Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Maren Bråthen Kristoffersen

Proof of the Equivalence of Weak Solutions of the Spatial and the Referential Formulations of Balance Laws

Master's thesis in Applied Physics and Mathematics
Supervisor: Helge Holden
June 2023

NTNU
Norwegian University of Science and Technology

Maren Bråthen Kristoffersen

Proof of the Equivalence of Weak Solutions of the Spatial and the Referential Formulations of Balance Laws

Master's thesis in Applied Physics and Mathematics
Supervisor: Helge Holden
June 2023

Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

Preface

This master thesis marks the end of the five-year study program Applied Physics and Mathematics at Norwegian University of Science and Technology (NTNU) in Trondheim with the specialization Industrial Mathematics. My supervisor for this thesis was Professor Helge Holden.

Firstly, I would like to give my acknowledgment to my supervisor Professor Helge Holden for an interesting topic for my master thesis and for every useful discussion and helpful guidance throughout my thesis work. This have increased my knowledge and interest for analysis of conservation laws. Secondly, I would like to thank my friends and family for all the support. Especially, my parents for both proof reading the thesis and always listening to me when I needed someone to talk to.

Maren Bråthen Kristoffersen
June 2023
Trondheim

Abstract

For an extensive physical variable we can formulate a general balance law based on the principle that the production of the physical variable inside a domain is balanced by the flux over the boundary of said domain. Furthermore, the balance law can be formulated for different coordinate systems. The most common choices for coordinates systems are Euler and Lagrange coordinates. The formulation of conservation of mass, momentum, and energy in Euler and Lagrange coordinates will result in the Euler and the Lagrange equations. The main objective of this master thesis is to increase the understanding about the proof of the equivalence between weak solutions of balance laws in Euler and Lagrange coordinates. These equations are known for admitting discontinuous solutions in finite time, even for smooth initial conditions. Thus, throughout the master thesis we work with weak solutions and distributions.

This master thesis consists of two main proofs. The first proof is that a general formulation of a balance law is preserved under a bi-Lipschitz change of coordinates. This proof consists of three main steps. First, we show that we can reduce a general balance law to a field equation. Next, we show that the field equation is preserved under a bi-Lipschitz change of coordinates. Lastly, we show that we can obtain the original formulation of the balance law from the field equation. The second main proof is the equivalence between the weak solutions of the one-dimensional Euler equations and the one-dimensional Lagrange equations. We start by assuming no vacuum, and use the theory proved in the first part of the thesis to show equivalence. Thereafter we show that the equivalence still holds for solutions with vacuum. As a part of this proof we have to strengthen the definition of a weak solution in Lagrange coordinates.

Sammendrag

Vi kan utlede en generell balanselov for alle ekstensive fysiske variabler ved å bruke prinsippet om at produksjonen av variabelen i et domene er balansert av fluksen av den samme variabelen over randen til domenet. Det er flere mulige valg for koordinatsystem når man utleder en balanselov, men de vanligste er Euler- og Lagrangekoordinater. Hvis man utleder masse-, impuls- og energibevarelse i Euler- og i Lagrangekoordinater vil det resultere i Eulerlikningene og Lagrangelikningene. Hovedmålet for denne masteroppgaven er å øke forståelsen om beviset av ekvivalensen mellom svake løsninger av balanselover i Euler- og Lagrangekoordinater. Disse likningene er kjent for å resultere i diskontinuerlige løsninger i løpet av endelig tid, selv for glatte initialbetingelser. Dermed jobber vi med svake løsninger og distribusjoner gjennom hele masteroppgaven.

Denne masteroppgaven består av to hovedbevis. Det første beviset er at en generell balanselov er bevart under et bi-Lipschitz variabelskift. Dette beviset består av tre hovedsteg. Vi starter med å vise at en generell balanselov kan reduseres til en feltlikning. Deretter viser vi at feltlikningen er bevart under et bi-Lipschitz variabelskifte. Til slutt viser vi at det er mulig å oppnå den opprinnelige formuleringen av balanseloven fra feltlikningen. Det andre hovedbeviset er ekvivalensen mellom de svake løsningene av de endimensjonale Euler- og Lagrangelikningene. Vi starter med å bruke teorien vist i det første beviset til å vise ekvivalensen når vi antar at løsningene er vakuumfrie. Deretter, viser vi at ekvivalensen fortsatt holder hvis vi tillater løsninger med vakuum. En viktig del av dette beviset er at vi må innføre en ny definisjon av svake løsninger i Lagrangekoordinater.

Table of Contents

Notation

ρ Mass density.

τ Specific volume.

u Velocity.

S Specific entropy.

e Specific internal energy.

p Pressure.

$\partial\Omega$ The boundary of Ω.

$\boldsymbol{N}$ Outward unit normal.

$\boldsymbol{E}_i$ The ith unit base-vector.

$\mathbf{K}$ Compact subspace.

$\mathcal{B}_r(\boldsymbol{x})$ Open ball in $\mathbb{R}^n$ with center at $\boldsymbol{x}$ and radius r, $\mathcal{B}_r(\boldsymbol{x}) = \{\boldsymbol{a} \in \mathbb{R}^n : |\boldsymbol{x} - \boldsymbol{a}| < r\}$. If the ball is centered at $\boldsymbol{x} = \boldsymbol{0}$ we write $\mathcal{B}_r$.

$\mathcal{B}$ An arbitrary ball in $\mathbb{R}^n$.

$\overline{A}$ Let A be a set, then $\overline{A}$ is the closure of that set.

A^c The complement of the set A, $A^c = \mathbb{R}^n \setminus A = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \notin A\}$.

$\mathbb{R}^+$ All the non-negative real numbers, $\mathbb{R}^+ = [0, \infty]$.

$\overline{\mathbb{R}}$ The set of all real numbers including infinity and negative infinity, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

$\mathbb{M}^{n\times k}$ The vector space of $n \times k$ matrices. When $k = 1$ will $\mathbb{M}^{n\times 1} = \mathbb{R}^n$.

$C^i(\Omega)$ The function space with i-differentiable functions defined over Ω. If $i = \infty$ this is the function space with infinitely differentiable functions, i.e., $C^\infty = \bigcap_{i=0}^{\infty} C^i$. If the set Ω is obvious we may only write C^i.

$C_c^i(\Omega)$ The function space with i-differentiable functions with compact support over Ω. If the set Ω is obvious we may only write C_c^i.

$L^p(\Omega; \mu)$ The function space with p-integrable functions, i.e., $f \in L^p(\Omega)$ if $\int_\Omega |f|^p d\mu < \infty$. For $p = \infty$, $f \in L^\infty(\Omega)$ if $\sup_{x\in\Omega} |f| < \infty$. If the set Ω and the measure μ is obvious we may only write L^p.

$L^1_{loc}(\Omega;\mu)$ The function space with locally integrable functions, i.e., $f \in L^1_{loc}(\Omega)$ if $\int_{\mathbf{K}} |f| d\mu < \infty$ for all $\mathbf{K}$ compact subset of Ω. If the set Ω and the measure μ is obvious we may only write L^1_{loc}.

$W^{k,p}(\Omega)$ The Sobolev space where k is an integer and $1 \leq p \leq \infty$, $W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \forall |c| \leq k, \frac{\partial^c f}{\partial^{c_1} x_1 \ldots \partial^{c_n} x_n} \in L^p(\Omega)\}$, where $c = (c_1, \ldots c_n)$, $|c| = \sum_{i=1}^n c_i$, and the derivation holds in the weak sense.

f_x The partial derivative of f with respect to x, $f_x = \partial_x f = \frac{\partial f}{\partial x}$.

$\text{div}_{\boldsymbol{x}}\, \boldsymbol{f}$ The divergence of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, $\text{div}_{\boldsymbol{x}} \boldsymbol{f} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$. If it is obvious what we differentiate with respect to we may write $\text{div}\boldsymbol{f}$.

$\nabla_{\boldsymbol{x}} f$ The gradient of f with respect to $\boldsymbol{x}$, $\nabla_{\boldsymbol{x}} f = \text{grad}_{\boldsymbol{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \ldots & \frac{\partial f}{\partial x_n} \end{bmatrix}$. If it is obvious what we differentiate with respect to we may write ∇f.

$\nabla^2_{\boldsymbol{x}} f$ The Hessian matrix of function f with respect to $\boldsymbol{x}$, $\left(\nabla^2_{\boldsymbol{x}} f\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. If it is obvious what we differentiate with respect to we may write $\nabla^2 f$.

$\text{curl}\, \boldsymbol{f}$ The vector operator describing the infinitesimal circulation of a vector field in $\mathbb{R}^3$, $\text{curl}\boldsymbol{f} = \nabla \times \boldsymbol{f} = [\frac{\partial f_3}{x_2} - \frac{\partial f_2}{x_3}, \frac{\partial f_1}{x_3} - \frac{\partial f_3}{x_1}, \frac{\partial f_2}{x_1} - \frac{\partial f_1}{x_2}]$.

JF The Jacobian matrix for the function F, $(JF)_{ij} = \frac{\partial F_i}{\partial x_j}$.

$\det JF$ The determinant of the Jacobian matrix for F.

$f|_{t=t_0}$ The function at $t = t_0$.

$(f \circ g)(x)$ The composition of f and g, $(f \circ g)(x) = f(g(x))$.

$(f * g)(x)$ The convolution of f and g, $(f * g)(x) = \int f(x-y)g(y)dy = \int g(x-y)f(y)dy$.

$\sup_{x \in \Omega} f(x)$ The supremum of the function f over the domain Ω, which is the least upper bound for $f(x)$ when $x \in \Omega$.

$\inf_{x \in \Omega} f(x)$ The infimum of the function f over the domain Ω, which is the greatest lower bound for $f(x)$ when $x \in \Omega$.

$\text{supp}\, f$ The support of the function f, i.e., the set of points x such that $f(x)$ is non-zero, $\text{supp} f = \{x : f(x) \neq 0\}$.

$f(A)$ The image of the function f, $f(A) = \{f(x) : x \in A\}$.

$f^{-1}(A)$ The preimage of the function f, $f^{-1}(A) = \{x : f(x) \in A\}$.

$\text{diam}(A)$ The diameter for the set A, $\text{diam}(A) = \sup\{|\boldsymbol{x} - \boldsymbol{y}| : \boldsymbol{x}, \boldsymbol{y} \in A\}$.

$l_E f(\boldsymbol{x}_0)$ The approximate limit of the function f in E, which means that for all $\varepsilon > 0$ the point $\boldsymbol{x}_0$ is a point of rarefaction of the set $\{\boldsymbol{x} \in E : |f(\boldsymbol{x}) - l_E f(\boldsymbol{x}_0)| > \varepsilon\}$. If $E = \mathbb{R}^n$ we denote the approximate limit by $lf(\boldsymbol{x}_0)$ and if E is the half space $(\boldsymbol{x} - \boldsymbol{x}_0) \cdot a > 0$ we denote the approximate limit by $l_a f(\boldsymbol{x}_0)$.

$\widehat{f}(u(x))$ The functional superposition of the function f, $\widehat{f}(u(x)) = \int_0^1 f\big(l_a u(x) t + l_{-a} u(x)(1-t)\big) dt$.

$|\cdot|$ The Euclidean norm in $\mathbb{R}^n$, $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$.

$\|\cdot\|_{L^p}$ The L^p norm of a function for $1 \leq p < \infty$, $\|f\|_{L^p} = \left(\int |f|^p dx\right)^{1/p}$.

$\|\cdot\|_\infty$ The L^∞ norm of a function, $\|f\|_\infty = \|f\|_{L^\infty} = \sup_x |f(x)|$.

$\boldsymbol{v} \cdot \boldsymbol{w}$ The scalar product between the vector $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$ and $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]$, $\boldsymbol{v} \cdot \boldsymbol{w} = \sum_i v_i w_i$.

$\alpha(n)$ The volume of the unit ball $\mathcal{B}_1$ in $\mathbb{R}^n$.

μ A general measure.

m_n The Lebesgue measure in $\mathbb{R}^n$.

$\mathcal{H}^s$ The s-dimensional Hausdorff measure in $\mathbb{R}^n$.

$\mu \llcorner A$ The measure μ restricted to A, $\mu \llcorner A(E) = \mu(A \cap E)$.

$\nu \ll \mu$ The measure ν is absolutely continuous with respect to the measure μ, i.e., $\nu(A) = 0$ for all A with $\mu(A)$.

$\nu \perp \mu$ The measures ν and μ are mutually singular, i.e., there exists a Borel set $B \subset \Omega$ such that $\mu(\Omega \setminus B) = \nu(B) = 0$.

$f_\# \mu$ The push forward measure constructed by the function f and measure μ, $f_\# \mu(A) = \mu(f^{-1}(A))$.

χ_A The characteristic function for the set A, $\chi_A(x) = 1$ if $x \in A$ and zero otherwise.

a.e. almost everywhere.

Chapter 1

Introduction

The purpose of this thesis is to increase the understanding of the proof of the equivalence between the weak solutions of the spatial and the referential formulation of a balance law.

1.1 Background

1.1.1 Short historical background

The theory of continuum physics and conservation laws has a rich history, with its beginning in the 18th century, and still to this day, it is an active topic of research. In the beginning most of the research was intertwined with gas dynamics. For instance, the oldest system of conservation laws is the Euler equations for barotropic gas flow [8, p. XVIII]. Furthermore, in this period of time, Euler introduced the referential description of motion, which later has been named Lagrange description. In addition, the spatial, or Euler, description of motion was conceived by d'Alembert and Daniel Bernoulli [8, Section 2.9]. In the beginning of the 19th century the principles of the theory of general balance laws were introduced and stemmed from the development of the theory of elasticity. Throughout the next 150 years the theory was thoroughly investigated by mathematicians, physicists, chemists, and engineers [8, p. XXX]. In 1948 Courant and Friedrichs gathered the theory about balance laws to write the book *Supersonic Flow and Shock Waves* [5] in the language of mathematics. Furthermore, in 1987 Wagner showed the equivalence of the spatial and the referential formulations of the one-dimensional Euler equations for a compressible, inviscid, and non-heat-conducting gas [31]. Thereafter, Dafermos wrote the article *Equivalence of referential and spatial field equations in continuum physics* [7] in 1993, where he showed the equivalence of Lagrange and Euler formulation of a multidimensional conservation law. Later, he continued his work in the book *Hyperbolic Conservation Laws in Continuum Physics* [8], where he, among other topics, proved that a general balance law is preserved under a bi-Lipschitz change of coordinates.

1.1.2 Spatial and referential formulation

In continuum physics, there are two main choices for point of view when deriving physical laws. The first one is the spatial, also called Euler, point of view, where we observe the physical system from a fixed external frame of reference. Thus, the independent variables will be the spatial coordinates and time [27, p. 4]. The other choice for point of view is the referential, also called Lagrange, point of view, where we describe the physical system from the perspective of particles in the system and every particle is labeled by the position at time $t = 0$. So, the independent variables will be the label of the particles and time [27, p. 4]. Both points of view have advantages and disadvantages, which mean that in some cases one point of view is more preferable than the other. For instance, Euler point of view is useful when we want to study the overall motion of a system as a whole. However, Lagrange point of view will be a better choice if we want to analyze local phenomena and understand the interaction between and behavior of individual particles. Furthermore, Newton's laws are formulated from the perspective of a particle and thus conservation laws derived from Newton's laws will be in Lagrange representation. However, most of the mathematical theory is developed in Euler coordinates. This is due to the Lagrange equations becoming highly non-linear in higher dimensions [11]. In addition, in Lagrange representation it is easy to retrieve the trajectories of a particle, and thus compute the acceleration by taking the second time derivative of the position. Using this and Newton's second law, we can easily find the force acting on the fluid [24, Chapter 2.1].

As we can see from the examples above, in some cases it is more preferable to use Euler point of view over Lagrange and sometimes the opposite is true. This is one reason it is interesting to prove the equivalence of the two points of view. The equivalence will allow us to use the description that works best for the particular problem we are facing. In addition, we can construct methods based on a combination of Euler and Lagrange description, where we use the strengths from both descriptions. Furthermore, the equivalence will give use an increased theoretical understanding of the physical properties, since Euler and Lagrange formulations give us different insight to the physical properties. It will result in a connection between the macroscopic understanding from Euler point of view and the microscopic understanding from Lagrange point of view.

1.1.3 Balance laws

We have two types of physical properties, extensive and intensive. The physical properties that are dependent on the amount of material are called extensive properties and for these we can formulate balance laws. The idea behind balance laws, is that nothing can be created or destroyed. Another way to formulate this is that the production of a quantity in a domain has to be balanced by the flux of said quantity over the boundary of the domain. We can write the general balance law for a physical quantity on a domain $\mathcal{D}$ as

$$Q_{\mathcal{D}}(\partial\mathcal{D}) = P(\mathcal{D}), \tag{1.1}$$

where $Q_{\mathcal{D}}$ describes the flux over the boundary and P describes the production within the domain [8, Chapter 1.1]. Furthermore, three of the most used balance laws are the conservation of mass, momentum, and energy. The derivation of these conservation laws is based on the following three principles. [3, Chapter 1.1]

- Mass can neither be destroyed nor created.
- The rate of change of the momentum on a fluid is equal to the forces applied to it.
- Energy can neither be destroyed nor created.

For a one-dimensional compressible, inviscid, non-heat-conducting gas the conservation of mass, momentum, and energy from Euler point of view will result in the Euler equations given by [31, Equations (1.1)]

$$\rho_t + (\rho u)_x = 0, \tag{1.2a}$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{1.2b}$$
$$\Big(\rho e + \frac{1}{2}\rho u^2\Big)_t + \Big(u\Big(\rho e + \frac{1}{2}\rho u^2 + p\Big)\Big)_x = 0. \tag{1.2c}$$

Furthermore, from Lagrange point of view the conservation of mass, momentum, and energy, for the same gas, will result in the Lagrange equations given by [31, Equation (1.3)]

$$\tau_t - u_y = 0, \tag{1.3a}$$
$$u_t + p_y = 0, \tag{1.3b}$$
$$\Big(e + \frac{1}{2}u^2\Big)_t + \Big(pu\Big)_y = 0, \tag{1.3c}$$

where $\tau = 1/\rho$.

The partial differential equations presented in this section are known for developing discontinuities. Thus, we often have to consider weak solutions for the systems of partial differential equations.

1.2 Research question

In the Chapters 1.1-1.3 in the book *Hyperbolic conservation laws in continuum physics* [8] Dafermos shows that we can reduce the general balance law (1.1) to a field equation

$$\operatorname{div} A = P. \tag{1.4}$$

Furthermore, he shows that the conservation law written as a field equation is preserved under a bi-Lipschitz change of coordinates. Lastly, he shows that we can retrieve the original general formulation of the balance law from a field equation.

In the article *Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions* by Wagner [31] he proves that the Euler and Lagrange equations for a one-dimensional compressible, inviscid, non-heat-conduction gas are equivalent. First, he proves the equivalence when he assumes that the there is no vacuum. Thereafter, he proves that if he strengthens the definition of a weak solution in Lagrange coordinates the equivalence still holds when he allows there to be vacuum.

In both the chapters by Dafermos and the article by Wagner most of the intermediate calculations are skipped. Understanding the intermediate calculations in detail will increase our understanding of the two formulations of the balance laws, and developing these calculations is our main research goal.

1.3 Proof outline

To answer the research question we are going to start by following Dafermos' proofs to show that a general balance law is conserved under a bi-Lipschitz coordinate. As a part of these we will use multiple properties of Radon and Hausdorff measures. Furthermore, we will show that the theory by Dafermos and the theory by Wagner coincide for the one-dimensional Euler equations when the solutions are without vacuum. Lastly, we will show that the equivalence still holds in some cases if we allow the solutions to include vacuum. Here we will follow the proof done by Wagner. Among other things we will introduce a stronger definition for weak solutions in Lagrange coordinates, which ensures the test function in Lagrange coordinates to be discontinuous in the vacuum set. Throughout both parts of the thesis we will use weak formulations and standard mollifiers since we will be considering non-differentiable solutions of the balance laws.

1.4 Thesis outline

This thesis is composed of the following chapters, of which Chapter 4 and 5 are the main contribution of this work. Some of the sections and chapter in this thesis are based on sections in my project thesis [20]. Which sections and chapters are specified in the list below.

- *Chapter 1: Introduction.* In this chapter we present some motivation for the importance of the results presented in this thesis, by providing some background information about Euler and Lagrange description and balance laws. Additionally, we present the research question for the thesis and the proof outline.
- *Chapter 2: Theoretical background.* This chapter includes some useful theorems and definitions from measure theory, about convergence, about mollifiers, about weak formulation of conservation laws and from the theory of functions of bounded variations. These theorems and definitions will be important for the main proofs in this thesis. Section 2.4 is based on Section 2.3.1 and Theorem 4.9 in [20]. However, it is rewritten to be consider a multidimensional balance law.
- *Chapter 3: Balance laws.* Here we derive the formulation of a general balance law and the conservation of mass, momentum, and energy for a compressible, inviscid, non-heat-conducting gas from both Euler and Lagrange point of view. This chapter is taken from Section 2.2 in [20] and is included for completeness.
- *Chapter 4: Multidimensional balance law.* The proof that a general formulation of a balance law can be reduced to a field equation and that this field equation is conserved under a bi-Lipschitz change of coordinates is found in this chapter. In addition, we show that we can retrieve the original general formulation of the balance law from a field equation.
- *Chapter 5: One-dimensional Euler and Lagrange equations.* In this chapter we start by showing that we can use the main theorem from Chapter 4 to show that the Euler and Lagrange equations are equivalent when we consider solutions without vacuum. Then we proceed to show that in some cases the equivalence still holds true when we allow there to be vacuum. Section 5.1.1 is based on Section 4.1 in [20] and included for completeness.
- *Chapter 6: Conclusion and future research.* This chapter includes some concluding remarks. In addition, it includes two suggestions for future research, which are BV transformations of a general balance law and entropy solutions.

Chapter 2

Theoretical background

In this chapter we present some mathematical theory that is useful to understand before tackling the main proofs in this thesis. It includes topics in measure theory, convergence, mollifiers, weak solutions, and functions of bounded variation. The section about weak solutions is based on sections from my project thesis [20].

2.1 Measure theory

We start this section with some fundamental definitions and theorems from measure theory, before defining Radon and Hausdorff measure and stating some useful theorems and lemmas.

Let X be a nonempty set and denote the set of all subsets of X by 2^X. We start by defining a σ-algebra. From [9, Definition 1.4] we have the following definition.

Definition 2.1 (σ-algebra)**.** A collection of subsets $\mathcal{A} \subseteq 2^X$ is called a *σ-algebra* if it satisfies

(i) $\varnothing, X \in \mathcal{A}$,

(ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$,

(iii) $A_k \in \mathcal{A}, k \in \mathbb{N}$ implies $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$,

(iv) $A_k \in \mathcal{A}, k \in \mathbb{N}$ implies $\bigcap_{k\in\mathbb{N}} A_k \in \mathcal{A}$.

The smallest σ-algebra containing all open subsets is called a *Borel σ-algebra* and a set in a *Borel σ-algebra* is called a *Borel set.* Next, we define some useful properties of set functions.

Definition 2.2 (Properties of a set function [13, p. 283])**.** Let μ be a set functions from $\mathcal{A} \subseteq 2^X$ to $\overline{\mathbb{R}}$. We say that

(i) μ is *monotone* if for every $A, B \in \mathcal{A}$ with $A \subset B$ we have $\mu(A) \leq \mu(B)$.

(ii) μ is *additive* if for all finite collections of pairwise disjoint subsets $A_k \in \mathcal{A}$, $k = 1, \dots, N$ we have $\mu\big(\cup_{k=1}^N A_k\big) = \sum_{k=1}^N A_k$. If the equality also holds for all countable infinite collection of subsets, we say that μ is *countable additive.*

(iii) μ is *countable subadditive* if for all countable collections of subsets $A_k \in \mathcal{A}$, $k \in \mathbb{N}$ we have $\mu\big(\cup_{k\in\mathbb{N}} A_k\big) \leq \sum_{k\in\mathbb{N}} \mu(A_k)$.

Furthermore, an outer measure is defined as follows.

Definition 2.3 (Outer measure)**.** A set function $\mu : 2^X \to \overline{\mathbb{R}}$ that satisfies $\mu(\varnothing) = 0$, and is monotone and subadditive is a *signed outer measure.* If μ only takes positive values, i.e., $\mu : 2^X \to [0, \infty]$, we say that μ is an *unsigned outer measure.*

For simplicity, in the rest of this thesis we will say measure when we are referring to an outer measure.

Definition 2.4 (Borel measure [23, Definition 1.5(2) and 1.5(3)])**.** We say that a measure μ is a *Borel measure* if every Borel set is a μ-measurable set. Furthermore, a Borel measure μ is *Borel regular* if there for every $A \subseteq \mathbb{R}^n$ exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

From [10, p. 54] we have the following criterion for a measurable subset.

Proposition 2.1 (Measurable subset)**.** *We say that a subset A is* measurable *if it satisfies*

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \quad \forall B \subseteq X.$$

The following formulation of Fubini's Theorem can be found in [10, Section 2.6.2].

Theorem 2.2 (Fubini's Theorem)**.** [1] *Suppose α is a measure on X, and β is a measure on Y. If f is an $\alpha \times \beta$ integrable, i.e., f has finite $\alpha \times \beta$ integral, then*

$$\int f d(\alpha \times \beta) = \iint f(x, y) d\alpha(x) d\beta(y) = \iint f(x, y) d\beta(y) d\alpha(x).$$

Proof. See [10, Section 2.6.2]. □

Lastly, we define simple functions and μ-measurable functions. In [9, Definition 1.10] measurable function is defined as

Definition 2.5 (Measurable function)**.** A function $f : X \to Y$ is said to be a *μ-measurable function* if for all open sets $U \subset Y$ the set $f^{-1}(U)$ is a μ-measurable set.

In [13, Section 5.2.1b] we find following definition of a simple function.

Definition 2.6 (Simple function)**.** A function $f : X \to \overline{\mathbb{R}}$ is called a *simple function* if it takes finitely many finite distinct values $a_1, \ldots, a_N$. Thus, we can write a simple function as

$$f(x) = \sum_{i=1}^{N} a_i \chi_{E_i},$$

where $E_i := \{x \in X : f(x) = a_i\}$ and χ_E is the characteristic function. A simple function is μ-measurable if and only if all E_i are μ-measurable sets.

Furthermore, we have the following useful lemma from [13, Lemma 5.45].

[1] We have only included (4) from the theorem in [10].

Lemma 2.3 (Approximation of a simple function)**.** *A non-negative function* $g : X \to \mathbb{R}^+ \cup \infty$ *is a* μ*-measurable function if and only if there exists a non-decreasing sequence of* μ*-measurable simple functions* $\{f_k\}$ *such that* $f_k(x) \to g(x)$ *pointwise.*

Proof. See [13, p. 308]. □

In this thesis we will consider the case where $X = \mathbb{R}^n$, thus for the rest of this chapter we will define Radon and Hausdorff measure and state theorems and lemmas for the case when $X = \mathbb{R}^n$.

2.1.1 Radon measure

In [10, Section 2.2.5] we find the following definition of a Radon measure.

Definition 2.7 (Radon measure)**.** A Borel measure μ on $\mathbb{R}^n$ is said to be a *Radon measure* on $\mathbb{R}^n$ if it satisfies that

(i) All compact subsets $\mathbf{K}$ have finite measure, i.e., $\mu(\mathbf{K}) < \infty$

(ii) For all $A \subset \mathbb{R}^n$ we have

$$\mu(A) = \inf\{\mu(U) : U \supset A, U \text{ open}\}.$$

This property is called *outer regularity.*

(iii) For all $U \subset \mathbb{R}^n$ open, U is a μ-measurable and

$$\mu(U) = \sup\{\mu(\mathbf{K}) : \mathbf{K} \subset U, \mathbf{K} \text{ compact}\}.$$

This property is called *inner regularity.*

Next, we define what we mean by a measure restricted to A.

Definition 2.8 (Restricted to A [9, Definition 1.2])**.** Let μ be a measure on $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. We say $\mu \llcorner A$ is μ *restricted to* A and define it by

$$(\mu \llcorner A)(B) := \mu(A \cap B) \quad \forall B \subseteq \mathbf{R}^n.$$

Furthermore, in [23, Theorem 1.9] we find the following theorem about $\mu \llcorner A$.

Theorem 2.4. *Let* $A \subseteq \mathbb{R}^n$*,* μ *be the measure and let* $\mu \llcorner A$ *be* μ *restricted to* A*. Then we have*

(i) If E *is a* μ*-measurable set,* E *is also a* $\mu \llcorner A$*-measurable set.*

(ii) If μ *is a Borel regular measure and* $\mu(A) < \infty$*, then* $\mu \llcorner A$ *is a Borel regular measure.*

Proof. See [23, Theorem 1.9]. □

To show that a Borel measure is a Radon measure if and only if it is Borel regular and finite for every compact set, we use the following theorem from [23, Theorem 1.10].

Theorem 2.5. *Let μ be a Borel regular measure on $\mathbb{R}^n$, A a μ-measurable set, and $\varepsilon > 0$.*

(i) If $\mu(A) < \infty$, there is a closed set $C \subset A$ such that $\mu(A \setminus C) < \varepsilon$.

(ii) If there are open sets $U_1, U_2, \ldots$, such that $A \subset \bigcup_{i=1}^{\infty} U_i$ and $\mu(U_i) < \infty$ for all i, then there is an open set U such that $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof. See [23, Theorem 1.10]. □

Proposition 2.6. *A Borel measure is a Radon measure if and only if it is Borel regular and finite for every compact set $\mathbf{K} \subset \mathbb{R}^n$.*

Proof. We start by assuming μ is a Borel regular measure such that $\mu(\mathbf{K}) < \infty$ for all compact $\mathbf{K}$. Clearly, property (i) in Definition 2.7 is satisfied.

To prove property (iii) we start by observing that since U is an open set, and thus a Borel set, U will be a μ-measurable set. To show inner regularity, we first assume $\mu(U) < \infty$ and use assertion (i) in Theorem 2.5, to conclude that for every $\varepsilon > 0$ there exists a closed set $C \subset U$ such that $\mu(U \setminus C) < \varepsilon$. C is a compact set, since $\mu(C) \leq \mu(U) < \infty$. So, we have proved inner regularity for open sets with finite measure. The next part of the proof, the proof of inner regularity for U with infinite measure, is based on the proof of Theorem 1.8 in [9]. Let $D_i = \{x \in \mathbb{R}^n : i-1 < |x| < i\}$ be the torus with inner radius $i-1$ and outer radius i. Since $D_i \subset \overline{D_i}$ and $\overline{D_i}$ is a compact set, $\mu(D_i) < \infty$. Thus, $\mu(A \cap D_i) < \infty$ and, by the above argument, for every $\varepsilon > 0$ there exists C_i closed such that $\mu\big((A \cap D_i) \setminus C_i\big) \leq \varepsilon$. In fact, there exists a C_i such that $\mu(A \cap D_i) - 1/2^i \leq \mu(C_i)$. Furthermore, $\infty = \mu(A) = \sum_{i=1}^{\infty} \mu(A \cap D_i)$ and $\cup_{i=1}^{\infty} C_i \subseteq A$. So,

$$\lim_{k \to \infty} \mu\Big(\bigcup_{i=1}^{k} C_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} C_i\Big) = \sum_{i=1}^{\infty} \mu(C_i) \geq \sum_{i=1}^{\infty} \mu(A \cap D_i) - 1/2^i = \infty.$$

In addition, $\bigcup_{i=1}^{k} C_i$ is closed for all k. Thus, for an open set U with infinity measure the following is true

$$\mu(U) = \sup\{\mu(C) : C \subset U, C \text{ closed}\}.$$

The last step of the proof of property (iii) is to show that

$$\sup\{\mu(C) : C \subset U, C \text{ closed}\} = \sup\{\mu(\mathbf{K}) : \mathbf{K} \subset U, \mathbf{K} \text{ compact}\}, \tag{2.1}$$

for all open subsets U. Let $\overline{\mathcal{B}}_r$ be the closed ball with center at $\boldsymbol{x} = \mathbf{0}$ and radius r. Then, $C \cap \overline{\mathcal{B}}_r$ will be a compact set and $\lim_{r \to \infty} C \cap \overline{\mathcal{B}}_r = C$. Thus Equation (2.1) holds true.

Furthermore, we prove property (ii) in Definition 2.7. Assume A is μ-measurable and let $\mathcal{B}_r$ be an open ball with center at $\boldsymbol{x} = \mathbf{0}$ and radius r. Then $A \subset \bigcup_{r=1}^{\infty} \mathcal{B}_r$ and $\mu(\mathcal{B}_r) < \infty$, since $\overline{\mathcal{B}}_r$ is a compact subset and $\mathcal{B}_r \subset \overline{\mathcal{B}}_r$. Thus, by assertion (ii) in Theorem 2.5 for every ε there exists an open set U such that $U \supset A$ and $\mu(U \setminus A) < \varepsilon$. Thus property (ii) in Definition 2.7 is proved for measurable A. For a subset A which is not measurable we use that μ is Borel regular, i.e., there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$. Furthermore, B is a measurable set and thus

$$\mu(B) = \inf\{\mu(U) : U \supset B, U \text{ open}\}.$$

In addition, since $B \supset A$,

$$\inf\{\mu(U) : U \supset B, U \text{ open}\} \geq \inf\{\mu(U) : U \supset A, U \text{ open}\}.$$

Using this we obtain

$$\mu(A) \geq \inf\{\mu(U) : U \supset A, U \text{ open}\},$$

and equality is due to μ being monotone.

Now, we prove that a Radon measure is Borel regular. Let A be a set and by property (ii) in Definition 2.7

$$\mu(A) = \inf\{\mu(U) : U \supset A, U \text{ open}\}.$$

So, for every $\varepsilon > 0$ an open set U exists such that

$$\mu(U \setminus A) < \varepsilon.$$

Using this we construct a sequence U_i such that $\mu(U_i) < \varepsilon + \mu(A)$. Furthermore, the set $\bigcap_{i=1}^{k} U_i \supset A$ for every k, since $U_i \supset A$ for all i. In addition, $\mu\big(\bigcap_{i=1}^{k} U_i\big) \to \mu(A)$ as $k \to \infty$. Thus, since the countable intersection of open sets is a Borel set, $B = \bigcap_{i=1}^{\infty} U_i$ will satisfy $A \subset B$ and $\mu(A) = \mu(B)$. Thus, a Radon measure is a Borel regular measure which is finite for all compact subsets. □

In addition, we can prove the following property of $\mu \llcorner A$.

Theorem 2.7. *If μ is a Borel regular measure and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Proof. By assertion (ii) in Theorem 2.4 we have that $\mu \llcorner A$ is a Borel regular measure. In addition, $\mu \llcorner A(\mathbf{K}) = \mu(\mathbf{K} \cap A) \leq \mu(A) < \infty$ for all compact subsets $\mathbf{K}$. Thus, by Proposition 2.6 $\mu \llcorner A$ is a Radon measure. □

Next, we want to state the Lebesgue's Decomposition Theorem [19, Theorem 7.33] and Radon-Nikodym Theorem [19, Corollary 7.34], but first we have to define *absolute continuity*, *mutual singularity*, *equivalent measures* and *density of a measure*. From [19, Definition 7.30] we have the following definition.

Definition 2.9. [2] Assume μ and ν are Borel measures on $\Omega \subseteq \mathbb{R}^n$.

(i) The measure ν is *absolutely continuous* with respect to μ, written $\nu \ll \mu$, if $\nu(A) = 0$ for all $A \subseteq \Omega$ with $\mu(A) = 0$.

(ii) The measures ν and μ are *mutually singular*, written $\nu \perp \mu$, if there exists a Borel subset $B \subseteq \mathbb{R}^n$ such that $\mu(\Omega \setminus B) = \nu(B) = 0$.

(iii) The measures ν and μ are *equivalent* if $\mu \ll \nu$ and $\nu \ll \mu$.

(iv) Let $f : \Omega \to \overline{\mathbb{R}}$ be a measurable map. Define the measure ν by

$$\nu(A) := \int_A f d\mu$$

for $A \subset \Omega$. We say that f is the *density* of $\nu := f\mu$ with respect to μ.

Theorem 2.8 (Lebesgue's Decomposition Theorem)**.** *Let μ and ν be two Radon measures on $\Omega \subseteq \mathbb{R}^n$. Then ν can be uniquely decomposed into an absolutely continuous part $\nu_{a.c.}$ and a singular part ν_s with respect to μ, i.e., $\nu = \nu_{a.c.} + \nu_s$, where $\nu_{a.c.} \ll \mu$ and $\nu_s \perp \mu$. $\nu_{a.c.}$ has a density with respect to μ, denoted $\frac{d\nu_{a.c.}}{d\mu}$, which is μ-measurable and finite μ-a.e.*

[2]Have included Definition 4.13 from [19] and rewritten to coincide with the notation in this thesis.

Proof. See [19, p. 157]. □

Theorem 2.9 (Radon-Nikodym Theorem)**.** *Let μ and ν be Radon measures on $\Omega \subseteq \mathbb{R}^n$. Then ν has a density with respect to μ if and only if $\nu \ll \mu$. In this case, $\frac{d\nu}{d\mu}$ is μ-measurable and finite μ-a.e. $\frac{d\nu}{d\mu}$ is called the* Radon-Nikodym derivative *of ν with respect to μ.*

Proof. See [19, p. 157]. □

Thus, if a measure ν is absolutely continuous with respect to μ, there exists a measurable function $f = \frac{d\nu}{d\mu}$ such that

$$\nu(A) = \int_A f d\mu \quad \forall A \subseteq \Omega.$$

Furthermore we have the Lebesgue Differentiation Theorem [9, Theorem 1.32].

Theorem 2.10 (Lebesgue Differentiation Theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n; \mu)$. Then*

$$\lim_{r \to 0} \frac{1}{\mu(\mathcal{B}_r(\boldsymbol{x}))} \int_{\mathcal{B}_r(\boldsymbol{x})} f d\mu = f(\boldsymbol{x}) \quad \text{for } \mu\text{-a.e.} \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{2.2}$$

Proof. See [9, Theorem 1.32]. □

A point $\boldsymbol{x}$ satisfying (2.2), is called a *Lebesgue point* for the Radon measure μ.

Corollary 2.11 (Lebesgue Differentiation Theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^n$, $f \in L^1_{loc}(\mathbb{R}^n; \mu)$ and $\{E_r\}$ a sequence of set which shrinks nicely when $r \downarrow 0$, i.e., there exist an α such that $\mu(E_r) = \alpha\mu(\mathcal{B}_r(\boldsymbol{x}))$ and $\mathcal{B}_r(\boldsymbol{x}) \supset E_r$. Then*

$$\lim \frac{1}{\mu(E_r)} \int_{E_r} f d\mu = f(\boldsymbol{x}) \quad \text{for } \mu\text{-a.e.} \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Proof. This proof is a rewritten version of the proof of Theorem 3.21 in [12], where we have used an arbitrary Radon measure instead of the Lebesgue measure. Consider

$$\begin{aligned} \Big| \frac{1}{\mu(E_r)} \int_{E_r} f(\boldsymbol{y}) d\mu(\boldsymbol{y}) - f(\boldsymbol{x}) \Big| &\leq \frac{1}{\mu(E_r)} \int_{E_r} |f(\boldsymbol{y}) - f(\boldsymbol{x})| d\mu(\boldsymbol{y}) \\ &\leq \frac{1}{\alpha\mu(\mathcal{B}_r(\boldsymbol{x}))} \int_{\mathcal{B}_r(\boldsymbol{x})} |f(\boldsymbol{y}) - f(\boldsymbol{x})| d\mu(\boldsymbol{y}) = 0, \end{aligned}$$

where we have used Theorem 2.10. □

The next lemma is taken from [26, Proposition 7.1.1].

Lemma 2.12 (Urysohn's Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$, F a closed subset of Ω, and U an open set containing F. Then there exists a continuous function f from Ω into $[0, 1]$, equal to 1 on F and to 0 on U^c.*

Proof. See [26, Proposition 7.1.1]. □

Before stating the Vitali Covering Theorem [13, Theorem 6.64] we have to define a fine cover [13, Definition 6.63].

Definition 2.10 (Fine cover)**.** Let $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}^n$. We say that $\mathcal{F}$ *finely covers* $A \subset \mathbb{R}^n$ if for any $\boldsymbol{x} \in A$ and for any $\varepsilon > 0$ there is an $F \in \mathcal{F}$ and $\operatorname{diam}(F) < \varepsilon$.

Here, $\operatorname{diam} A = \sup\{|\boldsymbol{x} - \boldsymbol{y}| : \boldsymbol{x}, \boldsymbol{y} \in A\}$.

Theorem 2.13 (Vitali Covering Theorem)**.** *Every Radon measure μ in $\mathbb{R}^n$ has the following property: If $A \subset \mathbb{R}^n$ is a bounded Borel set and $\mathcal{F}$ is a family of closed balls that finely covers A, then there is a disjoint countable subfamily $\mathcal{F}' \subset \mathcal{F}$ such that*

$$\mu\Big(A \setminus \bigcup_{B \in \mathcal{F}'} B\Big) = 0.$$

Proof. See [13, Theorem 6.64]. □

Lastly, we will state two theorems defined by Wagner in [31, Formulae 1 and 2] that will be useful to do a change of variables later in this thesis. He has obtained the formulae from [10]. First, we define the measure $f_{\#}(\mu)$.

Definition 2.11 (Push-forward of a measure by a function [10, p. 54])**.** Let $f : X \to Y$ be a function, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Then f induces a map $f_{\#}$ which associates the measure $f_{\#}(\mu)$ on Y for each measure μ on X by the formula

$$f_{\#}(\mu)\big(B\big) = \mu\big(f^{-1}(B)\big) \text{ for } B \subset Y,$$

where f^{-1} is the preimage of f.

Proposition 2.14. *$f^{-1}(B)$ is μ-measurable if and only if B is $f_{\#}(\mu \llcorner A)$-measurable for every $A \subset X$.*

Proof. First, assume that $f^{-1}(B)$ is μ-measurable, then by Proposition 2.1

$$\mu(E) = \mu\big(E \cap f^{-1}(B)\big) + \mu\big(E \setminus f^{-1}(B)\big) \quad \forall E \subset X. \tag{2.3}$$

Let $E = f^{-1}(C) \cap A$. Then we have

$$\mu\big(f^{-1}(C) \cap A\big) = \mu\big((f^{-1}(C) \cap A) \cap f^{-1}(B)\big) + \mu\big(f^{-1}(C) \cap A \setminus f^{-1}(B)\big). \tag{2.4}$$

Furthermore,

$$\begin{aligned} f^{-1}(C) \cap f^{-1}(B) &= \{x \in \mathbb{R}^n : f(x) \in C\} \cap \{x \in \mathbb{R}^n : f(x) \in B\} = \{x \in \mathbb{R}^n : f(x) \in C \cap B\} \\ &= f^{-1}(C \cap B). \end{aligned}$$

Similarly, $f^{-1}(C) \setminus f^{-1}(B) = f^{-1}(C \setminus B)$. Thus, Equation (2.4) becomes

$$f_{\#}(\mu \llcorner A)\big(C\big) = f_{\#}(\mu \llcorner A)\big(C \cap B\big) + f_{\#}(\mu \llcorner A)\big(C \setminus B\big).$$

For every A and C there exists a set E such that $E = f^{-1}(C) \cap A$. Thus, for every $A \subset X$ the above equation will hold for every $C \subset Y$, since (2.3) holds for every E. Hence, B is $f_{\#}(\mu \llcorner A)$ measurable for all $A \subset X$.

Next, assume that B is $f_{\#}(\mu \llcorner A)$ for all $A \subset X$. Then

$$f_{\#}(\mu \llcorner A)\big(C\big) = f_{\#}(\mu \llcorner A)\big(C \cap B\big) + f_{\#}(\mu \llcorner A)\big(C \setminus B\big) \quad \forall C \subset Y. \tag{2.5}$$

We can rewrite this as

$$\mu\big(f^{-1}(C) \cap A\big) = \mu\big((f^{-1}(C) \cap A) \cap f^{-1}(B)\big) + \mu\big((f^{-1}(C) \cap A) \setminus B\big).$$

For all E a set A and a set C exist such that $f^{-1}(C) \cap A = E$, and since for all A Equation (2.5) holds for all C

$$\mu\big(E\big) = \mu\big(E \cap f^{-1}(B)\big) + \mu\big(E \setminus B\big) \quad \forall E \subset X.$$

Thus, $f^{-1}(B)$ is μ-measurable and the proposition is proved. □

Next, we state the definition of a proper function.

Definition 2.12 (Proper function [10, Section 2.2.17])**.** We say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *proper* if for every compact subset $\mathbf{K} \subset \mathbb{R}^m$ the preimage of $\mathbf{K}$ is compact, i.e., $f^{-1}(\mathbf{K})$ is compact.

From [10, Section 2.2.17] we have the following theorem.

Theorem 2.15. *Let $X, Y \subset \mathbb{R}^n$ be countable unions of compact sets. If μ is a Radon measure on X and $T : X \to Y$ is a proper map, then $T_\#\mu$ is a Radon measure on Y.*

Proof. To prove $T_\#\mu$ is a Radon measure, we prove that

(i) $T_\#\mu(\mathbf{K}) < \infty$ for all compact subsets, $\mathbf{K} \subset Y$.

(ii) For all A
$$T_\#\mu(A) = \inf\{T_\#\mu(U) : U \supset A, U \text{ open}\}.$$

(iii) for all U open, U is measurable and
$$T_\#\mu(U) = \sup\{T_\#\mu(\mathbf{K}) : \mathbf{K} \subset U, \mathbf{K} \text{ compact}\}.$$

First, we observe that $T_\#\mu(\mathbf{K}) = \mu\big(T^{-1}(\mathbf{K})\big) < \infty$, where we have used that T is proper, thus $T^{-1}(\mathbf{K})$ is a compact subset of X. So, property (i) is proved.

Furthermore, in Section 2.2.17 in [10] they have proved that for every $A \subset Y$ and $\varepsilon > 0$ there exists an open subset $U \subset Y$ such that $A \subset U$ and

$$T_\#\mu(U) \le \varepsilon + T_\#\mu(A).$$

In addition, we always have $T_\#\mu(A) \le T_\#\mu(U)$ since $A \subset U$. Thus, by letting $\varepsilon \to 0$ we get that for every A the following holds true $T_\#\mu(A) = \inf\{T_\#\mu(U) : U \supset A, U \text{ open}\}$, and property (ii) is proved.

Lastly, to prove property (iii). Since T is a proper function, it is continuous and thus every preimage of open sets is open, and we have

$$\begin{aligned} T_\#\mu(U) = \mu\big(T^{-1}(U)\big) &= \sup\{\mu(\mathbf{K}) : \mathbf{K} \subset T^{-1}(U), \mathbf{K} \text{ compact}\} \\ &\le \sup\{\mu\big(T^{-1}(\mathbf{K})\big) : \mathbf{K} \subset U, \mathbf{K} \text{ compact}\} = \sup\{T_\#\mu(\mathbf{K}) : \mathbf{K} \subset U, \mathbf{K} \text{ compact}\}. \end{aligned}$$

We have equality by using that $\mathbf{K} \subset U$ implies that $T_\#\mu(\mathbf{K}) \le T_\#\mu(U)$.

Thus, we have proved that $T_\#\mu$ is a Radon measure. □

Furthermore, from [10, Theorem 2.4.18] we obtain the following theorem.

Theorem 2.16. [3] *Let $X, Y \subset \mathbb{R}^n$ be countable unions of compact sets. Suppose $T : X \to Y$ and f maps $T_{\#}\mu$-a.e. $y \in Y$ into $\overline{\mathbb{R}}$.*

(i) If $f \circ T$ is μ-measurable then f is $T_{\#}\mu$ measurable and

$$\int (f \circ T) d\mu = \int f d(T_{\#}\mu). \tag{2.6}$$

(ii) If f is $T_{\#}\mu$-measurable, μ is a Radon measure and T is proper, then $f \circ T$ is μ-measurable.

Proof. The proof is based on the proof of Theorem 2.4.18 in [10]. However, we generalize the proof to include the case when μ not necessarily satisfies $\mu(X) < \infty$, but instead we require that μ is a Radon measure, T is proper and the spaces X and Y are countable unions of compact sets.

Assume that $f \circ T$ is μ-measurable, i.e., for every μ-measurable set E the set $(f \circ T)^{-1}(E) = T^{-1}\big(f^{-1}(E)\big)$ is a μ-measurable set. By Proposition 2.14 this implies that $f^{-1}(E)$ is a $T_{\#}(\mu \llcorner A)$-measurable set for all A. Thus, by choosing $A = X$, f is a $T_{\#}(\mu)$-measurable function. Next, assume that $f = \chi_E$ is a characteristic function. We start by observing

$$\chi_E \circ T = \chi_E(T(x)) = \begin{cases} 1 & T(x) \in E \\ 0 & T(x) \notin E \end{cases} = \chi_{T^{-1}(E)}.$$

Thus, for characteristic functions we have

$$\int \chi_E d(T_{\#}\mu) = T_{\#}\mu(E) = \mu\big(T^{-1}(E)\big) = \int \chi_{T^{-1}(E)} d\mu = \int \chi_E \circ T d\mu.$$

By Lemma 2.3 for every non-negative f there exists an increasing sequence of simple functions g_k such that $g_k(x) \to f(x)$ pointwise. Thus, by using the Monotone Convergence Theorem 2.21 we can conclude that the (2.6) holds true for non-negative functions. Furthermore, to conclude that the equality holds for a general function, we use Theorem 2.4.4(6) in [10], which state that for a measure ν and measurable function f

$$\int f d\nu = \int f^+ d\nu - \int f^- d\nu,$$

where $f^+ = \sup\{f, 0\}$ and $f^- = -\inf\{f, 0\}$. Then by the fact that the equality holds for non-negative functions, the equality holds for general functions. Thus, assertion (i) is proved.

Next, we prove assertion (ii). We start by observing that from Theorem 2.15, $T_{\#}\mu$ is a Radon measure, since μ is a Radon measure and T is proper. Furthermore, assume that f is a $T_{\#}\mu$-measurable function, then $f^{-1}(E)$ is a $T_{\#}\mu$-measurable set for all measurable set E. Let $\mathbf{K} \subset Y$ be a compact set, then since T is proper $T^{-1}(\mathbf{K}) \subset X$ is compact. Additionally, from Section 2.1.5(4) in [10] we have the following statement *If $\mu(X) < \infty$, $T : X \to Y$ and C is an $T_{\#}\mu$-measurable set, then $f^{-1}(C)$ is μ-measurable.* So, if we consider $f|_{T^{-1}(\mathbf{K})} : T^{-1}(\mathbf{K}) \to \mathbf{K}$ we can use the previous statement to conclude that since $\big(f|_{T^{-1}(\mathbf{K})}\big)^{-1}(E)$ is $T_{\#}\mu$-measurable, then $T^{-1} \circ \big(f|_{T^{-1}(\mathbf{K})}\big)^{-1}(E) = \big(f|_{T^{-1}(\mathbf{K})} \circ T\big)^{-1}(E)$ is μ-measurable and thus $f|_{T^{-1}(\mathbf{K})} \circ T$ is a μ-measurable function. This result holds true for all compact subsets $\mathbf{K}$. Since both X and Y are countable unions of compact sets will the result extend to f by a straightforward partition of unity argument. □

[3] The theorem is slightly altered. We consider a Radon measure μ and a proper function T defined on a countable union of compact sets, instead of considering a measure μ satisfying $\mu(X) < \infty$.

The two previous theorems will result in Formula 1 from [31]. Next, we define the number $N(T, \boldsymbol{y})$.

Definition 2.13 ([10, Section 2.10.9])**.** Let $N(f, \boldsymbol{y})$ be the number, possibly infinite, of points $\boldsymbol{x}$ in $\mathbb{R}^n$ such that $f(\boldsymbol{x}) = \boldsymbol{y}$ and $N(f|_A, \boldsymbol{y})$ is the number of points $\boldsymbol{x} \in A$ such that $f(\boldsymbol{x}) = \boldsymbol{y}$.

Furthermore, in [10, Section 3.1.6] we have the following formulation of Rademacher's theorem.

Theorem 2.17 (Rademacher's theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then f is differentiable m_n-a.e. in $\mathbb{R}^n$.*

The second change of variables theorem [31, Formula 2] is obtained from [10, Section 3.2.3]

Theorem 2.18. *Let X and Y be subsets of $\mathbb{R}^n$ and let $T : X \to Y$ be Lipschitz. Then*

(i) If A is an m_n-measurable set, then

$$\int_A JT d\boldsymbol{x} = \int N(T|_A, \boldsymbol{y}) d\boldsymbol{y}.$$

(ii) If $u \in L^1(\mathbb{R}^n)$, then

$$\int u(\boldsymbol{x}) JT d\boldsymbol{x} = \int \sum \{u(\boldsymbol{x}) : T(\boldsymbol{x}) = \boldsymbol{y}\} d\boldsymbol{y}.$$

Proof. See [10, Section 3.2.3]. □

Corollary 2.19. *Let X and Y be subsets of $\mathbb{R}^n$, $T : X \to Y$ be Lipschitz and $u \in L^1(\mathbb{R}^n)$. Then*

$$\int u\big(T(\boldsymbol{x})\big) JT d\boldsymbol{x} = \int u(\boldsymbol{y}) N(T, \boldsymbol{y}) d\boldsymbol{y}.$$

Proof. From assertion (*ii*) in Theorem 2.18 we have

$$\int u\big(T(\boldsymbol{x})\big) JT d\boldsymbol{x} = \int \sum \{u\big(T(\boldsymbol{x})\big) : T(\boldsymbol{x}) = \boldsymbol{y}\} d\boldsymbol{y}.$$

Furthermore, we observe that

$$\sum \{u\big(T(\boldsymbol{x})\big) : T(\boldsymbol{x}) = \boldsymbol{y}\} = u(\boldsymbol{y}) N(T, \boldsymbol{y}),$$

since $N(T, \boldsymbol{y})$ is the number of points $\boldsymbol{x}$ such that $T(\boldsymbol{x}) = y$ and for each such $\boldsymbol{x}$ will $u\big(T(\boldsymbol{x})\big) = u(\boldsymbol{y})$. The corollary is proved. □

2.1.2 Hausdorff measure

In this section we will define Hausdorff measure and state some useful properties. In [9, Definition 2.1] we find the following definition of a Hausdorff measure.

Definition 2.14 (Hausdorff measure)**.** Let $A \subseteq \mathbb{R}^n$, $0 \le s < \infty$, $0 \le \delta \le \infty$. We write

$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{j=1}^{\infty} \alpha(s)\Big(\frac{\text{diam } C_j}{2}\Big)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, \text{diam } C_j \le \delta\Big\},$$

where

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma(s/2+1)}.$$

For A and s as above, define

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s$ *s-dimensional Hausdorff measure* on $\mathbb{R}^n$.

Next, we state some properties of Hausdorff measure

Lemma 2.20. *We consider the s-dimensional Hausdorff measure on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$. Then*

(i) The s-dimensional Hausdorff measure is a Borel regular measure for all $0 \le s < \infty$.

(ii) If $\mathcal{H}^t(A) < \infty$, then $\mathcal{H}^s(A) = 0$ for all $t < s$.

(iii) If $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^t(A) = \infty$ for all $t < s$.

(iv) In $\mathbb{R}^n$ the $\mathcal{H}^n = m_n$.

(v) If $0 \le s < n$ the s-dimensional Hausdorff measure is not *a Radon measure.*

(vi) If A is a set of finite $\mathcal{H}^s$-measure, i.e., $\mathcal{H}^s(A) < \infty$, the s-dimensional Hausdorff measure restricted to A, $\mathcal{H} \llcorner A$, is a Radon measure.

Proof. See proof of Theorem 2.1 in [9] for a proof of assertion (i).

For the proof of assertion (ii), see [12, Proposition 10.22].

The proof of assertion (iii) is the contrapositive of assertion (ii). However, we write out the proof, which is based on the proof of Proposition 10.22 in [12]. Assume $\mathcal{H}^t(A) < \infty$. From the definition of Hausdorff measure, we have that for every $\delta > 0$ there exists $\{C_j\}$ such that diam $C_j \le \delta$ and $A \subset \bigcup_j C_j$. Furthermore, $\sum_j \frac{\alpha(t)}{2^t}(\text{diam } C_j)^t \le \mathcal{H}^t(A) + 1$. Thus, we get

$$\mathcal{H}^s_\delta(A) \le \sum_j \frac{\alpha(s)}{2^s}(\text{diam } C_j)^s \le (2\delta)^{s-t}\frac{\alpha(s)}{\alpha(t)}\sum_j \frac{\alpha(t)}{2^t}(\text{diam } C_j^t) \le (2\delta)^{s-t}\frac{\alpha(s)}{\alpha(t)}\big(\mathcal{H}^t(A)+1\big),$$

where $2^{s-t}\alpha(s)/\alpha(t)$ is a constant. So,

$$\mathcal{H}^s_\delta(A) \le 2^{s-t}\frac{\alpha(s)}{\alpha(t)}\delta^{s-t}(\mathcal{H}^t(A)+1) \to 0 \quad \text{as} \quad \delta \to 0,$$

since we have assumed that $\mathcal{H}^t(A) < \infty$ and $s > t$. This is a contradiction since $\mathcal{H}^s(A) > 0$ and thus $\mathcal{H}^t(A) = \infty$ for all $t < s$.

The proof of assertion (iv) can be found in [13, Theorem 6.75].

Next, we will prove assertion (v), by using assertion (iii) to show that the s-dimensional Hausdorff measure it is not finite for all compact sets. Consider the closed unit ball $\overline{\mathcal{B}}_1 \subset \mathbb{R}^n$, which is a compact set in $\mathbb{R}^n$. For the n-dimensional Hausdorff measure this ball has finite measure, in fact $\mathcal{H}^n(\overline{\mathcal{B}}_1) = \alpha(n) < \infty$. By the claim above, $\mathcal{H}^s(\overline{\mathcal{B}}_1) = \infty$ for all $s < n$ and we have shown assertion (v).

Assertion (vi) is a direct result of assertion (i) and Theorem 2.7. □

2.2 Convergence

In this section we present some concepts and theorems regarding convergence. We start by defining L^∞ weak* convergence.

Definition 2.15 (L^∞ weak* convergence)**.** Let μ be a measure. We say that a sequence u_n converge to u in $L^\infty(\Omega)$ weak* if it satisfies

$$\lim_{n\to\infty} \int_\Omega u_n \phi d\mu = \int_\Omega u\phi\mu \quad \forall \phi \in L^1(\Omega).$$

Next, we define Monotone Convergence Theorem [10, Section 2.4.7] and Dominated Convergence Theorem [10, Section 2.4.9].

Theorem 2.21 (Monotone Convergence Theorem)**.** *Let f_i, $i = 1, 2, 3, \ldots$ be μ-measurable functions such that $0 \le f_i(x) \le f_{i+1}(x)$, for $i = 1, 2, 3, \ldots$, and for $x \in \mathbb{R}^n$, then*

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n(x) d\mu(x).$$

Proof. See [10, Section 2.4.7]. □

Theorem 2.22 (Dominated Convergence Theorem)**.** *Suppose h is a μ-integrable function, i.e., the μ-integral of h is finite. If f_i, $i = 1, 2, 3 \ldots$ and g are μ-measurable functions such that*

$$|f_i(x)| \le h(x) \quad \text{for} \quad i = 1, 2, 3, \ldots, \qquad f_n(x) \to g(x) \quad \text{as} \quad n \to \infty$$

whenever $x \in \mathbb{R}^n$, then

$$\int |f_i - g| d\mu \to 0, \quad \text{hence} \quad \int f_i d\mu \to \int g d\mu, \quad \text{as} \quad n \to \infty.$$

Proof. See [10, Section 2.4.9]. □

Furthermore, we state the Moore-Osgood Theorem [14, p. 100].

Theorem 2.23 (Moore-Osgood Theorem)**.** *Let $x \in X$ and $y \in Y$. Suppose that the functions $f(x, y)$, g(x) and h(y) are real-finite-valued and that*

$$\lim_{x\to a} f(x, y) = h(y) \text{ on } Y,$$
$$\lim_{y\to b} f(x, y) = g(x) \text{ uniformly on } X.$$

Then the limits

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y), \qquad \lim_{x\to a} g(x), \qquad \lim_{y\to b} h(y),$$

all exists and are equal and finite.

Proof. See [14, p. 101]. □

Lastly, we define a uniformly bounded family of functions and an equicontinuous family of functions before stating the Arzelà-Ascoli Theorem. In [32, Definition 1 in Section 16.4] we are given the following definition for a uniformly bounded family of functions.

Definition 2.16 (Uniformly bounded family of functions)**.** A family $\mathcal{F}$ of functions $f : X \to Y$ defined on a set $X \subseteq \mathbb{R}^n$ and assuming values in $Y \subseteq \mathbb{R}^m$ is *uniformly bounded on* X if the set of values

$$V = \{y \in Y \mid \exists f \in \mathcal{F}, \exists x \in X \quad s.t. \quad y = f(x)\}$$

of functions in the family is bounded in Y.

In addition, we have the following definition for an equicontinuous family of functions [32, Definition 2 in Section 16.4].

Definition 2.17 (Equicontinuous family of functions)**.** Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. A family of functions $f : X \to Y$ is *equicontinuous on* X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for any function f in the family and $x_1, x_2 \in X$ such that $|x_1 - x_2| < \delta$.

Additionally, [32, Theorem 1 in Section 16.4] then goes on to state the following formulation of the Arzelà-Ascoli Theorem.

Theorem 2.24 (The Arzelà-Ascoli Theorem)**.** [4] *Let $\mathcal{F}$ be a family of functions $f : \mathbf{K} \to \mathbb{R}^n$ defined on a compact metric space $\mathbf{K}$ with values in $\mathbb{R}^n$. A necessary and sufficient condition for every sequence $\{f_k \in \mathcal{F} \mid k \in \mathbb{N}\}$ to contain a uniformly convergent subsequence is that the family $\mathcal{F}$ be uniformly bounded and equicontinuous.*

Proof. See [32, Theorem 1 in Section 16.4]. □

2.3 Mollifier

We start by defining a mollifier.

Definition 2.18 (Mollifier)**.** A *mollifier* is a function $\omega \in C_c^\infty(\mathbb{R}^n)$ which satisfies the following conditions

- $0 \leq \omega \leq 1$,
- $\int_{\mathbb{R}^n} \omega d\boldsymbol{x} = 1$,
- supp $\omega = \mathcal{B}_1(0)$.

We define a *standard mollifier* to be

$$\omega_\varepsilon = \frac{1}{\varepsilon^n} \omega\left(\frac{\boldsymbol{x}}{\varepsilon}\right),$$

where $\varepsilon > 0$. Lastly, we define the *mollification* of f by

$$f_\varepsilon(\boldsymbol{x}) = (f * \omega_\varepsilon)(\boldsymbol{x}) = \int_{\mathbb{R}^n} f(\boldsymbol{x} - \boldsymbol{z})\omega_\varepsilon(\boldsymbol{z})d\boldsymbol{z} = \int_{\mathbb{R}^n} f(\boldsymbol{z})\omega_\varepsilon(\boldsymbol{x} - \boldsymbol{z})d\boldsymbol{z}.$$

[4] In *Mathematical Analysis II* by Zorich the statement is "... the family $\mathcal{F}$ be totally bounded and equicontinuous.", but when $Y = \mathbb{R}^n$ totally boundedness and uniformly boundedness are equivalent [32, Page 395]. Thus, we instead use the statement "... the family $\mathcal{F}$ be uniformly bounded and equicontinuous."

We have that a standard mollifier ω_ε also satisfies the two first conditions for a mollifier given in Definition 2.18. It is trivial that $\omega_\varepsilon \geq 0$ since both ε and ω are positive. Furthermore,

$$\int_{\mathbb{R}^n} \omega_\varepsilon d\boldsymbol{x} = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\omega\left(\frac{\boldsymbol{x}}{\varepsilon}\right) d\boldsymbol{x} = \int_{\mathbb{R}^n} \omega(\boldsymbol{a})d\boldsymbol{a} = 1,$$

where we have used $\boldsymbol{a} = \boldsymbol{x}/\varepsilon$, and hence $d\boldsymbol{x} = \varepsilon^n d\boldsymbol{a}$. Lastly, we have that supp $\omega_\varepsilon = \mathcal{B}_\varepsilon$ since $|\boldsymbol{x}/\varepsilon| < 1 \implies |\boldsymbol{x}| < \varepsilon$. Next, we state a theorem with some properties of a mollifier.

Theorem 2.25 (Properties of mollifiers [9, Theorem 4.1]). [5] *Let f_ε be the mollification of f and $\Omega_\varepsilon := \{x \in \Omega : \inf\{|x-a| : a \in \partial\Omega\} > \varepsilon\}$. Then*

(i) $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

(ii) If $f \in C(\Omega)$, then $f_\varepsilon \to f$ uniformly on compact subsets of Ω.

(iii) If $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$, then $f_\varepsilon \to f$ in $L^p_{loc}(\Omega)$.

(iv) $f_\varepsilon \to f$ pointwise m_n-a.e.

Proof. See [9, Theorem 4.1]. □

2.4 Weak formulation

Before we derive the weak formulation of a conservation law, we define a weak derivative.

Definition 2.19 (Weak derivative). Let $f \in L^1_{loc}(\Omega)$ be a function. The weak derivative of f with respect to x_i is the function g that satisfies

$$\int f\phi_{x_i} dx = -\int g\phi dx \quad \forall \phi \in C_c^\infty.$$

The first part of this section is a based on Section 2.3.1 in [20], but here the formulation of weak solutions is written in the multidimensional case. Let $\Omega \subset \mathbb{R}^n$ be an open domain. We can write a general multidimensional balance law on Ω as

$$\operatorname{div} A(\boldsymbol{X}) = P(\boldsymbol{X}), \tag{2.7}$$

where $A(\boldsymbol{X}) : \Omega \to \mathbb{R}^n$ and $P(\boldsymbol{X}) : \Omega \to \mathbb{R}$. First, we have to introduce the notion of test functions. A test function is a sufficiently smooth function with compact support. We often use $\phi \in C_0^\infty(\Omega)$ as a test function. Here, C_0^∞ is the set of all infinitely differentiable functions with compact support. We start by assuming that $A \in C^1(\Omega)$, multiply (2.7) by ϕ and integrate over Ω. We obtain

$$\int_\Omega \operatorname{div} A(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} = \int_\Omega P(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X}. \tag{2.8}$$

Using the product rule we have $\operatorname{div}(A\phi) = \phi \operatorname{div} A + \nabla\phi \cdot A$, where we have used that A is a vector field and ϕ is a function. Thus,

$$\begin{aligned}\int_\Omega \operatorname{div} A(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} &= \int_\Omega \operatorname{div}(A(\boldsymbol{X})\phi(\boldsymbol{X}))d\boldsymbol{X} - \int_\Omega \nabla\phi(\boldsymbol{X}) \cdot A(\boldsymbol{X})d\boldsymbol{X} \\ &= \int_{\partial\Omega} \phi(\boldsymbol{X})A(\boldsymbol{X}) \cdot \boldsymbol{N} d\mathcal{H}^{n-1}(\boldsymbol{X}) - \int_\Omega \nabla\phi(\boldsymbol{X}) \cdot A(\boldsymbol{X})d\boldsymbol{X},\end{aligned}$$

[5] We have only included the properties we will use later in the thesis.

where $\boldsymbol{N}$ is the outward unit normal and we have used the Divergence Theorem [22, Theorem 8.3] to obtain the last equality. We often consider $\phi(\boldsymbol{X})$ with compact support in the domain we are integrating over and thus $\phi(\boldsymbol{X})|_{\partial\Omega} = 0$ and the *weak formulation* is

$$\int_{\Omega} \nabla\phi(\boldsymbol{X}) \cdot A(\boldsymbol{X}) + P(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} = 0.$$

However, when $\boldsymbol{X} = (\boldsymbol{x}, t)$ we often include the initial condition in weak formulation. In these cases we consider $\phi(\boldsymbol{X})$ where the support includes $t = 0$ and we obtain the following *weak formulation*

$$\int_{\Omega} \nabla\phi(\boldsymbol{x}, t) \cdot A(\boldsymbol{x}, t) + P(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t)d\boldsymbol{x}dt + \int_{\partial\Omega} \phi(\boldsymbol{x}, 0)A(\boldsymbol{x}, 0) \cdot \boldsymbol{E}_n d\boldsymbol{x} = 0,$$

where $\boldsymbol{E}_n$ is the nth unit base-vector. From these formulations to be satisfied we no longer need to require that A is differentiable, we only need that A, P are locally integrable functions. In fact, we can allow P to merely be a Radon measure such that

$$\langle P, \phi\rangle = \int \phi dP.$$

Then we rewrite the weak formulation as

$$\int_{\Omega} \nabla\phi(\boldsymbol{x}, t) \cdot A(\boldsymbol{x}, t)d\boldsymbol{x}dt + \langle P, \phi\rangle_{\Omega} + \int_{\partial\Omega} \phi(\boldsymbol{x}, 0)A(\boldsymbol{x}, 0) \cdot \boldsymbol{E}_n d\boldsymbol{x} = 0.$$

If this equation holds, $\operatorname{div} A = P$ *in the sense of distributions.*

The rest of this section is based on [20, pp. 19–21], but we have rewritten Theorem 4.9 in [20] such that we are considering a general multidimensional balance law. We will show that it is sufficient that $\phi \in W_c^{1,\infty}$, i.e., ϕ is a Lipschitz function with compact support, for ϕ to be a test function. To do this we propose the following theorem.

Theorem 2.26. *Let $A \in L^1_{loc}(\Omega, \mathbb{M}^{1\times n})$ and P a Radon measure. We say that* $\operatorname{div} A = P$ *in the sense of distributions, if*

$$\int_{\Omega} \nabla\phi(\boldsymbol{x}, t) \cdot A(\boldsymbol{x}, t)d\boldsymbol{x}dt + \langle P, \phi\rangle_{\Omega} + \int_{\partial\Omega} \phi(\boldsymbol{x}, 0)A(\boldsymbol{x}, 0) \cdot \boldsymbol{E}_n d\boldsymbol{x} = 0 \quad \forall\phi \in C_c^{\infty}.$$

It is sufficient that $\phi \in W_c^{1,\infty}$ for the above equation represent $\operatorname{div} A = P$ *in the sense of distributions.*

Proof. This proof is based on the proof of Theorem 4.9 in [20]. However, it is rewritten to consider a general multidimensional balance law. We start by assuming that $\phi \in W_c^{1,\infty}$ and use a standard mollifier ω_ε to obtain $\phi_\varepsilon = \omega_\varepsilon * \phi$, $(\phi_{\boldsymbol{x}_i})_\varepsilon = \omega_\varepsilon * \phi_{\boldsymbol{x}_i}$ and $(\phi_t)_\varepsilon = \omega_\varepsilon * \phi_t$. We observe that

$$(\phi_\varepsilon)_t = \frac{d}{dt}\iint \phi(\boldsymbol{x} - \boldsymbol{z}, t - s)\omega_\varepsilon(\boldsymbol{z}, s)d\boldsymbol{z}ds = \iint (\phi(\boldsymbol{x} - \boldsymbol{z}, t - s))_t\omega_\varepsilon(\boldsymbol{z}, s)d\boldsymbol{z}ds = (\phi_t)_\varepsilon,$$

and, with an equivalent calculation, we get $(\phi_\varepsilon)_{\boldsymbol{x}_i} = (\phi_{\boldsymbol{x}_i})_\varepsilon$. From Theorem 2.25 property (i), and the fact that ϕ has compact support, $\phi_\varepsilon \in C_c^{\infty}$. So, we have that

$$\int_{\Omega} \nabla\phi_\varepsilon(\boldsymbol{x}, t) \cdot A(\boldsymbol{x}, t)d\boldsymbol{x}dt + \langle P, \phi_\varepsilon\rangle_{\Omega} + \int_{\partial\Omega} \phi_\varepsilon(\boldsymbol{x}, 0)A(\boldsymbol{x}, 0) \cdot \boldsymbol{E}_n d\boldsymbol{x} = 0.$$

The next step is to show that

$$\int_\Omega \nabla\phi_\varepsilon(\boldsymbol{x},t)\cdot A(\boldsymbol{x},t)d\boldsymbol{x}dt + \langle P,\phi_\varepsilon\rangle_\Omega + \int_{\partial\Omega}\phi_\varepsilon(\boldsymbol{x},0)A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n d\boldsymbol{x}$$
$$\to \int_\Omega \nabla\phi(\boldsymbol{x},t)\cdot A(\boldsymbol{x},t)d\boldsymbol{x}dt + \langle P,\phi\rangle_\Omega + \int_{\partial\Omega}\phi(\boldsymbol{x},0)A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n d\boldsymbol{x}$$

as $\varepsilon\to 0$. We start by showing that

$$\int \phi_\varepsilon(\boldsymbol{x},0)A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n d\boldsymbol{x} \to \int \phi(\boldsymbol{x},0)A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n d\boldsymbol{x} \text{ as } \varepsilon\to 0.$$

By property (ii) in Theorem 2.25 and that ϕ is a Lipschitz continuous function, we have that $\phi_\varepsilon\to\phi$ uniformly on compact subsets of Ω. Using this we get

$$\left|\int A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n\big[\phi_\varepsilon(\boldsymbol{x},0)-\phi(\boldsymbol{x},0)\big]d\boldsymbol{x}\right| \le \int \Big|A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n\Big|\Big|\phi_\varepsilon(\boldsymbol{x},0)-\phi(\boldsymbol{x},0)\Big|d\boldsymbol{x}$$
$$\le \delta\|A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n\|_{L^1_{loc}}.$$

Since $\|A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n\|_{L^1_{loc}}<\infty$, will $\delta\|A(\boldsymbol{x},0)\|_{L^1_{loc}}\to 0$ as $\delta\to 0$. Hence,

$$\int \phi_\varepsilon(\boldsymbol{x},0)A(\boldsymbol{x},0)\cdot\boldsymbol{E}_n d\boldsymbol{x} \to \int \phi(\boldsymbol{x},0)A(\boldsymbol{x},0)d\boldsymbol{x} \text{ as } \varepsilon\to 0.$$

Next, we show that

$$\iint \nabla\phi_\varepsilon\cdot A d\boldsymbol{x}dt \to \iint \nabla\phi\cdot A d\boldsymbol{x}dt \text{ as } \varepsilon\to 0.$$

To prove this we use the Dominated Convergence Theorem 2.22. We start by showing that $|\nabla\phi_\varepsilon|$ is bounded, by showing that $|(\phi_t)_\varepsilon|$ and $|(\phi_{x_i})_\varepsilon|$, $i=1,\dots,n-1$, are bounded.

$$|(\phi_t)_\varepsilon| = \left|\iint_{\mathbb{R}^n}\phi_t(\boldsymbol{x}-\boldsymbol{z},t-s)\omega_\varepsilon(\boldsymbol{z},s)d\boldsymbol{z}ds\right| \le \iint_{\mathbb{R}^n}\omega_\varepsilon(\boldsymbol{z},s)|\phi_t(\boldsymbol{x}-\boldsymbol{z},t-s)|d\boldsymbol{z}ds$$
$$\le \sup_{(\boldsymbol{x},t)\in\mathbb{R}^n}|\phi_t(\boldsymbol{x},t)|\iint_{\mathbb{R}^n}\omega(\boldsymbol{z},s)d\boldsymbol{z}ds = \|\phi_t\|_\infty \le M<\infty.$$

Here we have used the properties of ω_ε, and that $\phi\in W^{1,\infty}_c$ implies that $\phi_t\in L^\infty$. Similarly, we can show that $|(\phi_{x_i})_\varepsilon|$, $i=1,\dots,n-1$, are bounded. Using this we get

$$|\nabla\phi_\varepsilon\cdot A(\boldsymbol{x},t)| \le M\sum_{i=1}^n |A(\boldsymbol{x},t)\cdot\boldsymbol{E}_i|,$$

and $A(\boldsymbol{x},t)\cdot\boldsymbol{E}_i$ are integrable functions for all $i=1,\dots,n$. From (iv) in Theorem 2.25 we have that $(\phi_t)_\varepsilon$ converges to ϕ_t pointwise almost everywhere. Thus all the conditions in the Dominated Convergence Theorem 2.22 are fulfilled and we have that

$$\lim_{\varepsilon\to 0}\iint \nabla\phi_\varepsilon\cdot A d\boldsymbol{x}dt = \iint \nabla\phi\cdot A d\boldsymbol{x}dt.$$

Lastly, we show that $\langle P,\phi_\varepsilon\rangle$ converges to $\langle P,\phi\rangle$ as $\varepsilon\to 0$. Let $\mathbf{K}$ be the compact support of ϕ and consider,

$$|\langle P,\phi_\varepsilon\rangle - \langle P,\phi\rangle| \le \int_{\mathbf{K}}|\phi_\varepsilon-\phi|dP \le \delta P(\mathbf{K}),$$

where we have used that $\phi_\varepsilon\to\phi$ uniformly on compact subsets. This will converge to zero as $\varepsilon\to 0$ since P being a Radon measure implies $P(\mathbf{K})<\infty$ as $\mathbf{K}$ is compact.

So, we have shown that it is sufficient that $\phi\in W^{1,\infty}_c$ for ϕ to be a test function. □

2.5 BV functions

In this section we present the concept of functions with bounded variation and some useful properties of such functions. The definitions and theorems in this section are taken from [30], if not otherwise stated.

We start by defining a weak derivative as local measures and total variation for a measure or a function.

Definition 2.20 (Weak derivative as locally measure)**.** We say that the weak derivatives are *local measures* if there exists a vector-valued set function $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ which is defined for all bounded Borel subsets, B, which satisfies $\overline{B} \subset \Omega$ with the following properties

(i) $\boldsymbol{\mu}(B) < \infty$, for all B satisfying $\overline{B} \subset \Omega$,

(ii) countably additive, for all B satisfying $\overline{B} \subset \Omega$,

(iii) $\int_\Omega f(\boldsymbol{x}) \frac{\partial \psi(\boldsymbol{x})}{\partial x_i} d\boldsymbol{x} = -\int_\Omega \psi(\boldsymbol{x}) \mu_i(d\boldsymbol{x})$.

Remark. *A Radon measure will satisfy the properties in the definition above, and thus a Radon measure will be a local measure.*

Definition 2.21 (Total variation of a measure)**.** The *total variation* of a vector-valued measure, $\boldsymbol{\mu}$, on $\Omega \subset \mathbb{R}^n$ is defined as

$$T.V.(\boldsymbol{\mu})(E) = \sup \Big\{ \sum_{i \in \mathbb{N}} |\boldsymbol{\mu}(E_i)| : \bigcup_{i \in \mathbb{N}} E_i = E, \quad E_i \quad \boldsymbol{\mu} - \text{measurable} \Big\}.$$

Definition 2.22 (Total variation of a function)**.** The *total variation* of a function of one variable, f, is defined as

$$T.V.(f)([a,b]) = \sup \Big\{ \sum_{i \in \mathcal{I}} |f(x_i) - f(x_{i-1})| : \mathcal{I} \text{ finite partition of [a,b]} \Big\}.$$

The *total variation* of a function F of $n \geq 2$ variables is defined as

$$T.V.(F)(E) = \sup \Big\{ \int_E F \operatorname{div} \phi d\boldsymbol{x} : \phi \in C_c^1(E; \mathbb{R}^n), |\phi| \leq 1 \Big\}.$$

We are ready to define $BV(\Omega)$.

Definition 2.23 (Function space $BV(\Omega)$)**.** [6] The function space $BV_{loc}(\Omega)$ denotes the set of all the functions $u \in L^1_{loc}$ with weak derivatives that are locally measures. If $u \in L^1$ and the weak derivatives $\boldsymbol{\mu}$ are countably additive on all Borel subsets of Ω and satisfies $T.V.(\boldsymbol{\mu})(\Omega) < \infty$ we call the function space $BV(\Omega)$.

The next two concept we define are a point of density and a point of rarefaction.

Definition 2.24 (Point of density/rarefaction)**.** We say that $\boldsymbol{x}$ is a *point of density* for the set E if it satisfies

$$\lim_{r \to 0} \frac{m_n\big(E \cap \mathcal{B}_r(\boldsymbol{x})\big)}{m_n\big(\mathcal{B}_r(\boldsymbol{x})\big)} = 1.$$

[6]Note that when we say BV_{loc} we refer to the BV in [30, p. 226] and we say BV when we refer to $\overline{BV}$ in [30, p. 226].

A point $\boldsymbol{x}$ that satisfies

$$\lim_{r\to 0} \frac{m_n\big(E\cap \mathcal{B}_r(\boldsymbol{x})\big)}{m_n\big(\mathcal{B}_r(\boldsymbol{x})\big)} = 0,$$

is called a *point of rarefaction* for the set E.

Now we are ready to define the approximate limit and approximate continuity.

Definition 2.25 (Approximate limit)**.** Let $u(\boldsymbol{x})$ be a function defined on $E \subseteq \mathbb{R}^n$ and $\boldsymbol{x}_0$ not be a point of rarefaction for E. Then we say that $l_E u(\boldsymbol{x}_0)$ is the *approximate limit* if for every $\varepsilon > 0$ it satisfies

$$\lim_{r\to 0} \frac{m_n\big(\{\boldsymbol{x}\in E : |u(\boldsymbol{x}) - l_E u(\boldsymbol{x}_0)| > \varepsilon\} \cap \mathcal{B}_r(\boldsymbol{x}_0)\big)}{m_n\big(\mathcal{B}_r(\boldsymbol{x}_0)\big)} = 0,$$

i.e., $\boldsymbol{x}_0$ is a point of rarefaction of the set $\{\boldsymbol{x} \in E : |u(\boldsymbol{x}) - l_E u(\boldsymbol{x}_0)| > \varepsilon\}$. If $E = \mathbb{R}^n$ we denote the approximate limit by $lu(\boldsymbol{x}_0)$ and if E is the half space $(\boldsymbol{x} - \boldsymbol{x}_0)\cdot a > 0$ we denote the approximate limit by $l_a u(\boldsymbol{x}_0)$.

Definition 2.26 (Approximate continuity)**.** We say that a function, u, is approximately continuous at a point $\boldsymbol{x}_0 \in E$ if

$$l_E u(\boldsymbol{x}_0) = u(\boldsymbol{x}_0).$$

Furthermore, from [9, Theorem 1.37] we have the following useful theorem.

Theorem 2.27 (Measurability and approximate continuity)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is m_n-measurable. Then f is approximately continuous m_n-a.e.*

Proof. See [9, Theorem 1.37]. □

Next, we define regular points.

Definition 2.27 (Regular point)**.** Let $\boldsymbol{u}(\boldsymbol{x})$ be a vector-valued function defined in some neighborhood of the point $\boldsymbol{x}_0$. Then $\boldsymbol{x}_0$ is a *regular point* of $\boldsymbol{u}(\boldsymbol{x})$ if a unit vector a exists such that $l_a\boldsymbol{u}(\boldsymbol{x}_0)$ and $l_{-a}\boldsymbol{u}(\boldsymbol{x}_0)$ exist and are finite.

Furthermore, we have the following theorem.

Theorem 2.28. *Let $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \ldots, u_d(\boldsymbol{x})) \in BV_{loc}(\Omega)$ be a vector-valued function where each $u_i(\boldsymbol{x})$ is bounded. Furthermore let E be the set of all regular points of $\boldsymbol{u}(\boldsymbol{x})$. Then $\mathcal{H}^{n-1}(\Omega \setminus E) = 0$, i.e., $\mathcal{H}^{n-1}$-a.e. $\boldsymbol{x}$ is a regular point of $\boldsymbol{u}(\boldsymbol{x})$.*

Proof. See [30, Lemma 9.1]. □

Lastly, we want to study how we can differentiate a discontinuous function. To be able to do this we have to introduce functional superposition.

Definition 2.28 (Functional superposition)**.** Suppose $\boldsymbol{x}$ is a regular point of the vector-valued function $\boldsymbol{u}(\boldsymbol{x})$ and let $f(\boldsymbol{u})$ be a integrable function in $\mathbb{R}^n$ of n variables defined on the interval $l_a\boldsymbol{u}(\boldsymbol{x})t + l_{-a}\boldsymbol{u}(\boldsymbol{x})(1-t)$ for $t \in [0,1]$. Then, the *functional superposition* is given by

$$\widehat{f}(\boldsymbol{u}(\boldsymbol{x})) = \int_0^1 f\big(l_a\boldsymbol{u}(\boldsymbol{x})t + l_{-a}\boldsymbol{u}(\boldsymbol{x})(1-t)\big)dt.$$

At a point $\boldsymbol{x}$ of approximate continuity for $\boldsymbol{u}(\boldsymbol{x})$ we have that $\widehat{f}(\boldsymbol{u}(\boldsymbol{x})) = f(\boldsymbol{u}(\boldsymbol{x}))$. The following theorem state how we can differentiate a discontinuous function.

Theorem 2.29. *Let $\boldsymbol{u} = (u_1, \ldots, u_d) \in BV_{loc}(\Omega)$ and let $\partial\widehat{f}(\boldsymbol{u}(\boldsymbol{x}))/\partial u_k$ be locally integrable with respect to the measure $\partial u_k/\partial x_i$ in Ω $(k = 1, \ldots, d; i = 1, \ldots, n)$. Then $f(\boldsymbol{u}(\boldsymbol{x})) \in BV_{loc}(\Omega)$ and*

$$\frac{\partial}{\partial x_i} f(\boldsymbol{u}(\boldsymbol{x})) = \sum_{k=1}^{d} \frac{\partial\widehat{f}(\boldsymbol{u}(\boldsymbol{x}))}{\partial u_k} \frac{\partial u_k}{\partial x_i}, \quad (i = 1, \ldots, n).$$

Proof. See [30, Sections 13.3-13.5]. □

Lastly, we state the following useful theorem for functions of bounded total variation from [18, Corollary A.10].

Theorem 2.30 (Helly's Selection Theorem)**.** *Let $\{f_k\}$ be a sequence of functions defined on an interval $[a, b]$ and assume that this sequence satisfies*

$$T.V.(f_k) < M \qquad \textit{and} \qquad \|f_k\|_\infty < M,$$

where M is some constant independent of k. Then there exists a subsequence f_{k_i} that converges almost everywhere to some function f of bounded variation.

Proof. See [18, Corollary A.10]. □

Chapter 3

Balance laws

3.1 Multidimensional balance law

The formulation of the general balance law in this subsection is based on Section 1.1 in [8]. Let Ω be an open subset of $\mathbb{R}^n$ and $\mathcal{D} \subset \Omega$ be an open subset with a piecewise twice continuously differentiable boundary, which means that the graph of the boundary is a piecewise twice continuously differentiable function. We are going to derive the formulation of a general balance law on the domain $\mathcal{D}$ by using the concept that for any extensive quantity the flux over the boundary of a domain, $\partial\mathcal{D}$ is balanced by the production inside the domain $\mathcal{D}$. We start by introducing $P(\mathcal{D})$ which gives the value of the production inside $\mathcal{D}$. In this thesis we assume that P is a Radon measure which is absolutely continuous with respect to the Lebesgue measure, i.e., there exists a production density function $\tilde{p} \in L^1_{loc}\big(\Omega; m_n\big)$ such that

$$P(\mathcal{D}) = \int_{\mathcal{D}} \tilde{p}(\boldsymbol{X}) d\boldsymbol{X}.$$

Furthermore, we introduce $Q_{\mathcal{D}}$ which gives the value of the flux across the boundary of $\mathcal{D}$. We assume that $Q_{\mathcal{D}}$ is a measure, which is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure, $\mathcal{H}^{n-1}$. Again, this implies that we can write

$$Q_{\mathcal{D}}(\mathcal{C}) = \int_{\mathcal{C}} q_{\mathcal{D}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}), \tag{3.1}$$

where $q_{\mathcal{D}}(\boldsymbol{X}) \in L^1\big(\partial\mathcal{D}; \mathcal{H}^{n-1}\big)$ is the flux density function. Then for each $\mathcal{D}$ in Ω we have the following general balance law

$$Q_{\mathcal{D}}(\partial\mathcal{D}) = P(\mathcal{D}). \tag{3.2}$$

3.2 Specific conservation law

In Section 2.2 in my project thesis [20] I derived the conservation of mass, momentum, and energy for a compressible, inviscid, non-heat-conducting gas. This section is taken from [20, Section 2.2] and is included for completeness. We start by making some assumptions, and make the same assumptions as in the article *Equivalence of the Euler and Lagrange equations of gas dynamics for weak solutions* [31], which uses the book *Supersonic flow and shock waves* by Courant and Friedrichs [5] to obtain the equations for conservation of mass, momentum, and energy. They assume that the only force acting on the fluid is the pressure force, i.e., they neglect the forces due to gravity and assume that the fluid is non-heat-conducting. This means that there are no thermal forces. In addition, they assume that the fluid is compressible and inviscid. As mentioned in Section 1.1.2 a fluid can be described either by an Euler or by a Lagrange point of view. This will result in two different systems of partial differential equations. For each conservation law we are first going to derive it from the Euler point of view followed by the derivation of the Lagrange representation of the equation. This will result in the Euler and the Lagrange equations.

We consider a gas and define ρ to be the mass density, $\boldsymbol{u} = (u_1, \ldots, u_n)$ to be the velocity vector and S to be the entropy per mass unit. Furthermore, we define p to be the pressure and e to be the internal energy of the gas per unit mass. We assume both p and e to be functions of the mass density and entropy only. In addition, we denote the specific volume by $\tau = 1/\rho$.

3.2.1 Conservation of mass

Euler point of view

We start by considering the conservation of mass in Euler representation and obtain our derivation of the conservation law from [3, Section **i** in Chapter 1.1]. We choose a control volume $\Omega \in \mathbb{R}^n$ fixated in space and therefore independent of time. By the fact that mass neither can be created nor destroyed, we have that the change of mass in Ω has to be equal to the mass flux over the boundary of Ω, $\partial\Omega$. The mass in Ω is given by

$$M(\Omega) = \int_{\Omega} \rho(\boldsymbol{x}, t) \mathrm{d}\Omega,$$

where ρ is the mass density of the gas. We denote $\boldsymbol{N}$ to be the outward unit normal on $\partial\Omega$ and have that the volume flow rate over $\partial\Omega$ is given by $\boldsymbol{u} \cdot \boldsymbol{N}$, where $\boldsymbol{u}$ is the velocity vector. From this we get that the mass flow over the boundary is given by

$$\text{Mass flow over } \partial\Omega = \int_{\partial\Omega} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{N} \mathrm{d}\partial\Omega.$$

Thus, as the increase of mass is equal to the inward mass flow over the boundary, we obtain

$$\frac{d}{dt}M(\Omega) = \frac{d}{dt}\int_{\Omega} \rho(\boldsymbol{x}, t)\mathrm{d}\Omega(t) = \int_{\Omega} \frac{\partial}{\partial t}\rho(\boldsymbol{x}, t)\mathrm{d}\Omega(t) = -\int_{\partial\Omega} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{N}\mathrm{d}\partial\Omega.$$

We assume that the boundary of Ω, $\partial\Omega$, to be sufficiently smooth and $\rho, u_i \in C^1$ for $i = 1, \ldots, n$, so we can use the Divergence Theorem [22, Theorem 8.3]. We then get

$$\int_{\Omega} \left[\rho_t + \mathrm{div}_{\boldsymbol{x}}(\rho\boldsymbol{u})\right] \mathrm{d}\Omega = 0,$$

and since Ω was chosen arbitrary we get

$$\rho_t + \mathrm{div}_{\boldsymbol{x}}(\rho\boldsymbol{u}) = 0. \tag{3.3}$$

This is conservation of mass in Euler representation. In one dimension this equation becomes

$$\rho_t + (\rho u)_x = 0.$$

Lagrange point of view

The derivation of conservation of mass in Lagrange representation is obtained from [2, Chapter 3.1]. We start by denoting $\Omega(t)$ to be a volume element with the same fluid particles over time. No particles can leave or enter $\Omega(t)$, so the mass in $\Omega(t)$ is conserved over time, but the volume and shape of $\Omega(t)$ can change over time. To be able to represent conservation of mass on integral form in Lagrange coordinates we have to start in Euler coordinates. We can write the Euler representation of conservation of mass as

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x},t)\mathrm{d}\Omega(t) = 0.$$

To convert to the Lagrange representation, we do a change of variables where we define $\boldsymbol{x} = \hat{x}(\boldsymbol{y},t)$ and use that

$$J_x = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}. \tag{3.4}$$

We then get that mass conservation in Lagrange representation is given by

$$\frac{d}{dt}\int_{W} \rho(\boldsymbol{y},t)\det J_x \mathrm{d}W = 0,$$

where W is the domain of the particles in $\Omega(t)$ in Lagrange coordinates. In Lagrange representation we follow the particles, hence W will not be dependent on time and we get

$$\int_{W} \frac{\partial}{\partial t}\{\rho(\boldsymbol{y},t)\det J_x\}\mathrm{d}W = 0.$$

This equation holds for all W and hence we have that

$$(\rho \det J_x)_t = 0.$$

In one dimension we can create an easy description for the Lagrange coordinate y such that we can eliminate the Jacobian. There are different choices to obtain this, but Courant and Friedrichs [5, Page 30] suggest using conservation of mass to define y. The idea is that we define y by saying that a particle can neither appear nor disappear. To do this they consider a tube around the x-axis and define a start section of this tube where $y = 0$. This is moving with the fluid. Then they define y to be the mass of the fluid between this start section and the current position, i.e., they define y to be

$$y = \int_{x(0,t)}^{x(y,t)} \rho(s,t)ds. \tag{3.5}$$

Furthermore, they differentiate (3.5) with respect to y and obtain

$$1 = \rho(x(y,t),t)x_y(x(y,t)) \implies x_y = \tau.$$

Using that $J_x = x_y = \tau$, and that $x_t = u$, we get

$$0 = (\rho x_y)_t = \rho_t x_y + \rho x_{yt} = \rho^2 x_y\left(\frac{\rho_t}{\rho^2} + \frac{x_{ty}}{\rho x_y}\right) = \rho\left(-\left(\frac{1}{\rho}\right)_t + u_y\right),$$

where we have used that the mixed derivatives are equal, $(x_y)_t = x_{yt} = x_{ty} = (x_t)_y$ [22, Theorem 3.10]. By using the assumption of no vacuum, we can rewrite the Lagrange representation of mass in one dimension to be

$$\tau_t - u_y = 0.$$

3.2.2 Conservation of momentum

Euler point of view

Next, we want to derive the conservation of momentum. We start by defining $\Omega(t) \subset \mathbb{R}^n$ to be a domain of the gas that consists of the same amount of material over time. Recall that we assumed that the only forces acting on the gas are the pressure forces. Hence, the rate of change of momentum in $\Omega(t)$ will be equal to the pressure forces acting on the boundary of $\Omega(t)$. Again, we have that ρ is the mass density and $\boldsymbol{u}$ is the velocity, so that the total momentum in $\Omega(t)$ is given by

$$\text{Total momentum in } \Omega(t) = \int_{\Omega(t)} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) \mathrm{d}\Omega(t).$$

Since we are considering an inviscid gas, all forces acting on the gas are normal to the boundary. Thus, the pressure forces on the boundary can be written as

$$\text{Pressure forces on } \partial\Omega(t) = \int_{\partial\Omega(t)} p(\boldsymbol{x}, t)\boldsymbol{N} \mathrm{d}\partial\Omega(t),$$

where $\boldsymbol{N}$ is the outward unit normal on $\partial\Omega(t)$. The rate of change of momentum is equal to the pressure forces, where we have that an increase of momentum is due to pressure forces acting in the opposite direction of the normal vector. Thus, we get

$$\frac{d}{dt} \int_{\Omega(t)} \rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) \mathrm{d}\Omega(t) = -\int_{\partial\Omega(t)} p(\boldsymbol{x}, t)\boldsymbol{N} \mathrm{d}\partial\Omega(t). \tag{3.6}$$

Here, we get n equations, one for each spatial dimension. We first consider the right-hand side. To be able to use the Divergence Theorem [22, Theorem 8.3] we multiply the right-hand side by a fixed vector $\boldsymbol{v}$ [3, Section **ii** Chapter 1.1]

$$\boldsymbol{v} \cdot \int_{\partial\Omega(t)} p\boldsymbol{N} \mathrm{d}\partial\Omega(t) = \int_{\partial\Omega(t)} p\boldsymbol{v} \cdot \boldsymbol{N} \mathrm{d}\partial\Omega(t) = \int_{\Omega(t)} \operatorname{div}_{\boldsymbol{x}}(p\boldsymbol{v}) \mathrm{d}\Omega(t) = \boldsymbol{v} \cdot \int_{\Omega(t)} \nabla p \mathrm{d}\Omega(t),$$

and we get that

$$-\int_{\partial\Omega(t)} p\boldsymbol{N} \mathrm{d}\partial\Omega(t) = -\int_{\Omega(t)} \nabla p \mathrm{d}\Omega(t). \tag{3.7}$$

We consider one of the equations in (3.6), i.e.,

$$\frac{d}{dt} \int_{\Omega(t)} \rho u_i \mathrm{d}\Omega(t) = -\int_{\Omega(t)} \frac{\partial p}{\partial x_i} \mathrm{d}\Omega(t), \tag{3.8}$$

where we have used Equation (3.7) to obtain the right-hand side. Since the domain is time dependent, we have to be more careful when calculating the left-hand side of the equation. We use Reynolds Transport Theorem [21, Equation 3.35 in Chapter 3.6] which states that

$$\frac{d}{dt} \int_{\Omega(t)} f(\boldsymbol{x}, t) \mathrm{d}\Omega(t) = \int_{\Omega(t)} \frac{\partial f(\boldsymbol{x}, t)}{\partial t} \mathrm{d}\Omega(t) + \int_{\partial\Omega(t)} f(\boldsymbol{x}, t)\boldsymbol{u} \cdot \boldsymbol{N} \mathrm{d}\partial\Omega(t).$$

Using the Divergence Theorem [22, Theorem 8.3] we obtain the following representation of the Reynolds Transport Theorem

$$\frac{d}{dt} \int_{\Omega(t)} f(\boldsymbol{x}, t) \mathrm{d}\Omega(t) = \int_{\Omega(t)} \left[\frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(f(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t)) \right] \mathrm{d}\Omega(t). \tag{3.9}$$

Then by (3.9) we get

$$\frac{d}{dt}\int_{\Omega(t)} \rho u_i \mathrm{d}\Omega(t) = \int_{\Omega(t)} \left[(\rho u_i)_t + \mathrm{div}_{\boldsymbol{x}}(\rho u_i \boldsymbol{u})\right] \mathrm{d}\Omega(t),$$

and by inserting this in (3.8) we get

$$\int_{\Omega(t)} \left[(\rho u_i)_t + \mathrm{div}_{\boldsymbol{x}}(\rho u_i \boldsymbol{u}) + \frac{\partial p}{\partial x_i}\right] \mathrm{d}\Omega(t) = 0.$$

Again, since we chose $\Omega(t)$ arbitrary we have that

$$(\rho u_i)_t + \mathrm{div}_{\boldsymbol{x}}(\rho u_i \boldsymbol{u}) + \frac{\partial p}{\partial x_i} = 0,$$

for $i = 1, \ldots, n$, which is equivalent to

$$(\rho u_i)_t + \mathrm{div}_{\boldsymbol{x}}(\rho u_i \boldsymbol{u} + p\boldsymbol{E}_i) = 0,$$

where $\boldsymbol{E}_i = [0, \ldots, 1, \ldots, 0]$ is the ith unit base vector. This is the conservation of momentum in Euler representation. Thus, in one dimension we get one equation which is the following equation

$$(\rho u)_t + (\rho u^2 + p)_x = 0.$$

Lagrange point of view

Next, we want to derive the Lagrange representation of conservation of momentum. To do this we use Newton's second law

$$F = \frac{d}{dt}(\rho(\boldsymbol{x}, t)\boldsymbol{x}_t),$$

where F is the sum of the forces acting on the fluid. We have assumed that the only force acting on the fluid is the pressure force, which is given by the negative pressure gradient, i.e., $F = -\nabla_{\boldsymbol{x}} p$ [5, Section 7]. We get a negative sign since the force is acting inwards. We then obtain the following equation

$$\frac{d}{dt}\left(\rho(\boldsymbol{x}, t)\boldsymbol{x}_t\right) = -\nabla_{\boldsymbol{x}} p(\boldsymbol{x}, t).$$

This is written in Euler coordinates, and we have to convert it to Lagrange coordinates by using a similar technique as we did when deriving conservation of mass. In addition, we use conservation of mass and obtain

$$\rho(\boldsymbol{y}, t)\boldsymbol{x}_{tt} = -\nabla_{\boldsymbol{x}} p(\boldsymbol{y}, t).$$

Next, we want to expand the derivation using the chain rule, and we get

$$\rho(\boldsymbol{y}, t)\boldsymbol{x}_{tt} = -\nabla_{\boldsymbol{y}} p(\boldsymbol{y}, t) J_y,$$

where we have used that

$$J_y = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}. \tag{3.10}$$

In addition, if we use that $\boldsymbol{x}_t = \boldsymbol{u}$, we find the Lagrange representation of the conservation of momentum is the given by the following vector equation

$$\rho \boldsymbol{u}_t + \nabla_{\boldsymbol{y}} p J_y = 0.$$

In one dimension we can use the definition of the Lagrange coordinates given in (3.5), which means that we get $J_y = y_x = \rho$. Consequently,

$$\rho u_t + p_y \rho = 0 \implies u_t + p_y = 0,$$

is a rewritten version of the Lagrange representation of conservation of momentum.

3.2.3 Conservation of energy

Euler point of view

The total energy per mass unit is given by the sum of the internal energy per mass unit, e, and the kinetic energy per mass unit, which is given by $\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}$ [3, Section **iii** in Chapter 1.1]. Thus, the total energy in $\Omega(t)$ is

$$\text{Total energy in } \Omega(t) = \int_{\Omega(t)} \rho(\boldsymbol{x}, t) \left(e(\boldsymbol{x}, t) + \frac{1}{2}\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t) \right) \mathrm{d}\Omega(t).$$

To derive the conservation of energy equation we use the assumption that the rate of change in energy is only due to the work done by the pressure force on the boundary of $\Omega(t)$. The work done by the pressure force per area on the boundary is given by $(p\boldsymbol{u}) \cdot \boldsymbol{N}$ [3, Section **iii** in Chapter 1.1] and thus we get

$$\frac{d}{dt} \int_{\Omega(t)} \rho(\boldsymbol{x}, t) \left(e(\boldsymbol{x}, t) + \frac{1}{2}\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t) \right) \mathrm{d}\Omega(t) = \int_{\partial\Omega(t)} (p(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t)) \cdot \boldsymbol{N} \mathrm{d}\partial\Omega(t).$$

Using Reynolds Transport Theorem, as stated in Equation (3.9), on the left-hand side and the Divergence Theorem [22, Theorem 8.3] on the right-hand side we get

$$\int_{\Omega(t)} \left(\rho e + \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} \right)_t + \operatorname{div}_{\boldsymbol{x}} \left(\boldsymbol{u} \left(\rho e + \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} + p \right) \right) \mathrm{d}\Omega(t) = 0.$$

This holds for all $\Omega(t)$ and hence we have that the conservation of energy in Euler representation is given by

$$\left(\rho e + \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} \right)_t + \operatorname{div}_{\boldsymbol{x}} \left(\boldsymbol{u} \left(\rho e + \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} + p \right) \right) = 0.$$

In one dimension, this is equivalent to

$$\left(\rho e + \frac{1}{2}\rho u^2 \right)_t + \left(u \left(\rho e + \frac{1}{2}\rho u^2 + p \right) \right)_x = 0.$$

Lagrange point of view

To derive the Lagrange representation of the conservation of energy we again assume that the only force acting on the fluid is the pressure force. The work done by the pressure force per unit volume per time is $-\operatorname{div}_{\boldsymbol{x}}(p\boldsymbol{x}_t) = -\operatorname{div}_{\boldsymbol{x}}(p\boldsymbol{u})$ [5, Page 16]. Using this, and that the total energy per unit mass is given by $e + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}$, we get the following formulation for conservation of energy

$$\left(\rho(\boldsymbol{x},t)\left(e(\boldsymbol{x},t) + \frac{1}{2}\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)\right)\right)_t + \mathrm{div}_{\boldsymbol{x}}(p(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)) = 0.$$

The variables are given in Euler coordinates, and we again have to convert to Lagrange coordinates by a similar technique as before. Doing this, and again using conservation of mass we get

$$\rho(\boldsymbol{y},t)\left(e(\boldsymbol{y},t) + \frac{1}{2}\boldsymbol{u}(\boldsymbol{y},t)\cdot\boldsymbol{u}(\boldsymbol{y},t)\right)_t + \mathrm{div}_{\boldsymbol{x}}(p(\boldsymbol{y},t)\boldsymbol{u}(\boldsymbol{y},t)) = 0.$$

The next step is to expand the differentiation with respect to $\boldsymbol{x}$ such that the differentiation is with respect to $\boldsymbol{y}$. Using the chain rule, we get

$$\begin{aligned}\mathrm{div}_{\boldsymbol{x}}(p\boldsymbol{u}) &= \frac{\partial(pu_1)}{\partial x_1} + \ldots + \frac{\partial(pu_n)}{\partial x_n}\\ &= \left(\frac{\partial(pu_1)}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \ldots + \frac{\partial(pu_1)}{\partial y_n}\frac{\partial y_n}{\partial x_1}\right) + \ldots + \left(\frac{\partial(pu_n)}{\partial y_1}\frac{\partial y_1}{\partial x_n} + \ldots + \frac{\partial(pu_n)}{\partial y_n}\frac{\partial y_n}{\partial x_n}\right)\\ &= \sum_{i=1}^{n}\nabla_y(pu_i)\cdot y_{x_i},\end{aligned}$$

where $\nabla_y(pu_i) = [(pu_i)_{y_1}, \ldots, (pu_i)_{y_n}]$ and $y_{x_i} = [(y_1)_{x_i}, \ldots, (y_n)_{x_i}] = \mathrm{Col}_i(J_y)$, i.e., the ith column of the matrix J_y given in (3.10). This results in

$$\rho\left(e + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_t + \sum_{i=1}^{n}\nabla_y(pu_i)\cdot\mathrm{Col}_i(J_y) = 0.$$

Using that y can be defined by (3.5) and thus $J_y = y_x = \rho$, we get the following equation for the Lagrange representation of energy conservation in one dimension

$$\left(e + \frac{1}{2}u^2\right)_t + (pu)_y = 0.$$

Chapter 4

Multidimensional balance law

In this chapter we will show that a general multidimensional balance law is preserved under a coordinate change. This can be used to show that different balance laws are equivalent in spatial and referential coordinates. In [8] Dafermos shows that the divergence formulation of a balance law is conserved under a bi-Lipschitz homeomorphism. In fact he starts out with a general formulation of a balance law, then shows that this can be reduced to a field equation. Furthermore, he proves that the field equation is indifferent to a change of coordinates. Lastly, he shows that we can obtain the original general formulation of the balance law from a field equation. In this chapter we are going to present the same proof as Dafermos, but with additional calculations and some other assumptions. Thus, many of the theorems and the proofs in this section are based on [8].

In Section 3.1 we derived the following formulation of a general balance law in the open domain $\mathcal{D}$ with piecewise twice continuous differentiable boundary

$$Q_{\mathcal{D}}\big(\partial\mathcal{D}\big) = P\big(\mathcal{D}\big). \tag{4.1}$$

Throughout this chapter we assume that P is an absolutely continuous Radon measure with respect to the Lebesgue measure.

In the proof that the general formulation of the balance law can reduce to a field equation, we will use the following lemma.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathcal{C}$ a hyperplane with co-dimension one and $\mathcal{W} = \mathcal{C} \cap \Omega \subset \mathbb{R}^{n-1}$. The set*

$$\Big\{ \sum_i \chi_{\mathcal{B}_i} : \mathcal{B}_i \subset \mathcal{W} \text{ open ball} \Big\}$$

is dense in $L^1\big(\mathcal{W};\mathcal{H}^{n-1}\big)$.

Proof. This proof is divided into the proof of five claims.
Claim 1: The simple functions are dense in L^1.
Proof of Claim 1: Let f be an arbitrary non-negative function in L^1, i.e., f is $\mathcal{H}^{n-1}$-measurable and $\int_{\mathcal{W}} |f| d\mathcal{H}^{n-1} < \infty$. By Lemma 2.3 there exists an increasing sequence of simple functions, $\{h_k\}$ such that h_k converges to f pointwise, i.e., $\lim_{k\to\infty} h_k(\boldsymbol{X}) = f(\boldsymbol{X})$. Thus, by using Monotone Convergence Theorem 2.21 we get

$$\int f(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = \lim_{k\to\infty} \int h_k(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}),$$

so $\|h_k - f\|_{L^1} \to 0$ as $k \to \infty$. To extend this to an arbitrary L^1-function, define $f^+ = \sup\{f, 0\}$ and $f^- = -\inf\{f, 0\}$, and write $f = f^+ - f^-$. By the above calculations, there exists a sequence of simple functions, $\widehat{h}_k$, satisfying $\|\widehat{h}_k - f^+\|_{L^1} \to 0$ and a sequence of simple functions, $\tilde{h}_k$, satisfying $\|\tilde{h}_k - f^-\|_{L^1} \to 0$. Furthermore, the function $h_k = \widehat{h}_k - \tilde{h}_k$ will be a simple function and

$$\|f - h_k\|_{L^1} \leq \|f^+ - \widehat{h}_k\|_{L^1} + \|f^- - \tilde{h}_k\|_{L^1} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, the claim is proved.
Claim 2: The set of functions $\mathcal{R} = \{\sum_i a_i \chi_{R_i} : \quad R_i = [a_{i,1}, b_{i,1}] \times \ldots \times [a_{i,n-1}, b_{i,n-1}] \subset \mathcal{W}\}$ is dense in the set of simple functions.
Proof of Claim 2: The proof of this claim is based on the proof of Proposition 10 in Section 7.4 in [25]. Since every simple function is a finite linear combination of characteristic functions, we only need to show that for every measurable A with finite measure, there exists a sequence $g_k \in \mathcal{R}$ such that $\|g_k - \chi_A\|_\infty \to 0$ as $k \to \infty$. It is sufficient to consider the sets A with finite $\mathcal{H}^{n-1}$-measure, since the simple function $h_k = \cup_{i=1}^N a_i \chi_{A_i}$ should approximate an integrable f, i.e., $\int h_k d\mathcal{H}^{n-1} \to \int f d\mathcal{H}^{n-1} < \infty$ and $\int h_k d\mathcal{H}^{n-1} = \sum_{i=1}^N a_i \mathcal{H}^{n-1}(A_i) < \infty$. Thus, $\mathcal{H}^{n-1}(A_i) < \infty$ as long as $a_i \neq 0$.

By the Borel regularity of $\mathcal{H}^{n-1}$ for all measurable A and all $\varepsilon > 0$ there exists an open set $U \supset A$ such that $\mathcal{H}^{n-1}(U \setminus A) < \varepsilon$. Furthermore, we have that every open set can be written as a countable union of pairwise disjoint rectangles R_i, such that $\mathcal{H}^{n-1}(U) = \mathcal{H}^{n-1}(\cup_i R_i) = \sum_i \mathcal{H}^{n-1}(R_i)$. Let

$$g_k = \sum_{i=1}^{k} \chi_{R_i} = \chi_{\bigcup_{i=1}^k R_i},$$

where the last equality is due to $\{R_i\}$ being pairwise disjoint. For all $\varepsilon > 0$ we can choose $K > 0$ such that for all $k > K$

$$\mathcal{H}^{n-1}\Big(U \setminus \bigcup_{i=1}^{k} R_i\Big) < \varepsilon,$$

where $\varepsilon \to 0$ as $k \to \infty$. Now, let us consider

$$\|g_k - \chi_A\|_{L^1} \leq \|g_k - \chi_U\|_{L^1} + \|\chi_U - \chi_A\|_{L^1} = \mathcal{H}^{n-1}\Big(U \setminus \bigcup_{i=1}^{k} R_i\Big) + \mathcal{H}^{n-1}\big(U \setminus A\big) < 2\varepsilon.$$

Thus, $g_k \to \chi_A$ in $L^1(\mathcal{W}; \mathcal{H}^{n-1})$ as $k \to \infty$ and the claim is proved.
Claim 3: Let $\mathbf{B} = \{\text{All closed balls, } \overline{\mathcal{B}} \subset \mathcal{W}\}$. For every $R \subset A$ and $\varepsilon > 0$, there exists a disjoint countable subfamily of $\mathbf{B}$, $\{\overline{\mathcal{B}}_i\}$, such that $\cup_i \overline{\mathcal{B}}_i \subset R$ and $\mathcal{H}^{n-1}\big(R \setminus \cup_i \overline{\mathcal{B}}_i\big) < \varepsilon$.
Proof of Claim 3: To prove this claim we use Vitali Covering Theorem 2.13. First, observe that since $\mathcal{H}^{n-1}(A) < \infty$ and by Theorem 2.7, $\mathcal{H}^{n-1} \llcorner A$ is a Radon measure, and for all subset $E \subset A$, $\mathcal{H}^{n-1} \llcorner A(E) = \mathcal{H}^{n-1}(E)$. In addition, since $\mathbf{B}$ consist of all closed balls we can choose a family of closed balls $\tilde{\mathbf{B}} \subset \mathbf{B}$ that finely covers R such that $R \supset \cup_{\mathcal{B} \in \tilde{\mathbf{B}}} \mathcal{B}$. Thus, by Vitali Covering Theorem, we have that a countable disjoint subfamily $\tilde{\mathbf{B}}' \subset \tilde{\mathbf{B}}$ exists, such that

$$\mathcal{H}^{n-1}\Big(R \setminus \bigcup_{\mathcal{B} \in \tilde{\mathbf{B}}'} \mathcal{B}\Big) = \mathcal{H}^{n-1} \llcorner A\Big(R \setminus \bigcup_{\mathcal{B} \in \tilde{\mathbf{B}}'} \mathcal{B}\Big) = 0$$

and the claim is proved.
Claim 5: The set $\{\sum_i \chi_{\mathcal{B}_i} : \mathcal{B}_i \subset \mathcal{W} \text{ open ball}\}$ is dense in $L^1(\mathcal{W}; \mathcal{H}^{n-1})$.

Proof of Claim 5: Let $f \in L^1(\mathcal{W};\mathcal{H}^{n-1})$, ϕ be a simple function and consider

$$\Big\|\sum_i a_i\chi_{\mathcal{B}_i} - f\Big\|_{L^1} \le \Big\|\sum_i a_i\chi_{\mathcal{B}_i} - \sum_i a_i\chi_{\overline{\mathcal{B}}_i}\Big\|_{L^1} + \Big\|\sum_i a_i\chi_{\overline{\mathcal{B}}_i} - \sum_j b_j\chi_{R_j}\Big\|_{L^1} + \Big\|\sum_j b_j\chi_{R_j} - \phi\Big\|_{L^1} + \|\phi - f\|_{L^1} < \varepsilon.$$

We have used the four previous claims in addition to the fact that $\mathcal{H}^{n-1}(\overline{\mathcal{B}}) = \mathcal{H}^{n-1}(\mathcal{B})$. The lemma is thus proved. □

The following theorem states that we can reduce the general balance law to a field equation, i.e., the balance law can be written as $\operatorname{div} A = P$.

Theorem 4.2 (Adapted from [8, Theorem 1.2.1])**.** *Consider the balance law* (4.1) *on* Ω *where* P *is a signed Radon measure that is absolute continuous with respect to the Lebesgue measure and the* $Q_{\mathcal{D}}$ *are induced, through* (3.1)*, by density flux functions* $q_{\mathcal{D}}$*. Assume that* $q_{\mathcal{D}}$ *is bounded, i.e.,* $|q_{\mathcal{D}}(\boldsymbol{X})| \le C$*, for all domains* $\mathcal{D}$ *with piecewise twice continuously differentiable boundary and any* $\boldsymbol{X} \in \partial\mathcal{D}$*. Then,*

(i) Let $\boldsymbol{N}$ *be a unit normal vector in* $\mathbb{R}^{n-1}$*, which is associated with a bounded measurable function* $a_{\boldsymbol{N}}$ *on* Ω*, with the following property: Let* $\mathcal{D}$ *be any domain with piecewise twice continuously differentiable boundary in* Ω *and suppose* $\boldsymbol{X}$ *is some point on* $\partial\mathcal{D}$ *where the outward normal to* $\mathcal{D}$ *exists and is* $\boldsymbol{N}$*. Then* $q_{\mathcal{D}}(\boldsymbol{X}) = a_{\boldsymbol{N}}(\boldsymbol{X})$ *for* $\mathcal{H}^{n-1}$ *almost every* $\boldsymbol{X}$*.*

(ii) There exists a vector field $A \in L^\infty(\Omega, \mathbb{M}^{1\times n})$ *such that, for any fixed unit normal vector* $\boldsymbol{N}$*,* $a_{\boldsymbol{N}}(\boldsymbol{X}) = A(\boldsymbol{X})\boldsymbol{N}$*, a.e. on* Ω*.*

(iii) The function A *satisfies the field equation* $\operatorname{div} A = P$*, in the sense of distributions on* Ω*.*

Proof. Let $\boldsymbol{N}$ be a unit normal vector in $\mathbb{R}^{n-1}$ and define $\mathcal{C}$ to be a hyperplane with co-dimension one and normal vector $\boldsymbol{N}$. Furthermore, let the intersection with Ω be nonempty. In addition, let $\mathcal{B}_r(\boldsymbol{X})$ be the ball with center at $\boldsymbol{X}$ and radius r and let $\mathcal{B}_r^-(\boldsymbol{X}) = \{\boldsymbol{Y} \in \mathcal{B}_r(\boldsymbol{X}) : (\boldsymbol{X} - \boldsymbol{Y}) \cdot \boldsymbol{N} < 0\}$. Furthermore, we define

$$a_{\boldsymbol{N}}(\boldsymbol{X}) = \lim_{r\to 0} \frac{1}{\mathcal{H}^{n-1}\big(\mathcal{C} \cap \mathcal{B}_r(\boldsymbol{X})\big)} \int_{\mathcal{C}\cap\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{B}_r^-(\boldsymbol{X})}(\boldsymbol{Y})\, d\mathcal{H}^{n-1}(\boldsymbol{Y}). \tag{4.2}$$

Since $\mathcal{H}^{n-1}$ is not a Radon measure, as stated in Lemma 2.20(v), we cannot directly use the Lebesgue Differentiation Theorem 2.11. However, we can use assertion (vi) in Lemma 2.20 to construct a Radon measure. Let $R > 0$ and $\mathcal{B}_R(\boldsymbol{X})$ be the ball with radius R in $\mathbb{R}^n$. Since $\mathcal{C}$ is a hyperplane with co-dimension one, $\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}$ is the ball with radius $R_0 \le R$ in $\mathbb{R}^{n-1}$. Thus, $\mathcal{H}^{n-1}\big(\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}\big)$ is the volume of the ball $\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}$ in $\mathbb{R}^{n-1}$, which is finite for all finite R, i.e., $\mathcal{H}^{n-1}\big(\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}\big) < \infty$. Thus, $\mathcal{H}^{n-1} \llcorner \{\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}\}$ is a Radon measure. In addition,

$$\mathcal{H}^{n-1}\big(\mathcal{B}_r(\boldsymbol{X}) \cap \mathcal{C}\big) = \mathcal{H}^{n-1} \llcorner \{\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}\}\big(\mathcal{B}_r(\boldsymbol{X}) \cap \mathcal{C}\big) \quad \forall r < R,$$

since $\mathcal{B}_r(\boldsymbol{X}) \subset \mathcal{B}_R(\boldsymbol{X})$ for all $r < R$, So, we can use the Lebesgue Differentiation Theorem 2.11 for the measure $\mathcal{H}^{n-1} \llcorner \{\mathcal{B}_R(\boldsymbol{X}) \cap \mathcal{C}\}$ and if we consider all balls in Ω, we get that $a_{\boldsymbol{N}}$ exists for almost all $\boldsymbol{X} \in \mathcal{C} \cap \Omega$. Furthermore, we get that $a_{\boldsymbol{N}}$ is bounded and $\mathcal{H}^{n-1}$-measurable since $|q_{\mathcal{D}}| < \infty$ and $q_{\mathcal{D}} \in L^1\big(\partial\mathcal{D}; \mathcal{H}^{n-1}(\boldsymbol{X})\big)$ for all $\mathcal{D}$. Next, we want to consider the properties of $a_{\boldsymbol{N}}$. Fix a hyperplane $\mathcal{C}$ with the normal $\boldsymbol{N}$ and let $\boldsymbol{X}_0 \in \mathcal{C} \cap \Omega$ be the center of the fixed ball $\mathcal{B}$. Furthermore, define

$$A_\tau = \{\boldsymbol{X} : \boldsymbol{X} - \tau \boldsymbol{Y} \in \mathcal{C} \cap \mathcal{B}\}, \tag{4.3}$$

and the domain

$$\mathcal{D} = \bigcup_{-\beta<\tau<\alpha} A_\tau, \tag{4.4}$$

where α, β are small non-negative numbers. We write the balance law for the cylindrical domain $\mathcal{D}$ as

$$\int_{\mathcal{D}} q_{\mathcal{D}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = P(\mathcal{D}).$$

We denote the lateral side of the cylinder with base $\mathcal{C} \cap \mathcal{B}$ and height τ by Cyl_τ. Using that both the production inside of $\mathcal{D}$ and the flux over the boundary of $\mathcal{D}$ are additive over disjoint subsets we can rewrite the balance law as

$$\begin{aligned}\int_{A_\alpha} a_{\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{A_{-\beta}} a_{-\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{\mathrm{Cyl}_\alpha} q_{\mathcal{D}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) \\ + \int_{\mathrm{Cyl}_{-\beta}} q_{\mathcal{D}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = P(\mathcal{D}),\end{aligned}$$

where we have used that at A_α and $A_{-\beta}$, by definition, $q_{\mathcal{D}}$ is equal to $a_{\boldsymbol{N}}$ and $a_{-\boldsymbol{N}}$, respectively. The integrals over Cyl_α and $\mathrm{Cyl}_{-\beta}$ can be estimated by

$$\begin{aligned}\Big| \int_{\mathrm{Cyl}_\alpha} q_{\mathcal{D}} d\mathcal{H}^{n-1}(\boldsymbol{X}) \Big| &\leq \int_{\mathrm{Cyl}_\alpha} |q_{\mathcal{D}}(\boldsymbol{X})| d\mathcal{H}^{n-1}(\boldsymbol{X}) \leq C \int_{\mathrm{Cyl}_\alpha} d\mathcal{H}^{n-1} = C\mathcal{H}^{n-2}(\mathcal{C} \cap \partial\mathcal{B})\alpha \\ &= O(\alpha).\end{aligned}$$

We have used that the $\mathcal{H}^{n-1}(\mathrm{Cyl}_\alpha)$ is the surface area of the lateral side of the cylinder, which is given by the height of the cylinder times the surface area of the boundary of the base, i.e., $\mathcal{H}^{n-1}(\mathrm{Cyl}_\alpha) = \mathcal{H}^{n-2}(\mathcal{C} \cap \partial\mathcal{B})\alpha$. So, we have rewritten the balance law as

$$\int_{A_\alpha} a_{\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{A_{-\beta}} a_{-\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = P(\mathcal{D}) + O(\alpha) + O(\beta). \tag{4.5}$$

By letting $\beta = 0$ and $\alpha \downarrow 0$ we get

$$\lim_{\alpha \downarrow 0} \int_{A_\alpha} a_{\boldsymbol{N}}(\boldsymbol{Y}) d\mathcal{H}^{n-1}(\boldsymbol{Y}) + \int_{\mathcal{C}\cap\mathcal{B}} a_{-\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = \lim_{\alpha \downarrow 0} P(\mathcal{D}),$$

where we have used that $A_\beta = \mathcal{C} \cap \mathcal{B}$ when $\beta = 0$. By doing a change of variables in the first integral, given by $\boldsymbol{X} = \boldsymbol{Y} - \alpha \boldsymbol{N}$, we get

$$\lim_{\alpha \downarrow 0} \int_{\mathcal{C}\cap\mathcal{B}} a_{\boldsymbol{N}}(\boldsymbol{X} + \alpha \boldsymbol{N}) d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{\mathcal{C}\cap\mathcal{B}} a_{-\boldsymbol{N}}(\boldsymbol{X}) d\mathcal{H}^{n-1}(\boldsymbol{X}) = \lim_{\alpha \downarrow 0} P(\mathcal{D}).$$

The domain $\mathcal{D}$ converges to $\mathcal{C} \cap \mathcal{B}$ when $\alpha \downarrow 0$. Furthermore,

$$|P(\mathcal{C} \cap \mathcal{B})| = \Big| \int_{\mathcal{C}\cap\mathcal{B}} \tilde{p}(\boldsymbol{X}) d\boldsymbol{X} \Big| \leq C m_n(\mathcal{C} \cap \mathcal{B}) = 0,$$

where we have used that P is absolutely continuous with respect to the Lebesgue measure in addition to $m_n(\mathcal{C} \cap \mathcal{B}) = 0$ since $\mathcal{C} \cap \mathcal{B} \in \mathbb{R}^{n-1}$. Thus,

$$\lim_{\alpha \downarrow 0} P(\mathcal{D}) = 0.$$

Hence,

$$\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\mathcal{C}\cap\mathcal{B}}d\mathcal{H}^{n-1}(\boldsymbol{X}) = -\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\mathcal{C}\cap\mathcal{B}}d\mathcal{H}^{n-1}(\boldsymbol{X}), \quad \forall\mathcal{B}\subset\Omega. \tag{4.6}$$

Let $\tilde{\mathcal{B}} = \mathcal{C}\cap\mathcal{B}$, which is an open ball in $\mathbb{R}^{n-1}$, and let $\{\tilde{\mathcal{B}}_i\}$ be a disjoint family of open balls such that $\mathcal{H}^{n-1}(\cup_i\tilde{\mathcal{B}}_i) < \infty$. We want to show that

$$\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\sum_{i=1}^{\infty}\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) = -\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\sum_{i=1}^{\infty}\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}).$$

To do this we have to show that

$$\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) = \sum_{i=1}^{\infty}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) \tag{4.7}$$

and

$$\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) = \sum_{i=1}^{\infty}\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}). \tag{4.8}$$

First, let us consider (4.7) and start by showing that the sum on the right-hand side is finite.

$$\begin{aligned}
\Big|\sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X})\Big| &\le \int_{\mathcal{C}\cap\Omega}|a_{-\boldsymbol{N}}(\boldsymbol{X})|\sum_{i=1}^{K}\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1} \\
&\le \|a_{-\boldsymbol{N}}\|_\infty\int_{\mathcal{C}\cap\Omega}\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1} \\
&= \|a_{-\boldsymbol{N}}\|_\infty\mathcal{H}^{n-1}\big(\cup_{i=1}^{K}\tilde{\mathcal{B}}_i\big) \\
&\le \|a_{-\boldsymbol{N}}\|_\infty\mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i\big) < \infty, \quad \forall K.
\end{aligned}$$

Hence, we have

$$\lim_{K\to\infty}\sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) < \infty,$$

and we can write

$$\lim_{K\to\infty}\sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) = \sum_{i=1}^{\infty}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}).$$

We are now ready to show (4.7).

$$\begin{aligned}
\sum_{i=1}^{\infty}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) &= \lim_{K\to\infty}\sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}) \\
&= \lim_{K\to\infty}\int_{\mathcal{C}\cap\Omega}\sum_{i=1}^{K} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}d\mathcal{H}^{n-1}(\boldsymbol{X}),
\end{aligned}$$

where the last equality is due to the finite sum. To use the Dominated Convergence Theorem 2.22, we have to show that $\sum_{i=1}^{K} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})$ converges pointwise to a $\mathcal{H}^{n-1}$-measurable limit function and $\sum_{i=1}^{K} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})$ is pointwise bounded by an unsigned absolutely integrable function G for all K. We have

$$\lim_{K\to\infty}\sum_{i=1}^{K} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X}) = a_{-\boldsymbol{N}}(\boldsymbol{X})\lim_{K\to\infty}\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i}(\boldsymbol{X}) = a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X}),$$

which is a $\mathcal{H}^{n-1}$-measurable function since $a_{-\boldsymbol{N}}$ is measurable by definition and $\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X})$ is a characteristic function for a $\mathcal{H}^{n-1}$-measurable set and thus measurable. Furthermore,

$$\Big|\sum_{i=1}^{K} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})\Big| \leq \|a_{-\boldsymbol{N}}\|_\infty \chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X}) = G(\boldsymbol{X})$$

and

$$\int_{\mathcal{C}\cap\Omega} G(\boldsymbol{X})\mathcal{H}^{n-1}(\boldsymbol{X}) = \|a_{-\boldsymbol{N}}\|_\infty \int_{\mathcal{C}\cap\Omega} \chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X})\mathcal{H}^{n-1}(\boldsymbol{X}) = \|a_{-\boldsymbol{N}}\|_\infty \mathcal{H}^{n-1}\Big(\bigcup_{i=1}^{\infty}\tilde{\mathcal{B}}_i\Big) < \infty.$$

Thus, we can use the Dominated Convergence Theorem 2.22 to conclude (4.7). Next, we consider (4.8). Again, we have to start by showing that the sum is finite.

$$\begin{aligned}\Big|\sum_{i=1}^{K} \lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\Big| &\leq \lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} |a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})|\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\\ &\leq \lim_{\tau\to 0}\|a_{\boldsymbol{N}}\|_\infty \int_{\mathcal{C}\cap\Omega}\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\\ &= \|a_{\boldsymbol{N}}\|_\infty \mathcal{H}^{n-1}\big(\cup_{i=1}^{K}\tilde{\mathcal{B}}_i\big)\\ &\leq \|a_{\boldsymbol{N}}\|_\infty \mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i\big) < \infty, \quad \forall K.\end{aligned}$$

So, as above we can write

$$\lim_{K\to\infty}\sum_{i=1}^{K}\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}) = \sum_{i=1}^{\infty}\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}),$$

and are ready to prove (4.8). First, we use Moore-Osgood's Theorem 2.23 to show that we can interchange the limit order. Due to Equation (4.6) we have that for all $\varepsilon > 0$ a $\tau > 0$ exists such that

$$\begin{aligned}&\Big|\sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}) + \sum_{i=1}^{K}\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\Big|\\ &\leq \sum_{i=1}^{K}\Big|\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\Big| < \sum_{i=1}^{K}\varepsilon/K = \varepsilon.\end{aligned}$$

Since $\mathcal{H}^{n-1}\big(\cup_{i=1}^{K}\tilde{\mathcal{B}}_i\big) \leq \mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i\big) < \infty$ for every K, for every $\varepsilon > 0$ there exists a $K > 0$ such that $\mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i \setminus \cup_{i=1}^{K}\tilde{\mathcal{B}}_i\big) < \varepsilon$. We can use this to conclude

$$\begin{aligned}&\Big|\int_{\mathcal{C}\cap\Omega}\sum_{i=1}^{K} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}) - \int_{\mathcal{C}\cap\Omega}\sum_{i=1}^{\infty} a_{N}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})\Big|\\ &\leq \int_{\mathcal{C}\cap\Omega}|a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})|\big|\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i} - \chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}\big| d\mathcal{H}^{n-1}(\boldsymbol{X}) \leq \|a_{\boldsymbol{N}}\|_\infty \mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i \setminus \cup_{i=1}^{K}\tilde{\mathcal{B}}_i\big)\\ &< \varepsilon,\end{aligned}$$

where ε is independent of τ. So,

$$\int_{\mathcal{C}\cap\Omega}\sum_{i=1}^{K} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X}) \to \int_{\mathcal{C}\cap\Omega}\sum_{i=1}^{\infty} a_{N}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\tilde{\mathcal{B}}_i} d\mathcal{H}^{n-1}(\boldsymbol{X})$$

uniformly in τ. Thus, the conditions in Moore-Osgood Theorem 2.23 are satisfied, and we can interchange the order of the limits. Again, we use the Dominated Convergence Theorem 2.22 to change the order of integration and the limit. Since

$$\sum_{i=1}^{K} \chi_{\tilde{\mathcal{B}}_i} \to \sum_{i=1}^{\infty} \chi_{\tilde{\mathcal{B}}_i} \quad \text{as} \quad k \to \infty,$$

$a_{\boldsymbol{N}}$ is bounded and $\mathcal{H}^{n-1}\big(\cup_{i=1}^{\infty} \chi_{\tilde{\mathcal{B}}_i}\big) < \infty$ the limit function is a $\mathcal{H}^{n-1}$-measurable function. Furthermore, we have

$$\Big|a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\sum_{i=1}^{K}\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})\Big| = \Big|a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\cup_{i=1}^{K}\tilde{\mathcal{B}}_i}(\boldsymbol{X})\Big| \leq \Big|a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X})\Big|$$

with

$$\int_{\mathcal{C}\cap\Omega}\Big|a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\chi_{\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i}(\boldsymbol{X})\Big|d\mathcal{H}^{n-1}(\boldsymbol{X}) \leq \|a_{\boldsymbol{N}}\|_\infty \mathcal{H}^{n-1}(\cup_{i=1}^{\infty}\tilde{\mathcal{B}}_i) < \infty.$$

Thus, the conditions of Dominated Convergence Theorem 2.22 are satisfied, and we have shown (4.8). So, by using (4.6), (4.7) and (4.8) we can conclude that

$$\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\sum_{i=1}^{\infty}\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) = -\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\sum_{i=1}^{\infty}\chi_{\tilde{\mathcal{B}}_i}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}).$$

So, by Lemma 4.1 we get

$$\lim_{\tau\downarrow 0}\int_{\mathcal{C}\cap\Omega} a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N})\phi d\mathcal{H}^{n-1}(\boldsymbol{X}) = -\int_{\mathcal{C}\cap\Omega} a_{-\boldsymbol{N}}(\boldsymbol{X})\phi d\mathcal{H}^{n-1}(\boldsymbol{X}), \quad \forall\phi\in L^1(\mathcal{C}\cap\Omega), \tag{4.9}$$

and $a_{\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N}) \to -a_{-\boldsymbol{N}}(\boldsymbol{X})$ as $\tau\downarrow 0$ in $L^\infty(\mathcal{C}\cap\Omega)$ weak*.

If we now let $\alpha = 0$ and $\beta\downarrow 0$ we get that, by a similar argumentation, $a_{-\boldsymbol{N}}(\boldsymbol{X}+\tau\boldsymbol{N}) \to -a_{\boldsymbol{N}}(\boldsymbol{X})$ as $\tau\uparrow 0$ in $L^\infty(\mathcal{C}\cap\Omega)$ weak*. Thus, $a_{\boldsymbol{N}}$ is Lebesgue measurable.

Now, we let $\alpha\downarrow 0$ and $\beta\downarrow 0$ in (4.5) and get

$$\begin{aligned} 0 &= \lim_{\alpha,\beta\downarrow 0}\Big\{\int_{A_\alpha} a_{\boldsymbol{N}}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{A_{-\beta}} a_{-\boldsymbol{N}}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X})\Big\} \\ &= \lim_{\alpha,\beta\downarrow 0}\Big\{\int_{\mathcal{C}\cap\mathcal{B}} a_{\boldsymbol{N}}(\boldsymbol{X}+\alpha\boldsymbol{N})d\mathcal{H}^{n-1}(\boldsymbol{X}) + \int_{\mathcal{C}\cap\mathcal{B}} a_{-\boldsymbol{N}}(\boldsymbol{X}-\beta\boldsymbol{N})d\mathcal{H}^{n-1}(\boldsymbol{X})\Big\} \\ &= \int_{\mathcal{C}\cap\mathcal{B}}\{a_{\boldsymbol{N}}(\boldsymbol{X}) + a_{-\boldsymbol{N}}(\boldsymbol{X})\}d\mathcal{H}^{n-1}(\boldsymbol{X}), \end{aligned} \tag{4.10}$$

where the second equality is due to a same change of variables as above and the third equality is due to Equation (4.9). Since Equation (4.10) holds true for all $\mathcal{B}\subset\Omega$ we have that $a_{-\boldsymbol{N}}(\boldsymbol{X}) = -a_{\boldsymbol{N}}(\boldsymbol{X})$ $\mathcal{H}^{n-1}$-a.e. in $\mathcal{C}\cap\Omega$. We are now ready to show $q_{\mathcal{D}}(\boldsymbol{X}) = a_{\boldsymbol{N}}(\boldsymbol{X})$. We start by considering any domain $\mathcal{D}\subset\Omega$ with piecewise twice continuously differentiable boundary. Fix a point $\boldsymbol{X}\in\partial\mathcal{D}$. The outward normal at $\boldsymbol{X}$ is denoted $\boldsymbol{N}$ and $\mathcal{C}$ is the tangential hyperplane. Let $r > 0$ be small, $\mathcal{B}_r(\boldsymbol{X})$ the ball with center at $\boldsymbol{X}$ and radius r and $\mathcal{B}_r^-(\boldsymbol{X})$ be the semiball defined above. Furthermore, the balance law for $\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})$ is

$$\int_{\partial\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{D}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) + \int_{\mathcal{D}\cap\partial\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) = P\big(\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})\big) \tag{4.11}$$

and for $\mathcal{B}_r^-(\boldsymbol{X})$ the balance law is

$$\begin{aligned}\int_{\mathcal{C}\cap\mathcal{B}_r(\boldsymbol{X})} a_{\boldsymbol{N}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) + \int_{\mathcal{D}\cap\partial\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) \\ + \int_{\mathcal{B}_r^-(\boldsymbol{X})\setminus(\mathcal{C}\cup\mathcal{D})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) = P\big(\mathcal{B}_r^-(\boldsymbol{X})\big).\end{aligned} \tag{4.12}$$

If we subtract (4.12) from (4.11) we obtain

$$\begin{aligned}\int_{\partial\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{D}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) - \int_{\mathcal{C}\cap\mathcal{B}_r(\boldsymbol{X})} a_{\boldsymbol{N}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) \\ = \int_{\partial\mathcal{B}_r^-(\boldsymbol{X})\setminus(\mathcal{C}\cup\mathcal{D})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) + P\big(\mathcal{B}_r^-(\boldsymbol{X})\setminus\mathcal{D}\big),\end{aligned} \tag{4.13}$$

where we have used that the production is additive over disjoint subsets. To bound the right-hand side of the equation, we first consider

$$\begin{aligned}\Big|\int_{\partial\mathcal{B}_r^-\setminus(\mathcal{C}\cup\mathcal{D})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y})\Big| &\leq \int_{\partial\mathcal{B}_r^-\setminus(\mathcal{C}\cup\mathcal{D})} |q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})|d\mathcal{H}^{n-1}(\boldsymbol{Y}) \\ &\leq C\mathcal{H}^{n-1}\big(\partial\mathcal{B}_r^-\setminus(\mathcal{C}\cup\mathcal{D})\big) \\ &= C\text{Ratio}(r)\mathcal{H}^{n-1}\big(\partial\mathcal{B}_r\big) = C\text{Ratio}(r)O(r^{n-1}).\end{aligned}$$

To determine how Ratio(r) depend on r we start by considering the case when $\mathcal{D}$ is a ball. In Figure 4.1 we see the domain $\mathcal{D}$ and the ball $\mathcal{B}_r(\boldsymbol{X})$ in the two-dimensional case. First, we calculate the area of the triangle depicted in Figure 4.1 by using Heron's formula [6, p. 12]

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the sides of the triangle and $s = (a+b+c)/2$. Using this we get that

$$h = \frac{2A}{R} = \frac{r}{R}\sqrt{R^2 - \frac{r^2}{4}}.$$

Next, we want to determine the angle $\theta = \arccos\frac{h}{r} = \arccos\Big(\frac{1}{R}\sqrt{R^2 - \frac{r^2}{4}}\Big)$. So,

$$\text{Ratio}(r) = \frac{\theta}{2\pi} = \frac{1}{2\pi}\arccos\Big(\frac{1}{R}\sqrt{R^2 - \frac{r^2}{4}}\Big).$$

To determine the r-dependence we consider

$$\lim_{r\to 0}\frac{\text{Ratio}(r)}{\sqrt{r}} = \lim_{r\to 0}\frac{\arccos\Big(\frac{1}{R}\sqrt{R^2 - \frac{r^2}{4}}\Big)}{2\pi\sqrt{r}} = \lim_{r\to 0}\frac{2\sqrt{r}}{\sqrt{4R^2 - r^2}} = 0.$$

Thus, we have shown that in two-dimension Ratio$(r) = o(r^{1/2})$. This result can easily be extended to higher dimensions. Every piecewise twice differentiable function can locally be approximated by a circle a.e., in fact the best circle approximation of a curve at a given point is the osculating circle which has the same tangent and curvature [15, pp. 99–100]. Hence,

$$\Big|\int_{\partial\mathcal{B}_r^-\setminus(\mathcal{C}\cup\mathcal{D})} q_{\mathcal{B}_r(\boldsymbol{X})}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y})\Big| = o(r^{n-1}).$$

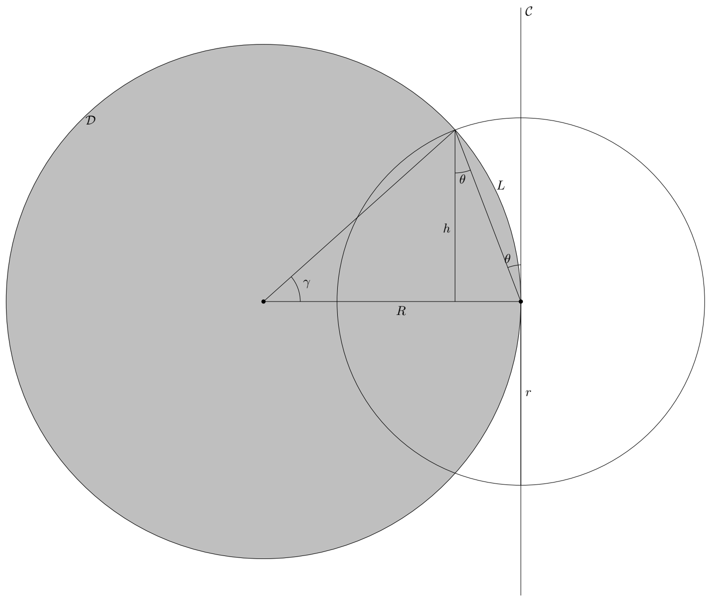

Figure 4.1: The figure illustrates the domain $\mathcal{D}$ as a circle with the tangent plane $\mathcal{C}$ in two dimensions. In addition, in the figure we can see the definition of the radii R and r, the angles γ and θ, the arc length L and lastly h.

Next, to bound $P\big(\mathcal{B}_r^-(\boldsymbol{X}) \setminus \mathcal{D}\big)$ we use Radon-Nikodym Theorem 2.9 to conclude that $|\tilde{p}(\boldsymbol{X})| = |\frac{dP}{dm_n}| \leq K < \infty$ m_n-a.e. Thus,

$$|P\big(\mathcal{B}_r^-(\boldsymbol{X}) \setminus \mathcal{D}\big)| = \Big|\int_{\mathcal{B}_r^-(\boldsymbol{X})\setminus\mathcal{D}} \tilde{p}(\boldsymbol{X}) d\boldsymbol{X}\Big| \leq \int_{\mathcal{B}_r^-(\boldsymbol{X})\setminus\mathcal{D}} \Big|\frac{dP}{dm_n}(\boldsymbol{X})\Big| d\boldsymbol{X} \leq K m_n\big(\mathcal{B}_r^-(\boldsymbol{X}) \setminus \mathcal{D}\big)$$
$$= O(r^n).$$

So, we have

$$\int_{\partial\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{D}}(\boldsymbol{Y}) d\mathcal{H}^{n-1}(\boldsymbol{Y}) - \int_{\mathcal{C}\cap\mathcal{B}_r(\boldsymbol{X})} a_{\boldsymbol{N}}(\boldsymbol{Y}) d\mathcal{H}^{n-1}(\boldsymbol{Y}) = o(r^{n-1}). \tag{4.14}$$

Let $c(n-1) = \mathcal{H}^{n-1}(\mathcal{B}_1)$, where $\mathcal{B}_1$ is the unit ball in $\mathbb{R}^{n-1}$, such that $\mathcal{H}^{n-1}\big(\mathcal{B}_r(\boldsymbol{X}) \cap \mathcal{C}\big) = c(n-1)r^{n-1}$. Furthermore, we can show that when r is small $\mathcal{H}^{n-1}(\partial\mathcal{D}\cap\mathcal{C}) \approx c(n-1)r^{n-1}$. We are going to prove this in the two-dimensional case, but as before we can extend the proof to higher dimensions. In two dimensions $c(1) = 2$ and $\mathcal{H}^{n-1}(\mathcal{B}_r(\boldsymbol{X}) \cap \mathcal{C}) = 2r$ Let L and ϕ be the arc length and angle depicted in Figure 4.1. By a similar computation as for θ, we get

$$\gamma = \arcsin\frac{h}{R} = \arcsin\Big(\frac{r}{R^2}\sqrt{R^2 - \frac{r^2}{4}}\Big),$$

and thus

$$L = R\gamma = R \arcsin\Big(\frac{r}{R^2}\sqrt{R^2 - \frac{r^2}{4}}\Big).$$

When r is small the series expansion of L is given by

$$L = R\Big(\frac{r}{R} + \frac{r^3}{24R^3} + O\big(r^5\big)\Big) = r + O\big(r^3\big),$$

where we have used that the series expansion of $\arcsin x$ at $x = 0$ is

$$\arcsin x = x + \frac{x^3}{6} + O\big(x^5\big).$$

So, in two dimensions, we have $\mathcal{H}^{n-1}\big(\partial\mathcal{D}\cap\mathcal{C}\big) = 2L \approx 2r = \mathcal{H}^{n-1}\big(B_r(\boldsymbol{X})\cap\mathcal{C}\big)$, for small r.

By dividing (4.14) by $c(n-1)r^{n-1}$ and letting $r\downarrow 0$ we obtain

$$\lim_{r\downarrow 0}\frac{1}{c(n-1)r^{n-1}}\Big\{\int_{\partial\mathcal{D}\cap\mathcal{B}_r(\boldsymbol{X})} q_{\mathcal{D}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y}) - \int_{\mathcal{C}\cap\mathcal{B}_r(\boldsymbol{X})} a_{\boldsymbol{N}}(\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{Y})\Big\}$$
$$= \lim_{r\downarrow 0}\frac{o(r^{n-1})}{c(n-1)r^{n-1}},$$

where the right-hand side is equal to zero by the definition of $o(r^{n-1})$. Thus, by Lebesgue Differentiation Theorem 2.11 $q_{\mathcal{D}} = a_{\boldsymbol{N}}$ $\mathcal{H}^{n-1}$-a.e.

We are going to use *Cauchy tetrahedron argument* [28, pp. 15–17] to prove assertion (*ii*). Let $\{\boldsymbol{E}_i\}_{i=1}^n$ be the standard orthonormal basis for $\mathbb{R}^n$ and define the vector field $A \in L^\infty\big(\Omega,\mathbb{M}^{1\times n}\big)$ by

$$A(\boldsymbol{X}) = \begin{bmatrix} a_{\boldsymbol{E}_i}(\boldsymbol{X}) & \dots & a_{\boldsymbol{E}_n}(\boldsymbol{X})\end{bmatrix}.$$

Furthermore, let $\boldsymbol{N}$ be a unit normal with components N_i. We assume that all of the components N_i are nonzero. In addition, let $\boldsymbol{X}$ be a Lebesgue point for $a_{\boldsymbol{E}_i}$, for all $i = 1,\dots,n$, and $a_{\boldsymbol{N}}$. Next, let $r>0$ and define

$$\mathcal{D} = \{Y : (Y_i - X_i)\operatorname{sgn} N_i > -r, i = 1,\dots,n; (\boldsymbol{Y}-\boldsymbol{X})\cdot\boldsymbol{N} < r\}.$$

The domain $\mathcal{D}$ has $n+1$ sides, given by $Y_i = -\operatorname{sgn} N_i r + X_i$ for $i = 1,\dots,n$ and $(\boldsymbol{Y}-\boldsymbol{X})\cdot\boldsymbol{N} = r$, and the ball $\mathcal{B}_r(\boldsymbol{X})$ is the inscribed sphere of $\mathcal{D}$. For instance, in two dimensions $\mathcal{D}$ is a triangle, and in three dimension a tetrahedron. We denote the side described by the equation $(\boldsymbol{Y}-\boldsymbol{X})\cdot\boldsymbol{N} = r$ and inequalities $(Y_i - X_i)\operatorname{sgn} N_i > -r$, for $i = 1,\dots,n$, by $\mathcal{C}$, with the normal $\boldsymbol{N}$, and, for a given i, $\mathcal{C}_i$ will denote the side described by the equation $Y_i = -\operatorname{sgn} N_i r + X_i$ and inequalities $(Y_j - X_j)\operatorname{sgn} N_j > -r$, for $j \neq i$, and $(\boldsymbol{Y}-\boldsymbol{X})\cdot\boldsymbol{N} < r$, with the normal $-\operatorname{sgn} N_i E_i$. We observe that $\mathcal{D}$ indeed have a piecewise twice continuously differentiable boundary and we can use the results proved above. Now, consider the balance law for the domain $\mathcal{D}$

$$\int_{\mathcal{D}} q_{\mathcal{D}}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) = P(\mathcal{D}).$$

Again, we use that the flux over the boundary of a domain is additive and divide the boundary of $\mathcal{D}$ in the sides $\mathcal{C}_i$, $i = 1,\dots,n$, and $\mathcal{C}$. We then get the following formulation of the balance law

$$\int_{\mathcal{C}} a_{\boldsymbol{N}}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) + \sum_{i=1}^n\int_{\mathcal{C}_i} a_{-\operatorname{sgn} N_i\boldsymbol{E}_i}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) = P(\mathcal{D}).$$

By using that $a_{-\boldsymbol{N}} = -a_{\boldsymbol{N}}$ and dividing by $\mathcal{H}^{n-1}(\mathcal{C})$, in addition to observing that $\mathcal{H}^{n-1}(\mathcal{C}_i) = |N_i|\mathcal{H}^{n-1}(\mathcal{C})$, we get

$$\frac{1}{\mathcal{H}^{n-1}(\mathcal{C})}\int_{\mathcal{C}} a_{\boldsymbol{N}}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) - \sum_{i=1}^n N_i\frac{1}{\mathcal{H}^{n-1}(\mathcal{C}_i)}\int_{\mathcal{C}_i} a_{E_i}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) = \frac{P(\mathcal{D})}{\mathcal{H}^{n-1}(\mathcal{C})},$$

where we have used that $N_i = \operatorname{sgn} N_i|N_i|$. Furthermore, we observe that $P(\mathcal{D}) = O(r^n)$ and $\mathcal{H}^{n-1}(\mathcal{C}) = O(r^{n-1})$, thus $\lim_{r\to 0} P(\mathcal{D})/\mathcal{H}^{n-1}(\mathcal{C}) = 0$. So, by letting $r\to 0$ and using Lebesgue Differential Theorem 2.11 we can conclude

$$a_{\boldsymbol{N}}(\boldsymbol{X}) = \sum_{i=1}^n N_i a_{\boldsymbol{E}_i}(\boldsymbol{X}) = A(\boldsymbol{X})\boldsymbol{N},$$

and assertion (*ii*) is proved.

Lastly, we prove assertion (*iii*), i.e., $\operatorname{div} A = P$ in the sense of distributions. Let ω_ε be a standard mollifier and let $A_\varepsilon = A * \omega_\varepsilon$ and $P_\varepsilon = P * \omega_\varepsilon$. We start by observing that, since P is absolutely continuous with respect to the Lebesgue measure,

$$P_\varepsilon(\boldsymbol{X}) = \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y}) dP(\boldsymbol{X}-\boldsymbol{Y}) = \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y})\tilde{p}(\boldsymbol{X}-\boldsymbol{Y})d\boldsymbol{Y} = \tilde{p}_\varepsilon(\boldsymbol{X}),$$

where we have used that $\tilde{p}$ is the production density function.

Let $\mathcal{D} \subset \Omega$ be a hypercube and consider the Divergence Theorem [22, Theorem 8.3] for A_ε on $\mathcal{D}$

$$\begin{aligned}
\int_{\mathcal{D}} \operatorname{div} A_\varepsilon(\boldsymbol{X})d\boldsymbol{X} &= \int_{\partial\mathcal{D}} A_\varepsilon(\boldsymbol{X})\boldsymbol{N}(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) \\
&= \int_{\partial\mathcal{D}}\int_{\mathbb{R}^n} A(\boldsymbol{X}-\boldsymbol{Y})\boldsymbol{N}(\boldsymbol{X}-\boldsymbol{Y})\omega_\varepsilon(Y)d\boldsymbol{Y}\, d\mathcal{H}^{n-1}(\boldsymbol{X}) \\
&= \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y}) \int_{\partial\mathcal{D}} A(\boldsymbol{X}-\boldsymbol{Y})\boldsymbol{N}(\boldsymbol{X}-\boldsymbol{Y})d\mathcal{H}^{n-1}(\boldsymbol{X})d\boldsymbol{Y} \\
&= \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y}) \int_{\partial\mathcal{D}_{\boldsymbol{Y}}} A(\boldsymbol{Z})\boldsymbol{N}(\boldsymbol{Z})d\mathcal{H}^{n-1}(\boldsymbol{Z})d\boldsymbol{Y}.
\end{aligned}$$

Here, $\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{Y}$ such that $\mathcal{D}_{\boldsymbol{Y}} = \{\boldsymbol{Z} : \boldsymbol{Z} + \boldsymbol{Y} \in \mathcal{D}\}$ is the $\boldsymbol{Y}$-transformation of $\mathcal{D}$. To arrive at the equation above we have used the definition of $A_\varepsilon = A * \omega_\varepsilon$ and Fubini's Theorem 2.2. Next, we use the balance law to conclude that

$$\int_{\partial\mathcal{D}} A(\boldsymbol{Z})N(\boldsymbol{Z})d\mathcal{H}^{n-1}(\boldsymbol{Z}) = \int_{\partial\mathcal{D}} a_{\boldsymbol{N}}(\boldsymbol{Z})d\mathcal{H}^{n-1}(\boldsymbol{Z}) = P(\mathcal{D}_{\boldsymbol{Y}}),$$

for almost all $\boldsymbol{Y}$ satisfying $|\boldsymbol{Y}| < \varepsilon$. Thus,

$$\begin{aligned}
\int_{\mathcal{D}} \operatorname{div} A_\varepsilon(\boldsymbol{X})d\boldsymbol{X} &= \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y})P(\mathcal{D}_{\boldsymbol{Y}})d\boldsymbol{Y} = \int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y}) \int_{\mathcal{D}_{\boldsymbol{Y}}} \tilde{p}(\boldsymbol{Z})d\boldsymbol{Z}d\boldsymbol{Y} \\
&= \int_{\mathcal{D}}\int_{\mathbb{R}^n} \omega_\varepsilon(\boldsymbol{Y})\tilde{p}(\boldsymbol{X}-\boldsymbol{Y})d\boldsymbol{Y}d\boldsymbol{X} = \int_{\mathcal{D}} P_\varepsilon(\boldsymbol{X})d\boldsymbol{X}
\end{aligned}$$

and $\operatorname{div} A_\varepsilon = P_\varepsilon$. So, for all test functions ϕ we can write

$$\int_{\mathbb{R}^n} \operatorname{div} A_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} = \int_{\mathbb{R}^n} P_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X},$$

which we will use to show that the equality holds in the sense of distributions when $\varepsilon \to 0$. We start by observing that $A \in L^\infty(\Omega, \mathbb{M}^{1\times n})$ implies that $A \in L^1_{loc}(\Omega, \mathbb{M}^{1\times n})$. Thus, from property (*iii*) in Theorem 2.25 we have that $A_\varepsilon \to A$ in L^1_{loc}, and

$$\phi \mapsto \int_{\mathbb{R}^n} A(\boldsymbol{X})\boldsymbol{E}_i\phi(\boldsymbol{X})d\boldsymbol{X}$$

defines a distribution. Furthermore, since P is absolutely continuous with respect to Lebesgue measure

$$\phi \mapsto \int_{\mathbb{R}^n} \tilde{p}(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X}$$

will also define a distribution. Here, $\tilde{p} \in L^1_{loc}$ is the production density function. Thus, if we show that

$$\int_{\mathbb{R}^n} \operatorname{div} A_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} \to -\int_{\mathbb{R}^n} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X}, \qquad \forall\phi \text{ test function}$$

and
$$\int_{\mathbb{R}^n} P_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} \to \int_{\mathbb{R}^n} \tilde{p}(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X}, \qquad \forall\phi \text{ test function}$$
we have proved the assertion. We have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \operatorname{div} A_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} + \int_{\mathbb{R}^n} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X}\Big| &= \Big| - \int_{\mathbb{R}^n} A_\varepsilon(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X} \\ &\quad + \int_{\mathbb{R}^n} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X}\Big| \\ &\le \int_{\mathbb{R}^n} \big|A_\varepsilon(\boldsymbol{X}) - A(\boldsymbol{X})\big||\nabla\phi(\boldsymbol{X})|d\boldsymbol{X} \\ &< \delta\|\nabla\phi\|_\infty,\end{aligned}$$

where we used that $\nabla\phi$ has compact support and $A_\varepsilon \to A$ in L^1_{loc}. $\delta\|\nabla\phi\|_\infty \to 0$ as $\delta \to 0$, since $\|\partial\phi/\partial X_i\|_\infty < \infty$ for all $i = 1, \dots, n$. In addition, we have

$$\Big|\int_{\mathbb{R}^n} \tilde{p}_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} - \int_{\mathbb{R}^n} p(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X}\Big| \le \int_{\mathbb{R}^n} |\tilde{p}_\varepsilon(\boldsymbol{X}) - p(\boldsymbol{X})|\phi(\boldsymbol{X})d\boldsymbol{X} \xrightarrow{\varepsilon\to 0} 0,$$

since ϕ has compact support, $\|\phi\|_\infty < \infty$ for all $i = 1, \dots, n$ and $\tilde{p}_\varepsilon \to \tilde{p}$ in L^1_{loc}, by proposition (iii) in Theorem 2.25. Thus, we have

$$\int_{\mathbb{R}^n} \operatorname{div} A_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} = \int_{\mathbb{R}^n} P_\varepsilon(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X} \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^n} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X} + \langle P, \phi\rangle = 0,$$

where $\langle P, \phi\rangle = \int \tilde{p}(\boldsymbol{X})\phi(\boldsymbol{X})d\boldsymbol{X}$. So, we have proved that $\operatorname{div} A = P$ in the sense of distributions. □

The next theorem state that the field equation $\operatorname{div} A = P$ is preserved under a bi-Lipschitz coordinate change. In this theorem, $\boldsymbol{X} \in \Omega$ and $\boldsymbol{Y} \in \Omega^*$. Furthermore, $A = A(\boldsymbol{X})$ and $A^* = A^*(\boldsymbol{Y})$, similarly for P, ϕ and a.

Theorem 4.3 (Adapted from [8, Theorem 1.3.1]). *Let Ω be an open subset of $\mathbb{R}^n$, $A \in L^1_{loc}(\Omega, \mathbb{M}^{1\times n})$ and P be a Radon measure on Ω. Furthermore, let A and P satisfy the field equation*
$$\operatorname{div} A = P, \tag{4.15}$$
in the sense of distributions on Ω. Consider any bi-Lipschitz homeomorphism T of Ω to a subset Ω^* of $\mathbb{R}^n$, with Jacobian matrix
$$J = \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}} \tag{4.16}$$
such that
$$\det J \ge a > 0, \quad \text{a.e. on } \Omega. \tag{4.17}$$
The, $A^* \in L^1_{loc}(\Omega^*, \mathbb{M}^{1\times n})$ and P^*, Radon measure, defined by
$$A^* \circ T = (\det J)^{-1} A J^\top, \tag{4.18}$$
$$\langle P^*, \phi^*\rangle = \langle P, \phi\rangle, \quad \text{where } \phi = \phi^* \circ T, \tag{4.19}$$
satisfy the field equation
$$\operatorname{div} A^* = P^*, \tag{4.20}$$
in the sense of distributions on Ω^*.

Proof. We start by stating the weak formulation of $\operatorname{div} A = P$. In Section 2.4 we have shown that it is sufficient that ϕ is a Lipschitz function with compact support, i.e., $\phi \in W_c^{1,\infty}(\Omega)$, for ϕ to be test function. Thus the weak formulation is

$$\int_\Omega A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X} + \langle P, \phi\rangle = 0, \qquad \forall \phi \in W_c^{1,\infty}(\Omega).$$

Let ϕ^* be a test function in Ω^*. Since T is a bi-Lipschitz function, we then have that $\phi = \phi^* \circ T$ is a test function as well. Furthermore, we have that

$$\begin{aligned}\nabla\phi = \nabla(\phi^*\circ T) &= \Big[\frac{\partial}{\partial X_1}(\phi^*\circ T)_1, \dots, \frac{\partial}{\partial X_n}(\phi^*\circ T)_n\Big] \\ &= \Big[\frac{\partial\phi_1^*}{\partial Y_1}\frac{\partial Y_1}{\partial X_1} + \dots + \frac{\partial\phi_1^*}{\partial Y_n}\frac{\partial Y_n}{\partial X_1}, \dots, \frac{\partial\phi_n^*}{\partial Y_1}\frac{\partial Y_1}{\partial X_n} + \dots + \frac{\partial\phi_n^*}{\partial Y_n}\frac{\partial Y_n}{\partial X_n}\Big] = J^\top\nabla\phi^*,\end{aligned}$$

and $d\boldsymbol{Y} = \det J d\boldsymbol{X}$. Using this we get,

$$\begin{aligned}\int_{\Omega^*} A^*(\boldsymbol{Y})\nabla\phi^*(\boldsymbol{Y})d\boldsymbol{Y} + \langle P^*, \phi^*\rangle &= \int_\Omega (A^*\circ T)(J^\top)^{-1}\nabla\phi \det J d\boldsymbol{X} + \langle P, \phi\rangle \\ &= \int_\Omega (\det J)^{-1} A J^\top (J^\top)^{-1}\nabla\phi \det J d\boldsymbol{X} + \langle P, \phi\rangle \\ &= \int_\Omega A\nabla\phi d\boldsymbol{X} + \langle P, \phi\rangle = 0,\end{aligned}$$

where we have used (4.18), (4.19) and that since T is invertible, so is J. □

Furthermore, we want to show that we can obtain the original balance law from a field equation. We start by showing this when we are considering planar surfaces. In this thesis we have assumed that P is absolutely continuous with respect to Lebesgue measure, thus we have rewritten Lemma 1.3.3 in [8] as the following lemma.

Lemma 4.4. *Let $A \in L^\infty(\mathcal{K}, \mathbb{M}^{1\times n})$ and P be a Radon measure which is absolutely continuous with respect to Lebesgue measure, such that $\operatorname{div} A = P$ in the sense of distributions, on a cylindrical domain $\mathcal{K} = \mathcal{B} \times (\alpha, \beta)$, where $\mathcal{B}$ is a ball in $\mathbb{R}^{n-1}$. Let $\boldsymbol{E}_n$ denote the nth unit base-vector in $\mathbb{R}^n$ and set $\boldsymbol{X} = (\boldsymbol{x}, t)$, with $\boldsymbol{x}$ in $\mathcal{B}$ and t in (α, β). Since, P is absolutely continuous with respect to Lebesgue measure, the function $\tau \mapsto a(\cdot, \tau)$ is continuous on (α, β), in the weak* topology of $L^\infty(\mathcal{B})$.*

Then, after one modifies, if necessary, A on a set of measure zero, the function $a(\boldsymbol{x}, t) = A(\boldsymbol{x}, t)\cdot\boldsymbol{E}_n$ satisfies

$$a(\boldsymbol{x}, \tau) = \lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{\tau-\varepsilon}^{\tau} A(\boldsymbol{x}, t)\boldsymbol{E}_n dt, \tag{4.21a}$$

$$a(\boldsymbol{x}, \tau) = \lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{\tau}^{\tau+\varepsilon} A(\boldsymbol{x}, t)\boldsymbol{E}_n dt. \tag{4.21b}$$

Furthermore, for any $\tau \in (\alpha, \beta)$ and any Lipschitz continuous function ϕ with compact support in $\mathcal{K}$,

$$\int_{\mathcal{B}} a(\boldsymbol{x}, \tau)\phi(\boldsymbol{x}, \tau)d\boldsymbol{x} = \int_{\mathcal{B}\times(\alpha,\tau)} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X} + \langle P, \phi\rangle_{\mathcal{B}\times(\alpha,\tau)}, \tag{4.22a}$$

$$-\int_{\mathcal{B}} a(\boldsymbol{x}, \tau)\phi(\boldsymbol{x}, \tau)d\boldsymbol{x} = \int_{\mathcal{B}\times(\tau,\beta)} A(\boldsymbol{X})\nabla\phi(\boldsymbol{X})d\boldsymbol{X} + \langle P, \phi\rangle_{\mathcal{B}\times(\tau,\beta)}. \tag{4.22b}$$

Proof. Let $\mathcal{B}$ be a ball in $\mathbb{R}^{n-1}$ with radius r and $\mathcal{B}_\varepsilon$ be the ball with the same center as $\mathcal{B}$ and radius $r-\varepsilon$. Furthermore, let ω_ε be a standard mollifier and define $A_\varepsilon = A * \omega_\varepsilon$ and $P_\varepsilon = P * \omega_\varepsilon$. Then A_ε and P_ε are smooth fields that satisfy $\operatorname{div} A_\varepsilon = P_\varepsilon$. Furthermore, we define $a_\varepsilon = A_\varepsilon \cdot \boldsymbol{E}_n$. By multiplying $\operatorname{div} A_\varepsilon = P_\varepsilon$ by a test function $\phi(\boldsymbol{x}) \in W_c^{1,\infty}\big(\mathcal{B}_\varepsilon, \mathbb{R}^{n-1}\big)$ and integrating over $\mathcal{B}_\varepsilon \times (r,s)$, $\alpha+\varepsilon < r < s < \beta - \varepsilon$ we obtain

$$\int_r^s \int_{\mathcal{B}_\varepsilon} \operatorname{div} A_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x} dt = \int_r^s \int_{\mathcal{B}_\varepsilon} P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x} dt.$$

Next, we do integration by parts on the left-hand side. This results in

$$\int_r^s \int_{\mathcal{B}_\varepsilon} \operatorname{div} A_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x} dt = \int_{\mathcal{B}_\varepsilon} A_\varepsilon(\boldsymbol{x},t) \cdot \boldsymbol{E}_n \phi(\boldsymbol{x}) d\boldsymbol{x}\Big|_{t=r}^{t=s} - \int_r^s \int_{\mathcal{B}_\varepsilon} A_\varepsilon(\boldsymbol{x},t) \cdot \nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) d\boldsymbol{x} dt,$$

where we have used that $\frac{\partial}{\partial t}\phi(\boldsymbol{x}) = 0$ and, due to the compact support of ϕ, $\phi(\boldsymbol{x})|_{\boldsymbol{x}\in\partial\mathcal{B}_\varepsilon} = 0$. If we use that $a_\varepsilon(\boldsymbol{x},t) = A_\varepsilon(\boldsymbol{x},t)\cdot \boldsymbol{E}_n$, we get

$$\begin{aligned} &\int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},s)\phi(\boldsymbol{x}) d\boldsymbol{x} - \int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x} \\ &\qquad = \int_r^s \int_{\mathcal{B}_\varepsilon} \Big\{ A_\varepsilon(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) \Big\} d\boldsymbol{x} dt. \end{aligned} \tag{4.23}$$

The next step is to show that the total variation of the function $t \mapsto \int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x}$, over the interval $(\alpha+\varepsilon, \beta-\varepsilon)$, is bounded, uniformly in $\varepsilon > 0$. From Definition 2.22 we have that

$$T.V.(f) = \sup_{\mathcal{I}} \sum_{i\in\mathcal{I}} |f(x_{i+1}) - f(x_i)|, \tag{4.24}$$

where the supremum is taken over all finite partitions, $\mathcal{I}$, of $(\alpha+\varepsilon, \beta-\varepsilon)$. Using this and Equation (4.23) we get

$$\begin{aligned} T.V.\Big(\int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) d\boldsymbol{x}\Big) &= \sup_{\mathcal{I}} \sum_{i\in\mathcal{I}} \Big| \int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t_{i+1})\phi(\boldsymbol{x}) d\boldsymbol{x} - \int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t_i)\phi(\boldsymbol{x}) d\boldsymbol{x} \Big| \\ &= \sup_{\mathcal{I}} \sum_{i\in\mathcal{I}} \Big| \int_{t_i}^{t_{i+1}} \int_{\mathcal{B}_\varepsilon} \big\{ A_\varepsilon(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x}) \big\} d\boldsymbol{x} dt \Big| \\ &\le \sup_{\mathcal{I}} \sum_{i\in\mathcal{I}} \int_{t_i}^{t_{i+1}} \int_{\mathcal{B}_\varepsilon} \big\{ |A_\varepsilon(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x})| \big\} d\boldsymbol{x} dt \\ &\le \sup_{\mathcal{I}} \sum_{i\in\mathcal{I}} (t_{i+1} - t_i) \Big\{ \sum_{j=1}^n \sup_{\substack{\boldsymbol{x}\in\mathcal{B} \\ t\in(\alpha,\beta)}} |A_\varepsilon(\boldsymbol{x},t)\cdot\boldsymbol{E}_j| \int_{\mathcal{B}_\varepsilon} |\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})\cdot\boldsymbol{E}_j| d\boldsymbol{x} \\ &\qquad\qquad + \sup_{\substack{\boldsymbol{x}\in\mathcal{B} \\ t\in(\alpha,\beta)}} |P_\varepsilon(\boldsymbol{x},t)| \int_{\mathcal{B}_\varepsilon} |\phi(\boldsymbol{x})| d\boldsymbol{x} \Big\} \\ &< K(\beta-\alpha) \end{aligned}$$

To show that there exists a constant $K < \infty$ independent of ε, such that the last inequality above holds true, we use that $A \in L^\infty\big(\mathcal{K}, \mathbb{M}^{1\times n}\big)$ and that P is absolutely continuous with respect to Lebesgue measure. We have that

$$|A_\varepsilon\cdot\boldsymbol{E}_n| = \Big| \int A(\boldsymbol{X}-\boldsymbol{Y})\cdot\boldsymbol{E}_n \omega_\varepsilon(\boldsymbol{Y}) d\boldsymbol{Y} \Big| \le \|A\cdot\boldsymbol{E}_n\|_\infty \Big| \int \omega_\varepsilon(\boldsymbol{Y}) d\boldsymbol{Y} \Big| = \|A\cdot\boldsymbol{E}_n\|_\infty < \infty,$$

and hence $\|A_\varepsilon\cdot\boldsymbol{E}_n\|_\infty \le K_1 < \infty$, where K_1 is independent of ε. Furthermore, we have that

$$|P_\varepsilon| = \Big| \int \omega_\varepsilon(\boldsymbol{X}-\boldsymbol{Y}) dP(\boldsymbol{X}) \Big| = \Big| \int \omega_\varepsilon(\boldsymbol{X}-\boldsymbol{Y}) \tilde{p}(\boldsymbol{Y}) d\boldsymbol{Y} \le \|\tilde{p}\|_\infty < \infty,$$

since $\tilde{p} \in L^1_{loc}(\Omega)$. Thus, $\|P_\varepsilon\|_\infty \leq \|P\|_\infty \leq K_2 < \infty$, K_2 independent of ε. In addition, we use that $\phi, \nabla_{\boldsymbol{x}}\phi \in L^1_{loc}$, since $\phi \in W^{1,\infty}_c$. We then get that $T.V.\Big(\int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x})d\boldsymbol{x}\Big)$ is uniformly bounded in $\varepsilon > 0$. In addition,

$$\Big\|\int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x})d\boldsymbol{x}\Big\|_\infty = \sup_{t\in(\alpha,\beta)}\Big|\int_{\mathcal{B}_\varepsilon} A_\varepsilon(\boldsymbol{x},t)\cdot\boldsymbol{E}_n\phi(\boldsymbol{x})d\boldsymbol{x}\Big|$$
$$\leq \sup_{t\in(\alpha,\beta),\boldsymbol{x}\in\mathcal{B}} |A(\boldsymbol{x},t)\cdot\boldsymbol{E}_n| \int_{\mathcal{B}}\phi(\boldsymbol{x})d\boldsymbol{x} \leq \|A\cdot\boldsymbol{E}_n\|_\infty C < \infty,$$

where $\int_{\mathcal{B}}\phi(\boldsymbol{x})d\boldsymbol{x} \leq C < \infty$. Let ϕ_l be a countable family of test functions that is dense in $L^1(\mathcal{B})$. We are going to use Helly's Selection Theorem 2.30 together with a diagonal argument to obtain a subsequence $\{\varepsilon_m\}$ such that $\int_{\mathcal{B}} a_{\varepsilon_m}\phi_l d\boldsymbol{x}$ converge for all ϕ_l. The conditions in Helly's Selection Theorem 2.30 are fulfilled and we have that there exist a subsequence $\int_{\mathcal{B}} a_{\varepsilon_k}\phi_1(\boldsymbol{x})d\boldsymbol{x}$ that is convergent for almost all $t \in (\alpha,\beta)$, except $t \in G$ with $m_1(G) = 0$. From this we extract a subsequence such that $\int_{\mathcal{B}} a_{\varepsilon_j}\phi_2(\boldsymbol{x})d\boldsymbol{x}$ converges. Continue this such that for all ϕ_l there exist a convergent subsequence. We now extract the diagonal sequence and obtain a subsequence $\{\int_{\mathcal{B}} a_{\varepsilon_m}\phi_l(\boldsymbol{x})d\boldsymbol{x}\}$ that is convergent, for almost all $t \in (\alpha,\beta)$, for all ϕ_l. For some function $t \mapsto a(\cdot,t)$ we define the limit function as $\int_{\mathcal{B}} a\phi_l(\boldsymbol{x})d\boldsymbol{x}$ for all $l = 1,2,\ldots$, which has bounded variation over (α,β). To prove that $a(\boldsymbol{x},t) = A(\boldsymbol{x},t)\cdot\boldsymbol{E}_n$ we consider

$$\lim_{m\to\infty}\int_{\mathcal{B}} a_{\varepsilon_m}(\boldsymbol{x},t)\phi_l(\boldsymbol{x})d\boldsymbol{x} = \lim_{m\to\infty}\int_{\mathcal{B}} A_{\varepsilon_m}(\boldsymbol{x},t)\cdot\boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathcal{B}} A(\boldsymbol{x},t)\cdot\boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x}.$$

The last inequality is due to $A_\varepsilon\cdot\boldsymbol{E}_n \to A\cdot\boldsymbol{E}_n$ in L^1_{loc} as $\varepsilon \to 0$, by proposition (iii) in Theorem 2.25, since $A\cdot\boldsymbol{E}_n \in L^1_{loc}$. Furthermore, we have that

$$\lim_{m\to\infty}\int_{\mathcal{B}} a_{\varepsilon_m}(\boldsymbol{x},t)\phi_l(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathcal{B}} a(\boldsymbol{x},t)\phi_l(\boldsymbol{x})d\boldsymbol{x}.$$

Thus,

$$\int_{\mathcal{B}} a(\boldsymbol{x},t)\phi_l(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathcal{B}} A(\boldsymbol{x},t)\cdot\boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x}, \quad \forall\phi_l,$$

and hence $a(\boldsymbol{x},t) = A(\boldsymbol{x},t)\cdot\boldsymbol{E}_n$ a.e. in $\mathcal{K}$. So, the function $a(\cdot,t)$ is independent of the choice of subsequence $\{\varepsilon_m\}$.

Let us again consider (4.23), but now to prove that, since P is absolutely continuous with respect to Lebesgue measure, $a(\cdot,t)$ is continuous on (α,β) in the weak* topology of $L^\infty(\mathcal{B})$. We start by proving that $\{\int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x},r)\phi(\boldsymbol{x})d\boldsymbol{x}\}$ is an equicontinuous family of functions.

$$\Big|\int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x},s)\phi(\boldsymbol{x})d\boldsymbol{x} - \int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x},r)\phi(\boldsymbol{x})d\boldsymbol{x}\Big| = \Big|\int_r^s\int_{\mathcal{B}}\big\{A_\varepsilon(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x})\big\}d\boldsymbol{x}dt\Big|$$
$$\leq |s-r|\Big(\sum_{j=1}^n\|A\cdot\boldsymbol{E}_j\|_\infty\int_{\mathcal{B}}\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})\cdot\boldsymbol{E}_j d\boldsymbol{x}$$
$$+ \|P\|_\infty\int_{\mathcal{B}}\phi(\boldsymbol{x})d\boldsymbol{x}\Big).$$

By a similar argument as in the proof of bounded variation of $\{\int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x})d\boldsymbol{x}\}$ we get

$$\sum_{j=1}^n\|A\cdot\boldsymbol{E}_j\|_\infty\int_{\mathcal{B}}\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})\cdot\boldsymbol{E}_j d\boldsymbol{x} + \|P\|_\infty\int_{\mathcal{B}}\phi(\boldsymbol{x})d\boldsymbol{x} \leq M < \infty,$$

where M is independent of ε. Thus, if we choose $\delta = \epsilon/M$ we get

$$\Big| \int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x}, s)\phi(\boldsymbol{x})d\boldsymbol{x} - \int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x}, r)\phi(\boldsymbol{x})d\boldsymbol{x}\Big| < \epsilon \quad \text{when} \quad |s - r| < \delta,$$

and by Definition 2.17 the family of functions $t \mapsto \int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x}, t)\phi(\boldsymbol{x})d\boldsymbol{x}$ is equicontinuous. Hence, $\int_{\mathcal{B}} a(\boldsymbol{x}, t)\phi(\boldsymbol{x})d\boldsymbol{x}$ is continuous on (α, β), for any $l = 1, 2, \ldots$. Thus, $t \mapsto a(\cdot, t)$ is continuous on (α, β), in $L^\infty(\mathcal{B})$ weak*.

To conclude (4.21a), we consider

$$\lim_{\delta\downarrow} \frac{1}{\delta} \int_{\tau-\delta}^{\delta} \int_{\mathcal{B}} a(\boldsymbol{x}, t)\phi_l(\boldsymbol{x})d\boldsymbol{x}dt = \lim_{\delta\downarrow} \frac{1}{\delta} \int_{\tau-\delta}^{\delta} \int_{\mathcal{B}} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x}dt.$$

We can use Lebesgue Differentiation Theorem 2.10 to conclude that the left-hand side is equal to $\int_{\mathcal{B}} a(\boldsymbol{x}, \tau)\phi_l(\boldsymbol{x})d\boldsymbol{x}$ since $a(\cdot, t)$ is continuous in the L^∞ weak* topology. Next, let us consider the right-hand side

$$RS = \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \int_{\mathcal{B}} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x}dt.$$

Since we have

$$\Big| \iint_{(\tau-\delta,\tau)\times\mathcal{B}} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})dtd\boldsymbol{x}\Big| \leq \|A \cdot \boldsymbol{E}_n\|_\infty \iint_{(\tau-\delta,\tau)\times\mathcal{B}} |\phi_l(\boldsymbol{x})|dtd\boldsymbol{x} \leq \|A \cdot \boldsymbol{E}_n\|_\infty C_l\delta,$$

where $\iint_{(\tau-\delta,\tau)\times\mathcal{B}} |\phi_l(\boldsymbol{x})|dtd\boldsymbol{x} \leq C_l < \infty$, we can use Fubini's Theorem 2.2 to interchange the order of integration. So,

$$RS = \lim_{\delta\downarrow 0} \int_{\mathcal{B}} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})d\boldsymbol{x}dt.$$

Furthermore, we observe that

$$\Big|\frac{1}{\delta} \int_{\tau-\delta}^{\tau} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})dt\Big| \leq \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \Big|A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})\Big|dt \leq \|A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\|_\infty|\phi_l(\boldsymbol{x})|.$$

In addition,

$$\int_{\mathcal{B}} \|A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\|_\infty|\phi_l(\boldsymbol{x})|d\boldsymbol{x} \leq \|A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\|_\infty C_l < \infty.$$

Thus, by Dominated Convergence Theorem 2.22 we have

$$RS = \int_{\mathcal{B}} \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})dtd\boldsymbol{x}.$$

So,

$$\int_{\mathcal{B}} a(\boldsymbol{x}, \tau)\phi_l(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathcal{B}} \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n\phi_l(\boldsymbol{x})dtd\boldsymbol{x}, \quad \forall \phi_l,$$

for all balls $\mathcal{B}$, and thus

$$a(\boldsymbol{x}, \tau) = \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{\tau-\delta}^{\tau} A(\boldsymbol{x}, t) \cdot \boldsymbol{E}_n dt.$$

We can show (4.21b) in a similar matter.

Lastly, we want to show (4.22a). Start by fixing a $\tau \in (\alpha, \beta)$ and again multiply $\operatorname{div} A_\varepsilon = P_\varepsilon$ by a test function ϕ and integrate over $(\alpha + \varepsilon, s) \times \mathcal{B}_\varepsilon$ where $s \in (\alpha + \varepsilon, \tau) \setminus G$. This yields

$$\int_{\alpha+\varepsilon}^{s} \int_{\mathcal{B}_\varepsilon} \operatorname{div} A_\varepsilon(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t) = \int_{\alpha+\varepsilon}^{s} \int_{\mathcal{B}_\varepsilon} P_\varepsilon(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t)d\boldsymbol{x}dt.$$

Doing integration by parts on the left-hand side yields

$$\int_{\alpha+\varepsilon}^{s}\int_{\mathcal{B}_\varepsilon} \operatorname{div} A_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x},t)d\boldsymbol{x}dt = \int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},s)\phi(\boldsymbol{x},s)d\boldsymbol{x} - \int_{\alpha+\varepsilon}^{s}\int_{\mathcal{B}_\varepsilon} A_\varepsilon(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt,$$

where we have used $\phi|_{\boldsymbol{x}\in\partial\mathcal{B}_\varepsilon} = 0$ and $\phi(\boldsymbol{x},\alpha+\varepsilon) = 0$, due to the compact support of ϕ. Thus, we get

$$\int_{\mathcal{B}_\varepsilon} a_\varepsilon(\boldsymbol{x},s)\phi(\boldsymbol{x},s)d\boldsymbol{x} = \iint_{(\alpha+\varepsilon,s)\times\mathcal{B}_\varepsilon} \Big\{A_\varepsilon(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\Big\}d\boldsymbol{x}dt.$$

Since $\phi(\boldsymbol{x},t)$ has compact support in $(\alpha+\varepsilon,s)\times\mathcal{B}_\varepsilon$, we have

$$\int_{\mathcal{B}} a_\varepsilon(\boldsymbol{x},s)\phi(\boldsymbol{x},s)d\boldsymbol{x} = \iint_{(\alpha,s)\times\mathcal{B}} \Big\{A_\varepsilon(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t) + P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\Big\}d\boldsymbol{x}dt.$$

Now, we let $\varepsilon\to 0$. In the proof of assertion (iii) in Theorem 4.2 we showed that

$$\iint_{(\alpha,s)\times\mathcal{B}} P_\varepsilon(\boldsymbol{x},t)\phi(\boldsymbol{x},t)d\boldsymbol{x}dt \to \langle P,\phi\rangle_{(\alpha,s)\times\mathcal{B}}$$

and

$$\iint_{(\alpha,s)\times\mathcal{B}} A_\varepsilon(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt \to \iint_{(\alpha,s)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)$$

as $\varepsilon\to 0$. Thus, for almost all $s\in(\alpha,\tau)$ we have

$$\int_{\mathcal{B}} a(\boldsymbol{x},s)\phi(\boldsymbol{x},s)d\boldsymbol{x} = \iint_{(\alpha,s)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt + \langle P(\boldsymbol{x},t),\phi(\boldsymbol{x},t)\rangle_{(\alpha,s)\times\mathcal{B}}.$$

Next, we let $s\uparrow\tau$. We first consider

$$\begin{aligned}\iint_{(\alpha,s)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt &= \iint_{(\alpha,\tau)\times\mathcal{B}} \chi_{(\alpha,s)\times\mathcal{B}}A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt\\ &\quad+\iint_{(\alpha,\tau)\times\mathcal{B}} \chi_{(\alpha,s)\times\mathcal{B}}p(\boldsymbol{x},t)\phi(\boldsymbol{x},t)d\boldsymbol{x}dt.\end{aligned}$$

If we let $s\uparrow\tau$, we have $\chi_{(\alpha,s)\times\mathcal{B}}\to\chi_{(\alpha,\tau)\times\mathcal{B}}$ and thus

$$\iint_{(\alpha,s)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt \to \iint_{(\alpha,\tau)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt.$$

Similarly, we can show that

$$\langle P(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\rangle_{(\alpha,s)\times\mathcal{B}} \to \langle P(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\rangle_{(\alpha,s)\times\mathcal{B}} \text{ as } s\uparrow\tau.$$

Thus, we get

$$\int_{\mathcal{B}} a(\boldsymbol{x},\tau)\phi(\boldsymbol{x},\tau)d\boldsymbol{x} = \iint_{(\alpha,\tau)\times\mathcal{B}} A(\boldsymbol{x},t)\cdot\nabla\phi(\boldsymbol{x},t)d\boldsymbol{x}dt + \langle P(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\rangle_{(\alpha,\tau)\times\mathcal{B}}, \tag{4.25}$$

where the limit on the left-hand side is due to the map $t\mapsto\int_{\mathcal{B}} a(\boldsymbol{x},t)\phi(\boldsymbol{x},t)d\boldsymbol{x}$ being continuous. The proof of (4.22b) is similar, but we integrate over $(s,\beta-\varepsilon)\times\mathcal{B}_\varepsilon$ where $s\in(\tau,\beta-\varepsilon)\setminus G$. □

The lemma above proved that it is possible to retrieve the density flux function $a_{\boldsymbol{E}_n}$ when we are considering planar surfaces. The next theorem state that we can retrieve the density flux function $q_{\mathcal{D}}$ even for arbitrary domains $\mathcal{D}$.

Theorem 4.5 (Adapted from [8, Theorem 1.3.4]). *Let $A \in L^\infty\big(\Omega, \mathbb{M}^{1\times n}\big)$ and P be a Radon measure which is absolutely continuous with respect to the Lebesgue measure, such that* $\operatorname{div} A = P$ *is satisfied in the sense of distributions, on an open subset Ω of $\mathbb{R}^n$. Then, for any proper domain $\mathcal{D}$ in Ω there exists a bounded $\mathcal{H}^{n-1}$-measurable function $q_{\mathcal{D}}$ on $\partial\mathcal{D}$ such that*

$$\int_{\partial\mathcal{D}} q_{\mathcal{D}}(\boldsymbol{X})\phi(\boldsymbol{X})d\mathcal{H}^{n-1}(\boldsymbol{X}) = \int_{\mathcal{D}} A(\boldsymbol{X})\cdot\nabla\phi(\boldsymbol{X})d\boldsymbol{X} + \langle P,\phi\rangle_{\mathcal{D}}, \tag{4.26}$$

for any Lipschitz continuous function ϕ on $\mathbb{R}^n$, with compact support in Ω.

Proof. Let $\mathcal{B}$ be the unit ball in $\mathbb{R}^{n-1}$. Furthermore, define the cylindrical domain

$$\mathcal{K}^* = \{\boldsymbol{Y} = (\boldsymbol{y}, t) : \boldsymbol{y} \in \mathcal{B}, t \in (-1,1)\}.$$

Let $\mathcal{D}$ be a domain with twice continuously differentiable boundary, i.e., the boundary is a Lipschitz boundary. Thus, there exists a bi-Lipschitz homeomorphism Q form $\mathcal{K}^*$ to some open subset $\mathcal{K}$ of Ω. This bi-Lipschitz homeomorphism satisfies $Q(0) = \overline{\boldsymbol{X}} \in \partial\mathcal{D}$, $Q(\mathcal{B}\times(-1,0)) = \mathcal{D}\cap\mathcal{K}$ and $Q(\mathcal{B}\times\{0\}) = \partial\mathcal{D}\cap\mathcal{K}$. Furthermore, let T be the inverse of Q with $J = \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}}$ as the Jacobian matrix, which satisfies $\det J \geq a > 0$. Next, we construct A^* such that it satisfies $A^*\circ T = (\det J)^{-1}AJ^\top$ and P^* such that $\langle P^*, \phi^*\rangle = \langle P, \phi\rangle$ for $\phi = \phi^*\circ T$, in the sense of distributions.

By Lemma 4.4 there exists $a^* = A^*\cdot \boldsymbol{E}_n$ that satisfies

$$\int_{\mathcal{B}} a^*(\boldsymbol{y},0)\phi^*(\boldsymbol{y},0)d\boldsymbol{y} = \iint_{\mathcal{B}\times(-1,0)} A^*(\boldsymbol{Y})\nabla\phi^*(\boldsymbol{Y})d\boldsymbol{Y} + \langle P^*,\phi^*\rangle.$$

Transforming the equation above to an equation on Ω, by transformation given by T and using that $\nabla\phi = J^\top\nabla\phi^*$ we get

$$\begin{aligned}\int_{\mathcal{B}} a^*(\boldsymbol{y},0)\phi^*(\boldsymbol{y},0)d\boldsymbol{y} &= \int_{\partial\mathcal{D}} (a^*\circ T)(\boldsymbol{x},0)(\phi^*\circ T)(\boldsymbol{x},0)\frac{dy}{d\mathcal{H}^{n-1}}d\mathcal{H}^{n-1}(\boldsymbol{x})\\ &= \int_{\partial\mathcal{D}} a(\boldsymbol{x},0)\phi(\boldsymbol{x},0)\frac{dy}{d\mathcal{H}^{k-1}}d\mathcal{H}^{n-1}(\boldsymbol{x}),\end{aligned}$$

and

$$\begin{aligned}\iint_{\mathcal{B}\times(-1,0)} A^*(\boldsymbol{Y})\nabla\phi^*(\boldsymbol{Y})d\boldsymbol{Y} + \langle P^*,\phi^*\rangle &= \iint_{\mathcal{D}\cap\mathcal{K}} (A^*\circ T)(J^\top)^{-1}\nabla\phi\det J d\boldsymbol{X} + \langle P,\phi\rangle_{\mathcal{D}\cap\mathcal{K}}\\ &= \iint_{\mathcal{D}\cap\mathcal{K}} A\cdot\nabla\phi d\boldsymbol{X} + \langle P,\phi\rangle_{\mathcal{D}\cap\mathcal{K}},\end{aligned}$$

where we have used that $A^*\circ T = (\det J)^{-1}AJ^\top$. Then, if we define

$$q_{\mathcal{D}} = a^*\circ T\frac{dy}{d\mathcal{H}^{n-1}} = a^*\circ T\frac{\det J}{E_n^\top JN}. \tag{4.27}$$

we get

$$\int_{\partial\mathcal{D}} q_{\mathcal{D}}d\mathcal{H}^{n-1}(\boldsymbol{x}) = \iint_{\mathcal{D}\cap\mathcal{K}} A\cdot\nabla\phi d\boldsymbol{X} + \langle P,\phi\rangle_{\mathcal{D}\cap\mathcal{K}}.$$

In the next lemma

$$\frac{dy}{d\mathcal{H}^{n-1}} = \frac{\det J}{E_n^\top JN}$$

is proved for the case $Q(y,s) = (ay+bs, cy+ds)$. We assume the result holds in higher dimensions and for more complex Q. We have now proved Equation (4.26), but only for test functions with compact support in the set $\mathcal{K}$. We can extend the result to an arbitrary test function with compact support in Ω. To do this, we start by observing that integrand on the right-hand side is independent of the homeomorphism, since A and P are given in the theorem. Thus, the values of $q_{\mathcal{D}}$ in $\mathcal{D} \cap \mathcal{K}$ are not dependent on the construction above, so they are not dependent on $\mathcal{K}$ or the homeomorphism. Thus, by a standard partition of unity argument we can conclude that the result holds for an arbitrary test function. □

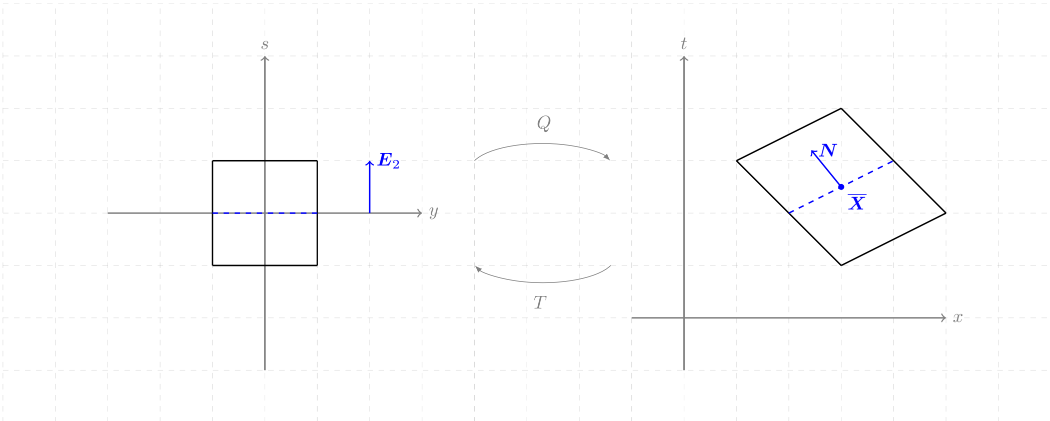

Figure 4.2: The figure illustrates how the unit square, $[-1,1] \times [-1,1]$, in the (y,s)-plane is mapped by $Q(\boldsymbol{Y}) = (ay+bs, cy+ds)$ into the (x,t)-plane. In addition, the figure depicts the standard unit normal in s-direction and $\boldsymbol{N}$, which is the normal at some point $\overline{\boldsymbol{X}}$.

Lemma 4.6. *If* $\boldsymbol{X} = (x,t) = Q(\boldsymbol{Y}) = (ay+bs, cy+ds)$, *where* $\boldsymbol{Y} = (y,s)$, *we have that*

$$\frac{dy}{d\mathcal{H}^1}(x,t) = \frac{\det J}{\boldsymbol{E}_2^\top J \boldsymbol{N}},$$

when $J = \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}}$, $\boldsymbol{E}_2$ *is the standard unit normal in* s*-direction and* $\boldsymbol{N}$ *is the normal at some point* $\overline{\boldsymbol{X}}$.

Proof. In Figure 4.2 we can see a possible transformation of the unit square in $\mathbb{R}^2$, using $(x,t) = Q(y,s) = (ay+bs, cy+ds)$. In this case, we have that the endpoints of the blue line segment in the right plot in Figure 4.2 are $(-a,-c)$ and (a,c). In addition, we have

$$\boldsymbol{E}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{N} = \frac{1}{\sqrt{\Delta x^2 + \Delta t^2}} \begin{bmatrix} -\Delta t \\ \Delta x \end{bmatrix} = \frac{1}{\sqrt{4a^2+4c^2}} \begin{bmatrix} -2c \\ 2a \end{bmatrix} = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} -c \\ a \end{bmatrix}.$$

As in the figure we denote Q^{-1} by T. The Jacobian matrix for Q will be the inverse Jacobian matrix for T, so we denote Jacobian matrix for Q by J^{-1}. Thus,

$$J^{-1} = \frac{\partial X}{\partial Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies J = \frac{\partial Y}{\partial X} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and $\det J = \frac{1}{ad-bc}$. So, we have

$$\boldsymbol{E}_2^\top J \boldsymbol{N} = \frac{1}{ad-bc} \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} -c \\ a \end{bmatrix} = \frac{1}{ad-bc} \frac{1}{\sqrt{a^2+c^2}} (a^2+c^2) = \frac{\sqrt{a^2+c^2}}{ad-bc}$$

and

$$\frac{\det J}{\boldsymbol{E}_2^\top J \boldsymbol{N}} = \frac{1}{\sqrt{a^2+c^2}}.$$

Next, we calculate $\frac{d\mathcal{H}^1(x,t)}{dy}$, with $x = ay$ and $t = cy$, by using the definition of derivative.

$$\begin{aligned}\frac{d\mathcal{H}^1(x,t)}{dy} &= \frac{d\mathcal{H}^1(ay,cy)}{dy} = \lim_{\Delta y\to 0}\frac{\mathcal{H}^1\big((a,c)(y+\Delta y)\big)-\mathcal{H}^1\big((a,c)y\big)}{\Delta y} = \lim_{\Delta y\to 0}\frac{\mathcal{H}^1\big((a,c)\Delta y\big)}{\Delta y}\\ &= \lim_{\Delta y\to 0}\frac{\sqrt{(a\Delta y)^2+(c\Delta y)^2}}{\Delta y} = \sqrt{a^2+c^2},\end{aligned}$$

where we have used that $\mathcal{H}^1$ is additive. Thus,

$$\frac{dy}{d\mathcal{H}^1(x,t)} = \frac{1}{\sqrt{a^2+c^2}} = \frac{\det J}{\boldsymbol{E}_2^\top J\boldsymbol{N}},$$

and the lemma is proved. □

Chapter 5

One-dimensional Euler and Lagrange equations

In the previous section we showed that the divergence form of a general multidimensional balance law is conserved under a bi-Lipschitz transformation. In this chapter we are going to consider a specific system of conservation laws in one dimension, namely the Euler and Lagrange equation for a compressible, inviscid, and non-heat-conducting gas. The Euler equations are given by

$$\rho_t + (\rho u)_x = 0, \tag{5.1a}$$

$$(\rho u)_t + (\rho u^2 + p(\rho, S))_x = 0, \tag{5.1b}$$

$$\left(\rho e(\rho, S) + \frac{\rho u^2}{2}\right)_t + \left(u\left(\rho e(\rho, S) + \frac{\rho u^2}{2} + p(\rho, S)\right)\right)_x = 0, \tag{5.1c}$$

and the Lagrange equations are given by

$$\tau_t - \tilde{u}_y = 0, \tag{5.2a}$$

$$\tilde{u}_t + \tilde{p}(\tau, \tilde{S})_y = 0, \tag{5.2b}$$

$$\left(\tilde{e}(\tau, \tilde{S}) + \frac{\tilde{u}^2}{2}\right)_t + (\tilde{u}\tilde{p}(\tau, \tilde{S}))_y = 0, \tag{5.2c}$$

where $\tilde{p}(\tau, S) = p(\tilde{\rho}, \tilde{S})$ and $\tilde{e}(\tau, S) = e(\tilde{\rho}, \tilde{S})$.

5.1 No vacuum

The equivalence of these two systems of partial differential equations, given that $\rho > 0$, is shown in detail in my project thesis [20]. However, we can easily use Theorem 4.3 to show that these two systems are equivalent. First, we include the proof of existence of a transformation function $T(x, t)$ done in Section 4 in [20] for completeness.

5.1.1 Existence of transformation function $T(x,t)$

We will need the following theorem to prove the existence of the transformation function $T(x,t)$ with the Jacobi matrix

$$JT = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} \rho & -\rho u \\ 0 & 1 \end{bmatrix}.$$

Theorem 5.1. *Suppose a vector field* $\mathbf{F}$ *is* C^1 *on an open simply connected set* Ω *in* $\mathbb{R}^3$ *on which* $\text{curl}\mathbf{F} = \mathbf{0}$. *Then there is a function* g *from* Ω *to* $\mathbb{R}$ *so that* $\nabla g = \mathbf{F}$.

We start by showing the existence of a function $y(x,t)$ that satisfies

$$\frac{\partial y}{\partial x} = \rho \quad \text{and} \quad \frac{\partial y}{\partial t} = -\rho u.$$

However, since ρ and u can be non-differentiable functions and $\rho_t + (\rho u)_x = 0$ only holds weakly we cannot directly use Theorem 5.1. Therefore, we will use standard mollifiers to prove the existence. We define $\rho_\varepsilon = \rho * \omega_\varepsilon$ and $(\rho u)_\varepsilon = (\rho u) * \omega_\varepsilon$, which are smooth functions and satisfy

$$(\rho_\varepsilon)_t + \big((\rho u)_\varepsilon\big)_x = 0.$$

We want to show that there exists a y_ε that satisfies $(y_\varepsilon)_x = \rho_\varepsilon$ and $(y_\varepsilon)_t = -(\rho u)_\varepsilon$. To do this we use that $(y_\varepsilon)_{xt} = (\rho_\varepsilon)_t = -\big((\rho u)_\varepsilon\big)_x = (y_\varepsilon)_{tx}$. Therefore, the conditions in Theorem 5.1 are satisfied and there exists a function y_ε satisfying our requirements.

Next, we want to show that there exists a function y such that $y_\varepsilon \to y$ satisfying $y_x = \rho$ and $y_t = -\rho u$ weakly. The first step is to show that for all y_ε are Lipschitz continuous functions with the same Lipschitz constant for all ε. We use the following theorem from [9, Theorem 4.5].

Theorem 5.2 (Lipschitz continuity and $W^{1,\infty}$)**.** *Let* $f : \Omega \to \mathbb{R}$. *Then* f *is locally Lipschitz in* Ω *if and only if* $f \in W^{1,\infty}_{loc}(\Omega)$.

By the assumption that $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ and $0 < \delta \le \rho \le M < \infty$, we have that

$$\begin{aligned} |(y_\varepsilon)_x| = |\rho_\varepsilon| &= \left| \iint_{\mathbb{R}\times\mathbb{R}^+} \rho(z,s)\omega_\varepsilon(x-z,t-s)dzds \right| \\ &\le \iint_{\mathbb{R}\times\mathbb{R}^+} |\rho(z,s)|\omega_\varepsilon(x-z,t-s)dzds \le M < \infty \end{aligned}$$

and

$$\begin{aligned} |(y_\varepsilon)_t| = |(\rho u)_\varepsilon| &= \left| \iint_{\mathbb{R}\times\mathbb{R}^+} (\rho u)(z,s)\omega_\varepsilon(x-z,t-s)dzds \right| \\ &\le \iint_{\mathbb{R}\times\mathbb{R}^+} |\rho(z,s)||u(z,s)|\omega_\varepsilon(x-z,t-s)dz \le M \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |u(x,t)| < \infty. \end{aligned}$$

Therefore, since ρ and u are locally integrable function, $y_\varepsilon \in W^{1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ for all ε. Thus, by Theorem 5.2, y_ε is locally Lipschitz continuous with the same Lipschitz constant for all $\varepsilon > 0$.

The next step is to show that y_ε has a convergent subsequence. Using that y_ε are locally Lipschitz continuous with the same Lipschitz constant for all ε, we can show that the family of functions $\mathcal{Y} = \{y_\varepsilon : \varepsilon > 0\}$ is equicontinuous on compact sets. We have that on a compact subset $\mathbf{K} \subset \mathbb{R}^n$, a locally Lipschitz function is a Lipschitz function. Hence for $(x_1, t_1), (x_2, t_2) \in \mathbf{K}$, if we assume that $|(x_1, t_1) - (x_2, t_2)| < \delta$, and choose $\delta = \epsilon/\tilde{K}$ for each ϵ, we get

$$|y_\varepsilon(x_1, t_1) - y_\varepsilon(x_2, t_2)| \le \tilde{K}|(x_1, t_1) - (x_2, t_2)| < \epsilon \quad \forall \epsilon > 0,$$

for all $y_\varepsilon \in \mathcal{Y}$. Thus by Definition 2.17 $\mathcal{Y}$ is an equicontinuous family of functions on compact sets. Furthermore, we want to show that $\mathcal{Y}$ is uniformly bounded on $\mathbf{K}$ when $\mathbf{K}$ is a compact subset of $\mathbb{R} \times \mathbb{R}^+$. When defining y_ε we can choose $y_\varepsilon(0, 0) = 0$, and from the fact that y_ε are Lipschitz continuous with the same Lipschitz constant we get

$$|y_\varepsilon(x, t)| \le \tilde{K}|(x, t)|.$$

For all compact sets, there exists an open ball with the radius r such that the compact set is a subset of said open ball. Thus, we have that $|(x, t)|$, which is the Euclidean distance from the origin in $\mathbb{R}^2$, is bounded by an $r < \infty$ for all $(x, t) \in \mathbf{K}$. So, $|y_\varepsilon(x, t)| \le r\tilde{K}$ for all $y_\varepsilon \in \mathcal{Y}$. Thus, by Definition 2.16 $\mathcal{Y}$ is uniformly bounded on $\mathbf{K}$.

We have shown that the family of functions $\mathcal{Y}$ satisfies the conditions in the Arzelà-Ascoli Theorem 2.24 and hence there exists a subsequence y_{ε_n}, $n = 1, 2, \ldots$, where $\varepsilon_n \to 0$, such that $y_{\varepsilon_n} \to y$ uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}^+$. So, we have a candidate for the transformation function. The next step is to show that $y_x = \rho$ weakly. Since $y_{\varepsilon_n} \to y$ uniformly on compact subsets, $\mathbf{K}$, we have that

$$\left| \iint_{\mathbf{K}} y dxdt - \iint_{\mathbf{K}} y_{\varepsilon_n} dxdt \right| \le \iint_{\mathbf{K}} |y - y_{\varepsilon_n}| \, dxdt \le \delta m_2(\mathbf{K}) \xrightarrow{n \to \infty} 0,$$

where we have used that $m_2(\mathbf{K})$, the Lebesgue measure of $\mathbf{K}$, is finite and that δ is independent of (x, t), due to the uniform convergence. Using this we get the following

$$\iint y\phi_x dxdt = \lim_{n \to \infty} \iint y_{\varepsilon_n} \phi_x dxdt = -\lim_{n \to \infty} \iint (y_{\varepsilon_n})_x \phi dxdt = -\lim_{n \to \infty} \iint \rho_{\varepsilon_n} \phi dxdt.$$

To finish showing that $y_x = \rho$ weakly, we have to show that

$$\lim_{n \to \infty} \iint \rho_{\varepsilon_n} \phi dxdt = \iint \rho \phi dxdt. \tag{5.3}$$

Since ρ is a locally integrable function, we can use property (iii) in Theorem 2.25 and get that $\rho_{\varepsilon_n} \to \rho$ in L^1_{loc}. Using this we have

$$\left| \iint_{\mathbb{R} \times \mathbb{R}^+} \rho_{\varepsilon_n} \phi dxdt - \iint_{\mathbb{R} \times \mathbb{R}^+} \rho \phi dxdt \right| \le \iint_{\mathbf{K}} |\rho_{\varepsilon_n} - \rho| \, |\phi| dxdt \le \|\phi\|_\infty \|\rho_{\varepsilon_n} - \rho\|_{L^1_{loc}},$$

where $\mathbf{K} = \text{supp}\phi$ is a compact set. Furthermore, $\|\phi\|_\infty < \infty$ since ϕ has compact support and is a continuous function. Thus, we have shown Equation (5.3) and consequently that $y_x = \rho$ weakly. With a similar argumentation, we can show that $y_t = -\rho u$ in the weak sense. Since $y \in L^1_{loc}$ and the weak derivative $y_x, y_t \in L^\infty$, we have that $y \in W^{1,\infty}_{loc}$ and thus, by Theorem 5.2, y is Lipschitz continuous.

By Rademacher's Theorem 2.17, and the fact that y is Lipschitz continuous on every compact set, y is differentiable a.e. We have shown that there exists a function $y(x, t) = y$ which is Lipschitz and satisfies $y_x = \rho$ and $y_t = -\rho u$ a.e. From this we can define a map $T(x, t) = (y(x, t), t) = (y, t)$ with the following Jacobian matrix

$$JT = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} \rho & -\rho u \\ 0 & 1 \end{bmatrix},$$

which holds a.e. JT will have maximal rank, whenever it exists, since the Jacobian determinant is given by $\det JT = \rho$ and we have assumed no vacuum. From [4, Definition 1] we have the following definition.

Definition 5.1 (Generalized Jacobian)**.** The *generalized Jacobian* of f at x_0, denoted $\partial f(x_0)$, is the convex hull of all matrices M of the form

$$M = \lim_{i \to \infty} Jf(x_i),$$

where x_i converges to x_0 and f is differentiable at x_i for each i.

When the function f is in $C^1(\Omega)$, the generalized Jacobian $\partial f(x_0)$ coincides with $Jf(x_0)$ [4]. Furthermore, we have the following theorem from [4, Theorem 1].

Theorem 5.3. *If $\partial f(x_0)$ is of maximal rank, then there exist neighborhoods U and V of x_0 and $f(x_0)$ respectively, and a Lipschitz function $g : V \to \mathbb{R}^n$ such that*

(a) $g(f(u)) = u$ for every $u \in U$,

(b) $f(g(v)) = v$ for every $v \in V$.

The Jacobian of the map T has maximal rank for almost every (x, t). Hence, by Theorem 5.3, T has a Lipschitz inverse function almost everywhere. So, the transformation $(x, t) = T^{-1}(y, t) = Q(y, t)$ exists a.e.

5.1.2 Using Theorem 4.3 to show equivalence

In this section we will show how we can use Theorem 4.3 to conclude that a weak solution of (5.1) also is a weak solution of (5.2), and vice versa. From the subsection above we have that there exists a transformation $T(x, t) = (y, t)$ from Euler to Lagrange coordinates with the following Jacobian matrix

$$JT = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} \rho & -\rho u \\ 0 & 1 \end{bmatrix},$$

even in the case where (5.1a) only holds weakly. If we assume that there is no vacuum, i.e., that $\rho > 0$, the conditions of Theorem 4.3 are satisfied and through tedious calculation we can show that a weak solution of the Euler equations also is a weak solution of the Lagrange equations. The calculations can be found in Appendix A. Furthermore, T has inverse function Q which is Lipschitz. Q is the transformation from Lagrange to Euler coordinates, given by $Q(y, t) = (x, t)$, and satisfies

$$JQ = \begin{bmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial t} \\ \frac{\partial t}{\partial y} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} \tau & \tilde{u} \\ 0 & 1 \end{bmatrix}.$$

Thus the conditions of Theorem 4.3 are again satisfied and we can show that the Lagrange equations imply the Euler equations.

5.2 Vacuum

In this section we will allow the mass density to be zero in some set, i.e., we allow there to be a set with vacuum. The idea of vacuum in Lagrange coordinates may sound nonphysical, but in [31] Wagner shows that if we strengthen the definition of weak solutions in Lagrange coordinates, we can prove that the equivalence between the Euler and the Lagrange equations still holds true for these weak solutions. Thus, the proofs in this section will be heavily based on the proofs in [31] by Wagner. However, it will include additional calculations to make the proofs more understandable. We will first assume that the Euler equations holds weakly and show how we have to strengthen the definition of a weak solution in Lagrange coordinates to conclude that the Lagrange equations holds weakly as well. Furthermore, we will show that given that the Lagrange equations hold weakly with the new definition of weak solutions the Euler equations will also hold weakly.

5.2.1 The Euler formulation implies the Lagrange formulation

We start by assuming that (5.1) holds weakly. Furthermore, define T as $T(x,t) = (y(x,t),t)$ where

$$y(x,t) = \int_{x(t)}^{x} \rho(s,t)ds, \tag{5.4}$$

where $x(t)$ is the trajectory of a particle. Then we get

$$JT = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} \rho & -\rho u \\ 0 & 1 \end{bmatrix},$$

as before. From Section 5.1.1 we have that there exists a $T(x,t) = (y(x,t),t)$ which is Lipschitz. However, we see that since $\det JT = \rho = 0$ in some subset of $\mathbb{R}^n$, T will no longer be invertible in this subset, i.e., there no longer exists an inverse function $Q(y,t)$. In addition, the condition $\det JT \geq a > 0$ in Theorem 4.3 is no longer satisfied so we cannot use this theorem. In fact, T may map a set of positive measure, the vacuum set, to a set of measure zero. Thus, we can no longer consider τ as a function, but it may be considered as a measure. It is natural to consider τ to be the following measure $\tau = T_{\#}m_2$.

Theorem 5.4 (Adapted from [31, Lemma 1]). *Assume that*

$$\int_0^\infty \rho(x,0)dx = \int_{-\infty}^0 \rho(x,0)dx = \infty,$$

i.e., no half-line has finite measure at $t = 0$. Then T is a proper and onto function and hence τ is a Radon measure.

Proof. We start by assuming $\int_0^\infty \rho(x,0)dx = \int_{-\infty}^0 \rho(x,0)dx = \infty$. Furthermore, we define the following test function $\varphi(x,t) = \psi(x)\phi(t)$, where $\psi, \phi \geq 0$, and $\psi(x) = 1$ when $x \in [a,b]$ for $a < b$ and $\phi(t) = 1$ when $t \in [0,t_1]$ for $t_1 > 0$. In addition, for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ we assume $\int_{\mathbb{R}} \psi(x)dx < b - a + \varepsilon_1$, $\int_{\mathbb{R}^+} \phi(t)dt < t_1 + \varepsilon_2$ and thus $\int_{\mathbb{R}} |\psi'(x)|dx = 2$ and $\int_{\mathbb{R}^+} |\phi'(t)|dt = 1$. By assumption that the Euler equations holds weakly,

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \psi(x)\phi'(t)\rho(x,t) + \psi'(x)\phi(t)(\rho u)(x,t)dxdt + \int_{\mathbb{R}} \psi(x)\rho(x,0)dx = 0. \tag{5.5}$$

From this, we want to conclude that given $\int_0^\infty \rho(x,0)dx = \infty$ we have that $\int_0^\infty \rho(x,t_1)dx = \infty$ for almost all t_1. We start by considering

$$\begin{aligned}\Big|\iint_{\mathbb{R}\times\mathbb{R}^+} \psi'(x)\phi(t)(\rho u)(x,t)dxdt\Big| &\leq \iint_{\mathbb{R}\times\mathbb{R}^+} |\psi'(x)\phi(t)(\rho u)(x,t)|dxdt \\ &\leq \|\rho u\|_\infty \int_{\mathbb{R}^+} \phi(t) \int_{\mathbb{R}} |\psi'(x)|dxdt \\ &\leq \|\rho u\|_\infty (t_1+\varepsilon_2) \int_{\mathbb{R}} |\psi'(x)|dx = 2\|\rho u\|_\infty (t_1+\varepsilon_2).\end{aligned} \tag{5.6}$$

Since we have assumed that $\rho, u \in L^\infty$, and thus $2\|\rho u\|_\infty(t_1+\varepsilon_2) < \infty$, we get

$$\Big|\iint_{\mathbb{R}\times\mathbb{R}^+} (\rho u)(x,t)\psi'(x)\phi(t)dxdt\Big| \leq 2\|\rho u\|_\infty t_1$$

as $\varepsilon_2 \to 0$. Next, we consider

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \psi(x)\phi'(t)\rho(x,t)dxdt$$

as $\varepsilon_2 \to 0$. First, we investigate what happens to ϕ as $\varepsilon_2 \to 0$. By construction, we have that $\phi \geq 0$, $\phi(t) = 1$ for $t \in [0,t_1]$ and $\int_0^\infty \phi(t) < t_1 + \varepsilon_2$. Thus,

$$\int_{\mathbb{R}^+} \phi(t)dt = \int_0^{t_1} \phi(t)dt + \int_{t_1}^\infty \phi(t)dt = t_1 + \int_{t_1}^\infty \phi(t)dt < t_1 + \varepsilon_2.$$

This implies $\int_{t_1}^\infty \phi(t)dt < \varepsilon_2$ and letting $\varepsilon_2 \to 0$ we get

$$\int_{t_1}^\infty \phi(t)dt = 0,$$

since $\phi \geq 0$. So, $\phi = 0$ for a.e. $t > t_1$ and we can conclude that $\phi(t) \to \chi_{[0,t_1]}(t)$ as $\varepsilon_2 \to 0$. However, we are interested in $\phi'(t)$. It is natural to think that $\phi'(t) \to -\delta_{t_1}(t)$, where δ_{t_1} is the Dirac delta function. To show this we start by calculating

$$\int_{\mathbb{R}^+} \phi'(t)dt = \int_{t_1}^{t_1+h} \phi'(t)dt = \phi(t_1+h) - \phi(t_1) = 0 - 1 = -1.$$

Here, h is the length of the support of $\phi'(t)$, thus $h \to 0$ as $\varepsilon \to 0$. We are ready to consider

$$\begin{aligned}\Big|\int_{\mathbb{R}^+} \phi'(t)f(x,t)dt + f(x,t_1)\Big| &= \Big|\int_{t_1}^{t_1+h} \phi'(t)\big(f(x,t) - f(x,t_1)\big)dt\Big| \\ &\leq \int_{t_1}^{t_1+h} |\phi'(t)||f(x,t) - f(x,t_1)|dt \\ &\leq \|\phi'\|_\infty h \frac{1}{h}\int_{t_1}^{t_1+h} |f(x,t) - f(x,t_1)|dt.\end{aligned}$$

Since $\int |\phi'|dt$ is finite and equal for all h, $\|\phi'\|_\infty h$ is bounded by some constant independent of h for all h. By Lebesgue Differential Theorem 2.10 we have

$$\frac{1}{h}\int_{t_1}^{t_1+h} |f(x,t) - f(x,t_1)|dt \to 0,$$

for all Lebesgue points t_1 for the function $f(\cdot,t)$. This implies that

$$\Big|\int_{\mathbb{R}^+} \phi'(t)f(x,t)dt + f(x,t_1)\Big| \to 0,$$

for all Lebesgue points t_1. Thus, we have

$$\Big|\iint_{\mathbb{R}\times\mathbb{R}^+}\phi'(t)\psi(x)\rho(x,t)dtdx+\int_{\mathbb{R}}\psi(x)\rho(x,0)dx\Big|\to\Big|\int_{\mathbb{R}}\psi(x)\big(\rho(x,0)-\rho(x,t_1)\big)dx\Big| \quad (5.7)$$

as $\varepsilon_2\to 0$. If we return to (5.5) and insert (5.6) we obtain

$$\begin{aligned}0&=\Big|\iint_{\mathbb{R}\times\mathbb{R}^+}\phi'(t)\psi(x)\rho(x,t)dxdt+\iint_{\mathbb{R}\times\mathbb{R}^+}\phi(t)\psi'(x)(\rho u)(x,t)dxdt+\int_{\mathbb{R}}\psi(x)\rho(x,0)dx\Big|\\&\geq\Big|\iint_{\mathbb{R}\times\mathbb{R}^+}\phi'(t)\psi(x)\rho(x,t)dtdx+\int_{\mathbb{R}}\psi(x)\rho(x,0)dx\Big|-\Big|\iint_{\mathbb{R}\times\mathbb{R}^+}\phi(t)\psi'(x)(\rho u)(x,t)dxdt\Big|\\&\geq\Big|\iint_{\mathbb{R}\times\mathbb{R}^+}\phi'(t)\psi(x)\rho(x,t)dtdx+\int_{\mathbb{R}}\psi(x)\rho(x,0)dx\Big|-2(t_1+\varepsilon_2)\|\rho u\|_\infty.\end{aligned}$$

By letting $\varepsilon_2\to 0$ and inserting (5.7) we get

$$\Big|\int_{\mathbb{R}}\psi(x)\big(\rho(x,t_1)-\rho(x,0)\big)dx\Big|\leq 2t_1\|\rho u\|_\infty, \quad (5.8)$$

for any Lebesgue point, t_1, for the function $t\to\int\psi(x)\rho(x,t)dx$. Next, we want to determine what happens to $\psi(x)$ as $\varepsilon_1\to 0$. By a similar argument as for ϕ, where we now use that $\psi(x)=1$ when $x\in[a,b]$ and $\int_{\mathbb{R}}\psi(x)dx<b-a+\varepsilon_1$, we get that $\psi(x)\to\chi_{[a,b]}(x)$ as $\varepsilon_1\to 0$. Thus, as $\varepsilon_1\to 0$ Equation (5.8) converges to

$$\int_a^b\big\{\rho(x,t_1)-\rho(x,0)\big\}dx\leq 2t_1\|\rho u\|_\infty.$$

Furthermore, if we fix a, we have

$$\sup_{b>a}\int_a^b\big\{\rho(x,t_1)-\rho(x,0)\big\}dx\leq 2t_1\|\rho u\|_\infty.$$

Thus, we have shown that

$$\int_0^\infty\big\{\rho(x,t_1)-\rho(x,0)\big\}dx\leq 2t_1\|\rho u\|_\infty<\infty,$$

and since we have assumed $\int_0^\infty\rho(x,0)dx=\infty$ we get that

$$\int_0^\infty\rho(x,t_1)dx=\infty,\ \text{for almost all } t_1. \quad (5.9)$$

Next, we want to use this to argue that $y(x,t_1)$ is proper and onto for almost all t_1. Since

$$y(x,t_1)=\int_{x(t_1)}^x\rho(s,t_1)ds$$

and ρ is bounded we have that $y^{-1}[a,b]$ is a compact set and thus y is proper for almost all t_1. Furthermore, since ρ is bounded, for each $\tilde{y}$ there will exist a x such that $y(x,t)\leq\tilde{y}$. Thus, since y is a continuous function, and due to (5.9), there has to exist a $\tilde{x}$ such that $y(\tilde{x},t)=\tilde{y}$. Hence y is onto. Since t is mapped to t in $T(x,t)$, T will also be a proper and onto map. Thus, by Theorem 2.15 τ is a Radon measure. □

Before we are ready to state the new definition of weak solutions in Lagrange coordinates and prove the equivalence between the weak solutions, we have to prove some relations between τ, ρ and the Lebesgue measure, m_2. We start by stating the following theorem.

Theorem 5.5 ([31, Lemma 2])**.** *$T_{\#}\rho = m_2$.*

To prove this theorem we will use the following lemma.

Lemma 5.6. *Let U be an open subset of $\mathbb{R}^n$ and χ_U be the characteristic function. Furthermore, let μ be a Radon measure. Then there exists a sequence of continuous functions f_k such that $f_k \to \chi_U$ in μ, i.e., $\mu(|f_k - \chi_U|) < 1/k$ for all $k \in \mathbb{N}$.*

Proof. By the regularity properties of a Radon measure we have that for every open set U there exists a sequence F_k of compact set such that $\mu(U \setminus F_k) < 1/k$ for all $k \in \mathbb{N}$. Furthermore, by Urysohn's Lemma 2.12 for each F_k there exists a continuous function f_k that is equal to 1 on F_k and 0 on U^c. We can write these functions as

$$f_k = \begin{cases} 1, & \text{on } F_k, \\ g_k, & \text{on } U \setminus F_k, \\ 0, & \text{on } U^c, \end{cases}$$

where g_k is a continuous function. So, we get

$$\mu\big(|f_k - \chi_U|\big) = \int_{\mathbb{R}^n} |f_k - \chi_U| d\mu = \int_{U \setminus F_k} |g_k| d\mu \leq \|g_k\|_\infty \mu(U \setminus F_k) < \frac{1}{k}.$$

Hence, the lemma is proved. □

Proof (Theorem 5.5). First, we observe that ρ is a Radon measure and, by the proof of Theorem 5.4, T is proper and onto. Thus, by Theorem 2.15, $T_{\#}\rho$ is a Radon measure. Let ϕ be a test function. Using assertion (i) in Theorem 2.16 we get

$$T_{\#}\rho(\phi) = \int \phi dT_{\#}\rho = \int \phi \circ T \rho dx dt = \int \phi \circ T \frac{\partial y}{\partial x} dx dt.$$

The last equality is due to the definition of y in Equation (5.4). Furthermore, we use Corollary 2.19 to conclude

$$T_{\#}\rho(\phi) = \int \phi \circ T \frac{\partial y}{\partial x} dx dt = \int \phi N(T, y) dy dt = m_2(\phi).$$

For the last equality to be true, we have to show that $N(T, (y, t)) = 1$ m_2-a.e. We start by defining

$$A = \big\{(y, t) : N(T, (y, t)) > 1\big\}$$

and $B = T^{-1}(A) = \big\{(x, t) : T(x, t) \in A\big\}$, i.e., if $(x, t) \in B$ there exists an $(x', t) \in B$ such that $x \neq x'$ and $T(x, t) = T(x', t)$. However, since $T(x, t) = T(x', t)$ for almost all t is only satisfied if $\partial y / \partial x(x, t) = \rho(x, t) = 0$ for almost all t and almost all $x < x < x'$, for each t we have

$$m_2\Big(B \cap \Big\{\frac{\partial y}{\partial x} \neq 0 \text{ or do not exists}\Big\}\Big) = 0.$$

So, $\partial y / \partial x = 0$ m_2-a.e. on B. Furthermore, we have

$$\iint_A N(T, (y, t)) dy dt = \iint_B \frac{\partial y}{\partial x} dx dt = 0.$$

From this equation we can conclude that $N(T, (y, t)) = 0$ a.e. on A and, by the definition of A, this implies that $m_2(A) = 0$. So, $N(T, (y, t)) = 1$ m_2-a.e., and we have proved that $T_{\#}\rho(\phi) = m_2(\phi)$ for all test functions ϕ. By the Lemma 5.6 the measures are equal on open sets. Thus, by the regularity of a Radon measure, given in Definition 2.7, we have that the measure of every subset $A \subset \mathbb{R}^2$ can be approximated by the measure of an open set. Consequently, the measures are equal as Radon measures. □

Furthermore, since $y(x_1,t) = y(x_2,t)$ for almost all t is only satisfied if $\rho(x,t) = 0$ for almost all t and almost all $x_1 < x < x_2$, we can define $\tilde{\rho}$ τ-a.e. such that $\tilde{\rho}(T(x,t)) = \rho(x,t)$. By assertion (i) in Theorem 2.16 and that $\rho = \tilde{\rho} \circ T$ is m_2-measurable we have that $\tilde{\rho}$ is $\tau = T_{\#}m_2$-measurable.

Lemma 5.7 (Adapted from [31, p. 128])**.** *m_2 is absolutely continuous with respect to τ, in fact $m_2 = \tilde{\rho}\tau$.*

Proof. We start by considering

$$\tilde{\rho}\tau(\phi) = \int \phi\tilde{\rho}d\tau = \int \phi\tilde{\rho}dT_{\#}m_2 = \int (\phi\circ T)(\tilde{\rho}\circ T)dm_2 = \int \phi\circ T d\rho = T_{\#}\rho(\phi) = m_2(\phi).$$

We have used that $\tau = T_{\#}m_2$, $m_2 = T_{\#}\rho$ and $\rho = \tilde{\rho}\circ T$, in addition to assertion (i) in Theorem 2.16. So, by a similar argumentation as in the proof of Theorem 5.5 we get that $m_2 = \tilde{\rho}\tau$. Consequently, m_2 is absolutely continuous with respect to τ. □

Due to the vacuum set, τ will not be absolutely continuous with respect to the Lebesgue measure. However, we can use the Lebesgue's Decomposition Theorem 2.8 to decompose τ into an absolutely continuous part, $\tau_{a.c.}$, and a singular part, τ_s, with respect to the Lebesgue measure. Since τ_s is singular with respect to Lebesgue measure, we can divide $\mathbb{R}\times\mathbb{R}^+$ into Borel sets V and V^c such that $m_2(V) = \tau_s(V^c) = 0$. In fact, V is the vacuum set in Lagrange coordinates. Furthermore, we denote the density of $\tau_{a.c.}$ with respect to m_2 by $\tilde{\tau}$, i.e.,

$$\tau_{a.c.}(A) = \int_A \tilde{\tau}dydt.$$

Lemma 5.8 (Adapted from [31, p. 128])**.** *$\tilde{\rho}\tilde{\tau} = 1$ m_2-a.e.*

Proof. Let ϕ be a test function. Then consider

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}^+}\phi\tilde{\rho}\tilde{\tau}dydt &= \iint_{V^c}\phi\tilde{\rho}\tilde{\tau}dydt = \iint_{V^c}\phi\tilde{\rho}d\tau_{a.c.} = \iint_{V^c}\phi\tilde{\rho}d\tau_{a.c} + \iint_{V^c}\phi\tilde{\rho}d\tau_s\\ &= \iint_{V^c}\phi\tilde{\rho}d\tau = \iint_{V^c}\phi dydt = \iint_{\mathbb{R}\times\mathbb{R}^+}\phi dydt,\end{aligned}$$

where we have used that $m_2(V) = \tau_s(V^c) = 0$, $\tilde{\tau}$ is the density of $\tau_{a.c.}$ with respect to m_2, $\tau = \tau_{a.c} + \tau_s$ and $\tilde{\rho}\tau = m_2$. Since

$$\iint_{\mathbb{R}\times\mathbb{R}^+}\phi\tilde{\rho}\tilde{\tau}dydt = \iint_{\mathbb{R}\times\mathbb{R}^+}\phi dydt$$

for all ϕ, we have that $\tilde{\rho}\tilde{\tau} = 1$ m_2-a.e. □

Next, we state the new definition of a weak solution in Lagrange coordinates. We will later illustrate the intuition behind the new definition. This definition is formulated by Wagner in [31, Definition 2].

Definition 5.2. We say that (τ, u, S) is a weak solution of (5.2), if τ is a Radon measure on $\mathbb{R}\times\mathbb{R}^+$, and u and S are bounded τ-measurable functions such that (5.2a) holds in the sense of distributions, and the weak formulation of (5.2b) and (5.2c) holds with all test functions ϕ with compact support such that $\phi_y = f\tau$, and $\phi_t = g$, with $f, g \in L^\infty(\tau)$.

From this definition we can observe that the density function in (5.2b) and (5.2c) will be integrated with respect to gm_2 and thus can be changed on a set of m_2-measure zero. In addition, from (5.2b) and (5.2c) we see that e only appears in a density and the value of e can be changed on a set of m_2-measure zero. Thus the value of e in the vacuum set is irrelevant. Furthermore, by the assumption that $p(\rho, S) = 0$ as $\rho = 0$ and that S only occurs as an argument of p or e, we can conclude that the value of S in the vacuum set is irrelevant as well. Lastly, we can conclude that the value of u also is irrelevant in the vacuum set since it either occurs as a part of a density function or multiplied by p and is bounded. From these observations we have that u, D and F in (5.2) can all be changed in the vacuum set, and thus be chosen to vanish in the vacuum set. So, similarly to ρ, we can define $(\tilde{u}, \tilde{D}, \tilde{F})$ τ-a.e. such that $(\tilde{u}, \tilde{D}, \tilde{F})(T(x,t)) = (u, D, F)$.

Theorem 5.9 ([31, Lemma 3]). *$\tau_t - \tilde{u}_y = 0$ in sense of distributions.*

Proof. We let $\tau_0 = T_{\#}m_1$. Since τ is a Radon measure and $u \in L^\infty$, $\tau_t - \tilde{u}_y = 0$ in sense of distributions is given by

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \phi_t d\tau - \tilde{u}\phi_y dydt + \int_{\mathbb{R}} \phi d\tau_0 = 0.$$

We start by considering

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}^+} \phi_t d\tau - \tilde{u}\phi_y dydt + \int_{\mathbb{R}} \phi d\tau_0 &= \iint_{t>0} \{\phi_t - \phi_y \tilde{\rho}\tilde{u}\} d\tau + \int_{t=0} \phi d\tau_0 \\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} \{\phi_t \circ T - \phi_y \circ T \rho u\} dxdt + \int_{\mathbb{R}} \phi \circ T dx \\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} (\phi \circ T)_x dxdt + \int_{\mathbb{R}} \phi \circ T dx.\end{aligned}$$

T is a Lipschitz continuous function, so $\phi \circ T$ is a test function, and thus

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (\phi \circ T)_t dxdt + \int_{\mathbb{R}} \phi \circ T dx = 0$$

is the weak formulation of the trivial conservation law $1_t + 0_x = 0$. Hence, $\tau_t - \tilde{u}_y = 0$ in the sense of distributions. □

Next, we want to prove that if we assume that $\tau_t - \tilde{u}_y = 0$ in the sense of distributions, then there exists a function $Q \in BV_{loc}$ with a unique Lipschitz left inverse T.

Lemma 5.10 (Adapted from [31, p. 129]). *Let τ be a positive Radon measure which dominates m_2, i.e., there exists a constant $k > 0$ such that $\tau(E) \geq km_2(E)$ for all sets E, and $u \in L^\infty$ which satisfies $\tau_t - \tilde{u}_y = 0$ in sense of distributions. Then there exists a map $x(y,t)$ such that*

$$\frac{\partial x}{\partial y} = \tau \quad \text{and} \quad \frac{\partial x}{\partial t} = u. \tag{5.10}$$

In addition, $x \in BV_{loc}$ and for each t the map x is monotone increasing in y. Furthermore, we can define a map Q such that $Q(y,t) = (x(y,t), t)$. Then there exists a unique left inverse T, i.e., $T(Q(y,t)) = (y,t)$, which is Lipschitz.

Proof. By using convolution and Theorem 5.1 it can be proved that there exists an x such that (5.10) is satisfied in the sense of distributions, see [1]. The function $x \in L^1_{loc}$, and x_y is a Radon measure and $x_t \in L^\infty$, thus, by Definition 2.23, $x \in BV_{loc}$. Furthermore, since τ is a positive Radon measure x will be monotone increasing in y for each fixed t. So, there will exist a unique left inverse $y(x,t) = x$. Next, we define the map Q such that $Q(y,t) = (x(y,t),t)$ and since t is map to t will Q have the same properties as x and we can construct the unique left inverse $T(x,t) = (y,t)$ such that $T(Q(y,t)) = (y,t)$.

Furthermore, to show that T is a Lipschitz function we use a standard mollifier ω_ε and define $Q_\varepsilon(y,t) = (x_\varepsilon(y,t),t)$ where $x_\varepsilon = x * \omega_\varepsilon$. In addition, let $\tau_\varepsilon = \tau * \omega_\varepsilon$ and $u_\varepsilon = u * \omega_\varepsilon$. Since $\tau(E) \geq km_2(E)$ for all E, the set of characteristic functions is dense in L^1 and $\omega_\varepsilon \in L^1$, we can conclude

$$\tau_\varepsilon = \int \omega_\varepsilon(x-z,t-s)d\tau(z,s) \geq \int \omega_\varepsilon(x-z,t-s)kdzds = k.$$

By the Mean Value Theorem [22, Theorem 3.2] we have

$$\begin{aligned}|(x_\varepsilon)_1 - (x_\varepsilon)_2| &= |x_\varepsilon(y_1,t_1) - x_\varepsilon(y_2,t_2)| \\ &= \Big|\frac{\partial x_\varepsilon}{\partial y}(y_0,t_0)\Big||y_1-y_2| + \Big|\frac{\partial x_\varepsilon}{\partial t}(y_0,t_0)\Big||t_1-t_2| \geq k|y_1-y_2|,\end{aligned}$$

where $y_1 \leq y_0 \leq y_2$ and $t_1 \leq t_0 \leq t_2$. Thus,

$$\begin{aligned}\big|T(x_1,t_1) - T(x_2,t_2)\big| &= |(y_1,t_1) - (y_2,t_2)| = \sqrt{(y_1-y_2)^2 + (t_1-t_2)^2} \\ &\leq \sqrt{k^2\big((x_\varepsilon)_1 - (x_\varepsilon)_2\big)^2 + (t_1-t_2)^2} \\ &\leq \min\{1,k\}\big|((x_\varepsilon)_1,t_1) - ((x_\varepsilon)_1,t_1)\big|.\end{aligned}$$

By property (iv) in Theorem 2.25, x_ε converges to x pointwise a.e. Thus, letting $\varepsilon \to 0$ we get

$$|T(x_1,t_1) - T(x_2,t_2)| \leq C|(x_1,t_1) - (x_2,t_2)| \quad \text{a.e.}$$

and we have shown that T is Lipschitz a.e. □

To show the intuition behind the new definition of a weak solution in Lagrange coordinates, we consider a test function ψ in Euler coordinates and see what happens when we transform this to Lagrange coordinates. Let $\phi = \psi \circ Q$. To calculate the derivatives, ϕ_t and ϕ_y, we have to use functional superposition, since Q is a discontinuous function. The functional superposition is only defined on regular points, so we first have to show that m_2-a.e. point is regular. Let A be the set of irregular points of Q. Since $Q \in BV_{loc}$ and bounded, we can use Theorem 2.28 to conclude that $\mathcal{H}^1$-a.e. point is a regular point, i.e., $\mathcal{H}^{n-1}(A) = 0$. Furthermore, by assertion (ii) and (iv) in Lemma 2.20, $\mathcal{H}^1(A) = 0$ will imply that $m_2 = \mathcal{H}^2(A) = 0$. So, almost every point with respect to the Lebesgue measure is a regular point. Thus, the functional superposition is defined m_2-a.e. and we can use Theorem 2.29 to calculate the partial derivative of ϕ. We get

$$\phi_t = (\psi \circ Q)_t = \widehat{\psi_t \circ Q} + \widehat{\psi_x \circ Q}x_t = \widehat{\psi_t \circ Q} + \widehat{\psi_x \circ Q}u$$

and

$$\phi_y = (\psi \circ Q)_y = \widehat{\psi_x \circ Q}x_y = \widehat{\psi_x \circ Q}\tau.$$

By using the definition of functional superposition given in Definition 2.28 we have

$$|\widehat{\psi_x \circ Q}| = \Big|\int_0^1 \psi_x\big(l_aQ(x,t)s + l_{-a}Q(x,t)(1-s)\big)ds\Big| \leq \|\psi_x\|_\infty < \infty.$$

The last inequality is due to ψ being a Lipschitz function and hence, by Theorem 5.2, $\psi_x \in L^\infty$. Thus, $\widehat{\psi_x \circ Q} \in L^\infty$ and by a similar argument $\widehat{\psi_t \circ Q} \in L^\infty$. So, we get that ϕ

no longer is a Lipschitz continuous function, but rather a BV function that satisfies $\phi_t = g$ and $\phi_y = f\tau$, where $f, g \in L^\infty$. Thus, in this illustrative example the test function will satisfy the conditions imposed in Definition 5.2. These test functions will be discontinuous in the vacuum set, which is reasonable, since in Lagrange coordinates, we can assume that the values of the densities or fluxes vanishes in the vacuum set.

Lemma 5.11 ([31, Lemma 4])**.** *If τ is a Radon measure on $\mathbb{R}\times\mathbb{R}^+$ such that $\tau(E) \geq km_2(E)$, for all $E \subset \mathbb{R}\times\mathbb{R}^+$, $u \in L^\infty(\tau)$, ϕ is a function with compact support, and $\phi_y = f\tau$, $\phi_t = g$, where $f, g \in L^\infty(\tau)$, then there is at least one function ψ such that ψ is Lipschitz with compact support, and $\psi \circ Q = \phi$ a.e.*

Proof. To prove this lemma we use a standard mollifier ω_ε. We let $Q_\varepsilon = Q * \omega_\varepsilon$, $\phi_\varepsilon = \phi * \omega_\varepsilon$, $x_\varepsilon = x * \omega_\varepsilon$, $\tau_\varepsilon = \tau * \omega_\varepsilon$, $u_\varepsilon = u * \omega_\varepsilon$, $f_\varepsilon = f * \omega_\varepsilon$, $g_\varepsilon = g * \omega_\varepsilon$, $(f\tau)_\varepsilon = (f\tau) * \omega_\varepsilon$ and $T_\varepsilon = Q_\varepsilon^{-1}$. Thus, we have $\partial x_\varepsilon/\partial y = \tau_\varepsilon$, $\partial x_\varepsilon/\partial t = u_\varepsilon$, $\partial \phi_\varepsilon/\partial y = (f\tau)_\varepsilon$ and $\partial\phi_\varepsilon/\partial t = g_\varepsilon$. Furthermore, we construct the sequence $\psi_\varepsilon = \phi_\varepsilon \circ T_\varepsilon$. We now show that the partial derivatives of ψ_ε is bounded. By using the chain rule we obtain

$$\frac{\partial \psi_\varepsilon}{\partial x} = \frac{\partial \phi_\varepsilon}{\partial y}\frac{\partial y}{\partial x} = \frac{(f\tau)_\varepsilon}{\tau_\varepsilon}$$

and

$$\frac{\partial \psi_\varepsilon}{\partial t} = \frac{\partial \phi_\varepsilon}{\partial t} + \frac{\partial \phi_\varepsilon}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x_\varepsilon}{\partial t} = g_\varepsilon + \frac{(f\tau)_\varepsilon u_\varepsilon}{\tau_\varepsilon}.$$

Next, we observe that

$$\begin{aligned} |(f\tau)_\varepsilon(y_0,t_0)| &= \Big| \iint f(y,t)\omega_\varepsilon(y_0-y,t_0-t)d\tau(y,t)\Big| \\ &\leq \|f(y,t)\|_\infty \iint \omega_\varepsilon(y_0-y,t_0-t)d\tau(y,t) = \|f\|_\infty |\tau_\varepsilon|, \end{aligned}$$

and hence

$$\Big|\frac{(f\tau)_\varepsilon(y_0,t_0)}{\tau_\varepsilon(y_0,t_0)}\Big| \leq \|f\|_\infty \implies \Big\|\frac{(f\tau)_\varepsilon}{\tau_\varepsilon}\Big\|_\infty \leq \|f\|_\infty.$$

Thus, we get

$$\Big\|\frac{\partial\psi_\varepsilon}{\partial x}\Big\|_\infty \leq \|f\|_\infty$$

and

$$\Big\|\frac{\partial\psi_\varepsilon}{\partial t}\Big\|_\infty \leq \|g\|_\infty + \|f\|_\infty\|u\|_\infty.$$

In the last equation we have used that $\|g_\varepsilon\|_\infty \leq \|g\|_\infty$, $\|u_\varepsilon\|_\infty \leq \|u\|_\infty$ and that

$$|(f\tau)_\varepsilon u_\varepsilon| \leq \|f\|_\infty |\tau_\varepsilon||u_\varepsilon| \implies \Big\|\frac{(f\tau)_\varepsilon u_\varepsilon}{\tau_\varepsilon}\Big\|_\infty \leq \|f\|_\infty\|u\|_\infty.$$

Next, we want to show that $\{\psi_\varepsilon : \varepsilon > 0\}$ is an equicontinuous family of functions. Let $x_1 \leq x_0 \leq x_2$ and $t_1 \leq t_0 \leq t_2$. Then by using the Mean Value Theorem [22, Theorem 3.2] we get

$$\begin{aligned} |\psi_\varepsilon(x_1,t_1) - \psi_\varepsilon(x_2,t_2)| &= \Big|\frac{\partial\psi_\varepsilon}{\partial x}(x_0,t_0)\Big||x_1-x_2| + \Big|\frac{\partial\psi_\varepsilon}{\partial t}(x_0,t_0)\Big||t_1-t_2| \\ &\leq \Big\|\frac{\partial\psi_\varepsilon}{\partial x}\Big\|_\infty |x_1-x_2| + \Big\|\frac{\partial\psi_\varepsilon}{\partial t}\Big\|_\infty |t_1-t_2| \\ &\leq \|f\|_\infty|x_1-x_2| + (\|g\|_\infty + \|f\|_\infty\|u\|_\infty)|t_1-t_2| \\ &\leq C\big(|x_1-x_2| + |t_1-t_2|\big), \end{aligned}$$

where $C = \|f\|_\infty + \|g\|_\infty + \|f\|_\infty \|u\|_\infty$ is independent of ε. Thus, by Definition 2.17 the family of functions $\{\psi_\varepsilon : \varepsilon > 0\}$ is equicontinuous. Furthermore, consider

$$\|\psi_\varepsilon\|_\infty = \|\phi_\varepsilon \circ T_\varepsilon\|_\infty = \sup_{x,t} |\phi_\varepsilon(T_\varepsilon(x,t))| \leq \sup_{y,t} |\phi_\varepsilon(y,t)| = \|\phi_\varepsilon\|_\infty \leq \|\phi\|_\infty,$$

and by Definition 2.16 $\{\psi_\varepsilon : \varepsilon > 0\}$ is uniformly bounded. Then all the conditions in Arzelà-Ascoli Theorem 2.24 are satisfied and there exists a subsequence ψ_n that converges uniformly to some function ψ on compact sets. Here we have defined $n = 1/\varepsilon$, so $n \to \infty$ when $\varepsilon \to 0$. By Theorem 5.2 and the fact that both ψ_n and its partial derivatives are uniformly bounded, the limit function ψ is a Lipschitz function. The support of ψ_n is uniformly bounded, since $\mathrm{supp}(\psi_n) = \mathrm{supp}(\phi_n \circ T_n) \subseteq \mathrm{supp}(\phi_n) \subseteq \mathrm{supp}(\phi) + \mathrm{supp}(\omega_n) < \infty$, where we have used that ϕ and ω_n have compact support. Thus, ψ has compact support. The next step is to show that $\psi \circ Q = \phi$, m_2-a.e. Since $\psi_n = \phi_n \circ T_n$ and $T_n = Q_n^{-1}$ we have that $\psi_n \circ Q_n = \phi_n$. Furthermore,

$$\begin{aligned}
|\psi_n(Q_n(y,t)) - \psi(Q(y,t))| &= |\psi_n(Q_n(y,t)) - \psi_n(Q(y,t)) + \psi_n(Q(y,t)) - \psi(Q(y,t))| \\
&\leq |\psi_n(Q_n(y,t)) - \psi_n(Q(y,t))| + |\psi_n(x,t) - \psi(x,t)| \\
&\leq C|Q_n(y,t) - Q(y,t)| + \varepsilon.
\end{aligned}$$

In the last inequality we have used that ψ_n is infinitely continuously differentiable and ψ_n uniformly converges to ψ. Furthermore, by property *(iv)* in 2.25, $Q_n \to Q$ pointwise a.e. Thus, $\psi_n \circ Q_n \to \psi \circ Q$ a.e. Similarly, ϕ_n converging to ϕ pointwise a.e., and we get

$$\phi = \lim_{n\to\infty} \phi = \lim_{n\to\infty} \psi_n \circ Q_n = \psi \circ Q \quad \text{a.e.}$$

□

Theorem 5.12 ([31, Lemma 5])**.** *Let* (D, F) *be a density-flux pair of* (5.1b) *or* (5.1c) *from a weak solution of* (5.1) *wherein* ρ, u *and* S *are bounded. Suppose* $p(0, S) = 0$ *and* $e(0, S)$ *is finite for finite* S*. Then* $\tilde{u}$, $\tilde{D}$, $\tilde{F}$, *satisfying* $(\tilde{u}, \tilde{D}, \tilde{F})\big(T(x,t)\big) = (u, D, F)$, *satisfy* $(\tau\tilde{D})_t + (\tilde{F} - \tilde{u}\tilde{D})_y = 0$ *in the sense of Definition 5.2.*

Proof. We start by calculating $(\psi \circ Q)_y$ and $(\psi \circ Q)_t$. Since $Q \in BV_{loc}$ and discontinuous, we use Theorem 2.29. We get

$$(\psi \circ Q)_y = (\widehat{\psi_x \circ Q})\frac{\partial x}{\partial y} = (\widehat{\psi_x \circ Q})\tau$$

and

$$(\psi \circ Q)_t = (\widehat{\psi_t \circ Q}) + (\widehat{\psi_x \circ Q})\frac{\partial x}{\partial t} = (\widehat{\psi_t \circ Q}) + (\widehat{\psi_x \circ Q})\tilde{u}.$$

Here $(\widehat{\psi_t \circ Q})$ denote the functional superposition of $\psi_t \circ Q$ given in Definition 2.28. Let ϕ be a test function that satisfies $\phi_y = f\tau$ and $\phi_t = g$. From Lemma 5.11 there exists a ψ such that $\phi = \psi \circ Q$ a.e. Using this we get

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{D}\phi_t d\tau + \iint_{\mathbb{R}\times\mathbb{R}^+} (\tilde{F} - \tilde{u}\tilde{D})\phi_y dydt + \int_{\mathbb{R}} \tilde{D}_0\phi_0 d\tau_0 \\
&\qquad = \iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{D}(\psi \circ Q)_t d\tau + \iint_{\mathbb{R}\times\mathbb{R}^+} (\tilde{F} - \tilde{u}\tilde{D})(\psi \circ Q)_y dydt \\
&\qquad\quad + \int_{\mathbb{R}} \tilde{D}_0(\psi_0 \circ Q) d\tau_0 \\
&\qquad = \iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{D}\big[(\widehat{\psi_t \circ Q}) + (\widehat{\psi_x \circ Q})\tilde{u}\big] + (\tilde{F} - \tilde{u}\tilde{D})(\widehat{\psi_x \circ Q}) d\tau \qquad (5.11) \\
&\qquad\quad + \int_{\mathbb{R}} \tilde{D}_0(\psi_0 \circ Q) d\tau_0 \\
&\qquad = \iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{D}(\widehat{\psi_t \circ Q}) + \tilde{F}(\widehat{\psi_x \circ Q}) d\tau + \int_{\mathbb{R}} \tilde{D}_0(\psi_0 \circ Q) d\tau_0 \\
&\qquad = \iint_{\mathbb{R}\times\mathbb{R}^+} D(\widehat{\psi_t \circ Q}) \circ T + F(\widehat{\psi_x \circ Q}) \circ T dxdt + \int_{\mathbb{R}} D_0(\psi_0 \circ Q) \circ T dx.
\end{aligned}$$

In the last equality we have used assertion (i) in Theorem 2.16. Moreover, we have to prove that $D(\widehat{\psi_t \circ Q}) \circ T = D\psi_t$ and $F(\widehat{\psi_x \circ Q}) \circ T = F\psi_x$ m_2-a.e., and $D_0(\psi_0 \circ Q) \circ T = D_0\psi_0$ m_1-a.e.

First, we recall that $p(0, S) = 0$, $e(0, S)$ is finite for finite S and $u \in L^\infty$. Thus for D, F in (5.1b) and (5.1c) we have $\rho = 0$ implies $D = F = 0$. Since both ψ_t and ψ_x are bounded, we get that equality holds.

Next, we show that the equality $D(\widehat{\psi_t \circ Q}) \circ T = D\psi_t$ holds m_2-a.e. also in the case when $\rho \neq 0$. The proof that $F(\widehat{\psi_x \circ Q}) \circ T = F\psi_x$ holds m_2-a.e. is equivalent. We start by observing that since Q is a m_2-measurable function, by Theorem 2.27, Q is approximately continuous m_2-a.e. Thus, by [30, Eq. (5.3) p. 246] $(\widehat{\psi_t \circ Q}) = (\psi_t \circ Q)$ m_2-a.e. Let E be the set where $(\widehat{\psi_t \circ Q}) \neq (\psi_t \circ Q)$, so $m_2(E) = 0$ Furthermore, if we use that $m_2 = \tilde{\rho}\tau$ and assertion (i) in Theorem 2.16 we get

$$0 = m_2(E) = \int_E dm_2 = \int_E \tilde{\rho} d\tau = \int_{T^{-1}(E)} \tilde{\rho} \circ T dxdt = \int_{T^{-1}(E)} \rho dxdt.$$

Thus, $(\widehat{\psi_t \circ Q}) = (\psi_t \circ Q)$ ρ-a.e. To continue the proof, we use mollifiers. Again, let $Q_\varepsilon = Q * \omega_\varepsilon$ and let $T_\varepsilon = Q_\varepsilon^{-1}$. At the points of approximate continuity of Q, there exists a subsequence Q_n that converges to Q. We have again defined $n = 1/\varepsilon$, such that $n \to \infty$ as $\varepsilon \to 0$. To conclude that there exists a subsequence of $\{T_n\}$ that converges uniformly to some function T on compacts we show that $\{T_n\}$ is uniformly Lipschitz and use Arzelà-Ascoli Theorem 2.24. Since T_n is defined as $T_n = Q_n^{-1} = (x_n^{-1}(x,t), t)$, let $y_n(x,t) = x_n^{-1}(x,t)$ and consider

$$\frac{\partial y_n}{\partial x} = \frac{1}{\frac{\partial x_n}{\partial y}} = \frac{1}{\tau_n} \qquad \text{and} \qquad \frac{\partial y_n}{\partial t} = \frac{\partial y_n}{\partial x}\frac{\partial x}{\partial t} = \frac{u_n}{\tau_n}.$$

Furthermore, we use that τ dominates m_2, i.e., $\tau(E) \geq k m_2(E)$ for all $E \subset \mathbb{R} \times \mathbb{R}^+$, to bound $1/\tau_n$. From the proof of Lemma 5.10 we have $\tau_n \geq k$. So, we get

$$\left\| \frac{\partial y_n}{\partial x} \right\|_\infty \leq \frac{1}{k},$$

and

$$\left\| \frac{\partial y_n}{\partial t} \right\|_\infty \leq \frac{\|u\|_\infty}{k}.$$

Thus, by using the Mean Value Theorem [22, Theorem 3.2] we obtain

$$\begin{aligned}|T_n(x_1,t_1)-T_n(x_2,t_2)| &\le \Big|\frac{\partial y_n}{\partial x}(x_0,t_0)\Big||x_1-x_2| + \Big(\Big|\frac{\partial y_n}{\partial t}(x_0,t_0)\Big|+1\Big)|t_1-t_2| \\ &\le \frac{1}{k}|x_1-x_2| + \Big(\frac{\|u\|_\infty}{k}+1\Big)|t_1-t_2| \le C|(x_1,t_1)-(x_2,t_2)|,\end{aligned}$$

where C is independent of n and hence $\{T_n\}$ is uniformly Lipschitz with Lipschitz constant L_T. By a similar argument as in the proof of Lemma 5.11 we can use the Arzelà-Ascoli Theorem 2.24 to conclude that there exists a subsequence that converges uniformly on compacts to some Lipschitz function $\tilde{T}$. Furthermore, we want to show that this limit function in fact is the left inverse of Q. We start by considering

$$\begin{aligned}|\tilde{T}(Q(y,t))-(y,t)| &\le |\tilde{T}(Q(y,t))-T_n(Q(y,t))| + |T_n(Q(y,t))-T_n(Q_n(y,t))| \\ &\le |\tilde{T}(x,t)-T_n(x,t)| + L_T|Q(y,t)-Q_n(y,t)| \\ &< \varepsilon + L_T|Q(y,t)-Q_n(y,t)|,\end{aligned}$$

since $T_n \to \tilde{T}$ uniformly on compacts. Furthermore, by property (iv) in Theorem 2.25, $Q_n(X) \to Q(X)$ a.e. Using this we conclude that $\tilde{T}(Q(y,t)) = (y,t)$ a.e. and thus $\tilde{T}$ is the left inverse of Q a.e. and by the uniqueness of inverse will $\tilde{T}$ be the left inverse T introduced in Lemma 5.10.

Furthermore, let $T(x_0,t_0) = (y_0,t_0)$ be a point of approximate continuity of Q. We wish to show that $Q \circ T(x_0,t_0) = (x_0,t_0)$. We start by considering

$$\begin{aligned}|Q_n(T_n(x_0,t_0)) - Q(T(x_0,t_0))| &= \Big|\iint_{\mathbb{R}\times\mathbb{R}^+} Q(y,t)\omega_n\big(T_n(x_0,t_0)-(y,t)\big)dydt - Q(y_0,t_0)\Big| \\ &= \Big|\iint_{\mathbb{R}\times\mathbb{R}^+} \big[Q(y,t)-Q(y_0,t_0)\big]\omega_n\big(T_n(x_0,t_0)-(y,t)\big)dydt\Big| \\ &\le \iint_{\mathbb{R}\times\mathbb{R}^+} \big|Q(y,t)-Q(y_0,t_0)\big|\omega_n\big(T_n(x_0,t_0)-(y,t)\big)dydt\end{aligned} \tag{5.12}$$

where we have used that $\iint \omega_n dydt = 1$. By Definition 2.25 we have that $lQ(y_0,t_0)$ is the approximate limit if for every $\delta_1 > 0$ it satisfies

$$\lim_{r\to 0} \frac{m_2\big(\{(y,t)\in\mathbb{R}\times\mathbb{R}^+ : |Q(y,t)-lQ(y_0,t_0)|>\delta_1\}\cap\mathcal{B}_r(x)\big)}{m_2(\mathcal{B}_r(x))} = 0.$$

Let $r = 1/n$. Then the above limit implies

$$\lim_{n\to\infty}\frac{n^2}{\pi}m_2\big(\{(y,t)\in\mathbb{R}\times\mathbb{R}^+ : |Q(y,t)-lQ(y_0,t_0)|>\delta_1, |(y_0-y,t_0-t)|<1/n\}\big) = 0,$$

where we have used that $m_2(\mathcal{B}_r(x)) = \pi r^2$. Since (y_0,t_0) is a point of approximate continuity of Q we have $lQ(y_0,t_0) = Q(y_0,t_0)$. Now, let R be small such that for all n satisfying $1/n < R$, we get can rewrite the above limit as

$$m_2\big(\{(y,t)\in\mathbb{R}\times\mathbb{R}^+ : |Q(y,t)-Q(y_0,t_0)|>\delta_1, |(y_0-y,t_0-t)|<1/n\}\big) < \delta_2\frac{\pi}{n^2}. \tag{5.13}$$

Furthermore, choose n large such that

$$|T_n(x_0,t_0)-(y_0,t_0)| < R/2 \qquad \text{and} \qquad \operatorname{supp}(\omega_n) \subset \{(y,t) : |(y,t)| < R/2\}. \tag{5.14}$$

We again consider (5.12), and divide the integration domain into two sets given by $A = \{(y,t) : |Q(y,t) - Q(y_0,t_0| > \delta_1\}$ and $A^c = \{(y,t) : |Q(y,t) - Q(y_0,t_0| \leq \delta_1\}$,

$$\begin{aligned}|Q_n(T_n(x_0,t_0)) - Q(T(x_0,t_0))| &\leq \iint_A |Q(y,t) - Q(y_0,t_0)|\omega_n\big(T_n(x_0,t_0) - (y,t)\big)dydt \\ &\quad + \iint_{A^c} |Q(y,t) - Q(y_0,t_0)|\omega_n\big(T_n(x_0,t_0) - (y,t)\big)dydt \\ &\leq \|Q(y,t) - Q(y_0,t_0)\|_{\infty,B_R(y_0,t_0)}\|\omega_n\|_\infty \iint_A dydt \\ &\quad + \delta_1 \iint_{A^c} \omega_n(\big(T_n(x_0,t_0) - (y,t)\big)dydt.\end{aligned}$$

By using (5.13), (5.14) and that $\iint_{\mathbb{R}\times\mathbb{R}^+} \omega_n dydt = 1$ we get

$$|Q_n(T_n(x_0,t_0)) - Q(T(x_0,t_0))| \leq \|Q(y,t) - Q(y_0,t_0)\|_{\infty,B_R(y_0,t_0)}\delta_2\|\omega_n\|_\infty/n^2 + \delta_1.$$

To conclude, the proof that $Q(T(x_0,t_0)) = (x_0,t_0)$, we show that $\|\omega_n\|_\infty/n^2$ is bounded independent of n. In fact,

$$\|\omega_n\|_\infty/n^2 = \|\omega n^2\|_\infty/n^2 = \|\omega\|_\infty,$$

since ω_n is defined by $\omega_n = n^2\omega(ny,nt)$. In addition, by Definition 2.18, $0 \leq \omega \leq 1$, hence $\|\omega\|_\infty \leq 1$ and $\|\omega_n\|_\infty/n^2$ is bounded independent of n. Thus, we are free to choose δ_1 and δ_2 arbitrary small and we get that $Q(T(x_0,t_0)) = (x_0,t_0)$ ρ-a.e. Then, we have shown that $D(\widehat{\psi_t \circ Q}) \circ T = D(\psi_t \circ Q) \circ T = D\psi_t$ ρ-a.e. Since $\rho = 0$ implies $D = 0$ will $D(\widehat{\psi_t \circ Q}) \circ T = D\psi_t$ hold m_2-a.e.

Next, we want to prove that $D_0(\psi_0 \circ Q) \circ T = D_0\psi_0$ m_1-a.e. First, we observe that $Q(y,0) = x(y,0)$ is monotone. Thus, $x(y,0)$ is discontinuous for at most a countable number of points and is a measurable function. So, by Theorem 2.27 $x(y,0)$ is approximate continuous a.e. and we have that

$$m_1\big(\{y \in \mathbb{R} : |x(y,0) - x(y_0,0)| > \delta_1, |y - y_0| < 1/n\}\big) \leq \delta_2 2/n. \tag{5.15}$$

Let Q_n and T_n be define as above and let $T(x_0,0) = (y_0,0)$ be a point of approximate continuity. Furthermore, we consider

$$\begin{aligned}|Q_n(T_n(x_0,0)) - Q(T(x_0,0))| &= \Big|\int \big(x(y,0) - x(y_0,0)\big)\omega_n\big(T_n(x_0,0) - (y,0)\big)dy\Big| \\ &\leq \int |x(y,0) - x(y_0,0)|\omega_n\big(T_n(x_0,0) - (y,0)\big)dy \\ &\leq \delta_2\|x(y,0) - x(y_0,0)\|_{\infty,B_R(y_0)}\|\omega_n\|_\infty 2/n + \delta_1.\end{aligned}$$

The last inequality is due to Equation (5.15) and a similar argument as above. Thus, we can again choose δ_1 and δ_2 arbitrary small and we get that $Q(T(x_0,0)) = x_0$ ρ_0-a.e. Since $\rho_0 = 0$ implies that $D_0 = 0$ we have that the equality holds m_1-a.e. and we get that $D_0(\psi_0 \circ Q) \circ T = D_0\psi_0$ m_1-a.e.

Now, returning to Equation (5.11) we have

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{R}^+} \tilde{D}\phi_t d\tau + \iint_{\mathbb{R}\times\mathbb{R}^+} (\tilde{F} - \tilde{u}\tilde{D})\phi_y dydt + \int_{\mathbb{R}} \tilde{D}_0\phi_0 d\tau_0 \\ &\quad = \iint_{\mathbb{R}\times\mathbb{R}^+} D(\widehat{\psi_t \circ Q}) \circ T + F(\widehat{\psi_x \circ Q}) \circ T dxdt + \int_{\mathbb{R}} D_0(\psi_0 \circ Q) \circ T dx \\ &\quad = \iint_{\mathbb{R}\times\mathbb{R}^+} D\psi_t + F\psi_x dxdt + \int_{\mathbb{R}} D_0\psi_0 dx = 0.\end{aligned}$$

The last equality is due to the assumption that (D, F) is a density-flux pair of (5.1b) or (5.1c) from a weak solution of (5.1). □

So, we have now proved that $\tilde{u}$, $\tilde{D}$, $\tilde{F}$ satisfy

$$(\tau\tilde{D})_t + (\tilde{F} - \tilde{u}\tilde{D})_y = 0 \tag{5.16}$$

in the sense of Definition 5.2. First, let $D = \rho u$ and $F = \rho u^2 + p$. Then, if we insert this in (5.16) and rewrite, we get

$$\tilde{u}_t + \tilde{p}_y = 0,$$

which is conservation of momentum in Lagrange coordinates. Furthermore, if we let $D = \rho e + \rho u^2/2$ and $F = u(\rho e + \rho u^2/2 + p)$ and rewrite (5.16) we get

$$\left(\tilde{e} + \frac{\tilde{u}^2}{2}\right)_t + (\tilde{u}\tilde{p})_y = 0.$$

This is conservation of energy in Euler coordinates. Thus, with the new definition for weak solutions in Lagrange coordinates, the Euler equations will still imply the Lagrange equations.

5.2.2 The Lagrange formulation implies the Euler formulation

Next, we want to show that the reverse implication also holds true. Let (τ, u, S) satisfy (5.2) in the sense of Definition 5.2 and τ dominate m_2. From Lemma 5.10 there exists a function $Q(y,t) = (x(y,t), t)$ satisfying

$$\frac{\partial x}{\partial y} = \tau \quad \text{and} \quad \frac{\partial x}{\partial t} = u,$$

with the unique monotone left inverse T, i.e., $T(Q(y,t)) = (y,t)$. T is Lipschitz continuous with respect to τ. By the uniqueness of inverse, T will be the function defined by (5.4). Furthermore, let $\rho = Q_{\#}m_2$.

Since we have assumed that τ dominates m_2, we have that m_2 is absolutely continuous with respect to τ. Let $\tilde{\rho}$ be the density of m_2 with respect to τ and $\tilde{\tau}$ the density of $\tau_{a.c.}$ with respect to m_2. Then, we can show that $\tilde{\rho}\tilde{\tau} = 1$ m_2-a.e. Let ϕ be a test function and consider

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}^+} \phi\tilde{\rho}\tilde{\tau}dydt &= \iint_{V^c} \phi\tilde{\rho}\tilde{\tau}dydt = \iint_{V^c} \phi\tilde{\rho}d\tau_{a.c.} + \iint_{V^c} \phi\tilde{\rho}d\tau_s = \iint_{V^c} \phi\tilde{\rho}d\tau \\ &= \iint_{V^c} \phi dydt = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi dydt,\end{aligned} \tag{5.17}$$

where we have used that $m_2(V) = \tau_s(V^c) = 0$. Since this holds true for all ϕ we have that $\tilde{\rho}\tilde{\tau} = 1$ m_2-a.e.

Lemma 5.13 ([31, Lemma 6])**.** $\rho = \frac{\partial y}{\partial x} = \tilde{\rho} \circ T$.

Proof. We first observe that

$$T_{\#}Q_{\#}m_2(E) = Q_{\#}m_2\big(T^{-1}(E)\big) = m_2\big(Q^{-1}(T^{-1}(E)\big),$$

and that

$$\begin{aligned}Q^{-1}\big(T^{-1}(E)\big) &= Q^{-1}\big(\{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : T(x,t) \in E\}\big) \\ &= \big\{(y,t) \in \mathbb{R}\times\mathbb{R}^+ : Q(y,t) \in \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : T(x,t) \in E\}\big\} \\ &= \big\{(y,t) \in \mathbb{R}\times\mathbb{R}^+ : T(Q(y,t)) \in E\big\} \\ &= \big\{(y,t) \in \mathbb{R}\times\mathbb{R}^+ : (y,t) \in E\big\} = E,\end{aligned} \tag{5.18}$$

where we have used that T is the left inverse of Q. Thus, we have $T_{\#}\rho = T_{\#}Q_{\#}m_2 = m_2$. Let ϕ be any test function, then

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \phi dT_{\#}\Big(\frac{\partial y}{\partial x}\Big) = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\circ T\frac{\partial y}{\partial x}dxdt = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi(y,t)N(T,(y,t))dydt,$$

where the first equality is due to assertion (i) in Theorem 2.16 and the second equality is due to Corollary 2.19. From the proof of Theorem 5.5 we have that $N(T,y) = 1$ m_2-a.e. So, we have proved that $T_{\#}\partial y/\partial x = m_2$ on test functions, and by a similar argumentation as in Theorem 5.5, $T_{\#}\partial y/\partial x = m_2 = T_{\#}\rho$ as measures. Using Definition 2.11, we get that $\partial y/\partial x(T^{-1}(E)) = \rho(T^{-1}(E))$, hence $\partial y/\partial x = \rho = \tilde{\rho}\circ T$ for all F on the form $T^{-1}(E)$. Now, let F be any other set and $(x,t)\in T^{-1}(T(F))\setminus F$, then there exists $x'\neq x$ such that $T(x',t) = T(x,t)$. Thus, $x\in B$ and consequently $T^{-1}(T(F))\subseteq B$. Using this and that $\partial y/\partial x(B) = 0$, we have

$$\frac{\partial y}{\partial x}\Big(T^{-1}\big(T(F)\big)\setminus F\Big) \leq \frac{\partial y}{\partial x}(B) \implies \frac{\partial y}{\partial x}\Big(T^{-1}\big(T(F)\big)\Big) \leq \frac{\partial y}{\partial x}(F).$$

Furthermore, $T^{-1}(T(F)) = \big\{(x,t) : T(x,t)\in T(F)\big\}$ and for $(x,t)\in F$ we have $T(x,t)\in T(F)$. Thus, $F\subseteq T^{-1}(T(F))$ and we can conclude that

$$\frac{\partial y}{\partial x}(F) = \frac{\partial y}{\partial x}\Big(T^{-1}\big(T(F)\big)\Big).$$

Furthermore, using that $\rho(F) = Q_{\#}m_2(F) = m_2(Q^{-1}(F))$ and $Q^{-1}\big(T^{-1}(F)\big) = F$ we have

$$\rho\big(T^{-1}(T(F))\big) = m_2\big(Q^{-1}(T^{-1}(T(F)))\big) = m_2\big(T(F)\big).$$

Next, we want to show that $T(F)\setminus A\subseteq Q^{-1}(F)\subseteq T(F)$. We have that

$$\begin{aligned} T(F) &= \big\{(y,t) : \exists (x,t)\in F \ s.t.\ T(x,t) = (y,t)\big\},\\ Q^{-1}(F) &= \big\{(y,t) : Q(y,t)\in F\big\}, \text{ and}\\ T(F)\setminus A &= \big\{(y,t) : \exists (x,t)\in F\ s.t.\ T(x,t) = (y,t),\ \nexists x'\ s.t.\ x'\neq x \text{ and } T(x,t) = T(x',t)\big\}. \end{aligned}$$

If $(y,t)\in Q^{-1}(F)$ we have that there exists $(x,t) = Q(y,t)\in F$ such that $T(x,t) = T(Q(y,t)) = (y,t)$, since T is the left inverse of Q. Thus, $(y,t)\in T(F)$ and $Q^{-1}(F)\subseteq T(F)$. Furthermore, if we let $(y,t)\in T(F)\setminus A$, we have that there exists a unique $(x,t)\in F$ such that $T(x,t) = (y,t)$. Thus, $Q(y,t) = (x,t)$ and $T(Q(y,t)) = (y,t)$ and we have $(y,t)\in Q^{-1}(F)$. Hence $T(F)\setminus A\subseteq Q^{-1}(F)$. So, we have $T(F)\setminus A\subseteq Q^{-1}(F)\subset T(F)$ and consequently, since $m_2(A) = 0$, we have $m_2(T(F)) = m_2(Q^{-1}(F))$. Hence, we have

$$\rho\big(T^{-1}(T(F))\big) = m_2\big(T(F)\big) = m_2\big(Q^{-1}(F)\big) = \rho(F).$$

We can conclude that $\rho(F) = \rho\big(T^{-1}(T(F))\big) = \partial y/\partial x\big(T^{-1}(T(F))\big) = \partial y/\partial x(F)$. Thus, $\rho = \partial y/\partial x = \tilde{\rho}\circ T$ are equal as Radon measures. □

Lemma 5.14 ([31, Lemma 7])**.** $\tau = T_{\#}m_2$.

Proof. Since T is onto and proper and m_2 is a Radon measure, $T_{\#}m_2$ is a Radon measure. For any test function ϕ we have

$$\begin{aligned} T_{\#}m_2(\phi) &= \iint_{\mathbb{R}\times\mathbb{R}^+} \phi dT_{\#}m_2 = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\circ T dxdt = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\circ T\frac{1}{\partial y/\partial x}\partial y/\partial x dxdt\\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\circ T\frac{1}{\tilde{\rho}\circ T}\partial y/\partial x dxdt = \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\frac{1}{\tilde{\rho}}dydt\\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} \phi\tilde{\tau}dydt = \tau(\phi), \end{aligned}$$

where we have used Theorem 2.16, Corollary 2.19, Lemma 5.13 and that $\tilde{\rho}\tilde{\tau} = 1$ m_2-a.e. In addition, the last equality is due to a similar argument as in (5.17). Furthermore, by a similar argument as in Theorem 5.5 we get that $T_{\#}m_2$ and τ are equal as Radon measures. □

Next, we want to show that (5.1a) holds in the sense of distributions. Here, u is the velocity in Euler coordinates, and will satisfy $u(x,t) = \tilde{u}(T(x,t))$. In addition, addition ρ is the mass density, satisfying $\rho = Q_{\#}m_2$.

Theorem 5.15. *$\rho_t + (\rho u)_x = 0$ in the sense of a distribution.*

Proof. Let $\rho_0 = Q_{\#}m_1$. We start by considering

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}^+} \psi_t + \psi_x u d\rho + \int_{\mathbb{R}} \psi d\rho_0 &= \iint_{\mathbb{R}\times\mathbb{R}^+} \psi_t \circ Q + \psi_x \circ Q\tilde{u} dydt + \int_{\mathbb{R}} \psi \circ Q dy \\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} (\psi \circ Q)_t dydt + \int_{\mathbb{R}} \psi \circ Q dy \\ &= \iint_{\mathbb{R}\times\mathbb{R}^+} \phi_t dydt + \int_{\mathbb{R}} \phi dy.\end{aligned}$$

We have used that $(\psi \circ Q)_t = (\psi_t \circ Q) + (\psi_x \circ Q)x_t = \psi_t \circ Q + (\psi_x \circ Q)\tilde{u}$ at each point of approximate continuity of Q, i.e., almost everywhere. Furthermore, we let $\phi = \psi \circ Q$. From previous calculations, see page 63, we have that $\phi_t = (\widehat{\psi_t \circ Q}) + (\widehat{\psi_x \circ Q})\tilde{u}$ and $\phi_y = (\widehat{\psi_x \circ Q})\tau$, where $\widehat{\psi_x \circ Q}, \widehat{\psi_t \circ Q} \in L^\infty$. Thus, ϕ is a discontinuous test function that satisfies $\phi_t = g$ and $\phi_y = f\tau$, for given $f, g \in L^\infty$ and

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \phi_t dydt + \int_{\mathbb{R}} \phi dy = 0$$

is the weak formulation of the trivial balance law $1_t + 0_y = 0$. So, we have proved that $\rho_t + (\rho u)_x = 0$ in the sense of a distribution. □

Lastly, we show that given a density-flux pair, $\tilde{D}, \tilde{F}$, satisfying (5.2b) or (5.2c) in the sense of Definition 5.2, the density-flux pair D, F satisfying $(D,F)(x,t) = (\tilde{D}, \tilde{F})(T(x,t))$, will satisfy (5.1) in the weak sense.

Theorem 5.16 (Adapted from [31, p. 134])**.** *Let (D,F) be a density-flux pair of* (5.2b) *or* (5.2c) *from a weak solution of* (5.2) *in the sense of Definition 5.2, wherein τ, u and S are bounded. Then $(u,D,F)(x,t) = (\tilde{u}, \tilde{D}, \tilde{F})(T(x,t))$, satisfy $(\rho D)_t + (F + \rho u D)_x = 0$ weakly.*

Proof. Let ψ be a test function. The goal is to show that for a density-flux pair D, F the following equation, holds true

$$\iint_{\mathbb{R}\times\mathbb{R}^+} (\psi_t D + \psi_x u D) d\rho + \iint_{\mathbb{R}\times\mathbb{R}^+} \psi_x F dxdt + \int_{\mathbb{R}} \psi D_0 d\rho_0 = 0 \quad \forall \psi \in W_c^{1,\infty}(\mathbb{R}\times\mathbb{R}^+).$$

To conclude this, we start by dividing the left-hand side of the equation into two, one considering the density function D and one considering the flux function F. First, let us consider D.

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}^+}(\psi_t D+\psi_x uD)d\rho+\int_{\mathbb{R}}\psi D_0 d\rho_0 &= \iint_{\mathbb{R}\times\mathbb{R}^+}(\psi_t D+\psi_x uD)dQ_{\#}m_2+\int_{\mathbb{R}}\psi D_0 dQ_{\#}m_1\\ &= \iint_{\mathbb{R}\times\mathbb{R}^+}\big\{(\psi_t\circ Q)\tilde{D}+(\psi_x\circ Q)\tilde{u}\tilde{D}\big\}dydt\\ &\quad+\int_{\mathbb{R}}(\psi\circ Q)\tilde{D}_0 dy\\ &= \iint_{\mathbb{R}\times\mathbb{R}^+}(\psi\circ Q)_t\tilde{D}dydt+\int_{\mathbb{R}}(\psi\circ Q)\tilde{D}_0 dy,\end{aligned} \tag{5.19}$$

where we have used assertion (i) in Theorem 2.16, and that $(u,D)\circ Q=(\tilde{u},\tilde{D})\circ T\circ Q=(\tilde{u},\tilde{D})$, since T is the left inverse of Q. In addition, we used that $(\psi\circ Q)_t=(\psi_t\circ Q)+(\psi_x\circ Q)x_t=(\psi_t\circ Q)+(\psi_x\circ Q)\tilde{u}$ at a point of approximate continuity of Q, i.e., a.e. Next, we consider F

$$\iint_{\mathbb{R}\times\mathbb{R}^+}\psi_x F dxdt=\iint_{\mathbb{R}\times\mathbb{R}^+}\psi_x(\tilde{F}\circ T)dxdt=\iint_{\mathbb{R}\times\mathbb{R}^+}(\tilde{F}\circ T)d\psi_x=\iint_{\mathbb{R}\times\mathbb{R}^+}\tilde{F}dT_{\#}\psi_x.$$

To evaluate $T_{\#}\psi_x$, we choose a smooth function σ with compact support

$$\begin{aligned}T_{\#}\psi_x(\sigma) &= \iint_{\mathbb{R}\times\mathbb{R}^+}\sigma dT_{\#}\psi_x=\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma\circ T)\psi_x dxdt=-\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma\circ T)_x\psi dxdt\\ &= -\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma_y\circ T)\psi d\partial y/\partial x=-\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma_y\circ T)\psi d\rho\\ &= -\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma_y\circ T)\psi dQ_{\#}m_2=-\iint_{\mathbb{R}\times\mathbb{R}^+}(\sigma_y\circ T\circ Q)(\psi\circ Q)dydt\\ &= -\iint_{\mathbb{R}\times\mathbb{R}^+}\sigma_y(\psi\circ Q)dydt=\iint_{\mathbb{R}\times\mathbb{R}^+}\sigma d(\psi\circ Q)_y=(\psi\circ Q)_y(\sigma),\end{aligned}$$

where we have done integration by parts, used that $T\circ Q$ is the identity map and assertion (i) in Theorem 2.16. Thus, by a similar argument as in Theorem 5.5, $T_{\#}\psi_x$ and $(\psi\circ Q)_y$ are equal as signed Radon measures, and we have

$$\iint_{\mathbb{R}\times\mathbb{R}^+}\phi_x F dxdt=\iint_{\mathbb{R}\times\mathbb{R}^+}\tilde{F}(\phi\circ Q)_y dydt. \tag{5.20}$$

If we combine (5.19) and (5.20) we get

$$\iint_{\mathbb{R}\times\mathbb{R}^+}(\psi_t D+\psi_x uD)d\rho+\psi_x F dxdt+\int_{\mathbb{R}}\psi D_0 d\rho_0=\iint_{\mathbb{R}\times\mathbb{R}^+}\phi_t\tilde{D}+\tilde{F}\phi_y dydt+\int_{\mathbb{R}}\phi\tilde{D}_0 dy,$$

where $\phi=\psi\circ Q$. From the proof of Theorem 5.15 we have that ϕ is a discontinuous test function satisfying $\phi_t=g$ and $\phi_y=f\tau$, for given $f,g\in L^\infty$. Thus,

$$\iint_{\mathbb{R}\times\mathbb{R}^+}\phi_t\tilde{D}+\tilde{F}\phi_y dydt+\int_{\mathbb{R}}\phi\tilde{D}_0 dy=0$$

is the weak formulation of $\tilde{D}_t+\tilde{F}_y=0$ in the sense of Definition 5.2 and consequently

$$\iint_{\mathbb{R}\times\mathbb{R}^+}(\psi_t D+\psi_x uD)d\rho+\iint_{\mathbb{R}\times\mathbb{R}^+}\psi_x F dxdt+\int_{\mathbb{R}}\psi D_0 d\rho_0=0.$$

So, we have proved that $(\rho D)_t+(F+\rho uD)_x=0$ weakly. □

We can use this theorem to show that a weak solution of the conservation of momentum or energy in Lagrange coordinates also is a weak solution in Euler coordinates. First, let $\tilde{D} = \tilde{u}$ and $\tilde{F} = \tilde{p}$. We then get

$$(\rho u)_t + (p + \rho u^2)_x = 0,$$

which is the conservation of momentum in Euler coordinates. Furthermore, if we let $\tilde{D} = \tilde{e} + \tilde{u}^2/2$ and $\tilde{F} = \tilde{u}\tilde{p}$, we get the conservation of Euler coordinates given by

$$(\rho e + \rho u^2/2)_t + (up + \rho ue + \rho u^3/2)_x = 0.$$

Thus in this chapter we have shown that, with a strengthen definition of the weak solutions in Lagrange coordinates the weak solutions of Euler and Lagrange equations are equivalent.

Chapter 6

Conclusion and future research

The main goal of this master thesis was to increase the understanding of the proofs of equivalence between spatial and referential formulations of a balance law. This has been achieved through first proving that a general formulation of a balance law is preserved under a bi-Lipschitz change of coordinates. Furthermore, this was used to show that the weak solutions of the one-dimensional Euler and Lagrange equations are equivalent, but only when we assume the solutions to be without vacuum. Furthermore, we showed that if we allow for vacuum, the equivalence still holds true, given a strengthened definition of the weak solutions in Lagrange coordinates. This definition imposes that the test functions in Lagrange coordinates are discontinuous in the vacuum set.

Thus, in this master thesis we have shown the equivalence between weak solutions of balance laws in different coordinate systems. However, a weak solution is not unique. In fact, some of the weak solutions may be nonphysical solutions. To remedy this, one should include an entropy condition to ensure that the weak solutions are physically relevant, and hopefully unique. Let us consider the following conservation law

$$(U(x,t))_t + \mathrm{div}_x F(U(x,t)) = 0, \tag{6.1}$$

where $x \in \mathbb{R}^n$ and $U : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^m$. To formulate an entropy condition for this system we often introduce an entropy density $\eta(U)$ and an associated entropy flux $q(U)$ such that

$$(\nabla q)_i = \nabla \eta J F_i, \quad i = 1, \ldots, n, \tag{6.2}$$

where JF is the Jacobi matrix of F and JF_i is the column vectors. We say that a solution of (6.1) is a weak entropy condition if it satisfies

$$\eta_t + \mathrm{div}_x q \leq 0, \tag{6.3}$$

in the sense of distributions. The existence of an entropy density η with an associated entropy flux q is dependent on the following condition

$$\nabla^2 \eta J F_i = J F_i \nabla^2 \eta \quad i = 1, \ldots, n. \tag{6.4}$$

For $m = 1$ this is trivially satisfied and for $m = 2$ and $n = 1$ the equation will reduce to a scalar, linear, second-order partial differential equation [16, Section 6.4]. However, for the remaining cases the function η is over-determined and we are not guaranteed that such a function exists, since Equation (6.4) imposes $\frac{1}{2}m(m-1)n$ conditions on η [8, Section 3.2]. Though, for some cases we still can find entropy densities. For instance, if (6.1) is symmetric, i.e., $JF^{\top} = JF$, then $\eta(U) = \frac{1}{2}|U|^2$ satisfies (6.4) in a non-trivial matter [16, Section 6.4]. All the hyperbolic conservation laws derived from continuum physics will be symmetrizable, and thus attain an entropy/entropy flux pair [16, Section 6.4]. So, for the Euler equations there exists an entropy condition. In [17] Harten et al. derive a family of entropy conditions for the Euler equations for $\rho > 0$, i.e., for solutions with no vacuum. Thus, it would be interesting to investigate if these are entropy conditions even when the solutions include a vacuum set. If we are given an entropy/entropy flux pair η, q satisfying (6.2), we could show that the weak formulation of (6.3) in Euler coordinates imply a weak formulation of an entropy condition in Lagrange coordinate, and vice versa, by a similar proof as in Theorems 5.12 and 5.16. Furthermore, in the multidimensional case we can see from the proof of Theorem 4.3 that an entropy condition like (6.3) will be preserved under a bi-Lipschitz change of coordinates. Thus, an interesting topic for future work, would be to research possible entropy conditions for general balance laws, since most of the developed theory is for specific systems [29]. However, recent research show that for a system of balance laws, the entropy condition will not guarantee uniqueness as first expected [29].

An additional proposition for further research is to see if there is a possibility to prove that the field equation of a general balance law is preserved under a different transformation, for instance a BV transformation. This would connect the theory for multidimensional balance laws with the theory for the one-dimensional Euler and Lagrange equations for weak solutions with vacuum. It would be interesting to see if there is a possibility to generalize the result from the Euler equations, or if the proof of equivalence under a BV transformation is dependent on the specific balance laws.

Bibliography

[1] G. Alberti, 'A Lusin type theorem for gradients', *Journal of Functional Analysis*, vol. 100, no. 1, pp. 110–118, 1991.

[2] A. Bennett, *Lagrangian Fluid Dynamics* (Cambridge Monographs on Mechanics). Cambridge: Cambridge University Press, 2006.

[3] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Texts in Applied Mathematics, volume 4), Third ed. New York, NY: Springer New York, 1993.

[4] F. H. Clarke, 'On the inverse function theorem', *Pacific Journal of Mathematics*, vol. 64, no. 1, pp. 97–102, 1976.

[5] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Pure and applied mathematics). New York, NY: Interscience Publishers, Inc., 1948, vol. 1.

[6] H. S. M. Coxeter, *Introduction to Geometry*, Second ed. New York, NY: John Wiley & Sons, Inc., 1969.

[7] C. M. Dafermos, 'Equivalence of referential and spatial field equations in continuum physics', in *Nonlinear hyperbolic problems: Theoretical, Applied, and Computational Aspects (Taormina, Italy, 1992)*, Wiesbaden: Vieweg+Teubner Verlag, 1993, pp. 179–183.

[8] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics* (Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]), Fourth ed. Heidelberg: Springer-Verlag, Berlin, 2016.

[9] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions* (Textbooks in Mathematics), Revised ed. Boca Raton, FL: CRC Press, 2015.

[10] H. Federer, *Geometric Measure Theory* (Classics in Mathematics). Berlin, Heidelberg: Springer, 1996.

[11] C. Foias, C. Guillopé and R. Temam, 'Lagrangian representation of a flow', *Journal of Differential Equations*, vol. 57, no. 3, pp. 440–449, 1985.

[12] G. B. Folland, *Real analysis: Modern techniques and their Applications* (Pure and Applied Mathematics). New York, NY: John Wiley & Sons, Inc., 1984.

[13] M. Giaquinta and G. Modica, *Mathematical Analysis: Foundations and Advanced Techniques for Functions of Several Variables.* Boston, MA: Birkhäuser Boston, Ltd, 2012.

[14] L. M. Graves, *The Theory of Functions of Real Variables*, Second ed. New York, NY: McGraw-Hill Book Co., Inc., 1956.

[15] A. Gray, E. Abbena and S. Salamon, *Modern Differential Geometry of Curves and Surfaces with Mathematica®* (Studies in Advanced Mathematics), Third ed. Boca Raton, FL: Chapman & Hall/CRC, 2006.

[16] A. Greven, G. Keller and G. Warnecke, *Entropy* (Princeton Series in Applied Mathematics). Princeton, NJ: Princeton University Press, 2003.

[17] A. Harten, P. D. Lax, C. D. Levermore and W. J. Morokoff, 'Convex Entropies and Hyperbolicity for General Euler Equations', *SIAM Journal on Numerical Analysis*, vol. 35, no. 6, pp. 2117–2127, 1998.

[18] H. Holden and N. H. Risebro, *Front Tracking for Hyperbolic Conservation Laws* (Applied Mathematical Sciences, volume 152), Second ed. Berlin, Heidelberg: Springer, 2015.

[19] A. Klenke, *Probability Theory: a Comprehensive Course* (Universitext). London: Springer-Verlag, 2008.

[20] M. B. Kristoffersen, 'Proof of the equivalence of the weak solutions for the Euler and the Lagrange equations for a compressible, inviscid, non-heat-conducting gas', Project thesis, NTNU, Trondheim, Dec. 2022.

[21] P. K. Kundu, I. M. Cohen and D. R. Dowling, *Fluid Mechanics*, Fifth ed. Boston, MA: Academic Press, 2012.

[22] P. D. Lax and M. S. Terrell, *Multivariable Calculus with Applications* (Undergraduate Texts in Mathematics). Cham: Springer, 2017.

[23] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability* (Cambridge Studies in Advanced Mathematics, volume 44). Cambridge: Cambridge University Press, 1995.

[24] J. F. Price, 'Lagrangian and Eulerian representations of fluid flow: Kinematics and the equations of motion', Jun. 2006. [Online]. Available: https://www.whoi.edu/science/PO/people/jprice/class/ELreps.pdf.

[25] H. L. Royden and P. Fitzpatrick, *Real Analysis*, Fourth ed. Boston, MA: Prentice Hall, 2010.

[26] M. Simonnet, *Measures and Probabilities* (Universitext). New York, NY: Springer, 1996.

[27] C. Truesdell and K. R. Rajagopal, *An Introduction to the Mechanics of Fluids.* Boston, MA: Birkhäuser Boston, Inc., 2000.

[28] C. Truesdell, 'Cauchy and the modern mechanics of continua', *Revue d'Histoire des Sciences*, vol. 45, pp. 5–24, 1992.

[29] A. Vasseur, 'A review of recent applications of the relative entropy method to discontinuous solutions of conservation laws', *Quarterly of Applied Mathematics*, vol. 81, no. 3, pp. 553–565, 2023.

[30] A. I. Vol'pert, 'The spaces BV and quasilinear equations', *Mathematics of The Ussr-sbornik*, vol. 2, pp. 225–267, 1967.

[31] D. H. Wagner, 'Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions', *Journal of Differential Equations*, vol. 68, no. 1, pp. 118–136, 1987.

[32] V. A. Zorich, *Mathematical Analysis. II* (Universitext), Second ed. Berlin, Heidelberg: Springer, 2016, Translated from the fourth and the sixth corrected (2012) Russian editions by Roger Cooke and Octavio Paniagua T.

Appendix A

Equivalence between Euler and Lagrange equations using Theorem 4.3

This appendix includes the calculations done to show the equivalence between the weak solutions of the Euler and the Lagrange equations using Theorem 4.3. We start by considering the Euler equations given by

$$\rho_t + (\rho u)_x = 0, \tag{A.1a}$$

$$(\rho u)_t + (\rho u^2 + p(\rho, S))_x = 0, \tag{A.1b}$$

$$\left(\rho e(\rho, S) + \frac{\rho u^2}{2}\right)_t + \left(u\left(\rho e(\rho, S) + \frac{\rho u^2}{2} + p(\rho, S)\right)\right)_x = 0, \tag{A.1c}$$

and the transformation $T(x,t) = (y,t)$ with the following Jacobian matrix

$$JT = \begin{bmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial x} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho u & \rho \end{bmatrix},$$

with $\det JT = \rho$. For the conservation laws in (A.1) the production measure P is zero. Thus, to obtain the Lagrange equations we use Theorem 4.3 to conclude that if

$$\operatorname{div} A = 0$$

in the sense of distributions, then $A^* \circ T = (\det J)^{-1} A J^\top$ satisfies

$$\operatorname{div} A^* = 0.$$

For (A.1a) we have $A = [\rho, \rho u]$ and thus

$$A^* \circ T = \frac{1}{\rho} \begin{bmatrix} \rho & \rho u \end{bmatrix} \begin{bmatrix} 1 & -\rho u \\ 0 & \rho \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

which is the trivial partial differential equation $1_t + 0_y = 0$. Next, we consider (A.1b), where $A = [\rho u, \rho u^2 + p]$ and we get

$$A^* \circ T = \frac{1}{\rho} \begin{bmatrix} \rho u & \rho u^2 + p \end{bmatrix} \begin{bmatrix} 1 & -\rho u \\ 0 & \rho \end{bmatrix} = \begin{bmatrix} u & -\rho u^2 + \rho u^2 + p \end{bmatrix} = \begin{bmatrix} u & p \end{bmatrix}.$$

This implies that the partial differential equation $\tilde{u}_t + \tilde{p}(\tau, \tilde{S})_y = 0$ holds in the sense of distributions. Furthermore, for (A.1c) $A = [\rho e + \rho u^2/2, u(\rho e + \rho u^2/2 + p)]$ and we get

$$\begin{aligned} A^* \circ T &= \frac{1}{\rho} \begin{bmatrix} \rho e + \frac{\rho u^2}{2} & u(\rho e + \frac{\rho u^2}{2} + p) \end{bmatrix} \begin{bmatrix} 1 & -\rho u \\ 0 & \rho \end{bmatrix} \\ &= \begin{bmatrix} e + \frac{u^2}{2} & -u\rho e - \frac{\rho u^3}{2} + u\rho e + \frac{\rho u^3}{2} + up \end{bmatrix} = \begin{bmatrix} e + \frac{u^2}{2} & up \end{bmatrix}. \end{aligned}$$

Thus, we have that $(\tilde{e}(\tau, \tilde{S}) + \frac{\tilde{u}^2}{2})_t + (\tilde{u}\tilde{p}(\tau, \tilde{S}))_y = 0$ holds in the sense of distributions. Lastly, we consider the trivial partial differential equation $1_t + 0_x = 0$, with $A = [1, 0]$ and thus

$$A * \circ T = \frac{1}{\rho} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\rho u \\ 0 & \rho \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} & -u \end{bmatrix},$$

which implies that $\tau_t - \tilde{u}_y = 0$ holds in the sense of distributions. Thus, we have shown that the Euler equations holds in the sense of distributions implies that the following equations hold in the sense of distributions

$$\tau_t - \tilde{u}_y = 0, \tag{A.2a}$$

$$\tilde{u}_t + \tilde{p}(\tau, \tilde{S})_y = 0, \tag{A.2b}$$

$$\left(\tilde{e}(\tau, \tilde{S}) + \frac{\tilde{u}^2}{2} \right)_t + (\tilde{u}\tilde{p}(\tau, \tilde{S}))_y = 0. \tag{A.2c}$$

This is the Lagrange equations. To obtain the opposite implication we start out by assuming that (A.2) holds in the sense of distributions and use the transformation $Q(y, t) = (x, t)$ with the Jacobian matrix

$$JQ = \begin{bmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial y} \\ \frac{\partial x}{\partial t} & \frac{\partial x}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u & \tau \end{bmatrix},$$

with determinate $\det JQ = \tau$. By a similar calculation as above we can show that (A.2a) implies that $1_t + 0_x = 0$ holds in the sense of distributions, (A.2b) implies that (A.1b) holds in the sense of distributions and (A.2c) implies that (A.1c) holds in the sense of distributions. Lastly, $1_t + 0_y = 0$ implies that (A.1a) holds in the sense of distributions.

NTNU

Norwegian University of
Science and Technology