, \infty)$.

We have now shown that $\sigma(b^*b) \subset (-\infty, 0] \cap [0, \infty) = \{0\}$. Since b^*b is self-adjoint, we have $\|b^*b\| = r(b^*b) = 0$ (Lemma 2.2.3), so $b^*b = 0$. By Equation (2.9), we now see that $(a^*a)_-^3 = 0$. By the continuous functional calculus, this means that z^3 and 0 are equal as functions on $\sigma((a^*a)_-)$, so $\sigma((a^*a)_-) = \{0\}$. This gives $(a^*a)_- = 0$ by the same reasoning that we just used for b^*b. This concludes the proof, for we have now shown that $a^*a = (a^*a)_+ \geq 0$. $\square$

We can now give an alternate description of positive elements, which does not appeal to spectra. This is an important result that we may use without reference.

2.2.24 Corollary (Alternate description of positivity)**.** *Let $\mathcal{A}$ be a C*-algebra. Then, $\mathcal{A}^+ = \{a^*a : a \in \mathcal{A}\}$.*

Proof. By Theorem 2.2.23, $\{a^*a : a \in \mathcal{A}\} \subset \mathcal{A}^+$. Conversely, let $a \in \mathcal{A}^+$. By Proposition 2.2.21, there is a unique $b \in \mathcal{A}^+$ such that $a = b^2 = b^*b$. This gives the other inclusion, so $\mathcal{A}^+ = \{a^*a : a \in \mathcal{A}\}$. □

Let $\mathcal{A}$ be a C*-algebra. By combining the fact that $a^*a \in \mathcal{A}^+$ for all $a \in \mathcal{A}$ and the fact that every positive element has a unique positive square root, we can define absolute values: for any $a \in \mathcal{A}$ we write $|a| \coloneqq (a^*a)^{1/2}$ and call $|a|$ the *absolute value* of a.

We can also use the set $\mathcal{A}^+$ to define a partial order on the set of all self-adjoint elements in $\mathcal{A}$, which we now introduce a symbol for:

$$\mathcal{A}_{\mathrm{sa}} \coloneqq \{a \in \mathcal{A} : a^* = a\}.$$

By a partial order, we mean a relation that is reflexive, antisymmetric and transitive.

2.2.25 Proposition (The partial order on $\mathcal{A}_{\mathrm{sa}}$)**.** *Let $\mathcal{A}$ be a C*-algebra. For $a, b \in \mathcal{A}_{\mathrm{sa}}$, we will write $a \geq b$ to signify that $a - b \in \mathcal{A}^+$. Then, the relation $\geq$ is a partial order on $\mathcal{A}_{\mathrm{sa}}$.*

Moreover, for any $a, b, c \in \mathcal{A}_{\mathrm{sa}}$ and $t \in [0, \infty)$, the following implications are true.

(i) $a \geq b \implies a + c \geq a + c$

(ii) $a \geq b \implies ta \geq tb$

(iii) $a \geq b \implies -b \geq -a$

Proof. Let $a, b, c \in \mathcal{A}_{\mathrm{sa}}$. We have $a - a = 0 \in \mathcal{A}^+$, so $\geq$ is reflexive. If $a - b \in \mathcal{A}^+$ and $b - a \in \mathcal{A}^+$, then $\sigma(a - b) \subset (-\infty, 0] \cap [0, \infty) = \{0\}$. Thus, $\|a - b\| = r(a - b) = 0$ (Lemma 2.2.3) and so $a = b$. This shows that $\geq$ is antisymmetric. If $a-b \in \mathcal{A}^+$ and $b-c \in \mathcal{A}^+$, then $a-c = (a-b)+(b-c) \in \mathcal{A}^+$, since the sum of positive elements is positive (Lemma 2.2.22). This shows that $\geq$ is transitive and concludes the proof that $\geq$ is a partial order on $\mathcal{A}_{\mathrm{sa}}$.

Properties (i), (ii) and (iii) are immediate: (i) follows from $a - b = (a+c) - (b+c)$; (ii) follows from the fact that $a - b \in \mathcal{A}^+$ implies $t(a-b) \in \mathcal{A}^+$; (iii) follows from $a - b = (-b) - (-a)$. □

We now prove some less obvious properties of this partial order.

2.2.26 Proposition. *Let $\mathcal{A}$ be a C*-algebra and let $a, b \in \mathcal{A}_{\mathrm{sa}}$. Then, the following statements are true.*

(i) If $a \geq b$, then $c^*ac \geq c^*bc$ for all $c \in \mathcal{A}$.

(ii) We have $\|a\|1_{\mathcal{A}} \geq a$. Moreover, if $a \geq b \geq 0$, then $\|a\| \geq \|b\|$.

(iii) If $a \geq b \geq 0$ and $a, b \in \mathrm{Inv}(\mathcal{A})$, then $b^{-1} \geq a^{-1} \geq 0$.

Proof. We begin with (i). If $a - b \in \mathcal{A}^+$, then $a - b$ has a positive square root (Proposition 2.2.21), and we find that

$$c^*(a-b)c = \big((a-b)^{1/2}c\big)^*\big((a-b)^{1/2}c\big) \geq 0,$$

which proves (i).

For (ii), we have $\sigma(a) \subset [-\|a\|, \|a\|]$ by Lemma 2.2.3, so $\|a\| - z \geq 0$ in $C(\sigma(a))$. Thus, $\|a\|1_{\mathcal{A}} - a \geq 0$ by point (iv) of Lemma 2.2.22, so we obtain $\|a\|1_{\mathcal{A}} \geq a$.

Assume now that $a \geq b \geq 0$. We have $\|a\|1_{\mathcal{A}} \geq a \geq b$, so $\|a\|1_{\mathcal{A}} - b \geq 0$. By points (iv) and (v) of Lemma 2.2.22, we have $\|a\| - z \geq 0$ in $C(\sigma(b))$ and $\|b\| \in \sigma(b)$. This gives $\|a\| - \|b\| \geq 0$, which concludes the proof of (ii).

We now turn to (iii). By Proposition 2.2.9, we have $\mathrm{Inv}(C^*(b)) = \mathrm{Inv}(\mathcal{A}) \cap C^*(b)$, so since b is invertible in $\mathcal{A}$, its inverse is in $C^*(b)$. By the continuous functional calculus at b, it is now clear that $b \geq 0$ implies $b^{-1} \geq 0$ (see point (iv) of Lemma 2.2.22). Thus, b^{-1} has a positive square root $b^{-1/2} \in C^*(b^{-1}) \subset C^*(b)$ which commutes with b. We now have

$$b^{-1/2}ab^{-1/2} \geq b^{-1/2}bb^{-1/2} = 1_{\mathcal{A}}$$

by (i). Note that $(b^{-1/2})^{-1} = b^{1/2}$, since $(z^{-1/2})^{-1} = z^{1/2}$ in $C(\sigma(b))$.

Consider now the element $c := b^{-1/2}ab^{-1/2}$. This is self-adjoint, invertible and satisfies $c \geq 1_{\mathcal{A}}$. This means that $z \geq 1$ in $C(\sigma(c))$, so z is invertible in $C(\sigma(c))$ and $1 \geq z^{-1}$. Thus, $1_{\mathcal{A}} \geq c^{-1}$ by point (iv) of Lemma 2.2.22. This means that

$$1_{\mathcal{A}} \geq c^{-1} = (b^{-1/2}ab^{-1/2})^{-1} = b^{1/2}a^{-1}b^{1/2},$$

from which another application of (i) gives

$$b^{-1} = b^{-1/2}1_{\mathcal{A}}b^{-1/2} \geq a^{-1}.$$

The fact that $a^{-1} \geq 0$ follows by the same reasoning that we used to conclude that $b^{-1} \geq 0$ in the previous paragraph. This concludes the proof. $\square$

We will close this section with an alternate characterization of positivity when our C*-algebra is an algebra of operators on a Hilbert space. To prove it, we need a simple lemma from the theory of Hilbert spaces. Our inner products are linear in the first entry.

2.2.27 Lemma. *Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and suppose that $T \in \mathcal{B}(H)$ satisfies*

$$\langle Tv, v\rangle = 0 \quad \text{for all } v \in H.$$

Then, $T = 0$.

Proof. Our assumption on T gives

$$0 = \langle T(v+w), v+w\rangle = \langle Tv, w\rangle + \langle Tw, v\rangle \quad \text{for all } v, w \in H,$$

and so

$$2\langle Tv, w\rangle = \langle Tv, w\rangle + \langle Tw, v\rangle + i\big(\langle Tv, iw\rangle + \langle T(iw), v\rangle\big) = 0 + i0 = 0$$

for all $v, w \in H$. With $v \in H$ arbitrary and $w = Tv$, this gives $\|Tv\|^2 = \langle Tv, Tv\rangle = 0$, so $T = 0$. $\square$

2.2.28 Proposition. *Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $T \in \mathcal{B}(H)$. Then,*

$$T \geq 0 \text{ in } \mathcal{B}(H) \quad \Longleftrightarrow \quad \langle Tv, v\rangle \in [0,\infty) \text{ for all } v \in H.$$

Proof. The forward implication is easy: if $T \geq 0$, then T has a positive square root in $\mathcal{B}(H)$ (Proposition 2.2.21), so

$$\langle Tv, v\rangle = \langle (T^{1/2})^2 v, v\rangle = \langle T^{1/2}v, T^{1/2}v\rangle = \|T^{1/2}v\|^2 \geq 0 \quad \text{for all } v \in H.$$

For the converse, assume that $\langle Tv, v\rangle \in [0,\infty)$ for all $v \in H$. Then,

$$\langle (T - T^*)v, v\rangle = \langle Tv, v\rangle - \langle T^*v, v\rangle = \langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = 0 \quad \text{for all } v \in H,$$

so we have $T = T^*$ by Lemma 2.2.27. Thus, $\sigma(T) \subset \mathbb{R}$ (Lemma 2.2.7), so it suffices to show that $t\mathrm{Id}_H - T$ is invertible in $\mathcal{B}(H)$ whenever $t < 0$, for then $\sigma(T) \subset [0,\infty)$.

Let $t < 0$. We first show that $t\mathrm{Id}_H - T$ is injective. For this, it suffices to show that there is some constant $C > 0$ such that

$$\|(t\mathrm{Id}_H - T)v\| \geq C\|v\| \quad \text{and all } v \in H. \tag{2.10}$$

We find that

$$\|(t\mathrm{Id}_H - T)v\|\|v\| \geq \langle (T - t\mathrm{Id}_H)v, v\rangle = \langle Tv, v\rangle - t\|v\|^2 \geq |t|\|v\|^2$$

since $\langle Tv, v\rangle \geq 0$ and $t < 0$. This gives Equation (2.10) with $C = |t|$.

We now show that $t\mathrm{Id}_H - T$ is surjective. It is clearly self-adjoint. Now, if $v \in (S(H))^{\perp}$ for some self-adjoint $S \in \mathcal{B}(H)$, then $0 = \langle S(Sv), v\rangle = \langle Sv, Sv\rangle$, so $v \in \mathrm{Ker}(S)$. Thus, we see that

$$\big((t\mathrm{Id}_H - T)(H)\big)^{\perp} \subset \mathrm{Ker}(t\mathrm{Id}_H - T) = \{0\}.$$

By Equation (2.10), the range of $t\mathrm{Id}_H - T$ is closed.[9] As a closed subspace whose orthogonal complement is $\{0\}$, $(t\mathrm{Id}_H - T)(H) = H$, so $t\mathrm{Id}_H - T$ is surjective.

We now know that $t\mathrm{Id}_H - T$ is bijective. It is always the case that a bounded linear bijection on a Hilbert space has a bounded inverse, but we need not rely on this powerful result here: if we let $w \in H$ be arbitrary and set $v = (t\mathrm{Id}_H - T)^{-1}w$ in Equation (2.10), we obtain

$$\|w\| \geq C\|(t\mathrm{Id}_H - T)^{-1}w\| \quad \text{for all } w \in H,$$

so $(t\mathrm{Id}_H - T)^{-1}$ is bounded, which concludes the proof. □

2.2.4 Ideals and Quotients of C*-Algebras

In the general theory of C*-algebras, where units are not assumed to exists, the results of this subsection are typically proved by way of approximate units. Approximate units are instances of nets, a notion we have avoided in order to make the treatment as accessible as possible.

The following lemma affords us with “local approximate units” in the form of sequences; these will be sufficient for our purposes. This is the approach taken by Arveson [2, Section 1.3]. This subsection is largely based on his exposition.

2.2.29 Lemma. *Let $\mathcal{A}$ be a C*-algebra. Suppose that $I \subset \mathcal{A}$ is a closed ideal of $\mathcal{A}$ and that $a \in I$. Then, there exists a sequence $(e_n) \subset I$ such that:*

(i) e_n *is self-adjoint and* $\sigma(e_n) \subset [0,1]$ *for all* $n \in \mathbb{N}_1$*,*

(ii) $\lim_{n\to\infty} e_n a = a$.

Proof. We first assume that $a \geq 0$. Then, $\sigma(a) \subset [0,\infty)$, and we are free to apply the functional calculus at a.

Define a sequence $(f_n)_{n\in\mathbb{N}_1} \subset C(\sigma(a))$ by

$$f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq 1/n \\ 1 & \text{if } 1/n < t \end{cases}$$

[9] If a sequence $((t\mathrm{Id}_H - T)v_n)_n$ is Cauchy, then so is (v_n), and if $v_n \to v$, then we have $(t\mathrm{Id}_H - T)v_n \to (t\mathrm{Id}_H - T)v \in (t\mathrm{Id}_H - T)(H)$ by continuity.

and let $e_n \coloneqq f_n(a)$ for all $n \in \mathbb{N}_1$.

Fix any $n \in \mathbb{N}_1$. We will show that $e_n \in I$. Since $f_n(0) = 0$, there is a sequence of polynomials (p_m) *without constant terms* such that $p_m \to f_n$ uniformly on the compact set $\sigma(a)$, which is precisely the statement that $p_m \to f_n$ in $C(\sigma(a))$. We have $(p_m(a)) \subset I$ since $a^k \in I$ for all integers $k \geq 1$ ($a \in I$ and I is an ideal). Now, $p_m(a) \to f_n(a)$ in $\mathcal{A}$ by continuity of the functional calculus at a, so $e_n \coloneqq f_n(a) \in I$ since I is closed.

Now, f_n is real-valued, so $e_n^* = f_n(a)^* = \overline{f_n}(a) = f_n(a) = e_n$, proving that e_n is self-adjoint. Moreover, we have $\sigma(e_n) = \sigma(f_n(a)) = f_n(\sigma(a))$ by the spectral mapping theorem (Theorem 2.2.18). Since $f_n(\sigma(a)) \subset [0, 1]$, we get $\sigma(e_n) \subset [0, 1]$.

We now show that $ae_n \to a$ in $\mathcal{A}$. Consider the functions

$$g_n(t) = t\big(1 - f_n(t)\big) \in C(\sigma(a)) \quad \text{for } n \in \mathbb{N}_1.$$

For arbitrary $n \in \mathbb{N}_1$, we have $g_n(0) = 0$ and $g_n(t) = 0$ for all $t \geq 1/n$. For $0 \leq t \leq 1/n$, we have $g_n(t) = t(1 - nt)$. Differentiating shows that g_n (which is always positive) has a global maximum at $t = 1/(2n)$ and that $g(1/(2n)) = 1/(4n)$. Thus, $\|g_n\|_\infty \leq 1/(4n)$. Since the continuous functional calculus is an isometry, we find that

$$\|a - ae_n\| = \|a(1_{\mathcal{A}} - f_n(a))\| = \|g_n(a)\| = \|g_n\|_\infty \leq \frac{1}{4n} \to 0$$

as $n \to \infty$. This concludes the proof in the case that $a \in I$ is positive.

For a general $a \in I$, we have $a^*a \in I$ and $a^*a \geq 0$. We can therefore find a sequence $(e_n) \subset I$ of self-adjoint elements with $\sigma(e_n) \subset [0, 1]$ such that $(a^*a)e_n \to a^*a$. Note that $\|e_n\| = r(e_n) \leq 1$ (Lemma 2.2.3). Now, the C*-equality gives

$$\begin{aligned} \|a - ae_n\|^2 = \|a(1_{\mathcal{A}} - e_n)\|^2 &= \|(1_{\mathcal{A}} - e_n)a^*a(1_{\mathcal{A}} - e_n)\| \\ &\leq 2\|(a^*a)(1_{\mathcal{A}} - e_n)\|, \end{aligned}$$

so $(a^*a)e_n \to a^*a$ implies that $ae_n \to a$, which concludes the proof. □

If we want to form quotient C*-algebras, we clearly need to quotient out closed $\star$-ideals. The following propositions shows that closed ideals of C*-algebras are $\star$-ideals by default.

2.2.30 Proposition. *Let $\mathcal{A}$ be a C*-algebra and let $I \subset \mathcal{A}$ be a closed ideal. Then, I is a $\star$-ideal of $\mathcal{A}$.*

Proof. Let $a \in I$. We need to show that $a^* \in I$. By Lemma 2.2.29, we can find a sequence $(e_n) \subset I$ of self-adjoint elements such that $e_n a \to a$. Now, we have $(e_n a)^* = a^* e_n \in I$ since $e_n \in I$. Moveover, $(e_n a)^* \to a^*$ by continuity of the involution, so $a^* \in I$ since I is closed. □

2.2.31 Theorem (Quotients of C*-algebras)**.** *Let $\mathcal{A}$ be a C*-algebra and let $I \subset \mathcal{A}$ be a closed ideal. Then, the quotient algebra $\mathcal{A}/I$ equipped with the quotient norm and the involution $(a+I)^* = a^* + I$ is a C*-algebra.*

Proof. By Proposition 2.1.21 on quotients of Banach algebras, we already know that the algebra $\mathcal{A}/I$ equipped with the quotient norm is a Banach algebra. Appealing to Proposition 2.2.30, it is straightforward to check that the involution on $\mathcal{A}/I$ is well-defined and indeed an involution; we omit the details. The only remaining verification is the C*-equality.

Define

$$E \coloneqq \big\{e \in I : e^* = e \text{ and } \sigma(e) \subset [0,1]\big\}$$

We claim that

$$\|a+I\|_q \coloneqq \inf\big\{\|a+c\| : c \in I\big\} = \inf\big\{\|a-ae\| : e \in E\big\} \tag{2.11}$$

for all $a \in \mathcal{A}$. Since $E \subset I$, we have $-ae \in I$ for all $e \in E$, which gives the inequality $\leq$ between the infima above. For the opposite inequality, fix $a \in \mathcal{A}$ and $c \in I$. Using Lemma 2.2.29, fix a sequence $(e_n) \subset E$ such that $ce_n \to c$.

Note that any $e \in E$ satisfies $\sigma(1_{\mathcal{A}} - e) = 1 - \sigma(e) \subset [0,1]$ by the spectral mapping theorem (Theorem 2.2.18) and hence $\|1_{\mathcal{A}} - e\| = r(1_{\mathcal{A}} - e) \leq 1$ by Lemma 2.2.3. Submultiplicativity now gives $\|(a+c)(1_{\mathcal{A}} - e_n)\| \leq \|a+c\|$ for all $n \in \mathbb{N}$, and hence:

$$\begin{aligned}\|a+c\| &\geq \liminf_{n\to\infty} \|(a+c)(1_{\mathcal{A}} - e_n)\| \\ &= \liminf_{n\to\infty} \|a(1_{\mathcal{A}} - e_n) + c(1_{\mathcal{A}} - e_n)\| \\ &= \liminf_{n\to\infty} \|a(1_{\mathcal{A}} - e_n)\| \geq \inf_{n\geq 1} \|a - ae_n\|,\end{aligned}$$

where the transition to the last line follows from the fact that $c(1_{\mathcal{A}} - e_n) \to 0$ by our choice of $(e_n) \subset E$. Since $c \in I$ was arbitrary, this shows that

$$\inf\big\{\|a+c\| : c \in I\big\} \geq \inf\big\{\|a - ae\| : e \in E\big\},$$

so we have now proven Equation (2.11).

Applying Equation (2.11) to a and a^*a, we obtain one inequality of the C*-equality:

$$\begin{aligned}\|a+I\|_q^2 &= \inf\big\{\|a(1_{\mathcal{A}} - e)\|^2 : e \in E\big\} \\ &= \inf\big\{\|(1_{\mathcal{A}} - e)a^*a(1_{\mathcal{A}} - e)\| : e \in E\big\} \\ &\leq \inf\big\{\|a^*a(1_{\mathcal{A}} - e)\| : e \in E\big\} = \|a^*a + I\|_q,\end{aligned}$$

where we again have used the fact that $\|1_{\mathcal{A}} - e\| \leq 1$ for all $e \in E$.

Finally, the inequality we just proved, along with submultiplicativity, gives

$$\|a + I\|_q^2 \leq \|a^*a + I\|_q = \|(a + I)^*(a + I)\|_q \leq \|a^* + I\|_q \|a + I\|_q,$$

so $\|a + I\|_q \leq \|a^* + I\|_q$ (assuming $a + I \neq I$; the C*-equality is trivial for the zero-element). Another application of submultiplicativity now gives the remaining inequality:

$$\|a^*a + I\|_q \leq \|a^* + I\|_q \|a + I\|_q \leq \|a + I_q\|^2.$$

Thus, the quotient norm satisfies the C*-equality and we are done. $\square$

We now come to the long promised result that injective $\star$-algebra homomorphism between C*-algebras are necessarily isometric.

2.2.32 Theorem. *Let $\Phi\colon \mathcal{A} \to \mathcal{A}'$ be an injective $\star$-algebra homomorphism between C*-algebras. Then, Φ is isometric.*

Proof. By Proposition 2.2.6, we already know that Φ is norm-decreasing. That is, $\|\Phi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$. We will show that if there is some $a \in \mathcal{A}$ such that $\|\Phi(a)\| < \|a\|$, then Φ cannot be injective. The result then follows by contraposition: Φ injective $\implies$ $\|\Phi(a)\| \geq \|a\|$ for all $a \in \mathcal{A}$.

Suppose that $\|\Phi(a)\| < \|a\|$. We know that both a^*a and $\Phi(a^*a) = \Phi(a)^*\Phi(a)$ are positive (Theorem 2.2.23). Invoking Lemma 2.2.3, we find that

$$\begin{aligned} \sigma(a^*a) &\subset [0, r] \quad \text{with } r := r(a^*a) = \|a^*a\| \\ \text{and} \quad \sigma(\Phi(a^*a)) &\subset [0, s] \quad \text{with } s := r(\Phi(a^*a)) = \|\Phi(a^*a)\|. \end{aligned}$$

Our assumption, along with the C*-equality, gives $s < r$:

$$s = \|\Phi(a^*a)\| = \|\Phi(a)\|^2 < \|a\|^2 = \|a^*a\| = r.$$

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous function such that $f(r) = 1$ and $f(t) = 0$ whenever $|t - r| \geq r - s > 0$. We have $r = \|a^*a\| \in \sigma(a^*a)$ by point (v) of Lemma 2.2.22, so this means that $f|_{\sigma(a^*a)} \neq 0$, while $f|_{\sigma(\Phi(a^*a))} = 0$.

The continuous functional calculus is injective, so we now have $f(a^*a) \neq 0$ and $f(\Phi(a^*a)) = 0$. By Proposition 2.2.19, we have $\Phi(f(a^*a)) = f(\Phi(a^*a))$, so $0 \neq f(a^*a) \in \operatorname{Ker}\Phi$. Thus, Φ is not injective, which is what we wanted to show. $\square$

We will now see that the first isomorphism theorem holds for C*-algebras. Indeed, the assumption of the theorem is purely algebraic: it states that if the domain and target of any $\star$-algebra homomorphism happen to be

C*-algebras, then the isomorphism afforded by the algebraic isomorphism theorem is necessarily an isometric isomorphism between C*-algebras.[10]

2.2.33 Theorem (The first isomorphism theorem for C*-algebras)**.** *Let* $\Phi\colon \mathcal{A} \to \mathcal{A}'$ *be a* $\star$*-algebra homomorphism between C*-algebras. Then,* $\operatorname{Ker}\Phi$ *is a closed ideal of* $\mathcal{A}$*,* $\Phi(\mathcal{A})$ *is a C*-subalgebra of* $\mathcal{A}'$*, and the map*

$$\begin{aligned} \widetilde{\Phi}\colon \mathcal{A}/\operatorname{Ker}\Phi &\to \Phi(\mathcal{A}) \\ a + \operatorname{Ker}\Phi &\mapsto \Phi(a) \end{aligned}$$

is an (isometric) $\star$*-algebra isomorphism between C*-algebras. In particular,* $\mathcal{A}/\operatorname{Ker}\Phi \cong \Phi(\mathcal{A})$ *as C*-algebras.*

Proof. Using the fact that Φ is a $\star$-algebra homomorphism, it is straightforward to check that $\operatorname{Ker}\Phi$ is a $\star$-ideal of $\mathcal{A}$, that $\Phi(\mathcal{A})$ is a $\star$-subalgebra of $\mathcal{A}'$ and that $\widetilde{\Phi}$ is an isomorphism of $\star$-algebras; we omit the details. This amounts to the first isomorphism theorem for $\star$-algebras (which we chose to mention but not state in Subsection 2.1.1).

Continuity of Φ (Proposition 2.2.6) implies that the ideal $\operatorname{Ker}\Phi \subset \mathcal{A}$ is closed, so the quotient $\mathcal{A}/\operatorname{Ker}\Phi$ is a C*-algebra by Theorem 2.2.31. The only remaining verification is that the $\star$-subalgebra $\Phi(\mathcal{A}) \subset \mathcal{A}'$ is closed (and hence a C*-subalgebra of $\mathcal{A}'$ and a C*-algebra in its own right), for then $\widetilde{\Phi}$ is isometric by Proposition 2.2.6.

Consider now the canonical map $\Phi_q\colon \mathcal{A}/\operatorname{Ker}\Phi \to \mathcal{A}'$ induced by Φ, i.e. $\widetilde{\Phi}\colon \mathcal{A}/\operatorname{Ker}\Phi \to \Phi(\mathcal{A})$ followed by the inclusion $\Phi(\mathcal{A}) \to \mathcal{A}'$. By the last two paragraphs, $\Phi_q\colon \mathcal{A}/\operatorname{Ker}\Phi \to \mathcal{A}'$ is an injective $\star$-algebra homomorphism between C*-algebras, so by Theorem 2.2.32, it is an isometry.

This means that $\Phi(\mathcal{A})$ is closed, for if $(\Phi(a_n)) \subset \Phi(\mathcal{A})$ is a Cauchy sequence (and hence convergent in $\mathcal{A}'$), then

$$\begin{aligned} \|\Phi(a_m) - \Phi(a_n)\| &= \|\Phi_q(a_m + \operatorname{Ker}\Phi) - \Phi_q(a_n - \operatorname{Ker}\Phi)\| \\ &= \|(a_m + \operatorname{Ker}\Phi) - (a_n - \operatorname{Ker}\Phi)\|_q \end{aligned}$$

shows that $(a_n + \operatorname{Ker}\Phi)$ is a Cauchy sequence in the C*-algebra $\mathcal{A}/\operatorname{Ker}\Phi$, and hence convergent, say with limit $a + \operatorname{Ker}\Phi$. Continuity of Φ_q now implies that

$$\lim_{n\to\infty} \Phi(a_n) = \lim_{n\to\infty} \Phi_q(a_n + \operatorname{Ker}\Phi) = \Phi_q(a + \operatorname{Ker}\Phi) = \Phi(a) \in \Phi(\mathcal{A}).$$

This concludes the proof that $\Phi(\mathcal{A})$ is closed and hence the proof as a whole. $\square$

[10] This is quite incredible: there is no analogous result for Banach algebras, even if require the algebra homomorphism to be norm-decreasing in the first place (its image need not be a Banach space). This is another reflection of the fact that the norm of a C*-algebra is entirely rooted in its algebraic structure.

2.2.5 The Gelfand-Naimark Theorem

Before we conclude our discussion of C*-algebras, there is one more foundational result we wish to establish, namely *the Gelfand-Naimark theorem.* We have motivated C*-algebras as an abstraction of certain algebras of bounded operators on Hilbert spaces. In particular, we have seen that for a Hilbert space H, closed subalgebras of $\mathcal{B}(H)$ which are also closed under the taking of adjoints are C*-algebras (Example 2.2.5). These are C*-algebras we understand particularly well, given the well-developed theory of Hilbert spaces. The Gelfand-Naimark theorem states that *any* C*-algebra can be realized as a C*-algebra of this form.

The Gelfand-Naimark theorem is a result that belongs to the representation theory of C*-algebras. A proper introduction to representation theory would take us too far afield, so we only introduce those parts of the theory that we will need in order to properly state and prove this result.

2.2.34 Definition (Representations of C*-algebras)**.** Let $\mathcal{A}$ be a C*-algebra. A *representation* of $\mathcal{A}$ is $\star$-algebra homomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}(H)$, where H is a Hilbert space.

Whenever we refer to a map $\Phi\colon \mathcal{A} \to \mathcal{B}(H)$ as a representation, it should be implicitly understood that $\mathcal{A}$ is a C*-algebra and that Φ is a representation of $\mathcal{A}$.

2.2.35 Definition (Unitary equivalence)**.** Two representations $\Phi_1\colon \mathcal{A} \to \mathcal{B}(H_1)$ and $\Phi_2\colon \mathcal{A} \to \mathcal{B}(H_2)$ are said to be *unitarily equivalent* if there exists a unitary transformation $U\colon H_1 \to H_2$ such that

$$\Phi_2(a) = U\Phi_1(a)U^* \quad \text{for all } a \in \mathcal{A}. \tag{2.12}$$

We will write $\Phi_1 \sim \Phi_2$ to signify that Φ_1 and Φ_2 are unitarily equivalent.

Equation (2.12) is equivalent to the statement that the diagram

$$\begin{tikzcd} H_1 \arrow[r, "U"] \arrow[d, "\Phi_1(a)"'] & H_2 \arrow[d, "\Phi_2(a)"] \\ H_1 \arrow[r, "U"'] & H_2 \end{tikzcd}$$

commutes for every $a \in \mathcal{A}$. Unitary equivalence is clearly an equivalence relation among the representations of $\mathcal{A}$.[11]

[11]The class of all Hilbert space does not form a set, so neither do the representations of $\mathcal{A}$. Thus, unitary equivalence is not a relation on a set. The point is that the usual conditions which define an equivalence relation hold for any three representations Φ_1, Φ_2 and Φ_3. The same situation arises when considering isomorphism as an equivalence relation among e.g. groups or vector spaces.

Suppose now that we have a representation $\Phi\colon \mathcal{A} \to \mathcal{B}(H)$ and fix any nonzero $h \in H$. We then obtain a linear map

$$\begin{aligned} \tau_h\colon \mathcal{A} &\to \mathbb{C} \\ a &\mapsto \langle \Phi(a)h, h\rangle. \end{aligned} \tag{2.13}$$

We will quickly sketch how to realize the *orbit* of h, i.e. the vector-subspace

$$\Phi(\mathcal{A})h := \big\{\Phi(a)h \in H : a \in \mathcal{A}\big\} \subset H,$$

as a vector space quotient of $\mathcal{A}$. We will also reconstruct the inner product and the maps $\Phi(a)$ (restricted and corestricted to $\Phi(\mathcal{A})h$) on this quotient, and we will see that the only information we really need for all of this is the map τ_h.

First of all, note that $\Phi(\mathcal{A})h$ is the image of the linear map $T\colon \mathcal{A} \to H$ defined by $T(a) = \Phi(a)h$. Thus, the linear map

$$\begin{aligned} \widetilde{T}\colon \mathcal{A}/\operatorname{Ker} T &\to \Phi(\mathcal{A})h \\ a + \operatorname{Ker} T &\mapsto T(a) = \Phi(a)h \end{aligned}$$

is an isomorphism of vector spaces. We can use this isomorphism to transfer the inner product and the maps $\Phi(a)$ on $\Phi(\mathcal{A})h$ to $\mathcal{A}/\operatorname{Ker} T$. For any $a, b \in \mathcal{A}$, we find that

$$\big\langle a + \operatorname{Ker} T, b + \operatorname{Ker} T\big\rangle := \big\langle \Phi(a)h, \Phi(b)h\big\rangle = \big\langle \Phi(b^*a)h, h\big\rangle = \tau_h(b^*a) \tag{2.14}$$

and, with $\Phi'(a) := \widetilde{T}^{-1}\Phi(a)\widetilde{T}$,

$$\Phi'(a)(b + \operatorname{Ker} T) = \widetilde{T}^{-1}\Phi(a)\big(\Phi(b)h\big) = \widetilde{T}^{-1}\big(\Phi(ab)h\big) = ab + \operatorname{Ker} T. \tag{2.15}$$

We see that $0 = T(a) = \Phi(a)h$ if and only if $0 = \langle \Phi(a)h, \Phi(a)h\rangle = \langle \Phi(a^*a)h, h\rangle = \tau_h(a^*a)$. This means that

$$\operatorname{Ker} T = \big\{a \in \mathcal{A} : \tau_h(a^*a) = 0\big\},$$

so indeed, the map τ_h is all we need to define $\operatorname{Ker} T$ and construct the quotient $\mathcal{A}/\operatorname{Ker} T$. Moreover, Equation (2.14) shows that the inner product on $\mathcal{A}/\operatorname{Ker} T$ depends only on τ_h and Equation (2.15) shows that the maps $\Phi'(a)$ depend only on the algebraic structure of $\mathcal{A}$.

If we are given only the map τ_h, this construction allows us to recover part of the representation Φ, namely the part corresponding to the orbit of h. The inner product space $\mathcal{A}/\operatorname{Ker} T \cong \Phi(\mathcal{A})h$ need not be complete (the inner product need *not* induce the quotient norm), so in order for Φ' to become an actual representation of $\mathcal{A}$, we must extend the operators $\Phi'(a) \in \mathcal{B}(\mathcal{A}/\operatorname{Ker} T)$ to the Hilbert space completion of $\mathcal{A}/\operatorname{Ker} T$.[12]

Said another way, the only representations we can hope to fully recover by this method are of a very particular form:

2.2.36 Definition (Cyclic vectors and representations)**.** Let $\Phi\colon \mathcal{A} \to \mathcal{B}(H)$ be a representation. A vector $h \in H$ is said to be *cyclic* for Φ if

$$H = \overline{\operatorname{span}}_{\mathbb{C}}\big\{\Phi(a)h : a \in \mathcal{A}\big\},$$

and the representation Φ is said to be *cyclic* if there is a vector in H that is cyclic for Φ.

We now define the class of functionals in $\mathcal{A}^*$ which will turn out to correspond to maps of the form given by Equation (2.13), with the additional (and harmless) condition that $\|h\| = 1$.

2.2.37 Definition (States)**.** Let $\mathcal{A}$ be a C*-algebra. A *state* on $\mathcal{A}$ is a linear functional $\tau \in \mathcal{A}^*$ such that

(i) τ is normalized: $\|\tau\| = 1$.

(ii) τ is *positive*: $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$ (or, equivalently: $\tau(a) \geq 0$ for all $a \in \mathcal{A}^+$).

We write $S(\mathcal{A})$ to denote the set of all states on $\mathcal{A}$.

Our goal now is to construct a representation from any given state $\tau \in S(\mathcal{A})$. This procedure is known as the *GNS-construction*, and the representation we obtain is known as the *GNS-representation* associated to τ. The idea is simply to follow the recipe outlined above.

We begin by showing that the map $(a, b) \mapsto \tau(b^*a)$, which will become our inner product, has some desired properties.

2.2.38 Lemma. *Let $\mathcal{A}$ be a C*-algebra and let $\tau \in S(\mathcal{A})$. Then, for all $a, b \in \mathcal{A}$, we have the following relations:*

(i) $\tau(b^*a) = \overline{\tau(a^*b)}$ *(conjugate symmetry)*

(ii) $|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$ *(the Cauchy-Schwarz inequality)*

[12] We will assume that the reader is familiar with Hilbert space completions. However, in Chapter 4, we will introduce a vast generalization of Hilbert spaces, and a detailed construction of their completions can be found in Appendix B.

Proof. We begin with (i). Let $\lambda \in \mathbb{C}$ be arbitrary. We find that

$$0 \le \tau\big((a+\lambda b)^*(a+\lambda b)\big) = \tau(a^*a) + \lambda\tau(a^*b) + \overline{\lambda}\tau(b^*a) + |\lambda|^2\tau(b^*b). \quad (2.16)$$

Since $\tau(a^*a) + |\lambda|^2\tau(b^*b) \in \mathbb{R}$, the imaginary part of $\lambda\tau(a^*b) + \overline{\lambda}\tau(b^*a)$ must vanish. Now,

$$\begin{aligned} \lambda = 1 \quad &\Longrightarrow \quad 0 = \mathrm{Im}\big(\tau(a^*b) + \tau(b^*a)\big) = \mathrm{Im}\big(\tau(a^*b) - \overline{\tau(b^*a)}\big), \\ \lambda = i \quad &\Longrightarrow \quad 0 = \mathrm{Im}\big(i\tau(a^*b) - i\tau(b^*a)\big) = \mathrm{Re}\big(\tau(a^*b) - \overline{\tau(b^*a)}\big), \end{aligned}$$

so $\tau(a^*b) = \overline{\tau(b^*a)}$, which proves (i).

For (ii), suppose first that $\tau(b^*b) = 0$. If we let $t \in \mathbb{R}$ and set $\lambda = t$ or $\lambda = it$ in Equation (2.16), we find that

$$0 \le \tau(a^*a) + 2t\mathrm{Re}(\tau(b^*a)) \quad \text{and} \quad 0 \le \tau(a^*a) + 2t\mathrm{Im}(\tau(b^*a))$$

for all $t \in \mathbb{R}$. For this to be possible, we must have $\tau(b^*a) = 0$. This proves (ii) in the case that $\tau(b^*b) = 0$. If $\tau(b^*b) \neq 0$, then

$$\lambda = -\frac{\tau(b^*a)}{\tau(b^*b)} \quad \Longrightarrow \quad 0 \le \tau(a^*a) + \frac{1}{\tau(b^*b)}\big(-2|\tau(b^*a)|^2 + |\tau(b^*a)|^2\big),$$

which after some slight rearranging gives (ii). □

We are now prepared to construct the GNS-representation associated to a state $\tau \in S(\mathcal{A})$. As remarked, we will be forced to consider Hilbert space completions. Now, Hilbert space completions are unique precisely up to unitary transformations. Thus, different explicit constructions of Hilbert space completions will yield different representations of $\mathcal{A}$, but they will all be unitarily equivalent. Thus, the GNS-representation associated to a state is really defined *up to unitary equivalence.*

2.2.39 Theorem (The GNS construction)**.** *Let $\mathcal{A}$ be a C^*-algebra, let $\tau \in S(\mathcal{A})$ and define $N_\tau \coloneqq \big\{a \in \mathcal{A} : \tau(a^*a) = 0\big\}$. Then, N_τ is a vector-subspace of $\mathcal{A}$ and the map*

$$\begin{aligned} \mathcal{A}/N_\tau \times \mathcal{A}/N_\tau &\to \mathbb{C} \\ (a + N_\tau, b + N_\tau) &\mapsto \tau(b^*a) \end{aligned}$$

is a well-defined inner product on the vector space quotient $\mathcal{A}/N_\tau$. Moreover, the map[13]

$$\begin{aligned} \Phi_\tau \colon \mathcal{A} &\to \mathcal{B}(\mathcal{A}/N_\tau) \\ a &\mapsto \big(\Phi_\tau(a) : b + N_\tau \mapsto ab + N_\tau\big) \end{aligned}$$

[13]We repeat that the norm on $\mathcal{A}/N_\tau$, which defines $\mathcal{B}(\mathcal{A}/N_\tau)$, is *not* the quotient norm, but the norm induced by the inner product.

is a well-defined algebra homomorphism such that $\Phi_\tau(a^*) = \Phi_\tau(a)^*$ *for all* $a \in \mathcal{A}$.[14]

Finally, if we let H_τ *be the Hilbert space completion of* $\mathcal{A}/N_\tau$*, then each* $\Phi_\tau(a) \in \mathcal{B}(\mathcal{A}/N_\tau)$ *extends to an operator* $\overline{\Phi_\tau(a)} \in \mathcal{B}(H_\tau)$ *and the map*

$$\begin{aligned} \overline{\Phi}_\tau \colon \mathcal{A} &\to \mathcal{B}(H_\tau) \\ a &\mapsto \overline{\Phi}_\tau(a) := \overline{\Phi_\tau(a)} \end{aligned}$$

is a representation of $\mathcal{A}$*. Moreover, the vector* $h := 1_{\mathcal{A}} + N_\tau \in H_\tau$ *is cyclic for* $\overline{\Phi}_\tau$*, and we have that* $\tau(a) = \langle \overline{\Phi}_\tau(a)h, h\rangle$ *for all* $a \in \mathcal{A}$.

Proof. We will drop the τ-subscripts for the duration of this proof.

The fact that $|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$ for all $a, b \in \mathcal{A}$ (Lemma 2.2.38) implies that, for any given $a \in \mathcal{A}$, we have $\tau(a^*a) = 0$ if and only if $\tau(b^*a) = 0$ for all $b \in \mathcal{A}$. Thus,

$$N = \big\{a \in \mathcal{A} : \tau(b^*a) = 0 \text{ for all } b \in \mathcal{A}\big\}. \tag{2.17}$$

Linearity of τ now immediately implies that N is a vector-subspace of $\mathcal{A}$.

If $a \in N$, so that $\tau(b^*a) = 0$ for all $b \in \mathcal{A}$, then for any $c \in \mathcal{A}$, we find that $\tau(b^*(ca)) = \tau((c^*b)^*a) = 0$ for all $b \in \mathcal{A}$. This means that

$$a \in N \quad\implies\quad ca \in N \text{ for all } c \in \mathcal{A}. \tag{2.18}$$

We now check that the claimed inner product is well-defined and indeed an inner product. If $a + N = a' + N$ and $b + N = b' + N$, then $a' - a, b' - b \in N$, so Equation (2.18) (along with Lemma 2.2.38) implies that

$$\begin{aligned} \tau(b^*a) &= \tau(b^*a) + \tau\big(b^*(a'-a)\big) + \overline{\tau\big((a')^*(b'-b)\big)} \\ &= \tau\big(b^*a + b^*(a'-a) + (b'-b)^*a'\big) = \tau\big((b')^*a'\big), \end{aligned}$$

which proves well-definition. Linearity in the first entry follows from linearity of τ, conjugate symmetry corresponds to point (ii) of Lemma 2.2.38, nonnegativity corresponds to positivity of τ and nondegeneracy is immediate from our definition of N_τ. Thus, the axioms of an inner product are all satisfied.

We now consider the map $\Phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{A}/N)$. We begin by showing well-definition. Fix any $a \in \mathcal{A}$. We first need to show that $\Phi(a) : b + N \mapsto ab + N$ is well-defined, but this follows immediately from Equation (2.18):

$$b - b' \in N \quad\implies\quad ab - ab' = a(b - b') \in N.$$

[14] Because of incompleteness of $\mathcal{A}/N_\tau$, we can't necessarily speak of $\mathcal{B}(\mathcal{A}/N_\tau)$ as a $\star$-algebra, hence the clumsy formulation.

It is clear that $\Phi(a)$ is linear; we will show that it is bounded.

Fix any $b \in \mathcal{A}$. By point (ii) of Proposition 2.2.26, we know that $\|a^*a\|1_{\mathcal{A}} - a^*a \geq 0$ in $\mathcal{A}$. By point (i) of the same proposition, we find that

$$\|a\|^2 b^*b - (ab)^*(ab) = b^*\|a^*a\|b - b^*(a^*a)b \geq 0.$$

Since τ is positive, this gives $\tau((ab)^*(ab)) \leq \|a\|^2\tau(b^*b)$, which is exactly the statement that

$$\big\langle \Phi(a)(b+N), \Phi(a)(b+N) \big\rangle \leq \|a\|^2 \langle b+N, b+N \rangle.$$

Thus, $\|\Phi(a)\| \leq \|a\|$, so $\Phi(a) \in \mathcal{B}(\mathcal{A}/N)$. This proves that $\Phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{A}/N)$ is well-defined.

It is quite immediate that $\Phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{A}/N)$ is a linear map preserving the product and the unit. With $a, b, c \in \mathcal{A}$, the calculation

$$\big\langle \Phi(a)(b+N), c+N \big\rangle = \tau\big(c^*(ab)\big) = \tau\big((a^*c)^*b\big) = \big\langle b+N, \Phi(a^*)(c+N) \big\rangle$$

shows that $\Phi(a)^* = \Phi(a^*)$.

By Corollary B.1.5, the "extension map" $\mathrm{Ext}\colon \mathcal{B}(\mathcal{A}/N) \to \mathcal{B}(H)$ which maps each $T \in \mathcal{B}(\mathcal{A}/N)$ to its unique bounded linear extension $\mathrm{Ext}(T) \coloneqq \overline{T} \in \mathcal{B}(H)$ is an algebra homomorphism. Thus, $\overline{\Phi} = \mathrm{Ext} \circ \Phi$ is an algebra homomorphism as well. To see that it preserves the involution, consider any $a \in \mathcal{A}$ and $g, h \in H$, choose sequences $(h_n), (g_n) \subset \mathcal{A}/N$ such that $h_n \to h$ and $g_n \to g$ and use continuity of the inner product on H to conclude that

$$\langle \overline{\Phi}(a)h, g \rangle = \lim_{n\to\infty} \langle \Phi(a)h_n, g_n \rangle = \lim_{n\to\infty} \langle h_n, \Phi(a^*)g_n \rangle = \langle h, \overline{\Phi}(a^*)g \rangle.$$

This proves that $\overline{\Phi}$ is a $\star$-algebra homomorphism and hence a representation of $\mathcal{A}$.

Since $\overline{\Phi}(a)(1_{\mathcal{A}} + N) = a + N$ for all $a \in \mathcal{A}$, we see that $\overline{\Phi}(\mathcal{A})(1_{\mathcal{A}} + N) = \mathcal{A}/N$, so $h \coloneqq 1_{\mathcal{A}} + N$ is cyclic for Φ. Finally, for any $a \in \mathcal{A}$, we find that

$$\tau(a) = \tau(1_{\mathcal{A}}^* a) = \langle a+N, 1_{\mathcal{A}} + N \rangle = \langle \overline{\Phi}(a)h, h \rangle,$$

which concludes the proof. □

Having shown that cyclic representations of a C*-algebra $\mathcal{A}$ can be constructed from states on $\mathcal{A}$, we now wish to show that there exists an abundance of states on $\mathcal{A}$. To do so, we need the following characterization of states.

2.2.40 Lemma. *Let $\mathcal{A}$ be a C*-algebra and suppose that $\tau \in \mathcal{A}^*$ is normalized. Then, $\tau \in S(\mathcal{A})$ if and only if $\tau(1_{\mathcal{A}}) = 1$.*

Proof. Assume first that $\tau \in S(\mathcal{A})$ and fix any $a \in \mathcal{A}$. By point (ii) of Proposition 2.2.26, we have $\|a\|^2 1_{\mathcal{A}} - a^*a = \|a^*a\| 1_{\mathcal{A}} - a^*a \geq 0$ in $\mathcal{A}$, which implies that $\|a\|^2 \tau(1_{\mathcal{A}}) \geq \tau(a^*a)$. By the Cauchy-Schwarz inequality for states (Lemma 2.2.38), we now find that

$$|\tau(a)|^2 = |\tau(1_{\mathcal{A}}^* a)|^2 \leq \tau(1_{\mathcal{A}}^* 1_{\mathcal{A}}) \tau(a^*a) \leq \|a\|^2 \tau(1_{\mathcal{A}}).$$

Since $a \in \mathcal{A}$ was arbitrary, this implies that $\tau(1_{\mathcal{A}}) \geq \sqrt{\|\tau\|} = 1$. Since $\tau(1_{\mathcal{A}}) \leq \|\tau\| = 1$, we can conclude that $\tau(1_{\mathcal{A}}) = 1$.

Conversely, suppose that $\tau \in \mathcal{A}^*$ satisfies $\|\tau\| = 1$ and $\tau(1_{\mathcal{A}}) = 1$. We will show that $\tau(a) \geq 0$ whenever $a \in \mathcal{A}^+$.

Fix $a \in \mathcal{A}^+$ and let $t \in \mathbb{R}$ be arbitrary. We find that

$$\begin{aligned} |\tau(it1_{\mathcal{A}} - a)|^2 &\leq \|it1_{\mathcal{A}} - a\|^2 = \|(it1_{\mathcal{A}} - a)^*(it1_{\mathcal{A}} - a)\| = \|t^2 1_{\mathcal{A}} + a^2\| \\ &\leq t^2 + \|a\|^2 \end{aligned}$$

Now, since $\tau(it1_{\mathcal{A}} - a) = it - \tau(a)$, we see that

$$\big(t - \operatorname{Im}\tau(a)\big)^2 + \big(\operatorname{Re}\tau(a)\big)^2 = |\tau(it1_{\mathcal{A}} - a)|^2 \leq t^2 + \|a\|^2,$$

which after some rearranging becomes:

$$-2t\big(\operatorname{Im}\tau(a)\big) \leq \|a\|^2 - \big(\operatorname{Re}\tau(a)\big)^2 - \big(\operatorname{Im}\tau(a)\big)^2.$$

Since this holds for all $t \in \mathbb{R}$, we must have $\operatorname{Im}\tau(a) = 0$.

With the knowledge that $\tau(a) \in \mathbb{R}$, we find that

$$\|a\| - \tau(a) = \tau\big(\|a\| 1_{\mathcal{A}} - a\big) \leq \big\| \|a\| 1_{\mathcal{A}} - a \big\| \leq \|a\|,$$

where the last inequality follows from point (ii) of Lemma 2.2.22 (and the assumption that $a \geq 0$). This gives $\tau(a) \geq 0$, so we are done. $\square$

We now obtain the promised existence result more or less for free.

2.2.41 Proposition. *Let $\mathcal{A}$ be a C*-algebra. Then, for any $a \in \mathcal{A}$, there exists some $\tau \in S(\mathcal{A})$ such that $\tau(a^*a) = \|a\|^2$. Moreover, the GNS-representation $\overline{\Phi}_\tau$ associated to τ satisfies $\|\overline{\Phi}_\tau(a)\| = \|a\|$.*

Proof. Fix any $a \in \mathcal{A}$ and consider the commutative C*-algebra $C^*(a^*a)$ and its spectrum $\mathcal{M}_{C^*(a^*a)}$. By Lemma 2.1.24, we know that

$$\sigma(a^*a) = \big\{\mu(a^*a) : \mu \in \mathcal{M}_{C^*(a^*a)}\big\}.$$

By point (v) of Lemma 2.2.22, we have $\|a^*a\| \in \sigma(a^*a)$, so there must exist some $\mu \in \mathcal{M}_{C^*(a^*a)}$ such that $\mu(a^*a) = \|a^*a\| = \|a\|^2$. Lemma 2.1.24 also tells us that $\|\mu\| = 1$, and we have $\mu(1_{\mathcal{A}}) = 1$ by our definition of functionals as algebra homomorphisms.

Now, μ is a bounded linear functional on $C^*(a^*a) \subset \mathcal{A}$. By the Hahn-Banach theorem (Theorem A.2.1), there exists an extension $\widetilde{\mu} \in \mathcal{A}^*$ such that $\|\widetilde{\mu}\| = \|\mu\| = 1$. As $\widetilde{\mu}$ extends μ, we have $\widetilde{\mu}(a^*a) = \|a\|^2$ and $\widetilde{\mu}(1_{\mathcal{A}}) = 1$. By Lemma 2.2.40, this implies that $\widetilde{\mu} \in S(\mathcal{A})$.

Finally, with $\tau \coloneqq \widetilde{\mu}$, let $\overline{\Phi}_\tau \colon \mathcal{A} \to \mathcal{B}(H_\tau)$ be the GNS-representation associated to τ. Then,

$$\|a\|^2 = \tau(a^*a) = \langle a + N_\tau, a + N_\tau \rangle = \|\overline{\Phi}_\tau(a)(1_{\mathcal{A}} + N_\tau)\|^2 \leq \|\overline{\Phi}_\tau(a)\|^2,$$

because $\|1_{\mathcal{A}} + N_\tau\|^2 = \tau(1_{\mathcal{A}}^* 1_{\mathcal{A}}) = 1$. The $\star$-algebra homomorphism $\overline{\Phi}_\tau$ is norm-decreasing by Proposition 2.2.6, so the displayed equation implies that $\|a\| = \|\overline{\Phi}_\tau(a)\|$, which concludes the proof. □

With this result, we can construct a representation $\Phi \colon \mathcal{A} \to \mathcal{B}(H)$ such that any given element of $\mathcal{A}$ is not mapped to zero. In order to obtain an injective representation, we will construct a direct sum of such representations. To do this, we must consider direct sums of a possibly uncountable collection of Hilbert spaces, so we briefly discuss this notion before stating the Gelfand-Naimark theorem.

Given an arbitrary family $\{H_i\}_{i\in I}$ of Hilbert spaces (meaning that I is an arbitrary index set), we quickly sketch how to construct their *direct sum* $\bigoplus_{i\in I} H_i$ *as Hilbert spaces*. Consider their direct product $\prod_{i\in I} H_i$ as vector spaces.[15] We define the subset

$$\bigoplus_{i\in I} H_i \coloneqq \Big\{ (h_i)_{i\in I} \in \prod_{i\in I} H_i : \sum_{i\in I} \|h_i\|^2 < \infty \Big\},$$

where the sum is defined as the supremum over all finite sums. In order for such a sum to be finite, $(h_i)_{i\in I}$ can only have countably many nonzero components.[16] By standard ℓ^2-norm considerations (indeed, this is just an ℓ^2-space over a possibly uncountable set), it is now fairly straightforward to show that $\bigoplus_{i\in I} H_i$ is a vector-subspace of $\prod_{i\in I} H_i$ and that

$$\big\langle (h_i)_{i\in I}, (h'_j)_{j\in I} \big\rangle \coloneqq \sum_{i\in I} \langle h_i, h'_i \rangle.$$

[15] This is the vector space of all maps $\phi \colon I \to \bigcup_{i\in I} H_i$ such that $\phi(i) \in H_i$ for all $i \in I$ with pointwise operations. We write $(h_i)_{i\in I}$ to denote the map $\phi \in \prod_{i\in I} H_i$ satisfying $\phi(i) = h_i$ for all $i \in I$.

[16] A quick proof: If the supremum over finite sums is to be finite, then there can only be a finite number of terms with values between $1/(n+1)$ and $1/n$, for any given $n \in \mathbb{N}_1$. For this to hold, there can only be countably many nonzero terms.

defines an inner product on $\bigoplus_{i\in I} H_i$ which turns it into a Hilbert space.

If $\{T_i\}_{i\in I}$ is a collection of operators such that $T_i \in \mathcal{B}(H_i)$ for each $i \in I$ and $\sup_{i\in I} \|T_i\| < \infty$, it is straightforward to see that we obtain an operator $\bigoplus_{i\in I} T_i \in \mathcal{B}(\bigoplus_{i\in I} H_i)$ defined by

$$\bigoplus_{i\in I} T_i\big((h_i)_{i\in I}\big) = (T_i h_i)_{i\in I}.$$

Moreover, one may easily verify the identities:

$$\Big(\bigoplus_{i\in I} T_i\Big)^* = \bigoplus_{i\in I} T_i^*, \quad \bigoplus_{i\in I}(\lambda T_i + S_i) = \lambda \bigoplus_{i\in I} T_i + \bigoplus_{i\in I} S_i,$$
$$\bigoplus_{i\in I}(T_i \circ S_i) = \Big(\bigoplus_{i\in I} T_i\Big) \circ \Big(\bigoplus_{i\in I} S_i\Big) \quad \text{and} \quad \mathrm{Id}_{\bigoplus_{i\in I} H_i} = \bigoplus_{i\in I} \mathrm{Id}_{\mathrm{H_i}},$$

where $\lambda \in \mathbb{C}$ and $\{S_i\}_{i\in I}$ is another such collection of operators.

The proof the Gelfand-Naimark theorem is now simple.

2.2.42 Theorem (The Gelfand-Naimark theorem)**.** *For any C*-algebra $\mathcal{A}$, there exists an injective representation $\Phi\colon \mathcal{A} \to \mathcal{B}(H)$. By Theorem 2.2.33, Φ is then isometric and $\mathcal{A} \cong \Phi(\mathcal{A}) \subset \mathcal{B}(H)$ as C*-algebras.*

Proof. Consider the direct sum $H := \bigoplus_{\tau\in S(\mathcal{A})} H_\tau$ of Hilbert spaces, where H_τ is the Hilbert space corresponding to the GNS-representation $\overline{\Phi}_\tau\colon \mathcal{A} \to \mathcal{B}(H_\tau)$ associated to $\tau \in S(\mathcal{A})$. For each $a \in \mathcal{A}$, we know that $\|\overline{\Phi}_\tau(a)\| \le \|a\|$ for all $\tau \in S(\mathcal{A})$ (by Proposition 2.2.6), so we obtain a bounded operator $\bigoplus_{\tau\in S(\mathcal{A})} \overline{\Phi}_\tau(a)$ on H. By our discussion prior to the theorem, the map

$$\begin{aligned} \Phi\colon \mathcal{A} &\to \mathcal{B}(H) \\ a &\mapsto \bigoplus_{\tau\in S(\mathcal{A})} \overline{\Phi}_\tau(a) \end{aligned}$$

is a $\star$-algebra homomorphism, hence a representation of $\mathcal{A}$. Finally, for each $a \in \mathcal{A}$, Proposition 2.2.41 tells us that there exists some $\tau \in S(\mathcal{A})$ such that $\overline{\Phi}_\tau(a) \neq 0$, and this straightforwardly leads to the conclusion that this representation is injective. $\square$

Chapter 3

Time-Frequency Analysis and Gabor Analysis

In this chapter, we introduce the main subject of the thesis, which is Gabor analysis. Gabor analysis is a subject in time-frequency analysis, which we therefore introduce first. In many ways, time-frequency analysis is a refinement of Fourier theory. The basic idea is to combine the time and frequency domains (which are treated as separate entities in ordinary Fourier theory) in such a manner that we can study the frequency content of a signal locally. We refer to Gröchenig [17] for a thorough and accessible introduction to both time-frequency analysis and Gabor analysis; our exposition of these topics is largely based on his book.

Subsection 3.2.2 is a pivotal part of the thesis: it is where we outline our main argument.

3.1 | Time-Frequency Analysis

The goal of this section is to give the reader a quick introduction to those parts of time-frequency analysis that we will need in order to discuss Gabor analysis properly.

3.1.1 Preliminaries

We will assume familiarity with basic Fourier theory and measure theory.

From measure theory, we will mainly need Fatou's lemma, the monotone and dominated convergence theorems as well as Fubini's and Tonelli's theorems. To make sure that we are on the same page: by Tonelli's theorem, we refer to the fact that we may freely change the order of integration as long as the integrand is nonnegative, and by Fubini's theorem, we refer to the fact that we may change the order of integration as long as the integrand is absolutely integrable. The conditions for these results to hold (such as σ-finiteness and measurability) will always be trivially satisfied in our setting. There are many great introductions to measure theory: Cohn [9] is a detailed and broad classic; Axler [5] is a recent and very accessible introduction.

Our convention for the Fourier transform on $\mathbb{R}^d$ will be that

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot\omega}\,\mathrm{d}t \quad \text{for all } \omega \in \mathbb{R}^d \text{ and } f \in L^1(\mathbb{R}^d).$$

Its unitary extension/restriction to $L^2(\mathbb{R}^d)$ will also be denoted by $\mathcal{F}$. For $f, g \in L^2(\mathbb{R}^d)$ we will write $\langle f, g\rangle$ for their standard inner product and $f \cdot g$ or fg for their pointwise product. Inner products will be linear in the first entry and complex conjugation will be denoted by an overline.

We will write $\mathscr{S}(\mathbb{R}^d)$ to denote the Schwartz space, which is the space of all infinitely differentiable functions whose every derivative decays faster than any polynomial. We will freely appeal to basic facts regarding $\mathscr{S}(\mathbb{R}^d)$, such as invariance under the Fourier transform and the fact that $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

As for general Fourier theory, we will assume knowledge of pretty much everything up to the Poisson summation formula (which we will prove). Whenever we invoke such results, we will try to point them out. Everything we need can be found in the accessible introduction [31] by Stein and Shakarchi.

3.1.2 Translations, Modulations and Time-Frequency Shifts

The most basic operations performed on functions in time-frequency analysis are translations and modulations. The latter is a translation of a functions frequency spectrum, i.e. its Fourier transform. This subsection serves to introduce these operators and their basic properties.

For any $x, \omega \in \mathbb{R}^d$, we define operators T_x and M_ω on $L^2(\mathbb{R}^d)$ by

$$(T_x f)(t) = f(t - x) \quad \text{and} \quad (M_\omega f)(t) = e^{2\pi i t\cdot\omega} f(t),$$

for all $t \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$. The dot denotes the standard inner product on $\mathbb{R}^d$. We refer to T_x as a *translation* operator and M_ω as a *modulation* operator. These are unitary operators with adjoints/inverses given by

$$M_\omega^* = M_\omega^{-1} = M_{-\omega} \quad \text{and} \quad T_x^* = T_x^{-1} = T_{-x}.$$

For any $z = (x, \omega) \in \mathbb{R}^{2d}$, we further define the composition

$$\pi(z) := M_\omega T_x,$$

which we refer to as a *time-frequency shift*. In this context, we will refer to $\mathbb{R}^{2d}$ as the *time-frequency plane*. Thus, the time-frequency plane parametrizes the set of all time-frequency shifts. We may also write $\pi(x, \omega)$ for $\pi(z)$, and whenever we write something like "$(x, \omega) \in \mathbb{R}^{2d}$", it should be understood that $x, \omega \in \mathbb{R}^d$.

Those parts of time-frequency analysis of relevance to us are built entirely upon these operators, so it is their algebraic structure that shapes the subject. We collect some basic identities in the following lemma. We will feel free to use these properties without reference. We may on occasion refer to property (iii) as *the basic commutation relation.*

3.1.1 Lemma (Basic identities)**.** *Let $z = (x, \omega), w = (y, \eta) \in \mathbb{R}^{2d}$. Then, the following relations hold.*

(i) $T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x$

(ii) $\pi(z + w) = e^{2\pi i \eta\cdot x}\pi(z)\pi(w)$

(iii) $\pi(z)\pi(w) = e^{2\pi i(\omega\cdot y - \eta\cdot x)}\pi(w)\pi(z) = e^{2\pi i \Omega_J(z,w)}\pi(w)\pi(z)$

(iv) $\pi(z)^* = \pi(z)^{-1} = e^{-2\pi i\omega\cdot x}\pi(-z)$

In (iii), Ω_J refers to the standard symplectic form on $\mathbb{R}^{2d}$, as defined by Equation (1.4)*.*

Proof. The first three relations are straightforward calculations. We find that

$$T_x M_\omega f(t) = e^{2\pi i(t-x)\cdot\omega} f(t - x) = e^{-2\pi i x\cdot\omega} M_\omega T_x f(t),$$

for all $t \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, which gives (i). Noting that $M_\omega M_\eta = M_{\omega+\eta}$ and $T_x T_y = T_{x+y}$, we use (i) to find that

$$\pi(z)\pi(w) = M_\omega T_x M_\eta T_y = e^{-2\pi i\eta\cdot x} M_\omega M_\eta T_x T_y = e^{-2\pi i\eta\cdot x}\pi(z + w),$$

which is (ii). Appealing to (ii) twice, we find that

$$\pi(z)\pi(w) = e^{-2\pi i\eta\cdot x}\pi(z + w) = e^{-2\pi i\eta\cdot x}\pi(w + z) = e^{-2\pi i\eta\cdot x}e^{2\pi i\omega\cdot y}\pi(w)\pi(z).$$

Noting that $\Omega_J(z, w) = \omega \cdot y - \eta \cdot x$, we obtain (iii).

As a composition of unitary operators, $\pi(z)$ is unitary. Using (ii), we find that

$$e^{-2\pi i\omega\cdot x}\pi(-z)\pi(z) = \pi(0) = e^{-2\pi i\omega\cdot x}\pi(z)\pi(-z).$$

Since $\pi(0) = \mathrm{Id}_{L^2(\mathbb{R}^d)}$, we see that $\pi(z)^* = \pi(z)^{-1} = e^{-2\pi i\omega\cdot x}\pi(-z)$, which gives (iv) and concludes the proof. □

The following lemma verifies that modulations are translation of a functions frequency spectrum.

3.1.2 Lemma. *For all $x, \omega \in \mathbb{R}^d$, we have that*

$$\mathcal{F}T_x = M_{-x}\mathcal{F}, \quad \mathcal{F}M_\omega = T_\omega\mathcal{F},$$

as well as $\mathcal{F}\pi(x,\omega) = e^{2\pi i x\cdot\omega}\pi(\omega,-x)\mathcal{F}$.

Proof. Let $f \in \mathscr{S}(\mathbb{R}^d)$ and $\eta \in \mathbb{R}^d$. We find that

$$\mathcal{F}T_x f(\eta) = \int_{\mathbb{R}^d} f(t-x)e^{-2\pi i t\cdot\eta}\,\mathrm{d}t = \int_{\mathbb{R}^d} f(s)e^{-2\pi i(s+x)\cdot\eta}\,\mathrm{d}s = M_{-x}\mathcal{F}f(\eta)$$

and that

$$\mathcal{F}M_\omega f(\eta) = \int_{\mathbb{R}^d} e^{2\pi i t\cdot\omega}f(t)e^{-2\pi i t\cdot\eta}\,\mathrm{d}t = \mathcal{F}f(\eta-\omega) = T_\omega\mathcal{F}f(\eta).$$

For any $f \in L^2(\mathbb{R}^d)$, we can choose a sequence $(f_n) \subset \mathscr{S}(\mathbb{R}^d)$ such that $f_n \to f$ in $L^2(\mathbb{R}^d)$. Appealing to continuity of all the operators involved, we then find that

$$\mathcal{F}T_x\big(\lim_{n\to\infty} f_n\big) = \lim_{n\to\infty}\mathcal{F}T_x f_n = \lim_{n\to\infty} M_{-x}\mathcal{F}f_n = M_{-x}\mathcal{F}\big(\lim_{n\to\infty} f_n\big),$$

so that $\mathcal{F}T_x f = M_{-x}\mathcal{F}f$. By a similar calculation, $\mathcal{F}M_\omega f = T_\omega\mathcal{F}f$ as well. This proves the displayed relations of the lemma.

Finally, using what we have proved, we find that

$$\mathcal{F}\pi(x,\omega) = \mathcal{F}M_\omega T_x = T_\omega M_{-x}\mathcal{F} = e^{2\pi i x\cdot\omega}M_{-x}T_\omega\mathcal{F} = e^{2\pi i x\cdot\omega}\pi(\omega,-x)\mathcal{F},$$

which completes the proof. □

Before we move on to the next topic, there is one more class of operators we wish to introduce. We define

$$\psi(z) \coloneqq M_{\omega/2}T_x M_{\omega/2} = e^{-\pi i x\cdot\omega}\pi(z) \quad \text{for } z = (x,w) \in \mathbb{R}^{2d}.$$

These are known as *Heisenberg-Weyl operators.* They can be thought of as "symmetrized" time-frequency shifts; a compromise between $T_x M_\omega$ and $M_\omega T_x$. Heisenberg-Weyl operators occur naturally in quantum mechanics.[1] For more on this topic, we refer the reader to de Gosson [15, Chapter 8].

In time, we will find that Heisenberg-Weyl operators are more convenient than time-frequenecy shifts for our purposes. Nevertheless, time-frequency

[1]We have chosen the symbol ψ mainly as a nod to this fact.

analysis and Gabor theory is traditionally phrased in terms of time-frequenecy shifts, so we will stick with those for now. As a sneak peek at the property that will make Heisenberg-Weyl operators preferable, we observe that Lemma 3.1.2 implies that

$$\mathcal{F}\psi(x,\omega) = e^{-\pi i x\cdot\omega}\mathcal{F}\pi(x,\omega) = e^{\pi i x\cdot\omega}\pi(\omega,-x)\mathcal{F} = \psi(\omega,-x)\mathcal{F},$$

so that the phase factor disappears when working with Heisenberg-Weyl operators!

3.1.3 The Short-Time Fourier Transform

We now introduce the short-time Fourier transform (the STFT for short).

3.1.3 Definition. Let $f, g \in L^2(\mathbb{R}^d)$. We define *the short-time Fourier transform* of f with *window* g to be the function $V_g f$ on $\mathbb{R}^{2d}$ defined by

$$V_g f(z) = \langle f, \pi(z)g\rangle = \mathcal{F}\big(f \cdot T_x\overline{g}\big)(\omega) \quad \text{for } z = (x,\omega) \in \mathbb{R}^{2d}.$$

We will also write $V_g f(z) = V_g f(x,\omega)$.

The STFT $V_g f$ is a complex-valued function on the time-frequency plane. As the rightmost expression suggest, for fixed $x \in \mathbb{R}^d$, we can think of $\omega \mapsto V_g f(x,\omega)$ as a Fourier transform of a weighted version of f. If g is a Gaussian centered at the origin, then $f \cdot T_x\overline{g}$ represents a “snippet” of f in the vicinity of $x \in \mathbb{R}^d$. Taking its Fouier transform gives us information about the frequency spectrum of f in the vicinity of x. Thus, the STFT allows us to obtain localized frequency information about f. The variable x represents the location at which we wish to probe f.

We may think of the STFT operationally in the following manner. First, pick a window function $g \in L^2(\mathbb{R}^d)$, which will be our “probe”. Then, apply the entire time-frequency plane worth of operators to g in order to produce the set

$$\{\pi(z)g : z \in \mathbb{R}^{2d}\} \subset L^2(\mathbb{R}^d) \tag{3.1}$$

consisting of time-frequency shifted version of our probe. We can now study an arbitrary function $f \in L^2(\mathbb{R}^d)$ by considering its “components” with respect to this set. If we fix x and let ω vary, the components $\langle f, \pi(x,\omega)g\rangle$ represent the frequency content of f in the vicinity of x (assuming that g is localized near the origin). Of course, some probes will obscure our original signal more than others; much of Gabor theory is concerned with finding suitable probes/windows.

We will now develop some basic properties of the short-time Fourier transform. We begin by exploring the interplay between the STFT, time-frequency shifts and the ordinary Fourier transform. Gröchenig refers to the first property of the following lemma as the *covariance property* of the STFT and to the second as the *fundamental identity of time-frequency analysis.* The latter shows that the Fourier transform is related to a 90 degree rotation of the time-frequency plane (which seems natural, as it should exchange the roles of time and frequency).

3.1.4 Lemma. *Let $f, g \in L^2(\mathbb{R}^d)$. Then,*

(i) $\big(V_g\pi(y,\eta)f\big)(x,\omega) = e^{-2\pi i y\cdot(\omega-\eta)}V_g f(x-y,\omega-\eta)$

(ii) $V_g f(x,\omega) = e^{-2\pi i x\cdot\omega}V_{\mathcal{F}g}\mathcal{F}f(\omega,-x)$

for all $(x,\omega),(y,\eta)\in\mathbb{R}^{2d}$.

Proof. Freely using the basic identities of Lemma 3.1.1, we find that

$$\begin{aligned}\big\langle \pi(y,\eta)f,\pi(x,\omega)g\big\rangle &= \big\langle f,\pi(y,\eta)^*\pi(x,\omega)g\big\rangle \\ &= \big\langle f, e^{-2\pi i y\cdot\eta}\pi(-y,-\eta)\pi(x,\omega)g\big\rangle \\ &= e^{2\pi i y\cdot\eta}\big\langle f, e^{-2\pi i\omega\cdot(-y)}\pi(x-y,\omega-\eta)g\big\rangle \\ &= e^{-2\pi i y\cdot(\omega-\eta)}\big\langle f,\pi(x-y,\omega-\eta)g\big\rangle.\end{aligned}$$

Recalling the definition of the STFT, we obtain (i). The fact that $\mathcal{F}$ is unitary, along with Lemma 3.1.2, implies that

$$\langle f,\pi(x,\omega)g\rangle = \langle \mathcal{F}f,\mathcal{F}\pi(x,\omega)g\rangle = \langle \mathcal{F}f, e^{2\pi i x\cdot\omega}\pi(\omega,-x)\mathcal{F}g\rangle,$$

which proves (ii). □

For any $f, g\in L^2(\mathbb{R}^d)$, the ability to evaluate $V_g f$ at points is immediate from its definition (in contrast to the Fourier transform of an $L^2(\mathbb{R}^d)$-function). The following lemma shows that $V_g f$ is uniformly continuous as a function on the time-frequency plane.

3.1.5 Proposition. *Let $f, g \in L^2(\mathbb{R}^d)$. Then, $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$.*

Proof. We first show that the function $\mathbb{R}^d \to L^2(\mathbb{R}^d)$ defined by $x\mapsto T_x g$ is uniformly continuous (with respect to the Euclidean norm on $\mathbb{R}^d$). Our proof of this fact imitates the proof given by Rudin [29, Theorem 9.5 on p. 182].

Fix any $\epsilon > 0$ and choose a continuous function $\tilde{g}$ with compact support that moreover satisfies $\|g-\tilde{g}\|_2 < \epsilon$. This can always be done, as such functions are dense in $L^2(\mathbb{R}^d)$. For any $r > 0$ and $t \in \mathbb{R}^d$, let $B_t(r) \subset \mathbb{R}^d$ denote the closed ball of radius r centred at t and let $|B_t(r)|$ denote its volume.

Let $R > 0$ be such that $\operatorname{supp}(\tilde{g}) \subset B_0(R)$. A continuous function with compact support is uniformly continuous, so there exists some $\delta \in (0, R)$ such that, for all $x, y \in \mathbb{R}^d$,

$$\|x - y\| < \delta \qquad \Longrightarrow \qquad |\tilde{g}(x) - \tilde{g}(y)| < \frac{\epsilon}{|B_0(3R)|^{1/2}}.$$

Assume from now on that $x, y \in \mathbb{R}^d$ satisfy $\|x - y\| < \delta$. Since, in particular, $\|x - y\| < R$, we find that

$$\operatorname{supp}(T_x\tilde{g} - T_y\tilde{g}) \subset B_x(R) \cup B_y(R) \subset B_x(3R). \tag{3.2}$$

The last two displayed equations imply that

$$\|(T_x - T_y)\tilde{g}\|_2^2 = \int_{\mathbb{R}^d} |\tilde{g}(t - x) - \tilde{g}(t - y)|^2 \,\mathrm{d}t \le |B_x(3R)| \frac{\epsilon^2}{|B_0(3R)|} = \epsilon^2,$$

which (along with our choice of $\tilde{g}$) gives:

$$\begin{aligned} \|T_xg - T_yg\|_2 &\le \|T_xg - T_x\tilde{g}\|_2 + \|T_x\tilde{g} - T_y\tilde{g}\|_2 + \|T_y\tilde{g} - T_yg\|_2 \\ &= 2\|g - \tilde{g}\|_2 + \|(T_x - T_y)\tilde{g}\|_2 < 3\epsilon. \end{aligned}$$

This proves that for any $\epsilon > 0$, we can find some $\delta > 0$ such that $\|x-y\| < \delta$ implies that $\|T_xg - T_yg\|_2 < \epsilon$. In other words, the map $x \mapsto T_xg$ is uniformly continuous. Since $g \in L^2(\mathbb{R}^d)$ was arbitrary, the same result applies to the function $\mathcal{F}f \in L^2(\mathbb{R}^d)$. Lemma 3.1.2 implies that

$$\|(T_{-\omega} - T_{-\eta})\mathcal{F}f\|_2 = \|\mathcal{F}(M_{-\omega} - M_{-\eta})f\|_2 = \|(M_{-\omega} - M_{-\eta})f\|_2,$$

so we also obtain uniform continuity of the map $\omega \mapsto M_{-\omega}f$.

Let now $z = (x, \omega), w = (y, \eta) \in \mathbb{R}^{2d}$. Then,

$$\begin{aligned} |V_gf(z) - V_gf(w)| &= \left|\left\langle f, (M_\omega T_x - M_\eta T_y)g \right\rangle\right| \\ &= \left|\left\langle f, (M_\omega T_x - M_\omega T_y + M_\omega T_y - M_\eta T_y)g \right\rangle\right| \\ &\le \left|\left\langle f, M_\omega(T_x - T_y)g \right\rangle\right| + \left|\left\langle f, (M_\omega - M_\eta)T_yg \right\rangle\right| \\ &= \left|\left\langle M_{-\omega}f, (T_x - T_y)g \right\rangle\right| + \left|\left\langle (M_{-\omega} - M_{-\eta})f, T_yg \right\rangle\right| \\ &\le \|f\|_2\|T_xg - T_yg\|_2 + \|M_{-\omega}f - M_{-\eta}f\|_2\|g\|_2. \end{aligned}$$

Thus, uniform continuity of V_gf follows from uniform continuity of the maps $x \mapsto T_xg$ and $\omega \mapsto M_{-\omega}f$. $\square$

The next result we wish to prove is known as Moyal's identity. It is a foundational and extremely useful result. In order to derive it, it will be convenient to have an alternate description of the STFT. The following paragraph introduces the necessary machinery.

For any $f, g \in L^2(\mathbb{R}^d)$, define their *tensor product*

$$\begin{aligned} f \otimes g \colon \mathbb{R}^{2d} &\to \mathbb{C} \\ (x,t) &\mapsto f(x)g(t) \quad (\text{where } x, t \in \mathbb{R}^d). \end{aligned}$$

By Tonelli's theorem, we find that $\|f \otimes g\|_2 = \|f\|_2 \|g\|_2$, hence $f \otimes g \in L^2(\mathbb{R}^{2d})$. Let $\mathcal{F}_2 \colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ denote the Fourier transform with respect to the last d variables only. That is, for any $F \in L^1(\mathbb{R}^{2d})$, we define

$$\mathcal{F}_2 F(x,\omega) = \int_{\mathbb{R}^d} F(x,t) e^{-2\pi i t \cdot \omega} \, \mathrm{d}t \quad \text{for all } (x,\omega) \in \mathbb{R}^{2d},$$

and then extend $\mathcal{F}_2$ to $L^2(\mathbb{R}^d)$ in the usual manner. Again, by Tonelli's theorem, along with the fact that the regular Fourier transform is unitary (and that if $F \in L^2(\mathbb{R}^{2d})$, then $F(x,-) \in L^2(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$), we find that

$$\begin{aligned} \|\mathcal{F}_2 F\|_2^2 &= \int_{\mathbb{R}^{2d}} |\mathcal{F}_2 F(x,\omega)|^2 \mathrm{d}(x,\omega) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}_2 F(x,\omega)|^2 \, \mathrm{d}\omega \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |F(x,t)|^2 \, \mathrm{d}t \, \mathrm{d}x = \|F\|_2^2, \end{aligned}$$

so that $\mathcal{F}_2$ is unitary (invertibility is also quite immediate from invertibility of the usual Fourier transform). Finally, let $T \colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ denote the transformation $(x,t) \mapsto (t, t-x)$ and define

$$\begin{aligned} T^t \colon L^2(\mathbb{R}^{2d}) &\to L^2(\mathbb{R}^{2d}) \\ F &\mapsto T^t F := F \circ T. \end{aligned}$$

Then, T^t is clearly unitary as well (it is just a coordinate transformation). Now, for any $f, g \in L^2(\mathbb{R}^d)$, we find that

$$\big(\mathcal{F}_2 T^t (f \otimes \overline{g})\big)(x,\omega) = \int_{\mathbb{R}^d} (f \otimes \overline{g})(t, t-x) e^{-2\pi i t \cdot \omega} \, \mathrm{d}t = V_g f(x,\omega),$$

so that $\mathcal{F}_2 T^t (f \otimes \overline{g}) = V_g f$.[2]

We are now prepared to prove Moyal's identity.

[2] We have been slightly sloppy with the domain of $\mathcal{F}_2$ in this paragraph. In order to be completely rigorous, one should use approximating sequences for these calculations.

3.1.6 Proposition (Moyal's identity)**.** *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then, $V_{g_1} f_1, V_{g_2} f_2 \in L^2(\mathbb{R}^{2d})$ and we have that*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \int_{\mathbb{R}^{2d}} V_{g_1} f_1(z) \overline{V_{g_2} f_2(z)} \, \mathrm{d}z = \overline{\langle g_1, g_2 \rangle} \langle f_1, f_2 \rangle.$$

Proof. By the discussion preceding the proposition, we find that

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle &= \left\langle \mathcal{F}_2 T^t (f_1 \otimes \overline{g_1}), \mathcal{F}_2 T^t (f_2 \otimes \overline{g_2}) \right\rangle \\ &= \langle f_1 \otimes \overline{g_1}, f_2 \otimes \overline{g_2} \rangle = \langle f_1, f_2 \rangle \langle \overline{g_1}, \overline{g_2} \rangle, \end{aligned}$$

where the last equality is a simple application of Fubini's theorem. □

We will now see how to recover any function in $L^2(\mathbb{R}^d)$ from its short-time Fourier transform (w.r.t. any nonzero window). We will do this via a *weak integral*, a notion we quickly introduce before stating the result.

Let H be a complex Hilbert space and let X be a measure space. We say that a function $\phi \colon X \to H$ is *weakly integrable* if the following two conditions are satisfied:

(i) for every $k \in H$, the function

$$\begin{aligned} \langle \phi(-), k \rangle \colon X &\to \mathbb{C} \\ x &\mapsto \langle \phi(x), k \rangle \end{aligned}$$

is integrable (i.e. a member of $L^1(X)$);

(ii) the conjugate-linear map

$$\begin{aligned} H &\to \mathbb{C} \\ k &\mapsto \int_X \langle \phi(x), k \rangle \, \mathrm{d}x \end{aligned}$$

is bounded. A sufficient but not necessary condition for this is that $\int_X \|\phi(x)\| \, \mathrm{d}x < \infty$.

If ϕ is weakly integrable, then the Riesz representation theorem guarantees the existence of a unique element $f \in H$ such that

$$\langle f, k \rangle = \int_X \langle \phi(x), k \rangle \, \mathrm{d}x \quad \text{for all } k \in H.$$

We then define the *weak integral* of ϕ:

$$\int_X \phi(x) \, \mathrm{d}x := f,$$

so that

$$\left\langle \int_X \phi(x)\,\mathrm{d}x, k \right\rangle = \int_X \langle \phi(x), k\rangle\,\mathrm{d}x \quad \text{for all } k \in H.$$

We are now ready to state the promised result, often referred to as the *reconstruction formula* or the *inversion formula.*

3.1.7 Corollary. *Let $g, h \in L^2(\mathbb{R}^d)$ be non-orthogonal, i.e. $\langle h, g\rangle \neq 0$. Then,*

$$f = \frac{1}{\langle h, g\rangle} \int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)h\,\mathrm{d}z$$

for all $f \in L^2(\mathbb{R}^d)$.

Proof. We are weakly integrating the map $\phi\colon \mathbb{R}^{2d} \to L^2(\mathbb{R}^d)$ defined by

$$\phi(z) = \frac{1}{\langle h, g\rangle} V_g f(z)\pi(z)h.$$

By Moyal's identity (Proposition 3.1.6), we have that

$$\langle f, k\rangle = \frac{1}{\langle h, g\rangle} \int_{\mathbb{R}^{2d}} V_g f(z)\langle \pi(z)h, k\rangle\,\mathrm{d}z \quad \text{for all } k \in L^2(\mathbb{R}^d).$$

Since $V_g f, V_h k \in L^2(\mathbb{R}^{2d})$, and hence $V_g f \cdot \overline{V_h k} \in L^1(\mathbb{R}^{2d})$, condition (i) for the definition of a weak integral is satisfied. This equation also immediately implies that condition (ii) is satisfied and that the conclusion of the corollary holds. □

There is another way to interpret the reconstruction formula, which we now describe. Fix a nonzero window $g \in L^2(\mathbb{R}^d)$ and consider the linear map

$$\begin{aligned} \mathcal{V}_g\colon L^2(\mathbb{R}^d) &\to L^2(\mathbb{R}^{2d}) \\ f &\mapsto V_g f \end{aligned}$$

By Moyal's identity (Proposition 3.1.6), this is a well-defined and bounded map whose norm equals $\|g\|_2$. Consider the (suggestively named) map

$$\begin{aligned} \mathcal{V}_g^*\colon L^2(\mathbb{R}^{2d}) &\to L^2(\mathbb{R}^d) \\ F &\mapsto \int_{\mathbb{R}^{2d}} F(z)\pi(z)g\,\mathrm{d}z. \end{aligned}$$

Since, for every $F \in L^2(\mathbb{R}^{2d})$ and $f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^{2d}} \big\langle F(z)\pi(z)g, f\big\rangle\,\mathrm{d}z = \int_{\mathbb{R}^{2d}} F(z)\overline{V_g f(z)}\,\mathrm{d}z = \langle F, \mathcal{V}_g f\rangle,$$

the weak integral defining $\mathcal{V}_g^* F$ is well-defined. Moreover, this shows that $\langle \mathcal{V}_g^* F, f\rangle = \langle F, \mathcal{V}_g f\rangle$, which means that $\mathcal{V}_g^*$ is the adjoint of $\mathcal{V}_g$.

We can now write the reconstruction formula as follows:

$$\mathrm{Id}_{L^2(\mathbb{R}^d)} = \frac{1}{\langle h, g\rangle} \mathcal{V}_h^* \mathcal{V}_g \quad \text{for all } h, g \in L^2(\mathbb{R}^d) \text{ s.t. } \langle h, g\rangle \neq 0.$$

In particular, for any nonzero $g \in L^2(\mathbb{R}^d)$, we find that the short-time Fourier transform $\mathcal{V}_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is injective.

3.2 | Gabor Analysis

We are finally prepared to introduce the main subject of this thesis: Gabor analysis. The first subsection introduces the basic notions, the second outlines our proposed approach to the subject, while the third and fourth will be spent proving more subtle (though still foundational) results.

3.2.1 Gabor Systems, Frames and Associated Operators

This subsection is not based on any particular reference, but it is inspired by Gröchenig [17], Gröchenig and Koppensteiner [19] as well as lecture notes by Luef.

Before we discuss Gabor systems, we must introduce the notion of a lattice. A *lattice* $\Lambda \subset \mathbb{R}^{2d}$ is an additive subgroup of $\mathbb{R}^{2d}$ of the form

$$\Lambda = A\mathbb{Z}^{2d} := \left\{Ak : k \in \mathbb{Z}^{2d}\right\} \quad \text{for some } A \in \mathrm{GL}(2d, \mathbb{R}).$$

In this context, we will call A a *lattice matrix* (for Λ) and say that the matrix A *determines* the lattice Λ. We also define the *volume* of the lattice: $\mathrm{vol}(\Lambda) := |\det A|$.

Distinct matrices can determine the same lattice. The following simple lemma clarifies when this is the case.

3.2.1 Lemma (Lattice matrices)**.** *Two matrices $A, B \in \mathrm{GL}(2d, \mathbb{R})$ determine the same lattice, i.e. $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$, if and only if $A^{-1}B \in \mathrm{GL}(2d, \mathbb{Z})$.*

Proof. For any $A, B \in \mathrm{GL}(2d, \mathbb{R})$, we have the following equivalences:

$$A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d} \iff \mathbb{Z}^{2d} = A^{-1}B\mathbb{Z}^{2d} \iff A^{-1}B \in \mathrm{GL}(2d, \mathbb{Z}).$$

The first equivalence holds because A is a bijection on $\mathbb{R}^{2d}$ and the second equivalence holds because $A^{-1}B$ is additive/$\mathbb{Z}$-linear (and hence additively invertible if and only if it is bijective). $\square$

As a simple corollary to this lemma, we observe that $\mathrm{vol}(\Lambda)$ depends only on Λ, meaning that it is independent of our choice of lattice matrix: if we have $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$, then $\det B = \det(A)\det(A^{-1}B) = \pm\det A$, for $A^{-1}B \in \mathrm{GL}(2d, \mathbb{Z})$ implies that $\det(A^{-1}B) = \pm 1$.[3]

A *Gabor system* for $L^2(\mathbb{R}^d)$ is a set of the form

$$\mathcal{G}(g, \Lambda) = \big\{\pi(\lambda)g : \lambda \in \Lambda\big\}$$

where $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ is a lattice. In connection to the STFT, we can think of this as a discrete analogue of the set in Equation (3.1). That is, we wish to replace the entirety of the time-frequency plane with a lattice, so that we may study the localized frequency content of a function $f \in L^2(\mathbb{R}^d)$ by considering only a countable number of components $V_g f(\lambda) = \langle f, \pi(\lambda)g\rangle$. This is of course a prerequisite for numerical time-frequency analysis.

Gabor analysis deals with the spanning properties of Gabor systems; the spanning properties of $\mathcal{G}(g, \Lambda)$ tell us how suitable g and Λ are for the purposes outlined above. Whenever we refer to a "Gabor system $\mathcal{G}(g, \Lambda)$", it should be implicitly understood that $g \in L^2(\mathbb{R}^d)$ and that Λ is a lattice on $\mathbb{R}^{2d}$.

3.2.2 Definition (Bessel sequences and Gabor frames)**.** Let $\mathcal{G}(g, \Lambda)$ be a Gabor system. We say that $\mathcal{G}(g, \Lambda)$ is a *Bessel sequence* if there exists a constant $B > 0$ (called a *Bessel bound*) such that

$$\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{3.3}$$

Similarly, we call $\mathcal{G}(g, \Lambda)$ a *Gabor frame* for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}^{2d}). \tag{3.4}$$

In this situation, we refer to A as a *lower frame bound* and to B as an *upper frame bound* for the Gabor frame $\mathcal{G}(g, \Lambda)$. A Gabor frame is said to be *tight* if we can choose $A = B$, and we call it a *Parseval frame* if we can choose $A = B = 1$.

Ultimately, we wish to identify lattices Λ and pairs of functions $g, h \in L^2(\mathbb{R}^d)$ such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)h \quad \text{for all } f \in L^2(\mathbb{R}^d)$$

[3] A matrix over a commutative ring R, such as $\mathbb{Z}$, is invertible if and only if its determinant is invertible in R. The only invertible elements in $\mathbb{Z}$ are 1 and -1.

(we will define such sums soon). This is a discrete analogue of the reconstruction formula (Corollary 3.1.7). In Proposition 3.2.8, we will see that whenever $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, there is a canonical choice for h such that this holds.

Given a Gabor system $\mathcal{G}(g,\Lambda)$, we can describe the passage from f to

$$\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)h. \tag{3.5}$$

in two steps. First, we extract the coefficients $(\langle f,\pi(\lambda)g\rangle)_{\lambda\in\Lambda}$ from f, and then we form the linear combination above. These processes are called *analysis* (extracting the coefficients) and *synthesis* (forming a function from a set of coefficients). They are discrete analogues of the STFT $\mathcal{V}_g$ and its adjoint $\mathcal{V}_g^*$, as described after Corollary 3.1.7. We will define them properly soon.

In the Bessel condition (3.3), the sum can be defined as the supremum over all finite sums. Alternatively, one can choose any enumeration of Λ and define it as an ordinary limit. Non-negativity of the terms implies that the enumeration doesn't matter, as absolute convergence implies unconditional convergence. In Equation (3.5), we are summing functions in $L^2(\mathbb{R}^d)$, so a precise definition requires more care. We will define the infinite sum as a limit. In general, convergence will depend on the enumeration of Λ. However, we will only consider such sums when $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(h,\Lambda)$ are assumed to be Bessel sequences, and the upcoming Lemma 3.2.4 shows that the result is independent of the enumeration in this case.

In the following definition, $\ell^2(\Lambda)$ denotes the usual Hilbert space of functions $a\colon\Lambda\to\mathbb{C}$ (with $a_\lambda \coloneqq a(\lambda)$) such that $\|a\|_2^2=\sum_{\lambda\in\Lambda}|a_\lambda|^2<\infty$.

3.2.3 Definition (Analysis, synthesis and frame operators)**.** Let $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(h,\Lambda)$ be Bessel sequences. We define the following operators.

(i) The *analysis operator*:

$$\begin{aligned} C_g^\Lambda\colon L^2(\mathbb{R}^d)&\to\ell^2(\Lambda)\\ f&\mapsto\big(\langle f,\pi(\lambda)g\rangle\big)_{\lambda\in\Lambda}.\end{aligned}$$

(ii) The *synthesis operator*:

$$\begin{aligned} D_h^\Lambda\colon \ell^2(\Lambda)&\to L^2(\mathbb{R}^d)\\ (a_\lambda)_{\lambda\in\Lambda}&\mapsto\sum_{\lambda\in\Lambda}a_\lambda\pi(\lambda)h,\end{aligned}$$

where the sum is defined by an arbitrary enumeration of Λ (see Lemma 3.2.4 for well-definition).

(iii) The *mixed-type frame operator*:

$$\begin{aligned} S_{g,h}^{\Lambda} \coloneqq D_h^{\Lambda} \circ C_g^{\Lambda} \colon L^2(\mathbb{R}^d) &\to L^2(\mathbb{R}^d) \\ f &\mapsto \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)h. \end{aligned}$$

The condition that $\mathcal{G}(g,\Lambda)$ be a Bessel sequence is precisely the condition that the analysis operator C_g^{Λ} is well-defined and bounded. Indeed, we can rewrite the Bessel condition (3.3) as:

$$\|C_g^{\Lambda} f\|_2 \leq \sqrt{B}\|f\|_2 \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

The smallest possible Bessel bound B will therefore be the square of the operator norm $\|C_g^{\Lambda}\|$.

3.2.4 Lemma. *Let $\mathcal{G}(g,\Lambda)$ be a Bessel sequence. Then, D_g^{Λ} is well-defined (i.e. independent of enumeration of Λ) and it is the adjoint of C_g^{Λ}. That is, we have that*

$$\sum_{\lambda\in\Lambda} a_\lambda \overline{(C_g^{\Lambda} f)_\lambda} = \langle a, C_g^{\Lambda} f\rangle = \langle D_g^{\Lambda} a, f\rangle = \int_{\mathbb{R}^d} D_g^{\Lambda} a(t)\overline{f(t)}\,\mathrm{d}t$$

for all $a = (a_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ and $f \in L^2(\mathbb{R}^d)$. In particular, $\|D_g^{\Lambda}\| = \|C_g^{\Lambda}\|$.

Proof. Let $c_{00}(\Lambda) \subset \ell^2(\Lambda)$ denote the space of finite sequences, i.e. sequences with only finitely many nonzero terms. If $a \in c_{00}(\Lambda)$, then

$$\langle a, C_g^{\Lambda} f\rangle = \sum_{\lambda\in\Lambda} a_\lambda \overline{\langle f, \pi(\lambda)g\rangle} = \sum_{\lambda\in\Lambda} \langle a_\lambda \pi(\lambda)g, f\rangle = \Big\langle \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda)g, f\Big\rangle$$

since the sums are finite. Moreover, since $\sum_{\lambda\in\Lambda} a_\lambda\pi(\lambda)g \in L^2(\mathbb{R}^d)$, we now find that

$$\begin{aligned} \Big\|\sum_{\lambda\in\Lambda} a_\lambda\pi(\lambda)g\Big\|_2 &= \sup\Big\{\Big|\Big\langle\sum_{\lambda\in\Lambda} a_\lambda\pi(\lambda)g, f\Big\rangle\Big| : f \in L^2(\mathbb{R}^d) \text{ with } \|f\|_2 = 1\Big\} \\ &= \sup\Big\{\big|\langle a, C_g^{\Lambda} f\rangle\big| : f \in L^2(\mathbb{R}^d) \text{ with } \|f\|_2 = 1\Big\} \\ &\leq \|a\|_2\|C_g^{\Lambda}\|. \end{aligned}$$

This shows that the linear map $D\colon c_{00}(\Lambda) \to L^2(\mathbb{R}^d)$ defined by $a \mapsto \sum_{\lambda\in\Lambda} a_\lambda\pi(\lambda)g$ is bounded (with operator norm $\|D\| \leq \|C_g^{\Lambda}\|$). Since $c_{00}(\Lambda)$ is dense in $\ell^2(\Lambda)$, this implies that D extends to a bounded linear operator $\overline{D}\colon \ell^2(\Lambda) \to L^2(\mathbb{R}^d)$ (Theorem B.1.2).

Suppose that $(\Lambda_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of Λ such that $\bigcup_{n\in\mathbb{N}} \Lambda_n = \Lambda$. In particular, we could choose $\Lambda_n = \{\lambda_j\}_{j=1}^n$ for some enumeration $(\lambda_n)_{n\in\mathbb{N}}$ of Λ. For any $a \in \ell^2(\Lambda)$, we can now define a sequence $(a_n)_{n\in\mathbb{N}} \subset c_{00}(\Lambda)$ such that $a_n \to a$ by setting $(a_n)_\lambda = a_\lambda$ if $\lambda \in \Lambda_n$ and $(a_n)_\lambda = 0$ otherwise. By continuity of $\overline{D}$, we find that

$$\overline{D}a = \lim_{n\to\infty} Da_n = \lim_{n\to\infty} \sum_{\lambda\in\Lambda_n} a_\lambda \pi(\lambda) g.$$

This proves that the enumeration doesn't matter, as $\overline{D}a$ is independent of our choice of subsets Λ_n. We also see that $\overline{D} = D_g^\Lambda$. By continuity of the inner product, we find that

$$\langle D_g^\Lambda a, f\rangle = \lim_{n\to\infty} \langle Da_n, f\rangle = \lim_{n\to\infty} \langle a_n, C_g^\Lambda f\rangle = \langle a, C_g^\Lambda f\rangle,$$

which shows that D_g^Λ is the adjoint of C_g^Λ. The fact that $\|D_g^\Lambda\| = \|C_g^\Lambda\|$ now follows from a foundational result from Hilbert space theory (which we essentially proved in the beginning of Section 2.2), so we are done. □

3.2.5 Corollary. *Let $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(h,\Lambda)$ be Bessel sequences. Then, the mixed-type frame operator $S_{g,h}^\Lambda$ is bounded and $(S_{g,h}^\Lambda)^* = S_{h,g}^\Lambda$.*

Proof. By Lemma 3.2.4 and the paragraph that preceded it, C_g^Λ, C_h^Λ, D_g^Λ and D_h^Λ are all bounded, and

$$(S_{g,h}^\Lambda)^* = (D_h^\Lambda \circ C_g^\Lambda)^* = (C_g^\Lambda)^* \circ (D_h^\Lambda)^* = D_g^\Lambda \circ C_h^\Lambda = S_{g,h}^\Lambda,$$

as desired. □

In particular, the corollary we just proved shows that $S_{g,g}^\Lambda$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ whenever $\mathcal{G}(g,\Lambda)$ is a Bessel sequence. We now show that the frame condition can be phrased in terms of conditions on the frame operator. For the rest of this subsection, we will use I to denote the identity operator on $L^2(\mathbb{R}^d)$.

3.2.6 Proposition. *Let $\mathcal{G}(g,\Lambda)$ be a Bessel sequence. Then, the following statements are equivalent.*

(a) $\mathcal{G}(g,\Lambda)$ is a Gabor frame.

(b) There exist constants $A, B > 0$ such that $AI \le S_{g,g}^\Lambda \le BI$ in the C^*-algebra $\mathcal{B}(L^2(\mathbb{R}^d))$.

(c) The frame operator $S_{g,g}^\Lambda$ is invertible in $\mathcal{B}(L^2(\mathbb{R}^d))$.

Proof. By Corollary 3.2.5, $S^\Lambda_{g,g}$ is bounded and self-adjoint. Let $f \in L^2(\mathbb{R}^d)$. By continuity of the inner product, we have that

$$\begin{aligned}\langle S^\Lambda_{g,g} f, f\rangle &= \Big\langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g, f\Big\rangle = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\langle \pi(\lambda)g, f\rangle \\ &= \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2.\end{aligned} \tag{3.6}$$

Thus, the condition that $\mathcal{G}(g,\Lambda)$ be a Gabor frame may equivalently be written as: there exist constants $A, B > 0$ such that

$$\langle Af, f\rangle \le \langle S^\Lambda_{g,g} f, f\rangle \le \langle Bf, f\rangle \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

By Proposition 2.2.28, this is equivalent to the condition that both $BI - S^\Lambda_{g,g}$ and $S^\Lambda_{g,g} - AI$ are positive elements of $\mathcal{B}(L^2(\mathbb{R}^d))$. Since $S^\Lambda_{g,g}$ is self-adjoint, we may write this condition as $AI \le S^\Lambda_{g,g} \le BI$, which proves that (a) and (b) are equivalent.

We now wish to prove the equivalence of (b) and (c). We see from Equation (3.6) that $S^\Lambda_{g,g}$ is positive (appealing again to Proposition 2.2.28). This implies that $S^\Lambda_{g,g} \le \|S^\Lambda_{g,g}\| I$ by point (ii) of Proposition 2.2.26, so we only need to prove that invertibility of $S^\Lambda_{g,g}$ is equivalent to the existence of a constant $A > 0$ such that $AI \le S^\Lambda_{g,g}$.

We know that $\sigma(S^\Lambda_{g,g}) \subset [0,\infty)$. By definition of spectra, $S^\Lambda_{g,g}$ is invertible if and only if $0 \notin \sigma(S^\Lambda_{g,g})$. By compactness of spectra, this happens if and only if there exists some constant $A > 0$ such that $\sigma(S^\Lambda_{g,g}) \subset [A,\infty)$.

If we apply the continuous functional calculus at $S^\Lambda_{g,g}$, we find that $AI \le S^\Lambda_{g,g}$ (for any constant $A \in \mathbb{R}$) if and only if $A \le z$ as functions on $\sigma(S^\Lambda_{g,g})$ (point (iv) of Lemma 2.2.22; order preservation of the continuous functional calculus). Since $A \le z$ on $\sigma(S^\Lambda_{g,g})$ if and only if $\sigma(S^\Lambda_{g,g}) \subset [A,\infty)$, we can conclude that $AI \le S^\Lambda_{g,g}$ if and only if $\sigma(S^\Lambda_{g,g}) \subset [A,\infty)$. By the previous paragraph, we obtain the equivalence between (b) and (c). □

The following is an important lemma for the structure of the theory. We will soon see an application.

3.2.7 Lemma. *Let $\mathcal{G}(g,\Lambda)$ be a Bessel sequence. Then, the frame operator $S^\Lambda_{g,g}$ commutes with $\pi(\lambda)$ for every $\lambda \in \Lambda$.*

Proof. Since time-frequency shifts are unitary, the claimed commutation is equivalent to the condition that $\pi(\lambda)S^\Lambda_{g,g}\pi(\lambda)^* = S^\Lambda_{g,g}$ for all $\lambda \in \Lambda$. This is what we will prove.

Fix any $\lambda \in \Lambda$ and let $f \in L^2(\mathbb{R}^d)$. We find that

$$\begin{aligned}\big(\pi(\lambda)S^\Lambda_{g,g}\pi(\lambda)^*\big)f &= \pi(\lambda)\Big(\sum_{\mu\in\Lambda}\langle \pi(\lambda)^* f, \pi(\mu)g\rangle \pi(\mu)g\Big)\\ &= \sum_{\mu\in\Lambda}\langle f, \pi(\lambda)\pi(\mu)g\rangle \pi(\lambda)\pi(\mu)g\\ &= \sum_{\mu\in\Lambda}\langle f, \pi(\lambda+\mu)g\rangle \pi(\lambda+\mu)g = S^\Lambda_{g,g}f,\end{aligned}$$

where the transition to the last line follows from point (ii) of Lemma 3.1.1 (the phase factors that arise immediately cancel, since the inner product in conjugate-linear in the second entry). This proves our claim. □

Recall that we wish to identify lattices Λ and pairs of functions $g, h \in L^2(\mathbb{R}^d)$ such that

$$f = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle \pi(\lambda)h \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

Supposing that $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(h,\Lambda)$ are Bessel sequences, we can now write this condition simply as $S^\Lambda_{g,h} = I$. Since $(S^\Lambda_{g,h})^* = S^\Lambda_{h,g}$, we now know that this happens if and only if $S^\Lambda_{h,g} = I$. We will refer to the statement that $S^\Lambda_{g,h} = I$ by saying that g and h are *dual atoms.*

We now show that every Gabor frame gives rise to both a dual pair of atoms and a Parseval frame. This result crucially relies on the commutation property of the previous lemma.

3.2.8 Proposition (Canonical dual atom and Parseval frame)**.** *Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame. Then, the following statements are true.*

(i) Let $h := (S^\Lambda_{g,g})^{-1}g$. Then, $\mathcal{G}(h,\Lambda)$ is a Gabor frame and $S^\Lambda_{h,h} = (S^\Lambda_{g,g})^{-1}$. Moreover, g and h are dual atoms.

(ii) The Gabor system $\mathcal{G}((S^\Lambda_{g,g})^{-1/2}g,\Lambda)$ is a Parseval frame.

Proof. Throughout this proof, we will simply write S for $S^\Lambda_{g,g}$.

We begin with (i). We know that S is invertible by Proposition 3.2.6. As the inverse of a self-adjoint operator, S^{-1} is self-adjoint. By Lemma 3.2.7, we know that $S^{-1}\pi(\lambda) = \pi(\lambda)S^{-1}$ for all $\lambda \in \Lambda$. Thus, with $f \in L^2(\mathbb{R}^d)$ and $\lambda \in \Lambda$, we find that

$$\langle f, \pi(\lambda)h\rangle = \langle f, \pi(\lambda)S^{-1}g\rangle = \langle f, S^{-1}\pi(\lambda)g\rangle = \langle S^{-1}f, \pi(\lambda)g\rangle. \tag{3.7}$$

Now, the fact that $\mathcal{G}(g,\Lambda)$ is a Bessel sequence implies that

$$\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)h\rangle|^2 = \sum_{\lambda\in\Lambda} |\langle S^{-1}f, \pi(\lambda)g\rangle|^2 \leq B\|S^{-1}f\|^2 \leq B\|S^{-1}\|^2\|f\|^2$$

for all $f \in L^2(\mathbb{R}^d)$, where B is a Bessel bound for $\mathcal{G}(g,\Lambda)$. This proves that $\mathcal{G}(h,\Lambda)$ is a Bessel sequence as well. Thus, we know that the frame operator $S^{\Lambda}_{h,h}$ is well-defined and bounded.

Using Equation (3.7) again, along with the definitions of S and $S^{\Lambda}_{h,h}$, we find that

$$\begin{aligned}\langle S^{-1}f, f\rangle = \langle f, S^{-1}f\rangle = \langle S(S^{-1}f), (S^{-1}f)\rangle &= \sum_{\lambda\in\Lambda} |\langle S^{-1}f, \pi(\lambda)g\rangle|^2 \\ &= \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)h\rangle|^2 = \langle S^{\Lambda}_{h,h}f, f\rangle\end{aligned}$$

for all $f \in L^2(\mathbb{R}^d)$ (see Equation (3.6) for more details). By Lemma 2.2.27, this proves that $S^{\Lambda}_{h,h} = S^{-1}$ $(= (S^{\Lambda}_{g,g})^{-1})$. Thus, since S^{-1} is invertible, $\mathcal{G}(h,\Lambda)$ is a Gabor frame by Proposition 3.2.6.

Finally, for duality of g and h, we again appeal to Equation (3.7) and find that

$$f = S(S^{-1}f) = \sum_{\lambda\in\Lambda}\langle S^{-1}f, \pi(\lambda)g\rangle\pi(\lambda)g = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)h\rangle\pi(\lambda)g$$

for all $f \in L^2(\mathbb{R}^d)$, which concludes the proof of (i).

For (ii), recall that $S^{-1/2}$ denotes the unique positive square root of the positive operator S^{-1}, as afforded by Proposition 2.2.21 (by (i), we know that S^{-1} is a frame operator for a Gabor frame, so it is positive by Proposition 3.2.6). Since $S^{-1/2} \in C^*(S^{-1}) = C^*(S)$,[4] the operator $S^{-1/2}$ commutes with both S and $\pi(\lambda)$ for every $\lambda \in \Lambda$. Using these facts, we find that

$$\begin{aligned}\|f\|_2^2 = \langle f, f\rangle = \langle S(S^{-1/2}f), (S^{-1/2}f)\rangle &= \sum_{\lambda\in\Lambda} |\langle S^{-1/2}f, \pi(\lambda)g\rangle|^2 \\ &= \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)(S^{-1/2}g)\rangle|^2\end{aligned}$$

for every $f \in L^2(\mathbb{R}^d)$. This proves that $\mathcal{G}(S^{-1/2}g,\Lambda)$ is a Parseval frame and concludes the proof. □

[4] We didn't explicitly state or prove the equality $C^*(S^{-1}) = C^*(S)$ anywhere. However, using the fact that 0 is not in the spectrum of an invertible element, this is a straightforward consequence of the continuous functional calculus, as z^{-1} is continuous on $\mathbb{C}\backslash\{0\}$.

3.2.2 Trading Lattices for Symplectic Forms

Suppose we fix a lattice $\Lambda \subset \mathbb{R}^{2d}$. We now ask the question: for which $g \in L^2(\mathbb{R}^d)$ will $\mathcal{G}(g,\Lambda)$ be a Gabor frame? We will refer to the supposed answer to this question as "the structure of Gabor frames over Λ" or as "the structure of Gabor frames supported by Λ". This is an extremely difficult question in general; much is known, but there are many open questions. For example, it is known that $\mathrm{vol}(\Lambda) \leq 1$ is a necessary and sufficient condition for the existence of some $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,\Lambda)$ is a frame. Gröchenig and Koppensteiner [19] offer an excellent and accessible overview of the current state of knowledge.

We will not concern ourselves with questions of existence. Rather, our goal is to present a powerful framework/method for grouping lattices which support identical structures of Gabor frames. The purpose of this subsection is to outline the basic idea behind this framework.

Consider the set $\pi(\Lambda) \coloneqq \{\pi(\lambda) : \lambda \in \Lambda\} \subset \mathcal{B}(L^2(\mathbb{R}^d))$ of time-frequency shifts. We have the following basic commutation relation (restricted to the lattice):

$$\pi(\lambda)\pi(\mu) = e^{2\pi i \Omega_J(\lambda,\mu)}\pi(\mu)\pi(\lambda) \quad \text{for } \lambda,\mu \in \Lambda, \tag{3.8}$$

where Ω_J denotes the standard symplectic form on $\mathbb{R}^{2d}$, as defined by Equation (1.4). If we choose a lattice matrix $A \in \mathrm{GL}(2d,\mathbb{R})$ for Λ, then this becomes:

$$\pi(Ak)\pi(Al) = e^{2\pi i \Omega_J(Ak,Al)}\pi(Al)\pi(Ak) \quad \text{for } k,l \in \mathbb{Z}^{2d}.$$

Now, the map $(z,w) \mapsto \Omega_J(Az,Aw)$ is also a symplectic form on $\mathbb{R}^{2d}$. Indeed, it is simply the pullback $A^*\Omega_J$ of Ω_J by A. Let's introduce the notation

$$\pi_A(k) \coloneqq \pi(Ak) \quad \text{for } k \in \mathbb{Z}^{2d}.$$

Then, we can write the basic commutation relation as follows:

$$\pi_A(k)\pi_A(l) = e^{2\pi i (A^*\Omega_J)(k,l)}\pi_A(l)\pi_A(k) \quad \text{for } k,l \in \mathbb{Z}^{2d}. \tag{3.9}$$

All of this amounts to a change of basis for the time-frequency plane $\mathbb{R}^{2d}$ such that the lattice Λ is mapped to the *standard lattice* $\mathbb{Z}^{2d}$. We can think of this transition, from Equation (3.8) to Equation (3.9), as trading a general lattice for a general symplectic form. Loosely stated, our goal is to prove the following statement:

> The structure of Gabor frames over a lattice $\Lambda = A\mathbb{Z}^{2d}$ depends *only* on the symplectic from $A^*\Omega_J$.

Consider the mapping[5]

$$(\Omega_J)_*\colon \mathrm{GL}(2d,\mathbb{R}) \to \mathrm{Symp}(\mathbb{R}^{2d}) := \{\text{Symplectic forms on } \mathbb{R}^{2d}\} \\ A \mapsto A^*\Omega_J. \tag{3.10}$$

We will say that A *determines* $A^*\Omega_J$. Note that a matrix *representing* a symplectic form is different from a matrix *determining* a symplectic form; if A determines Ω, then A^TJA represents Ω, and if $\theta \in \mathcal{S}_{2d}$ represents Ω, then any matrix $A \in \mathrm{GL}(2d,\mathbb{R})$ such that $A^TJA = \theta$ determines Ω (see Proposition 1.1.8). Under the isomorphism $\mathrm{Symp}(\mathbb{R}^{2d}) \cong \mathcal{S}_{2d}$ (which maps a symplectic form to the matrix representing it), this mapping becomes $A \mapsto A^TJA$. We may also identify θ with Ω_θ and say that A *determines* $\theta = A^TJA$.

Our claim is that the mapping (3.10) isolates the property of $A\mathbb{Z}^{2d}$ that determines the structure of Gabor frames over $A\mathbb{Z}^{2d}$. The usefulness of this result stems from the fact that this mapping is many-to-one. Moreover, it is simple to give a precise description of the resulting classification:

3.2.9 Proposition. *Let $A, B \in \mathrm{GL}(2d,\mathbb{R})$. Then, the following statements are equivalent.*

(a) A and B determine the same symplectic form: $A^*\Omega_J = B^*\Omega_J$

(b) We have $B \in \mathrm{Sp}(2d,\mathbb{R})A$.

(c) With $\Omega := A^*\Omega_J$, we have $B \in A\mathrm{Sp}_\Omega(2d,\mathbb{R})$.

Proof. The statement that $A^*\Omega_J = B^*\Omega_J$ is equivalent to:

$$w^T(A^TJA)z = w^T(B^TJB)z \quad \text{for all } z, w \in \mathbb{R}^{2d},$$

which is equivalent to $A^TJA = B^TJB$. In turn, this is equivalent to:

$$J = A^{-T}(B^TJB)A^{-1} = (BA^{-1})^TJ(BA^{-1}),$$

i.e. the statement that $BA^{-1} \in \mathrm{Sp}(2d,\mathbb{R})$ (or, equivalently: $B \in \mathrm{Sp}(2d,\mathbb{R})A$). This gives the equivalence of (a) and (b).

Finally, by Lemma 1.2.3, we have that

$$A\mathrm{Sp}_\Omega(2d,\mathbb{R}) = \mathrm{Sp}(2d,\mathbb{R})A, \quad \text{where } \Omega = A^*\Omega_J$$

This proves that (b) is equivalent to (c). □

[5]To the degree that the map $\Omega_J \mapsto A^*\Omega_J$ is pre-composition with A (see the discussion following Definition 1.1.6), the map $A \mapsto A^*\Omega_J$ is post-composition with Ω_J. This is our reason for choosing the notation $(\Omega_J)_*$, as it is commonly used for post-composition.

In other words, the foregoing proposition shows that, for any $A \in \mathrm{GL}(2d,\mathbb{R})$, we have that

$$(\Omega_J)_*^{-1}(A^*\Omega_J) = A\mathrm{Sp}_\Omega(2d,\mathbb{R}) = \mathrm{Sp}(2d,\mathbb{R})A.$$

That is, we can describe the set of matrices determining the same symplectic form as A both as a left coset of $\mathrm{Sp}_\Omega(2d,\mathbb{R})$ in $\mathrm{GL}(2d,\mathbb{R})$ and as a right coset of $\mathrm{Sp}(2d,\mathbb{R})$. In yet other words, $(\Omega_J)_*$ groups lattice matrices by their orbits under the symplectic group (acting from the left).

There is a slight complication which we wish to address. If $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$ for some $A, B \in \mathrm{GL}(2d,\mathbb{R})$, it does *not* follow that $A^*\Omega_J = B^*\Omega_J$. That is, even though a *lattice matrix* determines a unique symplectic form, a *lattice* by itself does not. However, it is also simple to characterize the set of symplectic forms determined by a fixed lattice:

3.2.10 Proposition. *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ and set $\Lambda = A\mathbb{Z}^{2d}$. Then,*

$$(\Omega_J)_*\Big(\big\{B \in \mathrm{GL}(2d,\mathbb{R}) : B\mathbb{Z}^{2d} = \Lambda\big\}\Big) = \big\{R^*(A^*\Omega_J) : R \in \mathrm{GL}(2d,\mathbb{Z})\big\}.$$

Moreover, with $\theta := A^T JA$, this is precisely the set of forms

(a) represented by $\{R^T\theta R \in \mathcal{S}_{2d} : R \in \mathrm{GL}(2d,\mathbb{Z})\}$, and

(b) equivalently: determined by $\{AR : R \in \mathrm{GL}(2d,\mathbb{Z})\} = A\mathrm{GL}(2d,\mathbb{Z})$.

Proof. By Lemma 3.2.1, we have that

$$\big\{B \in \mathrm{GL}(2d,\mathbb{R}) : B\mathbb{Z}^{2d} = \Lambda\big\} = \big\{AR : R \in \mathrm{GL}(2d,\mathbb{Z})\big\} = A\mathrm{GL}(2d,\mathbb{Z}),$$

from which $(\Omega_J)_*(AR) = (AR)^*\Omega_J = R^*(A^*\Omega_J)$ give the inclusion $\subset$ (for the displayed equation of sets in the proposition). For the converse inclusion, suppose that $\Omega = R^*(A^*\Omega_J)$ for some $R \in \mathrm{GL}(2d,\mathbb{Z})$ and set $B := AR$. This means that

$$\Omega = R^*(A^*\Omega_J) = (AR)^*\Omega_J = B^*\Omega_J = (\Omega_J)_*(B),$$

and since $R \in \mathrm{GL}(2d,\mathbb{Z})$, we have that $B\mathbb{Z}^{2d} = (AR)\mathbb{Z}^{2d} = A\mathbb{Z}^{2d} = \Lambda$. This proves the displayed equality of sets in the proposition. The first sentence of the proof now implies (b), and the simple equality $R^T\theta R = (AR)^T J(AR)$ gives the equivalence between (a) and (b). $\square$

We will now describe what we mean by the statement that the structure of Gabor frames over a lattice $\Lambda = A\mathbb{Z}^{2d}$ depends only on the symplectic from $A^*\Omega_J$. It turns out that if $A, B \in \mathrm{GL}(2d,\mathbb{R})$ are such that $A^*\Omega_J = B^*\Omega_J$, then there exists a unitary map $U \in \mathcal{B}(L^2(\mathbb{R}^d))$ such that $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Gabor frame if and only if $\mathcal{G}(Ug, B\mathbb{Z}^{2d})$ is a Gabor frame. Moreover, the mixed-type frame operators are related by[6]

$$S_{Ug,Uh}^{B\mathbb{Z}^{2d}} = US_{g,h}^{A\mathbb{Z}^{2d}}U^{-1}.$$

Thus, this correspondence is as precise as one could hope for. The maps U will be metaplectic transformations, which we will introduce in Chapter 6.

Now, this is not an original result. The case where $A = I$ (and hence $A^*\Omega_J = \Omega_J$) is included in Gröchenig's textbook [17, Proposition 9.4.4 on p. 199], and the general case is included in de Gosson's textbook [15, Proposition 163 on p. 113]. We will prove a slight extension of de Gosson's result in Chapter 6 (see Theorem 6.1.10). However, to the knowledge of the author and his advisor Franz Luef, this point of view has not been thoroughly explored in the literature on Gabor frames.

Moreover, Luef [21, 22] has initiated the use of equivalence bimodules for the study of Gabor frames, and we also wish to show this correspondence can be lifted to the setting of such bimodules. These bimodules are particular kinds of Hilbert C*-modules over noncommutative tori. Much of this thesis will be spent introducing these notions and explicitly constructing the relevant bimodules. Our goal is to show that if two such bimodules arise from lattice matrices which determine the same symplectic form, then they are isomorphic as bimodules. These isomorphisms will be extensions of metaplectic transformations. The idea of exploring in this direction has been suggested by Chakraborty and Luef in [8], where they are working with the same structures in a different setting and proving part of the desired result.

Another avenue of exploration afforded by this correspondence is the use of symplectic methods for Gabor analysis. We will not pursue this in any depth. However, we wish to state and prove one preliminary result showing how these subjects are related.

Some classes of lattices are easier to work with than others in Gabor theory. Particularly convenient are lattices of the form

$$\Lambda = A_1\mathbb{Z}^d \times A_2\mathbb{Z}^d \quad \text{for } A_1, A_2 \in \mathrm{GL}(d, \mathbb{R}).$$

These are called *separable* lattices. Many results in Gabor theory have been proven only for separable lattices. Again, we refer to Gröchenig and Koppensteiner [19, Section 4.6] for examples of such results. The following proposition characterizes those symplectic forms which are determined by separable lattices. One characterization is quite abstract and formulated in

[6] Don't worry; we will clean up our notation soon.

terms of symplectic geometry, while the other describes the explicit form of matrices representing such forms (and is a quite immediate consequence of Proposition 3.2.10). The idea of seeking a result of this form was suggested to the author by his advisor. To our knowledge, this is a novel result.

3.2.11 Proposition. *Let $\theta \in \mathcal{S}_{2d}$ and let Ω_θ be the symplectic form it represents (Definition 1.1.3). Then, the following statements are equivalent.*

(a) There exists some $A \in \mathrm{GL}(2d, \mathbb{R})$ such that $\Omega_\theta = A^*\Omega_J$ and

$$A\mathbb{Z}^{2d} = A_1\mathbb{Z}^d \times A_2\mathbb{Z}^d \quad \text{for some } A_1, A_2 \in \mathrm{GL}(d, \mathbb{R}).$$

(b) There exists a basis $\{b_1, \ldots, b_{2d}\}$ for $\mathbb{Z}^{2d}$ (as an abelian group) and an Ω_θ-polarization (L, L') of $\mathbb{R}^{2d}$ such that

$$\{b_1, \ldots, b_{2d}\} \subset L \cup L'.$$

(c) The matrix θ is of the form

$$\theta = R^T \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix} R$$

for some $K \in \mathrm{GL}(d, \mathbb{R})$ and $R \in \mathrm{GL}(2d, \mathbb{Z})$.

Proof. We will show the following chain of implications: (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). As always, $\{e_j\}_{j=1}^{2d}$ will denote the standard basis for $\mathbb{R}^{2d}$. We define

$$L_x := \big\{(x, 0) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d\big\} \quad \text{and} \quad L_\omega := \big\{(0, \omega) \in \mathbb{R}^{2d} : \omega \in \mathbb{R}^d\big\},$$

so that (L_x, L_ω) is a polarization of $\mathbb{R}^{2d}$.

Assume (a) holds, so that $A\mathbb{Z}^{2d} = A_1\mathbb{Z}^d \times A_2\mathbb{Z}^d = (A_1 \oplus A_2)\mathbb{Z}^{2d}$. The matrix $K' \in \mathcal{S}_{2d}$ representing $(A_1 \oplus A_2)^*\Omega_J$ is given by

$$\begin{aligned} K' = (A_1 \oplus A_2)^T J (A_1 \oplus A_2) &= \begin{pmatrix} A_1^T & 0 \\ 0 & A_2^T \end{pmatrix} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & A_1^T A_2 \\ -A_2^T A_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix} \quad \text{with } K := A_1^T A_2 \in \mathrm{GL}(d, \mathbb{R}). \end{aligned}$$

Thus, since $A\mathbb{Z}^{2d} = (A_1 \oplus A_2)\mathbb{Z}^{2d}$, (c) follows from Proposition 3.2.10. This proves that (a) implies (c).

Assume now that (c) holds. We will show that (b) follows. Set

$$K' := \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}.$$

Clearly L_x and L_ω are $\Omega_{K'}$-Lagrangian planes. Since $\theta = R^T K' R$, this implies that $L := R^{-1}(L_x)$ and $L' := R^{-1}(L_\omega)$ are Ω_θ-Lagrangian.

Since L_x and L_ω are transversal, it follows that L and L' are transversal as well:

$$\mathbb{R}^{2d} = R^{-1}(\mathbb{R}^{2d}) = R^{-1}(L_x \oplus L_\omega) = R^{-1}(L_x) \oplus R^{-1}(L_\omega) = L \oplus L'.$$

In other words, (L, L') is an Ω_θ-polarization of $\mathbb{R}^{2d}$. Since $R \in \mathrm{GL}(2d, \mathbb{Z})$, we find that

$$b_j := R^{-1} e_j \quad \text{for } 1 \leq j \leq 2d$$

defines a basis for $\mathbb{Z}^{2d}$. Moreover, we have that

$$b_j = R^{-1} e_j \in R^{-1}(L_x \cup L_\omega) = R^{-1}(L_x) \cup R^{-1}(L_\omega) = L \cup L'$$

for $1 \leq j \leq 2d$. Thus, we have shown that (c) implies (b).

Finally, we need to show that (b) implies (a), so assume that (b) holds. Since $\{b_1, \ldots, b_{2d}\}$ is a basis for $\mathbb{R}^{2d} = L \oplus L'$ that is contained in $L \cup L'$, and L and L' are both d-dimensional, there must exist some permutation $\sigma \in \mathrm{Sym}(2d)$ (the symmetric group on $2d$ elements) such that

$$L = \operatorname{span}_{\mathbb{R}}\big\{b_{\sigma(1)}, \ldots, b_{\sigma(d)}\big\} \quad \text{and} \quad L' = \operatorname{span}_{\mathbb{R}}\big\{b_{\sigma(d+1)}, \ldots, b_{\sigma(2d)}\big\}. \tag{3.11}$$

Choose $B \in \mathrm{GL}(2d, \mathbb{R})$ such that $\theta = B^T J B$, i.e. $\Omega_\theta = B^* \Omega_J$ (Proposition 1.1.8). This implies that

$$\Omega_\theta(b_{\sigma(i)}, b_{\sigma(j)}) = \Omega_J(Bb_{\sigma(i)}, Bb_{\sigma(j)}) \quad \text{for } 1 \leq j \leq 2d.$$

Since L and L' are Ω_θ-Lagrangian planes, this shows that $B(L)$ and $B(L')$ are Ω_J-Lagrangian (i.e. just Lagrangian). Since B is invertible and L and L' are transversal, $B(L)$ and $B(L')$ are transversal as well. Thus, $(B(L), B(L'))$ is a polarization of $\mathbb{R}^{2d}$.

By Lemma 1.2.7, there exists some $S \in \mathrm{Sp}(2d, \mathbb{R})$ such that

$$SB(L) = L_x \quad \text{and} \quad SB(L') = L_\omega.$$

By Equation (3.11), this implies that

$$\begin{aligned} L_x &= SB(L) = \operatorname{span}_{\mathbb{R}}\big\{SBb_{\sigma(1)}, \ldots, SBb_{\sigma(d)}\big\} \\ \text{and} \quad L_\omega &= SB(L') = \operatorname{span}_{\mathbb{R}}\big\{SBb_{\sigma(d+1)}, \ldots, SBb_{\sigma(2d)}\big\}. \end{aligned}$$

We can now find change-of-basis matrices $A_1, A_2 \in \mathrm{GL}(d, \mathbb{R})$ such that

$$e_j = (A_1^{-1} \oplus A_2^{-1}) SBb_{\sigma(j)} \quad \text{for } 1 \leq j \leq 2d.$$

Since both $\{e_1, \ldots, e_{2d}\}$ and $\{b_{\sigma(1)}, \ldots, b_{\sigma(2d)}\}$ are bases for $\mathbb{Z}^{2d}$, we must have that

$$R \coloneqq (A_1^{-1} \oplus A_2^{-1}) SB \in \mathrm{GL}(2d, \mathbb{Z}).$$

Since S is symplectic, we also know that $(SB)^T J (SB) = B^T J B = \theta$. If we now set $A \coloneqq SB \in \mathrm{GL}(2d, \mathbb{R})$, then $\Omega_\theta = A^* \Omega_J$ and

$$A\mathbb{Z}^{2d} = SB\mathbb{Z}^{2d} = (A_1 \oplus A_2) R\mathbb{Z}^{2d} = (A_1 \oplus A_2)\mathbb{Z}^{2d} = A_1\mathbb{Z}^d \times A_2\mathbb{Z}^d$$

(since $(A_1 \oplus A_2)R = SB$). This shows that (b) implies (a) and concludes the proof. □

Before we close this subsection, we introduce some notation that is better suited to our needs. Given a lattice matrix $A \in \mathrm{GL}(2d, \mathbb{R})$ and windows $g, h \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g, A\mathbb{Z}^{2d})$ and $\mathcal{G}(h, A\mathbb{Z}^{2d})$ are Bessel sequences, we define the following variants of the synthesis and analysis operators:

$$\begin{aligned} C_g^A \colon L^2(\mathbb{R}^d) &\to \ell^2(\mathbb{Z}^{2d}) & \qquad D_h^A \colon \ell^2(\mathbb{Z}^{2d}) &\to L^2(\mathbb{R}^d) \\ f &\mapsto \big(\langle f, \pi_A(k) g\rangle\big)_{k \in \mathbb{Z}^{2d}} & (a_k)_{k\in\mathbb{Z}^{2d}} &\to \sum_{k \in \mathbb{Z}^{2d}} a_k \pi_A(k) h \end{aligned}$$

as well as their composition $S_{g,h}^A \coloneqq D_h^A \circ C_g^A$. If we set $\Lambda = A\mathbb{Z}^{2d}$, then these are just the operators C_g^Λ and D_h^Λ up to an isomorphisms between $\ell^2(\Lambda)$ and $\ell^2(\mathbb{Z}^{2d})$, and so $S_{g,h}^A = S_{g,h}^\Lambda$. In particular, all the results of Subsection 3.2.1 hold for these operators as well.

3.2.3 The Feichtinger Algebra and the FIGA

In this subsection we introduce a space of particularly well-behaved windows, namely the Feichtinger algebra. This space was first introduced in 1981 by Feichtinger [12]. It will be central to our construction of Hilbert C*-modules over noncommutative tori.

Throughout this thesis, g_0 will always denote the normalized Gaussian on $\mathbb{R}^d$. That is, $g_0(t) = 2^{d/4} e^{-\pi t^2}$ for all $t \in \mathbb{R}^d$.

3.2.12 Definition. We define *the Feichtinger algebra*:

$$S_0(\mathbb{R}^d) = \big\{ f \in L^2(\mathbb{R}^d) : V_{g_0} f \in L^1(\mathbb{R}^{2d}) \big\},$$

and equip it with the norm $\|f\|_{S_0, g_0} \coloneqq \|V_{g_0} f\|_1$.

We quickly argue that $\|\cdot\|_{S_0,g_0}$ actually defines a norm on $S_0(\mathbb{R}^d)$. Since $f \mapsto V_{g_0}f$ is a linear mapping, $S_0(\mathbb{R}^d)$ is a vector space, and homogeneity and the triangle inequality are immediate. By Moyal's identity (Proposition 3.1.6), we have that

$$\|V_{g_0}f\|_2 = \|g_0\|_2\|f\|_2 \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

If $\|f\|_{S_0,g_0} = \|V_{g_0}f\|_1 = 0$, then $V_{g_0}f = 0$ almost everywhere on $\mathbb{R}^d$, so $\|V_{g_0}f\|_2 = 0$ as well. Thus, by the displayed equation, we obtain $\|f\|_2 = 0$. This gives nondegeneracy, so $\|\cdot\|_{S_0,g_0}$ is indeed a norm.

The following lemma shows that, once $S_0(\mathbb{R}^d)$ has been defined, the Gaussian no longer plays a privileged role.

3.2.13 Lemma (Equivalent norms). *Let $g \in S_0(\mathbb{R}^d)$ be arbitrary but nonzero. Then, there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|V_{g_0}f\|_1 \le \|V_gf\|_1 \le C_2\|V_{g_0}f\|_1 \quad \textit{for all } f \in L^2(\mathbb{R}^d).$$

Thus, the norms $\|\cdot\|_{S_0,g_0}$ and $f \mapsto \|f\|_{S_0,g} := \|V_gf\|_1$ are equivalent on $S_0(\mathbb{R}^d)$.

Proof. Fix any $f \in L^2(\mathbb{R}^d)$ and $z \in \mathbb{R}^{2d}$. Moyal's identity and the covariance property of Lemma 3.1.4 implies that

$$\begin{aligned} |\overline{\langle g, g\rangle}\langle f, \pi(z)g_0\rangle| &= |\langle V_gf, V_g\pi(z)g_0\rangle| \\ &\le \int_{\mathbb{R}^{2d}} |V_gf(w)||V_g\pi(z)g_0(w)|\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} |V_gf(w)||V_gg_0(w-z)|\,\mathrm{d}w. \end{aligned}$$

If we now integrate over z (and change the order of integration, which is permitted by Tonelli's theorem), we find that

$$\|g\|_2^2\|V_{g_0}f\|_1 \le \|V_gf\|_1\|V_gg_0\|_1$$

With $C_1 := \|g\|_2^2/\|V_gg_0\|_1$ ($V_gg_0 \neq 0$ by Moyal's identity), this gives the inequality $C_1\|V_{g_0}f\|_1 \le \|V_gf\|_1$. The same exact argument with the roles of g and g_0 interchanged gives the other inequality and concludes the proof. $\square$

The Feichtinger algebra turns out be a very well-behaved and natural space of functions for time-frequency analysis. In fact, it is a Banach space consisting entirely of continuous functions and it is closed under both convolutions and pointwise products. We will have no need for these facts, so we refer the interested reader to Gröchenig [17, Chapter 11]. Moreover, $S_0(\mathbb{R}^d)$ is invariant under the Fourier transform and $\mathscr{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d)$ (which we will show). As a Banach space of continuous functions, $S_0(\mathbb{R}^d)$ provides a simplifying alternative to $\mathscr{S}(\mathbb{R}^d)$ for the study of distributions with a well-defined Fourier transform.

3.2.14 Lemma. *The following statements are true.*

(i) $S_0(\mathbb{R}^d)$ *is mapped to itself by the Fourier transform.*

(ii) $S_0(\mathbb{R}^d)$ *is mapped to itself by all time-frequency shifts.*

(iii) We have the inclusion $\mathscr{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d)$. *In particular,* $S_0(\mathbb{R}^d)$ *is dense in* $L^2(\mathbb{R}^d)$.

Proof. Fix any $f \in S_0(\mathbb{R}^d)$ and $z \in \mathbb{R}^{2d}$. By point (i) of Lemma 3.1.4, we see that

$$|(V_{g_0}\pi(z)f)(w)| = |V_{g_0}f(w-z)| \quad \text{for all } w \in \mathbb{R}^{2d}.$$

Integrating over w, we find that $\|\pi(z)f\|_{S_0,g_0} = \|V_{g_0}\pi(z)f\|_1 = \|V_{g_0}f\|_1 = \|f\|_{S_0,g_0}$. Thus, $\pi(z)f \in S_0(\mathbb{R}^d)$, which proves (i).

For (ii), recall that $\mathcal{F}g_0 = g_0$ (this is a basic result in Fourier theory). Point (ii) of Lemma 3.1.4 implies that

$$|V_{g_0}f(x,\omega)| = |V_{\mathcal{F}g_0}\mathcal{F}f(\omega,-x)| = |V_{g_0}\mathcal{F}f(\omega,-x)| \quad \text{for all } (x,\omega) \in \mathbb{R}^{2d}.$$

Thus, $\|\mathcal{F}f\|_{S_0,g_0} = \|V_{g_0}\mathcal{F}f\|_1 = \|V_{g_0}f\|_1 = \|f\|_{S_0,g_0}$, so $\mathcal{F}f \in S_0(\mathbb{R}^d)$ as well, proving (ii).

Finally, for (iii), consider the description $V_g f = \mathcal{F}_2 T^t(f \otimes \overline{g})$ of the STFT introduced just prior to Proposition 3.1.6. If $f, g \in \mathscr{S}(\mathbb{R}^d)$, it is straightforward to check that $f \otimes \overline{g} \in \mathscr{S}(\mathbb{R}^{2d})$. Similarly, one checks that $\mathscr{S}(\mathbb{R}^{2d})$ is invariant under both T^t and $\mathcal{F}_2$ (the latter holds for the same reason that $\mathscr{S}(\mathbb{R}^{2d})$ is invariant under the usual Fourier transform). Thus, if $f, g \in \mathscr{S}(\mathbb{R}^d)$, then $V_g f \in \mathscr{S}(\mathbb{R}^{2d})$. Clearly $g_0 \in \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^{2d}) \subset L^1(\mathbb{R}^{2d})$, so (iii) follows. $\square$

We now introduce a notion of great importance, namely that of the adjoint lattice. This is the notion underlying the duality inherent to Gabor theory, which we will discuss more in the next subsection. The FIGA, which is right around the corner, is also a manifestation of this duality.

3.2.15 Lemma. *Let* $A \in \mathrm{GL}(2d,\mathbb{R})$ *and consider the lattice* $\Lambda = A\mathbb{Z}^{2d}$. *If we define*

$$\Lambda^\circ := \big\{ z \in \mathbb{R}^{2d} : \pi(\lambda)\pi(z) = \pi(z)\pi(\lambda) \textit{ for all } \lambda \in \Lambda \big\}, \tag{3.12}$$

then $\Lambda^\circ = -JA^{-T}\mathbb{Z}^{2d}$.

Proof. By the basic commutation relation, we see that $z \in \Lambda^\circ$ if and only if $e^{2\pi i \Omega_J(z,\lambda)} = 1$ for all $\lambda \in \Lambda$, which is equivalent to the condition that $\Omega_J(z,\lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$. Since $\Lambda = A\mathbb{Z}^{2d}$, this, in turn, is equivalent to:

$$k^T(A^T J)z = (Ak)^T Jz = \Omega_J(z, Ak) \in \mathbb{Z} \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

Taking k to be standard basis elements, we see that this happens if and only if $(A^T J)z \in \mathbb{Z}^{2d}$, or, equivalently, if and only if $z \in (A^T J)^{-1}\mathbb{Z}^{2d} = -JA^{-T}\mathbb{Z}^{2d}$. □

3.2.16 Definition (The adjoint lattice)**.** For a lattice $\Lambda \subset \mathbb{R}^{2d}$, we define the *adjoint lattice* Λ° by Equation (3.12). Similarly, for a lattice matrix $A \in \mathrm{GL}(2d, \mathbb{R})$, we define the *adjoint lattice matrix* $A^\circ := -JA^{-T}$. Thus, by Lemma 3.2.15: $\Lambda = A\mathbb{Z}^{2d} \implies \Lambda^\circ = A^\circ\mathbb{Z}^{2d}$.

Our main goal for the remainder of this subsection is to prove the following identity, referred to as the *fundamental identity of Gabor analysis*, or the *FIGA* for short.

3.2.17 Theorem (The fundamental identity of Gabor analysis)**.** *Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and let $f_1, f_2, g_1, g_2 \in S_0(\mathbb{R}^d)$. Then,*

$$\sum_{k\in\mathbb{Z}^{2d}} V_{g_1}f_1(Ak)\overline{V_{g_2}f_2(Ak)} = \frac{1}{|\det A|}\sum_{k\in\mathbb{Z}^{2d}} V_{g_1}g_2(A^\circ k)\overline{V_{f_1}f_2(A^\circ k)},$$

with absolute convergence of both sums.

Following Feichtinger and Luef [13], we will prove Theorem 3.2.17 via symplectic variants of the Fourier transform and the Poisson summation formula.

3.2.18 Definition (The symplectic Fourier transform)**.** Let $F \in L^1(\mathbb{R}^{2d})$. We define the *symplectic Fourier transform* $\mathcal{F}^s F$ of F by

$$\mathcal{F}^s F(z) = \int_{\mathbb{R}^{2d}} F(w)e^{2\pi i \Omega_J(z,w)}\,\mathrm{d}w = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(y,\eta)e^{2\pi i(\omega\cdot y - \eta\cdot x)}\,\mathrm{d}y\,\mathrm{d}\eta$$

for all $z = (x, \omega) \in \mathbb{R}^{2d}$.

Since $\Omega_J(z, w) = w^T Jz = w \cdot (Jz)$, we see that the symplectic Fourier transform of F is given by $\mathcal{F}^s F(z) = \mathcal{F}F(-Jz)$, where $\mathcal{F}F$ is the ordinary Fourier transform of F and J is the standard symplectic matrix.

Unlike the ordinary Fourier transform on $\mathbb{R}^{2d}$, the symplectic Fourier transform (extended/restricted to $L^2(\mathbb{R}^{2d})$) is its own inverse: if $F, \mathcal{F}F \in L^1(\mathbb{R}^{2d})$, then,

$$\begin{aligned}(\mathcal{F}^s)^2 F(z) = \mathcal{F}(\mathcal{F}^s F)(-Jz) &= \int_{\mathbb{R}^{2d}} \mathcal{F}^s F(w) e^{-2\pi i w\cdot(-Jz)}\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} \mathcal{F}^s F(w) e^{2\pi i(-Jw)\cdot z}\,\mathrm{d}w = \int_{\mathbb{R}^{2d}} \mathcal{F}^s F(Jw') e^{2\pi i w'\cdot z}\,\mathrm{d}w' \\ &= \int_{\mathbb{R}^{2d}} \mathcal{F}F(w') e^{2\pi i w'\cdot z}\,\mathrm{d}w',\end{aligned}$$

which equals $F(z)$ via the usual Fourier inversion formula. Our main motivation for introducing the symplectic Fourier transform is the following result.

3.2.19 Lemma. *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then,*

$$\mathcal{F}^s(V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}) = V_{g_1}g_2 \cdot \overline{V_{f_1}f_2}$$

pointwise.

Proof. Since $V_{g_1}f_1, V_{g_2}f_2 \in L^2(\mathbb{R}^{2d})$ by Moyal's identity (Proposition 3.1.6), we know that $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2} \in L^1(\mathbb{R}^{2d})$. Let $z \in \mathbb{R}^{2d}$. We find that

$$\begin{aligned}\mathcal{F}^s(V_{g_1}f_1 \cdot \overline{V_{g_2}f_2})(z) &= \int_{\mathbb{R}^{2d}} \langle f_1, \pi(w)g_1\rangle \overline{\langle f_2, \pi(w)g_2\rangle} e^{2\pi i \Omega_J(z,w)}\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} \langle \pi(z)f_1, \pi(z)\pi(w)g_1\rangle \overline{\langle f_2, \pi(w)g_2\rangle} e^{2\pi i \Omega_J(z,w)}\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} \langle \pi(z)f_1, \pi(w)\pi(z)g_1\rangle \overline{\langle f_2, \pi(w)g_2\rangle}\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} \big(V_{\pi(z)g_1}\pi(z)f_1\big)(w) \overline{V_{g_2}f_2(w)}\,\mathrm{d}w \\ &= \overline{\langle \pi(z)g_1, g_2\rangle} \langle \pi(z)f_1, f_2\rangle = \big(V_{g_1}g_2 \cdot \overline{V_{f_1}f_2}\big)(z),\end{aligned}$$

where we have used the basic commutation relations of Lemma 3.1.1 and Moyal's identity (Proposition 3.1.6). $\square$

Before introducing the symplectic Poisson summation formula, we prove the ordinary Poisson summation formula. There are many different variants of this result, with differing assumptions and notions of convergence; we will prove a variant that is convenient for our proof of the FIGA.

For any subset $S \subset \mathbb{R}^{2d}$, we will write χ_S for the characteristic function of S. That is, $\chi_S(z) = 1$ if $z \in S$ and $\chi_S(z) = 0$ otherwise.

3.2.20 Proposition (The Poisson summation formula)**.** *Let $F \in L^1(\mathbb{R}^{2d})$ be a continuous function. Assume that there exists some $(c_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that*

$$|F(z)| \leq \sum_{k\in\mathbb{Z}^{2d}} |c_k| \chi_{k+[0,1)^{2d}}(z) \quad \textit{for all } z \in \mathbb{R}^{2d}, \tag{3.13}$$

and that $\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}F(k)| < \infty$. Then,

$$\sum_{k\in\mathbb{Z}^{2d}} F(z+k) = \sum_{k\in\mathbb{Z}^{2d}} \mathcal{F}F(k) e^{2\pi i z\cdot k} \quad \textit{for all } z \in [0,1]^{2d},$$

with absolute pointwise convergence of both sums.

Proof. Fix any $k \in \mathbb{Z}^{2d}$. Since F is continuous and $|F(z)| \leq |c_k|$ for all $z \in k+[0,1)^{2d}$, we have that $|F(z)| \leq |c_k|$ for all $z \in k+[0,1]^{2d}$. Equivalently:

$$|F(z+k)| \leq |c_k| \quad \text{for all } z \in [0,1]^{2d}.$$

Let $(k_j)_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^{2d}$ and fix any $\epsilon > 0$. Since $\sum_{k\in\mathbb{Z}^{2d}} |c_k| < \infty$, there exists some $N \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{n} |c_{k_j}| < \epsilon, \quad \text{whenever } n > m \geq N.$$

By our observation in the previous paragraph, we find that

$$\left| \sum_{j=1}^{n} |F(z+k_j)| - \sum_{i=1}^{m} |F(z+k_i)| \right| = \sum_{j=m+1}^{n} |F(z+k_j)| \leq \sum_{j=m+1}^{n} |c_{k_j}| < \epsilon$$

for all $z \in [0,1]^{2d}$ and $n > m \geq N$. This shows that $\sum_{k\in\mathbb{Z}^{2d}} |F(z+k)|$ is pointwise Cauchy (for any enumeration of $\mathbb{Z}^{2d}$) and hence pointwise convergent. Taking the limit $n \to \infty$ in the last displayed equation, we also see that the convergence is uniform on $[0,1]^{2d}$. Since each term $|F(z+k_j)|$ is continuous on $[0,1]^{2d}$, the uniform convergence implies that $\sum_{k\in\mathbb{Z}^{2d}} |F(z+k)|$ is a continuous function on $[0,1]^{2d}$. Since

$$\left| \sum_{j=1}^{n} F(z+k_j) - \sum_{i=1}^{m} F(z+k_i) \right| \leq \sum_{j=m+1}^{n} |F(z+k_j)|,$$

the same argument shows that

$$G(z) := \sum_{k\in\mathbb{Z}^{2d}} F(z+k) \quad \text{for } z \in [0,1]^{2d}$$

defines a continuous function $G\colon [0,1]^{2d} \to \mathbb{C}$.

We now calculate the Fourier coefficients of G. We will use the dominated convergence theorem twice, and both instances will be justified after the calculation. Let $l \in \mathbb{Z}^{2d}$. Then,

$$\begin{aligned}
\hat{G}(l) &= \int_{[0,1]^{2d}} G(z) e^{-2\pi i l\cdot z}\,\mathrm{d}z \\
&= \sum_{k\in\mathbb{Z}^{2d}} \int_{[0,1]^{2d}} F(z+k) e^{-2\pi i l\cdot z}\,\mathrm{d}z \\
&= \sum_{k\in\mathbb{Z}^{2d}} \int_{\mathbb{R}^{2d}} \chi_{k+[0,1]^{2d}}(z') F(z') e^{-2\pi i l\cdot (z'-k)}\,\mathrm{d}z' \\
&= \int_{\mathbb{R}^{2d}} F(z') e^{-2\pi i l\cdot z'}\,\mathrm{d}z' = \mathcal{F}F(l)
\end{aligned}$$

where we used the fact that $e^{2\pi i l\cdot k} = 1$ for all $k \in \mathbb{Z}^{2d}$ in the last step, before applying the dominated convergence theorem for the second time. The first instance of the dominated convergence theorem (going from the first to the second line) is justified by $\sum_{k\in\mathbb{Z}^{2d}} |F(z+k)|$ being continuous on $[0,1]^{2d}$ and hence in $L^1([0,1]^{2d})$ (we proved continuity in the previous paragraph). The second instance of the dominated convergence theorem is justified since $|F| \in L^1(\mathbb{R}^{2d})$.

We have now shown that $\hat{G}(k) = \mathcal{F}F(k)$ for all $k \in \mathbb{Z}^{2d}$. Since G is continuous and $\sum_{k\in\mathbb{Z}^{2d}} |\hat{G}(k)| < \infty$ by our assumption on $\mathcal{F}F$, the Fourier series of G converges uniformly to G on $[0,1]^{2d}$ (this is a foundational result from Fourier theory). Thus, we have that

$$\sum_{k\in\mathbb{Z}^{2d}} F(z+k) = G(z) = \sum_{k\in\mathbb{Z}^{2d}} \hat{G}(k) e^{2\pi i z\cdot k} = \sum_{k\in\mathbb{Z}^{2d}} \mathcal{F}F(k) e^{2\pi i z\cdot k},$$

for every $z \in [0,1]^{2d}$. We have proved that the leftmost series converges absolutely (and uniformly) pointwise, and the same is true of the rightmost series since $\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}F(k)| < \infty$ by assumption. Thus, we are done. □

We now state and prove the symplectic Poisson summation formula over an arbitrary lattice $A\mathbb{Z}^{2d}$.

3.2.21 Proposition (The symplectic Poisson summation formula)**.** *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ and let $F \in L^1(\mathbb{R}^d)$ be a continuous function. Assume that there exists some $(c_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that*

$$|F(Az)| \le \sum_{k\in\mathbb{Z}^{2d}} |c_k| \chi_{k+[0,1)^{2d}}(z) \quad \textit{for all } z \in \mathbb{R}^{2d}, \tag{3.14}$$

and that $\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}^s F(A^\circ k)| < \infty$. *Then,*

$$\sum_{k\in\mathbb{Z}^{2d}} F(Az + Ak) = \frac{1}{|\det A|} \sum_{k\in\mathbb{Z}^{2d}} \mathcal{F}^s F(A^\circ k) e^{-2\pi i \Omega_J(A^\circ k, Az)}$$

for all $z \in [0,1]^{2d}$, *with absolute pointwise convergence of both sums.*

Proof. Let $k \in \mathbb{Z}^{2d}$. The idea is to apply the ordinary Poisson summation formula to the function F_A on $\mathbb{R}^{2d}$ defined by $z \mapsto F(Az)$. We have $F_A \in L^1(\mathbb{R}^{2d})$ by a simple change of variables calculation.

We calculate the Fourier transform of F_A evaluated at k:

$$\begin{aligned}
\mathcal{F}F_A(k) &= \int_{\mathbb{R}^{2d}} F(Az')e^{-2\pi i z'\cdot k}\,\mathrm{d}z' \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^{2d}} F(z)e^{-2\pi i (A^{-1}z)\cdot k}\,\mathrm{d}z \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^{2d}} F(z)e^{2\pi i z\cdot (J^2 A^{-T}k)}\,\mathrm{d}z \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^{2d}} F(z)e^{2\pi i \Omega_J(JA^{-T}k, z)}\,\mathrm{d}z = \frac{1}{|\det A|}\mathcal{F}^s F(-A^\circ k).
\end{aligned}$$

This shows that $\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}F_A(k)| < \infty$, since $\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}^s F(A^\circ k)| < \infty$ by assumption. Equation (3.14) is simply Equation (3.13) for F_A, so F_A satisfies all the conditions for the Poisson summation formula (Proposition 3.2.20).

Since

$$z \cdot w = (Az)\cdot(-J^2 A^{-T} w) = (Az)\cdot(JA^\circ w) = \Omega_J(A^\circ w, Az)$$

for all $z, w \in \mathbb{R}^{2d}$, we can write the Poisson summation formula for F_A as follows:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^{2d}} F_A(z+k) &= \sum_{k\in\mathbb{Z}^{2d}} \mathcal{F}F_A(k)e^{2\pi i z\cdot k} \\
&= \frac{1}{|\det A|}\sum_{k\in\mathbb{Z}^{2d}} \mathcal{F}^s F(-A^\circ k) e^{2\pi i \Omega_J(A^\circ k, Az)}.
\end{aligned}$$

The simple relabelling $k \mapsto -k$ now gives the desired result. □

If one recalls Lemma 3.2.19, one can now see that the FIGA will follow from the symplectic Poisson summation formula applied to $F = V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$ and evaluated at $z = 0$. However, we still need to show that symplectic Poisson summation formula holds for $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$ when all four functions are in $S_0(\mathbb{R}^d)$. To do this, we will need three additional lemmas, whose statements and proofs have all been adapted from Gröchenig [17]. First up is an inequality that we will have use for later as well.

3.2.22 Lemma. *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ and suppose that $\langle g_1, g_2\rangle \neq 0$. Then,*

$$\begin{aligned}|V_{f_2} f_1(z)| &\le \frac{1}{|\langle g_1, g_2\rangle|}\int_{\mathbb{R}^{2d}} |V_{g_1} f_1(w)||V_{f_2} g_2(z-w)|\,\mathrm{d}w\\ &= \frac{1}{|\langle g_1, g_2\rangle|}\big(|V_{g_1} f_1| * |V_{f_2} g_2|\big)(z),\end{aligned}$$

for all $z \in \mathbb{R}^{2d}$ (where the right hand side may be infinite).

Proof. By the reconstruction formula (Corollary 3.1.7 and the subsequent discussion), we find that

$$\begin{aligned}\langle f_1, \pi(z) f_2\rangle &= \frac{1}{\langle g_2, g_1\rangle}\langle \mathcal{V}_{g_2}^* \mathcal{V}_{g_1} f_1, \pi(z) f_2\rangle\\ &= \frac{1}{\langle g_2, g_1\rangle}\Big\langle \int_{\mathbb{R}^{2d}} V_g f_1(w)\pi(w) g_2\,\mathrm{d}w, \pi(z) f_2\Big\rangle\\ &= \frac{1}{\langle g_2, g_1\rangle}\int_{\mathbb{R}^{2d}} V_{g_1} f_1(w)\langle \pi(w) g_2, \pi(z) f_2\rangle\,\mathrm{d}w\\ &= \frac{1}{\langle g_2, g_1\rangle}\int_{\mathbb{R}^{2d}} V_{g_1} f_1(w)\langle g_2, \pi(w)^*\pi(z) f_2\rangle\,\mathrm{d}w\\ &= \frac{1}{\langle g_2, g_1\rangle}\int_{\mathbb{R}^{2d}} V_{g_1} f_1(w) V_{f_2} g_2(z-w) e^{2\pi i(\eta-\omega)\cdot y}\,\mathrm{d}w,\end{aligned}$$

where $w = (y, \eta)$ (as integration variables) and $z = (x, \omega) \in \mathbb{R}^{2d}$. In the last step, we used points (ii) and (iv) of Lemma 3.1.1. Taking absolute values gives the desired result. □

Next up is a simple lemma (with a tedious proof) regarding the STFT $V_{g_0} g_0$. The subsequent lemma will transfer this result to $V_g f$, for arbitrary $f, g \in S_0(\mathbb{R}^d)$, which will give us the last piece we need to prove the FIGA.

3.2.23 Lemma. *With $g_0(t) = 2^{d/4} e^{-\pi t^2}$, we find that*

$$V_{g_0} g_0(z) = e^{-\pi i x\cdot\omega} e^{-\pi z^2/2} \quad \textit{for all } z = (x, \omega) \in \mathbb{R}^{2d}.$$

Moreover, for any $A \in \mathrm{GL}(2d, \mathbb{R})$ there exists some $(c_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$|V_{g_0} g_0(Az)| = e^{-\pi (Az)^2/2} \le \sum_{k\in\mathbb{Z}^{2d}} |c_k| \chi_{k+[0,1)^{2d}}(z) \quad \textit{for all } z \in \mathbb{R}^{2d}.$$

Proof. Recall that $\mathcal{F}g_0 = g_0$, let $D_r f(t) = f(rt)$ for $r > 0$ and $f \in L^2(\mathbb{R}^d)$, and recall also that $\mathcal{F}D_r = r^{-d} D_{r^{-1}} \mathcal{F}$ (these are basic properties of the Fourier transform). With $z = (x, \omega) \in \mathbb{R}^{2d}$, we calculate:

$$\begin{aligned} V_{g_0} g_0(x,\omega) &= \int_{\mathbb{R}^d} 2^{d/2} e^{-\pi t^2} \overline{e^{2\pi i t\cdot\omega} e^{-\pi(t-x)^2}} \, \mathrm{d}t \\ &= 2^{d/4} e^{-\pi x^2} \int_{\mathbb{R}^d} 2^{d/4} e^{-2\pi(t^2 - t\cdot x)} e^{-2\pi i t\cdot\omega} \, \mathrm{d}t \\ &= 2^{d/4} e^{-\pi x^2/2} \int_{\mathbb{R}^d} 2^{d/4} e^{-2\pi(t-x/2)^2} e^{-2\pi i t\cdot\omega} \, \mathrm{d}t \\ &= 2^{d/4} e^{-\pi x^2/2} \big(\mathcal{F} T_{x/2} D_{2^{1/2}} g_0\big)(\omega) \\ &= 2^{d/4} e^{-\pi x^2/2} 2^{-d/2} \big(M_{-x/2} D_{2^{-1/2}} \mathcal{F} g_0\big)(\omega) \\ &= e^{-\pi x^2/2} e^{-\pi i x\cdot\omega} e^{-\pi\omega^2/2}. \end{aligned}$$

This proves the first claim of the lemma.

As for the second claim, we first show that there exists some $(d_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$e^{-\pi z^2/2} \le \sum_{k\in\mathbb{Z}^{2d}} |d_k| \chi_{k+[0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}, \tag{3.15}$$

The Gaussian $h(z) = e^{-\pi z^2/2}$ (for $z \in \mathbb{R}^{2d}$) decreases radially from the origin. Thus, for every $k \in \mathbb{Z}^{2d}$, there exists some vector $n_k \in \mathbb{R}^{2d}$ whose entries are only zeroes and ones such that

$$e^{-\pi z^2/2} \le e^{-\pi(k+n_k)^2/2} \quad \text{for all } z \in k + [0,1)^{2d};$$

simply choose n_k such that $k + n_k$ is the corner of $k + [0,1]^{2d}$ that is closest to the origin. With $\{n_k : k \in \mathbb{Z}^{2d}\}$ chosen in this manner, we have that $\|n_k\| \le \sqrt{2d}$ for all $k \in \mathbb{Z}^{2d}$ and that

$$e^{-\pi z^2/2} \le \sum_{k\in\mathbb{Z}^{2d}} e^{-\pi(k+n_k)^2/2} \chi_{k+[0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}. \tag{3.16}$$

Temporarily fix $k \in \mathbb{Z}^{2d}$. If $z \in k + [0,1)^{2d}$, then $\|z\| \le \|k\| + \sqrt{2d}$. Thus,

$$\|k + n_k\| \ge \|k\| - \|n_k\| \ge (\|z\| - \sqrt{2d}) - \|n_k\| \ge \|z\| - 2\sqrt{2d},$$

which implies that $e^{-\pi(k+n_k)^2/2} \le e^{-\pi(\|z\|-2\sqrt{2d})^2/2}$ whenever $\|z\| \ge 2\sqrt{2d}$ (and $z \in k + [0,1)^{2d}$). Thus, for some constant $C > 0$, we have that

$$\begin{aligned} \sum_{k\in\mathbb{Z}^{2d}} |e^{-\pi(k+n_k)^2/2}| &= \int_{\mathbb{R}^{2d}} \sum_{k\in\mathbb{Z}^{2d}} e^{-\pi(k+n_k)^2/2} \chi_{k+[0,1)^{2d}}(z) \\ &\le C + \int_{\mathbb{R}^{2d}} e^{-\pi(\|z\|-2\sqrt{2d})^2/2} \, \mathrm{d}z. \end{aligned}$$

Changing to polar coordinates, we obtain a one-dimensional integral over $\|z\|^{2d-1}e^{-\pi(\|z\|-2\sqrt{2d})^2/2}$, which is finite. By Equation (3.16), we have now established Equation (3.15) (with $d_k = e^{-\pi(k+n_k)^2/2}$).

Now, Equation (3.15) implies that

$$e^{-\pi(Az)^2/2} \leq \sum_{k\in\mathbb{Z}^{2d}} |d_k|\chi_{k+[0,1)^{2d}}(Az) = \sum_{k\in\mathbb{Z}^{2d}} |d_k|\chi_{A^{-1}(k+[0,1)^{2d})}(z) \tag{3.17}$$

for all $z \in \mathbb{R}^{2d}$. For every $l \in \mathbb{Z}^{2d}$, let

$$m_l := \max\Big\{|d_k| : k \in \mathbb{Z}^{2d} \text{ s.t. } \big(l + [0,1)^{2d}\big) \cap A^{-1}\big(k + [0,1)^{2d}\big) \neq \emptyset\Big\}.$$

Clearly there is a positive integer N, independent of $k \in \mathbb{Z}^{2d}$, such that $A^{-1}\big(k+[0,1)^{2d}\big)$ intersects at most N cubes of the form $l+[0,1)^{2d}$ for $l \in \mathbb{Z}^{2d}$. This implies that

$$\sum_{l\in\mathbb{Z}^{2d}} m_l \leq N \sum_{k\in\mathbb{Z}^{2d}} |d_k|, \tag{3.18}$$

because for any given l, there is some $k \in \mathbb{Z}^{2d}$ such that $m_l = |d_k|$, and each $k \in \mathbb{Z}^{2d}$ contributes (in this manner) to at most N terms in $\sum_{l\in\mathbb{Z}^{2d}} m_l$.

Similarly, there is a positive integer M, independent of $l \in \mathbb{Z}^{2d}$, such that $l+[0,1)^{2d}$ intersects at most M sets of the form $A^{-1}\big(k+[0,1)^{2d}\big)$ for $k \in \mathbb{Z}^{2d}$. It follows by pointwise evaluation that

$$\sum_{k\in\mathbb{Z}^{2d}} |d_k|\chi_{A^{-1}(k+[0,1)^{2d})}(z) \leq M \sum_{l\in\mathbb{Z}^{2d}} m_l\chi_{l+[0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}, \tag{3.19}$$

because if $z \in l + [0,1)^{2d}$, so that the right hand side is Mm_l, then there are at most M nonzero terms on the left hand side, and m_l is defined to be the largest of these terms.

Finally, equations (3.17) and (3.19) combine to give:

$$e^{-\pi(Az)^2/2} \leq M \sum_{l\in\mathbb{Z}^{2d}} m_l\chi_{l+[0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d},$$

from which Equation (3.18) concludes the proof (with $c_k = Mm_k$). $\square$

3.2.24 Lemma. *Let $f, g \in S_0(\mathbb{R}^d)$ and let $A \in \mathrm{GL}(2d,\mathbb{R})$. Then, there exists some $(c_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that*

$$|V_gf(Az)| \leq \sum_{k\in\mathbb{Z}^{2d}} |c_k|\chi_{k+[0,1)^{2d}}(z) \quad \textit{for all } z \in \mathbb{R}^{2d}.$$

In particular, $(V_gf(Ak))_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$.

Proof. Let g_0 be the normalized Gaussian on $\mathbb{R}^d$, as always. By Lemma 3.2.22, with $g_1 = g_0, g_2 = g_0, f_1 = f$ and $f_2 = g$, we know that

$$|V_g f(z)| \leq |V_{g_0} f| * |V_g g_0|(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Applying the same lemma again, with $g_1 = g_0, g_2 = g_0, f_1 = g_0$ and $f_2 = g$, we find that

$$|V_g f(z)| \leq |V_{g_0} f| * \big(|V_{g_0} g_0| * |V_g g_0|\big)(z) = \big(|V_{g_0} f| * |V_g g_0|\big) * |V_{g_0} g_0|(z)$$

for all $z \in \mathbb{R}^{2d}$. By our assumption that $f, g \in S_0(\mathbb{R}^d)$, along with the equivalence of norms shown in Lemma 3.2.13 (and the fact that $g_0 \in S_0(\mathbb{R}^d)$ by Lemma 3.2.23), all these STFTs are in $L^1(\mathbb{R}^{2d})$. This implies that

$$|V_g f| \leq F * |V_{g_0} g_0| \quad \text{for some } F = |F| \in L^1(\mathbb{R}^d) \tag{3.20}$$

(pointwise), since $L^1(\mathbb{R}^{2d})$ is closed under convolutions. The idea is now to use the fact that $|V_{g_0} g_0|$ decays sufficiently fast to show that $|V_g f|$ does as well.

For any $k \in \mathbb{Z}^{2d}$, choose $v_k \in k + [0,1]^{2d}$ such that

$$|V_g f(Av_k)| = \max\big\{|V_g f(Az)| : z \in k + [0,1]^{2d}\big\}.$$

We will show that $\sum_{k \in \mathbb{Z}^{2d}} |V_g f(Av_k)| < \infty$, from which the result follows immediately with $c_k := V_g f(Av_k)$. For the rest of this proof, we will exchange sums and integrals freely. Since all terms will be nonnegative, this is justified via the monotone convergence theorem applied to the partial sums. We will also write $|A| := |\det A|$ to save space.

By Lemma 3.2.23, there exists some $(d_k)_{k \in \mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$|V_{g_0} g_0|(Az) \leq \sum_{l \in \mathbb{Z}^{2d}} |d_l| \chi_{l + [0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Equation (3.20) implies that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^{2d}} |V_g f(Av_k)| &\leq \sum_{k \in \mathbb{Z}^{2d}} \big(F * |V_{g_0} g_0|\big)(Av_k) \\ &= \sum_{k \in \mathbb{Z}^{2d}} \int_{\mathbb{R}^{2d}} F(z') |V_{g_0} g_0|(Av_k - z') \, \mathrm{d}z' \\ &= \sum_{k \in \mathbb{Z}^{2d}} |A| \int_{\mathbb{R}^{2d}} F(Az) |V_{g_0} g_0|(Av_k - Az) \, \mathrm{d}z \\ &\leq |A| \sum_{k \in \mathbb{Z}^{2d}} \int_{\mathbb{R}^{2d}} F(Az) \sum_{l \in \mathbb{Z}^{2d}} |d_l| \chi_{l + [0,1)^{2d}}(v_k - z) \, \mathrm{d}z. \end{aligned} \tag{3.21}$$

Temporarily fix $k, l \in \mathbb{Z}^{2d}$. We have $v_k \in k + [0,1]^{2d}$, so $v_k - z \in l + [0,1)^{2d}$ implies that

$$z \in v_k - l - [0,1)^{2d} \subset k + [0,1]^{2d} - l - [0,1)^{2d} \subset k - l + (-1,1]^{2d}.$$

This means that

$$\chi_{l+[0,1)^{2d}}(v_k - z) \leq \chi_{k-l+(-1,1]^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Equation (3.21) thus implies that

$$\sum_{k \in \mathbb{Z}^{2d}} |V_g f(Av_k)| \leq |A| \sum_{k \in \mathbb{Z}^{2d}} \sum_{l \in \mathbb{Z}^{2d}} \int_{\mathbb{R}^{2d}} F(Az) |d_l| \chi_{k-l+(-1,1]^{2d}}(z) \, \mathrm{d}z. \tag{3.22}$$

Now, note that $(-1,1]^{2d}$ is the disjoint union of 2^{2d} standard-lattice-translates of $(0,1]^{2d}$, so that, for any $l \in \mathbb{Z}^{2d}$ and $z \in \mathbb{R}^{2d}$:

$$\sum_{k \in \mathbb{Z}^{2d}} \chi_{k-l+(-1,1]^{2d}}(z) = \sum_{k \in \mathbb{Z}^{2d}} \chi_{k+(-1,1]^{2d}}(z) = 2^{2d} \sum_{k \in \mathbb{Z}^{2d}} \chi_{k+(0,1]^{2d}}(z) = 2^{2d}.$$

Continuing where we left off and changing the order of summation (which we may, as all terms are nonnegative), we now find that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^{2d}} |V_g f(Av_k)| &\leq |A| \sum_{l \in \mathbb{Z}^{2d}} |d_l| \int_{\mathbb{R}^{2d}} F(Az) \sum_{k \in \mathbb{Z}^{2d}} \chi_{k-l+(-1,1]^{2d}}(z) \, \mathrm{d}z \\
&= |A| \sum_{l \in \mathbb{Z}^{2d}} |d_l| \int_{\mathbb{R}^{2d}} F(Az) 2^{2d} \, \mathrm{d}z \\
&= 2^{2d} \|F\|_1 \sum_{l \in \mathbb{Z}^{2d}} |d_l| < \infty,
\end{aligned}$$

which is what we needed to show. □

We are now finally ready to prove the fundamental identity of Gabor analysis.

Proof of Theorem 3.2.17. Let $f_1, f_2, g_1, g_2 \in S_0(\mathbb{R}^d)$ and let $A \in \mathrm{GL}(2d, \mathbb{R})$. Our goal is to show that the pointwise product $V_{g_1} f_1 \cdot \overline{V_{g_2} f_2}$ satisfies the conditions of Proposition 3.2.21 on the symplectic Poisson summation formula.

By Lemma 3.2.24, there exist $(c_k)_{k \in \mathbb{Z}^{2d}}$ and $(d_l)_{l \in \mathbb{Z}^{2d}}$ in $\ell^1(\mathbb{Z}^{2d})$ such that

$$|V_{g_1} f_1(Az)| \leq \sum_{k \in \mathbb{Z}^{2d}} |c_k| \chi_{k+[0,1)^{2d}}(z) \quad \text{and} \quad |V_{g_2} f_2(Az)| \leq \sum_{l \in \mathbb{Z}^{2d}} |d_l| \chi_{l+[0,1)^{2d}}(z)$$

for all $z \in \mathbb{R}^{2d}$. This implies that

$$\begin{aligned}|V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}|(Az) &\le \Big(\Big(\sum_{k\in\mathbb{Z}^{2d}} |c_k|\chi_{k+[0,1)^{2d}}\Big) \cdot \Big(\sum_{l\in\mathbb{Z}^{2d}} |d_l|\chi_{l+[0,1)^{2d}}\Big)\Big)(z)\\ &= \sum_{k\in\mathbb{Z}^{2d}} |c_k d_k|\chi_{k+[0,1)^{2d}}(z)\end{aligned}$$

for all $z \in \mathbb{R}^{2d}$. Since $\ell^1(\mathbb{Z}^{2d}) \subset \ell^2(\mathbb{Z}^{2d})$, and since the pointwise product of two $\ell^2(\mathbb{Z}^{2d})$-sequences is in $\ell^1(\mathbb{Z}^{2d})$ by the Cauchy-Schwarz inequality, we can conclude that $(c_k d_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$.

By Lemma 3.2.19, we know that

$$\mathcal{F}^s(V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}) = V_{g_1}g_2 \cdot \overline{V_{f_1}f_2},$$

pointwise. Applying Lemma 3.2.24 to $V_{g_1}g_2$ and $V_{f_1}f_2$, with A° in place of A, and arguing as we did in the previous paragraph, we find that there is some $(a_k)_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$|V_{g_1}g_2 \cdot \overline{V_{f_1}f_2}|(A^\circ z) \le \sum_{k\in\mathbb{Z}^{2d}} |a_k|\chi_{k+[0,1)^{2d}}(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Thus,

$$\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{F}^s(V_{g_1}f_1 \cdot \overline{V_{g_2}f_2})(A^\circ k)| = \sum_{k\in\mathbb{Z}^{2d}} |V_{g_1}g_2 \cdot \overline{V_{f_1}f_2}|(A^\circ k) \le \sum_{k\in\mathbb{Z}^{2d}} |a_k| < \infty.$$

We have now shown that $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$ satisfies all the requirements for the symplectic Poisson summation formula (Proposition 3.2.21). Evaluating this formula at $z = 0$ gives:

$$\sum_{k\in\mathbb{Z}^{2d}} \big(V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}\big)(Ak) = \frac{1}{|\det A|} \sum_{k\in\mathbb{Z}^{2d}} \big(V_{g_1}g_2 \cdot \overline{V_{f_1}f_2}\big)(A^\circ k),$$

with absolute convergence of both sums. This is precisely the conclusion of Theorem 3.2.17, i.e. the FIGA, so we are done. □

As a corollary of the FIGA, we obtain the fact that all Gabor systems with windows in $S_0(\mathbb{R}^d)$ are Bessel sequences. This means that, as long as we choose our windows from $S_0(\mathbb{R}^d)$, we can consider analysis, synthesis and mixed-frame type operators without worrying about whether they are well-defined or bounded.

3.2.25 Corollary. *If $g \in S_0(\mathbb{R}^d)$, then $\mathcal{G}(g, \Lambda)$ is a Bessel sequence for any lattice $\Lambda \subset \mathbb{R}^{2d}$.*

Proof. Let $\Lambda = A\mathbb{Z}^{2d}$ for some $A \in \mathrm{GL}(2d, \mathbb{R})$. Suppose first that $f \in S_0(\mathbb{R}^d)$. Then, the FIGA implies that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^{2d}} |\langle f, \pi(Ak)g\rangle|^2 &= \sum_{k\in\mathbb{Z}^{2d}} V_g f(Ak)\overline{V_g f(Ak)} \\ &= \frac{1}{|\det A|}\sum_{k\in\mathbb{Z}^{2d}} V_g g(A^\circ k)\overline{V_f f(A^\circ k)}\end{aligned}$$

and asserts that both sums converge absolutely. Since time-frequency shifts are unitary, we find that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^{2d}} |V_g g(A^\circ k)\overline{V_f f(A^\circ k)}| &= \sum_{k\in\mathbb{Z}^{2d}} |V_g g(A^\circ k)\langle \pi(A^\circ k)f, f\rangle| \\ &\le \|f\|_2^2 \sum_{k\in\mathbb{Z}^{2d}} |V_g g(A^\circ k)|.\end{aligned}$$

Combining the last two displayed equations, we find that

$$\sum_{k\in\mathbb{Z}^{2d}} |\langle f, \pi(Ak)g\rangle|^2 \le B\|f\|_2^2 \quad \text{with } B := \frac{1}{|\det A|}\sum_{k\in\mathbb{Z}^{2d}} |V_g g(A^\circ k)|$$

By Lemma 3.2.24, $(V_g g(A^\circ k))_{k\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$, so $B < \infty$.

Fix now any $f \in L^2(\mathbb{R}^d)$ and choose a sequence $(f_n) \subset S_0(\mathbb{R}^d)$ such that $f_n \to f$ (Lemma 3.2.14). Define the functions

$$\begin{aligned} T\colon \mathbb{Z}^{2d} &\to \mathbb{R} & \qquad T_n\colon \mathbb{Z}^{2d} &\to \mathbb{R} \\ k &\mapsto |\langle f, \pi(Ak)g\rangle|^2 & k &\mapsto |\langle f_n, \pi(Ak)g\rangle|^2 \end{aligned}$$

for $n \in \mathbb{N}$. By continuity of the inner product, we see that $T_n \to T$ pointwise. By Fatou's lemma (with the counting measure on $\mathbb{Z}^{2d}$), we find that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^{2d}} |\langle f, \pi(Ak)g\rangle|^2 &= \sum_{k\in\mathbb{Z}^{2d}} \lim_{n\to\infty} |\langle f_n, \pi(Ak)g\rangle|^2 \\ &\le \liminf_{n\to\infty} \sum_{k\in\mathbb{Z}^{2d}} |\langle f_n, \pi(Ak)g\rangle|^2 \\ &\le \liminf_{n\to\infty} B\|f_n\|_2^2 = B\|f\|_2^2.\end{aligned}$$

This shows that $\mathcal{G}(g, \Lambda)$ is a Bessel sequence and concludes the proof. □

3.2.4 Duality and the Janssen Representation

Duality in Gabor theory refers to the fact that the properties of Gabor systems over a lattice Λ are related to the properties of Gabor systems over its adjoint lattice Λ°. In particular, *the* duality theorem in Gabor analysis states that a Gabor system $\mathcal{G}(g, \Lambda)$ is a frame if and only if $\mathcal{G}(g, \Lambda^\circ)$ is something called a *Riesz sequence*. In many instances, the latter condition is easier to check than the former. Moreover, this interplay between Λ and Λ° greatly constrains the class of lattices over which there can exist Gabor frames. An example of this is the previously mentioned fact that $\mathrm{vol}(\Lambda) \leq 1$ is required for the existence of Gabor frames over Λ.

We will not have need for the duality theorem, so we have chosen not to include it. Nevertheless, it lies at the heart of Gabor analysis, so it should be mentioned. Indeed, a central message of the survey article [19] by Gröchenig and Koppensteiner is that most (if not all) known characterizations of Gabor frames for $L^2(\mathbb{R}^d)$ are consequences of duality.

There is one important manifestation of duality that we will need:

3.2.26 Theorem (The Janssen representation)**.** *Let $g, h \in S_0(\mathbb{R}^d)$ and let $A \in \mathrm{GL}(2d, \mathbb{R})$. Then, the mixed-type frame operator $S^A_{g,h}$ can be written as:*

$$S^A_{g,h} = \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}^{2d}} \langle h, \pi(A^\circ k) g\rangle \pi(A^\circ k),$$

where the sum converges absolutely in $\mathcal{B}(L^2(\mathbb{R}^d))$.

Proof. Let $f_1, f_2 \in S_0(\mathbb{R}^d)$. By definition of frame operators (and boundedness, which is afforded by Corollary 3.2.25), we find that

$$\sum_{k \in \mathbb{Z}^{2d}} V_g f_1(Ak)\overline{V_h f_2(Ak)} = \sum_{k \in \mathbb{Z}^{2d}} \langle f_1, \pi(Ak)g\rangle\langle \pi(Ak)h, f_2\rangle = \langle S^A_{g,h} f_1, f_2\rangle$$

and that

$$\sum_{k \in \mathbb{Z}^{2d}} V_g h(A^\circ k)\overline{V_{f_1} f_2(A^\circ k)} = \sum_{k \in \mathbb{Z}^{2d}} \langle h, \pi(A^\circ k)g\rangle\langle \pi(A^\circ k)f_1, f_2\rangle = \langle S^{A^\circ}_{g,f_1} h, f_2\rangle.$$

The FIGA now implies that

$$\left\langle S^A_{g,h} f_1 - \frac{1}{|\det A|} S^{A^\circ}_{g,f_1} h, f_2 \right\rangle = 0 \quad \text{for all } f_2 \in S_0(\mathbb{R}^d).$$

By density of $S_0(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ (Lemma 3.2.14), we can conclude that $S^A_{g,h} f_1 = S^{A^\circ}_{g,f_1} h / |\det A|$. Thus, we have shown that

$$S^\Lambda_{g,h} f_1 = \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}^{2d}} \langle h, \pi(A^\circ k)g\rangle \pi(A^\circ k) f_1 \quad \text{for all } f_1 \in S_0(\mathbb{R}^d). \tag{3.23}$$

Now, we know that $S_{g,h}^{\Lambda}$ is a bounded operator on $L^2(\mathbb{R}^d)$, and that

$$\frac{1}{|\det A|} \sum_{k\in\mathbb{Z}^{2d}} \langle h, \pi(A^\circ k)g\rangle \pi(A^\circ k)$$

is as well, because (by Lemma 3.2.24)

$$\sum_{k\in\mathbb{Z}^{2d}} \|\langle h, \pi(A^\circ k)g\rangle \pi(A^\circ k)\| \leq \sum_{k\in\mathbb{Z}^{2d}} |V_g h(A^\circ k)| < \infty.$$

Equation (3.23) shows that these two bounded operators agree on the dense subspace $S_0(\mathbb{R}^d)$, so they must be equal (Theorem B.1.2). $\square$

Chapter 4

Noncommutative Tori and Hilbert C*-Modules

In this chapter, we introduce noncommutative tori and develop the basic theory of Hilbert C*-modules. Noncommutative tori have an alternate description in terms of twisted group C*-algebras, which we will find it more convenient to work with. We will first construct twisted group C*-algebras in detail, show how they relate to Gabor theory, and then show that they are in fact noncommutative tori. These are the contents of the first two sections.

In the third section, we begin our development of Hilbert C*-modules. We will conclude this chapter by constructing those Hilbert C*-modules over noncommutative tori that appear in Gabor analysis. In the next chapter, Chapter 5, we will upgrade these to equivalence bimodules. As outlined in Subsection 3.2.2, one of our main goals is to construct isomorphism between such bimodules, which we will achieve in Chapter 6.

4.1 | Twisted Group C*-Algebras and Time-Frequency Shifts

The purpose of this section is to provide a detailed construction of twisted group C*-algebras and to show how they relate to Gabor analysis. We will obtain our twisted group C*-algebras as completions of purely algebraic twisted group algebras, which we therefore introduce first. No constructions or results in this section are original, but the exposition is not based on any particular reference.

In order to make the connection to Gabor analysis as explicit as possible, we will only consider group C*-algebras over the group $(\mathbb{Z}^{2d}, +)$. Our constructions and results generalize immediately to arbitrary discrete groups

$(G, \cdot)$ by mere notational changes. With some additional work, they can also be generalized to arbitrary locally compact (Hausdorff) groups by working with Haar measures. We refer the interested reader to Raeburn and Williams [24, Appendix C.3].

Before we get to work, we establish some notation and conventions. For any Hilbert space H, we will write $\mathcal{B}(H)$ to denote the C*-algebra of bounded operators on H. All Hilbert spaces are assumed to be nonzero and over the complex numbers. We will write $\mathcal{U}(H)$ for the group of unitary elements in $\mathcal{B}(H)$ and we will use $\mathbb{T}$ to denote the unit circle in $\mathbb{C}$.

4.1.1 Twisted Group Algebras

Twisted group algebras are intimately related to projective unitary representations, which is where our development begins. As a sneak peek at their relevance to us, we reminder the reader of the fundamental identity $\pi(z+w) = e^{2\pi i \eta \cdot x}\pi(z)\pi(w)$ from Lemma 3.1.1, which, after the introduction of a lattice matrix $A \in \mathrm{GL}(2d, \mathbb{R})$, implies that

$$\pi_A(k)\pi_A(k') = e^{-2\pi i P_2(k')\cdot P_1(k)}\pi_A(k+k') \quad \text{for all } k, k' \in \mathbb{Z}^{2d},$$

where $P_1, P_2 \colon \mathbb{R}^{2d} \to \mathbb{R}^d$ are projections (which will be defined properly in Subsection 4.1.4).

A *projective unitary representation* of $\mathbb{Z}^{2d}$ consists of a Hilbert space H and a map $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ such that

$$\phi(k)\phi(k') = \gamma(k,k')\phi(k+k') \quad \text{for some } \gamma(k,k') \in \mathbb{T} \tag{4.1}$$

for all $k, k' \in \mathbb{Z}^{2d}$. It follows from this that $\phi(0) = \gamma(0,0)\mathrm{Id}_H$, for

$$\phi(0) = \phi(0+0) = \overline{\gamma(0,0)}\phi(0)^2,$$

from which composition with $\phi(0)^{-1}$ gives the result. Now, $\phi' = \overline{\gamma(0,0)}\phi$ defines a projective unitary representation such that $\phi'(0) = \mathrm{Id}_H$. Thus, there is no real loss of generality in assuming that $\phi(0) = \mathrm{Id}_H$, which we will do from now on.

We now deduce two properties of the map $\gamma\colon \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{T}$ associated to a projective unitary representation of $\mathbb{Z}^{2d}$. Inserting either $k = 0$ or $k' = 0$ into Equation (4.1), we see that $\gamma(k, 0) = 1 = \gamma(0, k)$ for all $k \in \mathbb{Z}^{2d}$. The relations

$$\begin{aligned} &\big(\phi(k_1)\phi(k_2)\big)\phi(k_3) = \gamma(k_1+k_2, k_3)\gamma(k_1,k_2)\phi\big((k_1+k_2)+k_3\big) \\ \text{and}\quad &\phi(k_1)\big(\phi(k_2)\phi(k_3)\big) = \gamma(k_1, k_2+k_3)\gamma(k_2,k_3)\phi\big(k_1+(k_2+k_3)\big) \end{aligned}$$

imply that

$$\gamma(k_1, k_2)\gamma(k_1 + k_2, k_3) = \gamma(k_1, k_2 + k_3)\gamma(k_2, k_3) \quad \text{for all } k_1, k_2, k_3 \in \mathbb{Z}^{2d}.$$

We will eventually see that any map $\mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{T}$ satisfying these two properties gives rise to a projective unitary representation of $\mathbb{Z}^{2d}$.

4.1.1 Definition (2-cocycles)**.** A map $\gamma\colon \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{T}$ is called a *2-cocycle* on $\mathbb{Z}^{2d}$ if

(i) $\gamma(k, 0) = 1 = \gamma(0, k)$

(ii) $\gamma(k_1, k_2)\gamma(k_1 + k_2, k_3) = \gamma(k_1, k_2 + k_3)\gamma(k_2, k_3)$

for all $k, k_1, k_2, k_3 \in \mathbb{Z}^{2d}$.

In the example preceding the definition, the 2-cocycle $\gamma\colon \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{T}$ would be called the 2-cocycle *associated* to the representation $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$. We may at times refer to certain 2-cocycles simply as *cocycles*.

In order to motivate the coming construction, there is one simple piece of terminology we must introduce. Let H be a Hilbert space and consider the C*-algebra $\mathcal{B}(H)$. We define the $\star$-algebra *generated* by a subset $S \subset \mathcal{B}(H)$ to be the smallest $\star$-subalgebra of $\mathcal{B}(H)$ containing S. It may also be described as the intersection of all $\star$-subalgebras of $\mathcal{B}(H)$ containing S (see Definition 2.2.12 for the analogous notion for C*-algebras).

Now, fix a 2-cocycle γ on $\mathbb{Z}^{2d}$ and consider a projective unitary representation $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ with associated 2-cocycle γ. Consider moreover the $\star$-algebra generated by $\phi(\mathbb{Z}^{2d})$ in $\mathcal{B}(H)$. This $\star$-algebra will clearly depend on the specifics of the representation $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$. Nevertheless, we may ask what its "most general form" looks like (as ϕ varies).

The upcoming definition provides the answer to this question: the γ-twisted group algebra $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is an abstract realization of the most general $\star$-algebra generated by the image of a projective unitary representation of $\mathbb{Z}^{2d}$ with 2-cocycle γ (we will unravel this connection in detail). It is a variant of the ordinary group algebra from the representation theory of discrete groups, obtained by accounting for the 2-cocycle γ as well as the involution on $\mathcal{B}(H)$.

By the symbol $\mathbb{C}^{\mathbb{Z}^{2d}}$, we are referring to the set of all functions from $\mathbb{Z}^{2d}$ to $\mathbb{C}$.

4.1.2 Definition (γ-twisted group algebras)**.** Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. The *γ-twisted group algebra* $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is the $\star$-algebra

$$\mathbb{C}[\mathbb{Z}^{2d}, \gamma] = \left\{ a \in \mathbb{C}^{\mathbb{Z}^{2d}} : |\mathrm{supp}(a)| < \infty \right\}$$

with pointwise addition and scalar multiplication,[1] product $(a, b) \mapsto a *_\gamma b$ given by

$$\big(a *_\gamma b\big)(k) = \sum_{l \in \mathbb{Z}^{2d}} a(l) b(k-l) \gamma(l, k-l)$$

and involution $\star\colon a \mapsto a^*$ given by $a^*(k) = \overline{\gamma(k, -k) a(-k)}$, for all $k \in \mathbb{Z}^{2d}$.

As the definition claims, $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is a $\star$-algebra. We will obtain this as a simple corollary of the upcoming Lemma 4.1.3.

For any $k \in \mathbb{Z}^{2d}$, define the *δ-function* at k, $\delta_k\colon \mathbb{Z}^{2d} \to \mathbb{C}$, by

$$\delta_k(l) = \begin{cases} 1 & \text{if } l = k \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\delta_k \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. Moreover, $\mathcal{D} := \{\delta_k : k \in \mathbb{Z}^{2d}\}$ is a vector space basis for $C[\mathbb{Z}^{2d}, \gamma]$. Indeed, the ordinary group algebra $\mathbb{C}[\mathbb{Z}^{2d}] = \mathbb{C}[\mathbb{Z}^{2d}, 1]$ is designed to be a vector space with a basis isomorphic to $\mathbb{Z}^{2d}$ and a product which is a bilinear extension of the group operation on $\mathbb{Z}^{2d}$. The twist introduces an additional phase factor into the group operation: we have that

$$\delta_k *_\gamma \delta_{k'} = \gamma(k, k') \delta_{k+k'} \quad \text{for all } k, k' \in \mathbb{Z}^{2d}. \tag{4.2}$$

This follows from the simple calculation:

$$\begin{aligned} \delta_k *_\gamma \delta_{k'}(l) &= \sum_{l' \in \mathbb{Z}^{2d}} \delta_k(l') \delta_{k'}(l - l') \gamma(l', l - l') \\ &= \delta_{k'}(l-k) \gamma(k, l-k) = \gamma(k, k') \delta_{k+k'}(l). \end{aligned}$$

If $\gamma \equiv 1$, so that $\mathbb{C}[\mathbb{Z}^{2d}, \gamma] = \mathbb{C}[\mathbb{Z}^{2d}]$ is the ordinary group algebra, then Equation (4.2) shows that $(\mathcal{D}, *_1) \cong (\mathbb{Z}^{2d}, +)$ as groups. In the general case, $\mathcal{D}$ does not close under $*_\gamma$, but we have the following result.

4.1.3 Lemma. *With the notation of Definition 4.1.2 and the subsequent discussion, the set*

$$\mathbb{T}\mathcal{D} = \big\{\xi \delta_k : \xi \in \mathbb{T} \text{ and } k \in \mathbb{Z}^{2d}\big\} \subset \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$$

is a group w.r.t. the product $*_\gamma$ in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, which is given by

$$(\xi \delta_k) *_\gamma (\eta \delta_{k'}) = (\xi \eta) \delta_k *_\gamma \delta_{k'} = \xi \eta \gamma(k, k') \delta_{k+k'} \tag{4.3}$$

for all $\xi, \eta \in \mathbb{T}$ and $k, k' \in \mathbb{Z}^{2d}$. The identity of $\mathbb{T}\mathcal{D}$ is δ_0 and the inverse of $\xi \delta_k$ is its adjoint $(\xi \delta_k)^*$. In particular, all elements of $\mathbb{T}\mathcal{D}$ are unitary in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$.

[1] As a vector space, this is identical to the ordinary group algebra $\mathbb{C}[\mathbb{Z}^{2d}]$. Another description is that it is the vector space of all finite "sequences" $\mathbb{Z}^{2d} \to \mathbb{C}$, typically denoted by $c_{00}(\mathbb{Z}^{2d})$.

Proof. The definition of the product $*_\gamma$ in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ immediately implies that it is bilinear. Thus, Equation (4.3) follows from Equation (4.2).

The 2-cocycle properties of γ are precisely the properties required for Equation (4.3) to define an associative binary operation on $\mathbb{T}\mathcal{D}$ with δ_0 acting as the identity; property (i) of Definition 4.1.2 is equivalent to δ_0 acting as the identity, while property (ii) is equivalent to associativity.

Fix any $k \in \mathbb{Z}^{2d}$. Using the definition of the involution on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, we find that

$$\delta_k^*(l) = \overline{\gamma(l, -l)\delta_k(-l)} = \overline{\gamma(-k, k)}\delta_{-k}(l) \quad \text{for all } l \in \mathbb{Z}^{2d},$$

so that $\delta_k^* = \overline{\gamma(-k, k)}\delta_{-k}$. Thus, we see that

$$\delta_k^* *_\gamma \delta_k = \overline{\gamma(-k, k)}\delta_{-k} *_\gamma \delta_k = \overline{\gamma(-k, k)}\gamma(-k, k)\delta_{-k+k} = \delta_0.$$

Inserting $k_1 = k_3 = k$ and $k_2 = -k$ in point (ii) of Definition 4.1.1 and using point (i) of the same definition, one finds that $\gamma(k, -k) = \gamma(-k, k)$. Using this, a similar calculation to the one above show that $\delta_k *_\gamma \delta_k^* = \delta_0$. Thus, for any $\xi \in \mathbb{T}$, we find that

$$(\xi\delta_k)^{-1} = \overline{\xi}\delta_k^{-1} = \overline{\xi}\delta_k^* = (\xi\delta_k)^*.$$

We have now shown that $\mathbb{T}\mathcal{D}$ has inverses, hence is a group, and moreover that inverses are given by adjoints in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, so we are done. □

4.1.4 Corollary. *The γ-twisted group algebra $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is a $\star$-algebra.*

Proof. It is immediate from its definition that $*_\gamma$ is bilinear. Expanding $a, b \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ in terms of δ-functions, we therefore find that

$$\left(\sum_k a(k)\delta_k\right) *_\gamma \left(\sum_{k'} b(k')\delta_{k'}\right) = \sum_k \sum_{k'} a(k)b(k')\delta_k *_\gamma \delta_{k'}.$$

Associativity now follows easily from the associativity of $*_\gamma$ restricted to $\mathbb{T}\mathcal{D}$. Moreover, the fact that δ_0 is an identity in $\mathbb{T}\mathcal{D}$ implies that is serves as a unit for $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. This shows that $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is an algebra.

Properties (i) and (ii) of an involution (Definition 2.1.10) are simple to verify using the definition of the (claimed) involution on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, so we will only show that $(a *_\gamma b)^* = b^* *_\gamma a^*$ follows. We know that

$$(\delta_k *_\gamma \delta_{k'})^* = (\delta_k *_\gamma \delta_{k'})^{-1} = \delta_{k'}^{-1} *_\gamma \delta_k^{-1} = \delta_{k'}^* *_\gamma \delta_k^* \quad \text{for all } k, k' \in \mathbb{Z}^{2d}.$$

Thus, expanding a and b in terms of δ-functions as above and using conjugate-linearity of the involution, we find that

$$\begin{aligned}(a *_\gamma b)^* &= \sum_k \sum_{k'} \overline{a(k)b(k')}(\delta_k *_\gamma \delta_{k'})^* = \sum_k \sum_{k'} \overline{b(k')a(k)}\delta_{k'}^* *_\gamma \delta_k^* \\ &= \Big(\sum_{k'} \overline{b(k')}\delta_{k'}^*\Big) *_\gamma \Big(\sum_k \overline{a(k)}\delta_k^*\Big) = b^* *_\gamma a^*,\end{aligned}$$

which concludes the proof. □

Let $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ be any projective unitary representation of $\mathbb{Z}^{2d}$ with associated 2-cocycle γ. If $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is meant to represent the most general structure of the $\star$-algebra generated by $\phi(\mathbb{Z}^{2d})$ in $\mathcal{B}(H)$, we should be able to construct from ϕ a $\star$-algebra homomorphism $\mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$. Before proving that this is the case, we introduce some convenient terminology.

4.1.5 Definition (Representations)**.** Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$.

(i) A *γ-twisted representation* of $\mathbb{Z}^{2d}$ is a map $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$, where H is a Hilbert space, such that

$$\phi(k)\phi(k') = \gamma(k, k')\phi(k + k')$$

for all $k, k' \in \mathbb{Z}^{2d}$.

(ii) A *representation* of the γ-twisted group algebra $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is a $\star$-algebra homomorphism $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$, where H is a Hilbert space.

Note that a γ-twisted representation is nothing but a projective unitary representation with a notational emphasis on the associated 2-cocycle.

4.1.6 Theorem. *Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. If $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ is a γ-twisted representation of $\mathbb{Z}^{2d}$, then there is a unique representation $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$ such that $\phi(k) = \Phi(\delta_k)$ for all $k \in \mathbb{Z}^{2d}$. Conversely, if $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$ is a representation of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, then*

$$\begin{aligned}\phi\colon \mathbb{Z}^{2d} &\to \mathcal{U}(H) \\ k &\mapsto \Phi(\delta_k)\end{aligned}$$

defines a γ-twisted representation of $\mathbb{Z}^{2d}$.

These constructions are mutual inverses of each other, so there is a bijective correspondence between γ-twisted representations of $\mathbb{Z}^{2d}$ and representations of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$.

Proof. Let $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ be a γ-twisted representation of $\mathbb{Z}^{2d}$. Since $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$ is a vector space with basis $\mathcal{D} = \{\delta_k : k \in \mathbb{Z}^{2d}\}$, we can define a unique linear map $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)$ by $\Phi(\delta_k) = \phi(k)$. Clearly, $\Phi(\mathbb{T}\mathcal{D}) \subset \mathcal{U}(H)$. We also find that

$$\begin{aligned}\Phi(\delta_k *_\gamma \delta_{k'}) &= \Phi\big(\gamma(k,k')\delta_{k+k'}\big) = \gamma(k,k')\phi(k+k')\\ &= \phi(k)\phi(k') = \Phi(\delta_k)\Phi(\delta_{k'}),\end{aligned}$$

for all $k, k' \in \mathbb{Z}^{2d}$, so the restriction (and corestriction) $\Phi|_{\mathbb{T}\mathcal{D}}\colon \mathbb{T}\mathcal{D} \to \mathcal{U}(H)$ is a group homomorphism. Since adjoints are given by inverses in both $\mathbb{T}\mathcal{D}$ and $\mathcal{U}(H)$, the map $\Phi|_{\mathbb{T}\mathcal{D}}$ preserves the involution. Since the involution and the product on $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$ are conjugate-linear and bilinear extensions of their restrictions to $\mathcal{D}$, it follows that the linear map Φ is a $\star$-algebra homomorphism and hence a representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$.

Assume now that $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)$ is an arbitrary representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$. Since all elements of $\mathcal{D}$ are unitary and Φ is a $\star$-algebra homomorphism, it follows that $\Phi(\mathcal{D}) \subset \mathcal{U}(H)$. We can therefore define a map $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ by $\phi(k) = \Phi(\delta_k)$. Letting $k, k' \in \mathbb{Z}^{2d}$, the calculation

$$\begin{aligned}\phi(k)\phi(k') &= \Phi(\delta_k)\Phi(\delta_{k'}) = \Phi(\delta_k *_\gamma \delta_{k'}) = \gamma(k,k')\Phi(\delta_{k+k'})\\ &= \gamma(k,k')\phi(k+k')\end{aligned}$$

shows that ϕ is a γ-twisted representation of $\mathbb{Z}^{2d}$.

The passage from a γ-twisted representation ϕ to the representation Φ is achieved by a linear extension from the basis $\mathcal{D}$, while the other direction is achieved by restricting a linear map to $\mathcal{D}$. These are clearly mutually inverse operations. $\square$

We now define a canonical γ-twisted representation of $\mathbb{Z}^{2d}$ for any 2-cocycle γ. This proves the assertion preceding Definition 4.1.1 that any 2-cocycle on $\mathbb{Z}^{2d}$ gives rise to a projective unitary representation of $\mathbb{Z}^{2d}$, assuring us that 2-cocycles are the correct characterization of the maps $\mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{T}$ associated to projective unitary representations.

4.1.7 Definition (γ-twisted left regular representations)**.** Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. The *γ-twisted left regular representation* of $\mathbb{Z}^{2d}$ is the map

$$\begin{aligned}L_\gamma\colon \mathbb{Z}^{2d} &\to \mathcal{U}\big(\ell^2(\mathbb{Z}^{2d})\big)\\ k &\mapsto L_\gamma(k),\end{aligned}$$

where $L_\gamma(k)$ is defined by

$$\big(L_\gamma(k)f\big)(l) = \gamma(k, l-k)f(l-k)$$

for all $f \in \ell^2(\mathbb{Z}^{2d})$ and $l \in \mathbb{Z}^{2d}$.

It is immediate that $L_\gamma(k) \in \mathcal{U}(\ell^2(\mathbb{Z}^{2d}))$ for any $k \in \mathbb{Z}^{2d}$. We also find that, for all $l, k, k' \in \mathbb{Z}^{2d}$ and $f \in \ell^2(\mathbb{Z}^{2d})$,

$$\begin{aligned}\big(L_\gamma(k)L_\gamma(k')f\big)(l) &= \gamma(k, l-k)\gamma\big(k', (l-k)-k'\big)f\big((l-k)-k'\big)\\ &= \gamma(k,k')\gamma\big(k+k', l-(k+k')\big)f\big(l-(k+k')\big)\\ &= \gamma(k,k')\big(L_\gamma(k+k')f\big)(l)\end{aligned}$$

(setting $k_1 = k$, $k_2 = k'$ and $k_3 = l - k - k'$ in point (ii) of Definition 4.1.1 gives the transition to the second line). This shows that L_γ is a γ-twisted representation of $\mathbb{Z}^{2d}$, as the name claims.

We conclude this subsection with a simple observation that we will need in the next one.

4.1.8 Lemma. *Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. Then, the representation $\mathcal{L}_\gamma \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(\ell^2(\mathbb{Z}^{2d}))$ corresponding to the γ-twisted left regular representation L_γ (see Theorem 4.1.6) is injective.*

Proof. For any $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, write $a_k = a(k)$ for all $k \in \mathbb{Z}^{2d}$, so that $a = \sum_{k\in\mathbb{Z}^{2d}} a_k\delta_k$. Assume that

$$\sum_{k\in\mathbb{Z}^{2d}} a_k L_\gamma(k) = \mathcal{L}_\gamma(a) = 0.$$

Applying the operator $\mathcal{L}_\gamma(a)$ to $\delta_0 \in \ell^2(\mathbb{Z}^{2d})$, we find that

$$0 = \sum_{k\in\mathbb{Z}^{2d}} a_k\big(L_\gamma(k)\delta_0\big)(l) = \sum_{k\in\mathbb{Z}^{2d}} a_k\gamma(k, l-k)\delta_k(l) = \sum_{k\in\mathbb{Z}^{2d}} a_k\delta_k(l) = a_l$$

for every $l \in \mathbb{Z}^{2d}$. Thus, $\mathcal{L}_\gamma(a) = 0$ implies that $a = 0$, so $\mathcal{L}_\gamma$ is injective. □

4.1.2 Twisted Group C*-Algebras

Fix a 2-cocycle γ on $\mathbb{Z}^{2d}$ and let $\phi \colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ be any γ-twisted representation of $\mathbb{Z}^{2d}$. In this subsection, we will investigate the structure of the C*-algebra $C^*(\phi(\mathbb{Z}^{2d})) \subset \mathcal{B}(H)$ generated by $\phi(\mathbb{Z}^{2d})$.

Using the fact that the closure of a $\star$-subalgebra of $\mathcal{B}(H)$ is a $\star$-subalgebra (which follows by continuity of all operations), it is straightforward to check that the C*-algebra generated by $\phi(\mathbb{Z}^{2d})$ is the norm-closure of the $\star$-algebra generated by $\phi(\mathbb{Z}^{2d})$. Thus, we can abstract the most general structure of $C^*(\phi(\mathbb{Z}^{2d}))$ (as ϕ varies) by finding the largest possible C*-algebra in which $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is dense.

With the goal of making this precise, we introduce not-necessarily-complete versions of C*-algebras and consider their completions.

4.1.9 Definition (Pre-C*-algebras)**.** A *pre-C*-algebra* is a $\star$-algebra A equipped with a submultiplicative norm that satisfies the C*-equality.

As we observed right after the definition of C*-algebras (Definition 2.2.1), the C*-equality implies that the unit is normalized, so pre-C*-algebras satisfy all the defining requirements of C*-algebras except completeness.

A *C*-algebra completion* of a pre-C*-algebra A is a C*-algebra $\mathcal{A}$ along with an isometric $\star$-algebra homomorphism $i_A \colon A \to \mathcal{A}$ whose image is dense. We may also refer to either $\mathcal{A}$ or the map i_A by itself as the C*-algebra completion of A. The following proposition shows that C*-algebra completions exist and that they can be obtained from any Banach space completion of A. Moreover, it shows that such completions are essentially unique.

4.1.10 Proposition (C*-algebra completions)**.** *Let A be a pre-C*-algebra and let $i_A \colon A \to \mathcal{A}$ be a Banach space completion of A. Then, there is a unique product and a unique involution on $\mathcal{A}$ such that $i_A \colon A \to \mathcal{A}$ becomes a $\star$-algebra homomorphism and $\mathcal{A}$ becomes a C*-algebra. In other words: such that $i_A \colon A \to \mathcal{A}$ becomes a C*-algebra completion of A.*

Moreover, if $j_A \colon A \to \mathcal{A}'$ is any other C-algebra completion of A, then there exists a unique (isometric) $\star$-algebra isomorphism $\Phi \colon \mathcal{A} \to \mathcal{A}'$ such that $j_A = \Phi \circ i_A$.*

Proof. A thorough discussion of Banach space completions, C*-algebra completions and more can be found in Appendix B. For a proof of this result, see Proposition B.2.10. $\square$

With this terminology under our belt, our goal is to construct the largest possible (or *universal*) C*-algebra completion of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. Thus, we need to equip it with the largest possible norm that turns it into a pre-C*-algebra.

Let $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. We define $\|a\|_u$ (u for *universal*) to be the supremum over all $r \in \mathbb{R}$ such that there exists a representation $\Phi \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$ with $\|\Phi(a)\| = r$.

4.1.11 Lemma (The universal norm)**.** *For any 2-cocycle γ on $\mathbb{Z}^{2d}$, the map*

$$\begin{aligned} \|\cdot\|_u \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] &\to [0, \infty) \\ a &\mapsto \|a\|_u \end{aligned}$$

is a norm on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ that turns it into a pre-C-algebra.*

Proof. For any $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, let $N(a)$ denote the set over which we are taking the supremum. That is, $r \in N(a)$ if and only if there exists a representation $\Phi \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$ such that $\|\Phi(a)\| = r$.

The existence and the injectivity of the representation $\mathcal{L}_\gamma$ of Lemma 4.1.8 implies both that $N(a) \neq \emptyset$ for any $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ and that $\|a\|_u = \sup N(a) > 0$ whenever $a \neq 0$.

For any representation $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$, we find that

$$\|\Phi(a)\| = \Big\|\Phi\Big(\sum_{k\in\mathbb{Z}^{2d}} a(k)\delta_k\Big)\Big\| \leq \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\|\Phi(\delta_k)\| = \sum_{k\in\mathbb{Z}^{2d}} |a(k)|,$$

since $\Phi(\delta_k) \in \mathcal{U}(H)$ and unitaries are normalized. This shows that $N(a)$ is bounded (these sums are finite, so $\sup N(a) \leq \sum_{k\in\mathbb{Z}^{2d}} |a(k)| < \infty$). We have now proved well-definition and nondegeneracy.

Fix any $a, b \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. If $r \in N(a+b)$, then there is a representation $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$ such that

$$r = \|\Phi(a+b)\| \leq \|\Phi(a)\| + \|\Phi(b)\| \leq \|a\|_u + \|b\|_u.$$

This implies that $\|a+b\|_u = \sup N(a+b) \leq \|a\|_u + \|b\|_u$, so the triangle inequality holds. A similar arguments shows that $\|\cdot\|_u$ is submultiplicative.

For any representation $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$, we have that

$$\|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2.$$

Thus, for any $r \geq 0$ we have $r \in N(a)$ if and only if $r^2 \in N(a^*a)$. This gives:

$$\|a^*a\|_u = \sup N(a^*a) = (\sup N(a))^2 = \|a\|_u^2,$$

i.e. the C*-equality. Fixing $\lambda \in \mathbb{C}$, a similar argument shows that $N(\lambda a) = |\lambda|N(a)$, which gives homogeneity.

We have now shown that $\|\cdot\|_u$ is a well-defined submultiplicative norm satisfying the C*-equality, so we are done. □

We are now ready to define the γ-twisted group C*-algebra $C^*(\mathbb{Z}^{2d}, \gamma)$. To reiterate, this is an abstract realization of the most general C*-algebra generated by the images of γ-twisted representation of $\mathbb{Z}^{2d}$.

4.1.12 Definition (γ-twisted group C*-algebras)**.** Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. The *γ-twisted group C*-algebra* $C^*(\mathbb{Z}^{2d}, \gamma)$ is the C*-algebra completion of the pre-C*-algebra $(\mathbb{C}[\mathbb{Z}^{2d}, \gamma], \|\cdot\|_u)$.

As is customary, we will often identify $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ with its (isometric) image in $C^*(\mathbb{Z}^{2d}, \gamma)$.

We now upgrade Theorem 4.1.6 (on the correspondence of representations) from $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ to $C^*(\mathbb{Z}^{2d}, \gamma)$. As in Subsection 2.2.5, and in line with Definition 4.1.5, a *representation* of $C^*(\mathbb{Z}^{2d}, \gamma)$ is simply a $\star$-algebra homomorphism $\Phi\colon C^*(\mathbb{Z}^{2d}, \gamma) \to \mathcal{B}(H)$, where H is a Hilbert space.

4.1.13 Theorem. *Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. If $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ is γ-twisted representation of $\mathbb{Z}^{2d}$, then there is a unique representation $\Phi\colon C^*(\mathbb{Z}^{2d},\gamma) \to \mathcal{B}(H)$ such that $\phi(k) = \Phi(\delta_k)$ for all $k \in \mathbb{Z}^{2d}$. Conversely, if $\Phi\colon C^*(\mathbb{Z}^{2d},\gamma) \to \mathcal{B}(H)$ is a representation of $C^*(\mathbb{Z}^{2d},\gamma)$, then*

$$\begin{aligned} \phi\colon \mathbb{Z}^{2d} &\to \mathcal{U}(H) \\ k &\mapsto \Phi(\delta_k) \end{aligned}$$

defines a γ-twisted representation of $\mathbb{Z}^{2d}$.

These constructions are mutual inverses of each other, so there is a bijective correspondence between γ-twisted representations of $\mathbb{Z}^{2d}$ and representations of $C^*(\mathbb{Z}^{2d},\gamma)$.

Proof. With Theorem 4.1.6 in mind, we will establish a bijective correspondence between representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$ and of $C^*(\mathbb{Z}^{2d},\gamma)$. Any representation $\Phi\colon C^*(\mathbb{Z}^{2d},\gamma) \to \mathcal{B}(H)$ will clearly restrict to a representation $\Phi|_{\mathbb{C}[\mathbb{Z}^{2d},\gamma]}\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)$, so one direction of this correspondence is trivial.

For the other direction, suppose that $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)$ is an arbitrary representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$. By definition of the universal norm $\|\cdot\|_u$, we have that

$$\|\Phi(a)\| \leq \|a\|_u \quad \text{for all } a \in \mathbb{C}[\mathbb{Z}^{2d},\gamma].$$

Thus, $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)$ is bounded. By Proposition B.1.2, Φ has a unique bounded linear extensions $\overline{\Phi}\colon C^*(\mathbb{Z}^{2d},\gamma) \to \mathcal{B}(H)$. By Proposition B.2.7, $\overline{\Phi}$ is also a $\star$-algebra homomorphism. Thus, $\overline{\Phi}$ is a representation of $C^*(\mathbb{Z}^{2d},\gamma)$.

Now, $\overline{\Phi}$ is only unique as a bounded linear extensions of Φ. However, any $\star$-algebra homomorphism from $C^*(\mathbb{Z}^{2d},\gamma)$ to $\mathcal{B}(H)$ must be bounded by Proposition 2.2.6. Thus, $\overline{\Phi}$ is the only $\star$-algebra homomorphism $C^*(\mathbb{Z}^{2d},\gamma) \to \mathcal{B}(H)$ that restricts to Φ.

Restrictions and unique linear bounded extensions are clearly mutual inverses, so this establishes the claimed bijective correspondence between representations of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$ and of $C^*(\mathbb{Z}^{2d},\gamma)$. The fact that the resulting correspondence between representations of $C^*(\mathbb{Z}^{2d},\gamma)$ and γ-twisted representations of $\mathbb{Z}^{2d}$ takes the claimed form follows immediately from Theorem 4.1.6 and the fact that we are working with extensions and restrictions. □

We close this subsection with a useful lemma regarding isomorphisms between twisted group algebras.

4.1.14 Lemma. *Let γ_1 and γ_2 be two 2-cocycles on $\mathbb{Z}^{2d}$ and suppose that $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma_1] \to \mathbb{C}[\mathbb{Z}^{2d},\gamma_2]$ is a $\star$-algebra isomorphism. Then, with respect to the universal norms on $\mathbb{C}[\mathbb{Z}^{2d},\gamma_1]$ and $\mathbb{C}[\mathbb{Z}^{2d},\gamma_2]$, Φ is isometric. This means* *that Φ extends to an (isometric) $\star$-algebra isomorphism $\overline{\Phi}\colon C^*(\mathbb{Z}^{2d}, \gamma_1) \to C^*(\mathbb{Z}^{2d}, \gamma_2)$.*

Proof. Suppose that $\Phi'\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma_2] \to \mathcal{B}(H)$ is a representation of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma_2]$. Then, $\Phi' \circ \Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma_1] \to \mathcal{B}(H)$ is a representation of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma_1]$, which means that

$$\|\Phi'(\Phi(a))\| = \|\Phi' \circ \Phi(a)\| \leq \|a\|_u \quad \text{for all } a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma_1],$$

by definition of the universal norm on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma_1]$. Since Φ' was arbitrary, this implies that

$$\|\Phi(a)\|_u \leq \|a\|_u \quad \text{for all } a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma_1],$$

this time by definition of the universal norm on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma_2]$. This shows that Φ is norm-decreasing. Since we can apply the same argument to Φ^{-1}, Φ must be isometric: $\|a\|_u = \|\Phi^{-1}\Phi(a)\|_u \leq \|\Phi(a)\|_u \leq \|a\|_u$ for all $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma_1]$.

By Theorem B.1.2 and point (iv) of Proposition B.1.4, Φ extends to a bounded linear *bijection* $\overline{\Phi}\colon C^*(\mathbb{Z}^{2d}, \gamma_1) \to C^*(\mathbb{Z}^{2d}, \gamma_2)$. By Proposition B.2.7, $\overline{\Phi}$ is a $\star$-algebra homomorphism and hence a $\star$-algebra isomorphism between C*-algebras. We can use the isometry of Φ to conclude that $\overline{\Phi}$ is an isometry, but we can also just refer to Proposition 2.2.6, which means that we are done. □

4.1.3 Twisted Convolution Algebras

Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. In this short subsection, we will introduce an algebra that lies between $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ and $C^*(\mathbb{Z}^{2d}, \gamma)$. It will a completion of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ (w.r.t. another norm than $\|\cdot\|_u$) that maps continuously into $C^*(\mathbb{Z}^{2d}, \gamma)$, but it will not be a C*-algebra; it will be a Banach $\star$-algebra. This algebra will be central to our construction of Hilbert C*-modules for Gabor theory.

Consider the ℓ^1-norm on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$:

$$\|a\|_1 = \sum_{k \in \mathbb{Z}^{2d}} |a(k)| \quad \text{for } a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma].$$

If we take the Banach space completion of $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ with respect to this norm, we clearly obtain a vector space isomorphic to $\ell^1(\mathbb{Z}^{2d})$. It turns out that the product and involution on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ also extend to this completion and yields a Banach $\star$-algebra.

4.1.15 Definition (pre-Banach $\star$-algebras)**.** A *pre-Banach $\star$-algebra* is an algebra B equipped with a submultiplicative norm that normalizes the unit and an involution that is isometric.

A *Banach $\star$-algebra completion* of a pre-Banach $\star$-algebra B is a Banach $\star$-algebra $\mathcal{B}$ along with an isometric $\star$-algebra homomorphism $i_B \colon B \to \mathcal{B}$ whose image is dense. The following proposition does for pre-Banach $\star$-algebras exactly what Proposition 4.1.10 did for pre-C*-algebras.

4.1.16 Proposition (Banach $\star$-algebra completions)**.** *Let B be a pre-Banach $\star$-algebra and let $i_B \colon B \to \mathcal{B}$ be a Banach space completion of B. Then, there is a unique product and a unique involution on $\mathcal{B}$ such that $i_B \colon B \to \mathcal{B}$ becomes a $\star$-algebra homomorphism and $\mathcal{B}$ becomes a Banach $\star$-algebra. In other words: such that $i_B \colon B \to \mathcal{B}$ becomes a Banach $\star$-algebra completion of B.*

Moreover, if $j_B \colon B \to \mathcal{B}'$ is any other Banach $\star$-algebra completion of B, then there exists a unique isometric $\star$-algebra isomorphism $\Phi \colon \mathcal{B} \to \mathcal{B}'$ such that $j_B = \Phi \circ i_B$.

Proof. See Proposition B.2.6 and Corollary B.2.8. $\square$

4.1.17 Lemma. *For any 2-cocycle γ on $\mathbb{Z}^{2d}$, the γ-twisted group algebra $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ equipped with the ℓ^1-norm is a pre-Banach $\star$-algebra.*

Proof. Let $a, b \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. By definition of the product in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, we find that

$$\begin{aligned}\|a *_\gamma b\|_1 &= \sum_{k \in \mathbb{Z}^{2d}} \Big| \sum_{l \in \mathbb{Z}^{2d}} a(l) b(k-l) \gamma(l, k-l) \Big| \\ &\leq \sum_{k \in \mathbb{Z}^{2d}} \sum_{l \in \mathbb{Z}^{2d}} |a(l) b(k-l)| \\ &= \sum_{l \in \mathbb{Z}^{2d}} |a(l)| \sum_{k \in \mathbb{Z}^{2d}} |b(k-l)| = \|a\|_1 \|b\|_1,\end{aligned}$$

so $\|\cdot\|_1$ is submultiplicative. The unit in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is δ_0, which is clearly normalized. Similarly, it is immediate from its definition that the involution is an isometry:

$$\|a^*\|_1 = \sum_{k \in \mathbb{Z}^{2d}} |\overline{\gamma(k,-k)} a(-k)| = \sum_{k \in \mathbb{Z}^{2d}} |a(k)| = \|a\|_1.$$

$\square$

4.1.18 Definition (γ-twisted convolution algebras)**.** Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. The *γ-twisted convolution algebra* $\ell^1(\mathbb{Z}^{2d}, \gamma)$ is the Banach $\star$-algebra completion of the pre-Banach $\star$-algebra $(\mathbb{C}[\mathbb{Z}^{2d}, \gamma], \|\cdot\|_1)$.

Since $\ell^1(\mathbb{Z}^{2d}, \gamma) \cong \ell^1(\mathbb{Z}^{2d})$ as Banach spaces, all elements of $\ell^1(\mathbb{Z}^{2d}, \gamma)$ can be thought of as functions $\mathbb{Z}^{2d} \to \mathbb{C}$. This makes $\ell^1(\mathbb{Z}^{2d}, \gamma)$ much more manageable than the C*-algebra $C^*(\mathbb{Z}^{2d}, \gamma)$. In particular, this means that we can write explicit expressions for products and involutions in $\ell^1(\mathbb{Z}^{2d}, \gamma)$. Indeed, in terms of components, they are exactly the same as in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$.

4.1.19 Lemma. *For any 2-cocycle γ on $\mathbb{Z}^{2d}$, there is a norm-decreasing $\star$-algebra homomorphism $\ell^1(\mathbb{Z}^{2d}, \gamma) \to C^*(\mathbb{Z}^{2d}, \gamma)$ whose restriction (and corestriction) to $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is the identity.*

Proof. In the proof of Lemma 4.1.11, we showed that for any representation $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathcal{B}(H)$, we have that

$$\|\Phi(a)\| \le \sum_{k \in \mathbb{Z}^{2d}} |a(k)| = \|a\|_1 \quad \text{for all } a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma].$$

Thus, by definition of the universal norm: $\|a\|_u \le \|a\|_1$ for all $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$.

Consider now the identity map $\mathrm{Id}\colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to \mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. If we equip its domain with the ℓ^1-norm and its target with the universal norm, then it becomes a bounded linear map between normed spaces (because $\|\mathrm{Id}(a)\|_u = \|a\|_u \le \|a\|_1$). Thus, by Theorem B.1.2, it has a unique bounded linear extension $\overline{\mathrm{Id}}\colon \ell^1(\mathbb{Z}^{2d}, \gamma) \to C^*(\mathbb{Z}^{2d}, \gamma)$ that is also norm-decreasing. By Proposoition B.2.7, this extension is a $\star$-algebra homomorphism. By construction, the restriction to $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ is the identity. □

In those cases of relevance to Gabor theory (i.e. for specific choices of γ, which we are about to introduce), we will see that the map afforded by this lemma is injective. Thus, this inclusion of $\ell^1(\mathbb{Z}^{2d}, \gamma)$ into $C^*(\mathbb{Z}^{2d}, \gamma)$ affords us with a dense subset of $C^*(\mathbb{Z}^{2d}, \gamma)$ that is very convenient for explicit calculations and constructions.

4.1.4 Connection to Time-Frequency Shifts

In this subsection, we will explore in detail how 2-cocycles, twisted representations of $\mathbb{Z}^{2d}$ and twisted group C*-algebras occur in the context of time-frequency analysis and Gabor analysis.

Consider the canonical projections

$$\begin{aligned} P_1\colon \mathbb{R}^{2d} &\to \mathbb{R}^d \\ (x, \omega) &\mapsto x \end{aligned} \qquad \text{and} \qquad \begin{aligned} P_2\colon \mathbb{R}^{2d} &\to \mathbb{R}^d \\ (x, \omega) &\mapsto \omega. \end{aligned}$$

Recall from Lemma 3.1.1 that

$$\pi(z)\pi(w) = e^{-2\pi i \eta\cdot x}\pi(z+w) \quad \text{for } z=(x,\omega), w=(y,\eta)\in\mathbb{R}^{2d}. \tag{4.4}$$

Thinking of $\mathbb{R}^{2d}$ as $\mathbb{R}^d\times(\mathbb{R}^d)^*$, there is a natural bilinear form on $\mathbb{R}^{2d}$, namely

$$\big((x,\omega),(y,\eta)\big) = (z,w)\mapsto P_2(w)\cdot P_1(z) = \eta\cdot x,$$

and this is precisely the structure that determines the phase factor occurring in Equation (4.4). We will refer to the map

$$\begin{aligned} \beta\colon \mathbb{R}^{2d}\times\mathbb{R}^{2d} &\to \mathbb{T} \\ (z,w) &\mapsto \beta(z,w) = e^{-2\pi i P_2(w)\cdot P_1(z)} \end{aligned}$$

as the *Heisenberg cocycle* on $\mathbb{R}^{2d}$. We define 2-cocycles on $\mathbb{R}^{2d}$ precisely as for $\mathbb{Z}^{2d}$ (Definition 4.1.1); it is not difficult to verify that β is a 2-cocycle on $\mathbb{R}^{2d}$. Indeed, the imaginary exponential of any bilinear map $\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}$ is easily seen to be a 2-cocycle on $\mathbb{R}^{2d}$.

We can now write Equation (4.4) as follows:

$$\pi(z)\pi(w) = \beta(z,w)\pi(z+w) \quad \text{for } z,w\in\mathbb{R}^{2d}. \tag{4.5}$$

Letting Ω_J denote the standard symplectic form on $\mathbb{R}^{2d}$, as defined by Equation (1.4), we moreover find that

$$e^{2\pi i\Omega_J(z,w)} = \beta(z,w)\overline{\beta(w,z)} \quad \text{for } z,w\in\mathbb{R}^{2d}, \tag{4.6}$$

since $-P_2(w)\cdot P_1(z) + P_2(z)\cdot P_1(w) = -\eta\cdot x + \omega\cdot y = \Omega_J(z,w)$ for $z=(x,\omega)$ and $w=(y,\eta)$.

We now establish how these quantities appear under the lattice-induced changes of basis described in Subsection 3.2.2 (where we traded the variation of lattices for the variation of symplectic forms).

4.1.20 Definition (Heisenberg and symplectic 2-cocycles)**.** Let $A\in\mathrm{GL}(2d,\mathbb{R})$, set $\theta = A^TJA$ and let Ω_θ denote the symplectic form represented by θ, i.e. $\Omega_\theta(z,w) = w^T\theta z$. The *Heisenberg cocycle* β_A determined by A is the 2-cocycle

$$\begin{aligned} \beta_A\colon \mathbb{R}^{2d}\times\mathbb{R}^{2d} &\to \mathbb{T} \\ (z,w) &\mapsto \beta_A(z,w) \coloneqq \beta(Az,Aw) = e^{-2\pi i P_2(Aw)\cdot P_1(Az)}. \end{aligned}$$

We also introduce the 2-cocycle

$$\begin{aligned} \rho_A\colon \mathbb{R}^{2d}\times\mathbb{R}^{2d} &\to \mathbb{T} \\ (z,w) &\mapsto \rho_A(z,w) \coloneqq e^{2\pi i\Omega_\theta(z,w)}, \end{aligned}$$

which we will refer to as the *symplectic 2-cocycle* determined by A. We will not distinguish β_A and ρ_A from their restrictions to $\mathbb{Z}^{2d}\times\mathbb{Z}^{2d}$, which are 2-cocycles on $\mathbb{Z}^{2d}$.

4.1.21 Lemma. *With the notation of the last definition, we have that*

$$\pi_A(z)\pi_A(w) = \beta_A(z,w)\pi_A(z+w) \tag{4.7}$$

and that

$$\rho_A(z,w) = e^{2\pi i \Omega_\theta(z,w)} = \beta_A(z,w)\overline{\beta_A(w,z)} \tag{4.8}$$

for all $z, w \in \mathbb{R}^{2d}$.

Proof. Since $\beta_A(z,w) = \beta(Az, Aw)$ and $\Omega_\theta(z,w) = \Omega_J(Az, Aw)$, equations (4.7) and (4.8) follow immediately from equations (4.5) and (4.6). □

The following result is immediate, but it is the reason for introducing all of the machinery of this chapter, so we elevate its status to that of a proposition.

4.1.22 Proposition. *For any* $A \in \mathrm{GL}(2d, \mathbb{R})$*, the map*

$$\begin{aligned} \pi_A \colon \mathbb{Z}^{2d} &\to \mathcal{U}\big(L^2(\mathbb{R}^d)\big) \\ k &\mapsto \pi_A(k) \end{aligned}$$

is a β_A*-twisted representation of* $\mathbb{Z}^{2d}$.

Proof. Equation (4.7). □

Recall that $C^*(\pi_A(\mathbb{Z}^{2d}))$ denotes the C*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^d))$ generated by $\pi_A(\mathbb{Z}^{2d})$.

4.1.23 Corollary. *For any* $A \in \mathrm{GL}(2d, \mathbb{R})$*, there is a* $\star$*-algebra homomorphism*

$$\begin{aligned} \Pi_A \colon C^*(\mathbb{Z}^{2d}, \beta_A) &\to \mathcal{B}\big(L^2(\mathbb{R}^d)\big) \\ \delta_k &\mapsto \Pi_A(\delta_k) = \pi_A(k). \end{aligned}$$

Moreover, $\Pi_A(C^*(\mathbb{Z}^{2d}, \beta_A)) = C^*(\pi_A(\mathbb{Z}^{2d}))$.

Proof. By Theorem 4.1.13, the β_A-twisted representation of $\mathbb{Z}^{2d}$ described in Proposition 4.1.22 implies the existence of the $\star$-algebra homomorphism Π_A.

By continuity of Π_A (Proposition 2.2.6), we have that

$$\Pi_A\big(C^*(\mathbb{Z}^{2d}, \beta_A)\big) = \Pi_A\Big(\overline{\mathbb{C}[\mathbb{Z}^{2d}, \beta_A]}^{\|\cdot\|_u}\Big) \subset \overline{\Pi_A\big(\mathbb{C}[\mathbb{Z}^{2d}, \beta_A]\big)} \subset C^*\big(\pi_A(\mathbb{Z}^{2d})\big),$$

where the last inclusions follows from the fact that $\Pi_A(\mathbb{C}[\mathbb{Z}^{2d}, \beta_A])$ is included in the $\star$-algebra generated by $\pi_A(\mathbb{Z}^{2d})$ (as is straightforward to verify). By the first isomorphism theorem for C*-algebras (Theorem 2.2.33), the image of Π_A, which clearly contains $\pi_A(\mathbb{Z}^{2d})$, is a C*-algebra. Thus, $C^*(\pi_A(\mathbb{Z}^{2d})) \subset \Pi_A(C^*(\mathbb{Z}^{2d}, \beta_A))$, which concludes the proof. □

We now wish to show that Π_A is injective, from which Corollary 4.1.23 (along with the first isomorphism theorem for C*-algebras) implies that $C^*(\mathbb{Z}^{2d}, \beta_A) \cong C^*(\pi_A(\mathbb{Z}^{2d}))$ as C*-algebras. This is a deep result, which we will not be able to prove in full. We will reproduce an argument by Rieffel [27], which in turn relies on a result by Green [16]. For this, we will need the maps introduced in the following lemma.

4.1.24 Lemma. *Let* $A \in \mathrm{GL}(2d, \mathbb{R})$ *and let* $z \in \mathbb{R}^{2d}$*. Then, the map* $\Phi_{\rho_A(-,z)} \colon \mathbb{C}[\mathbb{Z}^{2d}, \beta_A] \to \mathbb{C}[\mathbb{Z}^{2d}, \beta_A]$ *defined by*

$$\big(\Phi_{\rho_A(-,z)} a\big)(k) = \rho_A(k, z) a(k) \quad \text{for all } a \in \mathbb{C}[\mathbb{Z}^{2d}, \beta_A] \text{ and } k \in \mathbb{Z}^{2d}$$

is an isometric $\star$-algebra isomorphism (w.r.t. the universal norm).

Proof. Every statement in this proof should be quantified over all $a \in \mathbb{C}[\mathbb{Z}^{2d}, \beta_A]$ and $k \in \mathbb{Z}^{2d}$ ($z \in \mathbb{R}^{2d}$ is fixed).

Since

$$\big(\Phi_{\rho_A(-,z)} \delta_0\big)(k) = \rho_A(k, z)\delta_0(k) = \rho_A(0, z)\delta_0(k) = \delta_0(k),$$

the map $\Phi_{\rho_A(-,z)}$ preserves the unit. Bilinearity of the symplectic form Ω_θ (with $\theta = A^T J A$) implies that $\overline{\rho_A(-k, z)} = \rho_A(k, z)$, so that

$$\begin{aligned}
\big(\Phi_{\rho_A(-,z)} a\big)^*(k) &= \overline{\beta_A(k, -k)\big(\Phi_{\rho_A(-,z)} a\big)(-k)} \\
&= \overline{\beta_A(k, -k)\rho_A(-k, z)a(-k)} \\
&= \rho_A(k, z)\overline{\beta_A(k, -k)a(-k)} \\
&= \rho_A(k, z)a^*(k) = \big(\Phi_{\rho_A(-,z)} a^*\big)(k),
\end{aligned}$$

which shows that $\Phi_{\rho_A(-,z)}$ preserves the involution. Bilinearity of Ω_θ also implies that

$$\rho_A(l, z)\rho_A(k - l, z) = \rho_A(l, z)\rho_A(k, z)\rho_A(-l, z) = \rho_A(k, z)$$

for any $l \in \mathbb{Z}^{2d}$. With this in mind, it is simple to verify that $\Phi_{\rho_A(-,z)}$ preserves the product. Thus, $\Phi_{\rho_A(-,z)}$ is a $\star$-algebra homomorphism. Bijectivity follows from the fact that $(\Phi_{\rho_A(-,z)})^{-1} = \Phi_{\rho_A(-,-z)}$, as is easy to verify. Finally, $\Phi_{\rho_A(-,z)}$ is isometric by Lemma 4.1.14. $\square$

The crux of our proof that Π_A is injective will be provided by the following lemma. This is the result of Green referenced by Rieffel.

4.1.25 Lemma. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$. For each $z \in \mathbb{R}^{2d}$, let*

$$\Phi_{\rho_A(-,z)} \colon C^*(\mathbb{Z}^{2d}, \beta_A) \to C^*(\mathbb{Z}^{2d}, \beta_A)$$

be the unique bounded extension of the $\star$-algebra isomorphism from Lemma 4.1.24. Suppose that $I \subset C^*(\mathbb{Z}^{2d}, \beta_A)$ is a closed ideal such that

$$\Phi_{\rho_A(-,z)}(I) \subset I \quad \textit{for every } z \in \mathbb{R}^{2d}.$$

Then, I is trivial: $I = \{0\}$ or $I = C^*(\mathbb{Z}^{2d}, \beta_A)$.

Proof. See Rieffel [27, Proposition 2.1 on p. 265] for context and Green [16, Proposition 34 on p. 239] for the proof. One must also be familiar with the basic representation theory of C*-algebras in order to understand how Rieffel's conclusion follows from Green's result. □

Now for the promised result.

4.1.26 Proposition. *For any $A \in \mathrm{GL}(2d, \mathbb{R})$, the $\star$-algebra homomorphism Π_A of Corollary 4.1.23 is injective. Thus, $C^*(\mathbb{Z}^{2d}, \beta_A) \cong C^*(\pi_A(\mathbb{Z}^{2d}))$ as C*-algebras, with Π_A as the isomorphism.*

Proof. This proof is based on Rieffel [27, Proposition 2.2 on p. 265]. Let $z \in \mathbb{R}^{2d}$ be arbitrary. We will show that $\Phi_{\rho_A(-,z)}(\operatorname{Ker} \Pi_A) \subset \operatorname{Ker} \Pi_A$, from which Lemma 4.1.25 implies that $\operatorname{Ker} \Pi_A = \{0\}$ and hence gives the result.

By the basic commutation relation, we have that

$$\pi_A(k)\pi_A(z) = \rho_A(k, z)\pi_A(z)\pi_A(k) \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

For any $a = \sum_{k \in \mathbb{Z}^{2d}} a(k)\delta_k \in \mathbb{C}[\mathbb{Z}^{2d}, \beta_A]$, we now find that

$$\begin{aligned} \pi_A(z)^*\Pi_A(a)\pi_A(z) &= \pi_A(z)^* \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k)\pi_A(z) \\ &= \sum_{k \in \mathbb{Z}^{2d}} \rho_A(k, z)a(k)\pi_A(k) = \Pi_A\big(\Phi_{\rho_A(-,z)}a\big). \end{aligned}$$

By continuity of Π_A and $\Phi_{\rho_A(-,w)}$, along with density of $\mathbb{C}[\mathbb{Z}^{2d}, \beta_A]$, we can conclude that

$$\pi_A(z)^*\Pi_A(a)\pi_A(z) = \Pi_A\big(\Phi_{\rho_A(-,z)}a\big) \quad \text{for all } a \in C^*(\mathbb{Z}^{2d}, \beta_A).$$

This implies that $\Phi_{\rho_A(-,z)}(\operatorname{Ker} \Pi_A) \subset \operatorname{Ker} \Pi_A$, which is what we needed to show. □

Next up, we prove the result that allows us to think of $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ as a $\star$-subalgebra of $C^*(\mathbb{Z}^{2d}, \beta_A)$. It is clearly dense, as $\mathbb{C}[\mathbb{Z}^{2d}, \beta_A] \subset \ell^1(\mathbb{Z}^{2d}, \beta_A)$. Thus, if we equip $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ with the universal norm (defined in the same manner as on $\mathbb{C}[\mathbb{Z}^{2d}, \beta_A]$, or simply restricted from $C^*(\mathbb{Z}^{2d}, \beta_A)$), then $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ becomes a pre-C*-algebra whose C*-algebra completion is $C^*(\mathbb{Z}^{2d}, \beta_A)$. This is pretty much the only norm we will consider on $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ going forward.

4.1.27 Corollary. *For any $A \in \mathrm{GL}(2d, \mathbb{R})$, the $\star$-algebra homomorphism $\ell^1(\mathbb{Z}^{2d}, \beta_A) \to C^*(\mathbb{Z}^{2d}, \beta_A)$ of Lemma 4.1.19 is injective.*

If we follow it with $\Pi_A \colon C^*(\mathbb{Z}^{2d}, \beta_A) \to \mathcal{B}(L^2(\mathbb{R}^d))$, we obtain the map

$$\begin{aligned} \ell^1(\mathbb{Z}^{2d}, \beta_A) &\to \mathcal{B}(L^2(\mathbb{R}^d)) \\ a &\mapsto \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k), \end{aligned}$$

where the sums (clearly) converge absolutely. Having identified $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ with the image of its inclusion into $C^*(\mathbb{Z}^{2d}, \beta_A)$, we will denote this map by Π_A as well.

Proof. Let's denote the $\star$-algebra homomorphism $\ell^1(\mathbb{Z}^{2d}, \beta_A) \to C^*(\mathbb{Z}^{2d}, \beta_A)$ in question by Φ. By Proposition 4.1.26, Φ is injective if and only if the composition $\Pi_A \circ \Phi \colon \ell^1(\mathbb{Z}^{2d}, \beta_A) \to \mathcal{B}(L^2(\mathbb{R}^d))$ is injective; we will show that the composition is injective.

We know that $\Phi|_{\mathbb{C}[\mathbb{Z}^{2d}, \beta_A]}$ is the identity (Lemma 4.1.19). Thus, by continuity of Φ and Π_A, we find that

$$\Pi_A \circ \Phi(a) = \sum_{k \in \mathbb{Z}^{2d}} a(k)\Pi_A(\delta_k) = \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k) \quad \text{for all } a \in \ell^1(\mathbb{Z}^{2d}, \beta_A),$$

where the sums converge absolutely. We need to show that

$$\Pi_A \circ \Phi(a) = \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k) = 0 \qquad \Longrightarrow \qquad a(k) = 0 \text{ for all } k \in \mathbb{Z}^{2d}.$$

Our proof of this fact is based on Gröchenig [18, Proposition 2.3 on p. 5].

Fix $f, g \in L^2(\mathbb{R}^d)$ and let $a \in \ell^1(\mathbb{Z}^{2d}, \beta_A)$ be such that $\Pi_A \circ \Phi(a) = 0$. Then,

$$0 = \sum_{k \in \mathbb{Z}^{2d}} a(k)\big\langle \pi_A(k)\pi_{A^\circ}(z)f, \pi_{A^\circ}(z)g\big\rangle \quad \text{for all } z \in \mathbb{R}^{2d}.$$

By the basic commutation relation and the fact that time-frequency shifts are unitary, we find that

$$\pi_{A^\circ}(z)^*\pi_A(k)\pi_{A^\circ}(z) = e^{-2\pi i \Omega_J(A^\circ z, Ak)}\pi_A(k) \quad \text{for all } k \in \mathbb{Z}^{2d}$$

(note that $z \in \mathbb{R}^{2d}$; commutativity of $\pi_A(k)$ and $\pi_{A^\circ}(z)$ only follows if $z \in \mathbb{Z}^{2d}$).

Now, $A^\circ = -JA^{-T}$, so $\Omega_J(A^\circ z, Ak) = (Ak)^T J(-JA^{-T}z) = k^T \cdot z$. Combining this with the last two displayed equations gives:

$$0 = \sum_{k\in\mathbb{Z}^{2d}} a(k)\langle \pi_A(k)f, g\rangle e^{-2\pi i k^T\cdot z} \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Now, $|\langle \pi_A(k)f, g\rangle| \leq \|f\|_2\|g\|_2$ for all $k \in \mathbb{Z}^{2d}$ and $\sum_{k\in\mathbb{Z}^{2d}} |a(k)| < \infty$, so the sum above is an absolutely convergent Fourier series on $[0, 1]^{2d}$. Thus, since it vanishes, we must have that

$$a(k)\langle \pi_A(k)f, g\rangle = 0 \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

Finally, $f, g \in L^2(\mathbb{R}^d)$ were arbitrary, so this means that $a(k) = 0$ for all $k \in \mathbb{Z}^{2d}$ (one may e.g. take $g = \pi_A(k')f$ for $k' \in \mathbb{Z}^{2d}$), which is what we wanted to show. □

4.2 | Noncommutative Tori

Noncommutative tori are defined as C*-algebras that are *universal* with respect to a set of relations. We will only concern ourself with what it means for a noncommutative torus to be universal; we refer to Blackadar [6, Section II.8.3] for a discussion of universality of C*-algebras in a broader context.

The noncommutative torus is a foundational example in the theory of C*-algebras and operator algebras in general, and much is known about its structure. Indeed, the central constructions of our text, the equivalence bimodules which are Hilbert C*-modules over noncommutative tori, were investigated by Rieffel [27] long before they were discovered to be of relevance to Gabor theory by Luef [21, 22]. Thus, it is a fruitful endeavour to understand the connection between time-frequency shifts and noncommutative tori in detail. The main takeaway of this section will be that our β_A-twisted group C*-algebras $C^*(\beta_A, \mathbb{Z}^{2d})$ are noncommutative tori.

4.2.1 Twisted Group C*-Algebras as Noncommutative Tori

Recall that $\mathcal{T}_{2d}$ denotes the set of antisymmetric matrices in $M_{2d}(\mathbb{R})$ (Definition 1.1.3), and that a subset S of a C*-algebra $\mathcal{A}$ is said to *generate* the smallest C*-subalgebra of $\mathcal{A}$ containing S (Definition 2.2.12).

4.2.1 Definition (Noncommutative tori)**.** Let $\theta \in \mathcal{T}_{2d}$. The *noncommutative torus* $\mathcal{A}_\theta$ is a C*-algebra that is generated by unitaries $U_1, \dots, U_{2d}$ satisfying the *NCT-relations*

$$U_j U_i = e^{2\pi i \theta_{ij}} U_i U_j \quad \text{for } 1 \le i, j \le 2d, \tag{4.9}$$

which moreover has the following universal property: if $\mathcal{A}$ is any C*-algebra generated by unitaries $V_1, \dots, V_{2d}$ satisfying the NCT-relations, then there exists a $\star$-algebra homomorphism $\Phi\colon \mathcal{A}_\theta \to \mathcal{A}$ such that $\Phi(U_j) = V_j$ for $1 \le j \le 2d$.

Assuming the existence of $\mathcal{A}_\theta$ (which we will prove), it is unique up to $\star$-algebra isomorphisms: if $\mathcal{A}_\theta$ and $\mathcal{A}'_\theta$ both satisfy the universal property of the definition, we are guaranteed $\star$-algebra homomorphisms $\Phi\colon \mathcal{A}_\theta \to \mathcal{A}'_\theta$ and $\Phi'\colon \mathcal{A}'_\theta \to \mathcal{A}_\theta$ that are mutual inverses because of how they map the generators (along with continuity of $\star$-algebra homomorphism between C*-algebras: Proposition 2.2.6). Moreover, to motivate the term *universal*, note that if $\mathcal{A}$ is any C*-algebra generated by unitaries satisfying the NCT-relations and if $\Phi\colon \mathcal{A}_\theta \to \mathcal{A}$ is the $\star$-algebra homomorphism guaranteed by the universality of $\mathcal{A}_\theta$, we have that

$$\mathcal{A} \cong \mathcal{A}_\theta / \operatorname{Ker} \Phi \quad \text{as C*-algebras}$$

by the first isomorphism theorem for C*-algebras (Theorem 2.2.33). Thus, all such C*-algebras are quotients of $\mathcal{A}_\theta$.

The aim of this subsection is to prove the following theorem. This is a well-known and commonly quoted result, but the author was unable to find an explicit proof. Thus, the given proof, though certainly not original, is the result of the author's struggle to connect the dots.

Recall that $\{e_j\}_{j=1}^{2d}$ denotes the standard basis for ($\mathbb{R}^{2d}$ and) $\mathbb{Z}^{2d}$.

4.2.2 Theorem. *Let $\theta \in \mathcal{T}_{2d}$ and let $B\colon \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{R}$ be any biadditive (i.e. $\mathbb{Z}$-bilinear) map such that the 2-cocycle $\gamma := e^{2\pi i B}$ satisfies*

$$e^{2\pi i l^T \theta k} = \gamma(k, l)\overline{\gamma(l, k)} \quad \text{for all } k, l \in \mathbb{Z}^{2d}.$$

Then, $C^*(\mathbb{Z}^{2d}, \gamma)$ is the noncommutative torus $\mathcal{A}_\theta$ with $\delta_{e_1}, \dots, \delta_{e_{2d}}$ as the unitary generators satisfying the NCT-relations.

We will prove Theorem 4.2.2 through a series of lemmas. Given unitaries $V_1, \dots, V_{2d}$ satisfying the NCT-relations, in any C*-algebra $\mathcal{A}$, the idea is to construct a γ-twisted representation ϕ of $\mathbb{Z}^{2d}$ such that $\phi(e_j) = V_j$, and then use Theorem 4.1.13 to extend this to a $\star$-algebra homomorphism $C^*(\mathbb{Z}^{2d}, \gamma) \to$

$\mathcal{A}$ such that $\delta_{e_j} \mapsto V_j$. By the Gelfand-Naimark theorem (Theorem 2.2.42) this is a valid application of Theorem 4.1.13, for we can take $\mathcal{A} \subset \mathcal{B}(H)$ for some Hilbert space H. This will prove that $C^*(\mathbb{Z}^{2d}, \gamma)$ has the desired universal property. However, figuring out how to define ϕ will take a bit of work.

We will have need for products of the form $V^k := V_1^{k_1} V_2^{k_2} \cdots V_{2d}^{k_{2d}}$ for $k = (k_1, \ldots, k_{2d}) \in \mathbb{Z}^{2d}$. We begin by seeing how the product $V^k V^l$ relates to V^{k+l} (for $k, l \in \mathbb{Z}^{2d}$).

4.2.3 Lemma. *Let $\theta \in \mathcal{T}_{2d}$ and let $V_1, \ldots, V_{2d}$ be unitaries (in any C*-algebra) satisfying the NCT-relations* (4.9)*. Define*

$$V^k := V_1^{k_1} V_2^{k_2} \cdots V_{2d}^{k_{2d}} \quad \textit{for } k = (k_1, \ldots, k_{2d}) \in \mathbb{Z}^{2d}.$$

Then,

$$V^k V^l = \exp\left(2\pi i \sum_{i=1}^{2d-1} \sum_{j=i+1}^{2d} l_i \theta_{ij} k_j\right) V^{k+l}$$

for all $k = (k_1, \ldots, k_{2d}), l = (l_1, \ldots, l_{2d}) \in \mathbb{Z}^{2d}$.

Proof. Note the relations

$$\begin{aligned} & V_j^{-1} V_i^{-1} = (V_i V_j)^{-1} = (e^{2\pi i \theta_{ji}} V_j V_i)^{-1} = e^{2\pi i \theta_{ij}} V_i^{-1} V_j^{-1}, \\ \text{and} \quad & V_j^{-1} V_i = V_j^{-1} (V_i V_j) V_j^{-1} = V_j^{-1} (e^{2\pi i \theta_{ji}} V_j V_i) V_j^{-1} = e^{-2\pi i \theta_{ij}} V_i V_j^{-1}. \end{aligned}$$

For $n \in \mathbb{Z}\backslash\{0\}$, write $s_n = \operatorname{sgn}(n)$. These relations then show that

$$V_j^{s_n} V_i^{s_m} = V_i^{s_m} V_j^{s_n} e^{2\pi i \theta_{ij} s_n s_m} \quad \text{for any } n, m \in \mathbb{Z}\backslash\{0\}, \tag{4.10}$$

which gives:

$$\begin{aligned} V_j^n V_i^m &= \left(V_j^{s_n}\right)^{|n|} \left(V_i^{s_m}\right)^{|m|} \\ &= V_i^{s_m} \left(V_j^{s_n}\right)^{|n|} \left(V_i^{s_m}\right)^{|m|-1} e^{2\pi i \theta_{ij} s_m s_n |n|} \\ &\vdots \\ &= V_i^m V_j^n e^{2\pi i \theta_{ij} s_m s_n |n| |m|}, \end{aligned}$$

since $s_n|n| = n$ and likewise for m. We can now observe that

$$V_j^n V_i^m = V_i^m V_j^n e^{2\pi i \theta_{ij} nm} \quad \text{for all } n, m \in \mathbb{Z},$$

because the cases $n = 0$ or $m = 0$ are easily seen to hold. Thus, with $k, l \in \mathbb{Z}^{2d}$, we find that

$$\begin{aligned} V^k V^l &= V_1^{k_1} \cdots V_{2d}^{k_{2d}} V_1^{l_1} \cdots V_{2d}^{l_{2d}} \\ &= \exp\left(2\pi i \sum_{j=2}^{2d} l_1 \theta_{1j} k_j\right) V_1^{k_1+l_1} V_2^{k_2} \cdots V_{2d}^{k_{2d}} V_2^{l_2} \cdots V_{2d}^{l_{2d}} \\ &\vdots \\ &= \exp\left(2\pi i \sum_{i=1}^{2d-1} \sum_{j=i+1}^{2d} l_i \theta_{ij} k_j\right) V_1^{k_1+l_1} \cdots V_{2d}^{k_{2d}+l_{2d}}, \end{aligned}$$

which is what we wanted to show. □

Let ϕ denote the γ-twisted representation we wish to construct. We need to define $\phi(k)$ for any $k \in \mathbb{Z}^{2d}$. We know that the extension of ϕ to the representation $\Phi \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma] \to C^*(V_1, \ldots, V_{2d})$ (Theorem 4.1.6) will satisfy $\phi(k) = \Phi(\delta_k)$. We also know that we want $\phi(e_j) = \Phi(\delta_{e_j}) = V_j$. Since Φ is a $\star$-algebra homomorphism, this means that

$$\Phi\big(\delta_{e_1}^{k_1} *_\gamma \ldots *_\gamma \delta_{e_{2d}}^{k_{2d}}\big) = V^k.$$

Thus, we can figure out how to define $\phi(k)$ in terms of V^k by writing δ_k in terms of $\delta_{e_1}^{k_1} *_\gamma \ldots *_\gamma \delta_{e_{2d}}^{k_{2d}}$. This is the content of the following lemma. We will not need really need this lemma in our proof, but it provides the motivation for everything that follows, so we include it. It will be convenient to represent the biadditive map B (defining the 2-cocycle $\gamma = e^{2\pi i B}$) by a matrix.

4.2.4 Lemma. *Let $B = (B_{ij}) \in M_{2d}(\mathbb{R})$ and define the 2-cocycle $\gamma(k, l) \coloneqq e^{2\pi i l^T B k}$. Then, for any $k = (k_1, \ldots, k_{2d}) \in \mathbb{Z}^{2d}$, we have that*

$$\delta_k = \exp\left(-2\pi i \sum_{i=1}^{2d-1} \sum_{j=i+1}^{2d} k_i (B^T)_{ij} k_j - \pi i \sum_{i=1}^{2d} k_i B_{ii}(k_i - 1)\right) \delta_{e_1}^{k_1} *_\gamma \cdots *_\gamma \delta_{e_{2d}}^{k_{2d}}$$

in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$ (where $\delta_{e_i}^{k_i} = \delta_{e_i} *_\gamma \cdots *_\gamma \delta_{e_i}$ denotes the k_ith power of δ_{e_i} in $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$).

Proof. We calculate (using Equation 4.2):

$$\begin{aligned}
\delta_k &= \delta_{k_1 e_1 + \sum_{j=2}^{2d} k_j e_j} \\
&= \delta_{k_1 e_1} *_\gamma \delta_{\sum_{j=2}^{2d} k_j e_j} \overline{\gamma}\Big(k_1 e_1, \sum_{j=2}^{2d} k_j e_j\Big) \\
&\vdots \\
&= \delta_{k_1 e_1} *_\gamma \cdots *_\gamma \delta_{k_{2d} e_{2d}} \prod_{i=1}^{2d-1} \overline{\gamma}\Big(k_i e_i, \sum_{j=i+1}^{2d} k_j e_j\Big) \\
&= \exp\Big(-2\pi i \sum_{i=1}^{2d-1} \sum_{j=i+1}^{2d} k_j B_{ji} k_i\Big) \delta_{k_1 e_1} *_\gamma \cdots *_\gamma \delta_{k_{2d} e_{2d}}.
\end{aligned} \tag{4.11}$$

Let now $1 \leq i \leq 2d$ and $n \in \mathbb{Z}$. Assume first that $n \geq 2$. Then,

$$\begin{aligned}
\delta_{ne_i} &= \delta_{e_i + (n-1)e_i} \\
&= \delta_{e_i} *_\gamma \delta_{(n-1)e_i} \overline{\gamma}\big(e_i, (n-1)e_i\big) \\
&\vdots \\
&= (\delta_{e_i})^n \prod_{j=1}^{n-1} \overline{\gamma}\big(e_i, (n-j)e_i\big) \\
&= \exp\Big(-2\pi i \sum_{j=1}^{n-1} (n-j) B_{ii}\Big) (\delta_{e_i})^n \\
&= \exp\Big(-\pi i B_{ii} n(n-1)\Big) (\delta_{e_i})^n,
\end{aligned}$$

where the last equality follows from the fact that

$$2\sum_{j=1}^{n-1} (n-j) B_{ii} = 2B_{ii} \sum_{j'=1}^{n-1} j' = B_{ii} n(n-1).$$

Suppose now that $n \leq 2$. Then, since $(\delta_{-ne_i})^* = \overline{\gamma(ne_i, -ne_i)}\delta_{ne_i}$ (almost by definition of the involution, see the proof of Lemma 4.1.3 for details) and

$-n \geq 2$, our previous result implies that

$$\begin{aligned}
\delta_{ne_i} &= \gamma(ne_i, -ne_i)\big(\delta_{-ne_i}\big)^* \\
&= \exp\Big(2\pi i(-n)B_{ii}n\Big)\overline{\exp\Big(- \pi i B_{ii}(-n)\big(- n - 1\big)\Big)}\big((\delta_{e_i})^{-n}\big)^* \\
&= \exp\Big(\pi i B_{ii}\big(- 2n^2 + n^2 + n\big)\Big)\big((\delta_{e_i})^{-n}\big)^{-1}. \\
&= \exp\Big(- \pi i B_{ii} n\big(n - 1\big)\Big)(\delta_{e_i})^n.
\end{aligned}$$

Since the cases $|n| \leq 1$ are readily verified by insertion (along with the fact that $(\delta_{e_i})^{-1} = (\delta_{e_i})^* = \overline{\gamma(-e_i, e_i)}\delta_{-e_i}$), we find that

$$\delta_{ne_i} = \exp\Big(- \pi i B_{ii} n(n-1)\Big)\delta_{e_i}^n \quad \text{for all } n \in \mathbb{Z}.$$

Combining this with Equation (4.11) gives the result. $\square$

We now know how to obtain our γ-twisted representations. The following lemma proves that everything works out as hoped.

4.2.5 Lemma. *Let $\theta \in \mathcal{T}_{2d}$ and let $B\colon \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \to \mathbb{R}$ be any biadditive map such that the 2-cocycle $\gamma := e^{2\pi i B}$ satisfies*

$$e^{2\pi i l^T \theta k} = \gamma(k,l)\overline{\gamma(l,k)} \quad \textit{for all } k, l \in \mathbb{Z}^{2d}. \tag{4.12}$$

Suppose moreover that H is a Hilbert space and that $V_1, \ldots, V_{2d}$ are unitaries in $\mathcal{B}(H)$ satisfying the NCT-relations (4.9)*. Then, if we set*

$$B_{ij} := B(e_j, e_i) \quad \textit{for } 1 \leq i, j \leq 2d, \tag{4.13}$$

the map

$$\begin{aligned}
&\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H) \\
&\qquad k \mapsto \exp\Bigg(- 2\pi i \sum_{i=1}^{2d-1} \sum_{j=i+1}^{2d} k_i (B^T)_{ij} k_j - \pi i \sum_{i=1}^{2d} k_i B_{ii}\big(k_i - 1\big)\Bigg) V^k,
\end{aligned}$$

defines a γ-twisted representation of $\mathbb{Z}^{2d}$.

Proof. First of all, Equation (4.12) implies that, for all $k, l \in \mathbb{Z}^{2d}$,

$$e^{2\pi i l^T (B - B^T) k} = e^{2\pi i l^T B k} e^{-2\pi i k^T B l} = \gamma(k,l)\overline{\gamma(l,k)} = e^{2\pi i l^T \theta k},$$

where we have identified the bilinear form B with the matrix representing it. Thus, we find that

$$(B - B^T)_{ij} \equiv \theta_{ij} \pmod 1 \quad \text{for } 1 \leq i, j \leq 2d. \tag{4.14}$$

Fix now $k, l \in \mathbb{Z}^{2d}$. We wish to show that

$$\phi(k)\phi(l) = \gamma(k,l)\phi(k+l) = e^{2\pi i l^T B k}\phi(k+l), \tag{4.15}$$

for then ϕ is a γ-twisted representation of $\mathbb{Z}^{2d}$.

By definition of ϕ, we have that $\phi(k+l) = e^{-\pi i a}V^{k+l}$, where

$$a \coloneqq 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d}(k_i+l_i)B^T_{ij}(k_j+l_j) + \sum_{i=1}^{2d}(k_i+l_i)B_{ii}\big(k_i+l_i-1\big),$$

and similarly that $\phi(k)\phi(l) = e^{-\pi i b}V^kV^l$, where

$$b \coloneqq 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d}\big(k_iB^T_{ij}k_j + l_iB^T_{ij}l_j\big) + \sum_{i=1}^{2d}\Big(k_iB_{ii}(k_i-1) + l_iB_{ii}(l_i-1)\Big).$$

Appealing to Lemma 4.2.3, we now find that

$$\begin{aligned}\phi(k)\phi(l) &= e^{-\pi i b}V^kV^l \\ &= e^{-\pi i b}\exp\left(2\pi i\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i\theta_{ij}k_j\right)V^{k+l} \\ &= e^{\pi i(a-b)}\exp\left(2\pi i\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i\theta_{ij}k_j\right)\phi(k+l).\end{aligned}$$

Equation (4.15) therefore follows if we can show that

$$a - b \equiv 2l^TBk - 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i\theta_{ij}k_j \pmod 2. \tag{4.16}$$

We take a deep breath and calculate:

$$\begin{aligned}
a - b &= 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d}\Big(k_i B^T_{ij} l_j + l_i B^T_{ij} k_j\Big) + 2\sum_{i=1}^{2d} k_i B_{ii} l_i \\
&= 2\bigg(\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} k_i B^T_{ij} l_j + \sum_{i=1}^{2d} k_i B_{ii} l_i\bigg) + 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i B^T_{ij} k_j \\
&= 2\sum_{i=1}^{2d}\sum_{j=i}^{2d} k_i B^T_{ij} l_j + 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i B^T_{ij} k_j \\
&= 2\bigg(\sum_{i=1}^{2d}\sum_{j=1}^{2d} k_i B^T_{ij} l_j - \sum_{i=2}^{2d}\sum_{j=1}^{i-1} k_i B^T_{ij} l_j\bigg) + 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i B^T_{ij} k_j \\
&= 2k^T B^T l - 2\bigg(\sum_{i=2}^{2d}\sum_{j=1}^{i-1} k_i B^T_{ij} l_j - \sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i B^T_{ij} k_j\bigg) \\
&= 2l^T B k - 2\sum_{i=1}^{2d-1}\sum_{j=i+1}^{2d} l_i (B - B^T)_{ij} k_j,
\end{aligned}$$

where we used the fact that $\sum_{i=2}^{2d}\sum_{j=1}^{i-1} = \sum_{j=1}^{2d-1}\sum_{i=j+1}^{2d}$ (and then relabelled) in the last step. Finally, Equation (4.14) now implies Equation (4.16) and hence Equation (4.15), which concludes the proof. □

We are now ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. We begin by showing that $C^*(\mathbb{Z}^{2d}, \gamma)$ is generated by the unitaries $\delta_{e_1}, \dots, \delta_{e_{2d}}$ and that theses satisfy the NCT-relations (4.9).

Clearly $\delta_{e_1}, \dots, \delta_{e_{2d}}$ generate $C^*(\mathbb{Z}^{2d}, \gamma)$ as a C*-algebra, for they form a vector space basis for $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$, the closure of which is $C^*(\mathbb{Z}^{2d}, \gamma)$. By Equation (4.2), we see that

$$\delta_k *_\gamma \delta_l = \gamma(k, l)\delta_{k+l} = \gamma(k, l)\overline{\gamma(l, k)}\delta_l *_\gamma \delta_k \quad \text{for } k, l \in \mathbb{Z}^{2d}.$$

Thus, we find that

$$\delta_{e_j} *_\gamma \delta_{e_i} = \gamma(e_j, e_i)\overline{\gamma(e_i, e_j)}\delta_{e_i} *_\gamma \delta_{e_j} \quad \text{for } 1 \le i, j \le 2d.$$

Since we are assuming that

$$\gamma(e_j, e_i)\overline{\gamma(e_i, e_j)} = e^{2\pi i (e_i)^T \theta e_j} = e^{2\pi i \theta_{ij}},$$

this shows that $\delta_{e_1}, \dots, \delta_{e_{2d}}$ satisfy the NCT-relations.

Suppose now that $\mathcal{A}$ is a C*-algebra generated by unitaries $V_1, \dots, V_{2d}$ satisfying the NCT-relations. We can appeal to the Gelfand-Naimark theorem (Theorem 2.2.42) and take $\mathcal{A}$ to be a C*-subalgebra of $\mathcal{B}(H)$, for some Hilbert space H. In other words, we can suppose that $\mathcal{A} = C^*(\{V_1, \dots, V_{2d}\}) \subset \mathcal{B}(H)$.

By Lemma 4.2.5, we obtain a γ-twisted representation $\phi\colon \mathbb{Z}^{2d} \to \mathcal{U}(H)$ such that

$$\phi(e_j) = \exp\big(-\pi i B_{jj}(1-1)\big)V_j = V_j \quad \text{for } 1 \leq j \leq 2d.$$

By Theorem 4.1.13, this implies the existence of a $\star$-algebra homomorphism $\Phi\colon C^*(\mathbb{Z}^{2d}, \gamma) \to \mathcal{B}(H)$ such that $\Phi(\delta_{e_j}) = V_j$ for $1 \leq j \leq 2d$.

Now, we need Φ to corestrict to $\Phi|^{\mathcal{A}}\colon C^*(\mathbb{Z}^{2d}, \gamma) \to \mathcal{A}$ in order to conclude that $C^*(\mathbb{Z}^{2d}, \gamma)$ has the universal property characterizing $\mathcal{A}_\theta$. This is afforded by continuity of Φ, for:

$$\Phi\big(C^*(\mathbb{Z}^{2d}, \gamma)\big) = \Phi\big(\overline{\mathbb{C}[\mathbb{Z}^{2d}, \gamma]}\big) \subset \overline{\Phi\big(\mathbb{C}[\mathbb{Z}^{2d}, \gamma]\big)} \subset C^*\big(\phi(\mathbb{Z}^{2d})\big) = \mathcal{A},$$

where the fact that $\Phi(\mathbb{C}[\mathbb{Z}^{2d}, \gamma])$ is included in the $\star$-algebra generated by $\phi(\mathbb{Z}^{2d})$ gives the last inclusion. In fact, since $\phi(\mathbb{Z}^{2d}) \subset \Phi(\mathbb{C}[\mathbb{Z}^{2d}, \gamma])$, the first isomorphism theorem for C*-algebras implies that the image of Φ must be all of $\mathcal{A}$ (this is the same exact argument that we gave in the proof of Corollary 4.1.23, just in a more general setting).

We have now shown that $C^*(\mathbb{Z}^{2d}, \gamma)$ satisfies the requirements of Definition 4.2.1, with unitary generators $\delta_1, \dots, \delta_{2d}$. Thus, $C^*(\mathbb{Z}^{2d}, \gamma)$ is the noncommutative torus and we have proved Theorem 4.2.2. □

Finally, we summarize the equivalences that tie together all of our work in this chapter up until now. This will be the main takeaway from the work we have done.

4.2.6 Corollary. *If $A \in \mathrm{GL}(2d, \mathbb{R})$ and $\theta = A^T J A$, then*

$$\mathcal{A}_\theta \cong C^*(\mathbb{Z}^{2d}, \beta_A) \cong C^*\big(\pi_A(\mathbb{Z}^{2d})\big) \quad \textit{as C*-algebras},$$

where the identifications are determined by

$$U_j \mapsto \delta_{e_j} \mapsto \pi_A(e_j) \quad \textit{for } 1 \leq j \leq 2d.$$

Proof. The second identification is provided by Proposition 4.1.26. Since

$$\beta_A(k,l)\overline{\beta_A(l,k)} = e^{2\pi i \Omega_\theta(k,l)} = e^{2\pi i l^T \theta k} \quad \text{for all } k, l \in \mathbb{Z}^{2d}$$

by Equation (4.8), the first identification is afforded by Theorem 4.2.2 with $\gamma = \beta_A$. Finally, we note that $\star$-algebra homomorphisms between C*-algebras are uniquely determined by where they map a generating set for their domain, because of continuity (Proposition 2.2.6). □

4.3 | Hilbert C*-Modules and the Feichtinger Algebra

In this section, we introduce Hilbert C*-modules. Theses are quite direct generalizations of Hilbert spaces, where the complex number field is replaced by a general C*-algebra. To a large extent, the first two subsections are based on the excellent exposition of Raeburn and Williams [24]. In the third subsection, we will follow Luef [21, 22] and construct Hilbert C*-modules from the Feichtinger algebra.

As in Chapter 2, we are assuming that all algebras and algebra homomorphisms are unital. In particular, all C*-algebras are assumed to be unital. For a complex vector space E, we will write $\mathrm{End}_{\mathbb{C}}(E)$ for the vector space of all linear maps $E \to E$. We also wish to alert the reader to a slight notational conflict: the letter A is used both for arbitrary pre-C*-algebras and for lattice matrices. This ultimately leads to the notation ${}_A\langle\cdot,\cdot\rangle$ having two distinct meanings. These two topics are confined to separate subsections, so the context will hopefully make matters clear.

4.3.1 Inner Product Modules and Hilbert C*-Modules

As the name suggests, we will require completeness of Hilbert C*-modules. We must first introduce their not-necessarily-complete counterparts: inner product modules. For the notion of a pre-C*-algebra, see Definition 4.1.9.

4.3.1 Definition (Inner product A-modules)**.** Let A be a pre-C*-algebra. A *left inner product A-module* is a (nonzero) complex vector space E equipped with the following structure.

- An algebra homomorphism $\Phi\colon A \to \mathrm{End}_{\mathbb{C}}(E)$, referred to as an *action* of A on E. We will write $a \cdot f \coloneqq \Phi(a)f$ for all $a \in A$ and $f \in E$.
- A map $\langle\cdot,\cdot\rangle\colon E \times E \to A$, referred to as an *$A$-valued inner product.*

Moreover, we require the following conditions to hold for all $\lambda \in \mathbb{C}$, $a \in A$ and $f, g, h \in E$:

(i) $\langle \lambda f + g, h\rangle = \lambda\langle f, h\rangle + \langle g, h\rangle$ ($\mathbb{C}$-linearity in the first entry)

(ii) $\langle a \cdot f, g\rangle = a\langle f, g\rangle$ (A-linearity in the first entry)

(iii) $\langle f, g\rangle^* = \langle g, f\rangle$ (conjugate symmetry)

(iv) $\langle f, f\rangle \geq 0$ in the C*-algebra completion of A (positivity)

(v) $\langle f, f\rangle = 0 \implies f = 0$ (nondegeneracy)

If E is a left inner product A-module, conjugate symmetry and A-linearity combine to yield:

$$\langle f, a \cdot g\rangle = \langle a \cdot g, f\rangle^* = (a\langle g, f\rangle)^* = \langle f, g\rangle a^*$$

for all $f, g \in E$ and $a \in A$. Similarly, we obtain (complex) conjugate-linearity in the second entry.

A *right inner product A-module* (for a pre-C*-algebra A) is defined analogously, but with the following modifications:

- The algebra homomorphism is replaced by $\Phi\colon A^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$, where A^{op} is the opposite algebra of A,[2] and we write $\Phi(a)f = f \cdot a$. We still refer to this as an action of A (as opposed to A^{op}).
- Condition (i) is replaced by $\mathbb{C}$-linearity in the second entry.
- Condition (ii) is replaced by $\langle f, g \cdot a\rangle = \langle f, g\rangle a$, which we refer to as A-linearity in the second entry.

The following is a simple but important observation, which is straightforward to verify. Suppose that A is a pre-C*-algebra and that E is a left inner product A-module, whose action and inner product we will denote by

$$\Phi_A\colon A \to \mathrm{End}_{\mathbb{C}}(E) \quad \text{and} \quad {}_A\langle\cdot,\cdot\rangle\colon E \times E \to A.$$

If A' is another pre-C*-algebra and $\Phi\colon A' \to A$ is an isometric $\star$-algebra isomorphism (so that it extends to an isomorphism of their C*-algebra completions), then we can turn E into a left inner product A'-module by defining

$$\begin{aligned} \Phi_{A'} &\coloneqq \Phi_A \circ \Phi\colon A' \to \mathrm{End}_{\mathbb{C}}(E) \\ \text{and} \quad {}_{A'}\langle\cdot,\cdot\rangle &\coloneqq \Phi^{-1}\big({}_A\langle\cdot,\cdot\rangle\big)\colon E \times E \to A'. \end{aligned}$$

We will have need for this construction later.

We now provide some basic examples. These are clear illustrations of the manner in which inner product modules generalize inner product spaces by replacing $\mathbb{C}$ with a pre-C*-algebra A.

[2] We will define opposite algebras properly when the need arises to consider right inner product modules in detail.

4.3.2 Example (Inner product modules)**.**

- With the convention that inner products are linear in the first entry, every inner product space over $\mathbb{C}$ is a left inner product $\mathbb{C}$-module. With linearity in the second entry, we obtain right inner product $\mathbb{C}$-modules.

- Let A be a pre-C*-algebra and let $n \geq 1$ be an integer. Then, $A^n = A \times \cdots \times A$ as a vector space with pointwise operations becomes a left inner product A-module when equipped with the following action and inner product:

$$a \cdot (a_j)_{j=1}^n = (aa_j)_{j=1}^n \quad \text{and} \quad \left\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \right\rangle = \sum_{j=1}^n a_j b_j^*,$$

  where $a, a_j, b_j \in A$ for $1 \leq j \leq n$. If we exchange aa_j for $a_j a$ and $a_j b_j^*$ for $a_j^* b_j$, we obtain a right inner product A-module.

Assume now that E is a left inner product A-module. We wish to use the A-valued inner product to define a norm on E. To do so, we need the following lemma. It may be thought of as a generalization of the Cauchy-Schwarz inequality (to which it reduces if $A = \mathbb{C}$).

4.3.3 Lemma (The Cauchy-Schwarz inequality)**.** *Let A be a pre-C*-algebra with C*-algebra completion $\mathcal{A}$ and let E be a left inner product A-module. Then,*

$$\langle g, f \rangle \langle f, g \rangle \leq \|\langle f, f \rangle\| \langle g, g \rangle$$

holds in $\mathcal{A}$ for all $f, g \in E$.

Proof. Fix $f, g \in E$ and let $a \in A$ be arbitrary. We find that

$$0 \leq \langle a \cdot f - g, a \cdot f - g \rangle = a\langle f, f \rangle a^* - a\langle f, g \rangle - \langle g, f \rangle a^* + \langle g, g \rangle.$$

By (i) and (ii) of Proposition 2.2.26: $a\langle f, f \rangle a^* \leq a(\|\langle f, f \rangle\| 1_A) a^*$. Thus,

$$0 \leq a \|\langle f, f \rangle\| a^* - a\langle f, g \rangle - \langle g, f \rangle a^* + \langle g, g \rangle.$$

The lemma clearly holds for $f = 0$, so we assume that $f \neq 0$. If we now set $a = \langle g, f \rangle / \|\langle f, f \rangle\|$, we find that

$$0 \leq -\frac{1}{\|\langle f, f \rangle\|} \langle g, f \rangle \langle f, g \rangle + \langle g, g \rangle,$$

from which the result follows (the required manipulations are all justified by Proposition 2.2.25). □

Whenever we refer to a norm on an inner product module, it will be the norm afforded by the following proposition. Note that the "ordinary" Cauchy-Schwarz inequality follows from its generalization.

4.3.4 Proposition. *Let A be a pre-C*-algebra and let E be a left inner product A-module. Then, the map $\|\cdot\|\colon E \to [0,\infty)$ defined by*

$$\|f\| \coloneqq \|\langle f, f\rangle\|^{1/2} \quad \textit{for all } f \in E$$

is a norm on E. Moreover, $\|\langle f, g\rangle\| \leq \|f\|\|g\|$ for all $f, g \in E$.

Proof. Let $f, g \in E$ and let $\lambda \in \mathbb{C}$. By nondegeneracy of the A-valued inner product and of the norm on A, we have that

$$f \neq 0 \quad \Longrightarrow \quad \langle f, f\rangle \neq 0 \quad \Longrightarrow \quad \|\langle f, f\rangle\| \neq 0.$$

This gives nondegeneracy of the claimed norm. Homogeneity similarly follows from homogeneity of the norm on A and the fact that $\langle \lambda f, \lambda f\rangle = |\lambda|^2\langle f, f\rangle$.

Before showing the triangle inequality, we show that $\|\langle f, g\rangle\| \leq \|f\|\|g\|$. Combining the Cauchy-Schwarz inequality of Lemma 4.3.3 with point (ii) of Proposition 2.2.26 we find that

$$\|\langle f, g\rangle\|^2 = \|\langle g, f\rangle\langle f, g\rangle\| \leq \|\langle f, f\rangle\|\|\langle g, g\rangle\| = \|f\|^2\|g\|^2,$$

which proves the claim. Finally, the triangle inequality in A and the isometry of the involution now implies that

$$\begin{aligned}\|f+g\|^2 &= \|\langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle\| \\ &\leq \|f\|^2 + 2\|\langle f, g\rangle\| + \|g\|^2 \leq \big(\|f\| + \|g\|\big)^2,\end{aligned}$$

which verifies the triangle inequality and completes the proof. □

4.3.5 Corollary (Continuity of the inner product)**.** *Let A be a pre-C*-algebra and let E be a left inner product A-module. Then, the inner product is continuous: if $f, g \in E$ and $(f_n), (g_n) \subset E$ are such that $f_n \to f$ and $g_n \to g$, then $\langle f_n, g_n\rangle \to \langle f, g\rangle$ in $\mathcal{A}$.*

Proof. Using Proposition 4.3.4, we have that

$$\begin{aligned}\|\langle f_n, g_n\rangle - \langle f, g\rangle\| &\leq \|\langle f_n, g_n - g\rangle\| + \|\langle f_n - f, g\rangle\| \\ &\leq \|f_n\|\|g_n - g\| + \|f_n - f\|\|g\|\end{aligned}$$

from which continuity of the norm ($\|f_n\| \to \|f\|$) gives the result. □

4.3.6 Lemma. *Let A be a pre-C*-algebra and let E be a left inner product A-module. Then, $\|a \cdot f\| \leq \|a\|\|f\|$ for all $a \in A$ and $f \in E$.*

Proof. We find that

$$\|a \cdot f\|^2 = \|\langle a \cdot f, a \cdot f\rangle\| = \|a\langle f, f\rangle a^*\| \leq \|a\|\|\langle f, f\rangle\|\|a^*\|,$$

from which $\|a^*\| = \|a\|$ gives the result. □

This means that the action $\Phi\colon A \to \mathrm{End}_{\mathbb{C}}(E)$ associated to a left inner-product A-module E corestricts to $\Phi\colon A \to \mathcal{B}(E)$ when we equip E with the norm determined by the A-valued inner product. In fact, we see that Φ becomes norm-decreasing.

We are now ready to define Hilbert C*-modules.

4.3.7 Definition (Hilbert C*-modules)**.** Let $\mathcal{A}$ be a C*-algebra. A *left Hilbert $\mathcal{A}$-module* is a left inner product $\mathcal{A}$-module that is complete with respect to the norm defined in Proposition 4.3.4. We will refer to such modules as *left Hilbert C*-modules* when we wish to leave the C*-algebra unspecified.

Right Hilbert $\mathcal{A}$-modules and right Hilbert C*-modules are defined analogously. Note that Hilbert C*-modules require completeness of both the pre-C*-algebra and the inner product module.

Given a pre-C*-algebra A and two left inner product A-modules E and F, a map $T\colon E \to F$ is said to be *A-linear* if

$$T(a \cdot f) = a \cdot Tf \quad \text{for all } a \in A \text{ and } f \in E,$$

and it is said to *preserve the inner product* if

$$\langle Tf, Tg\rangle = \langle f, g\rangle \quad \text{for all } f, g \in E.$$

If T is A-linear, then it is also $\mathbb{C}$-linear, for the fact that the action is an algebra homomorphism implies that

$$T(\lambda f) = T\big((\lambda \mathrm{Id}_E)f\big) = T\big((\lambda 1_A) \cdot f\big) = (\lambda 1_A) \cdot Tf = \lambda Tf$$

for all $\lambda \in \mathbb{C}$ and $f \in E$.

We now turn to completions. If A is a pre-C*-algebra with C*-algebra completion $\mathcal{A}$ and E is a left inner product A-module, a *(left) Hilbert C*-module completion* of E consists of a left Hilbert $\mathcal{A}$-module $\overline{E}$ and an A-linear map $j_E\colon E \to \overline{E}$ that preserves the inner product and has a dense image. For this to be the case, j_E must in particular be a Banach space completion of E (for preservation of the inner product makes j_E an isometry). We may also refer to either $\overline{E}$ or the map j_E by itself as the Hilbert C*-module completion of E.

The following proposition shows that Hilbert C*-module completions exist and that they may be constructed from any Banach space completion of E.

4.3.8 Proposition (Hilbert C*-module completions)**.** *Let A be a pre-C*-algebra with C*-algebra completion $i_A\colon A \to \mathcal{A}$. Suppose that E is a left inner product A-module with action $\Phi_A\colon A \to \mathcal{B}(E)$ and A-valued inner product ${}_A\langle\cdot,\cdot\rangle$. Suppose moreover that $j_E\colon E \to \overline{E}$ is a Banach space completion of E (w.r.t. the norm determined by ${}_A\langle\cdot,\cdot\rangle$).*

Then, there exists a unique action $\Phi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}(\overline{E})$ and a unique $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle\colon \overline{E} \times \overline{E} \to \mathcal{A}$ such that j_E becomes an A-linear map preserving the inner product and $\overline{E}$ becomes a left Hilbert $\mathcal{A}$-module. In other words: such that j_E becomes a Hilbert C-module completion of E. Moreover, the Banach space completion norm on $\overline{E}$ equals the norm determined by ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$.*

Proof. This result and its proof can be found in Appendix B.3. In particular, see Proposition B.3.1. □

We will consider uniqueness of such completions once the appropriate notion of isomorphism has been introduced.

4.3.2 Morphisms of Hilbert C*-Modules

In the theory of ordinary Hilbert spaces, any bounded linear map between Hilbert spaces has an adjoint. This is a foundational feature of the theory. Indeed, our development of C*-algebras has illustrated how powerful the structure of the adjoint really is, even in the absence of inner products. It turns our that a bounded linear map between two Hilbert $\mathcal{A}$-modules (for some C*-algebra $\mathcal{A}$) need *not* have an adjoint, even if we require $\mathcal{A}$-linearity.

We will bypass this complication simply by restricting our attention to those maps which do have adjoints. We will christen such maps "adjointable". The most general maps between Hilbert C*-modules we will consider will be bounded $\mathcal{A}$-linear adjointable maps. It turns out, however, that all we need to require is the presence of an adjoint, for this implies both boundedness and $\mathcal{A}$-linearity.

For any two sets E and F, we write F^E to denote the set of all functions from E to F. If E and F are normed spaces, then $\mathcal{B}(E, F)$ denotes the normed algebra of all bounded $\mathbb{C}$-linear maps from E to F.

4.3.9 Definition. Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. We say that a function $T \in F^E$ is *adjointable* if there exits a function $S \in E^F$ such that

$$\langle Tf, g\rangle = \langle f, Sg\rangle \quad \text{for all } f \in E \text{ and } g \in F.$$

In this case, we call S the *adjoint* of T (we will show uniqueness soon).

As promised:

4.3.10 Lemma. *Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. If $T \in F^E$ is adjointable, then T is both $\mathcal{A}$-linear and bounded. In particular, since $\mathcal{A}$-linearity implies $\mathbb{C}$-linearity, we have that $T \in \mathcal{B}(E, F)$.*

Proof. Suppose $T \in F^E$ is adjointable, with adjoint S. We begin by showing that T is $\mathcal{A}$-linear: for any $a \in \mathcal{A}$ and $f, g \in E$, we find that

$$\langle T(a \cdot f), g\rangle = \langle a \cdot f, Sg\rangle = a\langle f, Sg\rangle = a\langle Tf, g\rangle = \langle a \cdot Tf, g\rangle,$$

which implies (set $g = T(a \cdot f) - a \cdot Tf$) that

$$\big\langle T(a \cdot f) - a \cdot Tf, T(a \cdot f) - a \cdot Tf\big\rangle = 0.$$

Nondegeneracy of the inner product now gives $\mathcal{A}$-linearity. For the statement that $\mathcal{A}$-linearity implies $\mathbb{C}$-linearity, see the discussion following Definition 4.3.7.

It remains to show that T is bounded. We will show that if a sequence $(f_n) \subset E$ converges to some $f \in E$ and the sequence $(Tf_n) \subset F$ converges to some $g \in F$, then $Tf = g$. By the closed graph theorem (Theorem A.2.6), this implies that T is bounded.

If $f_n \to f \in E$ and $Tf_n \to g \in F$, then, by continuity of the $\mathcal{A}$-valued inner products, we find that

$$\langle g, h\rangle = \lim_{n\to\infty} \langle Tf_n, h\rangle = \lim_{n\to\infty} \langle f_n, Sh\rangle = \langle f, Sh\rangle = \langle Tf, h\rangle$$

for all $h \in F$. As before, nondegeneracy of the inner product now implies that $g = Tf$, which concludes the proof. $\square$

4.3.11 Definition (Adjointable operators)**.** Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. We write $\mathcal{L}(E, F)$ for the set of all adjointable maps from E to F. By Lemma 4.3.10, $\mathcal{L}(E, F) \subset \mathcal{B}(E, F)$ and $\mathcal{L}(E, F)$ consists entirely of $\mathcal{A}$-linear maps. We also write $\mathcal{L}(E) \coloneqq \mathcal{L}(E, E)$.

Whenever we speak of a norm on $\mathcal{L}(E, F)$, we will mean the operator norm restricted from $\mathcal{B}(E, F)$ (where the norms on E and F are defined via their $\mathcal{A}$-valued inner products).

In order to speak of *the* adjoint of an operator $T \in \mathcal{L}(E, F)$, we should verify that it is unique. If T has two adjoints, say $S_1, S_2 \in E^F$, then we find that $\langle f, (S_1 - S_2)g\rangle = \langle (T - T)f, g\rangle = 0$ for all $f \in E$ and $g \in F$, from which nondegeneracy of the inner product gives $S_1 = S_2$. Going forward, we will denote the adjoint of T by T^*.

It is simple to verify that

$$(T^*)^* = T, \quad (\lambda T_1 + T_2)^* = \overline{\lambda} T_1^* + T_2^* \quad \text{and} \quad (ST)^* = T^* S^*,$$

for any $T, T_1, T_2 \in \mathcal{L}(E, F)$, $\lambda \in \mathbb{C}$ and $S \in \mathcal{L}(F, G)$, for any left Hilbert $\mathcal{A}$-module G. Indeed, the proofs are exactly the same as for ordinary Hilbert spaces and adjoints. These simple identities imply that $\mathcal{L}(E, F)$ is a vector-subspace of $\mathcal{B}(E, F)$ and that $\mathcal{L}(E)$ is $\star$-algebra.

4.3.12 Proposition. *Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. Then, the following statements are true.*

(i) $\|T\| = \|T^*\|$ *and* $\|T\|^2 = \|T^*T\|$ *for all* $T \in \mathcal{L}(E, F)$.

(ii) $\mathcal{L}(E, F)$ *is a closed vector-subspace of* $\mathcal{B}(E, F)$.

(iii) $\mathcal{L}(E)$ *is a C*-algebra with the involution given by the adjoint.*

Proof. Noting that $\|\langle f, g\rangle\| \le \|f\|\|g\|$ for all $f, g \in E$ (Proposition 4.3.4), the proof of (i) is pretty much identical to the argument given in the beginning of Section 2.2, where we proved that $T \in \mathcal{B}(H)$ satisfies these identities for any Hilbert space H.

As for (ii), we have already remarked that $\mathcal{L}(E, F)$ is a vector-subspace of $\mathcal{B}(E, F)$. To see that it is closed, suppose that $T \in \mathcal{B}(E, F)$ is the limit of a sequence $(T_n) \subset \mathcal{L}(E, F)$. In particular, (T_n) is then Cauchy, so $(T_n^*) \subset \mathcal{L}(F, E)$ is as well, by (i). Thus, $(T_n)^*$ converges in $\mathcal{B}(F, E)$. By continuity of our inner products, we find that

$$\langle Tf, g\rangle = \langle \lim_{n\to\infty} T_n f, g\rangle = \lim_{n\to\infty} \langle T_n f, g\rangle = \lim_{n\to\infty} \langle f, T_n^* g\rangle = \langle f, \lim_{n\to\infty} T_n^* g\rangle,$$

for all $f \in E$ and $g \in F$. Thus, T is adjointable with $T^* = \lim_{n\to\infty} T_n^*$. This proves that $\mathcal{L}(E, F)$ is closed.

The proof of (iii) is now simple: we have already remarked that $\mathcal{L}(E)$ is a $\star$-algebra with the involution given by the adjoint, (ii) shows that $\mathcal{L}(E)$ is closed (in the Banach algebra $\mathcal{B}(E)$) and (i) shows that the C*-equality holds. □

Similarly to how we obtained a generalized Cauchy-Schwarz inequality in the form of an inequality in $\mathcal{A}$, we now obtain a generalized boundedness condition for adjointable operators.

4.3.13 Corollary. *Let $\mathcal{A}$ be a C*-algebra and let E be a left Hilbert $\mathcal{A}$-module. Then, we have that*

$$\langle Tf, Tf\rangle \leq \|T\|^2\langle f, f\rangle \quad \textit{for all } T \in \mathcal{L}(E) \textit{ and } f \in E.$$

Proof. Since T^*T is a self-adjoint element in the C*-algebra $\mathcal{L}(E)$, point (ii) of Proposition 2.2.26 implies that

$$\|T\|^2 \mathrm{Id}_E - T^*T = \|T^*T\| \mathrm{Id}_E - T^*T \geq 0 \quad \text{in } \mathcal{L}(E).$$

Since positive elements have positive square roots (Proposition 2.2.21), there exists some $S^* = S \in \mathcal{L}(E)$ such that $\|T\|^2 \mathrm{Id}_E - T^*T = S^*S$. Now,

$$\begin{aligned} 0 \leq \langle Sf, Sf\rangle = \langle S^*Sf, f\rangle &= \|T\|^2\langle f, f\rangle - \langle T^*Tf, f\rangle \\ &= \|T\|^2\langle f, f\rangle - \langle Tf, Tf\rangle \end{aligned}$$

gives the result. □

We now introduce a particularly simple kind of map between Hilbert C*-modules (we will quickly shows that they are adjointable).

4.3.14 Definition (Rank-one operators)**.** Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. For each $f \in E$ and $g \in F$, we define the map $K_{f,g} \colon E \to F$ by

$$K_{f,g}(h) = \langle h, f\rangle \cdot g \quad \text{for all } h \in E.$$

Operators of this form are referred to as *rank-one operators*.

The terminology of rank-one operators is borrowed from the theory of Hilbert spaces. However, rank-one operators need not have one-dimensional ranges over $\mathbb{C}$. Instead, they have "one-dimensional ranges over $\mathcal{A}$", in the sense that $K_{f,g}(E) \subset \mathcal{A} \cdot g$. In the special case where $\mathcal{A} = \mathbb{C}$ and Hilbert $\mathcal{A}$-modules are ordinary Hilbert spaces, the terminology coincides. We will always use this term in the sense of Definition 4.3.14.

We now show that rank-one operators are adjointable, and hence bounded and $\mathcal{A}$-linear as well.

4.3.15 Lemma. *Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. Then, all rank-one operators from E to F are adjointable. In fact, we have that $(K_{f,g})^* = K_{g,f}$ for all $f \in E$ and $g \in F$.*

Proof. Let $h \in E$ and $k \in F$ be arbitrary. Using only the basic properties of our inner products, we find that

$$\begin{aligned}\big\langle K_{f,g}(h), k\big\rangle &= \big\langle \langle h, f\rangle \cdot g, k\big\rangle = \langle h, f\rangle\langle g, k\rangle = \big\langle h, \langle g, k\rangle^* \cdot f\big\rangle \\ &= \big\langle h, \langle k, g\rangle \cdot f\big\rangle = \big\langle h, K_{g,f}(k)\big\rangle,\end{aligned}$$

which gives the result. □

We now consider the subspace of $\mathcal{L}(E, F)$ consisting of limits of linear combinations of rank-one operators.

4.3.16 Definition (Compact operators)**.** Let $\mathcal{A}$ be a C*-algebra and let E be a left Hilbert $\mathcal{A}$-module. We define

$$\mathcal{K}(E, F) := \overline{\operatorname{span}}_{\mathbb{C}}\big\{K_{f,g} : f \in E \text{ and } g \in F\big\} \subset \mathcal{L}(E, F),$$

and we refer to the elements of $\mathcal{K}(E, F)$ as *compact operators*. We will write $\mathcal{K}(E) := \mathcal{K}(E, E)$.

As with rank-one operators, the terminology of compact operators is borrowed from Hilbert space theory. Our compact operators need *not* be compact in the ordinary sense of the word (that is, they need not map bounded sets to relatively compact sets). We will always use this term in the sense of Definition 4.3.16.

The reader who is familiar with compact operators on Hilbert spaces might now anticipate the following result.

4.3.17 Proposition. *Let $\mathcal{A}$ be a C*-algebra and let E be a left Hilbert $\mathcal{A}$-module. Then, $\mathcal{K}(E)$ is a closed ideal in $\mathcal{L}(E)$.*

Proof. It is clear from its definition that $\mathcal{K}(E)$ is a closed vector-subspace of $\mathcal{L}(E)$. To see that it is an ideal, let $T \in \mathcal{L}(E)$ and $f, g, h \in E$. We find that

$$\begin{aligned}&(TK_{f,g})(h) = T\big(\langle h, f\rangle \cdot g\big) = \langle h, f\rangle \cdot Tg = K_{f,Tg}(h) \\ \text{and}\quad &(K_{f,g}T)(h) = \langle Th, f\rangle \cdot g = \langle h, T^*f\rangle \cdot g = K_{T^*f,g}(h)\end{aligned}$$

where we have used $\mathcal{A}$-linearity of T in the first line (Lemma 4.3.10). Bilinearity and continuity of the product in $\mathcal{L}(E)$ now implies that $\mathcal{K}(E)$ is an ideal, which concludes the proof. □

Before we close this subsection, we define the obvious notion of isomorphisms between Hilbert C*-modules. This is not a notion which appears (to our knowledge) in Raeburn and Williams [24].

4.3.18 Definition (Unitary operators)**.** Let $\mathcal{A}$ be a C*-algebra and let E and F be two left Hilbert $\mathcal{A}$-modules. A map $U\colon E \to F$ is called *unitary* if $U \in \mathcal{L}(E,F)$ and

$$U^*U = \mathrm{Id}_E \quad \text{and} \quad UU^* = \mathrm{Id}_F.$$

As is the case for all adjointable maps, we know that a unitary map $U \in \mathcal{L}(E,F)$ and its adjoint $U^* \in \mathcal{L}(F,E)$ are $\mathcal{A}$-linear. It is clear that both U and U^* preserve the inner product. Thus, a unitary operator is a bijection between two Hilbert C*-modules which preserves all of the relevant structure in both directions.

As a reflection of this fact, we see that Hilbert C*-module completions are unique precisely up to unique unitary maps (clearly any Hilbert C*-module completion $j_E\colon E \to \overline{E}$ followed by a unitary map U is another Hilbert C*-module completion of E):

4.3.19 Proposition. *Let A be a pre-C*-algebra and let E be a left inner product A-module. If $j_E\colon E \to \overline{E}$ and $k_E\colon E \to \widetilde{E}$ are two Hilbert C*-module completions of E, then there exists a unique unitary map $U \in \mathcal{L}(\overline{E}, \widetilde{E})$ such that $k_E = U \circ j_E$.*

Proof. This result and its proof can be found in Appendix B.3. In particular, see Corollary B.3.3. □

4.3.3 The Feichtinger Algebra as an Inner Product Module

We are now finally ready to construct the long promised Hilbert C*-modules over noncommutative tori. We will find that much of the work has already been done. In this subsection, we are following Luef [21, 22].

What we will explicitly construct is an inner product $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-module, for any $A \in \mathrm{GL}(2d, \mathbb{R})$ (recall that β_A denotes the Heisenberg cocycle determined by A; Defintion 4.1.20). Thus, we will consider $\ell^1(\mathbb{Z}^{2d}, \beta_A)$ as a pre-C*-algebra whose C*-algebra completion is $C^*(\mathbb{Z}^{2d}, \beta_A)$ (see Corollary 4.1.27 and the paragraph preceding it). The vector space underlying the module will be the Feichtinger algebra $S_0(\mathbb{R}^d)$ from Subsection 3.2.3. The completion of this inner product $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-module will be the desired Hilbert C*-module.

Note that it is quite natural that $S_0(\mathbb{R}^d)$ pairs with $\ell^1(\mathbb{Z}^{2d}, \beta_A)$: both are defined by an absolute integrability/summability condition, and Lemma 3.2.24 shows that sampling the STFT $V_g f$ for $f, g \in S_0(\mathbb{R}^d)$ over $A\mathbb{Z}^{2d}$ produces an element of $\ell^1(\mathbb{Z}^{2d})$.

4.3.20 Theorem (The Feichtinger algebra as an inner product module)**.** *For any $A \in \mathrm{GL}(2d, \mathbb{R})$, the Feichtinger algebra $S_0(\mathbb{R}^d)$ is a left inner product $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-module with respect to the action*

$$a \cdot f \coloneqq \Pi_A(a)f = \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k)f \quad \text{for } a \in \ell^1(\mathbb{Z}^{2d}, \beta_A) \text{ and } f \in S_0(\mathbb{R}^d),$$

and the $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-valued inner product ${}_A\langle \cdot, \cdot \rangle$ defined for $f, g \in S_0(\mathbb{R}^d)$ by

$${}_A\langle f, g \rangle(k) = \langle f, \pi_A(k)g \rangle = V_g f(Ak) \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

We will prove this theorem soon. But first, we wish to highlight how the canonical operators from Gabor theory appear in this construction. Let $f, g, h \in S_0(\mathbb{R}^d)$ and let $a \in \ell^1(\mathbb{Z}^{2d}, \beta_A)$. Recall the analysis, synthesis and mixed-type frame operators, C_g^A, D_h^A and $S_{g,h}^A$, introduced at the end of Subsection 3.2.2. We find that

$$C_g^A(f) = {}_A\langle f, g \rangle \quad \text{and} \quad D_h^A(a) = a \cdot h.$$

Thus, we see that the inner product represents the analysis operator (or the discrete STFT, if you will), while the action represents the synthesis operator. Moreover, combining these, we see that

$$S_{g,h}^A(f) = (D_h^A \circ C_g^A)f = {}_A\langle f, g \rangle \cdot h = K_{g,h}(f),$$

so the mixed-type frame operator $S_{g,h}$ is precisely the rank-one operator $K_{g,h}$! Of course, in this context, these are operators between/on $S_0(\mathbb{R}^d)$ and $\ell^1(\mathbb{Z}^{2d}, \beta_A)$, as opposed to $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^{2d})$. Nevertheless, this is a clear illustration of how these modules encapsulate and reflect the basic structure of Gabor theory.

Before we prove Theorem 4.3.20 in full, we capture the fact that the module action is well-defined in a lemma.

4.3.21 Lemma. *If $f \in S_0(\mathbb{R}^d)$ and $a \in \ell^1(\mathbb{Z}^{2d}, \beta_A)$, then*

$$\Pi_A(a)f = \sum_{k \in \mathbb{Z}^{2d}} a(k)\pi_A(k)f \in S_0(\mathbb{R}^d),$$

where the sum converges absolutely in $L^2(\mathbb{R}^d)$.

Proof. Absolute convergence is immediate since $\|\pi_A(k)f\|_2 = \|f\|_2$ for every $k \in \mathbb{Z}^{2d}$. We need to verify that $V_{g_0}(\Pi_A(a)f) \in L^1(\mathbb{R}^{2d})$ (recall that g_0 denotes the normalized Gaussian; see Definition 3.2.12).

The continuity of the STFT $\mathcal{V}_{g_0}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ implies that

$$V_{g_0}\Big(\sum_{k\in\mathbb{Z}^{2d}} a(k)\pi_A(k)f\Big)(z) = \sum_{k\in\mathbb{Z}^{2d}} a(k)\big(V_{g_0}\pi(Ak)f\big)(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Using the covariance property of Lemma 3.1.4, we now find that

$$\begin{aligned}\big\|V_{g_0}\big(\Pi_A(a)f\big)\big\|_1 &\leq \int_{\mathbb{R}^{2d}}\Big(\sum_{k\in\mathbb{Z}^{2d}} |a(k)V_{g_0}f(z-Ak)|\Big)\,\mathrm{d}z \\ &= \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\int_{\mathbb{R}^{2d}} |V_{g_0}f(z-Ak)|\,\mathrm{d}z = \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\|V_{g_0}f\|_1,\end{aligned}$$

which is finite because $f \in S_0(\mathbb{R}^d)$ and $a \in \ell^1(\mathbb{Z}^{2d},\beta_A)$. □

Finally, as a last note before we prove Theorem 4.3.20, we recall the explicit form of products and involutions in $\ell^1(\mathbb{Z}^{2d},\beta_A)$, since we will need them. We have:

$$\begin{aligned}(a *_{\beta_A} b)(k) &= \sum_{l\in\mathbb{Z}^{2d}} a(l)b(k-l)\beta_A(l,k-l) \\ \text{and}\quad a^*(k) &= \overline{\beta_A(k,-k)a(-k)}\end{aligned}$$

for all $a, b \in \ell^1(\mathbb{Z}^{2d},\beta_A)$ and $k \in \mathbb{Z}^{2d}$.

Proof of Theorem 4.3.20. The action of $\ell^1(\mathbb{Z}^{2d},\beta_A)$ on $S_0(\mathbb{R}^d)$ is the just the map $\Pi_A\colon \ell^1(\mathbb{Z}^{2d},\beta_A) \to \mathcal{B}(L^2(\mathbb{R}^d))$ afforded by Corollary 4.1.27, but with the operators in its image restricted to act on $S_0(\mathbb{R}^d)$ instead of $L^2(\mathbb{R}^d)$. Lemma 4.3.6 is the statement that this restriction is well-defined. We already know that Π_A is a $\star$-algebra homomorphism (Corollary 4.1.23), so there is nothing more to prove regarding the action. As for the $\ell^1(\mathbb{Z}^{2d},\beta_A)$-valued inner product, we have already proved that it is well-defined: see Lemma 3.2.24. All we need to do is to verify the axioms of the inner product.

Fix any $f, g \in S_0(\mathbb{R}^d)$. The $\mathbb{C}$-linearity of the first entry follows immediately from linearity of the STFT $\mathcal{V}_g\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$. For conjugate symmetry, point (iv) of Lemma 3.1.1 gives:

$$\pi_A(k)^* = \overline{\beta_A(k,-k)}\pi_A(-k) \quad \text{for all } k \in \mathbb{Z}^{2d},$$

so that

$$\begin{aligned}{}_A\langle f,g\rangle^*(k) &= \overline{\beta_A(k,-k)\langle f,\pi_A(-k)g\rangle} = \big\langle \overline{\beta_A(k,-k)}\pi_A(-k)g, f\big\rangle \\ &= \langle g,\pi_A(k)f\rangle = {}_A\langle g,f\rangle(k)\end{aligned}$$

for all $k \in \mathbb{Z}^{2d}$.

We now turn to $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-linearity. With the definition of β_A in mind, the covariance property of Lemma 3.1.4 implies that

$$\big(V_g \pi(Al) f\big)(Ak) = \beta_A(l, k-l) V_g f(Ak - Al) \quad \text{for all } k, l \in \mathbb{Z}^{2d}.$$

Thus, with $a \in \ell^1(\mathbb{Z}^{2d}, \beta_A)$, we find that

$$\begin{aligned}
{}_A\langle a \cdot f, g\rangle^*(k) &= \Big\langle \sum_{l \in \mathbb{Z}^{2d}} a(l) \pi_A(l) f, \pi_A(k) g \Big\rangle \\
&= \sum_{l \in \mathbb{Z}^{2d}} a(l) \big\langle \pi_A(l) f, \pi_A(k) g \big\rangle \\
&= \sum_{l \in \mathbb{Z}^{2d}} a(l) \big(V_g \pi(Al) f\big)(Ak) \\
&= \sum_{l \in \mathbb{Z}^{2d}} a(l) \beta_A(l, k-l) V_g f(Ak - Al) \\
&= \sum_{l \in \mathbb{Z}^{2d}} a(l) {}_A\langle f, g\rangle(k-l) \beta_A(l, k-l) = \big(a *_{\beta_A} {}_A\langle f, g\rangle\big)(k)
\end{aligned}$$

for all $k \in \mathbb{Z}^{2d}$, which proves $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-linearity.

All that remains is positivity and nondegeneracy. For positivity, recall that we require ${}_A\langle f, f\rangle \geq 0$ in the C*-algebra completion of $\ell^1(\mathbb{Z}^{2d}, \beta_A)$, i.e. in $C^*(\mathbb{Z}^{2d}, \beta_A) \cong C^*(\pi_A(\mathbb{Z}^{2d})) \subset \mathcal{B}(L^2(\mathbb{R}^d))$. Under the inclusion $\ell^1(\mathbb{Z}^{2d}, \beta_A) \to \mathcal{B}(L^2(\mathbb{R}^d))$ (Corollary 4.1.27 and Proposition 4.1.26), we have that

$${}_A\langle f, f\rangle \mapsto \sum_{k \in \mathbb{Z}^{2d}} {}_A\langle f, f\rangle(k) \pi_A(k) = \sum_{k \in \mathbb{Z}^{2d}} V_f f(Ak) \pi_A(k).$$

By Proposition 2.2.28, we know that, for any $T \in \mathcal{B}(L^2(\mathbb{R}^d))$,

$$\langle Tg, g\rangle \geq 0 \text{ for all } g \in L^2(\mathbb{R}^d) \quad \Longrightarrow \quad T \geq 0 \text{ in } \mathcal{B}(L^2(\mathbb{R}^d)).$$

Let now $T = \sum_{k \in \mathbb{Z}^{2d}} V_f f(Ak) \pi_A(k)$. We will show that $T \geq 0$ in $\mathcal{B}(L^2(\mathbb{R}^d))$, which is equivalent to $T \geq 0$ in $C^*(\pi_A(\mathbb{Z}^{2d}))$ by spectral permanence (Proposition 2.2.9).

By continuity of inner products and the fact that T is a continuous operator on $L^2(\mathbb{R}^d)$, it suffices to show that $\langle Tg, g\rangle \geq 0$ for all g stemming from a dense subspace of $L^2(\mathbb{R}^d)$, e.g. for $g \in S_0(\mathbb{R}^d)$ (Lemma 3.2.14). With $g \in S_0(\mathbb{R}^d)$, we find that

$$\begin{aligned}
\langle Tg, g\rangle &= \Big\langle \sum_{k \in \mathbb{Z}^{2d}} V_f f(Ak) \pi_A(k) g, g \Big\rangle = \sum_{k \in \mathbb{Z}^{2d}} V_f f(Ak) \overline{V_g g(Ak)} \\
&= \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}^{2d}} V_f g(A^\circ k) \overline{V_f g(A^\circ k)} \geq 0,
\end{aligned}$$

where we have used the FIGA (Theorem 3.2.17) in the last line. This proves positivity of ${}_A\langle \cdot, \cdot \rangle$.

Finally, if $f \in S_0(\mathbb{R}^d)$ is nonzero, then ${}_A\langle f, f \rangle$ is nonzero, because ${}_A\langle f, f \rangle(0) = \langle f, f \rangle = \|f\|_2^2$. This gives nondegeneracy and concludes the proof. □

By Proposition 4.3.8 on Hilbert C*-module completions, we may complete $S_0(\mathbb{R}^d)$ to a left Hilbert $C^*(\mathbb{Z}^{2d}, \beta_A)$-module. We will denote this completion by ${}_A\mathcal{E}$. We may also consider ${}_A\mathcal{E}$ as a left Hilbert $\mathcal{A}_\theta$-module (where $\theta = A^T J A$), via the isomorphism $\mathcal{A}_\theta \cong C^*(\mathbb{Z}^{2d}, \beta_A)$ of Corollary 4.2.6.

A priori, ${}_A\mathcal{E}$ is an abstract completion, and we cannot interpret its elements as functions. However, it has recently become clear that ${}_A\mathcal{E}$ embeds continuously into $L^2(\mathbb{R}^d)$. This was first shown by Austad and Luef [4] (in a preprint). Austad and Enstad [3] has used this insight to further explore how ${}_A\mathcal{E}$ fits into the framework of Gabor frames in the $L^2(\mathbb{R}^d)$-setting. Thus, there is an ongoing exploration and illumination of the naturality and usefulness of these modules in Gabor theory.

Chapter 5

Morita Equivalence and Lattices

The purpose of this chapter is twofold. First, we introduce the notion of Morita equivalence of C*-algebras via equivalence bimodules and show how to upgrade the Feichtinger algebra to an equivalence bimodule. This is the content of the first section. Afterwards, in the second section, we have included some tangential considerations about how Morita equivalence relates to duality in Gabor theory.

5.1 | Morita Equivalence and the Feichtinger Algebra

Except for a brief encounter in Subsection 2.2.5, on the Gelfand-Naimark theorem, we have not properly introduced representations of C*-algebras. Indeed, representation theory is a vast and important topic that is front and center in the deeper study of C*-algebras.

The notion of Morita equivalence originates in ring theory, where two rings are said to be Morita equivalent if their categories of modules are additively equivalent. We will not unpack this; we refer the interested reader to Anderson and Fuller [1] for an approachable introduction to these topics. Modules can be viewed as representations of rings, so, loosely stated, Morita equivalence equates rings with isomorphic structures of representations. It turns out that if two rings R and S are Morita equivalent, then there exists a certain R-S-bimodule which one can use to turn representations of R into representations of S and vice versa. The existence of such a bimodule is also sufficient for Morita equivalence, so one can phrase Morita equivalence entirely in terms of the existence of certain bimodules.

The notion of Morita equivalence for C*-algebras is similar to the notion of Morita equivalence for rings, but there are important differences. Two

C*-algebras are called Morita equivalent if there exists a certain Hilbert C*-bimodule that relates their categories of representations, known as an *equivalence bimodule*. However, there may exist C*-algebras whose categories of representations are equivalent but for which no such bimodule exists. Originally, C*-algebras with equivalent categories of representations were called Morita equivalent, and the term *strong Morita equivalence* was used to refer to the existence of an equivalence bimodule. These bimodules have become so central to the subject that it has become common to omit the adjective "strong", which is a convention we will adopt. Equivalence bimodules were developed by Rieffel [26], who called them *imprivity bimodules* (a term that is still very much in use).

We will not develop the machinery needed to transfer representations between Morita equivalent C*-algebras. We will simply introduce equivalence bimodules as an interesting topic in their own right. The rest of the story can be found in Raeburn and Williams [24], who we will follow in our development of equivalence bimodules.

5.1.1 Opposite Algebras and Equivalence Bimodules

Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. An $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule will be a complex vector space E that is both a left Hilbert $\mathcal{A}$-module and a right Hilbert $\mathcal{B}$-module that moreover satisfies a couple of conditions relating these structures. In order to properly introduce the topic, we need to discuss opposite algebras and right inner product modules in some detail.

5.1.1 Definition (The opposite algebra)**.** Let A be a $\star$-algebra with the product denoted by juxtaposition. We define the *opposite $\star$-algebra* of A to be the same vector space A equipped with the same involution $\star\colon A \to A$, but with the product $\diamond\colon A \times A \to A$ defined by

$$a \diamond b = \diamond(a, b) := ba \quad \text{for all } a, b \in A.$$

We will write A^{op} for this $\star$-algebra. Thus, $A = A^{\mathrm{op}}$ as vector spaces, but their products differ.

It is straightforward to check that A^{op} actually is a $\star$-algebra. In order to illustrate the notation, we include a couple of the required calculations:

- $(a \diamond b)^* = (ba)^* = a^* b^* = b^* \diamond a^*$ (Property (iii) of Def. 2.1.10),
- $(a \diamond b) \diamond c = c(ba) = (cb)a = a \diamond (b \diamond c)$ (associativity of the product).

Moreover, it is clear that $(A^{\mathrm{op}})^{\mathrm{op}} = A$ as algebras.

If A is a pre-C*-algebra, then A^{op} is as well, with respect to the same norm. Submultiplicativity of the norm is immediate, and

$$\|a^* \diamond a\| = \|aa^*\| = \|(a^*)^* a^*\| = \|a^*\|^2 = \|a\|^2$$

gives the C*-equality. Moreover, if $\mathcal{A}$ denotes the C*-algebra completion of A, then $\mathcal{A}^{\mathrm{op}}$ is the C*-algebra completion of A^{op}. To be more precise: if $i_A\colon A \to \mathcal{A}$ is a C*-algebra completion of A, then, the same map $i_A\colon A^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ ($A = A^{\mathrm{op}}$ and $\mathcal{A} = \mathcal{A}^{\mathrm{op}}$ as sets) is a C*-algebra completion of A^{op}. This is immediate: the only structure that differs is the product, and that calculation

$$i_A(a) \diamond i_A(b) = i_A(b) i_A(a) = i_A(ba) = i_A(a \diamond b) \quad \text{for all } a, b \in A^{\mathrm{op}} = A$$

shows that the product on $\mathcal{A}^{\mathrm{op}}$ extends the product on A^{op}.

We now show that right inner product A-modules correspond to left inner product A^{op}-modules and vice versa.

5.1.2 Lemma. *Let A be a pre-C*-algebra. Suppose that E is a right inner product A-module with action $\Phi\colon A^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$ and A-valued inner product $\langle \cdot, \cdot \rangle_\times$. Then, the same vector space E with the same action $\Phi\colon A^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$ and an A^{op}-valued inner product ${}_\times\langle \cdot, \cdot \rangle$ defined by*

$$_\times\langle g, f \rangle := \langle f, g \rangle_\times \quad \textit{for all } f, g \in E$$

is a left inner inner product A^{op}-module.

Conversely, if F is a left inner product A-module with action $\Phi\colon A \to \mathrm{End}_{\mathbb{C}}(F)$ and A-valued inner product ${}_\times\langle \cdot, \cdot \rangle$. Then, the same vector space F with the same action $\Phi\colon (A^{\mathrm{op}})^{\mathrm{op}} = A \to \mathrm{End}_{\mathbb{C}}(F)$ and an A^{op}-valued inner product $\langle \cdot, \cdot \rangle_\times$ defined by

$$\langle f, g \rangle_\times := {}_\times\langle g, f \rangle \quad \textit{for all } f, g \in F$$

is a right inner product A^{op}-module.

Proof. We show the first half of the statement, regarding right inner product A-modules. The proof of the second half is almost identical, one just needs to break apart the equations and shuffle the pieces around.

Let $\diamond$ denote the product in A^{op}. There is nothing to verify regarding the action, for it doesn't change. Note, however, that although the action of A in the right module and the action of A^{op} in the left module are given by the same algebra homomorphism $\Phi\colon A^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$, we write them differently:

$$a \cdot f = f \cdot a \quad \text{for all } a \in A = A^{\mathrm{op}} \text{ and } f \in E.$$

This is sensible, because

$$(a \diamond b) \cdot f = \Phi(a \diamond b)(f) = \big(\Phi(a)\Phi(b)\big)(f) = \Phi(a)\big(\Phi(b)(f)\big) = a \cdot (b \cdot f)$$

while

$$f \cdot (ab) = \Phi(ab)(f) = \Phi(b \diamond a)(f) = \Phi(b)\big(\Phi(a)(f)\big) = \Phi(b)(f \cdot a) = (f \cdot a) \cdot b$$

for all $a, b \in A = A^{\mathrm{op}}$ and $f \in E$.

We need to check that the inner product ${}_\times\langle \cdot, \cdot \rangle$ satisfies the required properties (see Definition 4.3.1). The following statements should be quantified over all $f, g, h \in E$, $\lambda \in \mathbb{C}$ and $a \in A$.

(i) $\mathbb{C}$-linearity in the first entry:

$${}_\times\langle \lambda f + g, h\rangle = \langle h, \lambda f + g\rangle_\times = \lambda \langle h, f\rangle_\times + \langle h, g\rangle_\times = \lambda {}_\times\langle f, h\rangle + {}_\times\langle g, h\rangle,$$

(ii) A^{op}-linearity in the first entry:

$${}_\times\langle a \cdot f, g\rangle = \langle g, a \cdot f\rangle_\times = \langle g, f \cdot a\rangle_\times = \langle g, f\rangle_\times a = {}_\times\langle f, g\rangle a = a \diamond {}_\times\langle f, g\rangle.$$

(iii) Conjugate symmetry: ${}_\times\langle f, g\rangle^* = \langle g, f\rangle_\times^* = \langle f, g\rangle_\times = {}_\times\langle g, f\rangle$.

(iv) Positivity: ${}_\times\langle f, f\rangle = \langle f, f\rangle_\times \geq 0$.

(v) Nondegeneracy: ${}_\times\langle f, f\rangle = 0 \implies \langle f, f\rangle_\times = 0 \implies f = 0$.

This concludes the proof. $\square$

This correspondence has the following wonderful consequence: for every result we have for left inner product modules, we obtain a corresponding result for right inner product modules. Indeed, given a right A-module E, we can import the desired result from the corresponding left A^{op}-module structure on E. For example, the Cauchy-Schwarz inequality of Lemma 4.3.3 holds without change, because

$$\begin{aligned} \langle g, f\rangle_\times \langle f, g\rangle_\times &= \langle f, g\rangle_\times \diamond \langle g, f\rangle_\times = {}_\times\langle g, f\rangle \diamond {}_\times\langle f, g\rangle \\ &\leq \|{}_\times\langle f, f\rangle\| {}_\times\langle g, g\rangle = \|\langle f, f\rangle_\times\| \langle g, g\rangle_\times. \end{aligned}$$

Similarly, we can translate our definitions for left Hilbert C*-modules into definitions for right Hilbert C*-modules by seeing how our definition for a left Hilbert $\mathcal{A}^{\mathrm{op}}$-module appears in the corresponding right Hilbert $\mathcal{A}$-module structure. For example, if we let ${}^{\times}K_{f,g}$ denote a rank-one operator in a left Hilbert $\mathcal{A}^{\mathrm{op}}$-module E (so $f, g \in E$), then

$${}^{\times}K_{f,g}(h) := {}_{\times}\langle h, f\rangle \cdot g = g \cdot \langle f, h\rangle_{\times},$$

shows that we should define the rank-one operator corresponding to $f, g \in E$ in the right Hilbert $\mathcal{A}$-module E by $h \mapsto g \cdot \langle f, h\rangle_{\times}$.

Now, similarly to how we change our notation for the action when passing from left to right modules, we wish to make a slight notational change for rank-one operators. It seems natural to define $K^{\times}_{g,f} := {}^{\times}K_{f,g}$, so that the order of the subscripts matches the order in which they appear in the defining expressions. That is, we take

$$K^{\times}_{f,g}(h) = f \cdot \langle g, h\rangle_{\times} \quad \text{for all } h \in E$$

as a definition of the rank-one operator corresponding to $f, g \in E$ in a right Hilbert $\mathcal{A}$-module E.

If we similarly include subscripts for the spaces of adjointable and compact operators, we find that ${}_{\times}\mathcal{L}(E) = \mathcal{L}_{\times}(E)$ and that ${}_{\times}\mathcal{K}(E) = \mathcal{K}_{\times}(E)$. That is, adjointable operators for a left Hilbert $\mathcal{A}^{\mathrm{op}}$-module are exactly the same as adjointable operators for the corresponding right Hilbert $\mathcal{A}$-module, and likewise for compact operators. There is no more substance to this than simple observations such as:

$$\langle Tf, g\rangle_{\times} = {}_{\times}\langle g, Tf\rangle = {}_{\times}\langle T^{*}g, f\rangle = \langle f, T^{*}g\rangle_{\times}.$$

This concludes our discussion of the left A^{op}-module and right A-module correspondence for now.

There is only one more notion we need before we are ready to study equivalence bimodules.

5.1.3 Definition (Fullness of inner-product modules)**.** Let A be a pre-C*-algebra. A left or right inner product A-module E is said to be *full* if

$$\overline{\operatorname{span}}_{\mathbb{C}}\{\langle f, g\rangle : f, g \in E\} = A.$$

We are now ready for the main definition of this section. We emphasize that the notion of equivalence bimodules, which we are about to introduce, is *entirely distinct* from the left A^{op}-module and right A-module correspondence we just discussed. Considering a left Hilbert $\mathcal{A}^{\mathrm{op}}$-module as a right Hilbert $\mathcal{A}$-module does *not* turn it into an $\mathcal{A}^{\mathrm{op}}$-$\mathcal{A}$-equivalence bimodule in general.

5.1.4 Definition (Equivalence bimodules)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. An $\mathcal{A}$-$\mathcal{B}$-*equivalence bimodule* is a complex vector space E with the following structure.

(i) E is a full left Hilbert $\mathcal{A}$-module and a full right Hilbert $\mathcal{B}$-module. We will denote the $\mathcal{A}$-valued inner product by ${}_\bullet\langle\cdot,\cdot\rangle$ and the $\mathcal{B}$-valued inner product by $\langle\cdot,\cdot\rangle_\bullet$.

(ii) $\mathcal{A}$ acts as adjointable operators w.r.t. $\langle\cdot,\cdot\rangle_\bullet$ and $\mathcal{B}$ acts as adjointable operators w.r.t. ${}_\bullet\langle\cdot,\cdot\rangle$, meaning that

$$\langle a\cdot f, g\rangle_\bullet = \langle f, a^*\cdot g\rangle_\bullet \quad \text{and} \quad {}_\bullet\langle f\cdot b, g\rangle = {}_\bullet\langle f, g\cdot b^*\rangle$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $f,g\in E$.

(iii) The inner products satisfy the following associativity condition:

$${}_\bullet\langle f,g\rangle\cdot h = f\cdot\langle g,h\rangle_\bullet$$

for all $f,g,h\in E$.

Whenever we speak of an "$\mathcal{A}$-$\mathcal{B}$-equivalence bimodule" it should be implicitly understood that $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras. We reiterate that two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* if there exists an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule.

We will write $\Phi_\mathcal{A}\colon \mathcal{A}\to \mathrm{End}_\mathbb{C}(E)$ and $\Phi_\mathcal{B}\colon \mathcal{B}^{\mathrm{op}}\to \mathrm{End}_\mathbb{C}(E)$ for the two actions that make up part of the bimodule structure. We also need to notationally distinguish the maps $T\colon E\to E$ that are adjointable with respect to the $\mathcal{A}$-valued inner product from those that are adjointable with respect to the $\mathcal{B}$-valued inner product. We will write $\mathcal{L}_\mathcal{A}(E)$ for those operators that are adjointable w.r.t. ${}_\bullet\langle\cdot,\cdot\rangle$ and $\mathcal{L}_\mathcal{B}(E)$ for those that are adjointable w.r.t. $\langle\cdot,\cdot\rangle_\bullet$. Note that a map may have one adjoint with respect to ${}_\bullet\langle\cdot,\cdot\rangle$ and another adjoint with respect to $\langle\cdot,\cdot\rangle_\bullet$, so we need to be careful with our notation.

With regard to rank-one and compact operators, we define, for $f,g\in E$:

$$K^{\mathcal{A}}_{f,g}(h) \coloneqq {}_\bullet\langle h,f\rangle\cdot g \quad \text{and} \quad K^{\mathcal{B}}_{f,g}(h) \coloneqq f\cdot\langle g,h\rangle_\bullet, \tag{5.1}$$

and we write $\mathcal{K}_\mathcal{A}(E)$ and $\mathcal{K}_\mathcal{B}(E)$ for the respective spaces of compact operators (see Definition 4.3.16).

It turns out that equivalence bimodules intertwine inner products and compact operators in a very particular manner:

5.1.5 Proposition. *Let E be an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule. Then, the associated actions $\Phi_\mathcal{A}\colon \mathcal{A}\to \mathrm{End}_\mathbb{C}(E)$ and $\Phi_\mathcal{B}\colon \mathcal{B}^{\mathrm{op}}\to \mathrm{End}_\mathbb{C}(E)$ satisfy*

$$\Phi_\mathcal{A}\big({}_\bullet\langle f,g\rangle\big) = K^{\mathcal{B}}_{f,g} \quad \textit{and} \quad \Phi_\mathcal{B}\big(\langle g,h\rangle_\bullet\big) = K^{\mathcal{A}}_{g,h}, \tag{5.2}$$

for all $f, g, h \in E$. Moreover,[1] $\mathcal{K}_\mathcal{A}(E) = \mathcal{L}_\mathcal{A}(E)$ *and* $\mathcal{K}_\mathcal{B}(E) = \mathcal{L}_\mathcal{B}(E)$, *and the actions corestrict to (isometric) $\star$-algebra isomorphisms*

$$\Phi_\mathcal{A} \colon \mathcal{A} \to \mathcal{L}_\mathcal{B}(E) \quad \text{and} \quad \Phi_\mathcal{B} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{L}_\mathcal{A}(E).$$

Proof. We will prove the claims that pertain to $\Phi_\mathcal{A}$. The claims for $\Phi_\mathcal{B}$ are proved analogously. Alternatively, one may note that an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule is exactly the same as a $\mathcal{B}^{\mathrm{op}}$-$\mathcal{A}^{\mathrm{op}}$-equivalence bimodule, so that exactly the same argument (applied to a $\mathcal{B}^{\mathrm{op}}$-$\mathcal{A}^{\mathrm{op}}$-equivalence bimodule) gives the result.

With our definition of rank-one operators in mind (Equation (5.1)), condition (iii) in Definition 5.1.4 becomes:

$$K^\mathcal{A}_{g,h}(f) = \Phi_\mathcal{A}\big({}_\bullet\langle f, g\rangle\big)(h) = \Phi_\mathcal{B}\big(\langle g, h\rangle_\bullet\big)(f) = K^\mathcal{B}_{f,g}(h)$$

for all $f, g, h \in E$, which gives Equation (5.2).

Condition (ii) in Definition 5.1.4 is precisely the statement that $\Phi_\mathcal{A} \colon \mathcal{A} \to \mathrm{End}_\mathbb{C}(E)$ satisfies $\Phi_\mathcal{A}(\mathcal{A}) \subset \mathcal{L}_\mathcal{B}(E)$ and that the corestriction $\Phi_\mathcal{A} \colon \mathcal{A} \to \mathcal{L}_\mathcal{B}(E)$ preserves the involution. This means that we have a $\star$-algebra homomorphism $\Phi_\mathcal{A} \colon \mathcal{A} \to \mathcal{L}_\mathcal{B}(E)$.

By fullness of the left Hilbert $\mathcal{A}$-module structure, along with continuity of $\Phi_\mathcal{A} \colon \mathcal{A} \to \mathcal{L}_\mathcal{B}(E)$ (Proposition 2.2.6), we now find that

$$\begin{aligned}\Phi_\mathcal{A}(\mathcal{A}) &= \Phi_\mathcal{A}\big(\overline{\mathrm{span}}_\mathbb{C}\big\{{}_\bullet\langle f, g\rangle : f, g \in E\big\}\big)\\ &\subset \overline{\mathrm{span}}_\mathbb{C}\big\{\Phi_\mathcal{A}({}_\bullet\langle f, g\rangle) : f, g \in E\big\}\\ &= \overline{\mathrm{span}}_\mathbb{C}\big\{K^\mathcal{B}_{f,g} : f, g \in E\big\} = \mathcal{K}_\mathcal{B}(E).\end{aligned}$$

Now, we know that $\mathrm{Id}_E = \Phi_\mathcal{A}(1_\mathcal{A}) \in \Phi_\mathcal{A}(\mathcal{A}) \subset \mathcal{K}_\mathcal{B}(E)$ (by our requirement that algebra homomorphisms be unital). Since $\mathcal{K}_B(E)$ is an ideal in $\mathcal{L}_\mathcal{B}(E)$ (Proposition 4.3.17), this implies that $\mathcal{K}_B(E) = \mathcal{L}_\mathcal{B}(E)$.

By the first isomorphism theorem for C*-algebras (Theorem 2.2.33), we know that $\Phi_\mathcal{A}(\mathcal{A})$ is a C*-subalgebra of $\mathcal{L}_\mathcal{B}(E)$. In particular, $\Phi_\mathcal{A}(\mathcal{A}) \subset \mathcal{L}_\mathcal{B}(E)$ is closed. Thus, since $\mathrm{span}_\mathbb{C}\{K^\mathcal{B}_{f,g} : f, g \in E\} \subset \Phi_\mathcal{A}(\mathcal{A})$ by Equation (5.2), we find that $\mathcal{L}_\mathcal{B}(E) = \mathcal{K}_\mathcal{B}(E) \subset \Phi_\mathcal{A}(\mathcal{A}) \subset \mathcal{L}_\mathcal{B}(E)$, from which we can conclude that $\Phi_\mathcal{A}(\mathcal{A}) = \mathcal{L}_\mathcal{B}(E)$.

All that remains is to show that $\Phi_\mathcal{A}$ is injective (it is then automatically isometric by Proposition 2.2.6). Suppose that $a \in \operatorname{Ker} \Phi_\mathcal{A}$ and let $\epsilon > 0$ be

[1]This part of the result depends crucially on our convention that all algebra homomorphisms are unital! In the nonunitial theory, we obtain $\star$-algebra isomorphisms $\Phi_\mathcal{A} \colon \mathcal{A} \to \mathcal{K}_\mathcal{B}(E)$ and $\Phi_\mathcal{B} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{K}_\mathcal{A}(E)$, which map inner products to compact operators in the same fashion, but it need *not* be the case that $\mathcal{K}_\mathcal{A}(E) = \mathcal{L}_\mathcal{A}(E)$ and $\mathcal{K}_\mathcal{B}(E) = \mathcal{L}_\mathcal{B}(E)$.

arbitrary. By fullness of the left Hilbert $\mathcal{A}$-module structure, we can find $f_i, g_i \in E$ (for $1 \leq i \leq N$, where N is some integer) such that

$$\Big\| 1_{\mathcal{A}} - \sum_{i=1}^{N} {}_{\bullet}\langle f_i, g_i \rangle \Big\| < \epsilon, \quad \text{and hence } \Big\| a - a \sum_{i=1}^{N} {}_{\bullet}\langle f_i, g_i \rangle \Big\| \leq \|a\| \epsilon.$$

But $a \in \operatorname{Ker} \Phi_{\mathcal{A}}$ implies that

$$a \sum_{i=1}^{N} {}_{\bullet}\langle f_i, g_i \rangle = \sum_{i=1}^{N} {}_{\bullet}\langle a \cdot f_i, g_i \rangle = \sum_{i=1}^{N} {}_{\bullet}\langle \Phi_{\mathcal{A}}(a) f_i, g_i \rangle = 0,$$

so we find that $\|a\| \leq \|a\| \epsilon$ for any $\epsilon > 0$. This means that $a = 0$, so $\Phi_{\mathcal{A}}$ is injective and we are done. $\square$

We now explore an alternative to condition (ii) in our definition of equivalence bimodules. Note that we have two potentially different norms on an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule E. We will write

$${}_{\bullet}\|f\|^2 = \|{}_{\bullet}\langle f, f \rangle\| \quad \text{and} \quad \|f\|_{\bullet}^2 = \|\langle f, f \rangle_{\bullet}\| \quad \text{for all } f \in E.$$

We will soon find that these norms must in fact be equal, but for now, we must distinguish them.

5.1.6 Lemma (Alternate condition)**.** *Consider Definition 5.1.4. In the presence of conditions (i) and (iii), condition (ii) is equivalent to the following:*

$$\langle a \cdot f, a \cdot f \rangle_{\bullet} \leq \|a\|^2 \langle f, f \rangle_{\bullet} \quad \text{and} \quad {}_{\bullet}\langle f \cdot b, f \cdot b \rangle \leq \|b\|^2 {}_{\bullet}\langle f, f \rangle \tag{5.3}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $f \in E$.

Proof. We first note that Equation (5.3) holds in any $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule E: Proposition 5.1.5 shows that $\Phi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(E)$ and $\Phi_{\mathcal{B}} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{L}_{\mathcal{A}}(E)$ are (isometric) $\star$-algebra isomorphisms, so Equation (5.3) follows immediately from Corollary 4.3.13.

We must now show that condition (ii) of Definition 5.1.4 follows from conditions (i) and (iii) along with Equation (5.3).

Let $f, g, h, k \in E$. Using only condition (iii) and the Hilbert C*-module axioms, we find that

$$\begin{aligned} \big\langle {}_{\bullet}\langle f, g \rangle \cdot h, k \big\rangle_{\bullet} &= \big\langle f \cdot \langle g, h \rangle_{\bullet}, k \big\rangle_{\bullet} = \langle g, h \rangle_{\bullet}^{*} \langle f, k \rangle_{\bullet} = \langle h, g \rangle_{\bullet} \langle f, k \rangle_{\bullet} \\ &= \big\langle h, g \cdot \langle f, k \rangle_{\bullet} \big\rangle_{\bullet} = \big\langle h, {}_{\bullet}\langle g, f \rangle \cdot k \big\rangle_{\bullet} = \big\langle h, {}_{\bullet}\langle f, g \rangle^{*} \cdot k \big\rangle_{\bullet}. \end{aligned}$$

This shows that condition (ii) holds for ${}_\bullet\langle f, g\rangle \in \mathcal{A}$. By linearity of $\Phi_{\mathcal{A}}$ and sesquilinearity of $\langle\cdot,\cdot\rangle_\bullet$, we can conclude that

$$\langle a \cdot h, k\rangle_\bullet = \langle h, a^* \cdot k\rangle_\bullet \quad \text{for all } a \in \operatorname{span}_{\mathbb{C}}\{{}_\bullet\langle f, g\rangle : f, g \in E\} \subset \mathcal{A}.$$

Now, $\operatorname{span}_{\mathbb{C}}\{{}_\bullet\langle f, g\rangle : f, g \in E\}$ is dense in $\mathcal{A}$ by condition (i), so for any $a \in \mathcal{A}$, we may choose a sequence (a_n) for which condition (ii) holds and such that $a_n \to a$.

By Equation (5.3) (and point (ii) of Proposition 2.2.26), we find that

$$\|a \cdot f\|_\bullet^2 = \|\langle a \cdot f, a \cdot f\rangle_\bullet\| \leq \|a\|^2 \|\langle f, f\rangle_\bullet\| \leq \|a\|^2 \|f\|_\bullet^2$$

for all $a \in \mathcal{A}$ and $f \in E$. This means that $a_n \cdot f \to a \cdot f$ in the right Hilbert $\mathcal{B}$-module structure on E whenever $a_n \to a$ in $\mathcal{A}$. If we now fix any $a \in \mathcal{A}$ and choose a sequence (a_n) for which condition (ii) holds and such that $a_n \to a$, we find that

$$\langle a \cdot h, k\rangle_\bullet = \lim_{n\to\infty} \langle a_n \cdot h, k\rangle_\bullet = \lim_{n\to\infty} \langle h, a_n^* \cdot k\rangle_\bullet = \langle h, a^* \cdot k\rangle$$

by continuity of $\langle\cdot,\cdot\rangle_\bullet$ and the involution on $\mathcal{A}$.

Finally, a very similar argument with the roles of $\mathcal{A}$ and $\mathcal{B}$ interchanged (or an appeal to left/right correspondence, as outlined in the beginning of the proof of Proposition 5.1.5) concludes the proof. □

When considering not-necessarily-complete versions of equivalence bimodules, as we now will, one finds that the alternate condition of Lemma 5.1.6 is stronger than condition (ii) of Defintion 5.1.4, and that one must assume the alternate condition in order for completion to work. The problem is that condition (ii) imposes no boundedness on the action of $\mathcal{A}$ with respect to the right $\mathcal{B}$-module norm, whereas Equation (5.3) does.

5.1.7 Definition (Pre-equivalence bimodules). Let A and B be two pre-C*-algebras. An *A-B-pre-equivalence bimodule* is a complex vector space E with the following structure.

(i) E is a full left inner product A-module and a full right inner product B-module. We will denote the A-valued inner product by ${}_\circ\langle\cdot,\cdot\rangle$ and the B-valued inner product by $\langle\cdot,\cdot\rangle_\circ$.

(ii) We have that

$$\langle a \cdot f, a \cdot f\rangle_\circ \leq \|a\|^2 \langle f, f\rangle_\circ \quad \text{and} \quad {}_\circ\langle f \cdot b, f \cdot b\rangle \leq \|b\|^2 {}_\circ\langle f, f\rangle$$

for all $a \in A$, $b \in B$ and $f \in E$, where the left inequality is an inequality in the C*-algebra completion of B and the right inequality is an inequality in the C*-algebra completion of A.

(iii) The inner products satisfy the following associativity condition:

$$_\circ\langle f, g\rangle \cdot h = f \cdot \langle g, h\rangle_\circ$$

for all $f, g, h \in E$.

Suppose we have an A-B-pre-equivalence module E, whose actions and inner products we will denote by

$$\Phi_A \colon A \to \mathrm{End}_{\mathbb{C}}(E), \quad \Phi_B \colon B^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E), \quad {}_A\langle\cdot,\cdot\rangle \quad \text{and} \quad \langle\cdot,\cdot\rangle_B.$$

Then, if $\Phi_1 \colon A' \to A$ and $\Phi_2 \colon B' \to B$ are two isometric $\star$-algebra isomorphism between pre-C*-algebras (which will extend to isomorphisms between their C*-algebra completions), we can turn E into an A'-B'-pre-equivalence bimodule by defining the actions

$$\Phi_{A'} = \Phi_A \circ \Phi_1 \colon A' \to \mathrm{End}_{\mathbb{C}}(E) \quad \text{and} \quad \Phi_{B'} = \Phi_B \circ \Phi_2 \colon (B')^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$$

and the inner products

$$_{A'}\langle\cdot,\cdot\rangle = \Phi_1^{-1}\big({}_A\langle\cdot,\cdot\rangle\big) \quad \text{and} \quad \langle\cdot,\cdot\rangle_{B'} = \Phi_2^{-1}\big(\langle\cdot,\cdot\rangle_B\big).$$

This is a simple but useful observation, which is straightforward to verify.

If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, Lemma 5.1.6 implies that an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule is an $\mathcal{A}$-$\mathcal{B}$-pre-equivalence bimodule as well. Thus, the following lemma implies that the two norms $_\bullet\|\cdot\|$ and $\|\cdot\|_\bullet$ on an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule coincide, as claimed.

5.1.8 Lemma. *Let E be an A-B-pre-equivalence bimodule. Then, the norm $_\circ\|\cdot\|$ on E determined by A agrees with the norm $\|\cdot\|_\circ$ determined by B. In other words, $_\circ\|f\| = \|_\circ\langle f, f\rangle\|^{1/2} = \|\langle f, f\rangle_\circ\|^{1/2} = \|f\|_\circ$ for all $f \in E$.*

Proof. We first note that condition (ii) in the definition of pre-equivalence bimodules implies that, for any $b \in B$ and $f \in E$:

$$_\circ\|f \cdot b\|^2 = \|_\circ\langle f \cdot b, f \cdot b\rangle\| \le \|b\|^2 \|_\circ\langle f, f\rangle\| = \|b\|^2 \big({}_\circ\|f\|\big)^2, \tag{5.4}$$

and hence that $_\circ\|f \cdot b\| \le \|b\|\big({}_\circ\|f\|\big)$ (where we again have used point (ii) of Proposition 2.2.26 to convert an inequality in the C*-algebra completion of B to an inequality of norms).

We also find that

$$\|_\circ\langle f, f\rangle\|^2 = \|_\circ\langle f, f\rangle_\circ\langle f, f\rangle\| = \big\|_\circ\big\langle {}_\circ\langle f, f\rangle \cdot f, f\big\rangle\big\| = \big\|_\circ\big\langle f \cdot \langle f, f\rangle_\circ, f\big\rangle\big\|.$$

Appealing to the Cauchy-Schwarz inequality for ${}_\circ\langle\cdot,\cdot\rangle$ and Equation (5.4) with $b = \langle f, f\rangle_\circ$, we can now conclude that

$$\|{}_\circ\langle f, f\rangle\|^2 \leq \big({}_\circ\|f \cdot \langle f, f\rangle_\circ\|\big)\big({}_\circ\|f\|\big) \leq \|\langle f, f\rangle_\circ\|\big({}_\circ\|f\|\big)^2.$$

By our definitions, this is the statement that ${}_\circ\|f\|^4 \leq (\|f\|_\circ)^2({}_\circ\|f\|)^2$, so we obtain ${}_\circ\|f\| \leq \|f\|_\circ$.

The same argument with the roles of A and B interchanged now gives the result. □

Before we consider completions, we introduce the appropriate notion of isomorphisms for equivalence bimodules. This is not a notion we have seen elsewhere, but it seems natural and convenient to introduce it. For the notion of a unitary map between Hilbert C*-modules (i.e. an isomorphism of Hilbert C*-modules), see Definition 4.3.18.

5.1.9 Definition (Biunitary maps)**.** Let E and F be two $\mathcal{A}$-$\mathcal{B}$-equivalence bimodules. A map $U\colon E \to F$ is called *biunitary* if it is unitary both as a map between left Hilbert $\mathcal{A}$-modules and as a map between right Hilbert $\mathcal{B}$-modules. In other words, a biunitary map $U\colon E \to F$ is an invertible map

$$U \in \mathcal{L}_{\mathcal{A}}(E, F) \cap \mathcal{L}_{\mathcal{B}}(E, F)$$

whose inverse equals its adjoint in both of these spaces.

A biunitary map $U\colon E \to F$ is both $\mathcal{A}$-linear, $\mathcal{B}$-linear and preserves both inner products. This implies that it preserves the equivalence bimodule conditions as well. For example, we see that

$$\begin{aligned} U\big({}_\bullet\langle f, g\rangle \cdot h\big) &= {}_\bullet\langle f, g\rangle \cdot Uh = {}_\bullet\langle Uf, Ug\rangle \cdot Uh \\ \text{and}\quad {}_\bullet\langle Uf \cdot b, Ug\rangle &= {}_\bullet\langle U(f \cdot b), Ug\rangle = \langle f \cdot b, g\rangle \end{aligned}$$

for all $f, g, h \in E$ and $b \in \mathcal{B}$.

We now turn to completions. Suppose that A and B are pre-C*-algebras with C*-algebra completions $\mathcal{A}$ and $\mathcal{B}$. If E is an A-B-pre-equivalence bimodule, then an *equivalence bimodule completion* of E consists of an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule $\overline{E}$ along with a map $j_E\colon E \to \overline{E}$ that makes $\overline{E}$ (along with j_E) a Hilbert C*-module completion of both module structures on E (meaning that j_E is both A- and B-linear, preserves both inner products and has a dense image). As always, we may refer to either $\overline{E}$ or the map j_E by itself as an equivalence bimodule completion of E.

The following proposition shows existence and uniqueness of equivalence bimodule completions. We have usually deferred proofs of such results to

Appendix B, but this result is so intertwined with the contents of this section that we give the proof here. Note that Lemma 5.1.8 allows us to speak of *the* Banach space completion of an equivalence bimodule E.

5.1.10 Proposition (Equivalence bimodule completions)**.** *Let A and B be pre-C*-algebras with C*-algebra completions $\mathcal{A}$ and $\mathcal{B}$. Suppose that E is an A-B-pre-equivalence bimodule and let $j_E\colon E \to \overline{E}$ be a Banach space completion of E. Then, there exist a unique $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule structure on $\overline{E}$ such that $j_E\colon E \to \overline{E}$ becomes a Hilbert C*-module completion of both inner product module structures on E. In other words: such that j_E becomes an equivalence bimodule completion of E.*

Moreover, if $k_E\colon E \to \widetilde{E}$ is another equivalence bimodule completion of E, then there exists a unique biunitary map $U\colon \overline{E} \to \widetilde{E}$ such that $k_E = U \circ j_E$.

Proof. By Proposition 4.3.8 (and its appropriate translation to the setting of right modules), we can give $\overline{E}$ the structure of a left Hilbert $\mathcal{A}$-module and a right Hilbert $\mathcal{B}$-module in such a manner that $j_E\colon E \to \overline{E}$ becomes a Hilbert C*-module completion of both module structures on E. Moreover, these structures are unique. We must verify that $\overline{E}$, equipped with this structure, satisfies the equivalence bimodule axioms of Definition 5.1.4.

Since E is full as a left inner product A-module and as a right inner product B-module, it is clear that $\overline{E}$ is full both as a left Hilbert $\mathcal{A}$-module and as a right Hilbert $\mathcal{B}$-module (for a set which is dense in A is dense in $\mathcal{A}$ and likewise for B and $\mathcal{B}$).

For the following, keep in mind that there is a single topology on $\overline{E}$. We know that the $\mathcal{A}$- and $\mathcal{B}$-valued inner products ${}_{\bullet}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\bullet}$ are continuous (Corollary 4.3.5) and that the actions of $\mathcal{A}$ and $\mathcal{B}$ are continuous (Lemma 4.3.6). Thus, the fact that

$$ {}_{\circ}\langle f, g\rangle \cdot h = f \cdot \langle g, h\rangle_{\circ} \quad \text{for all } f, g, h \in E $$

straightforwardly implies (via approximating sequences) that the corresponding equation holds in $\overline{E}$. This gives axiom (iii) of Definition 5.1.4.

Similarly, the fact that

$$ \langle a \cdot f, a \cdot f\rangle_{\circ} \le \|a\|^2 \langle f, f\rangle_{\circ} \quad \text{and} \quad {}_{\circ}\langle f \cdot b, f \cdot b\rangle \le \|b\|^2 {}_{\circ}\langle f, f\rangle \tag{5.5} $$

for all $a \in A$, $b \in B$ and $f \in E$ implies that the corresponding equation holds in $\overline{E}$. To see this, one must also invoke the fact that a limit of positive elements in a C*-algebra is positive.[2] This shows that the alternate condition

[2]A proof of this fact can be found in the third to last paragraph of the proof of Proposition B.3.1.

of Lemma 5.1.6 holds in $\overline{E}$. Thus, by the contents of that lemma, we have shown that $\overline{E}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule.

As for the claim of uniqueness, suppose that $k_E\colon E \to \widetilde{E}$ is another equivalence bimodule completion of E. Then, as j_E and k_E are both Banach space completions of E (the assumption that k_E preserves the inner product implies that it is an isometry), Corollary B.1.3 on the uniqueness of Banach space completions implies the existence of a unique isometric linear isomorphism $U\colon \overline{E} \to \widetilde{E}$ such that $k_E = U \circ j_E$. By Proposition 4.3.19 (and its appropriate translation to right modules), this map must be unitary with respect to both module structures, for otherwise U would not be unique. □

5.1.2 The Feichtinger Algebra as a Pre-Equivalence Bimodule

Let $A \in \mathrm{GL}(2d, \mathbb{R})$. The goal of this section is to turn the Feichtinger algebra $S_0(\mathbb{R}^d)$ into an $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule and to consider its completion. Throughout this section, we are following Luef [21, 22], who is translating a bimodule construction by Rieffel [27] into the language of Gabor theory and exploring the implications of this correspondence. Rieffel's constructions are in terms of Schwartz spaces, while Luef is working with the Feichtinger algebra.

We will obtain the right $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-module structure on $S_0(\mathbb{R}^d)$ from the left $\ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ})$-module structure afforded by Theorem 4.3.20. For this, we need the following lemma.

5.1.11 Lemma. *Let γ be a 2-cocycle on $\mathbb{Z}^{2d}$. Then,*

$$\begin{aligned} \Gamma\colon \mathbb{C}[\mathbb{Z}^{2d}, \overline{\gamma}] &\to \mathbb{C}[\mathbb{Z}^{2d}, \gamma]^{\mathrm{op}} \\ a &\mapsto \overline{a^*} \end{aligned}$$

(where the involution is taken in $\mathbb{C}[\mathbb{Z}^{2d}, \overline{\gamma}]$) is a $\star$-algebra isomorphism which extends to an (isometric) $\star$-algebra isomorphism $\Gamma\colon C^*(\mathbb{Z}^{2d}, \overline{\gamma}) \to C^*(\mathbb{Z}^{2d}, \gamma)^{\mathrm{op}}$.

Proof. For the duration of this proof, we will write $a(k) = a_k$ for all $k \in \mathbb{Z}^{2d}$ and any $a \in \mathbb{C}[\mathbb{Z}^{2d}, \overline{\gamma}]$ or $a \in \mathbb{C}[\mathbb{Z}^{2d}, \gamma]^{\mathrm{op}}$. Moreover, since we are dealing with two distinct involutions, we will denote the involution on $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]^{\mathrm{op}}$ by $^\bullet$, so that

$$(a^*)_k = \overline{\overline{\gamma}(k, -k) a_{-k}} = \gamma(k, -k)\overline{a_{-k}} \quad \text{and} \quad (a^\bullet)_k = \overline{\gamma(k, -k) a_{-k}}.$$

We find that

$$\Gamma(a)_k = (\overline{a^*})_k = \overline{\gamma(k, -k)\overline{a_{-k}}} = \overline{\gamma(k, -k)} a_{-k} \quad \text{for all } k \in \mathbb{Z}^{2d}. \tag{5.6}$$

It is quite clear that Γ is bijective ($\mathbb{C}[\mathbb{Z}^{2d},\overline{\gamma}]$ and $\mathbb{C}[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}}$ are the same as vector spaces, and both complex conjugation and the involution are bijections). We need to verify that Γ is a $\star$-algebra homomorphism.

Since δ_0 is the unit in both $\mathbb{C}[\mathbb{Z}^{2d},\overline{\gamma}]$ and $\mathbb{C}[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}}$, the calculation

$$\Gamma(\delta_0)_k = \overline{\gamma(k,-k)}\delta_0(-k) = \overline{\gamma(-0,0)}\delta_0(k) = (\delta_0)_k$$

shows that Γ preserves the unit. The calculations

$$\begin{aligned} \Gamma(a)^{\bullet}_k &= \overline{\gamma(k,-k)\Gamma(a)_{-k}} = \overline{\gamma(k,-k)\overline{\gamma(-k,k)}a_k} \\ \text{and}\quad \Gamma(a^*)_k &= \overline{\gamma(k,-k)}(a^*)_{-k} = \overline{\gamma(k,-k)}\gamma(-k,k)\overline{a_k}, \end{aligned}$$

show that Γ preserves the involution.

Preservation of the product requires a bit more work. Let $a,b\in\mathbb{C}[\mathbb{Z}^{2d},\overline{\gamma}]$ and $k\in\mathbb{Z}^{2d}$. Note that the identity to verify is:

$$\Gamma(a *_{\overline{\gamma}} b)_k = \big(\Gamma(b) *_{\gamma} \Gamma(a)\big)_k,$$

because of the opposite structure on the target. We find that

$$\Gamma(a *_{\overline{\gamma}} b)_k = \overline{\gamma(k,-k)}(a *_{\overline{\gamma}} b)_{-k} = \overline{\gamma(k,-k)}\sum_{l\in\mathbb{Z}^{2d}} a_l b_{-k-l}\overline{\gamma(l,-k-l)}$$

and that $\big(\Gamma(b) *_{\gamma} \Gamma(a)\big)_k$ equals

$$\begin{aligned} &\sum_{l\in\mathbb{Z}^{2d}} \Gamma(b)_l\Gamma(a)_{k-l}\gamma(l,k-l) \\ &= \sum_{l\in\mathbb{Z}^{2d}} \overline{\gamma(l,-l)}b_{-l}\overline{\gamma(k-l,l-k)}a_{l-k}\gamma(l,k-l) \\ &= \sum_{l'\in\mathbb{Z}^{2d}} \overline{\gamma(k+l',-k-l')}b_{-k-l'}\overline{\gamma(-l',l')}a_{l'}\gamma(k+l',-l'). \end{aligned}$$

Thus, preservation of the product follows if we can show that

$$\overline{\gamma(k+l,-k-l)\gamma(-l,l)}\gamma(k+l,-l) = \overline{\gamma(k,-k)\gamma(l,-k-l)} \tag{5.7}$$

for all $k,l\in\mathbb{Z}^{2d}$.

By the defining properties of 2-cocycles (Definition 4.1.1), we find that

$$\begin{aligned} \gamma(k+l,-k-l) &= \gamma(k+l,-k-l)\gamma(0,k) \\ &= \gamma(k+l,-k-l+k)\gamma(-k-l,k) \\ &= \gamma(k+l,-l)\gamma(-k-l,k) \\ \text{and that}\quad \gamma(l,-l) &= \gamma(l,-l)\gamma(0,-k) = \gamma(l,-l-k)\gamma(-l,-k). \end{aligned}$$

This gives (using also that γ maps into $\mathbb{T}$, so that conjugates are inverses):

$$\overline{\gamma(k+l,-k-l)\gamma(l,-l)}\gamma(k+l,-l) = \overline{\gamma(-k-l,k)\gamma(l,-l-k)\gamma(-l,-k)}.$$

Finally, noting that

$$\gamma(-l,-k)\gamma(-k-l,k) = \gamma(-l,-k+k)\gamma(-k,k) = \gamma(-k,k) = \gamma(k,-k)$$

gives Equation (5.7) (for the last equality, see the proof of Lemma 4.1.3). Thus, we have proved that Γ is a $\star$-algebra isomorphism.

To conclude that Γ extends to a $\star$-algebra isomorphism $\Gamma\colon C^*(\mathbb{Z}^{2d},\overline{\gamma}) \to C^*(\mathbb{Z}^{2d},\gamma)^{\mathrm{op}}$, it is tempting to appeal to Lemma 4.1.14, which states that $\star$-algebra isomorphisms between twisted group algebras are isometric w.r.t. their universal norms. We cannot use the lemma as stated, because of the opposite structure on the target of Γ. However, it turns out that we can use the exact same argument as we did for its proof.

The proof of Lemma 4.1.14 clearly carries through if we equip $\mathbb{C}[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}}$ with the universal norm (defined as the supremum over the norms of all representations), but we need to know that this equals the universal norm on $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$, which is the norm we use to form the completion $C^*(\mathbb{Z}^{2d},\gamma)$ (which defines $C^*(\mathbb{Z}^{2d},\gamma)^{\mathrm{op}}$, as well as its norm). The Gelfand-Naimark theorem (Theorem 2.2.42) saves the day: if $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}} \to \mathcal{B}(H)$ is any representation of $C[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}}$, we may interpret the same map Φ as a $\star$-algebra homomorphism $\Phi\colon \mathbb{C}[\mathbb{Z}^{2d},\gamma] \to \mathcal{B}(H)^{\mathrm{op}}$, and since $\mathcal{B}(H)^{\mathrm{op}}$ is a C*-algebra, we can consider $\mathcal{B}(H)^{\mathrm{op}}$ to be a C*-subalgebra of $\mathcal{B}(H')$, for some Hilbert space H'. This argument shows that every representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]^{\mathrm{op}}$ can be turned into a representation of $\mathbb{C}[\mathbb{Z}^{2d},\gamma]$ and vice versa. Thus, their universal norms agree, which means that the argument from Lemma 4.1.14 carries through and concludes the proof. $\square$

Consideration of the ℓ^1-norm that defines the $\star$-subalgebras $\ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}}) \subset C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$ and $\ell^1(\mathbb{Z}^{2d},\beta_{A^\circ})^{\mathrm{op}} \subset C^*(\mathbb{Z}^{2d},\beta_{A^\circ})^{\mathrm{op}}$ makes it clear that the isomorphism Γ from Lemma 5.1.11 restricts and corestricts to a $\star$-algebra isomorphism $\Gamma\colon \ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}}) \to \ell^1(\mathbb{Z}^{2d},\beta_{A^\circ})^{\mathrm{op}}$.

Invoking Lemma 5.1.2, we now immediately obtain a right $\ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}}) \cong \ell^1(\mathbb{Z}^{2d},\beta_{A^\circ})^{\mathrm{op}}$-module structure on $S_0(\mathbb{R}^d)$ from the left $\ell^1(\mathbb{Z}^{2d},\beta_{A^\circ})$-module structure afforded by Theorem 4.3.20. However, some consideration of the isomorphism $\Gamma\colon \ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}}) \to \ell^1(\mathbb{Z}^{2d},\beta_{A^\circ})^{\mathrm{op}}$ will be required.

5.1.12 Proposition. *For any $A \in \mathrm{GL}(2d,\mathbb{R})$, the Feichtinger algebra $S_0(\mathbb{R}^d)$ is a right inner product $\ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$-module with respect to the action*

$$f \cdot b = \sum_{k\in\mathbb{Z}^{2d}} b(k)\pi_{A^\circ}(k)^* f \quad \text{for } b \in \ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}}) \text{ and } f \in S_0(\mathbb{R}^d),$$

and the $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-valued inner product defined for $f, g \in S_0(\mathbb{R}^d)$ by

$$\langle f, g\rangle_{A^\circ}(k) = \frac{1}{|\det A|}\langle \pi_{A^\circ}(k)g, f\rangle \quad \textit{for all } k \in \mathbb{Z}^{2d}.$$

Proof. As discussed prior to the proposition, Lemma 5.1.2 turns $S_0(\mathbb{R}^d)$ into a right $\ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}}$-module with inner product

$$\langle f, g\rangle_\times(k) = {}_\times\langle g, f\rangle(k) = {}_{A^\circ}\langle g, f\rangle(k) = \langle g, \pi_{A^\circ}(k)f\rangle \quad \text{for } k \in \mathbb{Z}^{2d}$$

and action $\Pi_{A^\circ} \colon \ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ}) \to \mathrm{End}_{\mathbb{C}}(S_0(\mathbb{R}^d))$. By the construction outlined after Definition 4.3.1 (of inner product modules), we can use the (isometric, w.r.t. the pre-C*-algebra norms) $\star$-algebra isomorphism $\Gamma \colon \ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}}) \to \ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}}$ to transfer this to a right $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-module structure. The action and the inner product become

$$\Pi_{A^\circ} \circ \Gamma \quad \text{and} \quad \langle\cdot,\cdot\rangle_{A^\circ} = \Gamma^{-1}\big(\langle\cdot,\cdot\rangle_\times\big)$$

(note that we may equivalently consider the same map Γ as a $\star$-algebra isomorphism $\Gamma \colon \ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})^{\mathrm{op}} \to \ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ})$, which is more appropriate for the action). All we need to do is to verify that the action and the inner product take the claimed forms.

We begin with a couple of useful observations. We see that

$$\pi_{A^\circ}(-k)^* = \overline{\beta_{A^\circ}(k, -k)}\pi_{A^\circ}(k) \quad \text{for all } k \in \mathbb{Z}^{2d} \tag{5.8}$$

by point (iv) of Lemma 3.1.1. Appealing to Equation (5.6), we find that

$$(\Gamma b)(k) = \overline{\beta_{A^\circ}(k, -k)}b(-k) \quad \text{and} \quad (\Gamma^{-1}c)(k) = \beta_{A^\circ}(-k, k)c(-k)$$

for all $b \in \ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$, $c \in \ell^1(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}}$ and $k \in \mathbb{Z}^{2d}$.

Since $\Pi_{A^\circ} \circ \Gamma(b) = \Pi_{A^\circ}(\Gamma b)$, we now find that

$$\begin{aligned} f \cdot b = \Pi_{A^\circ}(\Gamma b)f &= \sum_{k\in\mathbb{Z}^{2d}} (\Gamma b)(k)\pi_{A^\circ}(k)f = \sum_{k\in\mathbb{Z}^{2d}} b(-k)\overline{\beta_{A^\circ}(k, -k)}\pi_{A^\circ}(k)f \\ &= \sum_{k\in\mathbb{Z}^{2d}} b(-k)\pi_{A^\circ}(-k)^* f, \end{aligned}$$

which gives the claimed form of the action (relabel $k \mapsto -k$). As for the inner product, we note that $\beta_{A^\circ}(k, -k) = \beta_{A^\circ}(-k, k)$ and find that

$$\begin{aligned} \langle f, g\rangle_{A^\circ}(k) &= \Gamma^{-1}(\langle f, g\rangle_\times)(k) = \beta_{A^\circ}(-k, k)\langle g, \pi_{A^\circ}(-k)f\rangle \\ &= \big\langle \beta_{A^\circ}(k, -k)\pi_{A^\circ}(-k)^* g, f\big\rangle = \langle \pi_{A^\circ}(k)g, f\rangle. \end{aligned}$$

In the proposition, we have scaled this inner product by a positive number. We are free do to so, because the inner product axioms will remain satisfied. $\square$

Before we state and prove that we actually obtain a pre-equivalence bimodule, we record the fullness of the left inner product $\ell^1(\mathbb{Z}^{2d},\beta_A)$-module structure on $S_0(\mathbb{R}^d)$ in a lemma. By the left/right correspondence of Lemma 5.1.2, this clearly implies that $S_0(\mathbb{R}^d)$ is full a as a right inner product $\ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$-module as well.

5.1.13 Lemma. *For any $A \in \mathrm{GL}(2d,\mathbb{R})$, the left inner product $\ell^1(\mathbb{Z}^{2d},\beta_A)$-module $S_0(\mathbb{R}^d)$ is full.*[3]

Proof. For this, we must again appeal to Lemma 4.1.25, which states that any ideal $I \subset C^*(\mathbb{Z}^{2d},\beta_A)$ satisfying $\Phi_{\rho_A(-,z)}(I) \subset I$ for all $z \in \mathbb{R}^{2d}$ is trivial.

Fix $z \in \mathbb{R}^{2d}$. The basic commutation relation implies that

$$\pi_A(z)\pi_A(k)\pi_A(z)^* = \rho_A(z,k)\pi_A(k) = \overline{\rho_A(k,z)}\pi_A(k) \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

Thus, for $f,g \in S_0(\mathbb{R}^d)$ and $k \in \mathbb{Z}^{2d}$, we find that

$$\begin{aligned}\big(\Phi_{\rho_A(-,z)}({}_A\langle f,g\rangle)\big)(k) &= \rho_A(k,z)\big\langle f,\pi_A(k)g\big\rangle = \big\langle f,\overline{\rho_A(k,z)}\pi_A(k)g\big\rangle \\ &= \big\langle \pi_A(z)^*f,\pi_A(k)\pi_A(z)^*g\big\rangle = {}_A\big\langle \pi_A(z)^*f,\pi_A(z)^*g\big\rangle(k),\end{aligned}$$

where we are allowed to write the last expression because $S_0(\mathbb{R}^d)$ is closed under time-frequency shifts (Lemma 3.2.14). Thus,

$$\Phi_{\rho_A(-,z)}({}_A\langle f,g\rangle) = {}_A\langle \pi_A(z)^*f,\pi_A(z)^*g\rangle \quad \text{for all } f,g \in S_0(\mathbb{R}^d). \tag{5.9}$$

Now, the vector-subspace

$$J := \operatorname{span}_{\mathbb{C}}\big\{{}_A\langle f,g\rangle : f,g \in S_0(\mathbb{R}^d)\big\} \subset \ell^1(\mathbb{Z}^{2d},\beta_A)$$

is an ideal by $\ell^1(\mathbb{Z}^{2d},\beta_A)$-sesquilinearity of the inner product. Let $a \in C^*(\mathbb{Z}^{2d},\beta_A)$ and $b \in \overline{J}$, where $\overline{J}$ denotes the closure of J in $C^*(\mathbb{Z}^{2d},\beta_A)$. Choose sequences $(a_n)_n \subset \ell^1(\mathbb{Z}^{2d},\beta_A)$ and $(b_n)_n \subset J$ such that $a_n \to a$ and $b_n \to b$ in $C^*(\mathbb{Z}^{2d},\beta_A)$. Then, $(a_nb_n)_n \subset J$, so $ab \subset \overline{J}$ by continuity of the product (and similarly $ba \in \overline{J}$). Thus, $\overline{J}$ is an ideal in $C^*(\mathbb{Z}^{2d},\beta_A)$.

By continuity of $\Phi_{\rho_A(-,z)}$ and Equation (5.9), we can now conclude that

$$\Phi_{\rho_A(-,z)}(\overline{J}) \subset \overline{\Phi_{\rho_A(-,z)}(J)} \subset \overline{J} \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Thus, $\overline{J} = C^*(\mathbb{Z}^{2d},\beta_A)$ by Lemma 4.1.25, which gives the result. □

[3] Keep in mind that we are considering $\ell^1(\mathbb{Z}^{2d},\beta_A)$ as a pre-C*-algebra, so that the fullness condition requires the linear span of inner products to be dense w.r.t. the norm $\ell^1(\mathbb{Z}^{2d},\beta_A)$ inherits from $C^*(\mathbb{Z}^{2d},\beta_A)$, i.e. the universal norm.

5.1.14 Theorem (The Feichtinger algebra as a pre-equivalence bimodule)**.** *Let $A \in \mathrm{GL}(2d, \mathbb{R})$. Then, $S_0(\mathbb{R}^d)$ equipped with the left inner product $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-module structure of Theorem 4.3.20 and the right inner product $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-module structure of Proposition 5.1.12 becomes an $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule.*

Proof. Lemma 5.1.13 verifies condition (i) of Definition 5.1.7, we need to verify conditions (ii) and (iii) of that definition.

We begin with (iii): we wish to show that

$$_A\langle f, g\rangle \cdot h = f \cdot \langle g, h\rangle_{A^\circ} \quad \text{for all } f, g, h \in S_0(\mathbb{R}^d).$$

If we unpack our notation, this becomes:

$$\sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi_A(k)g\rangle \pi_A(k)h = \frac{1}{|\det A|} \sum_{k\in\mathbb{Z}^{2d}} \langle \pi_{A^\circ}(k)h, g\rangle \pi_{A^\circ}(k)^* f. \tag{5.10}$$

Using Equation (5.8), we find that

$$\langle \pi_{A^\circ}(k)h, g\rangle \pi_{A^\circ}(k)^* = \langle h, \pi_{A^\circ}(k)^* g\rangle \pi_{A^\circ}(k)^* = \langle h, \pi_{A^\circ}(-k)g\rangle \pi_{A^\circ}(-k)$$

for all $k \in \mathbb{Z}^{2d}$. Thus, Equation (5.10) is equivalent to:

$$\sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi_A(k)g\rangle \pi_A(k)h = \frac{1}{|\det A|} \sum_{k\in\mathbb{Z}^{2d}} \langle h, \pi_{A^\circ}(k)g\rangle \pi_{A^\circ}(k)f,$$

which is precisely the Janssen representation of the frame operator $S^A_{g,h}$ (Theorem 3.2.26)! Thus, condition (iii) certainly holds.

Condition (ii) becomes:

$$\langle a \cdot f, a \cdot f\rangle_{A^\circ} \leq \|a\|^2 \langle f, f\rangle_{A^\circ} \tag{5.11}$$

for all $a \in \ell^1(\mathbb{Z}^{2d}, \beta_A)$ and $f \in S_0(\mathbb{R}^d)$, in addition to a similar statement regarding the action of $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$ and the inner product $_A\langle \cdot, \cdot\rangle$. These two statements are proved similarly, so we only prove the one displayed above.

Now, Equation (5.11) is an equation in $C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$. The action of $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$ extends to an action of $C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$ by Proposition 4.3.8. This action is defined by the composite map

$$C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})^{\mathrm{op}} \xrightarrow{\Gamma} C^*(\mathbb{Z}^{2d}, \beta_{A^\circ}) \xrightarrow{\Pi_{A^\circ}} \mathrm{End}_{\mathbb{C}}(S_0(\mathbb{R}^d)),$$

where Γ is the isomorphism from Lemma 5.1.11 and Π_{A° is the usual map $\Pi_{A^\circ} \colon C^*(\mathbb{Z}^{2d}, \beta_{A^\circ}) \to \mathcal{B}(L^2(\mathbb{R}^d))$, only with the operators in its image restricted to act on $S_0(\mathbb{R}^d)$. We know Π_{A° to be injective (Proposition 4.1.26), so the composite $\star$-algebra homomorphism

$$C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})^{\mathrm{op}} \xrightarrow{\Gamma} C^*(\mathbb{Z}^{2d}, \beta_{A^\circ}) \xrightarrow{\Pi_{A^\circ}} \mathcal{B}(L^2(\mathbb{R}^d))$$

is injective as well. If we apply this composition to Equation (5.11), we obtain the equation

$$\sum_{k\in\mathbb{Z}^{2d}} \langle a\cdot f, a\cdot f\rangle_{A^\circ}(k)\pi_{A^\circ}(k)^* \leq \sum_{k\in\mathbb{Z}^{2d}} \|a\|^2\langle f, f\rangle_{A^\circ}(k)\pi_{A^\circ}(k)^* \tag{5.12}$$

in $\mathcal{B}(L^2(\mathbb{R}^d))$. By the first isomorphism theorem for C*-algebras (Theorem 2.2.33), we can consider $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})^{\mathrm{op}}$ as a C*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^d))$. By spectral permanence, Equation (5.11) now holds in $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})^{\mathrm{op}}$ if and only if Equation (5.12) holds in $\mathcal{B}(L^2(\mathbb{R}^d))$. Finally, the spectrum of an element in $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})^{\mathrm{op}}$ equals its spectrum in $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$ (invertibility in one implies invertibility in the other), so the positivity condition is the same as well. Thus, we can conclude that Equation (5.11) holds in $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$ if we can prove Equation (5.12) in $\mathcal{B}(L^2(\mathbb{R}^d))$.

By Proposition 2.2.28 and the density of $S_0(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ (Lemma 3.2.14), it now suffices to check that

$$\big\langle g\cdot\langle a\cdot f, a\cdot f\rangle_{A^\circ}, g\big\rangle \leq \|a\|^2\big\langle g\cdot\langle f,f\rangle_{A^\circ}, g\big\rangle \quad \text{for all } g\in S_0(\mathbb{R}^d). \tag{5.13}$$

Applying condition (iii) twice, we find that[4]

$$\begin{aligned}\big\langle g\cdot\langle a\cdot f, a\cdot f\rangle_{A^\circ}, g\big\rangle &= \big\langle {}_A\langle g, a\cdot f\rangle\cdot(a\cdot f), g\big\rangle = \big\langle a\cdot f, {}_A\langle g, a\cdot f\rangle^*\cdot g\big\rangle \\ &= \big\langle a\cdot f, {}_A\langle a\cdot f, g\rangle\cdot g\big\rangle = \big\langle a\cdot f, (a\cdot f)\cdot\langle g,g\rangle_{A^\circ}\big\rangle.\end{aligned}$$

Now, $\langle g,g\rangle_{A^\circ}\geq 0$, so we can consider its unique positive square root afforded by Proposition 2.2.21. This square root will be an element of $C^*(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$, but need not be in $\ell^1(\mathbb{Z}^{2d},\overline{\beta_{A^\circ}})$. Thus, the action by $\langle g,g\rangle_{A^\circ}^{1/2}$ is part of the extended action. Continuing our calculation, we find that

$$\big\langle g\cdot\langle a\cdot f, a\cdot f\rangle_{A^\circ}, g\big\rangle = \big\langle (a\cdot f)\cdot\langle g,g\rangle_{A^\circ}^{1/2}, (a\cdot f)\cdot\langle g,g\rangle_{A^\circ}^{1/2}\big\rangle. \tag{5.14}$$

We claim that

$$(a\cdot f)\cdot\langle g,g\rangle_{A^\circ}^{1/2} = a\cdot(f\cdot\langle g,g\rangle_{A^\circ}^{1/2}),$$

which is equivalent to the operator $\Pi_A(a)$ commuting with the operator defining the action of $\langle g,g\rangle_{A^\circ}^{1/2}$. Now, anything that commutes with $\langle g,g\rangle_{A^\circ}$ commutes with its square root, and $\langle g,g\rangle_{A^\circ}$ acts as

$$\frac{1}{|\det A|}\sum_{k\in\mathbb{Z}^{2d}}\langle\pi_{A^\circ}(k)g,g\rangle\pi_{A^\circ}(k)^*, \quad \text{while} \quad \Pi_A(a) = \sum_{k\in\mathbb{Z}^{2d}} a(k)\pi_A(k),$$

[4] Note that we did *not* use condition (ii) for equivalence bimodules for the second equality, for this is the $L^2(\mathbb{R}^d)$ inner product, and we know that the action $\Pi_{A^\circ}\circ\Gamma$ is a $\star$-algebra homomorphism.

and these operators commute by definition of the adjoint lattice (see Lemma 3.2.15 and the subsequent definition).

Finally, continuing where we left off with Equation (5.14), and using the fact that $\|\Pi_A(a)\| = \|a\|$, we find that

$$\begin{aligned}\big\langle g \cdot \langle a \cdot f, a \cdot f\rangle_{A^\circ}, g\big\rangle &= \big\langle a \cdot (f \cdot \langle g, g\rangle_{A^\circ}^{1/2}), a \cdot (f \cdot \langle g, g\rangle_{A^\circ}^{1/2})\big\rangle \\ &\le \|a\|^2 \big\langle f \cdot \langle g, g\rangle_{A^\circ}^{1/2}, f \cdot \langle g, g\rangle_{A^\circ}^{1/2}\big\rangle \\ &= \|a\|^2 \big\langle f, f \cdot \langle g, g\rangle_{A^\circ}\big\rangle = \|a\|^2 \big\langle f, {}_A\langle f, g\rangle \cdot g\big\rangle \\ &= \|a\|^2 \big\langle {}_A\langle g, f\rangle \cdot f, g\big\rangle = \|a\|^2 \big\langle g \cdot \langle f, f\rangle_{A^\circ}, g\big\rangle.\end{aligned}$$

This gives Equation (5.13), which means that we are done. □

There is one part of the proof which we wish to emphasize, namely that the associativity condition for the inner product, i.e. the condition that

$${}_A\langle f, g\rangle \cdot h = f \cdot \langle g, h\rangle_{A^\circ} \quad \text{for all } f, g, h \in S_0(\mathbb{R}^d),$$

amounts the Janssen representation of the frame operator. This highlights how deeply related the concrete duality in Gabor theory is to these equivalence bimodules, which are quite abstract constructions motivated by representation theory. We also saw how the proof relied on the defining commutativity property between a lattice and its adjoint.

For any $A \in \mathrm{GL}(2d, \mathbb{R})$, we may now consider the equivalence bimodule completion of $S_0(\mathbb{R}^d)$ as an $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule. This yields a $C^*(\mathbb{Z}^{2d}, \beta_A)$-$C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-equivalence bimodule which we will denote by ${}_A\mathcal{E}_{A^\circ}$. As a left Hilbert $C^*(\mathbb{Z}^{2d}, \beta_A)$-module, ${}_A\mathcal{E}_{A^\circ}$ is precisely ${}_A\mathcal{E}$, which we discussed at the end of Subsection 4.3.3. Now, we have seen that we can consider ${}_A\mathcal{E}$ as a left Hilbert $\mathcal{A}_\theta$-module (where $\theta = A^T J A$), because of the isomorphism afforded by Corollary 4.2.6. The following proposition shows that we can think of ${}_A\mathcal{E}_{A^\circ}$ as an $\mathcal{A}_\theta$-$\mathcal{A}_{\theta^{-1}}$-equivalence bimodule.

5.1.15 Proposition. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and let $\theta = A^T J A$. Then,*

$$\mathcal{A}_{\theta^{-1}} \cong C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}}) \quad \text{as } C^*\text{-algebras}$$

via the identification $U_j \mapsto \delta_{e_j}$ for $1 \le j \le 2d$.

Proof. We find that

$$(A^\circ)^T J A^\circ = (-J A^{-T})^T J (-J A^{-T}) = A^{-1} J A^{-T} = -(A^T J A)^{-1} = -\theta^{-1},$$

so that the symplectic form determined by the lattice matrix A° is $-\theta^{-1}$. Certainly $\theta^{-1} \in \mathcal{T}_{2d}$, and we have that

$$e^{2\pi i l^T \theta^{-1} k} = e^{2\pi i \Omega_{\theta^{-1}}(k,l)} = \overline{\rho_{A^\circ}}(k, l) = \overline{\beta_{A^\circ}}(k, l)\overline{\overline{\beta_{A^\circ}}(l, k)}$$

for all $k, l \in \mathbb{Z}^{2d}$ (by Equation (4.8) and the fact that $\Omega_{\theta^{-1}} = -\Omega_{-\theta^{-1}}$). Thus, Theorem 4.2.2 gives the desired result. □

5.2 | Relating Morita Equivalence to Lattices

We have now seen how the duality in Gabor theory, based on the interplay between a lattice $\Lambda = A\mathbb{Z}^{2d}$ and its adjoint $\Lambda^\circ = A^\circ\mathbb{Z}^{2d}$, is reflected in the existence of an equivalence bimodule between the noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta^{-1}}$. This means that certain instances of Morita equivalence between noncommutative tori are connected to duality in Gabor theory. However, there are pairs of Morita equivalent noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ for which $\theta' \neq \theta^{-1}$. In this section we take a slight detour from our main objective and explore how the general notion of Morita equivalence of noncommutative tori may be connected to Gabor theory and to duality in particular.

5.2.1 Morita Equivalence of Noncommutative Tori

As we mentioned when introducing noncommutative tori in Section 4.2: they are well studied objects and much is known about their structure. This subsection provides an example of this: there is a good understanding of when two noncommutative tori are Morita equivalent. In particular, we will see a sufficient condition for Morita equivalence. This condition is *not* necessary, but it comes close to being necessary, for there is a strengthening of the notion of Morita equivalence of noncommutative tori for which it is both necessary and sufficient. More on this can be found in our reference for this subsection, Li [20], who is proving a conjecture of Rieffel and Schwarz [25].

We define the group $\mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$ to be the set of all matrices

$$g = \begin{pmatrix} K & L \\ M & N \end{pmatrix} \in M_{4d}(\mathbb{Z})$$

where $K, L, M, N \in M_{2d}(\mathbb{Z})$ satisfy the following conditions:

$$K^T M + M^T K = 0 = L^T N + N^T L \quad \text{and} \quad K^T N + M^T L = I. \tag{5.15}$$

Of course, based on this description, it is not obvious that such matrices form a group. We will not be concerned with the group structure, so we omit the verifications.

The advertised sufficient condition for Morita equivalence is based on an action of $\mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$ on the set $\mathcal{T}_{2d}$ of all antisymmetric matrices in $M_{2d}(\mathbb{R})$.

5.2.1 Theorem. *Let $\theta \in \mathcal{T}_{2d}$. Suppose that $g \in \mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$ (with blocks $K, L, M, N \in M_{2d}(\mathbb{Z})$ as above). If $M\theta + N \in \mathrm{GL}(2d, \mathbb{R})$, set*

$$g \cdot \theta := (K\theta + L)(M\theta + N)^{-1}.$$

Then, $g \cdot \theta \in \mathcal{T}_{2d}$ and $\mathcal{A}_{g\cdot\theta}$ is Morita equivalent to $\mathcal{A}_\theta$.

Proof. The fact that $g \cdot \theta \in \mathcal{T}_{2d}$ can be shown quite easily by using the generators for $\mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$ which we are about to introduce. For the claim about Morita equivalence, see Li [20, Theorem 1.1] (who uses the term *strong Morita equivalence* for what we are calling Morita equivalence). □

We now define three classes of elements in $\mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$:

$$\rho(R) := \begin{pmatrix} R^T & 0 \\ 0 & R^{-1} \end{pmatrix} \quad \text{for } R \in \mathrm{GL}(2d, \mathbb{Z}), \tag{5.16}$$

$$\nu(N) := \begin{pmatrix} I_{2d} & N \\ 0 & I_{2d} \end{pmatrix} \quad \text{for antisymmetric } N \in M_{2d}(\mathbb{Z}) \tag{5.17}$$

$$\text{and} \quad \sigma_{2p} := \begin{pmatrix} I_{2d} - P_{2p} & P_{2p} \\ P_{2p} & I_{2d} - P_{2p} \end{pmatrix} \quad \text{for } 1 \le p \le d, \tag{5.18}$$

where $P_{2p} = I_{2p} \oplus 0_{2(d-p)}$ denotes the projection onto the first $2p$ factors of $\mathbb{R}^{2d}$. In [25], Rieffel and Schwarz show that σ_2 ($p = 1$) together with all elements of the form $\rho(R)$ and $\nu(N)$ generate $\mathrm{SO}(2d, 2d \,|\, \mathbb{Z})$ as a group. We have slightly modified their definition of $\rho(R)$ so that it is more in line with our conventions (we have replaced R with R^T, which we may, since the transpose is a bijection of $\mathrm{GL}(2d, \mathbb{Z})$).

In the following two subsections, we will attempt to interpret the Morita equivalences afforded by Theorem 5.2.1 in terms of relations among lattice matrices. If we expand our conception of Gabor analysis and replace $\mathbb{R}^d$ by an arbitrary locally compact abelian group, then there already exists a very satisfactory explanation of how these Morita equivalences relate to lattices (see the discussion of *embedding maps* by Li [20], which is a notion that appears already in Rieffel's work [27]). However, this connection relates lattices in $\mathbb{R}^{2d}$ via lattices in larger groups. We wish to explore what can be said purely in terms of lattices on $\mathbb{R}^{2d}$. To our knowledge, this has not been done before.

Although Theorem 5.2.1 holds for all $\theta \in \mathcal{T}_{2d}$, we are concerned only with those matrices representing symplectic forms, so we will mostly restrict our attention to the case where $\theta, g \cdot \theta \in \mathcal{S}_{2d}$.

5.2.2 Linear Permutations and Combinations of Lattices

In this subsection we consider generators of the form $\rho(R)$ and $\nu(N)$ for $R \in \mathrm{GL}(2d, \mathbb{Z})$ and $N^T = -N \in M_{2d}(\mathbb{Z})$, as defined by equations (5.16) and

(5.17).

The first of these, the $\rho(R)$'s, correspond to an equivalence we have already met. If we let $\theta \in \mathcal{S}_{2d}$, then

$$\{\rho(R) \cdot \theta : R \in \mathrm{GL}(2d, \mathbb{Z})\} = \{R^T \theta R : R \in \mathrm{GL}(2d, \mathbb{Z})\} \subset \mathcal{S}_{2d},$$

and this is precisely the set of matrices representing those symplectic forms determined by a fixed *lattice* Λ; see Proposition 3.2.10. The action of $\rho(R)$ on $\theta = A^T JA$ corresponds to a linear permutation of the lattice $A\mathbb{Z}^{2d}$. Indeed, the equivalence between (a) and (b) in the following proposition is *precisely* the content of Proposition 3.2.10. However, since the notation and setting is slightly different and the proof is so simple, we include it here as well.

5.2.2 Proposition. *Let $\theta, \theta' \in \mathcal{S}_{2d}$. Then, the following statements are equivalent.*

(a) There exists some $R \in \mathrm{GL}(2d, \mathbb{Z})$ such that $\theta' = \rho(R) \cdot \theta$.

(b) There exist $A, B \in \mathrm{GL}(2d, \mathbb{R})$ such that $\theta = A^T JA$, $\theta' = B^T JB$ and $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$.

Moreover, if (b) holds, then (a) holds with $R = A^{-1}B$, we have that

$$\begin{aligned} \Phi_R \colon C^*(\mathbb{Z}^{2d}, \beta_B) &\to C^*(\mathbb{Z}^{2d}, \beta_A) \\ \delta_k &\mapsto \delta_{Rk} \end{aligned}$$

is an (isometric) $\star$-algebra isomorphism, and $\mathcal{A}_\theta \cong \mathcal{A}_{\theta'}$ as C^*-algebras.

Proof. Suppose that (a) holds. Choose $A \in \mathrm{GL}(2d, \mathbb{R})$ such that $\theta = A^T JA$ (by Proposition 1.1.8) and set $B = AR$. Then,

$$B^T JB = (AR)^T J(AR) = R^T \theta R = \rho(R) \cdot \theta = \theta',$$

and $A\mathbb{Z}^{2d} = A(R\mathbb{Z}^{2d}) = B\mathbb{Z}^{2d}$. Thus, (b) follows from (a).

Now, suppose that (b) holds. Then, since $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$, we must have $A^{-1}B \in \mathrm{GL}(2d, \mathbb{Z})$ (Lemma 3.2.1). Thus, with $R \coloneqq A^{-1}B$, we find that

$$\rho(R) \cdot \theta = R^T (A^T JA) R = (AR)^T J(AR) = B^T JB = \theta',$$

so (a) follows. This proves the equivalence of (a) and (b).

Suppose now that (b) holds and set $R = A^{-1}B$. Our proof that (b) implies (a) shows that (a) holds with this choice of R. Since $A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$, we have that

$$\pi_A(\mathbb{Z}^{2d}) = \{\pi(Ak) : k \in \mathbb{Z}^{2d}\} = \{\pi(Bk) : k \in \mathbb{Z}^{2d}\} = \pi_B(\mathbb{Z}^{2d})$$

and hence that $C^*(\pi_A(\mathbb{Z}^{2d})) = C^*(\pi_B(\mathbb{Z}^{2d}))$ (these are just two descriptions of the exact same C*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^d))$). Noting that $B = AR$ and recalling the $\star$-algebra isomorphisms of Proposition 4.1.26, the composition

$$\begin{array}{ccccc} C^*(\mathbb{Z}^{2d},\beta_B) & \xrightarrow{\Pi_B} & C^*(\pi_B(\mathbb{Z}^{2d})) = C^*(\pi_A(\mathbb{Z}^{2d})) & \xrightarrow{\Pi_A^{-1}} & C^*(\mathbb{Z}^{2d},\beta_A) \\ \delta_k \mapsto & & \pi_B(k) = \pi_A(Rk) & & \mapsto \delta_{Rk} \end{array}$$

provides an isomorphism of C*-algebras, as claimed. Finally, by Corollary 4.2.6, we also have that

$$\mathcal{A}_\theta \cong C^*(\pi_A(\mathbb{Z}^{2d})) = C^*(\pi_B(\mathbb{Z}^{2d})) \cong \mathcal{A}_{\theta'} \quad \text{as C*-algebras,}$$

which concludes the proof. □

We now see how the fact that a single lattice $\Lambda \subset \mathbb{R}^{2d}$ determines multiple symplectic forms fits into the framework of modules over noncommutative tori. The two modules that result from two arbitrary choices $A, B \in \mathrm{GL}(2d,\mathbb{R})$ of lattices matrices for $\Lambda = A\mathbb{Z}^{2d} = B\mathbb{Z}^{2d}$ are related by an isomorphism $\mathcal{A}_{A^TJA} \cong \mathcal{A}_{B^TJB}$ of C*-algebras. Thus: even though a fixed lattice does not determine a unique symplectic form, it determines a unique noncommutative torus, *up to isomorphisms* of C*-algebras. Moreover, we can easily go between the resulting modules via the transition procedure outlined shortly after Definition 4.3.1 of inner product modules.

As for generators of the form $\nu(N)$, with $N^T = -N \in M_{2d}(\mathbb{Z})$, as defined by Equation (5.17), it seems difficult to give a complete description in terms of lattices on $\mathbb{R}^{2d}$. To see why this is the case, note that the action of $\nu(N)$ on $\mathcal{T}_{2d}$ corresponds to the map

$$\theta \mapsto \nu(N)\cdot\theta = \theta + N,$$

which means that $e^{2\pi i\theta_{ij}} = e^{2\pi i(\nu(N)\cdot\theta)_{ij}}$ for all $1 \leq i,j \leq 2d$. While this changes the symplectic form, it does not change the noncommutative torus at all: $\mathcal{A}_\theta = \mathcal{A}_{\nu(N)\cdot\theta}$, since the NCT-relations only involve the exponentiated version of the symplectic form. Thus, this correspondence seems to have more to do with the fact that noncommutative tori only depend on exponentiated symplectic forms than it has to do with the symplectic forms themselves, and this is not true of lattices.

However, we can exploit this correspondence to derive some consequences for Gabor theory. To do so, we should introduce some terminology which we could have introduce in Chapter 1, on linear symplectic algebra. Given a symplectic form $\Omega\colon \mathbb{R}^{2d}\times\mathbb{R}^{2d} \to \mathbb{R}$, a subspace $V \subset \mathbb{R}^{2d}$ is called Ω-*isotropic* if $\Omega_{V\times V} = 0$ (i.e. if $\Omega(v,w) = 0$ for all $v,w \in V$). In particular,

Ω-Lagrangian subspaces are Ω-isotropic (and can in fact be characterized as maximal Ω-isotropic subspaces).

In the following proposition, we will write $\theta_A \coloneqq A^T JA$ for any $A \in M_{2d}(\mathbb{R})$. Note that in order for θ_A to represent a symplectic form, we must have $A \in \mathrm{GL}(2d, \mathbb{R})$. The value of the following proposition is that it gives us means to modify a lattice without changing the noncommutative torus it determines, which may have implications for the structure of Gabor frames over that lattice.

5.2.3 Proposition. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and let $K, L \in M_{2d}(\mathbb{Z})$. Then, we have that*

$$\theta_{AK+A^\circ L} = \nu(N) \cdot (\theta_{AK} + \theta_{A^\circ L}) \quad \textit{for some } N = -N^T \in M_{2d}(\mathbb{Z}).$$

In particular, $\mathcal{A}_{\theta_{AK+A^\circ L}} = \mathcal{A}_{\theta_{AK}+\theta_{A^\circ L}}$.

As a special case, we find that

$$\theta_{A+A^\circ L} = \nu(N) \cdot \theta_A \quad \textit{if } \theta_{A^\circ L} = 0.$$

Geometrically, $A + A^\circ L$ is obtained by adding to the columns of A a set of vectors from the adjoint lattice $A^\circ \mathbb{Z}^{2d}$ that belong to an Ω_J-isotropic subspace.

Proof. Expanding out, we find that

$$\begin{aligned} \theta_{AK+A^\circ L} &= (AK + A^\circ L)^T J(AK + A^\circ L) \\ &= \theta_{AK} + (AK)^T J(A^\circ L) + (A^\circ L)^T J(AK) + \theta_{A^\circ L}. \end{aligned}$$

Let now

$$\begin{aligned} N &\coloneqq (AK)^T J(A^\circ L) + (A^\circ L)^T J(AK) \\ &= K^T (A^T JA^\circ)L + L^T ((A^\circ)^T JA)K. \end{aligned}$$

Recalling that $A^\circ = -JA^{-T}$, we find that

$$A^T JA^\circ = A^T J(-JA^{-T}) = I \quad \text{and} \quad (A^\circ)^T JA = -(A^T JA^\circ)^T = -I,$$

and hence that $N = K^T L - L^T K = K^T L - (K^T L)^T$. The fact that K and L are integer-valued now implies that $N = -N^T \in M_{2d}(\mathbb{Z})$, and the first displayed equation of the proof shows that

$$\theta_{AK+A^\circ L} = \theta_{AK} + \theta_{A^\circ L} + N = \nu(N) \cdot (\theta_{AK} + \theta_{A^\circ L}),$$

as desired.

As for the geometric characterization of the special case, note that $\theta_{A^\circ L} = 0$ if and only if

$$0 = (\theta_{A^\circ L})_{ij} = e_i^T (A^\circ L)^T J (A^\circ L) e_j = (A^\circ L e_i)^T J (A^\circ L e_j),$$

or equivalently:

$$\Omega_J(A^\circ L e_i, A^\circ L e_j) = 0 \quad \text{for all } 1 \leq i, j \leq 2d.$$

Since $L \in M_{2d}(\mathbb{Z})$, the columns $\{A^\circ L e_j\}_{j=1}^{2d}$ of $A^\circ L$ belong to the set $A^\circ \mathbb{Z}^{2d}$, and the last displayed equation shows that L must be chosen so that these span an Ω_J-isotropic subspace. □

In particular, given any $A \in \mathrm{GL}(2d, \mathbb{R})$, this proposition shows that we can add any single vector from the adjoint lattice $\Lambda^\circ = A^\circ \mathbb{Z}^{2d}$ to any column of A without changing the torus $\mathcal{A}_{\theta_A}$, since all one-dimensional subspaces are Ω_J-isotropic by antisymmetry of Ω_J. Note, however, that the resulting matrix $A + A^\circ L$ could fail to be invertible.

5.2.3 Partially Adjoint Lattices

In this subsection we consider generators of the form σ_{2p}, as defined by Equation (5.18). We first consider the case where $p = d$, as this will give us some valuable insight.

5.2.4 Proposition. *Let $\theta \in \mathcal{S}_{2d}$. Then, $\sigma_{2d} \cdot \theta = \theta^{-1}$, and if $A \in \mathrm{GL}(2d, \mathbb{R})$ determines θ (i.e. if $\theta = A^T J A$), then A° determines $-\theta^{-1}$.*

Proof. The fact that $\sigma_{2d} \cdot \theta = \theta^{-1}$ is immediate from the definitions. We have already seen that $\theta = A^T J A \implies -\theta^{-1} = (A^\circ)^T J A^\circ$ in the proof of Proposition 5.1.15. □

We now see that this particular Morita equivalence is implemented by the $\mathcal{A}_\theta$-$\mathcal{A}_{\theta^{-1}}$-equivalence bimodule we constructed in Subsection 5.1.2. The action by σ_{2d} on $\mathcal{T}_{2d}$ therefore corresponds precisely to the duality between a lattice and its adjoint. The fact that it is $\mathcal{A}_{\theta^{-1}}$ and not $\mathcal{A}_{-\theta^{-1}}$ that is Morita equivalent to $\mathcal{A}_\theta$, even though A° determines $-\theta^{-1}$, stems from the fact that $\mathcal{A}_{\theta^{-1}}^{\mathrm{op}} \cong \mathcal{A}_{-\theta^{-1}}$. This last fact is obvious with regard to the NCT-relations, and may also be understood in terms of twisted group C*-algebras and 2-cocycles via the isomorphism $C^*(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}} \cong C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$ from Lemma 5.1.11.

We now consider the cases where $1 \leq p \leq d-1$. Let $\theta \in \mathcal{S}_{2d}$, set $q := d-p$ and write

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}, \tag{5.19}$$

where $\theta_{11} \in \mathcal{T}_{2p}$, $\theta_{22} \in \mathcal{T}_{2q}$ and θ_{12} is a $2p \times 2q$ matrix. Note that $\theta_{21} = -\theta_{12}^T$. For the rest of this subsection, unless otherwise stated, all matrices written in block form will have blocks of these sizes.

If θ_{11} is invertible, then there is a formulaic way of block-diagonalizing θ, namely:

$$\theta = \begin{pmatrix} I_{2p} & \theta_{11}^{-1}\theta_{12} \\ 0 & I_{2q} \end{pmatrix}^T \begin{pmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix} \begin{pmatrix} I_{2p} & \theta_{11}^{-1}\theta_{12} \\ 0 & I_{2q} \end{pmatrix}. \tag{5.20}$$

The main result of this subsection will be a straightforward consequence of this decomposition. We will refer to Equation (5.20) as the *block-diagonal decomposition* of θ.

5.2.5 Lemma. *Let $1 \leq p \leq d-1$ and partition $\theta \in \mathcal{S}_{2d}$ as in Eq.* (5.19). *Then, $\sigma_{2p} \cdot \theta$ exists if and only if θ_{11} is invertible. In this case, we have that*

$$\begin{aligned} \sigma_{2p} \cdot \theta &= \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12}. \end{pmatrix} \\ &= \begin{pmatrix} I_{2p} & -\theta_{12} \\ 0 & I_{2q} \end{pmatrix}^T \begin{pmatrix} \theta_{11}^{-1} & 0 \\ 0 & \theta_{22} \end{pmatrix} \begin{pmatrix} I_{2p} & -\theta_{12} \\ 0 & I_{2q} \end{pmatrix}. \end{aligned}$$

In particular, $\sigma_{2p} \cdot \theta$ is invertible if and only if θ_{22} is invertible.

Proof. The definitions of σ_{2p} and $\sigma_{2p} \cdot \theta$ imply that

$$\sigma_{2p} \cdot \theta = \begin{pmatrix} I_{2p} & 0 \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ 0 & I_{2q} \end{pmatrix}^{-1}.$$

With $A, B, C, D \in M_{2d}(\mathbb{R})$ arbitrary, we see that

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ 0 & I_{2q} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \theta_{11}A + \theta_{12}C & \theta_{11}B + \theta_{12}D \\ C & D \end{pmatrix}.$$

The right hand side is equal to the identity matrix if and only if $C = 0$, $D = I_{2q}$, θ_{11} is invertible, $A = \theta_{11}^{-1}$ and $B = -\theta_{11}^{-1}\theta_{12}$. Both expressions for $\sigma_{2p} \cdot \theta$ are now straightforward to verify. The second expression is just the block-diagonal decomposition of $\sigma_{2p} \cdot \theta$ determined by Equation (5.20).

Finally, one may check that

$$\begin{pmatrix} I_{2p} & -\theta_{12} \\ 0 & I_{2q} \end{pmatrix}^{-1} = \begin{pmatrix} I_{2p} & \theta_{12} \\ 0 & I_{2q} \end{pmatrix},$$

which gives the claim regarding invertibility. $\square$

In order to interpret the relation between θ and $\sigma_{2p} \cdot \theta$ in terms of lattices, we will change basis in $\mathbb{R}^{2d}$ such that J becomes

$$J_{2p} \oplus J_{2q} = \begin{pmatrix} J_{2p} & 0 \\ 0 & J_{2q} \end{pmatrix},$$

where J_{2p} and J_{2q} are the standard symplectic matrices of dimensions corresponding to their subscripts. Note that, by Proposition 5.2.4, it is natural to seek a relation between lattice matrices determining θ and $-\sigma_{2p} \cdot \theta$. Since the $d = p$ case corresponds to the relation between a lattice and its adjoint, the idea is that the $d < p$ case should be describable in terms of "partially adjoint" lattices (or a "partial dualization", if you will).

5.2.6 Proposition (Partially adjoint lattices)**.** *Let $1 \leq p \leq d-1$ and assume that $\sigma_{2p} \cdot \theta \in \mathcal{S}_{2d}$ for some $\theta \in \mathcal{S}_{2d}$. Write θ in block form as in Eq.* (5.19)*. Then,*

$$\theta_{11}, \quad \theta_{22} \quad \textit{and} \quad \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12}$$

are all invertible. Let A_1, A_2 and B_2 be invertible square matrices (guaranteed to exist by Proposition 1.1.8) such that

$$A_1^T J_{2p} A_1 = \theta_{11}, \quad A_2^T J_{2q} A_2 = \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \quad \textit{and} \quad B_2^T J_{2q} B_2 = \theta_{22},$$

and let X be any $2q \times 2q$ matrix such that $X^T J_{2q} X = -J_{2q}$, e.g. $X = (-I_q) \oplus I_q$.

Then, with respect to the basis defined just prior to this proposition,

$$A = \begin{pmatrix} A_1 & A_1^{\circ}\theta_{12} \\ 0 & A_2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} A_1^{\circ} & -A_1^{\circ}\theta_{12} \\ 0 & XB_2 \end{pmatrix}$$

(where $A_1^{\circ} = -J_{2p}A_1^{-T}$) determine the symplectic forms (represented by) θ and $-\sigma_{2p} \cdot \theta$, respectively. That is,

$$A^T (J_{2p} \oplus J_{2q}) A = \theta \quad \textit{and} \quad B^T (J_{2p} \oplus J_{2q}) B = -\sigma_{2p} \cdot \theta.$$

Proof. Our assumption is that $\theta \in \mathcal{S}_{2d}$ and that $\sigma_{2p} \cdot \theta$ exists and is invertible. By Lemma 5.2.5, we know that θ_{11} and θ_{22} are invertible. Since θ is invertible, the block-diagonal decomposition of θ given by Equation (5.20) implies that $\theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12}$ is invertible as well. The matrices θ_{11}, $\theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12}$ and θ_{22} are all in $\mathcal{S}_{2d}$ (recall that $\theta_{12}^T = -\theta_{21}$), so Proposition 1.1.8 guarantees the existence of A_1, A_2 and B_2, as claimed.

The block-diagonal decomposition of θ also shows that

$$A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I_{2p} & \theta_{11}^{-1}\theta_{12} \\ 0 & I_{2q} \end{pmatrix}$$

determines θ, since $A_1 \oplus A_2$ determines $\theta_{11} \oplus (\theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12})$. Since

$$(A_1^\circ)^T J_{2p} A_1^\circ = -\theta_{11}^{-1} \quad \text{and} \quad (XB_2)^T J_{2q}(XB_2) = -\theta_{22},$$

the block-diagonal decomposition of $\sigma_{2p} \cdot \theta$ given in Lemma 5.2.5 similarly shows that

$$B := \begin{pmatrix} A_1^\circ & 0 \\ 0 & XB_2 \end{pmatrix} \begin{pmatrix} I_{2p} & -\theta_{12} \\ 0 & I_{2q} \end{pmatrix}$$

determines $-\sigma_{2p} \cdot \theta$. Multiplying out these expressions for A and B and using the fact that $\theta_{11}^{-1} = A_1^{-1}(-J_{2p})A_1^{-T} = A_1^{-1}A_1^\circ$ yields the forms given in the proposition. □

We now attempt to formulate this result in more geometric terms and to show how everything connects to the passage from A_1 to A_1° and hence from θ_{11} to $-\theta_{11}^{-1}$ (which we will refer to as the *dualization* of θ_{11}).

Let θ and $\sigma_{2p} \cdot \theta$ be as in Proposition 5.2.6. If we introduce $\phi := \theta_{11}^{-1}\theta_{12}$, we can write the block-diagonal decomposition of θ as

$$\theta = \begin{pmatrix} I_{2p} & \phi \\ 0 & I_{2q} \end{pmatrix}^T \begin{pmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} - \phi^*\theta_{11} \end{pmatrix} \begin{pmatrix} I_{2p} & \phi \\ 0 & I_{2q} \end{pmatrix},$$

where $\phi^*\theta_{11} = \phi^T\theta_{11}\phi$. We use this notation because the matrix $\phi^*\theta_{11}$ represents the pullback by $\phi\colon \mathbb{R}^{2q} \to \mathbb{R}^{2p}$ of the form represented by θ_{11}. In other words: $\phi^*\Omega_{\theta_{11}} = \Omega_{\phi^*\theta_{11}}$. This decomposition shows that the symplectic form Ω_θ on $\mathbb{R}^{2p+2q}$ can be described geometrically by the restricted forms $\Omega_{\theta_{11}}$ and $\Omega_{\theta_{22}}$ on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q}$, along with the linear transformation $\phi\colon \mathbb{R}^{2q} \to \mathbb{R}^{2p}$.

The block-diagonal decomposition of $\sigma_{2p} \cdot \theta$ can be written as

$$\theta' = \begin{pmatrix} I_{2p} & \psi \\ 0 & I_{2q} \end{pmatrix}^T \begin{pmatrix} \theta_{11}^{-1} & 0 \\ 0 & \theta_{22} \end{pmatrix} \begin{pmatrix} I_{2p} & \psi \\ 0 & I_{2q} \end{pmatrix},$$

where $\psi := -\theta_{12}$. Now, the linear transformations $\phi, \psi\colon \mathbb{R}^{2q} \to \mathbb{R}^{2p}$ are related by:

$$\phi^*\theta_{11} = (\theta_{11}^{-1}\theta_{12})^T\theta_{11}(\theta_{11}^{-1}\theta_{12}) = \theta_{12}^T\theta_{11}^{-T}\theta_{12} = -\theta_{12}^T\theta_{11}^{-1}\theta_{12} = \psi^*(-\theta_{11}^{-1}).$$

We can also write this as

$$(A_1\phi)^*\Omega_J = \phi^*\Omega_{\theta_{11}} = \psi^*\Omega_{-\theta_{11}^{-1}} = (A_1^\circ\psi)^*\Omega_J.$$

Thus, we see that ϕ and ψ are directly related by the dualization of θ_{11}. Moreover, with this notation, the lattice matrices A and B of Proposition 5.2.6 become

$$A = \begin{pmatrix} A_1 & A_1\phi \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} A_1^\circ & A_1^\circ\psi \\ 0 & XB_2 \end{pmatrix},$$

and we can determine B_2 (to the degree that B_2 is determined by the proposition, namely up to symplectic transformations) via the equation:

$$(B_2)^*\Omega_J = (A_2)^*\Omega_J + (A_1\phi)^*\Omega_J = (A_2)^*\Omega_J + (A_1^\circ\psi)^*\Omega_J$$

(this is just the equation $\theta_{22} = (\theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12}) + \theta_{21}\theta_{11}^{-1}\theta_{12}$ in different notation).

Chapter 6

Metaplectic Transformations

6.1 | Metaplectic Transformations and Equivalence Bimodules

This final section consists of three subsections. In the first, we introduce metaplectic transformations and discover the convenience of Heisenberg-Weyl operators (introduced in Subsection 3.1.2) over time-frequency shifts. In the second, we show how to reformulate our equivalence bimodules from Chapter 5 in terms of Heisenberg-Weyl operators. In the third, we make good on our promises from Subsection 3.2.2 and show the structure of Gabor frames over a lattice depends only the symplectic form determined by any of its lattice matrices. This statement takes two forms: we state one result with regard to the general $L^2(\mathbb{R}^d)$-theory and one with regard to our equivalence bimodules. In particular, we will construct isomorphisms between equivalence bimodules arising from lattice matrices related by a symplectic transformation.

6.1.1 The Metaplectic Representation

The subject of the metaplectic group is beautiful and subtle, and deserves a proper introduction. This is not a simple task, so we will settle for a simpler presentation that suffices for our needs and refer the interested reader to de Gosson [15, Chapter 7], who is our main source for this topic.

Recall Proposition 1.2.6, which states that the symplectic group $\mathrm{Sp}(2d,\mathbb{R})$ is generated by all elements of the form

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}, \quad V_P = \begin{pmatrix} I_d & 0 \\ -P & I_d \end{pmatrix} \quad \text{and} \quad M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}, \tag{6.1}$$

where $P^T = P \in M_d(\mathbb{R})$ and $L \in \mathrm{GL}(d, \mathbb{R})$. Recall also the Heisenberg-Weyl operators from Subsection 3.1.2:

$$\psi(z) := M_{\omega/2} T_x M_{\omega/2} = e^{-\pi i x \cdot \omega} \pi(z) \quad \text{for } z = (x, w) \in \mathbb{R}^{2d}. \tag{6.2}$$

Shortly after introducing these operators, we noted the identity:

$$\mathcal{F}\psi(x, \omega) = \psi(\omega, -x)\mathcal{F} \quad \text{for all } z = (x, \omega) \in \mathbb{R}^{2d},$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^d$. One may now notice that we can write this identity as $\mathcal{F}\psi(z) = \psi(Jz)\mathcal{F}$. That is, we have the commutative diagram

$$\begin{tikzcd} L^2(\mathbb{R}^d) \arrow[r, "\mathcal{F}"] \arrow[d, "\psi(z)"'] & L^2(\mathbb{R}^d) \arrow[d, "\psi(Jz)"] \\ L^2(\mathbb{R}^d) \arrow[r, "\mathcal{F}"'] & L^2(\mathbb{R}^d) \end{tikzcd}$$

for each $z \in \mathbb{R}^{2d}$.

This is but the tip of an iceberg: for *any* symplectic transformation $S \in \mathrm{Sp}(2d, \mathbb{R})$, we can find a unitary operator U on $L^2(\mathbb{R}^d)$ such that

$$U\psi(z) = \psi(Sz)U \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Of course, if U satisfies this identity, then, for any $\xi \in \mathbb{T}$, ξU is another unitary operator satisfying this identity. It turns out that there is a way to choose exactly two such operators for each $S \in \mathrm{Sp}(2d, \mathbb{R})$ (differing only by a sign) in such a manner that their collection forms a double covering group of the symplectic group $\mathrm{Sp}(2d, \mathbb{R})$. This is the metaplectic group $\mathrm{Mp}(2d, \mathbb{R})$, realized as a subgroup of $\mathcal{U}(L^2(\mathbb{R}^d))$. This realization is also referred to as the *metaplectic representation*.

We will not go into the details of this construction; we will settle for an ad hoc definition that is sufficient for our purposes. We first introduce the metaplectic operators associated to each of the generators from Equation (6.1), and then we simply define the metaplectic group as the subgroup of $\mathcal{U}(L^2(\mathbb{R}^d))$ generated by these operators.[1]

We have already seen that the Fourier transform $\mathcal{F}$ satisfies the desired relation for J. In order to obtain the metaplectic group, we have to multiply it by a certain phase. The correction choice turns out to be $e^{-\pi i d/4}$.

[1]For a proof that this method gives the same result as the usual definition, see de Gosson [15, Corollary 112 on p. 83].

6.1.1 Definition (Generators of the metaplectic group)**.** Let $M^T = M \in M_d(\mathbb{R})$ and let $L \in \mathrm{GL}(d,\mathbb{R})$. Choose $m \in \{0,1,2,3\}$ so that $i^m\sqrt{|\det L|}$ is a square root of $\det L$ (that is, choose $m \in \{0,2\}$ if $\det L > 0$ and $m \in \{1,3\}$ if $\det L < 0$). We define the operators $\widehat{J}, \widehat{V}_P$ and $\widehat{M}_{L,m}$ on $L^2(\mathbb{R}^d)$ by

- $\widehat{J}f(t) = e^{-\pi i d/4}\mathcal{F}f(t) = e^{-\pi i d/4}\displaystyle\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot t}\,\mathrm{d}x$
- $\widehat{V}_P f(t) = e^{-\pi i (Pt)\cdot t} f(t)$
- $\widehat{M}_{L,m}f(t) = i^m\sqrt{|\det L|}f(Lt)$

for all $t \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$.

With the unitarity of $\mathcal{F}$ in mind, it is straightforward to see that all of these operators are unitary; we omit the details. We will show that they behave as advertised:

6.1.2 Lemma. *We have that*

- $\widehat{J}\psi(z) = \psi(Jz)\widehat{J}$
- $\widehat{V}_P\psi(z) = \psi(V_P z)\widehat{V}_P$
- $\widehat{M}_{L,m}\psi(z) = \psi(M_L z)\widehat{M}_{L,m}$

for all $z \in \mathbb{R}^{2d}$*,* $M^T = M \in M_d(\mathbb{R})$*,* $L \in \mathrm{GL}(d,\mathbb{R})$ *and* $m \in \{0,1,2,3\}$ *(where the* m*'s are chosen in accordance with Definition 6.1.1).*

Proof. Since $\widehat{J} = e^{-\pi i d/4}\mathcal{F}$, it follows immediately from our discussion in the opening of this subsection that $\widehat{J}\psi(z) = \psi(Jz)\widehat{J}$ holds. As for the remaining identities, we must calculate.

Let $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$. We first calculate with time-frequency shifts, because they involve less phase factors. With $z = (x,\omega) \in \mathbb{R}^{2d}$, we find that

$$\big(\widehat{V}_P\pi(z)f\big)(t) = \widehat{V}_P\big(e^{2\pi i\omega\cdot t}f(t-x)\big) = e^{-\pi i(Pt)\cdot t}e^{2\pi i\omega\cdot t}f(t-x)$$

and moreover that

$$\begin{aligned}\big(\pi(V_P z)\widehat{V}_P f\big)(t) &= \pi(x,\omega - Px)\big(e^{-\pi i(Pt)\cdot t}f(t)\big)\\ &= e^{2\pi i(\omega-Px)\cdot t}e^{-\pi i(Pt-Px)\cdot(t-x)}f(t-x)\\ &= e^{-2\pi i(Px)\cdot t}e^{\pi i(Pt)\cdot x}e^{\pi i(Px)\cdot(t-x)}\big(\widehat{V}_P\pi(z)f\big)(t).\end{aligned}$$

Since $P = P^T$,

$$-2(Px) \cdot t + (Pt) \cdot x + (Px) \cdot (t - x) = -(Px) \cdot x,$$

from which we can conclude that $\pi(V_P z)\widehat{V}_P = e^{-\pi i (Px)\cdot x}\widehat{V}_P \pi(z)$. Using the relation between time-frequency shifts and Heisenberg-Weyl operators (Equation (6.2)), we now see that

$$\begin{aligned}\psi(V_P z)\widehat{V}_P = \psi(x, \omega - Px)\widehat{V}_P &= e^{-\pi i x\cdot(\omega - Px)}\pi(V_P z)\widehat{V}_P \\ &= e^{-\pi i x\cdot\omega}\widehat{V}_P\pi(z) = \widehat{V}_P\psi(z),\end{aligned}$$

which proves the second identity.

When it comes to the last identity, the constant $i^m\sqrt{|\det L|}$ plays no role. Without this constant, $\widehat{M_{L,m}}$ is just the operator $L^t\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by $L^t f = f \circ L$. We find that

$$\big(L^t\psi(z)f\big)(t) = L^t\big(e^{-\pi i x\cdot\omega}e^{2\pi i\omega\cdot t}f(t - x)\big) = e^{-\pi i x\cdot\omega}e^{2\pi i\omega\cdot(Lt)}f(Lt - x)$$

and that

$$\begin{aligned}\big(\psi(M_L z)L^t f\big)(t) &= \psi(L^{-1}x, L^T\omega)\big(f(Lt)\big) \\ &= e^{-\pi i(L^{-1}x)\cdot(L^T\omega)}e^{2\pi i(L^T\omega)\cdot t}f\big(L(t - L^{-1}x)\big).\end{aligned}$$

The fact that $\omega \cdot (Lt) = (L^T\omega) \cdot t$ and $(L^{-1}x) \cdot (L^T\omega) = x \cdot \omega$ implies that these two expression are equal, so that $L^t\psi(z) = \psi(M_L z)L^t$. Multiplying by $i^m\sqrt{|\det L|}$ now gives the third identity and concludes the proof. □

6.1.3 Definition (The metaplectic group)**.** We define the metaplectic group $\mathrm{Mp}(2d, \mathbb{R})$ to be the subgroup of $\mathcal{U}(L^2(\mathbb{R}^d))$ generated by the set of operators from Definition 6.1.1.

6.1.4 Proposition. *For each $S \in \mathrm{Sp}(2d, \mathbb{R})$, there exists a metaplectic operator $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ such that*

$$\widehat{S}\psi(z) = \psi(Sz)\widehat{S} \quad \textit{for all } z \in \mathbb{R}^{2d}.$$

Equivalently: such that the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d) & \xrightarrow{\widehat{S}} & L^2(\mathbb{R}^d) \\ {\scriptstyle \psi(z)}\big\downarrow & & \big\downarrow{\scriptstyle \psi(Sz)} \\ L^2(\mathbb{R}^d) & \xrightarrow[\widehat{S}]{} & L^2(\mathbb{R}^d) \end{array}$$

commutes for each $z \in \mathbb{R}^{2d}$. We refer to this situation by saying that $\widehat{S}$ is a metaplectic operator associated to the symplectic matrix S.

Proof. Suppose that we have two symplectic matrices $S_1, S_2 \in \mathrm{Sp}(2d, \mathbb{R})$, and that we have metaplectic operators $\widehat{S}_1, \widehat{S}_2 \in \mathrm{Mp}(2d, \mathbb{R})$ associated to these matrices. Then, $\widehat{S}_1\widehat{S}_2$ is a metaplectic operators associated to the product S_1S_2, because

$$\widehat{S}_1\widehat{S}_2\psi(z) = \widehat{S}_1\psi(S_2z)\widehat{S}_2 = \psi(S_1S_2z)\widehat{S}_1\widehat{S}_2 \quad \text{for all } z \in \mathbb{R}^{2d}.$$

By Proposition 1.2.6, any $S \in \mathrm{Sp}(2d, \mathbb{R})$ can be written as a finite product of elements from the set

$$\{V_P : P^T = P \in M_d(\mathbb{R})\} \cup \{M_L : L \in \mathrm{GL}(d, \mathbb{R})\} \cup \{J\},$$

and by Lemma 6.1.2, every element of this set has an associated metaplectic operator. A simple inductive argument modelled on the previous paragraph now provides the proof. $\square$

We emphasize that the foregoing proposition does *not* hold for time-frequency shifts. Recalling the canonical projections

$$\begin{aligned} P_1 \colon \mathbb{R}^{2d} &\to \mathbb{R}^d \\ (x, \omega) &\mapsto x \end{aligned} \qquad \text{and} \qquad \begin{aligned} P_2 \colon \mathbb{R}^{2d} &\to \mathbb{R}^d \\ (x, \omega) &\mapsto \omega, \end{aligned}$$

we can write the relation between time-frequency shifts and Heisenberg-Weyl operators as follows:

$$\psi(z) = e^{-\pi i P_2(z)\cdot P_1(z)}\pi(z) \quad \text{for all } z \in \mathbb{Z}^{2d}.$$

For any $S \in \mathrm{Sp}(2d, \mathbb{R})$, with associated metaplectic operator $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$, we now find that

$$\pi(Sz)\widehat{S} = e^{\pi i P_2(Sz)\cdot P_1(Sz)}\psi(Sz)\widehat{S} = e^{\pi i P_2(Sz)\cdot P_1(Sz)}e^{-\pi i P_2(z)\cdot P_1(z)}\widehat{S}\pi(z),$$

and these phase factors will generally *not* cancel. Because of the explicit dependence on z, we cannot absorb these phase factors into our metaplectic operators either.

6.1.2 Heisenberg-Weyl Operators and Bimodules

Having begun to see the advantage of Heisenberg-Weyl operators over time-frequency shifts, we now wish to recast our equivalence bimodules from Chapter 5 in terms of Heisenberg-Weyl operators. This will give us a nice opportunity to recap the overall story as well.

Let $A \in \mathrm{GL}(2d, \mathbb{R})$. For time-frequency shifts, we have seen that the identity

$$\pi_A(z)\pi_A(w) = e^{-2\pi i P_2(Aw)\cdot P_1(Az)}\pi_A(z+w) = \beta_A(z,w)\pi_A(z+w) \tag{6.3}$$

is of fundamental importance: it lead us to β_A-twisted representations of $\mathbb{Z}^{2d}$, the β_A-twisted group C*-algebra $C^*(\mathbb{Z}^{2d}, \beta_A)$ and its dense subset $\ell^1(\mathbb{Z}^{2d}, \beta_A)$, which we used to imbue the Feichtinger algebra $S_0(\mathbb{R}^d)$ with the structure of an $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule.

We now tell the same story with Heisenberg-Weyl operators in place of time-frequency shifts. Using the relation $\psi(z) = e^{-\pi i x\cdot\omega}\pi(z)$, where $z = (x, \omega)$, we find that

$$\begin{aligned}\psi(z)\psi(w) &= e^{-\pi i\omega\cdot x}e^{-\pi i\eta\cdot y}e^{-2\pi i\eta\cdot x}\pi(z+w)\\ &= e^{-\pi i\omega\cdot x}e^{-\pi i\eta\cdot y}e^{-2\pi i\eta\cdot x}e^{\pi i(x+y)\cdot(\omega+\eta)}\psi(z+w)\\ &= e^{\pi i(y\cdot\omega-\eta\cdot x)}\psi(z+w) = e^{\pi i\Omega_J(z,w)}\psi(z+w)\end{aligned}$$

for all $z = (x, \omega), w = (y, \eta) \in \mathbb{R}^{2d}$. We now introduce the notation

$$\psi_A(z) := \psi(Az) \quad \text{for all } z \in \mathbb{R}^{2d},$$

and with $\theta := A^T J A$, we define the *Heisenberg-Weyl cocycle* determined by A to be the map

$$\begin{aligned}\gamma_\theta \colon \mathbb{R}^{2d} \times \mathbb{R}^{2d} &\to \mathbb{R}^d\\ (z,w) &\mapsto \gamma_\theta(z,w) = e^{\pi i\Omega_\theta(z,w)}.\end{aligned}$$

Then, for all $z, w \in \mathbb{R}^{2d}$, we have that

$$\psi_A(z)\psi_A(w) = e^{\pi i\Omega_J(Az,Aw)}\psi_A(z+w) = \gamma_\theta(z,w)\psi_A(z+w). \tag{6.4}$$

This it the analogue of Equation (6.3) for Heisenberg-Weyl operators. We see that the cocycle β_A has been replaced by $\gamma_\theta = \gamma_{A^TJA}$.

A convenient simplification has occurred already at this stage: the explicit dependence on A has disappeared from the cocycle. Indeed, the cocycle depends only on the symplectic form determined by A. Not only is this more in line with our philosophy of emphasizing symplectic forms over lattices, but it makes it much easier to consider maps between modules, for while $C^*(\mathbb{Z}^{2d}, \beta_A)$ and $C^*(\mathbb{Z}^{2d}, \beta_B)$ are isomorphic whenever $A^TJA = B^TJB$, the corresponding twisted group C*-algebras for Heisenberg-Weyl operators will be *equal*, since $\gamma_{A^TJA} = \gamma_{B^TJB}$.

Moreover, the isomorphism $C^*(\mathbb{Z}^{2d}, \beta_A) \to C^*(\mathbb{Z}^{2d}, \beta_B)$ is not as simple as one might hope, for $\delta_k \mapsto \delta_k$ is generally *not* an algebra homomorphism.

Indeed, it seems that the overall simplest isomorphism is obtained by relating each of these C*-algebras to their common Heisenberg-Weyl counterparts (via the upcoming Lemma 6.1.5). Because of the isomorphisms $C^*(\mathbb{Z}^{2d}, \beta_A) \cong \mathcal{A}_\theta \cong C^*(\mathbb{Z}^{2d}, \beta_B)$ from Corollary 4.2.6, we also know that there is an isomorphism $C^*(\mathbb{Z}^{2d}, \beta_A) \to C^*(\mathbb{Z}^{2d}, \beta_B)$ determined by $\delta_{e_j} \mapsto \delta_{e_j}$. However, if one wants to relate $\delta_k \in C^*(\mathbb{Z}^{2d}, \beta_A)$ to $\delta_k \in C^*(\mathbb{Z}^{2d}, \beta_B)$ under this isomorphism, one will encounter complicated phase factors such as those from Lemma 4.2.4.

Returning to the main story: Equation (6.4) shows that the map

$$\begin{aligned} \psi_A \colon \mathbb{Z}^{2d} &\to \mathcal{U}\big(L^2(\mathbb{R}^d)\big) \\ k &\mapsto \psi_A(k) \end{aligned}$$

defines a γ_θ-twisted representation of $\mathbb{Z}^{2d}$. By Theorem 4.1.13, this determines a unique $\star$-algebra homomorphism

$$\begin{aligned} \Psi_A \colon C^*(\mathbb{Z}^{2d}, \gamma_\theta) &\to \mathcal{B}\big(L^2(\mathbb{R}^d)\big) \\ \delta_k &\mapsto \Psi_A(\delta_k) = \psi_A(k). \end{aligned}$$

Like Π_A, this map will be injective, and we can use it to equip the Feichtinger algebra $S_0(\mathbb{R}^d)$ with a left inner product $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-module structure and then obtain an appropriate pre-equivalence bimodule structure. One can do all of this by retracing our steps from the previous chapters (and appropriately adapting the crucial Lemma 4.1.25 by Green). Indeed, not much changes: by cancellations of phase factors, the FIGA and the Janssen representation will continue to hold with Heisenberg-Weyl operators in place of time-frequency shifts. However, so as not to sweep too many details under the rug, we will build on the results we already have and show how to transfer the familiar $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule structure to the Heisenberg-Weyl setting.

The following lemma simply shows that the correspondence

$$\psi(z) = e^{-\pi i x \cdot \omega} \pi(z) = e^{-\pi i P_2(z) \cdot P_1(z)} \pi(z)$$

leads to the expected isomorphism $C^*(\mathbb{Z}^{2d}, \gamma_\theta) \cong C^*(\mathbb{Z}^{2d}, \beta_A)$.

6.1.5 Lemma. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and let $\theta = A^T J A$. Then, there is a $\star$-algebra isomorphism*

$$\begin{aligned} \Xi_A \colon C^*(\mathbb{Z}^{2d}, \gamma_\theta) &\to C^*(\mathbb{Z}^{2d}, \beta_A) \\ \delta_k &\mapsto e^{-\pi i P_2(Ak) \cdot P_1(Ak)} \delta_k. \end{aligned}$$

Moreover, $\Psi_A = \Pi_A \circ \Xi_A$.

Proof. Recall that $\mathcal{D} \coloneqq \{\delta_k : k \in \mathbb{Z}^{2d}\}$ is a vector space basis for any twisted group algebra $\mathbb{C}[\mathbb{Z}^{2d}, \gamma]$. Consider the linear map defined by

$$\begin{aligned}\Xi_A \colon \mathbb{C}[\mathbb{Z}^{2d}, \gamma_\theta] &\to \mathbb{C}[\mathbb{Z}^{2d}, \beta_A] \\ \delta_k &\mapsto e^{-\pi i P_2(Ak)\cdot P_1(Ak)}\delta_k.\end{aligned}$$

This is clearly a bijection, for we can eyeball its inverse. We will show that it is a $\star$-algebra homomorphism as well.

For any $k \in \mathbb{Z}^{2d}$, we find that

$$\begin{aligned}\Xi_A(\delta_k^*) &= \Xi_A\big(\overline{\gamma_\theta(-k,k)}\delta_{-k}\big) = \Xi_A(\delta_{-k}) = e^{-\pi i P_2(-Ak)\cdot P_1(-Ak)}\delta_{-k} \\ \text{and}\quad \Xi_A(\delta_k)^* &= \overline{\beta_A(-k,k)e^{-\pi i P_2(Ak)\cdot P_1(Ak)}}\delta_{-k},\end{aligned}$$

from which the fact that $\beta_A(-k,k) = e^{-2\pi i P_2(Ak)\cdot P_1(-Ak)}$ implies that $\Xi_A(\delta_k^*) = \Xi_A(\delta_k)^*$. Since the involutions on these algebras are conjugate-linear extensions of their restrictions to $\mathcal{D}$, this implies that Ξ_A preserves the involution.

For any $k, l \in \mathbb{Z}^{2d}$, we find that

$$\begin{aligned}\Xi_A(\delta_k *_{\gamma_\theta} \delta_l) &= \gamma_\theta(k,l)\Xi_A(\delta_{k+l}) \\ &= \gamma_\theta(k,l)e^{-\pi i P_2(Ak+Al)\cdot P_1(Ak+Al)}\delta_{k+l} \\ &= e^{\pi i (P_2(Ak)\cdot P_1(Al) - P_2(Al)\cdot P_1(Ak))}e^{-\pi i P_2(Ak+Al)\cdot P_1(Ak+Al)}\delta_{k+l} \\ &= e^{-2\pi i P_2(Al)\cdot P_1(Ak)}e^{-\pi i P_2(Ak)\cdot P_1(Ak)}e^{-\pi i P_2(Al)\cdot P_1(Al)}\delta_{k+l} \\ &= \Xi_A(\delta_k) *_{\beta_A} \Xi_A(\delta_l),\end{aligned}$$

where we used that $\Omega_\theta(k,l) = \Omega_J(Ak, Al) = P_2(Ak)\cdot P_1(Al) - P_2(Al)\cdot P_1(Ak)$ (Equation (1.4)). Since the products on these algebras are bilinear extensions of their restrictions to $\mathcal{D}$, this implies that Ξ_A preserves the product.

Preservation of the identity is clear, so we have proved that Ξ_A is a $\star$-algebra isomorphism. By Lemma 4.1.14, we can conclude that it extends to the desired $\star$-algebra isomorphism $\Xi_A \colon C^*(\mathbb{Z}^{2d}, \gamma_\theta) \to C^*(\mathbb{Z}^{2d}, \beta_A)$.

Finally, since

$$\Pi_A \circ \Xi_A(\delta_k) = \Pi_A(e^{-\pi i P_2(Ak)\cdot P_1(Ak)}\delta_k) = e^{-\pi i P_2(Ak)\cdot P_1(Ak)}\pi_A(k) = \psi_A(k)$$

for any $k \in \mathbb{Z}$, and since any $\star$-algebra homomorphism from $C^*(\mathbb{Z}^{2d}, \gamma_\theta)$ is determined by where it maps the generators $\delta_{e_1}, \ldots, \delta_{e_{2d}}$, we can conclude that $\Pi_A \circ \Xi_A = \Psi_A$. $\square$

The previous lemma will allow us to transfer the left $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-module structure on $S_0(\mathbb{R}^d)$ to the Heisenberg-Weyl setting. The following lemma does the same for the right $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-module structure.

6.1.6 Lemma. *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ and let $\theta = A^T JA$. Then, there exists a $\star$-algebra isomorphism*

$$\begin{aligned}\Xi'_{A^\circ} \colon C^*(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}}) &\to C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})\\ \delta_k &\mapsto e^{\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}\delta_k.\end{aligned}$$

Proof. Because $(A^\circ)^T JA^\circ = -\theta^{-1}$ (see the proof of Proposition 5.1.15), Lemma 6.1.5 gives us a $\star$-algebra isomorphism

$$\begin{aligned}\Xi_{A^\circ} \colon C^*(\mathbb{Z}^{2d}, \gamma_{-\theta^{-1}})^{\mathrm{op}} &\to C^*(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}}\\ \delta_k &\mapsto e^{-\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}\delta_k,\end{aligned}$$

(a $\star$-algebra homomorphism remains a $\star$-algebra homomorphism if we take the opposite algebra for both its domain and its target). By Lemma 5.1.11 (with $\gamma = \gamma_{-\theta^{-1}}$ and $\gamma = \overline{\beta_{A^\circ}}$), we have $\star$-algebra isomorphisms:

$$\begin{aligned}C^*(\mathbb{Z}^{2d}, \overline{\gamma_{-\theta^{-1}}}) &\to C^*(\mathbb{Z}^{2d}, \gamma_{-\theta^{-1}})^{\mathrm{op}} & C^*(\mathbb{Z}^{2d}, \beta_{A^\circ})^{\mathrm{op}} &\to C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})\\ \delta_k &\mapsto \overline{\gamma_{-\theta^{-1}}(-k,k)}\delta_{-k} & \delta_k &\mapsto \beta_{A^\circ}(-k,k)\delta_{-k},\end{aligned}$$

where the expressions for δ_k can be deduced from Equation (5.6). Now, $\gamma_{-\theta^{-1}}(-k,k) = 0$ by antisymmetry of symplectic forms, so the first of these maps reduces to $\delta_k \mapsto \delta_{-k}$. If we now compose these three maps (Ξ_{A° goes in the middle), we obtain a $\star$-algebra isomorphism $\Xi'_{A^\circ} \colon C^*(\mathbb{Z}^{2d}, \overline{\gamma_{-\theta^{-1}}}) \to C^*(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$ such that

$$\begin{aligned}\Xi'_{A^\circ}(\delta_k) &= e^{-\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}\beta_{A^\circ}(-k,k)\delta_k\\ &= e^{-\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}e^{-2\pi i P_2(A^\circ k)\cdot P_1(-A^\circ k)}\delta_k = e^{\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}\delta_k.\end{aligned}$$

Finally, we need only note that $\overline{\gamma_{-\theta^{-1}}} = \gamma_{\theta^{-1}}$ to conclude the proof. □

We are now prepared to transfer the bimodule structure.

6.1.7 Theorem. *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ and let $\theta = A^T JA$. Then, the Feichtinger algebra $S_0(\mathbb{R}^d)$ becomes an $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-$\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-pre-equivalence bimodule when equipped with the actions and inner products defined by*

$$a\cdot f = \sum_{k\in\mathbb{Z}^{2d}} a(k)\psi_A(k)f \quad \textit{and} \quad f\cdot b = \sum_{k\in\mathbb{Z}^{2d}} b(k)\psi_{A^\circ}(k)^* f$$

$${}_A\langle f,g\rangle(k) = \langle f, \psi_A(k)g\rangle \quad \textit{and} \quad \langle f,g\rangle_{A^\circ}(k) = \frac{1}{|\det A|}\langle \psi_{A^\circ}(k)g, f\rangle$$

for all $f,g \in S_0(\mathbb{R}^d)$, $a \in \ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$, $b \in \ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$ and $k \in \mathbb{Z}^{2d}$.

Proof. As advertised, the idea is to use the $\star$-algebra isomorphisms from lemmas 6.1.5 and 6.1.6 to transfer the $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule structure afforded by Theorem 5.1.14 to an $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-$\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-pre-equivalence bimodule structure. The isomorphism Ξ_A and Ξ'_{A° clearly restrict to isometric (Proposition 2.2.6) $\star$-algebra isomorphisms:

$$\begin{aligned} \Xi_A \colon \ell^1(\mathbb{Z}^{2d}, \gamma_\theta) &\to \ell^1(\mathbb{Z}^{2d}, \beta_A) & \qquad \Xi'_{A^\circ} \colon \ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}}) &\to \ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}}) \\ \delta_k &\mapsto e^{-\pi i P_2(Ak)\cdot P_1(Ak)}\delta_k & \delta_k &\mapsto e^{\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)}\delta_k. \end{aligned}$$

Thus, we may use the method of transfer that we outlined immediately after Definition 5.1.7.

Let ${}_A\langle \cdot, \cdot \rangle'$ and $\langle \cdot, \cdot \rangle'_{A^\circ}$ be the $\ell^1(\mathbb{Z}^{2d}, \beta_A)$- and $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-valued inner products, respectively. The $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$- and $\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-valued inner products will then be defined by

$${}_A\langle f, g \rangle = \Xi_A^{-1}\big({}_A\langle f, g\rangle'\big) \quad \text{and} \quad \langle f, g\rangle_{A^\circ} = (\Xi'_{A^\circ})^{-1}\big(\langle f, g\rangle'_{A^\circ}\big).$$

Similarly, denoting the "old" actions by $\Pi_A \colon \ell^1(\mathbb{Z}^{2d}, \beta_A) \to \mathrm{End}_{\mathbb{C}}(E)$ and $\Pi'_{A^\circ} \colon \ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E)$, the new actions will be defined by

$$\begin{aligned} \Pi_A \circ \Xi_A &\colon \ell^1(\mathbb{Z}^{2d}, \gamma_\theta) \to \mathrm{End}_{\mathbb{C}}(E) \\ \text{and} \quad \Pi'_{A^\circ} \circ \Xi'_{A^\circ} &\colon \ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})^{\mathrm{op}} \to \mathrm{End}_{\mathbb{C}}(E). \end{aligned}$$

Our task is to verify that these become the actions and inner products described in the statement of the theorem.

We have already seen (Lemma 6.1.5) that $\Pi_A \circ \Xi_A = \Psi_A$, so that

$$\Pi_A \circ \Xi_A(a) = \Psi_A(a) = \Psi_A\Big(\sum_{k\in\mathbb{Z}^{2d}} a(k)\delta_k\Big) = \sum_{k\in\mathbb{Z}^{2d}} a(k)\psi_A(k)$$

for all $a \in \ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$, as claimed. As for the right action, we find that

$$\begin{aligned} \Pi'_{A^\circ} \circ \Xi'_{A^\circ}(b) &= \Pi'_{A^\circ}\Big(\sum_{k\in\mathbb{Z}^{2d}} b(k) e^{\pi i P_2(A^\circ)\cdot P_1(A^\circ)}\delta_k\Big) \\ &= \sum_{k\in\mathbb{Z}^{2d}} b(k) e^{\pi i P_2(A^\circ)\cdot P_1(A^\circ)}\pi_{A^\circ}(k)^* = \sum_{k\in\mathbb{Z}^{2d}} b(k)\psi_{A^\circ}(k)^* \end{aligned}$$

for all $b \in \ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$ (see Proposition 5.1.12 for the form of the right action of $\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$, which we are denoting by Π'_{A° here). This verifies that the actions take the stated forms.

As for the inner products, we let $f, g \in S_0(\mathbb{R}^d)$ and $k \in \mathbb{Z}^{2d}$, and find that

$$\begin{aligned}
{}_A\langle f, g\rangle(k) &= \Xi_A^{-1}\big({}_A\langle f, g\rangle'\big)(k) = \Xi_A^{-1}\Big(\sum_{l\in\mathbb{Z}^{2d}} \langle f, \pi_A(l)g\rangle \delta_l\Big)(k) \\
&= \Big(\sum_{l\in\mathbb{Z}^{2d}} \langle f, \pi_A(l)g\rangle e^{\pi i P_2(Al)\cdot P_1(Al)}\delta_l\Big)(k) \\
&= \langle f, \pi_A(k)g\rangle e^{\pi i P_2(Ak)\cdot P_1(Ak)} = \langle f, \psi_A(k)g\rangle
\end{aligned}$$

and that

$$\begin{aligned}
\langle f, g\rangle_{A^\circ}(k) &= (\Xi'_{A^\circ})^{-1}\Big(\frac{1}{|\det A|}\sum_{l\in\mathbb{Z}^{2d}} \langle \pi_{A^\circ}(l)g, f\rangle \delta_l\Big)(k) \\
&= \Big(\frac{1}{|\det A|}\sum_{l\in\mathbb{Z}^{2d}} \langle \pi_{A^\circ}(l)g, f\rangle e^{-\pi i P_2(A^\circ l)\cdot P_1(A^\circ l)}\delta_l\Big)(k) \\
&= \frac{1}{|\det A|}\langle \pi_{A^\circ}(k)g, f\rangle e^{-\pi i P_2(A^\circ k)\cdot P_1(A^\circ k)} = \frac{1}{|\det A|}\langle \psi_{A^\circ}(k)g, f\rangle,
\end{aligned}$$

which concludes the proof. □

For any $A \in \mathrm{GL}(2d, \mathbb{R})$, recall that ${}_A\mathcal{E}_{A^\circ}$ denotes the completion of $S_0(\mathbb{R}^d)$ as an $\ell^1(\mathbb{Z}^{2d}, \beta_A)$-$\ell^1(\mathbb{Z}^{2d}, \overline{\beta_{A^\circ}})$-pre-equivalence bimodule (Theorem 5.1.14 and the subsequent discussion). We now do the same with our new pre-equivalence bimodules. The completions are afforded by Proposition 5.1.10.

6.1.8 Definition. Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and let $\theta = A^T JA$. We will write ${}_A\mathscr{E}_{A^\circ}$ to denote the equivalence bimodule completion of $S_0(\mathbb{R}^d)$ equipped with the $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-$\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-pre-equivalence bimodule structure of Theorem 6.1.7. We will view ${}_A\mathscr{E}_{A^\circ}$ as a $C^*(\mathbb{Z}^{2d}, \gamma_\theta)$-$C^*(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-equivalence bimodule.

6.1.3 Metaplectic Transformations for Gabor Analysis

We are now finally in a position to make good on our promises from Subsection 3.2.2. Recall from Proposition 3.2.9 that two lattice matrices $A, B \in \mathrm{GL}(2d, \mathbb{R})$ determine the same symplectic form $A^*\Omega_J = B^*\Omega_J$, represented by $\theta := A^T JA = B^T JB$, if and only if

$$B \in \mathrm{Sp}(2d, \mathbb{R})A = A\mathrm{Sp}_\theta(2d, \mathbb{R}), \quad \text{or, equivalently: } BA^{-1} \in \mathrm{Sp}(2d, \mathbb{R}).$$

There are two central results which we wish to prove. Both of these results will follow solely from the assumption that $A \in \mathrm{GL}(2d, \mathbb{R})$ and that $B \in \mathrm{Sp}(2d, \mathbb{R})A$. This is *precisely* the statement that A and B determine the same symplectic form, it is just a convenient way of formulating our results, as it becomes immediately apparent that $BA^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$, which will play an important role. The first result is a very explicit confirmation that the structure of Gabor frames over $A\mathbb{Z}^{2d}$ depends only on the symplectic form determined by A. The second result lifts this correspondence between "symplectically related" lattices to the setting of equivalence bimodules by providing isomorphisms between these modules.

First of all, we make a simple but important observation: symplectic transformations are compatible with duality. Despite its simplicity, we have not seen this observation elsewhere.

6.1.9 Lemma. *For any $A \in \mathrm{GL}(2d, \mathbb{R})$ and $S \in \mathrm{Sp}(2d, \mathbb{R})$, we have that $(SA)^\circ = SA^\circ$.*

Proof. Since $S \in \mathrm{Sp}(2d, \mathbb{R})$, we know that $S^T \in \mathrm{Sp}(2d, \mathbb{R})$ by Lemma 1.2.5. This means that $SJS^T = J$, and hence $SJ = JS^{-T}$. Thus,

$$(SA)^\circ = -J(SA)^{-T} = -JS^{-T}A^{-T} = -SJA^{-T} = SA^\circ,$$

as claimed. □

We are now fully prepared to prove the first of our central results. Except for the statement about duality, the essence of this result is contained in de Gosson's book [15, Proposition 163 on p. 113]. There is an unfortunate congregation of S'es in this theorem, seeing as we are dealing with both frame operators and symplectic/metaplectic transformations. The sub- and superscripts that go along with frame operators hopefully make matters clear.

6.1.10 Theorem (Metaplectic transformations for Gabor systems)**.** *Let $A \in \mathrm{GL}(2d, \mathbb{R})$, suppose that $B \in \mathrm{Sp}(2d, \mathbb{R})A$ and let $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ be a metaplectic operator associated to $S = BA^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$. Then, for any $g, h \in L^2(\mathbb{R}^d)$, the following statements are true.*

(i) $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Bessel sequence if and only if $\mathcal{G}(\widehat{S}g, B\mathbb{Z}^{2d})$ is a Bessel sequence. Moreover, their optimal Bessel bounds are equal.

(ii) $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Gabor frame if and only if $\mathcal{G}(\widehat{S}g, B\mathbb{Z}^{2d})$ is a Gabor frame. Moreover, their optimal frame bounds are equal.

(iii) If $\mathcal{G}(g, A\mathbb{Z}^{2d})$ and $\mathcal{G}(h, A\mathbb{Z}^{2d})$ are Bessel sequences, then

$$S^B_{\widehat{S}g, \widehat{S}h} = \widehat{S}(S^A_{g,h})\widehat{S}^{-1}.$$

Moreover, all of these statements are true with A° and B° in place of A and B, with the same metaplectic operator $\widehat{S}$.

Proof. We begin with a couple of observation. First of all, by Proposition 6.1.4, we have that

$$\psi_B(k)\widehat{S} = \psi_{SA}(k)\widehat{S} = \psi(SAk)\widehat{S} = \widehat{S}\psi(Ak) = \widehat{S}\psi_A(k) \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

Because a time-frequency shift differs from the corresponding Heisenberg-Weyl operator only by a phase factor, we find that, for all $f, g \in L^2(\mathbb{R}^d)$,

$$|\langle f, \pi_B(k)\widehat{S}g\rangle| = |\langle f, \widehat{S}\pi_A(k)g\rangle| = |\langle \widehat{S}^{-1}f, \pi_A(k)g\rangle| \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

If we now assume that $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Bessel sequence, with Bessel bound $C > 0$, we find that

$$\sum_{k\in\mathbb{Z}^{2d}} |\langle f, \pi_B(k)\widehat{S}g\rangle|^2 = \sum_{k\in\mathbb{Z}^{2d}} |\langle \widehat{S}^{-1}f, \pi_A(k)g\rangle| \leq C\|\widehat{S}^{-1}f\|^2 = C\|f\|^2,$$

for all $f \in L^2(\mathbb{R}^d)$, since $\widehat{S}$ is unitary. This shows that $\mathcal{G}(\widehat{S}g, B\mathbb{Z}^{2d})$ is a Bessel sequence. Moreover, it shows that any Bessel bound for $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Bessel bound for $\mathcal{G}(\widehat{S}g, B\mathbb{Z}^{2d})$. A very similarly argument shows that if $\mathcal{G}(g, A\mathbb{Z}^{2d})$ is a Gabor frame, then $\mathcal{G}(\widehat{S}g, B\mathbb{Z}^{2d})$ is a Gabor frame with the same frame bounds. Finally, since $AB^{-1} = (BA^{-1})^{-1} = S^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$, there is a metaplectic operator associated to AB^{-1} (indeed, we can choose $\widehat{S}^{-1}$), so the situation is symmetric with respect to A and B. This proves (i) and (ii).

We now turn to (iii). We first note that, for all $f, g, h \in L^2(\mathbb{R}^d)$:

$$\langle f, \pi(z)g\rangle\pi(z)h = \langle f, \psi(z)g\rangle\psi(z)h \quad \text{for all } z \in \mathbb{R}^{2d},$$

because of cancellation of phase factors. This means that mixed-type frame operators are unchanged if we replace all time-frequency shifts by Heisenberg-Weyl operators. Thus, assuming that $\mathcal{G}(g, A\mathbb{Z}^{2d})$ and $\mathcal{G}(h, A\mathbb{Z}^{2d})$ are Bessel sequences, so that the frame operators are well-defined and bounded, we find that

$$\begin{aligned} S^B_{\widehat{S}g,\widehat{S}h}(\widehat{S}f) &= \sum_{k\in\mathbb{Z}^{2d}} \big\langle \widehat{S}f, \psi_B(k)\widehat{S}g\big\rangle\psi_B(k)\widehat{S}h \\ &= \sum_{k\in\mathbb{Z}^{2d}} \big\langle \widehat{S}f, \widehat{S}\psi_A(k)g\big\rangle\widehat{S}\psi_A(k)h \\ &= \sum_{k\in\mathbb{Z}^{2d}} \big\langle f, \psi_A(k)g\big\rangle\widehat{S}\psi_A(k)h = \widehat{S}(S^A_{g,h}f) \end{aligned}$$

for all $f \in L^2(\mathbb{R}^d)$, which proves (iii).

Finally, for the remark regarding A° and B°, we use Lemma 6.1.9 to see that $B^\circ = (SA)^\circ = SA^\circ \in \mathrm{Sp}(2d, \mathbb{R})A^\circ$ and that

$$B^\circ (A^\circ)^{-1} = (SA)^\circ (A^\circ)^{-1} = SA^\circ (A^\circ)^{-1} = S,$$

so that A° and B° satisfy the conditions of the theorem with the exact same choice for $\widehat{S}$. □

We now turn to the task of constructing isomorphisms between our equivalence bimodules. They will be constructed from metaplectic transformations. In order to apply metaplectic transformations to our pre-equivalence bimodules, we must first show that the Feichtinger algebra is invariant under such transformations.

6.1.11 Lemma. *Let $S \in \mathrm{Sp}(2d, \mathbb{R})$ and let $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ be a metaplectic operator associated to S. Then, $\widehat{S}(S_0(\mathbb{R}^d)) = S_0(\mathbb{R}^d)$, so that $\widehat{S}$ is a linear bijection of $S_0(\mathbb{R}^d)$.*

Proof. Our proof of this fact is based on Gröchenig [17, Proposition 12.1.3 on p. 247].

Let $f \in S_0(\mathbb{R}^d)$. By Lemma 3.2.13 on the equivalence of norms for $S_0(\mathbb{R}^d)$, we have that $V_f f \in L^1(\mathbb{R}^{2d})$. We will use this lemma multiple times without mention in this proof. Since $\psi(z)$ is related to $\pi(z)$ by a phase factor, we have that

$$|V_{\widehat{S}f} \widehat{S}f(Sz)| = |\langle \widehat{S}f, \pi(Sz)\widehat{S}f\rangle| = |\langle \widehat{S}f, \widehat{S}\pi(z)f\rangle| = |\langle f, \pi(z)f\rangle| = |V_f f(z)|$$

for all $z \in \mathbb{R}^{2d}$. If we now integrate over z, we find that $V_{\widehat{S}f}\widehat{S}f \in L^1(\mathbb{R}^{2d})$.

By the density of $S_0(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ (Lemma 3.2.14), we can certainly choose some $g \in S_0(\mathbb{R}^d)$ such that $\langle g, \hat{S}f\rangle \neq 0$. By Lemma 3.2.22 (with $f_1 = \widehat{S}f$, $f_2 = f$, $g_1 = \widehat{S}f$ and $g_2 = g$), we have that

$$|V_f \widehat{S}f(z)| \leq \frac{1}{|\langle \widehat{S}f, g\rangle|} \big(|V_{\widehat{S}f}\widehat{S}f| * |V_f g|\big)(z), \quad \text{for all } z \in \mathbb{R}^{2d}.$$

The function on the right is a convolution of two functions which we know to be in $L^1(\mathbb{R}^{2d})$, so it is in $L^1(\mathbb{R}^{2d})$ as well. Thus, $V_f\widehat{S}f \in L^1(\mathbb{R}^{2d})$, which means that $\widehat{S}f \in S_0(\mathbb{R}^d)$.

This proves that $\widehat{S}(S_0(\mathbb{R}^d)) \subset S_0(\mathbb{R}^d)$. Since $\mathrm{Mp}(2d, \mathbb{R})$ is a group, we have $\widehat{S}^{-1} \in \mathrm{Mp}(2d, \mathbb{R})$. Since

$$\psi(z)\widehat{S}^{-1} = \widehat{S}^{-1}\psi(Sz) \quad \text{for all } z \in \mathbb{R}^{2d},$$

we find that

$$\psi(S^{-1}z)\widehat{S}^{-1} = \widehat{S}^{-1}\psi(SS^{-1}z) = \widehat{S}^{-1}\psi(z) \quad \text{for all } z \in \mathbb{R}^{2d}.$$

This shows that $\widehat{S}^{-1}$ is a metaplectic operator associated to $S^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$. By what we just proved, applied to S^{-1} instead of S, we can conclude that $\widehat{S}^{-1}(S_0(\mathbb{R}^d)) \subset S_0(\mathbb{R}^d)$. This implies that $\widehat{S}(S_0(\mathbb{R}^d)) = S_0(\mathbb{R}^d)$, which concludes the proof. □

We are now finally ready to prove the result we have been building towards for most of this thesis. For finite subgroups of $\mathrm{Sp}(2d, \mathbb{R})$, Chakraborty and Luef [8] have proved this result with regard to the left module structure. The result regarding the right module structure is, to our knowledge, entirely new. The crucial observation is Lemma 6.1.9. Without this lemma, it is not even clear that $B^\circ = (SA)^\circ$ and A° will determine the same symplectic form and hence that ${}_A\mathscr{E}_{A^\circ}$ and ${}_B\mathscr{E}_{B^\circ}$ are equivalence bimodules over the same C*-algebras. For the notion of a *biunitary* map (an isomorphism of equivalence bimodules), see Definition 5.1.9.

6.1.12 Theorem (Metaplectic transformations for equivalence bimodules)**.** *Let $A \in \mathrm{GL}(2d, \mathbb{R})$ and set $\theta = A^T JA$. Suppose that $B \in \mathrm{Sp}(2d, \mathbb{R})A$ and let $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ be a metaplectic operator associated to $S = BA^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$. Then, $\widehat{S} \colon S_0(\mathbb{R}^d) \to S_0(\mathbb{R}^d)$ extends to a biunitary map $\mathscr{S} \colon {}_A\mathscr{E}_{A^\circ} \to {}_B\mathscr{E}_{B^\circ}$. In other words,*

$$\mathscr{S} \in \mathcal{L}_{C^*(\mathbb{Z}^{2d},\gamma_\theta)}\big({}_A\mathscr{E}_{A^\circ}, {}_B\mathscr{E}_{B^\circ}\big) \cap \mathcal{L}_{C^*(\mathbb{Z}^{2d},\gamma_{\theta^{-1}})}\big({}_A\mathscr{E}_{A^\circ}, {}_B\mathscr{E}_{B^\circ}\big)$$

is an invertible map whose inverse equals its adjoint in both of these spaces.

Proof. By Lemma 6.1.11, the restriction and corestriction $\widehat{S} \colon S_0(\mathbb{R}^d) \to S_0(\mathbb{R}^d)$ is well-defined. We wish to obtain an extension $\mathscr{S} \colon {}_A\mathscr{E}_{A^\circ} \to {}_B\mathscr{E}_{B^\circ}$, so we should of course equip the domain of $\widehat{S}$ with the $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-$\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-bimodule structure determined by A and its target with the $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-$\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-bimodule structure determined by B (via Theorem 6.1.7; note that $\theta = A^T JA = B^T JB$ and $-\theta^{-1} = (A^\circ)^T JA^\circ = (B^\circ)^T JB^\circ$).

We begin by verifying that $\widehat{S}$ is $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-linear and that it preserves the $\ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$-valued inner product. By Proposition 6.1.4, we have that

$$\psi_B(k)\widehat{S} = \psi_{SA}(k)\widehat{S} = \psi(SAk)\widehat{S} = \widehat{S}\psi(Ak) = \widehat{S}\psi_A(k) \quad \text{for all } k \in \mathbb{Z}^{2d}.$$

With $f, g \in S_0(\mathbb{R}^d)$ and $a \in \ell^1(\mathbb{Z}^{2d}, \gamma_\theta)$, we find that

$$\begin{aligned}\widehat{S}(a \cdot f) &= \widehat{S}\Big(\sum_{k \in \mathbb{Z}^{2d}} a(k)\psi_A(k)f\Big) = \sum_{k \in \mathbb{Z}^{2d}} a(k)\widehat{S}\psi_A(k)f \\ &= \sum_{k \in \mathbb{Z}^{2d}} a(k)\psi_B(k)\widehat{S}f = a \cdot (\widehat{S}f)\end{aligned}$$

and that

$$\begin{aligned}{}_B\langle \widehat{S}f, \widehat{S}g\rangle(k) &= \langle \widehat{S}f, \psi_B(k)\widehat{S}g\rangle = \langle \widehat{S}f, \widehat{S}\psi_A(k)g\rangle \\ &= \langle f, \psi_A(k)g\rangle = {}_A\langle f, g\rangle(k)\end{aligned}$$

for all $k \in \mathbb{Z}^{2d}$. This verifies that the left inner product module structure is preserved by $\widehat{S}$.

We now verify that $\widehat{S}$ is $\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-linear and that it preserves the $\ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-valued inner product. We first note that $|\det S| = 1$, so that $|\det A| = |\det B|$.[2] This follows from the calculation:

$$1 = |\det J| = |\det(S^T JS)| = |\det S^T||\det S| = |\det S|^2.$$

Recall from Lemma 6.1.9 that $B^\circ = (SA)^\circ = SA^\circ$, so that Proposition 6.1.4 gives

$$\psi_{B^\circ}(k)\widehat{S} = \psi(SA^\circ k)\widehat{S} = \widehat{S}\psi(A^\circ k) = \widehat{S}\psi_{A^\circ}(k) \quad \text{for all } k \in \mathbb{Z}^{2d}. \tag{6.5}$$

Thus, for $f, g \in S_0(\mathbb{R}^d)$, we find that

$$\begin{aligned}\langle \widehat{S}f, \widehat{S}g\rangle_{B^\circ}(k) &= \frac{1}{|\det A|}\langle \psi_{B^\circ}(k)\widehat{S}g, \widehat{S}f\rangle = \frac{1}{|\det A|}\langle \widehat{S}\psi_{A^\circ}(k)g, \widehat{S}f\rangle \\ &= \frac{1}{|\det A|}\langle \psi_{A^\circ}(k)g, f\rangle = \langle f, g\rangle_{A^\circ}(k)\end{aligned}$$

for all $k \in \mathbb{Z}^{2d}$. Now, Equation (6.5) implies that

$$\widehat{S}^{-1}\psi_{B^\circ}(k)^* = \big(\psi_{B^\circ}(k)\widehat{S}\big)^* = \big(\widehat{S}\psi_{A^\circ}(k)\big)^* = \psi_{A^\circ}(k)^*\widehat{S}^{-1},$$

and hence that $\psi_{B^\circ}(k)^*\widehat{S} = \widehat{S}\psi_{A^\circ}(k)^*$, for all $k \in \mathbb{Z}^{2d}$. Thus, with $f \in S_0(\mathbb{R}^d)$ and $b \in \ell^1(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$, we now find that

$$\begin{aligned}\widehat{S}(f \cdot b) &= \widehat{S}\Big(\sum_{k\in\mathbb{Z}^{2d}} b(k)\psi_{A^\circ}(k)^* f\Big) = \sum_{k\in\mathbb{Z}^{2d}} b(k)\widehat{S}\psi_{A^\circ}(k)^* f \\ &= \sum_{k\in\mathbb{Z}^{2d}} b(k)\psi_{B^\circ}(k)^*\widehat{S}f = (\widehat{S}f)\cdot b.\end{aligned}$$

This verifies that the right inner product module structure is preserved by $\widehat{S}$ as well.

By Lemma 6.1.11, $\widehat{S}$ is bijective on $S_0(\mathbb{R}^d)$. By Proposition B.3.2, the unique bounded linear extension of $\widehat{S}$, which we will denote by $\mathscr{S} : {}_A\mathscr{E}_{A^\circ} \to$

[2] In fact, we always have $\det S = 1$, so that $\mathrm{Sp}(2d, \mathbb{R}) \subset \mathrm{SL}(2d, \mathbb{R})$, but we will not have need for this fact – which is trickier to prove.

${}_B\mathscr{E}_{B^\circ}$, is unitary as a map between left $C^*(\mathbb{Z}^{2d}, \gamma_\theta)$-modules. Using the left-right correspondence of Lemma 5.1.2, or just adapting the proof of Proposition B.3.2 to the right module setting, we find that $\mathscr{S}$ is unitary as a map between $C^*(\mathbb{Z}^{2d}, \gamma_{\theta^{-1}})$-modules as well, which concludes the proof. □

We can summarize our result as follows:

6.1.13 Corollary. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$. Then, whenever $B \in \mathrm{Sp}(2d, \mathbb{R})A$, we have that*

$$ {}_A\mathscr{E}_{A^\circ} \cong {}_B\mathscr{E}_{B^\circ} \quad \textit{as equivalence bimodules,} $$

where the isomorphism is given by the bounded linear extension of a metaplectic operator associated to $BA^{-1} = B^\circ(A^\circ)^{-1} \in \mathrm{Sp}(2d, \mathbb{R})$.

Proof. Immediate from Theorem 6.1.12. □

Using Theorem 6.1.12, we can now easily prove a result similar to Theorem 6.1.10 in the context of these bimodules. Given $A \in \mathrm{GL}(2d, \mathbb{R})$, let's denote the two rank-one operators associated to $g, h \in {}_A\mathscr{E}_{A^\circ}$ by

$$ {}^AK_{g,h} : f \mapsto {}_A\langle f, g \rangle \cdot h \quad \text{and} \quad K^{A^\circ}_{g,h} : f \mapsto g \cdot \langle h, f \rangle_{A^\circ}. $$

Note that we are now working in the completion ${}_A\mathscr{E}_{A^\circ}$ of $S_0(\mathbb{R}^d)$ (without modifying our notation). When $g, h \in S_0(\mathbb{R}^d)$, we have seen that ${}^AK_{g,h}$ corresponds to the ordinary frame operator $S^A_{g,h}$:

$$ {}^AK_{g,h}f = {}_A\langle f, g \rangle \cdot h = \sum_{k \in \mathbb{Z}^{2d}} \langle f, \psi_A(k)g \rangle \psi_A(k)h = S^A_{g,h}f \quad \text{for all } f \in S_0(\mathbb{R}^d). $$

We repeat that the frame operators are unchanged if we replace time-frequency shifts with Heisenberg-Weyl operators (because of cancellations of phase factors). Similarly, we find that

$$ \begin{aligned} K^{A^\circ}_{g,h}f = g \cdot \langle h, f \rangle_{A^\circ} &= \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}^{2d}} \langle \psi_{A^\circ}(k) f, h \rangle \psi_{A^\circ}(k)^* g \\ &= \frac{1}{|\det A|} \sum_{k \in \mathbb{Z}^{2d}} \langle f, \psi_{A^\circ}(k)^* h \rangle \psi_{A^\circ}(k)^* g = \frac{1}{|\det A|} S^{A^\circ}_{h,g} f \end{aligned} $$

for all $f \in S_0(\mathbb{R}^d)$, so that $K^{A^\circ}_{g,h}$ corresponds to the frame operator $S^{A^\circ}_{h,g}$ (up to a constant).

With this correspondence between rank-one operators and frame operators in mind, the following corollary recovers much of Theorem 6.1.10, for windows in $S_0(\mathbb{R}^d)$ (or its completion, if one takes into account the embedding ${}_A\mathcal{E}_{A^\circ} \to L^2(\mathbb{R}^d)$ discussed towards the end of Subsection 4.3.3, which seems simple to recast in terms of ${}_A\mathscr{E}_{A^\circ}$, based on the method of proof in Luef and Austad [4] and Austad and Enstad [3]). The main takeaway of this corollary is how simple the proof becomes with the machinery of Theorem 6.1.12.

6.1.14 Corollary. *Let $A \in \mathrm{GL}(2d, \mathbb{R})$, suppose that $B \in \mathrm{Sp}(2d, \mathbb{R})A$ and let $\mathscr{S} : {}_A\mathscr{E}_{A^\circ} \to {}_B\mathscr{E}_{B^\circ}$ be one of the isomorphisms afforded by Corollary 6.1.13. Then, for any $g, h \in {}_A\mathscr{E}_{A^\circ}$, we have that*

$$\mathscr{S}({}^A K_{g,h})\mathscr{S}^{-1} = {}^B K_{\mathscr{S}g,\mathscr{S}h} \quad \textit{and} \quad \mathscr{S}(K^{A^\circ}_{g,h})\mathscr{S}^{-1} = K^{B^\circ}_{\mathscr{S}g,\mathscr{S}h}.$$

Proof. For any $f \in {}_A\mathscr{E}_{A^\circ}$, we find that

$$\big(\mathscr{S}({}^A K_{g,h})\big)(f) = \mathscr{S}\big({}_A\langle f, g\rangle \cdot h\big) = {}_B\langle \mathscr{S}f, \mathscr{S}g\rangle \cdot \mathscr{S}h = ({}^B K_{\mathscr{S}g,\mathscr{S}h}\mathscr{S})(f)$$

and that

$$\big(\mathscr{S}K^{A^\circ}_{g,h}\big)(f) = \mathscr{S}\big(g \cdot \langle h, f\rangle_{A^\circ}\big) = \mathscr{S}g \cdot \langle \mathscr{S}h, \mathscr{S}f\rangle_{B^\circ} = (K^{B^\circ}_{\mathscr{S}g,\mathscr{S}h}\mathscr{S})(f),$$

which is what the corollary claims. □

Finally, before we bow out, we note that Theorem 6.1.12 likely provides a concrete example of a general framework developed by Combes [10]. Further insights may be gained by exploring this connection.

Appendix A

Functional Analysis

This appendix contains two short sections. The first section is devoted to the Stone-Weierstrass theorem, which plays an important role in the construction and interpretation of the continuous functional calculus in Subsection 2.2.2. The second section is a hodgepodge of foundational results from functional analysis. These are appealed to at various points in the thesis, but mostly in Chapter 2.

A.1 | The Stone-Weierstrass Theorem

For any topological space X, we write $C(X)$ to denote the algebra of all continuous functions on X (see Example 2.1.7). If X is compact, then $C(X)$ is a C*-algebra with respect to the supremum norm (see Example 2.1.15) and with the involution given by pointwise complex conjugation. The supremum norm determines the topology of uniform convergence on X.

A.1.1 Theorem (Stone-Weierstrass). *Let X be a compact topological space and let A be a $\star$-subalgebra of the C*-algebra $C(X)$ with the following properties:*

- *A separates points of X: if $x, y \in X$ are two distinct points, then there is some $f \in A$ such that $f(x) \neq f(y)$.*
- *A vanishes at no point of X: there is no $x \in X$ such that $f(x) = 0$ for all $f \in A$.*

If this is the case, then A is dense in $C(X)$.

Proof. See Rudin [28, Theorem 7.33 on p. 165]. □

We will now apply the Stone-Weierstrass theorem to compact subsets of $\mathbb{C}$. Let z denote the identity $z\colon \mathbb{C} \to \mathbb{C}$ and let $\overline{z}$ denote the complex-conjugation map $\overline{z} : \lambda \mapsto \overline{\lambda}$. Clearly, z and $\overline{z}$ are in the algebra $C(\mathbb{C})$. We will refer to member of the set

$$P(z,\overline{z}) \coloneqq \operatorname{span}_{\mathbb{C}}\big\{z^n\overline{z}^m : (n,m) \in \mathbb{N}_0 \times \mathbb{N}_0\big\} \subset C(\mathbb{C})$$

as *polynomials in z and $\overline{z}$.*

A.1.2 Corollary. *Let $K \subset \mathbb{C}$ be a compact set and let $f \in C(K)$. Then, there exists a sequence (p_n) of polynomials in z and $\overline{z}$ such that $p_n \to f$ uniformly on K.*

Proof. The set $P(z,\overline{z})$ (viewed as a set of functions on K, by restriction) is a $\star$-subalgebra of $C(K)$ satisfying the assumptions of the Stone-Weierstrass theorem. Thus, $P(z,\overline{z})$ is dense in $C(K)$, which is exactly what the corollary claims. □

A.2 | Results from Functional Analysis

In this section we state four foundational results from functional analysis and derive the consequences we need. We have by no means provided their most general forms. In this subsection, as in Chapter 2, we assume that all vector spaces are over $\mathbb{C}$. If V is a normed space, then V^* denotes the space of bounded linear functionals on V, i.e. its continuous dual space.

We begin with the Hahn-Banach theorem, which lets us extend bounded linear functionals defined on subspaces of normed spaces.

A.2.1 Theorem (The Hahn-Banach theorem)**.** *Let V be a normed vector space. If $W \subset V$ is a vector-subspace and $\tau\colon W \to \mathbb{C}$ is a bounded linear functional, then there exists a bounded linear functional $\widetilde{\tau}\colon V \to \mathbb{C}$ such that $\widetilde{\tau}|_W = \tau$ and $\|\widetilde{\tau}\| = \|\tau\|$.*

Proof. See Bowers and Kalton [7, Subsection 3.2]. □

The following corollary states that the dual space of a normed space separates points.

A.2.2 Corollary. *Let V be a normed vector space and suppose that $v_1, v_2 \in V$ are such that $\tau(v_1) = \tau(v_2)$ for all $\tau \in V^*$. Then, $v_1 = v_2$.*

Proof. Consider the subspace $\mathbb{C}(v_2 - v_1) \subset V$. We can define a linear functional $\tau\colon \mathbb{C}(v_2 - v_1) \to \mathbb{C}$ by $\tau(\lambda(v_2 - v_1)) = \lambda\|v_2 - v_1\|$ for all $\lambda \in \mathbb{C}$. The functional τ is bounded since

$$|\tau(\lambda(v_2 - v_1))| = |\lambda|\|v_2 - v_1\| = \|\lambda(v_2 - v_1)\| \quad \text{for all } \lambda \in \mathbb{C}.$$

By the Hahn-Banach theorem, we can now find some $\widetilde{\tau} \in V^*$ such that $\widetilde{\tau}(v_2 - v_1) = \tau(v_2 - v_1)$. By assumption, we know that $\widetilde{\tau}(v_2) = \widetilde{\tau}(v_1)$, so then $\|v_2 - v_1\| = \widetilde{\tau}(v_2 - v_1) = 0$. □

Next up is the Banach-Alaoglu theorem, which we need to prove that the spectrum of a Banach algebra is compact.

A.2.3 Theorem (The Banach-Alaoglu theorem)**.** *Let V be a Banach space. Then, the closed unit ball $\{\tau \in V^* : \|\tau\| \leq 1\} \subset V^*$ is compact in the weak* topology (we say that it is weak*-compact).*

Proof. See Bowers and Kalton [7, Theorem 5.39 on p. 105]. □

We now turn to the principle of uniform boundedness.

A.2.4 Theorem (The uniform boundedness principle)**.** *Let V and W be Banach spaces and suppose that $\{T_i\}_{i\in I} \subset \mathcal{B}(V, W)$ is a collection of bounded linear maps from V to W (where I is an arbitrary index set). If the collection is pointwise bounded, in the sense that for every $v \in V$ we have that*

$$\sup\big\{\|T_i(v)\| : i \in I\big\} < \infty,$$

then it is bounded in operator norm: $\sup\{\|T_i\| : i \in I\} < \infty$.

Proof. See Bowers and Kalton [7, Theorem 4.10 on p. 65]. □

We will need the following corollary.

A.2.5 Corollary (Weakly bounded implies bounded in norm)**.** *Let V be a Banach space and let $S \subset V$ be any subset. If S is weakly bounded, in the sense that for every $\tau \in V^*$ we have that*

$$\sup\big\{|\tau(s)| : s \in S\big\} < \infty,$$

then S is bounded in norm: $\sup\{\|s\| : s \in S\} < \infty$.

Proof. For each $s \in S$, let $\mathrm{ev}_s \in (V^*)^*$ denote the evaluation functional $\tau \mapsto \mathrm{ev}_s(\tau) = \tau(s)$. We quickly argue that $\|\mathrm{ev}_s\| = \|s\|$ for each $s \in S$: the inequality $\|\mathrm{ev}_s\| \leq \|s\|$ follows from the fact that $|\mathrm{ev}_s(\tau)| = |\tau(s)| \leq \|\tau\|\|s\|$, while the Hahn-Banach theorem affords us with functionals $\tau_s \in V^*$ such that $\|\tau_s\| = \|s\|$ and $\mathrm{ev}_s(\tau_s) = \tau_s(s) = \|s\|^2$ (extend linear functionals $\mathbb{C}s \to \mathbb{C}$ defined by $\lambda s \mapsto \lambda\|s\|^2$).

Our assumption becomes:

$$\sup\left\{|\mathrm{ev}_s(\tau)| : s \in S\right\} = \sup\left\{|\tau(s)| : s \in S\right\} < \infty \quad \text{for all } \tau \in V^*.$$

By the uniform boundedness principle, applied to the Banach space V^* and the collection $\{\mathrm{ev}_s\}_{s\in S} \subset (V^*)^* = \mathcal{B}(V^*, \mathbb{C})$, we find that

$$\sup\left\{\|s\| : s \in S\right\} = \sup\left\{\|\mathrm{ev}_s\| : s \in S\right\} < \infty,$$

which concludes the proof. □

The last result we will need is the closed graph theorem.

A.2.6 Theorem (The closed graph theorem)**.** *Let V and W be Banach spaces and let $T\colon V \to W$ be any linear map. Suppose that, whenever we have a sequence $(v_n) \subset V$ such that both (v_n) and (Tv_n) converge (in V and W, respectively), it follows that $\lim_{n\to\infty} Tv_n = T(\lim_{n\to\infty} v_n)$. If this is the case, then T is bounded.*

Proof. See Bowers and Kalton [7, Theorem 4.35 on p. 76]. □

Appendix B

Completions and Extensions

The contents of this appendix, completions and extensions, are often relegated to exercises (our treatment might reveal why – these are simple but tedious arguments). Since completions play such a central role in our constructions, we have chosen to included a detailed treatment.

In the first section, we construct Banach space completions and extensions of bounded linear maps and prove that they are unique. In the second section, we show how to furnish Banach algebra, Banach $\star$-algebra and C*-algebra completions from ordinary Banach space completions. Finally, in the third and last subsection, we show how to complete inner product modules. Completions of pre-equivalence bimodules are so intertwined with the surrounding theory that we have chosen to include this topic in the main text (Proposition 5.1.10).

It is quite difficult to prove all of these results without either resorting to extremely repetitive calculations with approximating sequences or omitting most of the required calculations. With the goal of providing fairly exhaustive proofs, we have attempted to trade approximating sequences (where possible) for arguments by uniqueness of bounded extensions. The hope is that this will make the proofs slightly more interesting.

B.1 | Banach Space Completions and Bounded Linear Extensions

This section covers Banach space completions as well as extensions of bounded linear (and bilinear) maps. As it is the only case of interest to us, we will assume that all of our normed spaces are nonzero and over the complex numbers. We will frequently use the letter i to denote an inclusion; we will never have need for the imaginary unit in this appendix.

Given a normed space V, a *Banach space completion* of V is a Banach space $\overline{V}$ along with a linear isometry $i\colon V \to \overline{V}$ such that $i(V)$ is dense in $\overline{V}$. We may refer to either $\overline{V}$ or the map i by itself as a Banach space completion of V. The following theorem shows that Banach space completions exist. We will soon see that they are unique up to unique isometric linear isomorphisms (Corollary B.1.3).

B.1.1 Theorem (Banach space completions)**.** *Let V be a normed space. Then, there exists a Banach space $\overline{V}$ and a linear isometry $i\colon V \to \overline{V}$ such that $i(V)$ is dense in $\overline{V}$. In other words: such that $i\colon V \to \overline{V}$ is a Banach space completion of V.*

Proof. Let $\|\cdot\|_0$ denote the norm on V and consider the vector space $V^{\mathbb{N}}$ of all sequences in V with addition and scalar multiplication defined pointwise.

Let $C \subset V^{\mathbb{N}}$ denote the set of all Cauchy sequences in V. If $(v_n), (w_n) \in C$ and $\lambda \in \mathbb{C}$, then

$$\|(\lambda v_m + w_m) - (\lambda v_n + w_n)\|_0 \leq |\lambda|\|v_m - v_n\|_0 + \|w_m - w_n\|_0$$

shows that $\lambda(v_n) + (w_n) = (\lambda v_n + w_n) \in C$ as well. This shows that C is a vector-subspace of $V^{\mathbb{N}}$. A similar argument shows that

$$C_0 \coloneqq \big\{(v_n) \in C : \lim_{n\to\infty} \|v_n\|_0 = 0\big\}$$

is a vector-subspace of C. The quotient $\overline{V} \coloneqq C/C_0$ with the norm defined by

$$\|(v_n) + C_0\| \coloneqq \lim_{n\to\infty} \|v_n\|_0 \quad \text{for } (v_n) \in C$$

will be our Banach space completion of V. We first show that this is well-defined and indeed a norm.

The norm of a Cauchy sequence converges by the reverse triangle inequality, $|\|v\|_0 - \|w\|_0| \leq \|v - w\|_0$, so $\lim_{n\to\infty} \|v_n\|_0$ exists for all $(v_n) \in C$. It is also independent of representative: if $(v_n') + C_0 = (v_n) + C_0$, this means that $(v_n' - v_n) \in C_0$, so that

$$\begin{aligned}\lim_{n\to\infty} \|v_n'\|_0 &= \lim_{n\to\infty} \|v_n' - v_n + v_n\|_0 \\ &\leq \lim_{n\to\infty} \|v_n' - v_n\|_0 + \lim_{n\to\infty} \|v_n\|_0 = \lim_{n\to\infty} \|v_n\|_0,\end{aligned}$$

and the same argument with the roles of (v_n) and (v_n') interchanged gives the opposite inequality. Positivity, homogeneity and the triangle inequality are simple verifications; we omit the details. Nondegeneracy follows by our definition of C_0.

We now define the isometry $i\colon V \to \overline{V}$. For $v \in V$, we let $i(v)$ be the equivalence class of the constant sequence at v, i.e. $i(v) = (v)_{n\in\mathbb{N}} + C_0$. Constant sequences are certainly Cauchy, so i is well-defined. It is clearly linear, and $\|i(v)\| = \lim_{n\to\infty} \|v\|_0 = \|v\|_0$ shows that it is an isometry.

To see that $i(V) \subset \overline{V}$ is dense, let $(v_n) \in C$ and fix any $\epsilon > 0$. Since (v_n) is Cauchy, we can find some $N \in \mathbb{N}$ such that $\|v_n - v_N\|_0 < \epsilon$ for all $n \geq N$. We then find that

$$\big\|\big((v_n) + C_0\big) - i(v_N)\big\| = \lim_{n\to\infty} \|v_n - v_N\|_0 \leq \epsilon,$$

which shows that $i(V)$ is dense in $\overline{V}$.

We now show that $\overline{V}$ is complete. Let $(\nu^k)_k \subset \overline{V}$ be a Cauchy sequence (of equivalence classes of sequences in V) and write

$$\nu^k = (v_n^k)_n + C_0 \quad \text{for each } k \in \mathbb{N}.$$

Since $i(V)$ is dense in $\overline{V}$, we can find $w_k \in V$ such that $\|\nu^k - i(w_k)\| \leq 1/k$. We now claim that

$$(w_n)_n \in C \quad \text{and} \quad \nu^k = (v_n^k)_n + C_0 \to (w_n)_n + C_0 \text{ in } \overline{V}.$$

To see that $(w_n)_{n\in\mathbb{N}}$ is Cauchy, note that

$$\|w_m - w_n\|_0 \leq \|w_m - v_k^m\|_0 + \|v_k^m - v_k^n\|_0 + \|v_k^n - w_n\|_0$$

for all $m, n, k \in \mathbb{N}$, and that each of these terms are arbitrarily small for sufficiently large m, n, k, because

$$\begin{aligned} \lim_{k\to\infty} \|v_k^m - v_k^n\|_0 &= \|\nu^m - \nu^n\|, \\ \lim_{k\to\infty} \|w_m - v_k^m\|_0 &= \|i(w_m) - \nu^m\| \leq \frac{1}{m}, \\ \text{and} \quad \lim_{k\to\infty} \|w_n - v_k^n\|_0 &= \|i(w_n) - \nu^n\| \leq \frac{1}{n} \end{aligned}$$

(and $(\nu^k)_k \subset \overline{V}$ is Cauchy). To see that $(w_n)_n + C_0$ is the limit of $(\nu^k)_k$ in $\overline{V}$, note that

$$\begin{aligned} \big\|\nu^k - \big((w_n)_n + C_0\big)\big\| &= \lim_{n\to\infty} \|v_n^k - w_n\|_0 \\ &\leq \lim_{n\to\infty} \big(\|v_n^k - w_k\|_0 + \|w_k - w_n\|_0\big) \\ &= \|\nu^k - i(w_k)\| + \lim_{n\to\infty} \|w_k - w_n\|_0 \to 0 \end{aligned}$$

as $k \to \infty$ (the first term is $\leq 1/k$ by our choice of w_k, and the last term goes to zero because $(w_n)_n$ is Cauchy). This concludes the proof that $\overline{V}$ is complete and the proof as a whole. $\square$

We now turn our attention to extensions of bounded linear maps.

B.1.2 Theorem (Bounded linear extensions)**.** *Let $T\colon V \to W$ be a bounded linear map between normed spaces and let $i\colon V \to \overline{V}$ and $j\colon W \to \overline{W}$ be Banach space completions. Then, there exists a unique bounded linear map $\overline{T}\colon \overline{V} \to \overline{W}$ such that $j \circ T = \overline{T} \circ i$. Moreover, $\|\overline{T}\| = \|T\|$.*

Proof. Suppose first that $\overline{T}\colon \overline{V} \to \overline{W}$ is any continuous function such that $j \circ T = \overline{T} \circ i$. For any $v \in \overline{V}$, choose a sequence $(v_n) \subset V$ such that $i(v_n) \to v$. Then, by continuity of $\overline{T}$,

$$\overline{T}v = \lim_{n\to\infty} \overline{T}i(v_n) = \lim_{n\to\infty} jT(v_n),$$

so these conditions determine $\overline{T}$ uniquely. We will verify that the map $\overline{T}v \coloneqq \lim_{n\to\infty} jT(v_n)$ is well-defined, linear, bounded with norm $\|T\|$ and that it satisfies $j \circ T = \overline{T} \circ i$. In the following, we will frequently use the fact that i and j are isometries without mentioning it.

If $i(v_n) \to v \in \overline{V}$, then the sequence (v_n) is Cauchy. Since T is bounded, (Tv_n) is Cauchy as well, so $\lim_{n\to\infty} j(Tv_n)$ exists in the complete space $\overline{W}$. If we have another sequence $(v_n') \subset V$ such that $i(v_n') \to v$, then $i(v_n' - v_n) = i(v_n') - i(v_n) \to 0$, so that $v_n' - v_n \to 0$ in V. Since T is bounded, this implies that $jT(v_n') - jT(v_n) = jT(v_n' - v_n) \to 0$. We have now shown that $\overline{T}$ is well-defined and independent of our choice of sequences.

Knowing that $\overline{T}$ is independent of our choice of sequences, it is straightforward to verify that it is linear:

$$\begin{aligned}\overline{T}(\lambda v + w) &= \lim_{n\to\infty} jT(\lambda v_n + w_n) \\ &= \lambda \lim_{n\to\infty} jT(v_n) + \lim_{n\to\infty} jT(w_n) = \lambda \overline{T}v + \overline{T}w.\end{aligned}$$

The following calculation shows that $\overline{T}$ is bounded, with $\|\overline{T}\| \le \|T\|$:

$$\begin{aligned}\|\overline{T}v\| &= \lim_{n\to\infty} \|jT(v_n)\| = \lim_{n\to\infty} \|Tv_n\| \\ &\le \lim_{n\to\infty} \|T\|\|v_n\| = \|T\| \lim_{n\to\infty} \|i(v_n)\| = \|T\|\|v\|.\end{aligned}$$

Now, for any $v \in V$, let $(v_n)_n = (v)_n \subset V$ be the constant sequence at v. Then, since $i(v_n) = i(v) \to i(v)$ in $\overline{V}$, we find that

$$\overline{T}(i(v)) = \lim_{n\to\infty} jT(v) = jT(v),$$

which shows that $\overline{T} \circ i = j \circ T$. If we now choose a sequence $(v_n)_n \subset V$ of unit vectors such that $\|Tv_n\| \to \|T\|$, then $(i(v_n))_n \subset \overline{V}$ is a sequence of unit vectors such that $\|\overline{T}i(v_n)\| = \|jT(v_n)\| \to \|T\|$. Thus, $\|\overline{T}\| = \|T\|$ and we are done. □

We now obtain the claimed uniqueness of Banach space completions.

B.1.3 Corollary (Uniqueness of Banach space completions)**.** *Let V be a normed space and suppose that $i\colon V \to \overline{V}$ and $j\colon V \to \widetilde{V}$ are two Banach space completions of V. Then, there exists a unique isometric linear isomorphism $T\colon \overline{V} \to \widetilde{V}$ such that $j = T \circ i$.*

Proof. Consider the identity map $\mathrm{Id}_V\colon V \to V$. When applying Theorem B.1.2 to Id_V, we may use either $i\colon V \to \overline{V}$ or $j\colon V \to \widetilde{V}$ for the completion of its domain, and similarly for its target. All in all, there are four possible choices. Thus, we obtain four bounded linear maps

$$T_1\colon \overline{V} \to \widetilde{V}, \quad T_2\colon \widetilde{V} \to \widetilde{V}, \quad T_3\colon \widetilde{V} \to \overline{V} \quad \text{and} \quad T_4\colon \overline{V} \to \overline{V}$$

satisfying

$$j = T_1 \circ i, \quad j = T_2 \circ j, \quad i = T_3 \circ j \quad \text{and} \quad i = T_4 \circ i,$$

and we are assured that these are unique. By uniqueness, we can immediately conclude that $T_2 = \mathrm{Id}_{\widetilde{V}}$ and that $T_4 = \mathrm{Id}_{\overline{V}}$. However, we also find that

$$j = T_1 \circ i = T_1 \circ (T_3 \circ j) = (T_1 \circ T_3) \circ j \quad \text{and} \quad i = (T_3 \circ T_1) \circ i,$$

from which we can conclude that $T_1 \circ T_3 = \mathrm{Id}_{\widetilde{V}}$ and that $T_3 \circ T_1 = \mathrm{Id}_{\overline{V}}$ (again by uniqueness). Thus, $T := T_1$ is an isomorphism, and it is the unique bounded linear map such that $j = T \circ i$.

Finally, $\|T\| = 1$ and $\|T^{-1}\| = \|T_3\| = 1$, since both maps are bounded linear extensions of Id_V. This immediately implies that T is isometric: $\|v\| = \|T^{-1}T(v)\| \leq \|Tv\| \leq \|v\|$ for all $v \in V$. Thus, we are done. □

The following proposition shows that bounded linear extensions act as one would hope and expect.

B.1.4 Proposition (Properties of extensions)**.** *Let*

$$T_1\colon V \to W, \quad T_2\colon V \to W, \quad S\colon W \to Z$$

be bounded linear maps between normed spaces. The following statements are true.

(i) Extension is linear: $\overline{\lambda T_1 + T_2} = \lambda \overline{T_1} + \overline{T_2}$ *for all* $\lambda \in \mathbb{C}$.

(ii) Extension preserves compositions: $\overline{S \circ T_1} = \overline{S} \circ \overline{T_1}$.

(iii) Extension preserves identities: $\overline{\mathrm{Id}_V} = \mathrm{Id}_{\overline{V}}$.

(iv) *Extension preserves inverses: if T_1 is invertible and T_1^{-1} is bounded, then $\overline{T_1}$ is invertible and $\big(\overline{T_1}\big)^{-1} = \overline{T_1^{-1}}$.*

Proof. Let $i\colon V \to \overline{V}$, $j\colon W \to \overline{W}$ and $k\colon Z \to \overline{Z}$ be Banach space completions. Then, $j \circ T_1 = \overline{T_1} \circ i$, $j \circ T_2 = \overline{T_2} \circ i$ and $k \circ S = \overline{S} \circ j$. Thus,

$$j \circ (\lambda T_1 + T_2) = \lambda(j \circ T_1) + j \circ T_2 = \lambda\overline{T_1} \circ i + \overline{T_2} \circ i = (\lambda\overline{T_1} + \overline{T_2}) \circ i$$
$$\text{and} \quad k \circ (S \circ T_1) = (k \circ S) \circ T_1 = \overline{S} \circ (j \circ T_1) = (\overline{S} \circ \overline{T_1}) \circ i.$$

The fact that $\lambda\overline{T_1} + \overline{T_2}$ and $\overline{S} \circ \overline{T_1}$ are bounded linear maps satisfying these equations implies (i) and (ii) by uniqueness of bounded linear extensions.

Since $i \circ \mathrm{Id}_V = i = \mathrm{Id}_{\overline{V}} \circ i$, we see that $\mathrm{Id}_{\overline{V}} = \overline{\mathrm{Id}_V}$, which gives (iii). Finally, (iv) follows by combining (ii) and (iii):

$$\mathrm{Id}_{\overline{V}} = \overline{T_1^{-1} \circ T_1} = \overline{T_1^{-1}} \circ \overline{T_1} \quad \text{and} \quad \mathrm{Id}_{\overline{W}} = \overline{T_1 \circ T_1^{-1}} = \overline{T_1} \circ \overline{T_1^{-1}},$$

and so $\big(\overline{T_1}\big)^{-1} = \overline{T_1^{-1}}$ (note that we need boundedness of T_1^{-1} for the existence of $\overline{T_1^{-1}}$). □

For the case where $V = W$, we can summarize the proposition we just proved as follows.

B.1.5 Corollary. *Let V be a normed space. Then, the bounded linear extension map $\mathcal{B}(V) \to \mathcal{B}\big(\overline{V}\big)$ defined by $T \mapsto \overline{T}$ is an isometric algebra homomorphism.*

Proof. Points (i), (ii) and (iii) in Proposition B.1.4 (along with the fact that $\|T\| = \|\overline{T}\|$, as shown in Theorem B.1.2). □

We now turn to extensions of bilinear maps between normed spaces. These will be useful for extending inner products, algebra products and more. For normed spaces V, W and Z, we will say that a bilinear map $B\colon V \times W \to Z$ is *bounded* if there exists some $M \in [0, \infty)$ such that

$$\|B(v, w)\| \leq M\|v\|\|w\| \quad \text{for all } v \in V \text{ and } w \in W,$$

and we will write $\|B\|$ for the infimum over all such M.

B.1.6 Theorem (Bounded bilinear extensions)**.** *Let V, W and Z be normed spaces with Banach space completions $i\colon V \to \overline{V}$, $j\colon W \to \overline{W}$ and $k\colon Z \to \overline{Z}$. Suppose that $B\colon V \times W \to Z$ is a bounded bilinear map. Then, there exists a unique bounded bilinear map $\overline{B}\colon \overline{V} \times \overline{W} \to \overline{Z}$ that extends B, in the sense that*

$$k\big(B(v, w)\big) = \overline{B}\big(i(v), j(w)\big) \quad \textit{for all } v \in V \textit{ and } w \in W.$$

Moreover, $\|\overline{B}\| = \|B\|$.

Proof. Fix any $v \in V$. Then, the map

$$\begin{aligned} B(v,-)\colon W &\to Z \\ w &\mapsto B(v,-)(w) \coloneqq B(v,w) \end{aligned}$$

is a linear map with operator norm $\|B(v,-)\| \le \|B\|\|v\|$. By Theorem B.1.2, there is a unique linear extension $\overline{B(v,-)}\colon \overline{W} \to \overline{Z}$ with the same operator norm $\|\overline{B(v,-)}\| = \|B(v,-)\| \le \|B\|\|v\|$.

Consider now the map

$$\begin{aligned} B_V\colon V &\to \mathcal{B}(\overline{W},\overline{Z}) \\ v &\mapsto B_V(v) \coloneqq \overline{B(v,-)}. \end{aligned}$$

This is a linear map, because

$$\overline{B(\lambda v + w,-)} = \overline{\lambda B(v,-) + B(w,-)} = \lambda\overline{B(v,-)} + \overline{B(w,-)}$$

by Proposition B.1.4. We find that $\|B_V\| \le \|B\|$ because $\|B_V(v)\| = \|\overline{B(v,-)}\| \le \|B\|\|v\|$ for all $v \in V$. Since $\overline{Z}$ is a Banach space, $\mathcal{B}(\overline{W},\overline{Z})$ is a Banach space as well. Thus, we get a unique bounded linear extension

$$\overline{B_V}\colon \overline{V} \to \mathcal{B}(\overline{W},\overline{Z})$$

which satisfies $\|\overline{B_V}\| = \|B_V\| \le \|B\|$, and hence

$$\|\overline{B_V}(v)(w)\| \le \|\overline{B_V}(v)\|\|w\| \le \|\overline{B_V}\|\|v\|\|w\| \le \|B\|\|v\|\|w\| \tag{B.1}$$

for all $v \in \overline{V}$ and $w \in \overline{W}$.

We now define

$$\begin{aligned} \overline{B}\colon \overline{V} \times \overline{W} &\to \overline{Z} \\ (v,w) &\mapsto \overline{B}(v,w) \coloneqq \overline{B_V}(v)(w), \end{aligned}$$

and claim that this our desired extensions of B. Equation (B.1) shows that $\|\overline{B}\| \le \|B\|$. The fact that $\overline{B(v,-)}$ extends $B(v,-)$ and $\overline{B_V}$ extends B_V means that

$$\begin{aligned} &\overline{B(v,-)}\big(j(w)\big) = k\big(B(v,w)\big) && \text{for all } v \in V \text{ and } w \in W \\ \text{and} \quad &\overline{B_V}\big(i(v)\big) = B_V(v) = \overline{B(v,-)} && \text{for all } v \in V, \end{aligned}$$

and these combine to give

$$\overline{B}\big(i(v), j(w)\big) = \overline{B_V}\big(i(v)\big)\big(j(w)\big) = \overline{B(v,-)}\big(j(w)\big) = k\big(B(v,w)\big),$$

as desired. By this equation (and the fact that i, j and k are isometries), the norm of $\overline{B}$ cannot be less than that of B, so we must have $\|\overline{B}\| = \|B\|$.

We have had uniqueness at each step of our construction, but it is not a prior obvious that we could not have arrived at another extension by another method, so we will give a quick proof of uniqueness.

Fix any $v \in V$. Suppose we have a (potentially different) bounded bilinear map $\widetilde{B}\colon \overline{V} \times \overline{W} \to \overline{Z}$ satisfying

$$\widetilde{B}\big(i(v), j(w)\big) = k\big(B(v, w)\big) \text{ for all } v \in V \text{ and } w \in W.$$

Then, for each $v \in V$, the maps

$$\begin{aligned} \overline{W} &\to \overline{Z} \\ w &\mapsto \overline{B}\big(i(v), w\big) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{W} &\to \overline{Z} \\ w &\mapsto \widetilde{B}\big(i(v), w\big) \end{aligned}$$

are both bounded linear extensions of the map $W \to Z$ defined by $w \mapsto B(v, w)$. Thus, they must be equal:

$$\overline{B}\big(i(v), w\big) = \widetilde{B}\big(i(v), w\big) \quad \text{for all } v \in V \text{ and } w \in \overline{W}.$$

Finally, for each $w \in \overline{W}$, the maps

$$\begin{aligned} \overline{V} &\to \overline{Z} \\ v &\mapsto \overline{B}(v, w) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{V} &\to \overline{Z} \\ w &\mapsto \widetilde{B}(v, w) \end{aligned}$$

are both bounded linear extensions of the map $V \to \overline{Z}$ defined by $v \mapsto \overline{B}(i(v), w) = \widetilde{B}(i(v), w)$, so they must be equal as well. This shows uniqueness and concludes the proof. □

We close this section with a note on conjugate-linearity. Given a complex vector space V, we define the *complex conjugate* of V, which we will denote by V^c: this is the same set as V with the same abelian group structure, but with scalar multiplication defined by $(\lambda, v) \mapsto \overline{\lambda}v$ (where the juxtaposition denotes scalar multiplication in V). If V is a normed space, then V^c is a normed space as well (with the same norm), and V^c is clearly complete if and only if V is complete. If $i\colon V \to \overline{V}$ is a Banach space completion of V, then the exact same map $i\colon V^c \to (\overline{V})^c$ is a linear isometry with a dense image, so $\overline{V^c} \cong (\overline{V})^c$.

Now, a conjugate-linear map $T\colon V \to W$ is exactly the same as a linear map $T\colon V^c \to W$. The unique linear extension $\overline{T}\colon (\overline{V})^c \cong \overline{V^c} \to \overline{W}$ will therefore be a unique conjugate-linear extension $\overline{T}\colon \overline{V} \to \overline{W}$. The same holds for sesquilinear and conjugate-bilinear maps (i.e. maps which are conjugate linear in both entries). Thus, our proposition on the unique bounded extensions of bilinear maps gives us unique bounded extensions of sesquilinear maps and conjugate-bilinear maps as well.

B.2 | Completions of Various Algebras

In this section we will consider not-necessarily-complete versions of Banach algebras, Banach $\star$-algebras and C*-algebras, and show how to complete them. We will also consider bounded extensions of bounded algebra and $\star$-algebra homomorphisms. The appropriate place to read this section is during or after Chapter 2, where the relevant concepts are introduced. The results of this section are not needed until Chapter 4.

Consider a normed algebra A and let $i_A\colon A \to \overline{A}$ be a Banach space completion of A. If we want to turn $\overline{A}$ into an algebra and think of $\overline{A}$ as an algebra completion of A, the natural requirement is that i_A should be an algebra homomorphism. That is, i_A should preserve the unit and the product.

Now, let $\mu\colon A \times A \to A$ denote the product on A and $\overline{\mu}\colon \overline{A} \times \overline{A} \to \overline{A}$ denote the (suggestively named) product on $\overline{A}$. Preservation of the product by i_A, i.e. the condition that

$$i_A(a)i_A(b) = i_A(ab) \quad \text{for all } a, b \in A$$

is equivalent to the condition that

$$\overline{\mu}\big(i_A(a), i_A(b)\big) = i_A\big(\mu(a,b)\big) \quad \text{for all } a, b \in A, \tag{B.2}$$

which is precisely the condition that $\overline{\mu}$ be a bilinear extensions μ (as in Theorem B.1.6).

We begin with not-necessarily-complete version of Banach algebras.

B.2.1 Definition (Pre-Banach algebras)**.** A *pre-Banach algebra* is an algebra B equipped with a submultiplicative norm that normalizes the unit.

Given a pre-Banach algebra B, a *Banach algebra completion* of B is a Banach algebra $\mathcal{B}$ along with an isometric algebra homomorphism $i_B\colon B \to \mathcal{B}$ whose image is dense. We may refer to either $\mathcal{B}$ or the map i_B itself as a Banach algebra completion of B.

The following proposition shows that Banach algebra completions exist and that they can be obtained from any Banach space completion of B. Note also that any Banach algebra completion of B is, in particular, a Banach space completion of B.

B.2.2 Proposition (Banach algebra completions)**.** *Let B be a pre-Banach algebra and let $i_B\colon B \to \mathcal{B}$ be a Banach space completion of B. Then, there is a unique product on $\mathcal{B}$ such that $i_B\colon B \to \mathcal{B}$ becomes an algebra homomorphism and $\mathcal{B}$ becomes a Banach algebra. In other words: such that i_B becomes a Banach algebra completion of B.*

Proof. By Theorem B.1.6 (and submultiplicativity of the norm), the algebra product $\mu\colon B \times B \to B$ extends uniquely to a bilinear map $\overline{\mu}\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ satisfying

$$\|\overline{\mu}(a, b)\| \leq \|a\|\|b\| \quad \text{for all } a, b \in \mathcal{B}.$$

We first prove that $\overline{\mu}$ is associative. By the opening of this section, the fact that $\overline{\mu}$ extends μ is equivalent to i_B preserving the product. Thus, we have that

$$i_B(a)\big(i_B(b)i_B(c)\big) = i_B(a)i_B(bc) = i_B(abc) = \big(i_B(a)i_B(b)\big)i_B(c)$$

for all $a, b, c \in B$. This shows that, if we fix $c \in B$, the maps

$$\begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (a, b) &\mapsto (ab)i_B(c) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (a, b) &\mapsto a\big(bi_B(c)\big) \end{aligned}$$

are both bounded bilinear extensions of the bounded bilinear map $B \times B \to B$ defined by $(a, b) \mapsto abc$. By uniqueness of such extensions, they must be equal. If we now fix $a, b \in \mathcal{B}$, we may conclude that the maps

$$\begin{aligned} \mathcal{B} &\to \mathcal{B} \\ c &\mapsto (ab)c \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{B} &\to \mathcal{B} \\ c &\mapsto a(bc) \end{aligned}$$

are both bounded linear extensions of the bounded linear map $B \to \mathcal{B}$ defined by $c \mapsto (ab)i_B(c) = a(bi_B(c))$. Thus, they must be equal, which proves associativity.

Similarly, one finds that

$$i_B(1_B)i_B(a) = i_B(1_B a) = i_B(a) = i_B\big(\mathrm{Id}_B(a)\big) = i_B(a)i_B(1_B)$$

for all $a \in B$. This means that the maps

$$\begin{aligned} \mathcal{B} &\to \mathcal{B} \\ a &\mapsto i_B(1_B)a \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{B} &\to \mathcal{B} \\ a &\mapsto ai_B(1_B) \end{aligned}$$

are both bounded linear extensions of the identity on B, so they must be the identity on $\mathcal{B}$ (Proposition B.1.4). Thus, $1_{\mathcal{B}} = i_B(1_B)$ and $\|1_{\mathcal{B}}\| = \|1_B\| = 1$.

We have now verified that $\overline{\mu}$ turns $\mathcal{B}$ into a Banach algebra and that i_B is an algebra homomorphism, so we are done. $\square$

It is useful to know that bounded linear extensions of bounded algebra homomorphisms are algebra homomorphisms.

B.2.3 Proposition. *Let $\Phi\colon B \to B'$ be a bounded algebra homomorphism between pre-Banach algebras and let $i_B\colon B \to \mathcal{B}$ and $i_{B'}\colon B' \to \mathcal{B}'$ be Banach algebra completions. Then, the unique bounded linear extension $\overline{\Phi}\colon \mathcal{B} \to \mathcal{B}'$ is also an algebra homomorphism.*

Proof. For any maps $f\colon X \to Y$ and $g\colon X \to Y$ between sets X and Y, we will write $f \times g\colon X \times X \to Y \times Y$ for the natural map defined by $(x_1, x_2) \mapsto (f(x_1), g(x_2))$.

Let μ_B and $\mu_{B'}$ denote the products on the pre-Banach algebras indicated by their subscripts and let $\overline{\mu_B}$ and $\overline{\mu_{B'}}$ denote the products on their Banach algebra completions (then, $\overline{\mu_B}$ is a bounded bilinear extension of μ_B, and likewise for B', as we have seen).

We already know that $\overline{\Phi}$ preserves the unit:

$$\overline{\Phi}(1_{\mathcal{B}}) = (\overline{\Phi} \circ i_B)(1_B) = (i_{B'} \circ \Phi)(1_B) = i_{B'}(1_{B'}) = 1_{\mathcal{B}'}.$$

The statement that $\overline{\Phi}$ preserves the product is equivalent to the statement that

$$\overline{\mu_{B'}} \circ (\overline{\Phi} \times \overline{\Phi}) = \overline{\Phi} \circ \overline{\mu_B}$$

as maps $\mathcal{B} \times \mathcal{B} \to \mathcal{B}'$. Now, the calculations

$$\begin{aligned}
\overline{\mu_{B'}} \circ (\overline{\Phi} \times \overline{\Phi}) \circ (i_B \times i_B) &= \overline{\mu_{B'}} \circ \big((\overline{\Phi} \circ i_B) \times (\overline{\Phi} \circ i_B)\big) \\
&= \overline{\mu_{B'}} \circ \big((i_{B'} \circ \Phi) \times (i_{B'} \circ \Phi)\big) \\
&= \overline{\mu_{B'}} \circ (i_{B'} \times i_{B'}) \circ (\Phi \times \Phi) \\
&= i_{B'} \circ \mu_{B'} \circ (\Phi \times \Phi)
\end{aligned}$$

and

$$\overline{\Phi} \circ \overline{\mu_B} \circ (i_B \times i_B) = \overline{\Phi} \circ i_B \circ \mu_B = i_{B'} \circ \Phi \circ \mu_B$$

show that $\overline{\mu_{B'}} \circ (\overline{\Phi} \times \overline{\Phi})$ and $\overline{\Phi} \circ \overline{\mu_B}$ are both bounded bilinear extensions of the bounded bilinear map $\mu_{B'} \circ (\Phi \times \Phi) = \Phi \circ \mu_B$. Thus, by uniqueness of such extensions, they are equal. This proves that $\overline{\Phi}$ preserves the product and concludes the proof. □

B.2.4 Corollary (Uniqueness of Banach algebra completions)**.** *Let B be a pre-Banach algebra and suppose that $i_B\colon B \to \mathcal{B}$ and $j_B\colon B \to \mathcal{B}'$ are two Banach algebra completions of B. Then, there exists a unique isometric algebra isomorphism $\Phi\colon \mathcal{B} \to \mathcal{B}'$ such that $j_B = \Phi \circ i_B$.*

Proof. Both i_B and j_B are Banach space completions, so Corollary B.1.3 guarantees the existence of a unique isometric linear isomorphism $\Phi\colon \mathcal{B} \to \mathcal{B}'$ such that $j_B = \Phi \circ i_B$. The equation $j_B = \Phi \circ i_B$ means that Φ is a linear extension of the identity $\mathrm{Id}_B\colon B \to B$ (where i_B is taken to be the completion of its domain and j_B is taken to be the completions of its target). Since Id_B is an algebra homomorphism, so is Φ, by Proposition B.2.3. □

We now turn to not-necessarily-complete versions of Banach $\star$-algebras.

B.2.5 Definition (pre-Banach $\star$-algebras)**.** A *pre-Banach $\star$-algebra* is an algebra B equipped with a submultiplicative norm that normalizes the unit and an involution that is isometric.

Given a pre-Banach $\star$-algebra B, a *Banach $\star$-algebra completion* of B is a Banach $\star$-algebra $\mathcal{B}$ along with an isometric $\star$-algebra homomorphism $i_B\colon B \to \mathcal{B}$ whose image is dense. The usual variants of the terminology apply here as well.

B.2.6 Proposition (Banach $\star$-algebra completions)**.** *Let B be a pre-Banach $\star$-algebra and let $i_B\colon B \to \mathcal{B}$ be a Banach space completion of B. Then, there is a unique product and a unique involution on $\mathcal{B}$ such that $i_B\colon B \to \mathcal{B}$ becomes a $\star$-algebra homomorphism and $\mathcal{B}$ becomes a Banach $\star$-algebra. In other words: such that i_B becomes a Banach $\star$-algebra completion of B.*

Proof. By Proposition B.2.2, we already know that there is a unique product on $\mathcal{B}$ turning $i_B\colon B \to \mathcal{B}$ into an algebra homomorphism and $\mathcal{B}$ into a Banach algebra. Thus, we only need to verify the existence of a unique involution on $\mathcal{B}$ that turns $\mathcal{B}$ into a $\star$-Banach algebra and i_B into a $\star$-algebra homomorphism.

By Theorem B.1.2, the involution $\star\colon B \to B$ extends to a unique conjugate-linear map $\overline{\star}\colon \mathcal{B} \to \mathcal{B}$ satisfying $\|\overline{\star}(a)\| \leq \|a\|$ for all $a \in \mathcal{B}$. Since $\star$ is its own inverse, Proposition B.1.4 guarantees that $\overline{\star}$ is as well, so $(a^{\overline{\star}})^{\overline{\star}} = a$ for all $a \in \mathcal{B}$. Note that $\overline{\star}$ extending $\star$ means that $i_B(a^*) = i_B(a)^{\overline{\star}}$ for all $a \in B$, so we can conclude that i_B is a $\star$-algebra homomorphism once we have shown that $\overline{\star}$ defines an involution.

The map $\star \circ \mu\colon B \times B \to B$ defined by $(a,b) \mapsto (ab)^* = b^*a^*$ is a bounded conjugate-bilinear map. The maps

$$\begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (a,b) &\mapsto (ab)^{\overline{\star}} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (a,b) &\mapsto b^{\overline{\star}}a^{\overline{\star}} \end{aligned}$$

are both bounded conjugate-bilinear extensions of this map, as

$$(i_B(a)i_B(b))^{\overline{\star}} = i_B(ab)^{\overline{\star}} = i_B\big((ab)^*\big) = i_B(b^*a^*) = i_B(b)^{\overline{\star}} i_B(a)^{\overline{\star}}.$$

Thus, Theorem B.1.2 guarantees that they are equal. This proves that $\overline{\star}$ is an involution on $\mathcal{B}$. Finally, since $\overline{\star}$ is norm-decreasing and is its own inverse, it is isometric: $\|a\| = \|\overline{\star}^2(a)\| \leq \|\overline{\star}(a)\| \leq \|a\|$ for all $a \in \mathcal{B}$. $\square$

B.2.7 Proposition. *Let $\Phi\colon B \to B'$ be a bounded $\star$-algebra homomorphism between pre-Banach $\star$-algebras and let $i_B\colon B \to \mathcal{B}$ and $i_{B'}\colon B' \to \mathcal{B}'$ be Banach $\star$-algebra completions. Then, the unique bounded linear extension $\overline{\Phi}\colon \mathcal{B} \to \mathcal{B}'$ is also a $\star$-algebra homomorphism.*

Proof. Because of Proposition B.2.3, all we need to check is that $\overline{\Phi}$ preserves the involution. Write $\star_B$ and $\star_{B'}$ for the involutions on B and B' and write $\overline{\star_B}$ and $\overline{\star_{B'}}$ for their extensions. The statement that $\overline{\Phi}$ preserves the involution is equivalent to the statement that $\overline{\star_{B'}} \circ \overline{\Phi} = \overline{\Phi} \circ \overline{\star_B}$. The calculations

$$\begin{aligned} \overline{\star_{B'}} \circ \overline{\Phi} \circ i_B &= \overline{\star_{B'}} \circ i_{B'} \circ \Phi = i_{B'} \circ \star_{B'} \circ \Phi \\ \text{and} \quad \overline{\Phi} \circ \overline{\star_B} \circ i_B &= \overline{\Phi} \circ i_B \circ \star_B = i_{B'} \circ \Phi \circ \star_B \end{aligned}$$

show that both $\overline{\star_{B'}} \circ \overline{\Phi}$ and $\overline{\Phi} \circ \overline{\star_B}$ are bounded conjugate-linear extensions of the bounded conjugate-linear map $\star_{B'} \circ \Phi = \Phi \circ \star_B$. Thus, they are equal, so $\overline{\Phi}$ preserves the involution. $\square$

B.2.8 Corollary (Uniqueness of Banach $\star$-algebra completions)**.** *Let B be a pre-Banach $\star$-algebra and suppose that $i_B\colon B \to \mathcal{B}$ and $j_B\colon B \to \mathcal{B}'$ are two Banach $\star$-algebra completions of B. Then, there exists a unique isometric $\star$-algebra isomorphism $\Phi\colon \mathcal{B} \to \mathcal{B}'$ such that $j_B = \Phi \circ i_B$.*

Proof. All we need to check is that the unique isometric algebra isomorphism $\Phi\colon \mathcal{B} \to \mathcal{B}'$ afforded by Corollary B.2.4 preserves the involution. Since it is a bounded linear extension of the identity $\mathrm{Id}_B\colon B \to B$, this follows from Proposition B.2.7 (see the proof of Corollary B.2.4 for details). $\square$

Finally, we consider not-necessarily-complete versions of C*-algebras. Most of the work is already done, so this will be swift.

B.2.9 Definition (Pre-C*-algebras)**.** A *pre-C*-algebra* is a $\star$-algebra A equipped with a submultiplicative norm that satisfies the C*-equality.

A *C*-algebra completion* of a pre-C*-algebra A is a C*-algebra $\mathcal{A}$ along with an isometric $\star$-algebra homomorphism $i_A\colon A \to \mathcal{A}$ whose image is dense. Again, the usual variants of the terminology apply here as well.

As we observed immediately after the definition of C*-algebras (Definition 2.2.1), the C*-equality implies that the unit is normalized. Moreover, if one examines the proof of point (iii) of Lemma 2.2.3, where we showed that the involution on a C*-algebra is isometric, one will see that we only used the C*-equality and submultiplicativity. Thus, the proof carries over to the setting of pre-C*-algebras. By these observations, a pre-C*-algebra is, in particular, a pre-Banach $\star$-algebra. This fact makes the following proposition almost immediate.

B.2.10 Proposition (C*-algebra completions and uniqueness)**.** *Let A be a pre-C*-algebra and let $i_A\colon A \to \mathcal{A}$ be a Banach space completion of A. Then, there is a unique product and a unique involution on $\mathcal{A}$ such that $i_A\colon A \to \mathcal{A}$ becomes a $\star$-algebra homomorphism and $\mathcal{A}$ becomes a C*-algebra. In other words: such that i_A becomes a C*-algebra completion of A.*

Moreover, if $j_A\colon A \to \mathcal{A}'$ is any other C-algebra completion of A, then there exists a unique (isometric) $\star$-algebra isomorphism $\Phi\colon \mathcal{A} \to \mathcal{A}'$ such that $j_A = \Phi \circ i_A$.*

Proof. By Proposition B.2.6 and Corollary B.2.8, we only need to verify that the C*-equality holds in $\mathcal{A}$, where $\mathcal{A}$ is a Banach $\star$-algebra completion of A. The rest of the proposition follows from the fact that pre-C*-algebras are pre-Banach $\star$-algebras.

To do this, we resort to approximating sequences: let $a \in \mathcal{A}$ be arbitrary and choose a sequence $(a_n) \subset A$ such that $i_A(a_n) \to a$ in $\mathcal{A}$. Using the fact that i_A is an isometric $\star$-algebra homomorphism, along with continuity of the norm and the algebraic operations on $\mathcal{A}$, we find that

$$\begin{aligned}\|a^*a\| &= \|\lim_{n\to\infty} i_A(a_n^*a_n)\| = \lim_{n\to\infty}\|i_A(a_n^*a_n)\| = \lim_{n\to\infty}\|a_n^*a_n\| \\ &= \lim_{n\to\infty}\|a_n\|^2 = \lim_{n\to\infty}\|i_A(a_n)\|^2 = \|\lim_{n\to\infty} i_A(a_n)\|^2 = \|a\|^2,\end{aligned}$$

which verifies that the C*-equality holds on $\mathcal{A}$ in virtue of it holding on A. □

B.3 | Completions of Inner Product Modules

In this section we consider completions of inner product modules to Hilbert C*-modules. The appropriate place to read this section is during or after Section 4.3, where the relevant concepts are introduced.

Suppose that A is a pre-C*-algebra with C*-algebra completion $\mathcal{A}$. Given a left inner product A-module E, a *(left) Hilbert C*-module completion* of E is a left Hilbert $\mathcal{A}$-module $\overline{E}$ along with a map $j_E\colon E \to \overline{E}$ that is A-linear, preserves the inner product and whose image is dense. By preservation of the inner product, j_E must be an isometry, and A-linearity implies $\mathbb{C}$-linearity, so j_E is necessarily a Banach space completion of E.

B.3.1 Proposition (Hilbert C*-module completions)**.** *Let A be a pre-C*-algebra with C*-algebra completion $i_A\colon A \to \mathcal{A}$. Suppose that E is a left inner product A-module with action $\Phi_A\colon A \to \mathcal{B}(E)$ and A-valued inner product ${}_A\langle\cdot,\cdot\rangle$. Suppose moreover that $j_E\colon E \to \overline{E}$ is a Banach space completion of E (w.r.t. the norm determined by ${}_A\langle\cdot,\cdot\rangle$).*

Then, there exists a unique action $\Phi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}(\overline{E})$ and a unique $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle\colon \overline{E} \times \overline{E} \to \mathcal{A}$ such that j_E becomes an A-linear map preserving the inner product and $\overline{E}$ becomes a left Hilbert $\mathcal{A}$-module. In other words: such that j_E becomes a Hilbert C-module completion of E. Moreover, the Banach space completion norm on $\overline{E}$ equals the norm determined by ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$.*

Proof. Part 1: Extending the action

By Corollary B.1.5, the extension map $\mathrm{Ext}\colon \mathcal{B}(E) \to \mathcal{B}(\overline{E})$ defined by $\mathrm{Ext}(T) = \overline{T}$ is an isometric algebra homomorphism. Since $\mathcal{B}(\overline{E})$ is a Banach space, we can think of the map $\mathrm{Ext}\colon \mathcal{B}(E) \to \mathcal{B}(\overline{E})$ as a Banach space completion of $\mathcal{B}(E)$ followed by an inclusion of its image into $\mathcal{B}(\overline{E})$.

Lemma 4.3.6 shows that $\|\Phi_A(a)\| \leq \|a\|$ for all $a \in A$, so Φ_A is certainly bounded. By Proposition B.2.3, we now obtain a unique bounded linear extension (followed by an inclusion) $\Phi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}(\overline{E})$ that is also an algebra homomorphism.

With our choices of completions, the fact that $\Phi_{\mathcal{A}}$ extends Φ_A means that $\mathrm{Ext} \circ \Phi_A = \Phi_{\mathcal{A}} \circ i_A$. That is, $\mathrm{Ext}(\Phi_A a) = \Phi_{\mathcal{A}}(i_A(a))$ for all $a \in A$. By definition of the extension map, we now find that

$$j_E \circ (\Phi_A a) = \mathrm{Ext}(\Phi_A a) \circ j_E = \Phi_{\mathcal{A}}(i_A(a)) \circ j_E \quad \text{for all } a \in A,$$

which is precisely the statement that j_E is A-linear (we must identify A with $i_A(A)$ for A-linearity to be a sensible notion here).

As for uniqueness of the action $\Phi_{\mathcal{A}}$, we note that j_E-linearity, i.e. the requirement that

$$j_E \circ (\Phi_A a) = \Phi_{\mathcal{A}}(i_A(a)) \circ j_E \quad \text{for all } a \in A,$$

along with the fact that $\Phi_{\mathcal{A}}$ must be bounded (by Lemma 4.3.6), implies that $\Phi_{\mathcal{A}}(i_A(a)) = \mathrm{Ext}(\Phi_A a)$ for all $a \in A$. This shows that our definition of $\Phi_{\mathcal{A}}$ is forced upon us.

Part 2: Extending the inner product

The inner product ${}_A\langle\cdot,\cdot\rangle\colon E \times E \to A$ is a sesquilinear map that is bounded by Proposition 4.3.4, so it has a unique bounded sesquilinear extension $\overline{E} \times \overline{E} \to \mathcal{A}$ (see Theorem B.1.6 and the subsequent discussion).

The requirement that $j_E\colon E \to \overline{E}$ should preserve the inner product is precisely the requirement that ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ should be an extension of ${}_{A}\langle\cdot,\cdot\rangle$:

$$ {}_{\mathcal{A}}\langle j_E(f), j_E(g)\rangle = i_A\big({}_A\langle f, g\rangle\big) \quad \text{for all } f, g \in E. $$

Since any $\mathcal{A}$-valued inner product on $\overline{E}$ must be bounded by Proposition 4.3.4, our inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ must be the extensions afforded by Theorem B.1.6. We need to verify that it satisfies the axioms of an $\mathcal{A}$-valued inner product (Definition 4.3.1).

We begin with A-linearity (this will be a two-step procedure similar to our proof of associativity for Banach algebra completions). Fix any $a \in A$. The maps

$$ \begin{aligned} B_1\colon \overline{E}\times\overline{E} &\to \mathcal{A} \\ (f,g) &\mapsto i_A(a)\,{}_{\mathcal{A}}\langle f, g\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} B_2\colon \overline{E}\times\overline{E} &\to \mathcal{A} \\ (f,g) &\mapsto {}_{\mathcal{A}}\big\langle i_A(a)\cdot f, g\big\rangle \end{aligned} $$

are both bounded sesquilinear extensions of the bounded sesquilinear map $E \times E \to A$ defined by $(f, g) \mapsto a\,{}_A\langle f, g\rangle = {}_A\langle a \cdot f, g\rangle$, because

$$ \begin{aligned} B_1\big(j_E(f), j_E(g)\big) &= i_A(a)\,{}_{\mathcal{A}}\big\langle j_E(f), j_E(g)\big\rangle = i_A(a) i_A\big({}_A\langle f, g\rangle\big) \\ &= i_A\big(a\,{}_A\langle f, g\rangle\big) \\ \text{and}\quad B_2\big(j_E(f), j_E(g)\big) &= {}_{\mathcal{A}}\big\langle i_A(a)\cdot j_E(f), j_E(g)\big\rangle = {}_{\mathcal{A}}\big\langle j_E(a\cdot f), j_E(g)\big\rangle \\ &= i_A\big({}_A\langle a\cdot f, g\rangle\big). \end{aligned} $$

By Theorem B.1.6, we can conclude that $B_1 = B_2$. This only gives us $i_A(A)$-linearity. However, if we now fix $f, g \in \overline{E}$, then the maps

$$ \begin{aligned} \mathcal{A} &\to \mathcal{A} \\ a &\mapsto {}_{\mathcal{A}}\langle a\cdot f, g\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{A} &\to \mathcal{A} \\ a &\mapsto a\,{}_{\mathcal{A}}\langle f, g\rangle \end{aligned} $$

are two bounded linear extensions of the bounded linear map $A \to \mathcal{A}$ defined by $a \mapsto {}_{\mathcal{A}}\langle i_A(a)\cdot f, g\rangle = i_A(a)\,{}_{\mathcal{A}}\langle f, g\rangle$, so they are equal, which concludes the proof of $\mathcal{A}$-linearity.

To show conjugate symmetry, we apply the standard argument to the bounded sesquilinear map $E \times E \to A$ defined by $(f, g) \to {}_A\langle g, f\rangle = {}_A\langle f, g\rangle^*$ and its two bounded sesquilinear extensions:

$$ \begin{aligned} \overline{E}\times\overline{E} &\to \mathcal{A} \\ (f,g) &\mapsto {}_{\mathcal{A}}\langle g, f\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{E}\times\overline{E} &\to \mathcal{A} \\ (f,g) &\mapsto {}_{\mathcal{A}}\langle f, g\rangle^*. \end{aligned} $$

For positivty, we resort to approximating sequences: let $f \in \overline{E}$ and choose a sequence $(f_n) \subset E$ such that $j_E(f_n) \to f$. We know that

$$ {}_{\mathcal{A}}\big\langle j_E(f_n), j_E(f_n)\big\rangle = i_A\big({}_A\langle f_n, f_n\rangle\big) \geq 0 \quad \text{for all } n \in \mathbb{N} $$

by positivity in the A-module E (recall that we require positivity in the C*-algebra completion of A). Moreover, we know that ${}_{\mathcal{A}}\langle j_E(f_n), j_E(f_n)\rangle \to {}_{\mathcal{A}}\langle f, f\rangle$ by boundedness of the claimed inner product (the proof is identical to that of Corollary 4.3.5; we only need sesquilinearity and boundedness). Thus, positivity follows if we can show that a limit of positive elements in $\mathcal{A}$ is positive.

Suppose that $(a_n) \subset \mathcal{A}^+$ and that $a_n \to a \in \mathcal{A}$. By point (ii) of Lemma 2.2.22, we know that

$$\big\| \|a_n\| 1_{\mathcal{A}} - a_n \big\| \leq \|a_n\| \quad \text{for all } n \in \mathbb{N},$$

from which taking the limit gives $\| \|a\| 1_{\mathcal{A}} - a\| \leq \|a\|$. We already know that $a^* = a$ by continuity of the involution, so by point (i) of the same lemma, we find that $a \geq 0$, as desired.

We will show nondegeneracy, but first we show that the Banach space completion norm on $\overline{E}$, say $\|\cdot\|_B$, agrees with the norm determined by ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$. Appealing again to the continuity of ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$, as well as the continuity of all norms involved, we find that (whenever $j_E(f_n) \to f \in \overline{E}$)

$$\begin{aligned} \|f\|_B^2 &= \lim_{n\to\infty} \|j_E(f_n)\|_B^2 = \lim_{n\to\infty} \|f_n\|^2 = \lim_{n\to\infty} \|{}_A\langle f_n, f_n\rangle\| \\ &= \lim_{n\to\infty} \big\| i_A\big({}_A\langle f_n, f_n\rangle\big)\big\| = \lim_{n\to\infty} \big\|{}_{\mathcal{A}}\big\langle j_E(f_n), j_E(f_n)\big\rangle\big\| = \|{}_{\mathcal{A}}\langle f, f\rangle\|, \end{aligned}$$

which proves the claimed equality of norms.

Nondegeneracy of the Banach space completion norm on $\overline{E}$ and the norm on $\mathcal{A}$ now gives:

$$f \neq 0 \quad \Longrightarrow \quad \|{}_{\mathcal{A}}\langle f, f\rangle\| = \|f\|_B^2 \neq 0 \quad \Longrightarrow \quad {}_{\mathcal{A}}\langle f, f\rangle \neq 0.$$

This shows nondegeneracy of ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ and concludes the proof. □

There are many results one could formulate with regard to extensions of bounded maps between inner product modules. All we have need for is the following result, which essentially asserts that isomorphisms between inner product modules extend to isomorphisms between Hilbert C*-modules. For the notion of an isomorphism between Hilbert C*-modules, i.e. a *unitary* map, see Definition 4.3.18.

B.3.2 Proposition. *Let A be a pre-C*-algebra and let E and F be two left inner product A-modules. Suppose that $U\colon E \to F$ is an A-linear bijection preserving the inner product. Let $\overline{E}$ and $\overline{F}$ be Hilbert C*-module completions of E and F. Then, $U \in \mathcal{B}(E, F)$, and the unique bounded linear extension of U is a unitary map $\overline{U} \in \mathcal{L}(\overline{E}, \overline{F})$.*

Proof. Since U preserves the inner product, it must be isometric: $\|Uf\|^2 = \|\langle Uf, Uf\rangle\| = \|\langle f, f\rangle\| = \|f\|^2$ for all $f \in E$. Since A-linearity implies $\mathbb{C}$-linearity, we can conclude that $U \in \mathcal{B}(E, F)$. Thus, U has a unique bounded linear extension $\overline{U} \colon \overline{E} \to \overline{F}$. As a bijective isometry, its inverse has a bounded linear extension as well.

Preservation of the inner product also implies that

$$\langle f, U^{-1}g\rangle = \langle Uf, U(U^{-1}g)\rangle = \langle Uf, g\rangle \quad \text{for all } f \in E \text{ and } g \in F.$$

This means that the bounded sesquilinear maps

$$\begin{aligned} \overline{E} \times \overline{F} &\to \mathcal{A} \\ (f, g) &\mapsto \langle f, \overline{U^{-1}}g\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{E} \times \overline{F} &\to \mathcal{A} \\ (f, g) &\mapsto \langle \overline{U}f, g\rangle \end{aligned}$$

are both extensions of the bounded sesquilinear map $E \times F \to A$ defined by $(f, g) \mapsto \langle f, U^{-1}g\rangle = \langle Uf, g\rangle$, so they must be equal. This proves that $\overline{U}$ is an adjointable map with adjoint $\overline{U^{-1}}$ and $\mathcal{A}$-linearity now follows from Lemma 4.3.10. Finally, $\overline{U}^* = \overline{U^{-1}} = \overline{U}^{-1}$ by Proposition B.1.4. This shows that $\overline{U} \in \mathcal{L}(E, F)$ is unitary and concludes the proof. □

As a corollary, we obtain uniqueness of Hilbert C*-module completions.

B.3.3 Corollary (Uniqueness of Hilbert C*-module completions)**.** *Let A be a pre-C*-algebra and let E be a left inner product A-module. If $j_E \colon E \to \overline{E}$ and $k_E \colon E \to \widetilde{E}$ are two Hilbert C*-module completions of E, then there exists a unique unitary map $U \in \mathcal{L}(\overline{E}, \widetilde{E})$ such that $k_E = U \circ j_E$.*

Proof. The argument is the same as always. A unique linear isometry $U \colon \overline{E} \to \widetilde{E}$ such that $k_E = U \circ j_E$ exists by uniqueness of Banach space completions. The equation $k_E = U \circ j_E$ means that U is an extension of the identity $\mathrm{Id}_E \colon E \to E$ (with appropriate choices of completions) and Proposition B.3.2 then implies that U is unitary. □

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