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Master's thesis in Applied Physics and Mathematics
Supervisor: Mats Ehrnström
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Abstract

Building on work by Ehrnström, Mæhlen and Varholm, we prove the existence of limits at the origin for all higher-order derivatives of a highest, cusped, travelling-wave solution of the Whitham equation. We prove the exact values of the limits for the second and third derivative, and show that the exact value of the limit for the n-th derivative is as conjectured in the above-mentioned work if a certain expression equals zero for all integers $k \in \{2, ..., n-1\}$. We confirm that the expression equals zero for $k = 2$, and also for $k \in \{3, ..., 100\}$ using a computer-aided approach, however the complexity of the expression has prevented us from completing the calculation for the case of a general k. We expect further considerations to yield an analytic solution to this step.

Sammendrag

Basert på arbeid av Ehrnström, Mæhlen og Varholm, beviser vi eksistensen av grenseverdier ved origo for alle høyere ordens deriverte av en høyeste, spiss, reisende-bølge løsning av Whitham-likningen. For den andrederiverte og tredjederiverte finner vi de eksakte verdiene for grensene, og vi viser at den nøyaktige verdien av grensen til den n-te deriverte er som de ovenfornevnte forfatterne forventer så lenge verdien av et visst uttrykk er lik null for alle heltall $k \in \{2, ..., n-1\}$. Vi bekrefter at uttrykket er lik null for $k = 2$, og også for $k \in \{3, ..., 100\}$ ved å bruke dataprogram, men uttrykkets kompleksitet har hindret oss fra å fullføre beregningen for en vilkårlig k. Vi forventer at videre arbeid vil føre til et analytisk svar på dette punktet.

Preface

This thesis marks the conclusion of my five years as a student in the Applied Physics and Mathematics study program at the Norwegian University of Science and Technology (NTNU). It was written in the spring semester of 2023 at the Department of Mathematical Sciences at NTNU under the supervision of Professor Mats Ehrnström.

I would like to express my deepest gratitude to Mats for suggesting the topic of my thesis, and for his help along the way. His guidance and support have been instrumental in shaping the direction and success of my research, and I am immensely thankful to have had the opportunity to learn from him.

Robin Østern Lien
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1 Introduction

The Whitham Equation was introduced in [1] as a non-local, non-linear dispersive shallow water wave model of the form

$$\partial_t \phi + \partial_x (K * \phi + \phi^2) = 0, \tag{1.1}$$

where $\phi = \phi(t, x)$ is the surface profile, and $K = K(x)$ is defined by its Fourier transform

$$\hat{K}(\xi) = \int_{\mathbb{R}} K e^{-ix\xi} \mathrm{d}x := \sqrt{\frac{\tanh \xi}{\xi}}. \tag{1.2}$$

Whitham conjectured that the interplay between the linear dispersion and the nonlinear effects would give rise to smooth periodic and solitary waves, but also wave breaking and surface singularities, which are properties that the famous KdV equation lacks. Indeed, solutions of the Whitham equation and Whitham-type equations possessing these features have later been shown to exist [2, 3, 4, 5, 6, 7, 8, 9, 10].

Of particular interest to us is the existence of a highest, cusped, and even periodic traveling-wave solution φ of the Whitham equation, which was proved in [5] along with many other qualitative properties of this solution. Among the properties that the solution φ was shown to possess is smoothness away from the cusps and 1/2-Hölder continuity at the cusps, in particular at the cusp at the origin. That is, it was shown that $\varphi \in C^\infty(\mathbb{R}\backslash P\mathbb{Z})$, where P denotes the period of φ, and that $c_1|x|^{1/2} \leq \varphi(0) - \varphi(x) \leq c_2|x|^{1/2}$ for constants $0 < c_1 \leq c_2$ and $x \ll 1$. The authors of [5] conjectured that $c_1 = c_2 = \sqrt{\pi/8}$, but were not able to prove this result.

Recently, building on the work of [5], this question regarding the exact leading-order asymptotics at the origin for the highest, cusped wave φ was settled in [11]. Letting $u(x) := \varphi(0) - \varphi(x)$, the authors managed to show that the limit at the origin of $u(x)/x^{1/2}$ exists, and that

$$\lim_{x \to 0} \frac{u(x)}{x^{1/2}} = \sqrt{\frac{\pi}{8}},$$

confirming the conjecture posed in [5]. Furthermore, by using this newly determined limit at the origin of $u(x)/x^{1/2}$ they also managed to show the corresponding limit

$$\lim_{x \to 0} \frac{u'(x)}{x^{-1/2}} = \frac{1}{2}\sqrt{\frac{\pi}{8}}$$

for the derivative. Throughout this text, we will refer to these limits as the u-limit and the u'-limit respectively, and so on for higher order derivatives.

The procedure for determining the u'-limit in [11] consists of four main steps:

Step 1: Take the central difference of a certain integral equation containing u. We refer to this new equation as *the central difference equation* satisfied by u.

Step 2: Use the central difference equation to find a *first estimate* of the central difference $|u(x + h) - u(x - h)|$.

Step 3: Use the first estimate to find an *improved estimate* of $|u(x + h) - u(x - h)|$.

Step 4: Use the two estimates to show that the dominated convergence theorem applies for the integral in the central difference equation, whence we can calculate the u'-limit.

In steps two, three and four one divides the integral in the central difference equation into multiple smaller integrals by splitting the domain at certain "nice" points, and analyze each of these integrals independently. When calculating the u'-limit in step four, the u-limit is needed.

The authors of [11] state that the above steps may be performed inductively by replace u with $u^{(i)}$ (the i-th derivative of u), allowing one to prove corresponding limits for all higher-order derivatives of u. They conjecture that the n-th derivative of u, denoted by $u^{(n)}$, satisfies

$$u^{(n)}(x) = \left(\sqrt{\frac{\pi}{8}} + o(1)\right) \frac{\mathrm{d}^n}{\mathrm{d}x^n} x^{1/2},$$

but refrain from pursuing the question further.

Our goal is to do exactly this; Apply the methods of [11] and develop them further to study the asymptotic behavior at the origin of $u^{(n)}$. As we will see, the procedure used in [11] does not seem to translate directly to the study of higher order derivatives of u, so parts of the approach, particularly the first of the above-mentioned steps, need to be altered somewhat.

We are able to determine the u''-limit without any drastic changes to the approach for the u'-limit, however parts of the proof require more care. The reason is that the role played by u in the analysis of the u'-limit is played by u' in the analysis of the u''-limit, and there is a major qualitative difference between the behavior of u and u' near the origin; while $|u(x)|$ behaves like $|x|^{1/2}$, an *increasing* function, near the origin, $|u'(x)|$ behaves like $|x|^{-1/2}$, a *decreasing* function. In addition, $|x|^{-1/2}$ has a singularity at the origin, but the procedure *just* manages to work since $|x|^{-1/2}$ is integrable on a finite domain containing the singularity. There are two main consequences of the differences between u and u': The first is that the two estimates of Step 1 and Step 2 now need an extra factor to compensate for the singularity from $|x|^{-1/2}$. The second is that we must split the integral in the central difference equation into one more smaller integral, so as to isolate the new $|x|^{-1/2}$-type singularity, in steps three and four.

When moving on to the u'''-limit, however, the role of u' is replaced by u''. This is a problem as $|u''(x)|$ behaves like $|x|^{-3/2}$ near the origin, which is *not* integrable on a finite domain containing the singularity. We are, however, able to determine the u'''-limit if make the following change to Step 1: we return to the central difference equation satisfied by u and split the integral over multiple domains in such a way as to isolate the singularities *before* transforming this equation to one suitable for studying the u'''-limit. As we will see, this altered approach for Step 1 lends itself nicely to an inductive proof. There is, however, a problematic expression that appears in the Main Theorem 7.6 of this paper, which covers the inductive proof for the general $u^{(n)}$-limit. This expression, expression (7.2), has thus far managed to survive all of our attempts at simplification, forcing us to include a condition on when the value of the $u^{(n)}$-limit in the statement of the Main Theorem holds. We expect that (7.2) equals zero for all integers $n \geq 1$, and expect that further considerations will yield an analytical proof of this fact.

The rest of this paper is divided into six sections, of which the first three is a summary of the relevant parts from papers [5] and [11], while the final three is our own work. We begin with some setup where we cover and combine some important parts from both [5] and [11] that will be used throughout this paper. We also cover some notational conventions.

Then we look at paper [5] in more detail. Our focus will be on the proof for the smoothness of φ away from the cusps, and the 1/2-Hölder continuity at the cusps. The main result of the paper, the existence of a highest, cusped, even periodic travelling-wave solution φ, is a fact we will take for granted.

Next, we take a closer look at [11], where our focus will be on presenting the second part of what is referred to as the main Theorem, where the u- and u'-limits that are stated above are shown. The results in [11] are proved for a general class of functions of which the highest, cusped even periodic travelling-wave solution φ of the Whitham equation from [5] is a member. As we are focusing solely on the Whitham equation in this paper we will not require the same level of generality, and therefore present the relevant results from [11] for the special case of the solution φ. In our presentation of the relevant results from [5] and [11] we will elaborate somewhat where we deem it helpful and show some of the steps in the calculations in more detail.

The remaining three sections cover our study of the u''-limit, the u'''-limit and the $u^{(n)}$-limit respectively. The Main Theorem 7.6 and Corollary 7.7 in Section 7 are the main results of this paper, establishing the exact value of the $u^{(n)}$-limit is established if a certain condition holds, and the existence of the $u^{(n)}$-limit, respectively. Outside of some new lemmas needed for the inductive proof of the Main Theorem, Section 7 is, however, very similar to Section 6 where we study the u'''-limit. The new way of finding a usable central difference equation from our study of the u'''-limit, along with the new form of the two estimates and the new splitting of the integral from our study of the u''-limit, constitute the three most important contributions of this paper to the framework laid out in [11].

2 Setup

The search for a highest, cusped wave in [5] begins by looking for steady solutions $\phi(t,x) = \varphi(x-ct)$, i.e. waves that retain their shape as they travel, of the Whitham equation (1.1). This can be thought of as shifting our reference frame to that of the wave by "matching its speed and moving along side it", essentially letting us consider only the spatial coordinate. Here c denotes the speed of the wave. By substituting $\phi(t,x)$ with $\varphi(x-ct)$ in (1.1), we arrive at

$$-c\varphi'(x-ct) + (K * \varphi')(x-ct) + 2\varphi(x-ct)\varphi'(x-ct) = 0,$$

which becomes

$$(K * \varphi)(\tilde{x}) = c\varphi(\tilde{x}) - \varphi(\tilde{x})^2 + A, \tag{2.1}$$

where $\tilde{x} = x - ct$ and $A \in \mathbb{R}$ is a constant, after integrating the expression and rearranging [5]. The integration constant A can be set to zero without loss of generality by Galilean transformation as was done in [5], giving us the equivalent formulation

$$(K * \varphi)(x) = c\varphi(x) - \varphi(x)^2 \tag{2.2}$$

for steady solutions of the Whitham equation. This is the formulation that was used throughout [5]. As in [5], by a solution to (2.2) we mean a real-valued, continuous and bounded function φ that satisfies (2.2). We repeat that the travelling-wave solution $\varphi(x)$ of the Whitham equation (2.2) that was shown to exist in [5] is a highest wave (that is, the height of the wave reaches a height of $c/2$), is even, and is periodic with period P.

Following the steps of [11], we take a detour through a more general formulation of (2.2) to arrive at another equivalent formulation that will be of great importance. To this end, denote the right-hand side of (2.2) as $f(t) := ct - N(t)$, where we have replaced the square with a general nonlinear operator N [11]. That is, we are now considering the more general equation

$$(K * \varphi)(x) = f(\varphi(x)). \tag{2.3}$$

Suppose we have a largest wave of height γ, achieved at the origin $\varphi(0) = \gamma$, and that f has a nondegenerate local maximum at this γ, increasing to the left of it [11]. Also assuming that f is sufficiently smooth to Taylor expand around γ [11], we can write

$$f(t) = f(\gamma) + f'(\gamma)(t-\gamma) + \frac{1}{2}f''(\gamma)(t-\gamma)^2 + \frac{1}{3!}f'''(\gamma)(t-\gamma)^3 + \dots$$

Since f was assumed to be a local maximum at γ, the f'-term vanishes. We rearrange the equation and pull a factor of $(\gamma - t)^2$ out of the right-hand side, giving us

$$f(\gamma) - f(t) = \Big(-\frac{1}{2}f''(\gamma) + g(\gamma - t)\Big)(\gamma - t)^2,$$

after grouping the derivatives of order three and higher into a function $g(\gamma - t)$ [11]. Note from the Taylor expansion that $g(0) = 0$. We now introduce $u(x) := \gamma - \varphi(x)$ (this u will end up being the same as the one from the introduction). Inserting u into (2.3) together with the expression for $f(\gamma) - f(t)$, and noting that $(K * u)(0) = 0$, we see that u is a solution to

$$(K * u)(x) - (K * u)(0) = \Big(-\frac{1}{2}f''(\gamma) + g(u(x))\Big)u(x)^2,$$

which is non-negative if φ is a solution to (2.3) that achieves $\varphi(0) = \gamma$ from below, and that u is exactly zero at the origin [11]. This motivates the formulation

$$(1 + n(x))u(x)^2 = \int_{\mathbb{R}} (K(y-x) - K(y))u(y)dy, \tag{2.4}$$

where $n(0) = 0$ [11]. As $\hat{K}(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}$ is even, K is even, and as φ is even, u is even [11]. Accordingly, we can rewrite (2.4) as

$$\int_{\mathbb{R}} (K(y-x) - K(y))u(y)dy = \int_0^{\infty} (K(y+x) + K(y-x) - 2K(y))u(y)dy$$

in a straightforward manner [11]. We recognize the first part of the final integrand as a second-order central difference, which we denote as

$$\delta_x^2 K(y) := K(y+x) + K(y-x) - 2K(y), \tag{2.5}$$

allowing us to succinctly express (2.4) as

$$(1+n(x)))u(x)^2 = \int_0^\infty \delta_x^2 K(y) u(y) dy. \tag{2.6}$$

We now return to the steady Whitham equation (2.2), which is a special case special case of (2.3) where the nonlinear operator N is a square. The right-hand side of (2.2) is a quadratic polynomial in φ that achieves a maximum of $\varphi = c/2$, increasing to the left of this maximum. As it is a quadratic polynomial it is smooth. Its second derivative with respect to φ is $(c\varphi - \varphi^2)'' = -2$, and the higher order derivatives are zero. Thus, in this particular case we see that all of the assumptions on f in (2.3) are satisfied, with $-\frac{1}{2}f''(\gamma) = -\frac{1}{2}\cdot(-2) = 1$ and $n(x) = 0$. We therefore have that $u = c/2 - \varphi$ (the same u as in the introduction), where φ solves the steady Whitham equation (2.2), satisfies the equation

$$u(x)^2 = \int_0^\infty \delta_x^2 K(y) u(y) dy \tag{2.7}$$

[11]. This formulation is the foundation of the method for determining the u- and u'-limits in [11].

We also mention some important properties of the Whitham kernel K (defined in (1.2)) that were proved in [5]. Most importantly, it was shown that the integral kernel K can be decomposed into a singular and regular part,

$$\begin{aligned} K(x) &= K_{\text{sing}}(x) + K_{\text{reg}}(x), \\ K_{\text{sing}}(x) &= \frac{1}{|2\pi x|^{1/2}}, \end{aligned} \tag{2.8}$$

where $K_{\text{reg}}(x)$ is real analytic on $\mathbb{R}$ [5]. This can be seen by writing the right-hand side of (1.2) as

$$\sqrt{\frac{\tanh \xi}{\xi}} = \frac{1}{|\xi|^{1/2}} + \frac{(\tanh|\xi|)^{1/2} - 1}{|\xi|^{1/2}}$$

and noting that the inverse Fourier transform of the first term is exactly the first term in (2.8), and that the second term is integrable and exponentially decaying, so its inverse Fourier transform is real analytic [5]. This decomposition will be heavily used. Note that K is smooth for $x \neq 0$. We will also use [5, Proposition 2.1], which states that $K(x)$ and all of its derivatives are exponentially decaying. Since $K_{\text{reg}}(x) = K(x) - K_{\text{sing}}(x)$, it is necessarily decaying, which, together with the fact that K_{reg} is smooth, implies that $||K_{\text{reg}}^{(n)}||_{L^\infty} < \infty$ for all nonnegative integers n. We also recall from above that K is even, and it is also positive, strictly decreasing and strictly convex for $x > 0$ [5, Proposition 2.23].

We will also encounter the periodized Whitham kernel

$$K_P(x) = \sum_{n\in\mathbb{Z}} K(x+nP),$$

for $P \in (0,\infty)$ [5]. Note that this sum is absolutely convergent, as K has rapid decay [5]. Many of the properties of K translate directly to the periodization K_P, such as evenness, and the decomposition into a singular and regular part,

$$K_P(x) = \frac{1}{\sqrt{2\pi|x|}} + K_{P,\text{reg}}(x), \tag{2.9}$$

where $K_{P,\text{reg}}$ is real analytic in $(-P,P)$ [5]. For convenience we accept $P = \infty$, with the convention $K_\infty = K$ [5]. As in [5], we shall presuppose that any solution φ is P-periodic.

Throughout this paper we will let $\lesssim$ and $\gtrsim$ indicate inequalities that hold up to a uniform positive factor. If this factor is dependant on some parameter or function μ, this will be indicated by a subscript such as $\lesssim_\mu$. Similarly, we will also use $\eqsim$ to indicate that two expressions are equal up to some uniform positive factor.

3 Smoothness and 1/2-Hölder continuity of the highest, cusped wave

We now repeat some important results from [5], culminating in a proof for the smoothness of the highest wave solution φ of the Whitham equation away from the cusps, and its 1/2-Hölder regularity at the origin. At this point in [5] we do not yet know that a highest, cusped even periodic travelling wave solution of the Whitham equation exists, so the contents in this section is an a priori analysis of the highest wave. Again, we will not repeat the proof for the existence of such a wave, and instead refer the reader to [5] for the proof of this fact. We begin with some definitions.

For a nonnegative integer k we let $BUC^k(\mathbb{R})$ be the space of k times continuously differentiable functions on $\mathbb{R}$, whose derivatives of order less than or equal to k are bounded and uniformly continuous on $\mathbb{R}$.

Definition 3.1 (Hölder continuity). *We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is Hölder continuous of regularity $\alpha \in (0,1)$ at a point $x \in \mathbb{R}$ if*

$$|\varphi|_{C^\alpha_x} := \sup_{h \neq 0} \frac{|\varphi(x+h) - \varphi(x)|}{|h|^\alpha} < \infty,$$

and let

$$C^\alpha(\mathbb{R}) = \{\varphi \in BUC(\mathbb{R} : \sup_x |\varphi|_{C^\alpha_x} < \infty\},$$
$$C^{k,\alpha}(\mathbb{R}) = \{\varphi \in BUC^k(\mathbb{R} : \varphi^{(k)} \in C^\alpha_x < \infty\}.$$

Definition 3.2 (Besov spaces). *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $1 \leq q < \infty$. Using the Littlewood-Paley decomposition, the Besov spaces $B^s_{p,q}(\mathbb{R})$ are then defined by*

$$\left\{ f \in \mathcal{S}'(\mathbb{R}) : ||f||_{B^s_{p,q}(\mathbb{R})} := \Big[\sum_{j=0}^{\infty} (2^{sj} ||\gamma_j(D) f||_{L^p(\mathbb{R})})^q \Big]^{\frac{1}{q}} < \infty \right\}.$$

For $q = \infty$, the Besov space $B^s_{p,\infty}(\mathbb{R})$ is instead defined by

$$\left\{ f \in \mathcal{S}'(\mathbb{R}) : ||f||_{B^s_{p,q}(\mathbb{R})} := \sup_{j \geq 0} 2^{sj} ||\gamma_j(D) f||_{L^p(\mathbb{R})} < \infty \right\}.$$

Definition 3.3 (Zygmund spaces). *The Zygmund spaces $\mathcal{C}^s$, where $s \in \mathbb{R}$, are defined as $\mathcal{C}^s(\mathbb{R}) = B^s_{\infty,\infty}(\mathbb{R})$.*

Recall that $\mathcal{C}^s = C^{\lfloor s \rfloor, s - \lfloor s \rfloor}$ for $s \in \mathbb{R}_{>0} \backslash \mathbb{N}$ [5]. We will also need the following lemma which was used without proof in [5]. We have decided to include the proof here for clarity.

Lemma 3.4. *$L^\infty(\mathbb{R}) \subset B^0_{\infty,\infty}$, and $B^s_{\infty,\infty} \subset L^\infty(\mathbb{R})$ for $s > 0$.*

Proof. For the first statement, let $f \in L^\infty(\mathbb{R})$, i.e $||f||_{L^\infty(\mathbb{R})} < \infty$. Since γ_j is defined such that $f = \sum_{j=0}^\infty \gamma_j(D) f$, we certainly have that $\sup_{j \geq 0} ||\gamma_j(D) f||_{L^\infty(\mathbb{R})} \leq ||f||_{L^\infty(\mathbb{R})} < \infty$. But this is exactly the $B^0_{\infty,\infty}$-norm of f, so $L^\infty(\mathbb{R}) \subset B^0_{\infty,\infty}$. For the second statement, note that

$$\begin{aligned} ||f||_{L^\infty(\mathbb{R})} = || \sum_{j=0}^\infty \gamma_j(D) f \cdot 2^{sj} \cdot 2^{-sj} ||_{L^\infty(\mathbb{R})} &\leq \sum_{j=0}^\infty 2^{sj} ||\gamma_j(D) f||_{L^\infty(\mathbb{R})} 2^{-sj} \\ &\leq \sup_{j \geq 0} 2^{sj} ||\gamma_j(D) f||_{L^\infty(\mathbb{R})} \sum_{j=0}^\infty 2^{-sj} \lesssim ||f||_{B^s_{\infty,\infty}(\mathbb{R})}, \end{aligned}$$

where $s > 0$ ensures that final sum converges. Since the L^∞-norm is controlled by the $B^s_{\infty,\infty}$-norm, a function in $B^s_{\infty,\infty}$ is automatically in L^∞. In other words, $L^\infty \subset B^s_{\infty,\infty}$. □

In the following we let the operator L denote the action by convolution with K, that is $L : f \mapsto K*f$, defined via duality on the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions [5]. L defines a bounded operator

$$L : B^s_{p,q}(\mathbb{R}) \to B^{s+\frac{1}{2}}_{p,q}(\mathbb{R}),$$

as stated in [5, p. 18]. In particular,

$$L : \mathcal{C}^s(\mathbb{R}) \to \mathcal{C}^{s+\frac{1}{2}}(\mathbb{R}).$$

For a continuous periodic function f, L is given by $\int_{-P/2}^{P/2} K_P(x-y)f(y)\,\mathrm{d}y$, and for a bounded continuous function f, L is given by $\int_{\mathbb{R}} K(x-y)f(y)\,\mathrm{d}y$ [5].

We have now covered the necessary preliminaries and are ready to look at the relevant theorems from [5]. The first theorem we will look at establishes the smoothness of φ if its height is strictly less than $c/2$.

Theorem 3.5. *[5] Let $\varphi \leq \frac{c}{2}$ be a solution of the steady Whitham equation* (2.2)*. Then the following statements are true:*

(i) If $\varphi < \frac{c}{2}$ uniformly on $\mathbb{R}$, then $\varphi \in C^\infty(\mathbb{R})$ and with derivatives uniformly bounded on $\mathbb{R}$ [5].

(ii) φ is smooth on any open set where $\varphi < c/2$ [5].

Proof. We follow the proof from [5], but explain some of the steps in more detail. Let $\varphi < c/2$ uniformly on $\mathbb{R}$. Recall that L maps $B^s_{p,q}(\mathbb{R})$ into $B^{s+\frac{1}{2}}_{p,q}(\mathbb{R})$, and (by Theorem 3.4) $L^\infty \subset B^0_{\infty,\infty}(\mathbb{R})$ into $\mathcal{C}^{1/2}(\mathbb{R}) = B^{1/2}_{\infty,\infty}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Now we introduce the Nemytskii operator $g(v) := c/2 - \sqrt{c^2/4 - v}$, which maps $B^s_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ into itself for $v < c^2/4$ and $s > 0$ [12, Theorem 2.87]. All three maps are continuous. Since we assumed $\varphi < c/2$, it follows that $L\varphi < c^2/4$. This can be seen by noting that (2.2) is equivalent to the formulation

$$\varphi = \frac{c}{2} - \sqrt{c^2/4 - L\varphi}, \tag{3.1}$$

from which $L\varphi < c^2/4$ follows easily. Since $\varphi \in L^\infty(\mathbb{R}) \subset B^0_{\infty,\infty}$, we have that $L\varphi \in B^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Composing the maps L and g gives us that $g(L\varphi) \in B^{1/2}_{\infty,\infty} \cap L^\infty(\mathbb{R}) = B^{1/2}_{\infty,\infty}$ (the equality follows from Lemma 3.4). But (3.1) tells us that $\varphi = g(L\varphi)$, so $\varphi \in B^{1/2}_{\infty,\infty}$. Composing the maps again, now with the knowledge that $\varphi \in B^{1/2}_{\infty,\infty}$, implies that $\varphi = g(L\varphi) \in B^1_{\infty,\infty}$, since $B^s_{\infty,\infty} \subset L^\infty(\mathbb{R})$ when $s > 0$. Continuing iteratively, it is clear that $\varphi \in B^s_{\infty,\infty}$ for all $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ...\}$. Recall now the Zygmund and Hölder space introduced earlier, and in particular that for any $s \in \mathbb{N}$ we have $\varphi \in B^{s+1/2}_{\infty,\infty} = \mathcal{C}^{s+1/2} = C^{s,1/2} = \{f \in BUC^s(\mathbb{R}) : f^{(s)} \in C^{1/2}(\mathbb{R})\}$. In particular, $\varphi \in C^\infty(\mathbb{R})$ and each derivative is uniformly bounded.

For (ii), let φ be in L^∞ and $\mathcal{C}^s_{\text{loc}}$ on an open set U in the sense that $\psi\varphi \in \mathcal{C}^s$ for any $\psi \in C^\infty_0(U)$. We now want to show that for such a φ we in fact have that $L\varphi \in \mathcal{C}^{s+1/2}_{\text{loc}}$, as this allows us to again use the above iteration argument to infer that φ is smooth (on any open set U on which $\varphi < c/2$). To this end, let $\psi \in C^\infty_0(U)$, and let $\tilde{\psi} \in C^\infty_0(U)$ be a smooth cut-off function with $\tilde{\psi} = 1$ in a neighborhood $V \Subset U$ of supp ψ (that is, V is compactly contained in U). Then

$$\psi L\varphi = \psi L(\tilde{\psi}\varphi) + \psi L((1-\tilde{\psi})\varphi).$$

Since $\tilde{\psi}\varphi \in \mathcal{C}^s(\mathbb{R})$ and $L : \mathcal{C}^s(\mathbb{R}) \to \mathcal{C}^{s+1/2}(\mathbb{R})$, the first term on the right-hand side is in $\mathcal{C}^{s+1/2}(\mathbb{R})$. The second term is given by

$$\psi(x)(L((1-\tilde{\psi})\varphi))(x) = \int_{-\infty}^{\infty} K(x-y)\psi(x)(1-\tilde{\psi}(y))\varphi(y)dy.$$

Since $\tilde{\psi}$ is defined to equal one in a neighborhood of the support of ψ, the integrand vanishes for all y near x where $\psi(x) \neq 0$. If x instead is such that $\psi(x) = 0$, the integrand is also equal to zero. The second term is therefore smooth. Hence, $L\varphi$ is $\mathcal{C}^{s+1/2}_{\text{loc}}$ in U. Thus, if $\varphi \in L^\infty(\mathbb{R})$ and $\varphi < c/2$ on an open set U, we indeed have that $L\varphi \in \mathcal{C}^{1/2}_{\text{loc}}$ in U. Applying the above iteration argument used to prove (i) gives the final result. □

The following lemma will be used for the lower bound when we show the 1/2-Hölder continuity at the origin of φ.

Lemma 3.6. *[5] Let $P < \infty$, and let φ be an even, nonconstant solution of the steady Whitham equation* (2.2) *such that φ is nondecreasing on $(-P/2, 0)$ with $\varphi \leq c/2$. Then there exists a universal constant $\lambda_{K,P} > 0$, depending only on the kernel K and the period P, such that*

$$\frac{c}{2} - \varphi(P/2) \geq \lambda_{K,P}. \tag{3.2}$$

More generally,

$$\frac{c}{2} - \varphi(x) \gtrsim_{K,P} |x_0|^{1/2}, \tag{3.3}$$

uniformly for all $x \in [-P/2, x_0]$, with $x_0 < 0$.

Remark 3.7. Inspecting the proof of the lemma, one finds that estimate (3.3) is uniform in $P \gg 1$ and that it also holds in the limiting case $P = \infty$ (for $x \in (-\infty, x_0]$) [5].

Proof. We follow the proof from [5], but show the calculations in greater detail. Assume first that $\varphi(0) < c/2$. Then φ is smooth with bounded derivatives by Theorem 3.5. This means that we do not run into problems when differentiating the identity

$$\Big(\frac{c}{2} - \varphi\Big)^2 = c^2/4 - L\varphi,$$

which comes from rearranging and squaring (3.1). We get

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{c}{2} - \varphi(x)\Big)^2 = -2\Big(\frac{c}{2} - \varphi(x)\Big)\varphi'(x)$$

on the left hand side, and

$$\frac{\mathrm{d}}{\mathrm{d}x}(c^2/4 - L\varphi) = -\int_{-P/2}^{P/2} K_P(y)\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x-y)dy = -\int_{-P/2}^{P/2} K_P(x-t)\varphi'(t)dt$$

on the right-hand side, where we make the change of variables $t = x - y$. Letting $x \in [-\frac{3P}{8}, -\frac{P}{8}]$, this gives the estimate

$$\begin{aligned}
\Big(\frac{c}{2} - \varphi(P/2)\Big)\varphi'(x) &\geq \Big(\frac{c}{2} - \varphi(x)\Big)\varphi'(x) \\
&= \frac{1}{2}\int_{-P/2}^{P/2} K_P(x-y)\varphi'(y)dy \\
&= \frac{1}{2}\int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi'(y)dy \\
&\geq \frac{1}{2}\int_{-3P/8}^{-P/8} (K_P(x-y) - K_P(x+y))\varphi'(y)dy.
\end{aligned} \tag{3.4}$$

In the last step we have used that $K_P(x-y) > K_P(x+y)$ for $x, y \in (-P/2, 0)$. In fact, there exists a universal constant $\tilde{\lambda}_{K,P} > 0$ depending only on K and $P < \infty$ such that

$$\min\Big\{K_P(x-y) - K_P(x+y) : x, y \in \Big[\frac{-3P}{8}, \frac{-P}{8}\Big]\Big\} \geq \tilde{\lambda}_{K,P}.$$

Integrating (3.4) over $(-3P/8, -P/8)$ in x, we get

$$\begin{aligned}
&\Big(\frac{c}{2} - \varphi(P/2)\Big)(\varphi(-P/8) - \varphi(-3P/8)) \\
&\qquad \geq \frac{1}{2}\int_{-3P/8}^{-P/8}\Big(\int_{-3P/8}^{-P/8} (K_P(x-y) - K_P(x+y))dx\Big)\varphi'(y)dy \\
&\qquad \geq \frac{P}{8}\tilde{\lambda}_{K,P}(\varphi(-P/8) - \varphi(-3P/8).
\end{aligned}$$

From [5, Theorem 4.9], we have that $\varphi(-3P/8) < \varphi(-P/8)$ for φ as in the assumptions. Thus, we can divide by $\varphi(-3P/8) - \varphi(-P/8)$, arriving at

$$\frac{c}{2} - \varphi(P/2) \geq \frac{P}{8}\tilde{\lambda}_{K,P} := \lambda_{K,P}.$$

This proves (3.2) when $\varphi(0) < c/2$.

For the proof of (3.3), still assuming $\varphi(0) < c/2$, fix x_2, x_1 such that $-P/4 < x_2 < x_1 < 0$, let $x \in (x_2, x_1)$ and consider $\xi \in [-P/2, x_2]$. Then we have

$$\begin{aligned}\left(\frac{c}{2} - \varphi(\xi)\right)\varphi'(x) &\geq \frac{1}{2}\int_{x_2}^{x_1} (K_P(x-y) - K_P(x+y))\varphi'(y)dy && (3.5)\\ &\geq \frac{1}{2}\int_{x_2}^{x_1} (-2y)K_P'(y+\zeta)\varphi'(y)dy && (3.6)\\ &\geq -x_1 K_p'(2x_2)(\varphi(x_1) - \varphi(x_2)), && (3.7)\end{aligned}$$

where $|\zeta| < |x|$ comes from the mean value theorem. Integrating over (x_2, x_1) in x and dividing, we get

$$\frac{c}{2} - \varphi(\xi) \geq |x_1(x_2 - x_1)|K_p'(2x_2).$$

Letting $x_2 = x_0$, $x_1 = x_0/2$ and using that $K_P'(x) \sim |x|^{-3/2}$ for $0 < -x \ll 1$ (this comes from the fact that K_P can be written as (2.9)), we get that

$$\frac{c}{2} - \varphi(\xi) \geq \frac{1}{4}x_0^2 K_P'(2x_0) \gtrsim_{K,P} |x_0|^{1/2}.$$

This proves (3.3) for $\varphi(0) < c/2$.

Now, let instead $\varphi(0) = c/2$. The problem is that we no longer know if the assumptions on φ that are required for [5, Theorem 4.9] to hold, are true. Thus we need a new strategy to show that φ is strictly increasing on $(-P/2, 0)$ to ensure that we are not dividing by zero. To show the strict monotonicity of φ on the interval we instead use the double symmetrization formula

$$(L\varphi)(x+h) - (L\varphi)(x-h) = \int_{-P/2}^{0} (K_P(y-x) - K_P(y+x))(\varphi(y+h) - \varphi(y-h))dy, \quad (3.8)$$

which is a result of the fact that K_P and φ are even and periodic. Note that both the factors in the integrand are nonnegative for $x \in (-P/2, 0)$ and $h \in (0, P/2)$. In fact, from the properties of K_P we have that the first factor in the integrand is strictly greater than zero for such x and h. Also, as we assumed that φ is nonconstant and nondecreasing, the second factor must be strictly positive on some sub-interval of $(-P/2, 0)$. Thus, the integral is strictly greater than zero. From (2.2) we have the identity

$$\begin{aligned}L\varphi(x) - L\varphi(y) &= c\varphi(x) - \varphi(x)^2 - c\varphi(y) + \varphi(y)^2\\ &= c\varphi(x) - \varphi(x)^2 - \varphi(x)\varphi(y) + \varphi(x)\varphi(y) - c\varphi(y) + \varphi(y)^2 \qquad (3.9)\\ &= (c - \varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)),\end{aligned}$$

which shows that $L\varphi(x) = L\varphi(y)$ if $\varphi(x) = \varphi(y)$. Here, the first factor is nonzero for x and y different from zero. Combining this identity and (3.8) yields that φ is strictly increasing on $(-P/2, 0)$ (recall that φ is nonconstant by assumption). Now we only need to justify the differentiation under the integral sign in (3.4), which we do by applying Fatou's Lemma to $(c/2 - \varphi(x))\varphi'(x) = \lim_{h\to\infty}(L\varphi(x+h) - L\varphi(x-h))/4h$. Using (3.8), we get

$$\begin{aligned}(c/2 - \varphi(x))\varphi'(x) &= \lim_{h\to\infty} \frac{1}{4h}(L\varphi)(x+h) - (L\varphi)(x-h)\\ &= \lim_{h\to\infty}\int_{-P/2}^{0} (K_P(y-x) - K_P(y+x))\frac{\varphi(y+h) - \varphi(y-h)}{4h}dy\\ &\geq \frac{1}{2}\int_{-P/2}^{0} (K_P(y-x) - K_P(y+x))\lim_{h\to\infty}\frac{\varphi(y+h) - \varphi(y-h)}{2h}dy\end{aligned}$$

$$= \frac{1}{2}\int_{-P/2}^{0} (K_P(y-x) - K_P(y+x))\varphi'(y)dy.$$

The existence of the final limit, which comes from the fact that φ is smooth on $(-P/2, 0)$, ensures that the limit coincides with the limit infimum. The rest of the proof remains unchanged. □

Before we can prove the final theorem of this section, we need one more lemma. The statement of this lemma is used without proof in [5], but we have decided to include the proof here for clarity's sake.

Lemma 3.8. *Let $f \in C^\alpha$, $\alpha \in (0,1)$. If $\sqrt{f}$ is well-defined, then $\sqrt{f} \in C^{\alpha/2}$.*

Proof. We need to show that $|\sqrt{f(x)} - \sqrt{f(y)}| \lesssim |x-y|^{\alpha/2}$. To this end, notice that the left-hand side can be written as

$$\begin{aligned}
|\sqrt{f(x)} - \sqrt{f(y)}| &= |\sqrt{f(x)} - \sqrt{f(y)}| \frac{\sqrt{f(x)} + \sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} \\
&= \frac{|\sqrt{f(x)}^2 + \sqrt{f(x)}\sqrt{f(y)} - \sqrt{f(y)}\sqrt{f(x)} - \sqrt{f(y)}^2|}{\sqrt{f(x)} + \sqrt{f(y)}} \\
&= \frac{|f(x) - f(y)|}{\sqrt{f(x)} + \sqrt{f(y)}}.
\end{aligned}$$

Rewriting the numerator as $\sqrt{|f(x)-f(y)|} \cdot \sqrt{|f(x)-f(y)|}$, we see that the right-hand side is bounded by

$$\sqrt{|f(x)-f(y)|} \frac{\sqrt{f(x)} + \sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} = \sqrt{|f(x)-f(y)|}.$$

Now we use that $f \in C^\alpha$, which gives the bound

$$\begin{aligned}
\sqrt{|f(x)-f(y)|} &\lesssim \sqrt{|x-y|^\alpha} \\
&= |x-y|^{\alpha/2}.
\end{aligned}$$

This concludes the proof. □

Now we are ready for the main theorem of this section.

Theorem 3.9. *[5] Let $\varphi \leq c/2$ be a solution of the steady Whitham equation (2.2), which is even, nonconstant, and nondecreasing on $(-P/2, 0)$ with $\varphi(0) = c/2$. Then:*

(i) φ is smooth on $(-P, 0)$.

(ii) φ has Hölder regularity precisely $1/2$ at $x = 0$, that is, there exist constants $0 < c_1 < c_2$ such that $c_1|x|^{1/2} < \frac{c}{2} - \varphi(x) < c_2|x|^{1/2}$ for $|x| \ll 1$.

Remark 3.10. Note that the period P can be infinite in Theorem 3.9 [5].

Proof. We follow the proof from [5], but go through the computations and some of the arguments in greater detail. Note that *(ii)* is different in the corresponding Theorem statement in [5], but we have included it here to more clearly separate the steps of the proof. For *(i)*, note from the proof of Lemma 3.6 that a φ satisfying the assumptions is strictly increasing on $(-P/2, 0)$. Since φ is strictly increasing on $(-P/2, 0)$ and achieves $c/2$ exactly at the origin, we have that $\varphi < c/2$ on $(-P/2, 0)$. In fact, due to the periodicity and evenness of φ, we have that $\varphi < c/2$ on $(-P, 0)$. The claim then follows from Theorem 3.5.

Next we show that $\varphi \in C^\alpha(\mathbb{R})$ for all $0 \leq \alpha < 1/2$. The equality (3.9) implies that φ and $L\varphi$ have the same Hölder regularity at any point where $\varphi(x) < c/2$, because for such points we have that the factor $(c - \varphi(x) - \varphi(y)) > 0$. Consider now a point x_0 where $\varphi(x_0) = c/2$. For such a point, (3.9) reduces to

$$(\varphi(x_0) - \varphi(x))^2 = L\varphi(x_0) - L\varphi(x).$$

So, if $L\varphi$ is 2α-Hölder continuous at x_0, then φ is α-Hölder continuous at the same point by Lemma 3.8. Assume now that $\varphi \in C^\alpha$. We recall that L sends elements of the Zygmund space with regularity s (which corresponds to the Hölder-space $C^{\lfloor s \rfloor, s - \lfloor s \rfloor}$ at positive noninteger regularities) to an element in the Zygmund space with regularity $s + 1/2$. So applying L to $\varphi \in C^\alpha$ gives $L\varphi \in C^{\alpha+1/2}(\mathbb{R})$, which implies that φ in-fact has Hölder regularity $\frac{1}{2}(\alpha + \frac{1}{2})$ at x_0. Recall that L in particular maps $B^0_{\infty,\infty} = \mathcal{C}^0(\mathbb{R})$ continuously into $\mathcal{C}^{1/2}(\mathbb{R}) = C^{1/2}(\mathbb{R})$. Then, since we have that $\varphi \in L^\infty$ (which is a subset of $B^0_{\infty,\infty}$), we get $L\varphi \in B^{1/2}_{\infty,\infty} = \mathcal{C}^{1/2} = C^{1/2}$, which gives us that φ is in-fact in $C^{1/4}$. Since we now know that $\varphi \in C^{1/4}$, applying L again gives us that $L\varphi \in C^{3/4}$, which means that we in fact have $\varphi \in C^{3/8}$. Continuing this process, we can increase the Hölder-exponent more and more to get that $\varphi \in C^\alpha$ for all $\alpha < 1/2$ at x_0, as $\frac{1}{2}(\alpha + \frac{1}{2}) > \alpha$ for all $\alpha < 1/2$. We extend this to a global argument as follows. From the fact that $\varphi \leq c/2$, it is clear that $\varphi(x) - \varphi(y) \leq c - \varphi(x) - \varphi(y)$. Together with (3.9), this shows that

$$(\varphi(x) - \varphi(y))^2 \leq |L\varphi(x) - L\varphi(y)|,$$

for all $x, y \in \mathbb{R}$. Thus $\varphi \in C^\alpha(\mathbb{R})$ for all $\alpha < 1/2$.

To prove (ii), we first recall from (2.7) that u solves

$$(u(x))^2 = \frac{1}{2} \int_{\mathbb{R}} (K(x+y) + K(x-y) - 2K(y))u(y)\mathrm{d}y. \tag{3.10}$$

We now make the following claim: There is a constant c_2, independent of α, such that

$$\frac{1}{2} \int_{\mathbb{R}} |K(x+y) + K(x-y) - 2K(y)|(w(y))^\alpha \mathrm{d}y \leq c_2 (w(x))^{2\alpha}, \quad 0 \leq \alpha \leq \frac{1}{2}, \tag{3.11}$$

where

$$w(x) = \min\{|x|, 1\}.$$

For $|x| \geq 1$ the claim is certainly true, as in this case the right-hand side reduces to c_2, and since K is integrable and $||w(x)||_\infty \leq 1$. For $|x| \leq 1$ we split K into its singular and a regular part as in (2.8).

For the regular part, note that

$$\begin{aligned}
&\int_{\mathbb{R}} |K_{\text{reg}}(x+y) + K_{\text{reg}}(x-y) - 2K_{\text{reg}}(y)|(w(y))^\alpha \, \mathrm{d}y \\
&\leq \int_{\mathbb{R}} |K_{\text{reg}}(x+y) + K_{\text{reg}}(y-x) - 2K_{\text{reg}}(y)| \, \mathrm{d}y \\
&\lesssim \int_{\mathbb{R}} \frac{|x|^2}{(1+|y|^{5/2}}) \, \mathrm{d}y \\
&\lesssim |x|^2,
\end{aligned}$$

for $|x| \leq 1$. In the third line we have used a Taylor expansion around y, followed by the estimate

$$|K''_{\text{reg}(y)}| = |K''(y) - \frac{3}{4\sqrt{2\pi}|y|^{5/2}}| \lesssim \frac{1}{(1+|y|)^{5/2}}$$

(which follows from K_{reg} being smooth and the exponential decay of K and its derivatives). For the singular part we use the identity

$$\begin{aligned}
&\int_{\mathbb{R}} \Big| \frac{1}{\sqrt{|x+y|}} + \frac{1}{\sqrt{|y-x|}} - \frac{2}{\sqrt{|y|}} \Big| |y|^\alpha \mathrm{d}y \\
&\qquad = |x|^{\frac{1}{2}+\alpha} \int_{\mathbb{R}} \Big| \frac{1}{\sqrt{|1+\tau|}} + \frac{1}{\sqrt{|\tau-1|}} - \frac{2}{\sqrt{|\tau|}} \Big| |\tau|^\alpha \mathrm{d}\tau,
\end{aligned} \tag{3.12}$$

where we have made the change of variables $y = x\tau$. The integral on the right-hand side converges, since we have

$$\Big|\frac{1}{\sqrt{|1+\tau|}} + \frac{1}{\sqrt{|\tau-1|}} - \frac{2}{\sqrt{|\tau|}}\Big| \lesssim \tau^{-\frac{5}{2}}, \qquad |\tau| \gg 1.$$

Using this identity we see that the singular part of the integral converges,

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}}\Big|&\frac{1}{\sqrt{2\pi|x+y|}} + \frac{1}{\sqrt{2\pi|y-x|}} - \frac{2}{\sqrt{2\pi|y|}}\Big|(w(y))^{\alpha}\mathrm{d}y \\
&\leq \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}\Big|\frac{1}{\sqrt{|x+y|}} + \frac{1}{\sqrt{|y-x|}} - \frac{2}{\sqrt{|y|}}\Big||y|^{\alpha}\mathrm{d}y \qquad (3.13)\\
&= \frac{1}{2\sqrt{2\pi}}|x|^{\frac{1}{2}+\alpha}\int_{\mathbb{R}}\Big|\frac{1}{\sqrt{|1+\tau|}} + \frac{1}{\sqrt{|\tau-1|}} - \frac{2}{\sqrt{|\tau|}}\Big||\tau|^{\alpha}\mathrm{d}\tau < \infty.
\end{aligned}$$

Since $|x|^{\frac{1}{2}+\alpha} \leq |x|^{2\alpha}$ for $|x| \leq 1$ and $0 \leq \alpha \leq 1/2$, (3.11) follows. We can now combine (3.11) and (3.10) to finish the proof. Multiplying (3.10) by $1/(w(x))^{2\alpha}$ we get

$$\begin{aligned}
\Big(\frac{|u(x)|}{(w(x))^{\alpha}}\Big)^2 &= \frac{1}{2(w(x))^{2\alpha}}\int_{\mathbb{R}}(K(x+y)+K(x-y)-2K(y))u(y)\mathrm{d}y \\
&= \frac{1}{2(w(x))^{2\alpha}}\int_{\mathbb{R}}(K(x+y)+K(x-y)-2K(y))u(y)\frac{(w(y))^{\alpha}}{(w(y))^{\alpha}}\mathrm{d}y \qquad (3.14)\\
&\leq \frac{1}{2(w(x))^{2\alpha}}\sup_{y\in\mathbb{R}}\Big|\frac{u(y)}{(w(y))^{\alpha}}\Big|\int_{\mathbb{R}}|K(x+y)+K(x-y)-2K(y)|(w(y))^{\alpha}\mathrm{d}y.
\end{aligned}$$

We now use the claim on the final integral,

$$\Big(\frac{|u(x)|}{(w(x))^{\alpha}}\Big)^2 \leq \frac{1}{(w(x))^{2\alpha}}\sup_{y\in\mathbb{R}}\Big|\frac{u(y)}{(w(y))^{\alpha}}\Big|c_2(w(x))^{2\alpha} = c_2\sup_{y\in\mathbb{R}}\Big|\frac{u(y)}{(w(y))^{\alpha}}\Big|.$$

Since the right hand side now is independent of x, we in fact have that

$$\Big(\sup_{x\in\mathbb{R}}\frac{|u(x)|}{(w(x))^{\alpha}}\Big)^2 \leq c_2\sup_{y\in\mathbb{R}}\Big|\frac{u(y)}{(w(y))^{\alpha}}\Big|.$$

From earlier, where we showed that $\varphi \in C^{\alpha}(\mathbb{R})$ for all $\alpha < 1/2$, we know that the right-hand side is bounded for all $\alpha < 1/2$. Thus, for $\alpha < 1/2$, we have that

$$\sup_{x\in\mathbb{R}}\frac{|u(x)|}{(w(x))^{\alpha}} \leq c_2.$$

In particular, for $|x| \leq 1$ we get

$$|u(x)| \leq c_2|x|^{\alpha},$$

whereupon we can let $\alpha \to 1/2$ to achieve

$$|u(x)| \leq c_2|x|^{\frac{1}{2}}$$

for all $|x| \leq 1$. Combining this upper bound with the lower bound from Lemma 3.6, we arrive at (ii). □

As stated in the introduction, the authors were not able to determine the optimal constants c_1 and c_2 from Theorem 3.9 in [5], but they conjectured that $c_1 = c_2 = \sqrt{\pi/8}$. This leads us to the next section.

4 The exact leading-order asymptotics at the origin

In this section we summarize the important results we will need from [11], where our goal is to establish the u- and u'-limit mentioned in the introduction. We repeat that our summary of [11] in this section is not written in as general a setting as is done in the paper, but adapted for the specific case of the highest cusped wave solution φ of the Whitham equation (2.1) whose existence was proved in [5]. We will not work with φ directly, but with u, which we recall is defined as $u(x) := \varphi(0) - \varphi(x)$. For simplicity's sake, throughout this and subsequent sections we only consider the limit of u and its derivatives as x *decreases* towards zero (as is done in [11]).

Following the steps of [11], let us start by taking a closer look at Theorem 3.9 to see exactly what stops us from determining the optimal constant c_2 in (ii). To this end, we start with a heuristic argument to show what we *expect* the limit of $u(x)/|x|^{1/2}$ as x tends to zero to be. We return to (2.7), but only consider the singular part of the kernel, $K_{\text{sing}}(x) = 1/\sqrt{|2\pi x|}$,

$$u(x)^2 = \int_0^\infty \delta_x^2 K_{\text{sing}}(y) u(y) \mathrm{d}y. \tag{4.1}$$

We have the identity

$$\begin{aligned} \delta_x^2 K_{\text{sing}}(\tau x) &= \frac{1}{\sqrt{|2\pi(\tau x + x)|}} + \frac{1}{\sqrt{|2\pi(\tau x - x)|}} - \frac{2}{\sqrt{|2\pi\tau x|}} \\ &= \frac{1}{\sqrt{|2\pi x|}} \Big(\frac{1}{\sqrt{|1+\tau|}} + \frac{1}{\sqrt{|\tau - 1|}} - \frac{2}{\sqrt{|\tau|}} \Big) \\ &= K_{\text{sing}}(x) \delta_1^2 \Big(\frac{1}{\sqrt{|\tau|}} \Big), \end{aligned} \tag{4.2}$$

which we will use in the proof of the following theorem.

Theorem 4.1. *[11] Equation* (4.1) *admits the solution*

$$u(x) = \sqrt{\frac{\pi}{8}} |x|^{\frac{1}{2}}. \tag{4.3}$$

Proof. We follow the proof from [11], but for the specific case of the highest, cusped solution φ of the Whitham equation from [5]. We also show the calculations in greater detail. Making the change of variables $y = \tau x$ in (4.1) and using the identity (4.2), we arrive at

$$\begin{aligned} u(x)^2 &= \int_0^\infty \delta_x^2 K_{\text{sing}}(y) u(y) \mathrm{d}y = \int_0^\infty \delta_x^2 K_{\text{sing}}(\tau x) u(\tau x) |x| \mathrm{d}\tau \\ &= |x| \int_0^\infty K_{\text{sing}}(x) \delta_1^2 \Big(\frac{1}{\sqrt{|\tau|}} \Big) u(\tau x) \mathrm{d}\tau = |x| K_{\text{sing}}(x) \int_0^\infty \delta_1^2 \Big(\frac{1}{\sqrt{|\tau|}} \Big) u(\tau x) \mathrm{d}\tau \\ &= \frac{|x|^{\frac{1}{2}}}{\sqrt{2\pi}} \int_0^\infty \delta_1^2 \Big(\frac{1}{\sqrt{|\tau|}} \Big) u(\tau x) \mathrm{d}\tau. \end{aligned}$$

Inserting the proposed solution (4.3) for u, we get that

$$\frac{\pi}{8} |x| = \frac{|x|}{4} \int_0^\infty \delta_1^2 \big(|\tau|^{-\frac{1}{2}} \big) |\tau|^{\frac{1}{2}} \mathrm{d}\tau. \tag{4.4}$$

To verify that proposed solution is indeed correct, we need to find that the integral on the right-hand side equals $\pi/2$. We will do this by rewriting the integral in terms of the beta function. To this end, we first need to replace the $-1/2$ and $1/2$ exponents with $s-1$ and s, where $s \in (0, \frac{1}{2})$, respectively, as some of the integrals in the following calculation start diverging exactly at $s = 1/2$ (replacing the square root with s in this manner follows from doing the above calculations with the more general singular kernel $|2\pi x|^{s-1}$, instead of $K_{\text{sing}}(x) = |2\pi x|^{-1/2}$). Thankfully, we can then use analytic continuation to extend the result to the $s = 1/2$-case, which is the one we are interested in.

$$\int_0^\infty \delta_1^2 (|\tau|^{s-1}) |\tau|^s \mathrm{d}\tau = \int_0^1 \delta_1^2 (|\tau|^{s-1}) |\tau|^s \mathrm{d}\tau + \int_1^\infty \delta_1^2 (|\tau|^{s-1}) |\tau|^s \mathrm{d}\tau$$

$$= \int_0^1 |1-\tau|^{s-1}|\tau|^s \mathrm{d}\tau - \int_0^1 |\tau|^{s-1}|\tau|^s \mathrm{d}\tau$$
$$+ \int_0^\infty \Big(|1+\tau|^{s-1} - |\tau|^{s-1}\Big)|\tau|^s \mathrm{d}\tau + \int_1^\infty \Big(|\tau-1|^{s-1} - |\tau|^{s-1}\Big)|\tau|^s \mathrm{d}\tau = (\star)$$

Notice that we can drop the absolute value signs in all of the four final integrals, as the inside of all the absolute values are always positive on their respective domains of integration. In the very last integral we can do the change of variables $\tau \mapsto (\tau+1)$ and integrate by parts,

$$\int_1^\infty \Big((\tau-1)^{s-1} - \tau^{s-1}\Big)\tau^s \mathrm{d}\tau = \int_0^\infty \Big(\tau^{s-1} - (\tau+1)^{s-1}\Big)(\tau+1)^s \mathrm{d}\tau$$
$$= \Big[\Big(\frac{1}{s}\tau^s - \frac{1}{s}(\tau+1)^s\Big)(\tau+1)^s\Big]_0^\infty - \int_0^\infty \Big(\frac{1}{s}\tau^s - \frac{1}{s}(\tau+1)^s\Big)s(\tau+1)^{s-1}\mathrm{d}\tau$$
$$= \frac{1}{s} - \int_0^\infty \Big(\tau^s - (\tau+1)^s\Big)(\tau+1)^{s-1}\mathrm{d}\tau,$$

where we have used the binomial theorem to expand the terms inside the boundary term-brackets. Continuing the above calculation and inserting this new expression for the final integral, we see that all except the first integral cancel each other,

$$(\star) = \int_0^1 (1-\tau)^{s-1}\tau^s \mathrm{d}\tau - \int_0^1 \tau^{2s-1}\mathrm{d}\tau$$
$$+ \int_0^\infty \Big((1+\tau)^{s-1} - \tau^{s-1}\Big)\tau^s \mathrm{d}\tau + \frac{1}{s} - \int_0^\infty \Big(\tau^s - (\tau+1)^s\Big)(\tau+1)^{s-1}\mathrm{d}\tau$$
$$= \int_0^1 (1-\tau)^{s-1}\tau^s \mathrm{d}\tau.$$

We recognize this final integral as the beta function, so by analytic continuation the equality is also true for $s = 1/2$. Using the properties of the beta function, which we denote by B, we therefore have for $s = 1/2$ that

$$\int_0^\infty \delta_1^2(|\tau|^{-\frac{1}{2}})|\tau|^{\frac{1}{2}}\mathrm{d}\tau = \mathrm{B}\Big(\frac{1}{2}+1, \frac{1}{2}\Big) = \frac{\pi}{2}. \tag{4.5}$$

Thus, the integral indeed evaluates to $\pi/2$, so the equality in (4.4) is true. Consequently, the proposed solution (4.3) for the equation (4.1) is indeed a solution. □

As the full kernel K behaves like K_{sing} near the origin (recall (2.8)), we expect a solution u of the full equation (2.7) to still admit the limit

$$\lim_{x\to 0} \frac{u(x)}{|x|^{\frac{1}{2}}} = \sqrt{\frac{\pi}{8}} \tag{4.6}$$

[11]. So what prevented the ascertainment of the optimal constant c_2 in Theorem 3.9? Had $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ inside the integrand on the right hand side of (3.12) been non-negative, (4.5) would immediately give us that

$$\frac{1}{2}\int_{\mathbb{R}} \Big|\frac{1}{\sqrt{2\pi|x+y|}} + \frac{1}{\sqrt{2\pi|y-x|}} - \frac{2}{\sqrt{2\pi|y|}}\Big|(w(y))^\alpha \mathrm{d}y \leq \frac{\pi}{8}|x|$$

as $\alpha \to 1/2$ in the calculation (3.13) [11]. Continuing the proof from here would indeed result in the bound $|u(x)| \leq \sqrt{\frac{\pi}{8}}|x|^{\frac{1}{2}}$, which we expect to be the optimal one based on the heuristic argument (and as we will soon see, is indeed the optimal bound). Unfortunately $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ *does* change sign from negative to positive at a point $\tau_0 \in (0,1)$, so more work is required. This indicates that the root $\tau_0 \in (0,1)$ is important. Also, had we known that the u-limit existed, we could have chosen $\alpha = 1/2$ in the first line of (3.14), let $x \to 0$, and used dominated convergence to find (4.6) [11]. Unfortunately, however, proving that the limit exists is exactly what is difficult.

Before we are ready to show the existence and value of the u-limit, we need to establish some more properties of the kernel K.

Lemma 4.2. *[11] The second difference $\delta_x^2 K$ is non-negative on (x, ∞), and satisfies*

$$0 \leq \int_{\nu}^{\infty} \delta_x^2 K(y)\, dy \leq -K'(\nu - x)x^2, \tag{4.7}$$

for any $0 \leq x < \nu$.

Proof. We follow the proof from [11], but show the calculations in greater detail. For the non-negativity claim, note by the convexity of K that

$$K(y) = K\Big(\frac{1}{2}(y+x) + \frac{1}{2}(y-x)\Big) \leq \frac{1}{2}K(y+x) + \frac{1}{2}K(y-x),$$

which, after multiplication by two and rearranging, becomes

$$0 \leq K(y+x) + K(y-x) - 2K(y) = \delta_x^2 K(y).$$

For the upper bound, recall that K is smooth away from the origin. We can therefore write

$$\begin{aligned}\int_0^x \int_0^x K''(y + t_1 - t_2)\mathrm{d}t_1 \mathrm{d}t_2 &= \int_0^x K'(y + x - t_2) - K'(y - t_2)\mathrm{d}t_2 \\ &= K(y+x) + K(y-x) - 2K(y) \\ &= \delta_x^2 K(y).\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_{\nu}^{\infty} \delta_x^2 K(y)\mathrm{d}y &= \int_{\nu}^{\infty} \int_0^x \int_0^x K''(y + t_1 - t_2)\mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}y \\ &= \int_0^x \int_0^x -K'(\nu + t_1 - t_2)\mathrm{d}t_1 \mathrm{d}t_2 \\ &\leq -K'(\nu - x)x^2,\end{aligned}$$

where we have used that $-K'$ is nonincreasing on $\mathbb{R}^+$. □

Lemma 4.3. *[11] The second difference $\delta_x^2 K_{reg}$ is integrable and satisfies*

$$||\delta_x^2 K_{reg}||_{L^1} \leq x^2 ||K''_{reg}||_{L^1}$$

for all $x \in \mathbb{R}_0^+$. Moreover, K'_{reg} admits the bound $||K'_{reg}||_{L^\infty} \leq ||K''_{reg}||_{L^1}$.

Proof. We follow the proof from [11]. Since K''_{reg} is integrable, we have that

$$\begin{aligned}||\delta_x^2 K_{\mathrm{reg}}||_{L^1} &= \int_{\mathbb{R}} \Big| \int_0^x \int_0^x K''_{\mathrm{reg}}(y + t_1 - t_2)\ \mathrm{d}t_1\ \mathrm{d}t_2 \Big|\ \mathrm{d}y \\ &\leq \int_0^x \int_0^x \int_{\mathbb{R}} \big|K''_{\mathrm{reg}}(y + t_1 - t_2)\big|\ \mathrm{d}y\ \mathrm{d}t_1\ \mathrm{d}t_2 = x^2 ||K''_{\mathrm{reg}}||_{L^1}\end{aligned}$$

for all $x \geq 0$. For the second part, note that K'_{reg} is odd as $K_{\mathrm{reg}} = K - K_{\mathrm{sing}}$ is necessarily even. Since K'_{reg} is absolutely continuous, we have that

$$||K'_{\mathrm{reg}}||_{L^\infty} = \sup_{x \in \mathbb{R}} |K'_{\mathrm{reg}}(x)| = \sup_{x \in \mathbb{R}} \Big| \int_0^x K''_{\mathrm{reg}}(y)\ \mathrm{d}y \Big| \leq \int_{\mathbb{R}} \big|K''_{\mathrm{reg}}(y)\big|\ \mathrm{d}y = ||K''_{\mathrm{reg}}||_{L^1}.$$

□

We also need some properties related to $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ (which appeared in (4.2)).

Lemma 4.4. *[11] The function $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is increasing on the interval $(0,1)$, where it has a unique root $\tau \in (\frac{1}{2}, \frac{2}{3})$. In addition, $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is positive on $(1,\infty)$,*

$$\int_0^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau = 0, \qquad \int_0^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau = \frac{\pi}{2}, \tag{4.8}$$

and

$$\frac{1}{2} < -\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau < \frac{3}{5}. \tag{4.9}$$

Proof. We follow the proof from [11], but show some of the calculations in greater detail. For the first integral in (4.8) we have that

$$\begin{aligned}
&\int_0^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau \\
&\quad= \int_0^1 (\tau+1)^{-\frac{1}{2}} + (1-\tau)^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}}\ \mathrm{d}\tau + \int_1^\infty (\tau+1)^{-\frac{1}{2}} + (\tau-1)^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}}\ \mathrm{d}\tau \\
&\quad= \left[2(\tau+1)^{\frac{1}{2}} - 2(1-\tau)^{\frac{1}{2}} - 4\tau^{\frac{1}{2}}\right]_0^1 + \left[2(\tau+1)^{\frac{1}{2}} + 2(\tau-1)^{\frac{1}{2}} - 4\tau^{\frac{1}{2}}\right]_1^\infty \\
&\quad= 2\sqrt{2} - 4 + \Big(\lim_{T\to\infty} \big(2(T+1)^{\frac{1}{2}} + 2(T-1)^{\frac{1}{2}} - 4T^{\frac{1}{2}}\big) - \big(2\sqrt{2}-4\big)\Big) \\
&\quad= \lim_{T\to\infty} \big(2(T^{\frac{1}{2}} + O(T^{-\frac{1}{2}})) + 2(T^{\frac{1}{2}} + O(T^{-\frac{1}{2}})) - 4T^{\frac{1}{2}}\big) \\
&\quad= 0,
\end{aligned}$$

where we have used the binomial theorem. The second integral in (4.8) is the same as (4.5), which was proved in Theorem 4.1. To show that $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is increasing on $(0,1)$, we differentiate on $(0,1)$ to see that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\Big((\tau+1)^{-\frac{1}{2}} + (1-\tau)^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}}\Big) = -\frac{1}{2}(\tau+1)^{-\frac{3}{2}} + \frac{1}{2}(1-\tau)^{-\frac{3}{2}} + \tau^{-\frac{3}{2}} > \frac{1}{2}\tau^{-\frac{3}{2}} > 0.
\end{aligned}$$

Since we now know that $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is increasing on $(0,1)$, it follows from the explicit evaluations

$$\delta_1^2\left(\left|\frac{1}{2}\right|^{-\frac{1}{2}}\right) < 0 < \delta_1^2\left(\left|\frac{2}{3}\right|^{-\frac{1}{2}}\right)$$

that there is a unique root $\tau_0 \in (\frac{1}{2}, \frac{2}{3})$. For the positivity on $(1,\infty)$ we use the strict convexity of $|\tau|^{-\frac{1}{2}}$ on $\mathbb{R}^+$,

$$|\tau|^{-\frac{1}{2}} = \big|\frac{1}{2}(\tau+1) + \frac{1}{2}(\tau-1)\big|^{-\frac{1}{2}} < \frac{1}{2}|\tau+1|^{-\frac{1}{2}} + \frac{1}{2}|\tau-1|^{-\frac{1}{2}},$$

which, after multiplication by two and rearranging, becomes

$$0 < |\tau+1|^{-\frac{1}{2}} + |\tau-1|^{-\frac{1}{2}} - 2|\tau|^{-\frac{1}{2}} = \delta_1^2(|\tau|^{-\frac{1}{2}})$$

for $\tau > 1$.

To establish (4.9) we note that

$$-\int_0^t \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau = 2t - \frac{2t^{\frac{3}{2}}}{(1+t)^{\frac{1}{2}} + (1-t)^{\frac{1}{2}}} + \operatorname{arsinh}(t^{\frac{1}{2}}) - \arcsin(t^{\frac{1}{2}}), \tag{4.10}$$

for all $t \in (0,1)$. Since $\tau^{\frac{1}{2}}$ is always positive on the domain of integration, the integral is maximized at τ_0 where $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ changes sign. A lower bound can therefore be established by evaluation at any other point, for instance at $2/3$, which gives us that

$$-\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau > -\int_0^{2/3} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau > \frac{1}{2}.$$

For the upper bound, notice by (4.10) and straightforward algebra that

$$-\int_0^t \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau + t^{\frac{3}{2}}\delta_1^2\big(|t|^{-\frac{1}{2}}\big) = \frac{1}{(t^{-1}-1)^{\frac{1}{2}}} - \frac{1}{(1+t^{-1})^{\frac{1}{2}}} + \operatorname{arsinh}(t^{\frac{1}{2}}) - \arcsin(t^{\frac{1}{2}}),$$

which is increasing on $(0,1)$. We can use this to see that

$$\begin{aligned}
&-\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau \\
&= -\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau + \tau_0^{\frac{3}{2}}\delta_1^2\big(|\tau_0|^{-\frac{1}{2}}\big) < -\int_0^{2/3} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ \mathrm{d}\tau + \Big(\frac{2}{3}\Big)^{\frac{3}{2}}\delta_1^2\Big(\Big|\frac{2}{3}\Big|^{-\frac{1}{2}}\Big) < \frac{3}{5},
\end{aligned}$$

where the first equality follows from τ_0 being a root of $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$, after which we evaluate at $2/3 > \tau_0$ to establish the inequality. ☐

For the following, define

$$g(x) := \frac{u(x)}{x^{\frac{1}{2}}}, \tag{4.11}$$

for $x > 0$. The following lemma is an asymptotic rephrasing of (2.7) for g. In this lemma we split the integrals that appear into three parts, whence we can exploit properties of the integrand on each of the domains. We will split the integral at the point x, and at the half-period point $P/2$ which allows us to exploit the fact that $u(x)$ is increasing on $[0, P/2]$ (the exact point at which we split the integral is not important, as long as $u(x)$ is increasing between the origin and this point). We are therefore working with $x \in (0, P/2]$ in the following lemmas. We also recall that $u(x)$ is non-negative, and smooth on $(0, P)$.

Lemma 4.5. *[11] With g as in* (4.11)*, there is a function $\lambda : (0,1) \to (0,1)$ so that*

$$g(x)^2 = \left(\int_{\lambda(x)}^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big) + o(1)\right) g(x) + \int_x^{P/2} \left[\frac{\delta_x^2 K(y) y^{1/2}}{x}\right] g(y)\ \mathrm{d}y + \int_{P/2}^{\infty} \frac{\delta_x^2 K(y)}{x} u(y)\ \mathrm{d}y \tag{4.12}$$

as $x \to 0$. Moreover, the square bracket is positive and satisfies

$$\lim_{x\to 0} \int_x^{P/2} \frac{\delta_x^2 K(y) y^{1/2}}{x}\ \mathrm{d}y = \int_1^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau, \tag{4.13}$$

while the final term admits the bound

$$0 \le \int_{P/2}^{\infty} \frac{\delta_x^2 K(y)}{x} u(y)\ \mathrm{d}y \le o(1) \tag{4.14}$$

as $x \to 0$.

Remark 4.6. The lemma also holds true when $P = \infty$. To see this, one can instead split the integral at some $\nu > 0$ small enough so that $u(x)$ is increasing on $[0, \nu]$ (and let $x \in (0, \nu]$). The rest of the proof remains unchanged.

Proof. We follow the proof from [11]. We divide each side of (2.7) by x and split the integral,

$$g(x)^2 = \frac{1}{x}\left[\int_0^x + \int_x^{P/2} + \int_{P/2}^{\infty}\right] \delta_x^2 K(y) u(y)\ \mathrm{d}y,$$

observing that both the left-hand side and the third integral matches equation (4.12). For the second integral, simply use that $u(y) = y^{1/2} g(y)$ to arrive at the form in equation (4.12).

For the first integral, start by recalling that $\delta_x^2 K_{\mathrm{sing}}(y)$ changes sign at $\tau_0 x$ (due to the identity (4.2) and $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ changing sign at τ_0). As u is increasing and non-negative on $[0, P/2]$, we can use the second mean value theorem for integrals on the first integral to conclude that, for every $x \in (0, P/2]$, we have

$$\int_0^x \delta_x^2 K(y) u(y)\ \mathrm{d}y$$

$$= u(0)\int_0^{\lambda(x)x} \delta_x^2 K(y)\ \mathrm{d}y + u(x)\int_{\lambda(x)x}^{x} \delta_x^2 K(y)u(y)\ \mathrm{d}y = u(x)\int_{\lambda(x)x}^{x} \delta_x^2 K(y)u(y)\ \mathrm{d}y,$$

for some $\lambda(x) \in (0,1)$. From here we use the identity (4.2) and Lemma 4.3 to find that

$$\int_{\lambda(x)x}^{x} \delta_x^2 K(y)u(y)\ \mathrm{d}y = x^{1/2}\int_{\lambda(x)}^{1} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + O(x^2),$$

as $x \to 0$, which matches the first integral in (4.12).

The positivity of the square brackets in (4.12) for $y \in (x, P/2]$ follows directly from the second difference $\delta_x^2 K(y)$ being non-negative, which was shown in Lemma 4.2. The limit (4.13) follows directly from the identity (3.12) together with Lemma 4.3. Finally, (4.14) follows directly from Lemma 4.2. □

We define

$$\underline{g}(x) := \min_{y\in[x,P/2]} g(y) \qquad \text{and} \qquad \overline{g}(x) := \max_{y\in[x,P/2]} g(y), \tag{4.15}$$

where g is as in (4.11) (again, replace $P/2$ with $\nu > 0$ if $P = \infty$). Note that $\underline{g}$ and $\overline{g}$ are certainly non-decreasing and non-increasing respectively, as the minimum and maximum values of g on the domain $[x, P/2]$ will never be decreased or increased respectively by increasing x (as increasing x makes the domain smaller).

Lemma 4.7. *[11] With g as in (4.11), we have*

$$m := \liminf_{x\to 0} g(x) > 0 \qquad \textit{and} \qquad M := \limsup_{x\to 0} g(x) < \infty,$$

for which the inequalities

$$M^2 \le \frac{m}{\sqrt{2\pi}}\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ d\tau + \frac{M}{\sqrt{2\pi}}\int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ d\tau, \tag{4.16}$$

$$m^2 \ge \frac{1}{\sqrt{2\pi}}\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\min(m, M\tau^{\frac{1}{2}})\ d\tau + \frac{m}{\sqrt{2\pi}}\int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}}\ d\tau, \tag{4.17}$$

hold.

Proof. We follow the proof from [11], but include a short explanation for why the assumption $m = 0$ allows us to pick a realizing sequence in the following. We also show some of the calculations in greater detail. In the following, $\underline{g}$ and $\overline{g}$ is as in (4.15). Our strategy will be to establish that $m > 0$ and $M < \infty$, before moving on to the stricter bounds. First we prove that $m > 0$. From (4.12) we can deduce that

$$g(x)^2 \ge \left(\int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + o(1)\right)g(x) + \underline{g}(x)\int_x^{P/2}\left[\frac{\delta_x^2 K(y)y^{1/2}}{x}\right]\ \mathrm{d}y, \tag{4.18}$$

by using that $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is monotonically increasing on $(0,1)$ and switches sign from negative to positive at some $\tau_0 \in (0,1)$ from Lemma 4.4. That is, by replacing $\lambda(x)$ with zero in the lower bound of the first integral in (4.12) we are certainly decreasing the value of the integral. We are also using the bounds from (4.14) to drop the third integral in (4.12), and the second term on the right-hand side of (4.18) is certainly smaller than the second integral in (4.12) as we are replacing $g(y)$ with its smallest value on the domain of integration.

Assume, for the sake of contradiction, that $m = 0$. We can then pick a realizing sequence $\{x_k\}_{k\in\mathbb{N}} \subset (0, P/2]$ for m (that is, x_k tends to zero) in such a way that $g = \underline{g}$ along that sequence. In other words, we can choose a sequence such that $g(x_k) = \underline{g}(x_k)$ for each $x_k \in \{x_k\}_{k\in\mathbb{N}}$. Note that the assumption that $m = 0$ is necessary to pick such a realizing sequence; Since $g(x)$ is non-negative, assuming $m = 0$ guarantees that for any point $\tilde{x}$ such that $\underline{g}(\tilde{x}) = g(\tilde{x})$, no matter how close $\tilde{x}$ is to the origin, we can always find a new point $x' \in (0, \tilde{x})$ such that $\underline{g}(x') = g(x')$. Then we can include x' in our realzing sequence. If instead $m > 0$ such a point in $(0, \tilde{x})$ would not necessarily exist. With the realizing sequence at hand, for each $x_k \in \{x_k\}_{k\in\mathbb{N}}$ we then have in (4.18) that

$$\begin{aligned} g(x_k)^2 &\geq \left(\int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + o(1)\right) g(x_k) + \underline{g}(x_k) \int_{x_k}^{P/2} \left[\frac{\delta_{x_k}^2 K(y) y^{1/2}}{x_k}\right] \mathrm{d}y \\ &= \left(\int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + o(1)\right) g(x_k) + g(x_k) \int_{x_k}^{P/2} \left[\frac{\delta_{x_k}^2 K(y) y^{1/2}}{x_k}\right] \mathrm{d}y, \end{aligned}$$

which, after dividing by $g(x_k)$, becomes

$$g(x_k) \geq \int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + o(1) + \int_{x_k}^{P/2} \left[\frac{\delta_{x_k}^2 K(y) y^{1/2}}{x_k}\right] \mathrm{d}y,$$

as $k \to \infty$. Going to the limit and using (4.13) for the second integral on the right-hand side, we find that

$$m \geq \int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + \int_1^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau = \int_1^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)(\tau^{1/2} - 1)\ \mathrm{d}\tau > 0.$$

The equality follows from the first integral in (4.8), and the final inequality follows from the positivity of the integrand on $(1, \infty)$ which was proved in Lemma 4.4. Here we have arrived at our contradiction, proving that $m > 0$.

Analogous to the above lower bound, we find that

$$g(x)^2 \leq \left(\int_{\tau_0}^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + o(1)\right) g(x) + \overline{g}(x) \int_x^{P/2} \left[\frac{\delta_x^2 K(y) y^{1/2}}{x}\right] \mathrm{d}y,$$

as $x \to 0$. Again for the sake of contradiction, we assume $M = \infty$ and can choose a realizing sequence along which $g = \overline{g}$, resulting in

$$M \leq \int_{\tau_0}^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\ \mathrm{d}\tau + \int_1^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau < \infty.$$

Here we have arrived at our contradiction, proving that $M < \infty$.

Now that we know that $0 < m \leq M < \infty$, we can derive the sharper inequalities (4.16) and (4.17). Knowing that g is bounded allows us to replace (4.12) with the simpler

$$g(x)^2 = \frac{1}{\sqrt{2\pi}} \int_0^{P/2x} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + o(1) \tag{4.19}$$

as $x \to 0$. We arrive at (4.19) as follows. For the third integral in (4.12) we split the integral into the singular and regular part,

$$\begin{aligned} \int_{P/2}^\infty \frac{\delta_x^2 K(y)}{x} u(y)\ \mathrm{d}y &= \int_{P/2}^\infty \frac{\delta_x^2 K_{\mathrm{sing}}(y)}{x} u(y)\ \mathrm{d}y + \int_{P/2}^\infty \frac{\delta_x^2 K_{\mathrm{reg}}(y)}{x} u(y)\ \mathrm{d}y \\ &\leq \int_{\frac{P}{2x}}^\infty \frac{\delta_x^2 K_{\mathrm{sing}}(\tau x)}{x} u(\tau x) x\ \mathrm{d}\tau + \frac{\sup_{y\in\mathbb{R}^+} u(y)}{x} \int_{P/2}^\infty \delta_x^2 K_{\mathrm{reg}}(y)\ \mathrm{d}y \\ &\leq \int_{\frac{P}{2x}}^\infty \delta_x^2 K_{\mathrm{sing}}(\tau x) u(\tau x)\ \mathrm{d}\tau + \frac{\sup_{y\in\mathbb{R}^+} u(y)}{x} x^2 ||K''_{\mathrm{reg}}||_{L^1} \\ &= \int_{\frac{P}{2x}}^\infty K_{\mathrm{sing}}(x)\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big) g(\tau x)(\tau x)^{1/2}\ \mathrm{d}\tau + o(1) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{P}{2x}}^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + o(1) \\ &= o(1), \end{aligned}$$

as $x \to 0$. Since g is bounded the integrand in the final integral is integrable, and hence the integral tends to zero as $x \to 0$. We also used that u is bounded, and Lemma 4.3. Similarly for the second integral in (4.12), we find that

$$\begin{aligned}\int_x^{P/2} \left[\frac{\delta_x^2 K(y) y^{1/2}}{x}\right] g(y)\ \mathrm{d}y &= \int_1^{P/2x} \delta_x^2 K_{\text{sing}}(\tau x)\tau^{1/2} x^{1/2} g(\tau x)\ \mathrm{d}\tau + \frac{1}{x}\int_x^{P/2} \delta_x^2 K_{\text{reg}}(y) u(y)\ \mathrm{d}y \\ &\leq \frac{1}{\sqrt{2\pi}} \int_1^{P/2x} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + \frac{\sup_{y\in\mathbb{R}^+} u(y)}{x} x^2 \|\delta_x^2 K''_{\text{reg}}\|_{L^1} \\ &= \frac{1}{\sqrt{2\pi}} \int_1^{P/2x} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + o(1),\end{aligned}$$

as $x \to 0$. We recall from the proof of Lemma 4.5 that the first integral in (4.12) is equivalent to the first integral in the following calculation. This formulation is easier to use in this case as the integrand is then exactly the same as for the second integral, but now with different bounds.

$$\begin{aligned}\int_0^x \frac{\delta_x^2 K(y)}{x} u(y)\ \mathrm{d}y &= \int_0^x \frac{\delta_x^2 K_{\text{sing}}(y)}{x} u(y)\ \mathrm{d}y + \int_0^x \frac{\delta_x^2 K_{\text{reg}}(y)}{x} u(y)\ \mathrm{d}y \\ &\leq \frac{1}{\sqrt{2\pi}} \int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + \frac{\sup_{y\in\mathbb{R}^+} u(y)}{x} x^2 \|\delta_x^2 K''_{\text{reg}}\|_{L^1} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^1 \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} g(\tau x)\ \mathrm{d}\tau + o(1),\end{aligned}$$

as $x \to 0$. Thus, we can indeed can simplify (4.12) to (4.19) now that we know g is bounded. Now we use this simpler formulation to show the stricter inequalities.

Recalling from Lemma (4.4) that $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is negative on $(0, \tau_0)$ and positive on (τ_0, ∞), we see that

$$M^2 \leq \Big(\inf_{y\in(0,\nu]} g(y)\Big) \frac{1}{\sqrt{2\pi}} \int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau + \Big(\sup_{y\in(0,\nu]} g(y)\Big) \frac{1}{\sqrt{2\pi}} \int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau$$

when we go to the limit. Here ν is simply introduced as some small positive variable (such that $u(x)$ is increasing on $(0, \nu]$, and with $x \in (0, \nu]$), as we then have that $\liminf_{x\to 0} g(x) \geq \inf_{y\in(0,\nu]} g(y)$ (remember, the first integral is negative) and $\limsup_{x\to 0} g(x) \leq \sup_{y\in(0,\nu]} g(y)$, giving us the bound. Letting $\nu \to 0$, we arrive at (4.16).

For (4.17), start by noting that

$$\tau^{1/2} g(\tau x) = \frac{u(\tau x)}{x^{1/2}} \leq \frac{u(x)}{x^{1/2}} = g(x)$$

for every $\tau \in (0, 1)$ and $x \in (0, \nu]$. Then we have that

$$\tau^{1/2} g(\tau x) \leq \min\Big(g(x), \tau^{1/2}\Big(\sup_{y\in(0,\nu]} g(y)\Big)\Big)$$

for all such x and τ. Then, from (4.19), we obtain the lower bound

$$\begin{aligned}g(x)^2 \geq \frac{1}{\sqrt{2\pi}} \int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big) \min\Big(g(x), \tau^{1/2}\Big(\sup_{y\in(0,\nu]} g(y)\Big)\Big)\ \mathrm{d}\tau + \\ \Big(\inf_{y\in(0,\nu]} g(y)\Big) \frac{1}{\sqrt{2\pi}} \int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2}\ \mathrm{d}\tau + o(1),\end{aligned}$$

as $x \to 0$. We take the limit along a sequence realizing m, then we let $\nu \to 0$, after which we arrive at (4.17). □

Now we are ready to prove the u-limit.

Proposition 4.8. *[11] The solution u enjoys the limit*

$$\lim_{x\to 0} \frac{u(x)}{x^{1/2}} = \sqrt{\frac{\pi}{8}}$$

Proof. We follow the proof from [11]. Let M and m be as in Lemma 4.7. Introducing $\sigma := M/m \geq 1$, we rewrite (4.16) and (4.17) purely in terms of σ and m,

$$\sqrt{2\pi}m \leq \sigma^{-2} \int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau + \sigma^{-1} \int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau, \tag{4.20}$$

$$\sqrt{2\pi}m \geq \int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\min(1, \sigma\tau^{\frac{1}{2}}) \, \mathrm{d}\tau + \int_{\tau_0}^{\infty} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau, \tag{4.21}$$

where we have moved the factor $1/\sqrt{2\pi}$ to the other side of the inequalities. When $\sigma = 1$, both right-hand sides simplify to $\int_0^\infty \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{1/2} \, \mathrm{d}\tau$, which equals $\pi/2$ by Lemma 4.4, giving us that $\sqrt{\pi/8} = m = M/\sigma = M$. Our strategy will therefore be to prove that no $m > 0$ simultaneously satisfies both of the above inequalities when $\sigma > 1$. We begin by introducing

$$f(\sigma) := \int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\min(1, \sigma\tau^{\frac{1}{2}}) \, \mathrm{d}\tau + \sigma^{-2}b + \Big(\frac{\pi}{2} + b\Big)(1 - \sigma^{-1}) \tag{4.22}$$

for $\sigma \geq 1$, where

$$b := -\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau.$$

Note that $f(\sigma)$ is the right-hand side of (4.21) minus the right-hand side of (4.20). If we can demonstrate that f is positive on $(1, \infty)$, we will have shown that there is no $m > 0$ satisfying both (4.21) and (4.20) simultaneously. The reason for this is that a positive f on $(1, \infty)$ would mean that the right-hand side of (4.20) is *smaller* than the right-hand side of (4.21) which clearly makes it impossible to satisfy the inequalities simultaneously.

We use the trivial inequality $\min(1, \sigma\tau^{1/2}) \leq \sigma\tau^{1/2}$ together with the fact that $\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)$ is negative on $(0, \tau_0)$ to see that

$$f(\sigma) \geq -\sigma b + \sigma^{-2}b + \Big(\frac{\pi}{2} + b\Big)(1 - \sigma^{-1}) = (1 - \sigma^{-1})\Big(\frac{\pi}{2} - b(\sigma + \sigma^{-1})\Big)$$

for all $\sigma \geq 1$. The first factor, $(1 - \sigma^{-1})$, is positive for all $\sigma > 1$, while differentiation shows that the second factor is a decreasing function of σ. Recalling from (4.9) that $b < 3/5$, straightforward calculation shows that

$$f(2) = (1 - 2^{-1})\Big(\frac{\pi}{2} - b(2 + 2^{-1})\Big) \geq (1 - 2^{-1})\Big(\frac{\pi}{2} - \frac{3}{5}(2 + 2^{-1})\Big) = \frac{3\pi}{4} - \frac{9}{4} > 0.$$

Thus, we have that $f(\sigma) > 0$ on $(1, 2]$.

Suppose now that $\sigma > 2^{1/2}$. We then have, by Lemma 4.4, that $\sigma^{-2} < 1/2 < \tau_0$, from which it follows that

$$\int_0^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\min(1, \sigma\tau^{\frac{1}{2}}) \, \mathrm{d}\tau = \sigma \int_0^{\sigma^{-2}} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau + \int_{\sigma^{-2}}^{\tau_0} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big) \, \mathrm{d}\tau.$$

Inserting this into the definition of f in (4.22), we can differentiate to get that

$$f'(\sigma) = \int_0^{\sigma^{-2}} \delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} \, \mathrm{d}\tau - 2\sigma^{-3}b + \Big(\frac{\pi}{2} + b\Big)\sigma^{-2},$$

for all $\sigma > 2^{1/2}$. Considering the integrand in this expression, by the convexity of $|\tau|^{-1/2}$ on $\mathbb{R}^+$ we have that

$$\delta_1^2\big(|\tau|^{-\frac{1}{2}}\big)\tau^{\frac{1}{2}} = \Big(\frac{1}{(1+\tau)^{1/2}} + \frac{1}{(1-\tau)^{1/2}}\Big)\tau^{1/2} - 2 \geq 2\tau^{1/2} - 2,$$

where we can drop the absolute values due to the domain of integration. Inserting this inequality into the expression, we get that

$$\begin{aligned} f'(\sigma) &\geq 2\int_0^{\sigma^{-2}} (\tau^{1/2} - 1) \, \mathrm{d}\tau - 2\sigma^{-3}b + \Big(\frac{\pi}{2} + b\Big)\sigma^{-2} \\ &= \Big(\frac{\pi}{2} + b - 2\Big)\sigma^{-2} + \Big(\frac{4}{3} - 2b\Big)\sigma^{-3} \\ &> \frac{\pi - 3}{2}\sigma^{-2} + \frac{2}{15}\sigma^{-3}, \end{aligned}$$

for all $\sigma > 2^{1/2}$ by using the bounds for b given in (4.9). This shows that f is increasing on $(2^{1/2}, \infty)$. Since we had that f is positive on $(1, 2]$, this means that f is in fact positive on $(1, \infty)$, which concludes the proof. □

4.1 The limit for the derivative

We now move on to the limit for the derivative. This subsection is also a summary of relevant content from [11], but we have restructured it somewhat so the four main steps that were listed in the introduction are as distinct as possible. This will make it easier for us to correlate the corresponding steps in later sections with this subsection. Also, as many of the proofs in subsequent sections contain calculations that are completely analogous to the ones we will encounter here, we write out the calculations in the proofs below in excruciating detail and simply refer back to them in the corresponding calculations of later proofs.

In this and subsequent sections we will make heavy use of the first-order central difference, so we introduce the notation

$$\delta_{2x} f = f(\cdot + x) - f(\cdot - x).$$

The identity

$$\delta_{2x} K(\tau x) = \delta_{2x} K_{\text{sing}}(\tau x) + \delta_{2x} K_{\text{reg}}(\tau x) = K_{\text{sing}}(x)\delta_2\left(|\tau|^{-1/2}\right) + \delta_{2x} K_{\text{reg}}(\tau x),$$

which follows from

$$\begin{aligned}\delta_{2x} K_{\text{sing}}(\tau x) &= \frac{1}{\sqrt{|2\pi(\tau x + x)|}} - \frac{1}{\sqrt{|2\pi(\tau x - x)|}} \\ &= \frac{1}{\sqrt{|2\pi x|}\sqrt{|\tau+1|}} - \frac{1}{\sqrt{|2\pi x|}\sqrt{|\tau-1|}} = K_{\text{sing}}(x)\delta_2\left(|\tau|^{-1/2}\right),\end{aligned}$$

will also be useful. We also state now that throughout this and later sections, we will repeatedly make these changes of variables $y = \tau x$ or $y = \tau h$. We will not always mention explicitly when we make these change of variables, but it should be clear from the calculations.

This section, and all subsequent sections of this paper, are structured to follow the main steps form the introduction. We begin with the first one.

Step 1. By the integral equation satisfied by u, (2.7), one finds that

$$\begin{aligned}u(x+h)^2 - u(x-h)^2 &= \int_0^\infty (\delta^2_{x+h} K(y) - \delta^2_{x-h} K(y))u(y)\ \mathrm{d}y \\ &= -\int_0^\infty \delta_{2h} K(y)\delta_{2x} u(y)\ \mathrm{d}y.\end{aligned} \tag{4.23}$$

This equation is what we refer to as the central difference equation satisfied by u.

Step 2. Now we show the first estimate for $\delta_{2h} u(x)$, which proves that we have more than 1/2-Hölder continuity only at the origin. Note that the following lemma and the lemma in Step 3 below are stated as a single lemma in [11], which is based on [5, Theorem 5.4 (*ii*)]. The ν that appears in the results of this and later sections is some small positive number $\nu \ll 1$ (importantly, $\nu < P/2$ so that $u(x)$ is increasing on $x \in [0, \nu]$), which we will shrink whenever necessary.

Lemma 4.9. *[11] There is some* $\nu > 0$ *so that*

$$|u(x+h) - u(x-h)| \lesssim |h|^{1/2},$$

i.e. u *is* 1/2*-Hölder continuous, uniformly on* $[0, \nu]$*.*

Remark 4.10. In fact, once the statement of the lemma is established we will have proved that $u \in C^{1/2}(\mathbb{R})$, as u is smooth on (ν, P) [5, Theorem 5.4 (*ii*)]. However, we only require this and following estimates on $[0, \nu]$.

Proof. We follow the main beats of the proof from [11], but show the calculations in great detail so we do not have to write out completely analogous calculations in later sections (we will simply refer back to the ones here instead). The proof is also restructured to ease comparisons with corresponding proofs in later sections, and we explain the arguments in great detail as they will be reused in the sections that follow.

Consider $0 < h \le x \le \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. We first note that the statemenet of the lemma follows directly from the limit at the origin for "large" h, say $h \in [x/2, x]$. Indeed, for such h we have that

$$
\begin{aligned}
&|u(x+h) - u(x-h)| \\
&\quad \le |u(x+h)| + |u(x-h)| \lesssim |x+h|^{1/2} + |x-h|^{1/2} \le |2h+h|^{1/2} + |2h-h|^{1/2} \simeq h^{1/2},
\end{aligned}
$$

where we used the u-limit in the second inequality. We now show that it also holds for $h < x/2$.

We consider the central difference equation satisfied by u, (4.23), and split the integral on the right-hand side in the following way:

$$
\begin{aligned}
u(x+h)^2 - u(x-h)^2 &= -\int_0^\infty \delta_{2h}K(y)\delta_{2x}u(y)\ \mathrm{d}y \\
&= -\left(\int_0^x + \int_x^\nu\right)\delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\ \mathrm{d}y \\
&\qquad - \int_0^\nu \delta_{2h}K_{\text{reg}}(y)\delta_{2x}u(y)\ \mathrm{d}y - \int_\nu^\infty \delta_{2h}K(y)\delta_{2x}u(y)\ \mathrm{d}y,
\end{aligned} \tag{4.24}
$$

for all $0 < 2h < x \le \nu$. We will use this formulation to arrive at claimed estimate by bounding each of the integrals on the right-hand side.

Since $u(x \pm h)$ behaves like $x^{1/2}$ for $2h < x \le \nu$ (after possibly shrinking ν), we can rewrite the left-hand side as

$$
|u(x+h)^2 - u(x-h)^2| = |\big(u(x+h) + u(x-h)\big)\delta_{2h}u(x)| \simeq |x^{1/2}\delta_{2h}u(x)|. \tag{4.25}
$$

As we will see, all the integral bounds will contain a factor $x^{1/2}$ which will cancel with the one appearing here.

Using the triangle inequality on $\delta_{2x}u(y)$, followed by the estimate for u at the origin, 4.8, we get

$$
\begin{aligned}
\left|\int_0^x \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\ \mathrm{d}y\right| &\lesssim \int_0^x \left|\delta_{2h}K_{\text{sing}}(y)\right|x^{1/2}\ \mathrm{d}y \\
&= x^{1/2}\int_0^{x/h} \left|\delta_{2h}K_{\text{sing}}(\tau h)\right| h\ \mathrm{d}\tau \\
&= x^{1/2}h\int_0^{x/h} \left|K_{\text{sing}}(h)\delta_2\left(|\tau|^{-1/2}\right)\right|\ \mathrm{d}\tau \\
&\simeq (xh)^{1/2}\int_0^{x/h} \left|\delta_2\left(|\tau|^{-1/2}\right)\right|\ \mathrm{d}\tau \lesssim (xh)^{1/2},
\end{aligned} \tag{4.26}
$$

for the first integral. The final inequality follows from the fact that the final integral converges on $\mathbb{R}^+$. Also note that we have bounded $u(y \pm x)$ by $x^{1/2}$ and not $y^{1/2}$, since x is the larger of the two variables.

For the third integral we use that K_{reg} is real analytic and all derivatives decay (so the derivative is bounded), together with the fact that $u(y)$ is bounded, non-negative and increasing on the domain of integration. Then we find that

$$
\begin{aligned}
\left|\int_0^\nu \delta_{2h}K_{\text{reg}}(y)\delta_{2x}u(y)\ \mathrm{d}y\right| &\le ||u||_{L^\infty}\int_0^\nu |K_{\text{reg}}(y+h) - K_{\text{reg}}(y-h)|\ \mathrm{d}y \\
&= 2h||u||_{L^\infty}\int_0^\nu \frac{|K_{\text{reg}}(y+h) - K_{\text{reg}}(y-h)|}{2h}\ \mathrm{d}y \\
&\le 2h\nu||u||_{L^\infty}||K'_{reg}||_{L^\infty} \simeq h \le (hx)^{1/2}.
\end{aligned} \tag{4.27}
$$

For the fourth integral, we have that

$$
\begin{aligned}
\left|\int_\nu^\infty \delta_{2h}K(y)\delta_{2x}u(y)\ \mathrm{d}y\right| &= \left|\int_\nu^\infty \int_{-h}^h K'(y+t)\ \mathrm{d}t\ \delta_{2x}u(y)\ \mathrm{d}y\right| \\
&\le ||u||_{L^\infty}\int_\nu^\infty \int_{-h}^h |K'(y+t)|\ \mathrm{d}t\ \mathrm{d}y
\end{aligned}
$$

$$= ||u||_{L^\infty} \int_{-h}^{h} \int_{\nu}^{\infty} |K'(y+t)| \text{ d}y \text{ d}t \tag{4.28}$$

$$= ||u||_{L^\infty} \int_{-h}^{h} |-K(\nu+t)| \text{ d}t$$

$$\leq ||u||_{L^\infty} 2hK(\nu - h) \lesssim hK\big(\nu - \frac{\nu}{2}\big) \simeq h \leq (hx)^{1/2},$$

where we have used that K is smooth away from the origin. The second-to-last inequality comes from the fact that K is decreasing.

We will not show the bound for the second integral, and simply state that the second integral can also be bounded by $(xh)^{1/2}$. The reason for this is that it requires a bit more work than the bounds for the integrals so far, work which will not be necessary when we show corresponding estimates for the derivatives of u later. The bound for the second integral is established through an interpolation argument, and we refer the read to [5, Theorem 5.4 (ii)] or [11, Lemma 5.6] for the proof.

Combining the left-hand side with the bounds for the integrals on the right-hand side, we see that the factor $x^{1/2}$ can be divided away, and we arrive at the claim. □

Step 3. With the first estimate at hand, we are ready to show the improved estimate. The reason we also need the improved estimate, is that the first estimate is not good enough to show that all the integrals we will encounter in Step 4 are dominated by integrable functions.

Lemma 4.11. *[11] The improved estimate*

$$|x^{1/2}\big(u(x+h) - u(x-h)\big)| \lesssim |h|$$

holds uniformly on $[0, \nu]$, for some $\nu > 0$.

Proof. We follow the main beats of the proof from [11], but now show the calculations in excruciating detail so we can skip the analogous calculations in later sections. This proof is also restructured to ease comparisons with the corresponding proof in later sections, and we explain the arguments in great detail as they will be reused in the sections that follow.

Consider $0 < h \leq x \leq \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. It is only difficult to show the improved estimate when h becomes infinitesimally small, as it follows directly from Lemma 4.9 for "large" h, say $h \in [x/2, x]$. For such $h \in [x/2, x]$ we indeed have that

$$|u(x+h) - u(x-h)| \lesssim h^{1/2} = \frac{h}{h^{1/2}} \leq \frac{h}{(x/2)^{1/2}} \lesssim \frac{h}{x^{1/2}}.$$

We now show that it also holds for $0 < 2h < x \leq \nu$.

We consider the same central difference equation satisfied by u as we did in the previous lemma. Notice from the proof of Lemma (4.9), that had all the integrals on the right-hand side been bounded by h, not $(xh)^{1/2}$, we would have our improved estimate as the factor $x^{1/2}$ would not cancel with the one on the left-hand side. But as we saw, the integrals over $(0, \nu)$ and (ν, ∞) were in-fact bounded by h, so we only need to show that integrals over $(0, x)$ and (x, ν) are also bounded by h. To do this, we will bound these integrals again, but now using Lemma (4.9) instead of the estimate at the origin.

Note that since u is an even function on $\mathbb{R}$, we have the following symmetry for the first-order central difference,

$$\delta_{2x}u(y) = u(y+x) - u(y-x) = u(x+y) - u(-(x-y)) = u(x+y) - u(x-y) = \delta_{2y}u(x).$$

We write the integral over $(0, x)$, as

$$\int_0^x \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y) \text{ d}y = \int_0^{2h} \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y) \text{ d}y + \int_{2h}^x \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y) \text{ d}y.$$

For the integral over $(0, 2h)$, we have that

$$\left| \int_0^{2h} \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y) \text{ d}y \right| \leq \int_0^{2h} \left|\delta_{2h}K_{\text{sing}}(y)\right|\left|\delta_{2y}u(x)\right| \text{ d}y$$

$$
\begin{aligned}
&\lesssim \int_0^{2h} \Big|\delta_{2h} K_{\text{sing}}(y)\Big| y^{1/2} \, \mathrm{d}y \\
&= \int_0^{2} \Big|\delta_{2h} K_{\text{sing}}(\tau h)\Big| (\tau h)^{1/2} h \, \mathrm{d}\tau \\
&= h^{3/2} \int_0^{2} \Big|K_{\text{sing}}(h)\delta_2 \Big(|\tau|^{-1/2}\Big)\Big| \tau^{1/2} \, \mathrm{d}\tau \\
&\simeq h \int_0^{2} \Big|\delta_2 \Big(|\tau|^{-1/2}\Big)\Big| \tau^{1/2} \, \mathrm{d}\tau \\
&\simeq h,
\end{aligned}
\tag{4.29}
$$

where we have used Lemma 4.9 to bound $\delta_{2y}u(x)$.

For the integral over (x,ν), using Lemma 4.9 results in the bound

$$
\begin{aligned}
&\left|\int_x^{\nu} \delta_{2h} K_{\text{sing}}(y)\delta_{2x}u(y) \, \mathrm{d}y\right| \le \int_x^{\nu} \Big|\delta_{2h} K_{\text{sing}}(y)\Big|\Big|\delta_{2x}u(y)\Big| \, \mathrm{d}y \lesssim \int_x^{\nu} \Big|\delta_{2h} K_{\text{sing}}(y)\Big| x^{1/2} \, \mathrm{d}y \\
&\quad = \int_{x/h}^{\nu/h} \Big|\delta_{2h} K_{\text{sing}}(\tau h)\Big| x^{1/2} h \, \mathrm{d}\tau \\
&\quad = \int_{x/h}^{\nu/h} \Big|K_{\text{sing}}(h)\delta_2 \Big(|\tau|^{-1/2}\Big)\Big| x^{1/2} h \, \mathrm{d}\tau \\
&\quad \simeq (xh)^{1/2} \int_{x/h}^{\nu/h} \Big|\delta_2 \Big(|\tau|^{-1/2}\Big)\Big| \, \mathrm{d}\tau \\
&\quad = (xh)^{1/2} \int_{x/h}^{\nu/h} \left(\frac{1}{(\tau-1)^{1/2}} - \frac{1}{(\tau+1)^{1/2}}\right) \mathrm{d}\tau \\
&\quad = (xh)^{1/2} 2\Big[(\tau-1)^{1/2} - (\tau+1)^{1/2}\Big]_{x/h}^{\nu/h} \\
&\quad \simeq (xh)^{1/2} \Big(\big(\frac{\nu}{h}-1\big)^{1/2} - \big(\frac{\nu}{h}+1\big)^{1/2} - \big(\frac{x}{h}-1\big)^{1/2} + \big(\frac{x}{h}+1\big)^{1/2}\Big) \\
&\quad = (xh)^{1/2} \left(\frac{\Big(\big(\frac{\nu}{h}-1\big)^{1/2} - \big(\frac{\nu}{h}+1\big)^{1/2}\Big)\Big(\big(\frac{\nu}{h}+1\big)^{1/2} + \big(\frac{\nu}{h}-1\big)^{1/2}\Big)}{\big(\frac{\nu}{h}+1\big)^{1/2} + \big(\frac{\nu}{h}-1\big)^{1/2}}\right. \\
&\qquad\qquad\qquad \left. - \frac{\Big(\big(\frac{x}{h}-1\big)^{1/2} - \big(\frac{x}{h}+1\big)^{1/2}\Big)\Big(\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}\Big)}{\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}}\right) \\
&\quad = (xh)^{1/2} \left(\frac{\big(\frac{\nu}{h}-1\big) - \big(\frac{\nu}{h}+1\big)}{\big(\frac{\nu}{h}+1\big)^{1/2} + \big(\frac{\nu}{h}-1\big)^{1/2}} - \frac{\big(\frac{x}{h}-1\big) - \big(\frac{x}{h}+1\big)}{\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}}\right) \\
&\quad = (xh)^{1/2} \left(\frac{-2}{\big(\frac{\nu}{h}+1\big)^{1/2} + \big(\frac{\nu}{h}-1\big)^{1/2}} - \frac{-2}{\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}}\right) \\
&\quad \simeq (xh)^{1/2} \left(\frac{1}{\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}} - \frac{1}{\big(\frac{\nu}{h}+1\big)^{1/2} + \big(\frac{\nu}{h}-1\big)^{1/2}}\right) \\
&\quad \le (xh)^{1/2} \frac{1}{\big(\frac{x}{h}+1\big)^{1/2} + \big(\frac{x}{h}-1\big)^{1/2}} \le (xh)^{1/2} \frac{1}{\big(\frac{x}{h}\big)^{1/2} + \big(\frac{x}{h}-\frac{x}{2h}\big)^{1/2}} \simeq (xh)^{1/2} \frac{1}{\big(\frac{x}{h}\big)^{1/2}} = h.
\end{aligned}
$$

For the integral over $(2h,x)$, using Lemma 4.9 gives us

$$
\begin{aligned}
&\left|\int_{2h}^{x} \delta_{2h} K_{\text{sing}}(y)\delta_{2x}u(y) \, \mathrm{d}y\right| \le \int_{2h}^{x} \Big|\delta_{2h} K_{\text{sing}}(y)\Big|\Big|\delta_{2x}u(y)\Big| \, \mathrm{d}y \\
&\quad \lesssim \int_{2h}^{x} \Big|\delta_{2h} K_{\text{sing}}(y)\Big| y^{1/2} \, \mathrm{d}y \\
&\quad = \int_{2}^{x/h} \Big|\delta_{2h} K_{\text{sing}}(\tau h)\Big| (\tau h)^{1/2} h \, \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&= h^{3/2}\int_2^{x/h}\Big|K_{\mathrm{sing}}(h)\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\tau^{1/2}\ \mathrm{d}\tau\\
&\simeq h\int_2^{x/h}\Big|\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\tau^{1/2}\ \mathrm{d}\tau\\
&= h\int_2^{x/h}\left(\frac{1}{(\tau-1)^{1/2}}-\frac{1}{(\tau+1)^{1/2}}\right)\tau^{1/2}\ \mathrm{d}\tau\\
&= h\int_2^{x/h}\frac{(\tau+1)^{1/2}-(\tau-1)^{1/2}}{(\tau-1)^{1/2}(\tau+1)^{1/2}}\tau^{1/2}\ \mathrm{d}\tau \qquad (4.30)\\
&\simeq h\int_2^{x/h}\frac{(\tau+1)^{1/2}-(\tau-1)^{1/2}}{(\tau-1)^{1/2}(\tau+1)^{1/2}(\tau+1-\tau+1)}\tau^{1/2}\ \mathrm{d}\tau\\
&= h\int_2^{x/h}\frac{(\tau+1)^{1/2}-(\tau-1)^{1/2}}{(\tau-1)^{1/2}(\tau+1)^{1/2}\Big((\tau+1)^{1/2}+(\tau-1)^{1/2}\Big)\Big((\tau+1)^{1/2}-(\tau-1)^{1/2}\Big)}\tau^{1/2}\ \mathrm{d}\tau\\
&= h\int_2^{x/h}\frac{1}{(\tau-1)^{1/2}(\tau+1)^{1/2}\Big((\tau+1)^{1/2}+(\tau-1)^{1/2}\Big)}\tau^{1/2}\ \mathrm{d}\tau\\
&\leq h\int_2^{x/h}\frac{\tau^{1/2}}{(\tau-1)^{1/2}(\tau-1)^{1/2}(\tau-1)^{1/2}}\ \mathrm{d}\tau\\
&= h\int_1^{\frac{x}{h}-1}\frac{(s+1)^{1/2}}{s^{3/2}}\ \mathrm{d}s\\
&\leq h\int_1^{\frac{x}{h}}\frac{s^{1/2}}{s^{3/2}}\ \mathrm{d}s + h\int_1^{\infty}\frac{1}{s^{3/2}}\ \mathrm{d}s\\
&\simeq h\ln\Big(\frac{x}{h}\Big)+h = h\ln\left(\Big(\frac{h}{x}\Big)^{-1}\right)+h = h\Big(-\ln\Big(\frac{h}{x}\Big)\Big)+h = h\Big(1+\Big|\ln\Big(\frac{h}{x}\Big)\Big|\Big).
\end{aligned}
$$

This estimate is not quite good enough yet. However, we can combine all the estimates we have so far to get the estimate

$$
|u(x+h)-u(x-h)| \lesssim \frac{h(1+|\ln(\frac{h}{x})|)}{x^{1/2}}.
$$

We use this new estimate to bound the integral on $(2h, x)$ one more time, to find that

$$
\begin{aligned}
&\left|\int_{2h}^{x}\delta_{2h}K_{\mathrm{sing}}(y)\delta_{2x}u(y)\ \mathrm{d}y\right| \leq \int_{2h}^{x}\Big|\delta_{2h}K_{\mathrm{sing}}(y)\Big|\Big|\delta_{2y}u(x)\Big|\ \mathrm{d}y\\
&\lesssim \int_{2h}^{x}\Big|\delta_{2h}K_{\mathrm{sing}}(y)\Big|\frac{y(1+|\ln(\frac{y}{x})|)}{x^{1/2}}\ \mathrm{d}y\\
&= \int_{2h/x}^{1}\Big|\delta_{2h}K_{\mathrm{sing}}(\tau x)\Big|\frac{\tau x(1+|\ln\tau|)}{x^{1/2}}x\ \mathrm{d}\tau\\
&= x^{3/2}\int_{2h/x}^{1}\Big|\frac{1}{|2\pi(\tau x+h)|^{1/2}}-\frac{1}{|2\pi(\tau x-h)|^{1/2}}\Big|\tau(1+|\ln\tau|)\ \mathrm{d}\tau\\
&\simeq x^{3/2}\int_{2h/x}^{1}\Big(\frac{1}{(\tau x-h)^{1/2}}-\frac{1}{(\tau x+h)^{1/2}}\Big)\tau(1+|\ln\tau|)\ \mathrm{d}\tau \qquad (4.31)\\
&\simeq x\int_{2h/x}^{1}\Big(\frac{1}{(\tau-\frac{h}{x})^{1/2}}-\frac{1}{(\tau+\frac{h}{x})^{1/2}}\Big)\tau(1+|\ln\tau|)\ \mathrm{d}\tau\\
&\simeq x\int_{\frac{2h}{x}}^{1}\frac{(\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}}{(\tau-\frac{h}{x})^{1/2}(\tau+\frac{h}{x})^{1/2}\big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\big)\big((\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}\big)}\frac{h}{x}\tau(1+|\ln\tau|)\ \mathrm{d}\tau\\
&= h\int_{\frac{2h}{x}}^{1}\frac{1}{(\tau-\frac{h}{x})^{1/2}(\tau+\frac{h}{x})^{1/2}\big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\big)}\tau(1+|\ln\tau|)\ \mathrm{d}\tau\\
&\leq h\int_{\frac{2h}{x}}^{1}\frac{1}{(\tau-\frac{h}{x})^{1/2}(\tau-\frac{h}{x})^{1/2}(\tau-\frac{h}{x})^{1/2}}\tau(1+|\ln\tau|)\ \mathrm{d}\tau
\end{aligned}
$$

$$= h \int_{\frac{2h}{x}}^{1} \frac{\tau(1 + |\ln \tau|)}{(\tau - \frac{h}{x})^{3/2}} \, \mathrm{d}\tau \lesssim h \int_{\frac{2h}{x}}^{1} \frac{1 + |\ln \tau|}{\tau^{1/2}} \, \mathrm{d}\tau \leq h \int_{0}^{1} \frac{1 + |\ln \tau|}{\tau^{1/2}} \simeq h.$$

Thus we have shown that all of the right-hand side integrals are bounded by h, which we combine with the left-hand side (4.25) to arrive at the improved estimate. □

Note that since $u(x)$ is smooth on $(0, P)$, we can take the limit as h tends to zero in this improved estimate to get the bound

$$u'(x) \lesssim x^{-1/2}$$

on $(0, \nu]$ [11]. This leads us to the final step.

Step 4. With the first estimate and the improved estimate at hand, we are ready for the final result of this section, where the u'-limit is determined.

Proposition 4.12. *[11] The derivative of the solution u enjoys the limit*

$$\lim_{x \to 0} \frac{u'(x)}{x^{-1/2}} = \frac{1}{2} \sqrt{\frac{\pi}{8}}$$

Proof. We follow the proof from [11], but show the calculations in excruciating detail so we can refer back to them later. We consider the same central difference equation satisfied by u as in the previous lemmas, but now we divide both sides by $2h$:

$$\begin{aligned} \frac{u(x+h)^2 - u(x-h)^2}{2h} &= -\frac{1}{2h} \int_0^{\infty} \delta_{2h} K(y) \delta_{2x} u(y) \, \mathrm{d}y \\ &= -\frac{1}{2h} \left(\int_0^{2h} + \int_{2h}^{x} + \int_{x}^{\nu} \right) \delta_{2h} K_{\text{sing}}(y) \delta_{2x} u(y) \, \mathrm{d}y \\ &\quad - \frac{1}{2h} \int_0^{\nu} \delta_{2h} K_{\text{reg}}(y) \delta_{2x} u(y) \, \mathrm{d}y - \frac{1}{2h} \int_{\nu}^{\infty} \delta_{2h} K(y) \delta_{2x} u(y) \, \mathrm{d}y, \end{aligned} \tag{4.32}$$

for all $0 < 2h < x \leq \nu$.

Taking the limit as h tends to zero, followed by the limit as x tends to zero, we get

$$\lim_{x \to 0} \lim_{h \to 0} \frac{u(x+h)^2 - u(x-h)^2}{2h} = \lim_{x \to 0} 2u(x)u'(x) = 2\sqrt{\frac{\pi}{8}} \lim_{x \to 0} \frac{u'(x)}{x^{-1/2}}$$

for the left-hand side. Here we used the u-limit in the last equality. We now consider the limits of the right-hand side integrals to show that this limit does indeed exists (which we do not know at the moment), and to find its exact value. Our strategy will be to prove that each of the five integrands in (4.32) are dominated by integrable functions, independently of x and h, which allows us to use the dominated convergence theorem to interchange limits and integrals.

For the first integral we see that

$$\begin{aligned} -\frac{1}{2h} \int_0^{2h} \delta_{2h} K_{\text{sing}}(y) \delta_{2y} u(x) \, \mathrm{d}y &= -\frac{1}{2h} \int_0^{2} \delta_{2h} K_{\text{sing}}(\tau h) \delta_{2\tau h} u(x) h \, \mathrm{d}\tau \\ &= -\frac{1}{2h} \int_0^{2} K_{\text{sing}}(h) \delta_2 \Big(|\tau|^{-1/2} \Big) \delta_{2\tau h} u(x) h \, \mathrm{d}\tau \\ &= \int_0^{2} \frac{-1}{\sqrt{8\pi} h^{1/2}} \delta_2 \Big(|\tau|^{-1/2} \Big) \delta_{2\tau h} u(x) \, \mathrm{d}\tau. \end{aligned} \tag{4.33}$$

for all $0 < 2h < x \leq \nu$. Notice for the integrand that we have

$$\Big| - \frac{1}{\sqrt{8\pi} h^{1/2}} \delta_2 \Big(|\tau|^{-1/2} \Big) \delta_{2\tau h} u(x) \Big| = \frac{\sqrt{2}}{\sqrt{8\pi}} \Big| \delta_2 \Big(|\tau|^{-1/2} \Big) \Big| \tau^{1/2} \frac{|\delta_{2\tau h} u(x)|}{|2\tau h|^{1/2}} \lesssim \Big| \delta_2 \Big(|\tau|^{-1/2} \Big) \Big| \tau^{1/2},$$

where we use Lemma 4.9 to bound $|\delta_{2\tau h} u(x)|$. The right-hand side is integrable on $[0, 2]$, so dominated convergence allows us to interchange the limit and integral. Thus, for each $0 < x \leq \nu$,

$$\lim_{h \to 0} \frac{-1}{\sqrt{8\pi} h^{1/2}} \delta_2 \Big(|\tau|^{-1/2} \Big) \delta_{2\tau h} u(x) = \lim_{h \to 0} \frac{-2h^{1/2}}{\sqrt{8\pi}} \delta_2 \Big(|\tau|^{-1/2} \Big) \tau \frac{\delta_{2\tau h} u(x)}{2\tau h} = 0.$$

The final equality follows from $u(x)$ being differentiable for $0 < x < P$ (which tells us that the final fraction is finite in the limit), so the limit tends to zero due to the $h^{1/2}$ in the numerator.

For the second integral have that

$$-\frac{1}{2h}\int_{2h}^{x}\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u(x)\ \mathrm{d}y = -\int_{2h/x}^{1}\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u(x)x\ \mathrm{d}\tau.$$

The integrand can be written as

$$\begin{aligned}
-\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u(x)x &= -\frac{1}{2h}\left(\frac{1}{|2\pi(\tau x+h)|^{1/2}} - \frac{1}{|2\pi(\tau x-h)|^{1/2}}\right)x\delta_{2\tau x}u(x) \\
&= -\frac{1}{\sqrt{8\pi}h}\left(\frac{1}{|x(\tau+\frac{h}{x})|^{1/2}} - \frac{1}{|x(\tau-\frac{h}{x})|^{1/2}}\right)x\delta_{2\tau x}u(x) \\
&= -\frac{1}{\sqrt{8\pi}h}\left(\frac{1}{(\tau+\frac{h}{x})^{1/2}} - \frac{1}{(\tau-\frac{h}{x})^{1/2}}\right)x^{1/2}\delta_{2\tau x}u(x) \\
&= \frac{1}{\sqrt{8\pi}h}\Big(\frac{(\tau+\frac{h}{x})^{1/2}}{(\tau-\frac{h}{x})^{1/2}(\tau+\frac{h}{x})^{1/2}} - \frac{(\tau-\frac{h}{x})^{1/2}}{(\tau+\frac{h}{x})^{1/2}(\tau-\frac{h}{x})^{1/2}}\Big)x^{1/2}\delta_{2\tau x}u(x) \\
&= \frac{1}{\sqrt{8\pi}h}\frac{(\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}}x^{1/2}\delta_{2\tau x}u(x) \qquad (4.34)\\
&= \frac{1}{\sqrt{8\pi}h}\frac{(\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}(\tau+\frac{h}{x}-\tau+\frac{h}{x})}\frac{2h}{x}x^{1/2}\delta_{2\tau x}u(x) \\
&= \frac{1}{\sqrt{2\pi}}\frac{(\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)\Big((\tau+\frac{h}{x})^{1/2}-(\tau-\frac{h}{x})^{1/2}\Big)}\frac{1}{x^{1/2}}\delta_{2\tau x}u(x) \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}\frac{\delta_{2\tau x}u(x)}{x^{1/2}}.
\end{aligned}$$

Using the improved estimate to bound the integrand, we get that

$$\begin{aligned}
&\left|\frac{1}{\sqrt{2\pi}}\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}\frac{x^{1/2}\delta_{2\tau x}u(x)}{x}\right| \\
&\qquad\simeq \left|\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}\right|\frac{|x^{1/2}\delta_{2\tau x}u(x)|}{x} \qquad (4.35)\\
&\qquad\lesssim \left|\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}\right|\frac{\tau x}{x} \\
&\qquad\leq \left|\frac{\tau}{\big(\tau^2/4\big)^{1/2}\tau^{1/2}}\right| \simeq \frac{1}{\tau^{1/2}},
\end{aligned}$$

for all $0 < 2h < x \leq \nu$ and $2h/x \leq \tau \leq 1$. Thus, the integrand is dominated by $1/\tau^{1/2}$, which is integrable on $(0,1)$, so we can apply the dominated convergence theorem. The h-limit evaluates to

$$\begin{aligned}
\lim_{h\to 0}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u(x)x &= \lim_{h\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}\frac{\delta_{2\tau x}u(x)}{x^{1/2}} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\big(\tau^2\big)^{1/2}\big(\tau^{1/2}+\tau^{1/2}\big)}\frac{\delta_{2\tau x}u(x)}{x^{1/2}} \qquad (4.36)\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\frac{\delta_{2\tau x}u(x)}{x^{1/2}},
\end{aligned}$$

where the first equality comes rewriting the integrand as in (4.34). For the x-limit we then get

$$\begin{aligned}\lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\frac{\delta_{2\tau x}u(x)}{x^{1/2}} &= \lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\frac{u(x+\tau x)-u(x-\tau x)}{x^{1/2}}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\left(\frac{u(x(1+\tau))(1+\tau)^{1/2}}{(x(1+\tau))^{1/2}}-\frac{u(x(1-\tau))(1-\tau)^{1/2}}{(x(1-\tau))^{1/2}}\right)\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\left((1+\tau)^{1/2}\sqrt{\frac{\pi}{8}}-(1-\tau)^{1/2}\sqrt{\frac{\pi}{8}}\right)\\ &= \frac{1}{8\tau^{3/2}}\Big((1+\tau)^{1/2}-(1-\tau)^{1/2}\Big),\end{aligned} \tag{4.37}$$

for each $\tau\in(0,1)$. We thus have that

$$\lim_{x\to 0}\lim_{h\to 0}\int_{2h/x}^{1}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u(x)x\ \mathrm{d}\tau = \int_0^1\frac{1}{8\tau^{3/2}}\Big((1+\tau)^{1/2}-(1-\tau)^{1/2}\Big)\ \mathrm{d}\tau.$$

Moving on to the third integral, we see it can be written as

$$-\frac{1}{2h}\int_x^{\nu}\delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\ \mathrm{d}y = -\frac{1}{2h}\int_1^{\nu/x}\delta_{2h}K_{\text{sing}}(\tau x)\delta_{2x}u(\tau x)x\ \mathrm{d}\tau.$$

This integrand can be expressed in the same way as the integrand of the second integral. For the bound, we use the improved estimate to find that

$$\begin{aligned}\Big|-\frac{1}{2h}\delta_{2h}K_{\text{sing}}(\tau x)\delta_{2x}u(\tau x)x\Big| &= \left|\frac{1}{\sqrt{2\pi}}\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\frac{\delta_{2x}u(\tau x)}{x^{1/2}}\right|\\ &\lesssim \left|\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\right|\frac{|(\tau x)^{1/2}\delta_{2x}u(\tau x)|}{\tau^{1/2}x}\\ &\lesssim \left|\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\right|\frac{x}{\tau^{1/2}x}\\ &= \frac{\tau^{-1/2}}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\lesssim\frac{\tau^{-1/2}}{\tau^{3/2}}=\frac{1}{\tau^2}\end{aligned}$$

for all $0<2h<x\le\nu$ and $1\le\tau\le\nu/x$. Since $1/\tau^2$ is integrable on $[1,\infty)$, we can interchange the limits and integral. For the h-limit, we get

$$\lim_{h\to 0}-\frac{1}{2h}\delta_{2h}K_{\text{sing}}(\tau x)\delta_{2x}u(\tau x)x = \frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\frac{\delta_{2x}u(\tau x)}{x^{1/2}}$$

for each fixed $x\in(0,\nu]$ and $\tau\in(1,\nu/x)$. Similarly, we furthermore get

$$\lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}\frac{\delta_{2x}u(\tau x)}{x^{1/2}} = \frac{1}{8\tau^{3/2}}\Big((1+\tau)^{1/2}-(\tau-1)^{1/2}\Big)$$

for every $\tau>0$.

For the fourth integral we use that K_{reg} is smooth, and each of its derivatives is decaying, together with the fact that u is bounded, to see that

$$\Big|\frac{\delta_{2h}K_{\text{reg}}(y)}{2h}\delta_{2x}u(y)\Big|\lesssim ||K'_{\text{reg}}||_{L^\infty}||u||_{L^\infty}\simeq 1,$$

for all $0<2h<x\le\nu$ and $0<y<\nu$. For the limits we then have

$$\lim_{h\to 0}\frac{\delta_{2h}K_{\text{reg}}(y)}{2h}\delta_{2x}u(y) = K'_{\text{reg}}(y)\delta_{2x}u(y)$$

for $0 < x \le \nu$ and $0 < y < \nu$, and

$$\lim_{x\to 0} K'_{\text{reg}}(y)\delta_{2x}u(y) = 0$$

for all $0 < y < \nu$. Thus, the integral evaluates to

$$\lim_{x\to 0}\lim_{h\to 0} \frac{-1}{2h}\int_0^{\nu} \delta_{2h}K_{\text{reg}}(y)\delta_{2x}u(y)\ \mathrm{d}y = 0.$$

For the fifth and final integral we bound the integral in the same way as we did in the proof of Lemma (4.9),

$$\Big|\frac{1}{2h}\delta_{2h}K(y)\delta_{2x}u(y)\Big| \lesssim \frac{1}{2h}\Big|\int_{-h}^{h} K'(y+t)\mathrm{d}t\Big| \lesssim -K'(\nu/2)$$

for all $0 < 2h < x \le \nu$ and $y > \nu$. For the limits we then get

$$\lim_{h\to 0}\frac{1}{2h}\delta_{2h}K(y)\delta_{2x}u(y) = K'(y)\delta_{2x}u(y), \tag{4.38}$$

for every $0 < x \le \nu < y$, and

$$\lim_{x\to 0} K'(y)\delta_{2x}u(y) = 0$$

for all $y > \nu$. That is, the fifth integral also vanishes in the limit,

$$\lim_{x\to 0}\lim_{h\to 0} \frac{-1}{2h}\int_{\nu}^{\infty} \delta_{2h}K(y)\delta_{2x}u(y)\ \mathrm{d}y = 0.$$

Thus, we have that each of the five limits exists. Combining the limits of the left-hand side and right-hand side give us that

$$2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u'(x)}{x^{-1/2}} = \int_0^{\infty} \frac{1}{8\tau^{3/2}}\Big((1+\tau)^{1/2} - |1-\tau|^{1/2}\Big)\ \mathrm{d}\tau = \frac{\pi}{8}. \tag{4.39}$$

After rearranging, this becomes

$$\lim_{x\to 0}\frac{u'(x)}{x^{-1/2}} = \frac{1}{2}\sqrt{\frac{\pi}{8}},$$

which concludes the proof. □

5 The limit for the second derivative

We are now ready to consider higher order derivatives of u. We follow the ideas proposed by the authors in [11], but all the work in this and subsequent sections is otherwise our own.

The aim of this section is to determine the u''-limit. To this end, we will follow the four main steps from the introduction and try to mirror the procedure for the u'-limit in Subsection 4.1. As stated in the introduction, our most important contributions from this section is the extra factor need on the left-hand side of the two estimates, and the new splitting of the integral in the central difference equation, which, as we will see, are are a consequence of the $|x|^{-1/2}$-type singularity from u'. This extra care that is required due to the $|x|^{-1/2}$-type singularity, when compared to Subsection 4.1, is the most difficult part of this section.

Step 1. We are searching for a central difference equation satisfied by u' that corresponds to (4.23) from the previous section. Recall that we took the central difference of an integral equation satisfied by u to arrive at (4.23), so we begin by looking for a corresponding integral equation satisfied by u'.

As suggested in [11], we can find the integral equation satisfied by u' by taking *only* the h limit in (4.32). First, however, we need to move the central difference in h over to u, so we end up with u' when taking the h-limit. If we do not move h over to u, we instead end up with K' inside the integral. This would cause problems, as $K'_{\text{sing}}(x) \simeq |x|^{-3/2}$ which is non-integrable on a finite domain containing the singularity, so we would not be able to bound the integrals in a similar manner to what we did in the previous section (as we will see, this is exactly what causes problems with the u'''-limit, as we end up with a non-integrable singularity inside the domain of integration no matter where we put h). Through some changes of variables and straightforward calculations, one can write the integral in (4.32) as

$$-\int_0^\infty \delta_{2h}K(y)\delta_{2x}u(y)\,\mathrm{d}y = -\int_0^\infty \delta_{2x}K(y)\delta_{2h}u(y)\,\mathrm{d}y.$$

Using this equality, taking only the h-limit in (4.32) gives us the integral equation

$$2u(x)u'(x) = -\int_0^\infty \delta_{2x}K(y)u'(y)\,\mathrm{d}y,$$

which is satisfied by u'. Taking the central difference of this integral equation, we get

$$\begin{aligned} &2u(x+h)u'(x+h) - 2u(x-h)u'(x-h) \\ &\quad = -\int_0^\infty \delta_{2(x+h)}K(y)u'(y)\,\mathrm{d}y + \int_0^\infty \delta_{2(x-h)}K(y)u'(y)\,\mathrm{d}y \\ &\quad = -\int_0^\infty \delta_{2h}K(y)\delta_{2y}u'(x)\,\mathrm{d}y, \end{aligned} \tag{5.1}$$

the central difference equation satisfied by u'. The final equality in (5.1) follows from straightforward calculation. Note that due to u' being odd, we no longer have the same symmetry for $\delta_{2y}u'(x)$ that we had for the central difference of u, $\delta_{2x}u(y) = \delta_{2y}u(x)$. Recognizing that (5.1) is similar to (4.24), we split the integral in a similar way:

$$\begin{aligned} 2u(x+h)u'(x+h) - 2u(x-h)u'(x-h) &= -\int_0^\infty \delta_{2h}K(y)\delta_{2y}u'(x)\,\mathrm{d}y \\ = -\Big(\int_0^x + \int_x^{2\nu}\Big)&\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)\,\mathrm{d}y \\ &- \int_0^{2\nu} \delta_{2h}K_{\text{reg}}(y)\delta_{2y}u'(x)\,\mathrm{d}y - \int_{2\nu}^\infty \delta_{2h}K(y)\delta_{2y}u'(x)\,\mathrm{d}y. \end{aligned} \tag{5.2}$$

We split the integral at 2ν instead of ν to make some of the calculations in the following proofs easier, shrinking ν if necessary.

Considering the left-hand side of (5.2), we get

$$\lim_{h\to 0} \frac{2u(x+h)u'(x+h) - 2u(x-h)u'(x-h)}{2h} = 2(u'(x))^2 + 2u(x)u''(x) \tag{5.3}$$

for the h-limit. Before taking the x-limit, we need to multiply both sides of the equation by x, as this allows us to use the u- and u'-limits. We then get

$$\begin{aligned} \lim_{x\to 0}\big(2x(u'(x))^2 + 2xu(x)u''(x)\big) &= \lim_{x\to 0}\Big(2(x^{1/2}u'(x))^2 + 2\frac{u(x)}{x^{1/2}}x^{3/2}u''(x)\Big) \\ &= 2\bigg(\frac{1}{2}\sqrt{\frac{\pi}{8}}\bigg)^2 + 2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u''(x)}{x^{-3/2}} \\ &= \frac{\pi}{16} + \sqrt{\frac{\pi}{2}}\lim_{x\to 0}\frac{u''(x)}{x^{-3/2}}, \end{aligned} \tag{5.4}$$

for the left-hand side. Again, we do not know yet if this limit exists; this we will show by considering the integrals on the right-hand side. First, however, we need the estimates for $|\delta_{2y}u'(x)|$.

Step 2. During the proof for the first estimate in Subsection 4.1, we repeatedly used the fact that $u(x)$ behaves like $x^{1/2}$, an increasing function, near the origin. As we can see in (5.1), the role of u is now replaced by u', which behaves like the $x^{-1/2}$ near the origin. Pay close attention to how this affects the following proofs when compared to Subsection 4.1 (this is for instance why we need the factor $(x-h)$ in the following lemmas, which was not present in the corresponding lemmas in Subsection 4.1).

Lemma 5.1. *There is some* $\nu > 0$ *so that*

$$|(x-h)(u'(x+h)-u'(x-h))| \lesssim h^{1/2}$$

uniformly on $[0,\nu]$.

Proof. Consider $0 < h \le x \le \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. As in the proof of Lemma 4.9, the claim follows directly from u'-limit for large h, say $h \in [x/2, x]$. Indeed, for such h we have that

$$\begin{aligned}&|(x-h)(u'(x+h)-u'(x-h))|\\ &\quad\lesssim |x-h|(|x+h|^{-1/2}+|x-h|^{-1/2}) \lesssim |x-h||x-h|^{-1/2} \le |2h-h|^{1/2} = |h|^{1/2},\end{aligned}$$

where Proposition (4.12) was used in the first inequality. We now show that the claim also holds for $h < x/2$.

Consider the expression (5.2),

$$\begin{aligned}&2(x-h)u(x+h)u'(x+h) - 2(x-h)u(x-h)u'(x-h)\\ &\quad= -\Big(\int_0^x + \int_x^{2\nu}\Big)\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\ \mathrm{d}y\\ &\qquad\qquad - \int_0^{2\nu}\delta_{2h}K_{\text{reg}}(y)\delta_{2y}u'(x)(x-h)\ \mathrm{d}y - \int_{2\nu}^{\infty}\delta_{2h}K(y)\delta_{2y}u'(x)(x-h)\ \mathrm{d}y,\end{aligned}$$

for all $0 < 2h < x \le \nu$. Notice that we have multiplied the equation through by $(x-h)$, and that taking the h-limit of the left-hand side divided by $2h$ gives us the same expression as the one on the left-hand side of (5.4).

Using that $u(x \pm h) \simeq x^{1/2}$ for $2h < x \le \nu$ (after possibly shrinking ν), we can rewrite the left-hand side as

$$\begin{aligned}&|2(x-h)u(x+h)u'(x+h) - 2(x-h)u(x-h)u'(x-h)|\\ &\qquad\simeq |x^{1/2}(x-h)(u'(x+h)-u'(x-h))|.\end{aligned}$$

The bounds of the integrals on the right-hand side will all contain a factor $x^{1/2}$ which will cancel with the one on the left-hand side.

For the first integral, we get that

$$\begin{aligned}\Big|\int_0^x \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\ \mathrm{d}y\Big| &\le |x-h|\int_0^x \big|\delta_{2h}K_{\text{sing}}(y)\big|\big|\delta_{2y}u'(x)\big|\ \mathrm{d}y\\ &\lesssim x\int_0^x \big|\delta_{2h}K_{\text{sing}}(y)\big||x-y|^{-1/2}\ \mathrm{d}y\\ &= x\int_0^{x/h}\big|\delta_{2h}K_{\text{sing}}(\tau h)\big||x-\tau h|^{-1/2}h\ \mathrm{d}\tau\\ &= xh\big|K_{\text{sing}}(h)\big|\int_0^{x/h}\Big|\delta_2\big(|\tau|^{-1/2}\big)\Big|\Big|1-\tau\frac{h}{x}\Big|^{-1/2}x^{-1/2}\ \mathrm{d}\tau\\ &\simeq (xh)^{1/2}\int_0^{x/h}\Big|\delta_2\big(|\tau|^{-1/2}\big)\Big|\Big|1-\tau\frac{h}{x}\Big|^{-1/2}\ \mathrm{d}\tau\\ &\lesssim (xh)^{1/2},\end{aligned}$$

The final integral converges on $\mathbb{R}^+$ since $\big|\delta_2\big(|\tau|^{-1/2}\big)\big|$ behaves like $\tau^{-3/2}$ for large τ (recall calculation (4.30)), and because the singularities (which are integrable) from the two factors in the integrand do not coincide; the singularity from the first factor is located at $\tau = 1$, while the singularity in the second factor is located at $\tau = x/h \ge 2$.

Luckily, the second integral does not require as much work as the second integral from the proof of Lemma 4.9, due to the fact that $u'(x)$ now is bounded by the decreasing function $x^{-1/2}$ on $(0, 2\nu)$ (again, shrinking ν if necessary):

$$\Big|\int_x^{2\nu}\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\ \mathrm{d}y\Big|$$

$$
\begin{aligned}
&\lesssim |x| \int_x^{2\nu} |\delta_{2h} K_{\text{sing}}(y)||x-y|^{-1/2} \,\mathrm{d}y \\
&\lesssim |x| h \int_x^{2\nu} \frac{(y-x)^{-1/2}}{(y-\xi h)^{3/2}_{\xi\in(-1,1)}} \,\mathrm{d}y \\
&\leq |x| h \int_x^{2\nu} \frac{(y-x)^{-1/2}}{(y-h)^{3/2}} \,\mathrm{d}y \\
&\lesssim |x| h \int_x^{2\nu} \frac{(y-x)^{-1/2}}{y^{3/2}} \,\mathrm{d}y \\
&= |x| h \int_1^{2\nu/x} \frac{(\tau x-x)^{-1/2}}{(\tau x)^{3/2}} x \,\mathrm{d}\tau \\
&= h \int_1^{2\nu/x} \frac{(\tau-1)^{-1/2}}{\tau^{3/2}} \,\mathrm{d}\tau \lesssim h \leq (hx)^{1/2},
\end{aligned}
$$

since the final integral converges on $[1,\infty)$. The second inequality is simply the mean value theorem.

For the third integral we previously used that u was bounded. As we now are dealing with u', which is not bounded, we proceed slightly differently:

$$
\begin{aligned}
&\Big| \int_0^{2\nu} \delta_{2h} K_{\text{reg}}(y) \delta_{2y} u'(x)(x-h) \,\mathrm{d}y \Big| \\
&\quad \lesssim xh \int_0^{2\nu} \frac{\Big|\delta_{2h} K_{\text{reg}}(y)\Big|}{2h} |x-y|^{-1/2} \,\mathrm{d}y \\
&\quad \leq xh \int_0^{2\nu} ||K'_{reg}||_{L^\infty} \frac{1}{|x-y|^{1/2}} \,\mathrm{d}y \\
&\quad \simeq xh \Big(\int_0^x \frac{1}{(x-y)^{1/2}} \,\mathrm{d}y + \int_x^{2\nu} \frac{1}{(y-x)^{1/2}} \,\mathrm{d}y \Big) \\
&\quad = xh \Big(\Big[-(x-y)^{1/2} \Big]_0^x + \Big[(y-x)^{1/2} \Big]_x^{2\nu} \Big) \\
&\quad \lesssim h \leq (hx)^{1/2}.
\end{aligned}
$$

For the fourth integral we need to consider two separate cases. First we consider only consider solitary wave solutions (i.e. $P = \infty$), that is solutions that decay, meaning that u has only a single cusp at the origin. This means that $u'(x)$ is bounded when x is strictly away from the origin. For solitary wave solutions, we therefore get that

$$
\begin{aligned}
&\Big| \int_{2\nu}^{\infty} \delta_{2h} K(y) \delta_{2y} u'(x)(x-h) \,\mathrm{d}y \Big| \\
&\quad \leq |x| \sup_{y\in[\nu,\infty)} |u'(y)| \int_{2\nu}^{\infty} \Big| \delta_{2h} K(y) \Big| \,\mathrm{d}y \lesssim \int_{2\nu}^{\infty} \Big| \delta_{2h} K(y) \Big| \,\mathrm{d}y \lesssim h \leq (hx)^{1/2},
\end{aligned}
$$

where the final integral is treated exactly as was done in (4.28). If instead u is a periodic solution, there will be a cusp, behaving exactly as the one at the origin, at every integer multiple of the period P. Therefore, we here need to use that $K(y)$ and its derivatives decay exponentially (recall the properties of K listed at the end of Section 2). We begin by splitting the integral in such a way that isolates the singularities at the integer multiples of P:

$$
\begin{aligned}
&\Big| \int_{2\nu}^{\infty} \delta_{2h} K(y) \delta_{2y} u'(x)(x-h) \,\mathrm{d}y \Big| \\
&\lesssim x \int_{2\nu}^{\frac{P}{2}} \Big| \delta_{2h} K(y) \Big| \Big| \delta_{2y} u'(x) \Big| \,\mathrm{d}y + x \sum_{i=1}^{\infty} \Big(\int_{Pi-\frac{P}{2}}^{Pi-2\nu} + \int_{Pi-2\nu}^{Pi+2\nu} + \int_{Pi+2\nu}^{Pi+\frac{P}{2}} \Big) \Big| \delta_{2h} K(y) \Big| \Big| \delta_{2y} u'(x) \Big| \,\mathrm{d}y.
\end{aligned}
$$

The first term can be bounded as in the solitary wave-case above. To bound the sum, notice that the domains of the first and last integrals inside the sum are bounded strictly away from the multiples of P. Consequently, we can pull out the supremum of u' over these integral domains. In the middle integral, we use the behavior of u' near the multiples of P. That is, we have the bound

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\left(\int_{Pi-\frac{P}{2}}^{Pi-2\nu}+\int_{Pi-2\nu}^{Pi+2\nu}+\int_{Pi+2\nu}^{Pi+\frac{P}{2}}\right)\left|\delta_{2h}K(y)\right|\left|\delta_{2y}u'(x)\right|\,\mathrm{d}y\\
&\lesssim \sup_{y\in[Pi-\frac{P}{2}-\nu,Pi-\nu)}|u'(y)|\sum_{i=1}^{\infty}\int_{Pi-\frac{P}{2}}^{Pi-2\nu}\left|\delta_{2h}K(y)\right|\,\mathrm{d}y\\
&\quad+\sum_{i=1}^{\infty}\int_{Pi-\nu}^{Pi+\nu}\left|\delta_{2h}K(y)\right|\left|x\pm y\mp Pi\right|^{-1/2}\,\mathrm{d}y+\sup_{y\in[Pi+2\nu,Pi+\frac{P}{2})}|u'(y)|\sum_{i=1}^{\infty}\int_{Pi+2\nu}^{Pi+\frac{P}{2}}\left|\delta_{2h}K(y)\right|\,\mathrm{d}y\\
&\lesssim \sum_{i=1}^{\infty}\int_{Pi-\frac{P}{2}}^{Pi-2\nu}\int_{-h}^{h}\left|K'(y+t)\right|\,\mathrm{d}t\,\mathrm{d}y+\sum_{i=1}^{\infty}\int_{Pi+2\nu}^{Pi+\frac{P}{2}}\int_{-h}^{h}\left|K'(y+t)\right|\,\mathrm{d}t\,\mathrm{d}y\\
&\quad+\sum_{i=1}^{\infty}\int_{Pi-\nu}^{Pi+\nu}\int_{-h}^{h}\left|K'(y+t)\right|\,\mathrm{d}t\left|x\pm y\mp Pi\right|^{-1/2}\,\mathrm{d}y.
\end{aligned}
$$

Finally, by [5, Proposition 2.1] on the exponential decay of K, this expression can be further bounded by

$$
\begin{aligned}
&h\sum_{i=1}^{\infty}\int_{Pi-\frac{P}{2}}^{Pi-2\nu}e^{-y+h}\,\mathrm{d}y+h\sum_{i=1}^{\infty}\int_{Pi+2\nu}^{Pi+\frac{P}{2}}e^{-y+h}\,\mathrm{d}y+h\sum_{i=1}^{\infty}\int_{Pi-\nu}^{Pi+\nu}e^{-y+h}\left|x\pm y\mp Pi\right|^{-1/2}\,\mathrm{d}y\\
&\lesssim h\sum_{i=1}^{\infty}e^{-Pi+\nu}\int_{Pi-\nu}^{Pi+\nu}\left|x\pm y\mp Pi\right|^{-1/2}\,\mathrm{d}y\\
&\lesssim h\le (xh)^{1/2}.
\end{aligned}
$$

Combining the right-hand sides with the left-hand side and dividing away the factor $x^{1/2}$ gives the estimate. ☐

Step 3. Now we use Lemma 5.1 to get the improved estimate.

Lemma 5.2. *The improved estimate*

$$|x^{1/2}(x-h)(u'(x+h)-u'(x-h))|\lesssim h$$

holds uniformly on $[0,\nu]$*, for some* $\nu>0$*.*

Proof. Consider $0<h\le x\le\nu\ll 1$ for some $\nu>0$ which we shrink whenever necessary. For $h\in[x/2,x]$ we have

$$|(x-h)(u'(x+h)-u'(x-h))|\lesssim h^{1/2}=\frac{h}{h^{1/2}}\le\frac{h}{(x/2)^{1/2}}\lesssim\frac{h}{x^{1/2}},$$

where the first inequality is simply Lemma 5.1. We now show that it also holds for $h<x/2$, and again consider the central difference equation satisfied by u' from Lemma 5.1.

Notice from the proof of Lemma 5.1 that all of the right-hand side integrals, except the one over $(0,x)$, were in fact bounded by h. We therefore only need to show that this first integral is also bounded by h to arrive at estimate in the statement of the lemma. To this end, we split the integral from 0 to x into three integrals, one on $(0,4h/3)$, one from $(4h/3,2x/3)$, and final one on $(2x/3,x)$. The idea behind splitting the domain in this way, is that only one of the factors from the integrand will have a singularity on that domain at a time, making them easier to work with. This difference in how we have to split the integral over $(0,x)$, as compared to corresponding integral in the proof of Lemma 4.11, is a consequence of the singularity $|x-y|^{-1/2}$ from $u'(x-y)$, which we did not have to deal with in the previous section.

We begin with the integral over $(0, 4h/3)$. Using Lemma 5.1 to bound $|\delta_{2y}u'(x)|$, and that $h < x/2$, we get

$$\begin{aligned}
&\left|\int_0^{4h/3} \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\,\mathrm{d}y\right| \\
&\qquad \lesssim (x-h)\int_0^{4h/3} \left|\delta_{2h}K_{\text{sing}}(y)\right| \frac{y^{1/2}}{(x-y)}\,\mathrm{d}y \\
&\qquad \le x\int_0^{4h/3} \left|\delta_{2h}K_{\text{sing}}(y)\right| \frac{y^{1/2}}{(x-\frac{2x}{3})}\,\mathrm{d}y \\
&\qquad \simeq \int_0^{4h/3} \left|\delta_{2h}K_{\text{sing}}(y)\right| y^{1/2}\,\mathrm{d}y \\
&\qquad \simeq h,
\end{aligned}$$

where the final equality follows from the same calculation as in (4.29).

To bound the integral over $(2x/3, x)$ we use the estimate at the origin,

$$\begin{aligned}
&\left|\int_{2x/3}^{x} \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\,\mathrm{d}y\right| \\
&\qquad \le (x-h)\int_{2x/3}^{x} \left|\delta_{2h}K_{\text{sing}}(y)\right|\left|\delta_{2y}u'(x)\right|\,\mathrm{d}y \\
&\qquad \lesssim x\int_{2x/3}^{x} \left|\delta_{2h}K_{\text{sing}}(y)\right||x-y|^{-1/2}\,\mathrm{d}y \\
&\qquad = x\int_{\frac{2x}{3h}}^{x/h} \left|\delta_{2h}K_{\text{sing}}(\tau h)\right|\frac{1}{(x-\tau h)^{1/2}}h\,\mathrm{d}\tau \\
&\qquad \simeq (xh)^{1/2}\int_{\frac{2x}{3h}}^{x/h} \left|\delta_2\left(|\tau|^{-1/2}\right)\right|\frac{1}{(1-\tau\frac{h}{x})^{1/2}}\,\mathrm{d}\tau \\
&\qquad \simeq (xh)^{1/2}\int_{\frac{2x}{3h}}^{x/h} \frac{1}{(\tau-1)^{3/2}}\frac{1}{(1-\tau\frac{h}{x})^{1/2}}\,\mathrm{d}\tau \\
&\qquad \le (xh)^{1/2}\frac{1}{(\frac{2x}{3h}-1)^{3/2}}\int_{\frac{2x}{3h}}^{x/h} \frac{1}{(1-\tau\frac{h}{x})^{1/2}}\,\mathrm{d}\tau \\
&\qquad = (xh)^{1/2}\frac{1}{(\frac{x}{h})^{3/2}(\frac{2}{3}-\frac{h}{x})^{3/2}}\left[\frac{-2x}{h}(1-\tau\frac{h}{x})^{1/2}\right]_{\frac{2x}{3h}}^{x/h} \\
&\qquad \lesssim h.
\end{aligned}$$

The third equality follows from rewriting $|\delta_2(|\tau|^{-1/2})|$ as was done in (4.30).

For the integral over $(4h/3, 2x/3)$, we use Lemma 5.1 to find that

$$\begin{aligned}
&\left|\int_{4h/3}^{2x/3} \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\,\mathrm{d}y\right| \\
&\lesssim (x-h)\int_{4h/3}^{2x/3} \left|\delta_{2h}K_{\text{sing}}(y)\right|\frac{y^{1/2}}{(x-y)}\,\mathrm{d}y \lesssim \int_{4h/3}^{2x/3} \left|\delta_{2h}K_{\text{sing}}(y)\right| y^{1/2}\,\mathrm{d}y \lesssim h\Big(1+\left|\ln\Big(\frac{h}{x}\Big)\right|\Big).
\end{aligned}$$

The second inequality follows from y being bounded strictly away from x, and the final inequality follows from the same calculation as in (4.30) (the domain of integration is slightly different, but not in any way that affects the calculation). As was the case in Lemma (4.11), this bound is not quite good enough yet, but combining the bounds so far gives us

$$|x^{1/2}(x-h)(u'(x+y)-u'(x-y))| \lesssim h\Big(1+\left|\ln\Big(\frac{h}{x}\Big)\right|\Big),$$

which we insert back into calculations above to find that

$$\left|\int_{4h/3}^{2x/3} \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)(x-h)\,\mathrm{d}y\right| \lesssim h,$$

by a calculation similar to (4.31).

Thus we have arrived at the improved estimate, as we have managed to show that the integral over $(0, x)$ from the central difference equation satisfied by u' is also bounded by h. $\square$

Step 4. We are now ready to show the u''-limit.

Proposition 5.3. *The second derivative of the solution u enjoys the limit*

$$\lim_{x\to 0} \frac{u''(x)}{x^{-3/2}} = -\frac{1}{4}\sqrt{\frac{\pi}{8}}. \tag{5.5}$$

Proof. We consider the central difference equation satisfied by u', divided by $2h$ and multiplied by x,

$$\begin{aligned}
\frac{2xu(x+h)u'(x+h) - 2xu(x-h)u'(x-h)}{2h} &= -\frac{1}{2h}\int_0^\infty \delta_{2h}K(y)\delta_{2y}u'(x)x \,\mathrm{d}y \\
= -\frac{1}{2h}\Big(\int_0^{4h/3} + \int_{4h/3}^{2x/3} + \int_{2x/3}^{x} + \int_x^{2\nu}\Big)&\delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u'(x)x \,\mathrm{d}y \\
-\frac{1}{2h}\int_0^{2\nu} \delta_{2h}K_{\mathrm{reg}}(y)\delta_{2y}u'(x)x \,\mathrm{d}y &- \frac{1}{2h}\int_{2\nu}^\infty \delta_{2h}K(y)\delta_{2y}u'(x)x \,\mathrm{d}y.
\end{aligned} \tag{5.6}$$

for all $0 < 2h < x \le \nu$. We know from (5.3) and (5.4) at the beginning of this section that

$$\lim_{x\to 0}\lim_{h\to 0} \frac{2xu(x+h)u'(x+h) - 2xu(x-h)u'(x-h)}{2h} = \frac{\pi}{16} + \sqrt{\frac{\pi}{2}} \lim_{x\to 0}\frac{u''(x)}{x^{-3/2}}.$$

Our new estimates will allow us to bound the right-hand side integrands by integrable functions, so that we can move the limits inside the integrals by dominated convergence.

The first integral can be written as

$$-\frac{1}{2h}\int_0^{4h/3} \delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u'(x)x \,\mathrm{d}y = \int_0^{4/3} \frac{-1}{\sqrt{8\pi}h^{1/2}}\delta_2\Big(|\tau|^{-1/2}\Big)\delta_{2\tau h}u'(x)x \,\mathrm{d}\tau,$$

for all $0 < 2h < x \le \nu$, by the same calculation as in (4.33). Using Lemma 5.1 to bound $|\delta_{2h}u'(x)|$, the integrand can be bounded by

$$\begin{aligned}
\Big| -\frac{1}{\sqrt{8\pi}h^{1/2}}\delta_2\Big(|\tau|^{-1/2}\Big)\delta_{2\tau h}u'(x)x\Big| &\lesssim x\Big|\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\frac{(\tau h)^{1/2}}{(x-\tau h)} \\
&= \Big|\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\frac{x}{x}\frac{\tau^{1/2}}{(1-\tau\frac{h}{x})} \\
&\le \Big|\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\frac{\tau^{1/2}}{(1-\frac{4}{3}\cdot\frac{1}{2})} \\
&\simeq \Big|\delta_2\Big(|\tau|^{-1/2}\Big)\Big|\tau^{1/2}.
\end{aligned}$$

The right-hand side is integrable on $(0, 4/3)$, so dominated convergence allows us to interchange the limits and integral. Thus, for each $0 < x \le \nu$ and $\tau \in (0, 4/3)$,

$$\lim_{h\to 0}\frac{-1}{\sqrt{8\pi}h^{1/2}}\delta_2\Big(|\tau|^{-1/2}\Big)\delta_{2\tau h}u'(x)x = \lim_{h\to 0}\frac{-2h^{1/2}}{\sqrt{8\pi}}\delta_2\Big(|\tau|^{-1/2}\Big)\tau x\frac{\delta_{2\tau h}u'(x)}{2\tau h} = 0,$$

since $u'(x)$ is differentiable for $0 < x < P$.

The second integral can be written as

$$-\frac{1}{2h}\int_{4h/3}^{2x/3} \delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u'(x)x \,\mathrm{d}y = \int_{\frac{4h}{3x}}^{2/3} \frac{-\delta_{2h}K_{\mathrm{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2 \,\mathrm{d}\tau.$$

Similarly to calculation (4.34), the integrand can be expressed as

$$\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2 = \frac{1}{\sqrt{2\pi}}\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}x^{1/2}\delta_{2\tau x}u'(x),$$

which, similarly to calculation (4.35), is bounded by

$$\begin{aligned}
&\left|\frac{1}{\sqrt{2\pi}}\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}x^{1/2}\delta_{2\tau x}u'(x)\right|\\
&\quad\lesssim\left|\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\right|x^{1/2}\frac{\tau x}{x^{1/2}(x-\tau x)}\\
&\quad\lesssim\frac{1}{\tau^{1/2}(1-\tau)},
\end{aligned}$$

for all $0 < 2h < x \leq \nu$ and $\frac{4h}{3x} < \tau < 2/3$, where we have used the improved estimate of Lemma 5.2 to bound $|\delta_{2\tau x}u'(x)|$. Thus, we have that the original integrand is dominated by the final expression, which is integrable on $(0, 2/3)$, allowing us to move the limits inside. For each $x \in (0, \nu]$ and $\tau \in (0, 2/3)$, we then have that

$$\lim_{h\to 0}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2 = \frac{1}{\sqrt{2\pi}}\frac{x^{1/2}\delta_{2\tau x}u'(x)}{2\tau^{3/2}},$$

and for each $\tau \in (0, 2/3)$ we get

$$\lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2\tau^{3/2}}x^{1/2}\delta_{2\tau x}u'(x) = \frac{1}{16\tau^{3/2}}\left(\frac{1}{(1+\tau)^{1/2}}-\frac{1}{(1-\tau)^{1/2}}\right).$$

Both limits follow from calculations similar to (4.36) and (4.37) respectively. In summary, the second integral can be expressed as

$$\lim_{x\to 0}\lim_{h\to 0}\int_{2h/x}^{2/3}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2\ \mathrm{d}\tau = \int_0^{2/3}\frac{1}{16\tau^{3/2}}\left((1+\tau)^{-1/2}-(1-\tau)^{-1/2}\right)\mathrm{d}\tau.$$

For the third integral we also make the change of variables $y = \tau x$,

$$-\frac{1}{2h}\int_{2x/3}^{x}\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)x\ \mathrm{d}y = \int_{2/3}^{1}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2\ \mathrm{d}\tau.$$

As the integrand is the same as for the second integral above, it can be expressed in the same way. However, when bounding the integrand we now need to use the estimate at the origin for the final bound to be integrable on $(2/3, 1)$:

$$\begin{aligned}
&\left|\frac{1}{\sqrt{2\pi}}\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}x^{1/2}\delta_{2\tau x}u'(x)\right|\\
&\quad\lesssim\left|\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}\right|x^{1/2}\frac{1}{(x-\tau x)^{1/2}}\\
&\quad\lesssim\frac{1}{\tau^{3/2}}\frac{1}{(1-\tau)^{1/2}},
\end{aligned}$$

for all $0 < 2h < x \leq \nu$ and $2/3 < \tau < 1$. Thus, we have that the original integrand is dominated by the final expression, which is integrable on $(2/3, 1)$, allowing us to move the limits inside. As $\tau < 1$ and the integrand is exactly the same as for the second integral, only now with $\tau \in (2/3, 1)$, we get the same limits. That is, for the third integral we have

$$\lim_{x\to 0}\lim_{h\to 0}\int_{2/3}^{1}\frac{-\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u'(x)x^2\ \mathrm{d}\tau = \int_{2/3}^{1}\frac{1}{16\tau^{3/2}}\left((1+\tau)^{-1/2}-(1-\tau)^{-1/2}\right)\mathrm{d}\tau.$$

Moving on to the fourth integral, we see it can be written as

$$-\frac{1}{2h}\int_x^{\nu} \delta_{2h}K_{\rm sing}(y)\delta_{2y}u'(x)x\ \mathrm{d}y = -\frac{1}{2h}\int_1^{\nu/x} \delta_{2h}K_{\rm sing}(\tau x)\delta_{2\tau x}u'(x)x^2\ \mathrm{d}\tau,$$

by the change of variables $y=\tau x$. Once more using the estimate at the origin to bound $|\delta_{2\tau x}u'(x)|$, the same calculation as for the third integral above gives the bound

$$\Big|-\frac{1}{2h}\delta_{2h}K_{\rm sing}(\tau x)\delta_{2x}u(\tau x)x^2\Big| \lesssim \frac{1}{\tau^{3/2}}\frac{1}{(\tau-1)^{1/2}},$$

for all $0<2h<x\leq\nu$ and $1<\tau<\nu/x$ (notice that we now have $(\tau-1)^{1/2}$ instead of $(1-\tau)^{1/2}$, as we are working with $\tau>1$). Since the right-hand side is integrable on $[1,\infty)$, we can interchange the limits and integral. The h-limit remains the same as for the third integral, but now for each fixed $x\in(0,\nu]$ and $\tau\in(1,\nu/x)$. The x-limit is similar to that of the third integral, except that we now have $(\tau-1)^{-1/2}$ instead of $(1-\tau)^{-1/2}$ due to $\tau>1$. In total, the fourth integral can be expressed as

$$\lim_{x\to 0}\lim_{h\to 0}\int_1^{\nu/x} \frac{-\delta_{2h}K_{\rm sing}(\tau x)}{2h}\delta_{2\tau x}u'(x)(x-h)x\ \mathrm{d}\tau = \int_1^{\infty}\frac{1}{16\tau^{3/2}}\Big((1+\tau)^{-1/2}+(\tau-1)^{-1/2}\Big)\ \mathrm{d}\tau.$$

The integrand of the fifth integral is dominated by a function that is integrable on $(0,2\nu)$ by the same argument as for the third integral in Lemma 5.1, whence

$$\lim_{x\to 0}\lim_{h\to 0}\int_0^{2\nu} \frac{-\delta_{2h}K_{\rm reg}(y)}{2h}\delta_{2y}u'(x)x\ \mathrm{d}y = 0$$

follows.

Similarly, for the final integral we find that

$$\lim_{x\to 0}\lim_{h\to 0}\int_{2\nu}^{\infty} \frac{-\delta_{2h}K(y)}{2h}\delta_{2y}u'(x)x\ \mathrm{d}y = 0,$$

as the dominated convergence theorem holds by same argument used for the final integral in the proof of Lemma 5.1.

Thus, each of the six right-hand side limits exist, and combining the integrals gives us

$$\frac{\pi}{16}+\sqrt{\frac{\pi}{2}}\lim_{x\to 0}\frac{u''(x)}{x^{-3/2}} = \int_0^{\infty}\frac{1}{16\tau^{3/2}}\Big((1+\tau)^{-1/2}+\mathrm{sgn}(\tau-1)|\tau-1|^{-1/2}\Big)\ \mathrm{d}\tau = 0. \tag{5.7}$$

After rearranging, this becomes

$$\lim_{x\to 0}\frac{u''(x)}{x^{-3/2}} = -\frac{1}{4}\sqrt{\frac{\pi}{8}},$$

which concludes the proof. □

6 The limit for the third derivative

With the u''-limit at hand, we now wish to determine the u'''-limit. As we have alluded to earlier, we will encounter some problems that prevent us from following the procedure of the previous sections. As a consequence we will need to alter the first of the four main steps listed in the introduction, that is, we will need to change our approach for finding a central difference equation satisfied by u''. This new way of finding a "good" central difference equation is the most important idea presented in this paper, and was also the most difficult part of the research process. Still, before we take a look at the altered approach, let us try to proceed as in Section 4.1 and 5 to get a better sense of the issues that arise.

Step 1 (Original). To get the central difference equation, we first need an integral equation satisfied by u''. In a similar manner to Step 1 in Section 5, we take only the h-limit of (5.6) to get the integral equation

$$\begin{aligned} & 2(u'(x))^2 + 2u(x)u''(x) \\ & = -\int_0^\infty K'(y)\delta_{2y}u'(x)\ \mathrm{d}y \qquad (6.1) \\ & = \int_0^\infty \big(K(y+x) + K(y-x)\big)u''(y)\ \mathrm{d}y. \qquad (6.2) \end{aligned}$$

In (6.1) we have left the central difference in h on K to get K' after taking the limit, while in (6.2) we moved h over to u' before we took the limit to get u'' (it is (6.2) here that corresponds to (5.1)). As we will see, both expressions cause problems. The central difference equations are then

$$\begin{aligned} & 2(xu'(x+h))^2 + 2x^2u(x+h)u''(x+h) - 2(xu'(x-h))^2 - 2x^2u(x-h)u''(x-h) \\ & = \int_0^\infty \delta_{2h}K'(y)\delta_{2x}u'(y)x^2\ \mathrm{d}y \qquad (6.3) \\ & = -\int_0^\infty \delta_{2h}K(y)\delta_{2x}u''(y)x^2\ \mathrm{d}y, \qquad (6.4) \end{aligned}$$

where (6.3) is the central difference of (6.4), and (6.4) is the central difference of (6.2). Notice that we have multiplied the equation by x^2.

To show how things go wrong, we split the integral in (6.3) into a part over $(0, 4h/3)$ (where $0 < 2h < x$). To show the first estimate we would have to bound this part of the integral by $(xh)^{1/2}$, but when we try to do so we get

$$\begin{aligned} \left|\int_0^{4h/3} \delta_{2h}K'_{\mathrm{sing}}(y)\delta_{2x}u'(y)x^2\ \mathrm{d}y\right| & \lesssim \int_0^{4h/3} \left|\frac{1}{|y+h|^{3/2}} - \frac{\mathrm{sgn}(y-h)}{|y-h|^{3/2}}\right|(x-y)^{-3/2}x^2\ \mathrm{d}y \qquad (6.5) \\ & \lesssim \left(\frac{x}{h}\right)^{1/2} \int_0^{4/3} \left|\frac{1}{|\tau+1|^{3/2}} - \frac{\mathrm{sgn}(\tau-1)}{|\tau-1|^{3/2}}\right|\ \mathrm{d}\tau, \end{aligned}$$

which does not converge. Now consider the integral (6.3) instead. If we split the integral in (6.3) into a part over $(2x/3, x)$, we would have to show that this integral is bounded by $(xh)^{1/2}$ in the proof for the first estimate. However, if we try to bound this integral using the estimate at the origin for u'', mirroring what was done in the preceding sections, we would find that

$$\begin{aligned} \left|-\int_{2x/3}^{x} \delta_{2h}K(y)\delta_{2x}u''(y)x^2\ \mathrm{d}y\right| & \lesssim (xh)^{1/2} \int_{\frac{2x}{3h}}^{x/h} \left|\delta_2\left(|\tau|^{-1/2}\right)\right| \frac{1}{(1-\tau\frac{h}{x})^{3/2}}\ \mathrm{d}\tau \qquad (6.6) \\ & \le (xh)^{1/2} \frac{1}{\left(\frac{2x}{3h}-1\right)^{3/2}} \left[\frac{2x}{h}\left(1-\tau\frac{h}{x}\right)^{-1/2}\right]_{\frac{2x}{3h}}^{x/h}, \end{aligned}$$

which also does not converge. As we can see, no matter which formulation we choose, there is a non-integrable singularity of type $|x|^{-3/2}$ at $y = x$ or $y = h$ that prevents us from proceeding as in the previous section. Of course, these calculations do not show that that the integrals above are not bounded; they only show that a potential bound can not be established as easily as in the previous sections.

One might try integration by parts on the integrals on the left-hand side of (6.5) and (6.6) to move the non-integrable singularity over to the factor that does not contain its singularity inside the domain. Unfortunately, this does still not work. To get a sense for why this is, consider the integrals on the right-hand side of the final equations we encountered when determining the u' and u''-limits, (4.39) and (5.7). If we were to formally derive the corresponding integral for the general $u^{(n+1)}$-limit using the same procedure, we would find it to be

$$\int_0^\infty \tau^{-3/2}\left((\tau+1)^{\frac{1}{2}-n} - (\mathrm{sgn}(1-\tau))^n|\tau-1|^{\frac{1}{2}-n}\right)\ \mathrm{d}\tau, \qquad (6.7)$$

times some constant factor which we ignore. One can try to split this integral at a point to isolate the singularities, say 2/3. If we then and formally integrate by parts n times the integral over

$(2/3, \infty)$, disregarding the boundary terms, we would find that

$$\begin{aligned}&\int_0^\infty \tau^{-3/2}\Big((\tau+1)^{\frac12-n} - (\operatorname{sgn}(1-\tau))^n|\tau-1|^{\frac12-n}\Big)\,\mathrm{d}\tau\\ &\simeq \int_0^{\frac23} \tau^{-\frac32}\Big((\tau+1)^{\frac12-n} - (1-\tau)^{\frac12-n}\Big)\,\mathrm{d}\tau + \int_{\frac23}^\infty \tau^{-n-\frac32}\Big((-1)^{n-1}(\tau+1)^{\frac12} + |\tau-1|^{\frac12}\Big)\,\mathrm{d}\tau.\end{aligned} \tag{6.8}$$

These integrals do indeed converge, so why does approach not work? Unfortunately, when we take the boundary terms into consideration one immediately sees that these explode, so we are not allowed to integrate by parts. Thus, it is clear that we need a somewhat altered approach to make headway with the u'''-limit.

Step 1 (Altered). For the u''-limit, we first found the central difference equation satisfied by u', *then* we split the integral in this equation in a way that isolated the singularities (recall (5.6)). The idea that will allow us to make progress on the u'''-limit, is to reverse this order; we split the previous central difference equation in such a way as to isolate the singularities *before* we move on to the new central difference equation. When reversing the order, we can move h over to the factor with a non-integrable singularity that is strictly outside of the domain of integration before taking any derivatives. In fact, even though we are interested in the u'''-limit, we will return all the way to the central difference equation (4.23) satisfied by u.

Still considering $0 < 2h < x \le \nu$, we split the integral in (4.23) as

$$\begin{aligned}&u(x+h)^2 - u(x-h)^2 = -\int_0^\infty \delta_{2h}K(y)\delta_{2x}u(y)\,\mathrm{d}y\\ &= -\int_0^{4h/3} \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\,\mathrm{d}y - \int_{4h/3}^{2x/3} \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\,\mathrm{d}y\\ &\quad - \int_{2x/3}^{2\nu} \delta_{2h}K_{\text{sing}}(y)\delta_{2x}u(y)\,\mathrm{d}y - \int_0^{2\nu} \delta_{2h}K_{\text{reg}}(y)\delta_{2x}u(y)\,\mathrm{d}y - \int_{2\nu}^\infty \delta_{2h}K(y)\delta_{2x}u(y)\,\mathrm{d}y.\end{aligned} \tag{6.9}$$

To get an equation we can use to study the u'''-limit, we want a central difference equation satisfied by u''. To achieve this, we will differentiate the above central difference equation with respect to x twice (we will differentiate formally under the integral sign, and justify this later). But before we differentiate with respect to x twice, we need to rewrite some of the integrals so that the derivatives end up on the correct factors. A derivative will appear from both the differentiation with respect to x, as well as when we later divide by $2h$ and let h tend to zero. We therefore need to make sure that both h and x are moved to the correct factor in the above integrals.

The exact splitting of the integral in (6.9) is inspired by the splitting of the integral in (5.6)[1], and the plan is now to transform the integrals in (6.9) as follows: In the integral over $(0, 4h/3)$, $\delta_{2h}K_{\text{sing}}(y)$ has a singularity at h. In this integral we therefore want to move h over to u so we later end up with the derivative on u, whose singularity located at $y = x$ (which is bounded away from the domain of integration). We keep the integral over $(4h/3, 2x/3)$ exactly as it is, as the factors do not contain any singularities in this domain. In the integral over $(2x/3, 2\nu)$ we want to move x over to K_{sing}, so that the derivatives are placed on K_{sing} when differentiating with respect to x, to avoid the non-integrable singularity at $y = x$ in the derivatives of u. In the integral on $(0, 2\nu)$ we move x to K_{reg}, as K_{reg} is smooth with decaying derivatives. Similarly, in the final integral on $(2\nu, \infty)$ we move x over to K, as $\delta_{2h}K(y)$ is smooth away from $y = h$ and since all the derivatives of K are exponentially decaying.

Following this plan one can show, after many changes of variables and tedious, but straightforward, calculations, that (6.9) can be rewritten as

$$\begin{aligned}u(x+h)^2 - u(x-h)^2 = &\int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u(x)\,\mathrm{d}y\\ &+ \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u(x)\,\mathrm{d}y\end{aligned}$$

[1] We note that there might be different ways of splitting the integral in (6.9) that make the following calculations less tedious. However, this splitting, where we not only isolate the singularities, but also decompose K into the singular and regular part, has been the easiest to work with of all expressions that we have experimented with, as it allows us to reuse much of the calculations from earlier sections.

$$
\begin{aligned}
&+ \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K_{\text{sing}}(x+y) + \delta_{2h}K_{\text{sing}}(x-y)\big)\ \mathrm{d}y \\
&+ \int_{-x/3}^{5x/3} \delta_{2h}K_{\text{sing}}(x+y)u(y)\ \mathrm{d}y \\
&+ \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K_{\text{reg}}(x+y) + \delta_{2h}K_{\text{reg}}(x-y)\big)\ \mathrm{d}y \\
&+ \int_{-x}^{x} \delta_{2h}K_{\text{reg}}(x+y)u(y)\ \mathrm{d}y \\
&+ \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K(x+y) + \delta_{2h}K(x-y)\big)\ \mathrm{d}y.
\end{aligned}
$$

Taking the derivative of this equation with respect to x (which we will justify later) gives us

$$
\begin{aligned}
&2u(x+h)u'(x+h) - 2u(x-h)u'(x-h) \\
&\qquad = \int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u'(x)\ \mathrm{d}y \\
&\qquad + \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x)\ \mathrm{d}y \\
&\qquad + \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K'_{\text{sing}}(x+y) + \delta_{2h}K'_{\text{sing}}(x-y)\big)\ \mathrm{d}y \\
&\qquad + \int_{-x/3}^{5x/3} \delta_{2h}K'_{\text{sing}}(x+y)u(y)\ \mathrm{d}y \\
&\qquad + \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K'_{\text{reg}}(x+y) + \delta_{2h}K'_{\text{reg}}(x-y)\big)\ \mathrm{d}y \\
&\qquad + \int_{-x}^{x} \delta_{2h}K'_{\text{reg}}(x+y)u(y)\ \mathrm{d}y \\
&\qquad + \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K'(x+y) + \delta_{2h}K'(x-y)\big)\ \mathrm{d}y \\
&\qquad + \delta_{2h}K_{\text{sing}}\Big(\frac{2}{3}x\Big)\bigg(u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big)\bigg),
\end{aligned}
$$

where the final term comes from the boundary terms. Differentiating once more gives us

$$
\begin{aligned}
&2u'(x+h)^2 + 2u(x+h)u''(x+h) - 2u'(x-h)^2 - 2u(x-h)u''(x-h) \\
&\qquad = \int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u''(x)\ \mathrm{d}y \\
&\qquad + \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)\ \mathrm{d}y \\
&\qquad + \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\text{sing}}(x+y) + \delta_{2h}K''_{\text{sing}}(x-y)\big)\ \mathrm{d}y \\
&\qquad + \int_{-x/3}^{5x/3} \delta_{2h}K''_{\text{sing}}(x+y)u(y)\ \mathrm{d}y \qquad (6.10)\\
&\qquad + \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\text{reg}}(x+y) + \delta_{2h}K''_{\text{reg}}(x-y)\big)\ \mathrm{d}y \\
&\qquad + \int_{-x}^{x} \delta_{2h}K''_{\text{reg}}(x+y)u(y)\ \mathrm{d}y \\
&\qquad + \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K''(x+y) + \delta_{2h}K''(x-y)\big)\ \mathrm{d}y \\
&\qquad + \delta_{2h}K_{\text{sing}}\Big(\frac{2}{3}x\Big)\bigg(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big)\bigg)
\end{aligned}
$$

$$+\,\delta_{2h}K'_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big(-u\Big(\frac{5}{3}x\Big)+u\Big(\frac{x}{3}\Big)\Big),$$

which is the central difference equation we will use to study the u'''-limit. The right-hand side now has seven integrals as well as some boundary terms we need to consider, which is somewhat more tedious than in the previous section, though still manageable.

Consider the left-hand side of (6.10). Multiplying by $1/2h$ and letting h tend to zero, we get

$$\lim_{h\to 0}\frac{2u'(x+h)^2+2u(x+h)u''(x+h)-2u'(x-h)^2-2u(x-h)u''(x-h)}{2h} \tag{6.11}$$
$$= 6u'(x)u''(x)+2u(x)u'''(x).$$

Multiplying this by x^2 (for the same reason we had to multiply by x to get (5.4)) and letting x tend to zero, we get

$$\begin{aligned}\lim_{x\to 0}6x^2u'(x)u''(x)+2x^2u(x)u'''(x) &= \lim_{x\to 0}6x^{1/2}u'(x)x^{3/2}u''(x)+2x^{-1/2}u(x)x^{5/2}u'''(x)\\ &= 6\frac{1}{2}\sqrt{\frac{\pi}{8}}\frac{-1}{4}\sqrt{\frac{\pi}{8}}+2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u'''(x)}{x^{-5/2}}\\ &= -\frac{3}{4}\frac{\pi}{8}+2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u'''(x)}{x^{-5/2}}.\end{aligned} \tag{6.12}$$

Remark 6.1. So far we have been studying the limits for u, u', u'' and u'''. Consider the h-limits the respective central difference equations used to study their limits:

$$\begin{aligned}
n=0:&\quad u(x)^2 &&= u(x)^2,\\
n=1:&\quad 2u(x)u'(x) &&= u(x)u'(x)+u'(x)u(x),\\
n=2:&\quad 2u'(x)^2+2u(x)u''(x) &&= u(x)u''(x)+2u'(x)^2+u''(x)u(x),\\
n=3:&\quad 6u'(x)u''(x)+2u(x)u'''(x) &&= u(x)u'''(x)+3u'(x)u''(x)+3u''(x)u'(x)+u'''(x)u(x),
\end{aligned}$$

where n denotes the order of the derivative of u we are studying. The right-hand sides seem to suggest that the general left-hand side to study the $u^{(n)}$-limit should be

$$\sum_{i=0}^{n}\binom{n}{i}u^{(i)}u^{(n-i)}.$$

Before we can divide the right-hand side of (6.10) by $2h$ and take h- and x-limit, we again need two estimates.

Step 2. We begin by showing the first estimate for the central difference of u''.

Lemma 6.2. *There is some $\nu > 0$ so that*

$$|(x-h)^2(u''(x+h)-u''(x-h))| \lesssim h^{1/2}$$

uniformly on $[0,\nu]$.

Proof. Consider $0 < h \le x \le \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. The estimate follows directly from the u''-limit for large h, say $h \in [x/2, x]$. Indeed, for such h we have that

$$\begin{aligned}&|(x-h)^2(u''(x+h)-u''(x-h))|\\ &\lesssim |x-h|^2(|x+h|^{-3/2}+|x-h|^{-3/2}) \lesssim |x-h|^2|x-h|^{-3/2} \le |2h-h|^{1/2} = |h|^{1/2},\end{aligned}$$

after possibly shrinking ν. We now show that the claim also holds for $h < x/2$.

Consider the central difference equation (6.10) from above multiplied by $(x-h)^2$:

$$2(x-h)^2u'(x+h)^2+2(x-h)^2u(x+h)u''(x+h)$$

$$
\begin{aligned}
&\qquad - 2(x-h)^2 u'(x-h)^2 - 2(x-h)^2 u(x-h)u''(x-h) \\
&= \int_{-h/3}^{7h/3} -K_{\mathrm{sing}}(y)\delta_{2(y-h)}u''(x)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{4h/3}^{2x/3} -\delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u''(x)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\mathrm{sing}}(x+y) + \delta_{2h}K''_{\mathrm{sing}}(x-y)\big)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{-x/3}^{5x/3} \delta_{2h}K''_{\mathrm{sing}}(x+y)u(y)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\mathrm{reg}}(x+y) + \delta_{2h}K''_{\mathrm{reg}}(x-y)\big)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{-x}^{x} \delta_{2h}K''_{\mathrm{reg}}(x+y)u(y)(x-h)^2 \,\mathrm{d}y \\
&\quad + \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K''(x+y) + \delta_{2h}K''(x-y)\big)(x-h)^2 \,\mathrm{d}y \\
&\quad + \delta_{2h}K_{\mathrm{sing}}\Big(\frac{2}{3}x\Big)\Big(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big)\Big)(x-h)^2 \\
&\quad + \delta_{2h}K'_{\mathrm{sing}}\Big(\frac{2}{3}x\Big)\Big(-u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big)\Big)(x-h)^2
\end{aligned}
$$

Using that $u(x \pm h) \simeq x^{1/2}$ and $u'(x \pm h) \simeq x^{-1/2}$ for $h < x/2$ (after possibly shrinking ν), the left-hand side can be written as

$$
\begin{aligned}
&|2(x-h)^2 u'(x+h)^2 + 2(x-h)^2 u(x+h)u''(x+h) \\
&\qquad - 2(x-h)^2 u'(x-h)^2 - 2(x-h)^2 u(x-h)u''(x-h)| \\
&\simeq |x^{1/2}(x-h)^2(u''(x+h) - u''(x-h))|.
\end{aligned}
$$

Now we consider the right-hand side.

Using the binomial theorem to expand $\delta_{2h}K_{\mathrm{sing}}$ and $\delta_{2h}K'_{\mathrm{sing}}$ shows that both of the boundary terms tend to zero on the order of h.

Using the estimate at the origin for u'', the first integral can be bounded by

$$
\begin{aligned}
&\left| \int_{-h/3}^{7h/3} - K_{\mathrm{sing}}(y)\delta_{2(y-h)}u''(x)(x-h)^2 \,\mathrm{d}y \right| \\
&\qquad \le x^2 \int_{-h/3}^{7h/3} |K_{\mathrm{sing}}(y)|(|x+y-h|^{-3/2} + |x-y+h|^{-3/2}) \,\mathrm{d}y \\
&\qquad \lesssim x^2 \int_{-h/3}^{7h/3} |K_{\mathrm{sing}}(y)|\Big|x - \frac{2x}{3}\Big|^{-3/2} \,\mathrm{d}y \\
&\qquad \simeq (xh)^{1/2} \int_{-1/3}^{7/3} |K_{\mathrm{sing}}(\tau)| \,\mathrm{d}\tau \\
&\qquad \simeq (xh)^{1/2},
\end{aligned}
$$

since the final integral converges.

Using the estimate at the origin for u'', the second integral can be bounded by

$$
\begin{aligned}
&\left| \int_{4h/3}^{2x/3} - \delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u''(x)(x-h)^2 \,\mathrm{d}y \right| \\
&\qquad \le x^2 \int_{4h/3}^{2x/3} |\delta_{2h}K_{\mathrm{sing}}(y)|(|x+y|^{-3/2} + |x-y|^{-3/2}) \,\mathrm{d}y \\
&\qquad \lesssim x^2 \int_{4h/3}^{2x/3} |\delta_{2h}K_{\mathrm{sing}}(y)|\Big|x - \frac{2x}{3}\Big|^{-3/2} \,\mathrm{d}y \qquad (6.13)
\end{aligned}
$$

$$
\begin{aligned}
&\simeq (xh)^{1/2} \int_{4/3}^{\frac{2x}{3h}} \left|\delta_2\left(|\tau|^{-1/2}\right)\right| \, \mathrm{d}\tau \\
&\lesssim (xh)^{1/2},
\end{aligned}
$$

since the final integral converges on $(4/3, \infty)$.

Using the estimate at the origin for u, the third integral can be bounded by

$$
\begin{aligned}
&\left| \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\mathrm{sing}}(x+y) + \delta_{2h}K''_{\mathrm{sing}}(x-y)\big)(x-h)^2 \, \mathrm{d}y \right| \\
&\qquad \lesssim x^2 \int_{5x/3}^{2\nu+x} y^{1/2}\Big(\frac{h}{(x+y-\xi h)^{7/2}_{\xi\in(-1,1)}} + \frac{h}{|x-y-\xi h|^{7/2}_{\xi\in(-1,1)}}\Big) \, \mathrm{d}y \\
&\qquad \lesssim x^2 h \int_{5x/3}^{2\nu+x} \frac{y^{1/2}}{(y-x-h)^{7/2}} \, \mathrm{d}y \\
&\qquad \leq x^2 h \int_{5x/3}^{2\nu+x} \frac{y^{1/2}}{(y-\frac{9}{10}y)^{7/2}} \, \mathrm{d}y \\
&\qquad \simeq x^2 h \int_{5/3}^{\frac{2\nu}{x}+1} \frac{1}{(x\tau)^3} x \, \mathrm{d}\tau \\
&\qquad \lesssim h \leq (xh)^{1/2},
\end{aligned}
$$

since the final integral converges on $(5/3, \infty)$. In the first inequality we have used the mean value theorem on $\delta_{2h}K''_{\mathrm{sing}}(x \pm y)$.

By a similar argument as for the third integral, we can bound the fourth integral by

$$
\begin{aligned}
&\left| \int_{-x/3}^{5x/3} \delta_{2h}K''_{\mathrm{sing}}(x+y)u(y)(x-h)^2 \, \mathrm{d}y \right| \\
&\qquad \lesssim \frac{h}{x} \int_{-x/3}^{5x/3} 1 \, \mathrm{d}y = \frac{h}{x}\Big(\frac{5x}{3} - \frac{-x}{3}\Big) \simeq h \leq (xh)^{1/2}.
\end{aligned}
$$

Since K_{reg} is smooth with decaying derivatives, and K is smooth away from its singularity with exponentially decaying derivatives, we can bound the fifth, sixth and final integral by

$$
\begin{aligned}
&\left| \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\mathrm{reg}}(x+y) + \delta_{2h}K''_{\mathrm{reg}}(x-y)\big)(x-h)^2 \, \mathrm{d}y \right| \lesssim h \leq (xh)^{1/2}, \\
&\left| \int_{-x}^{x} \delta_{2h}K''_{\mathrm{reg}}(x+y)u(y)(x-h)^2 \, \mathrm{d}y \right| \lesssim h \leq (xh)^{1/2}, \\
&\left| \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K''(x+y) + \delta_{2h}K''(x-y)\big)(x-h)^2 \, \mathrm{d}y \right| \lesssim h \leq (xh)^{1/2},
\end{aligned}
$$

through calculations similar to (4.27), (4.27) and (4.28), respectively. Here we see one advantage of the central difference equation (6.10) from our altered approach; when bounding the final integral we can simply use that u is bounded, so we do not need to go through the same tedious calculations as we did for the final integral in the proof of Lemma 5.1 when u is a periodic solution.

Having shown that all the right-hand side integrals are bounded by $(xh)^{1/2}$, we combine the right-hand side estimates with the left-hand side to arrive at the final result. □

With the proof of Lemma 6.2 at hand, we can justify the differentiation under the integral sign we did above. As the justification for all of the integrals can be performed in the same manner, we only show the procedure for one of them, say the derivative of the second integral in the central difference equation satisfied by u'. To this end, define the operator

$$
G(x) := \int_{4h/3}^{2x/3} -\delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u'(x) \, \mathrm{d}y,
$$

and consider

$$\frac{G(x+t)-G(x)}{t}$$
$$= \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\frac{\delta_{2y}u'(x+t)-\delta_{2y}u'(x)}{t}\,\mathrm{d}y + \int_{2x/3}^{\frac{2x}{3}+\frac{2t}{3}} -\frac{1}{t}\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x+t)\,\mathrm{d}y.$$

The first of these integrals can be written as

$$\begin{aligned}
&\int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\frac{\delta_{2y}u'(x+t)-\delta_{2y}u'(x)}{t}\,\mathrm{d}y\\
&= \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\frac{u'(x+y+t)-u'(x+y)-\big(u'(x-y+t)-u'(x-y)\big)}{t}\,\mathrm{d}y\\
&= \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\Big(u''(x+y+\xi_1 t)_{\xi_1\in(0,1)} - u''(x-y+\xi_2 t)_{\xi_2\in(0,1)}\Big)\,\mathrm{d}y,
\end{aligned}$$

where we have used the mean value theorem. By the same calculation as (6.13) in the proof of Lemma 6.2, this integral is dominated by an integrable function on the domain in question (of course, the final bound for this integral would be $h^{1/2}/x^{3/2}$ as we currently do not have the factor x^2 in the numerator to cancel with, but this is not an issue as $h^{1/2}/x^{3/2} < \infty$ for a fixed h and fixed x), and hence we can use the dominated convergence theorem. But before we do so, let us consider the second integral. Through two changes of variables, first $y = s + \frac{2x}{3}$, then $z = ts$, we can rewrite this integral as

$$\begin{aligned}
\int_{2x/3}^{\frac{2x}{3}+\frac{2t}{3}} -\frac{1}{t}\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u'(x+t)\,\mathrm{d}y &= \int_0^{\frac{2t}{3}} -\frac{1}{t}\delta_{2h}K_{\text{sing}}\Big(s+\frac{2x}{3}\Big)\delta_{2\left(s+\frac{2x}{3}\right)}u'(x+t)\,\mathrm{d}s\\
&= \int_0^{\frac{2}{3}} -\delta_{2h}K_{\text{sing}}\Big(tz+\frac{2x}{3}\Big)\delta_{2\left(tz+\frac{2x}{3}\right)}u'(x+t)\,\mathrm{d}z.
\end{aligned}$$

As u' in the final integrand is bounded strictly away from its singularities, the integrand is dominated by an integrable function on the domain in question. Consequently, we can move the limit inside. In total, by dominated convergence we have that

$$\lim_{t\to 0}\frac{G(x+t)-G(x)}{t} = \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)\,\mathrm{d}y + \frac{2}{3}\Big(-\delta_{2h}K_{\text{sing}}\Big(\frac{2x}{3}\Big)\delta_{2\left(\frac{2x}{3}\right)}u'(x)\Big).$$

This is indeed the same expression as when we differentiate under the integral sign directly, as the second term on the right-hand side will, together with the boundary terms from the other integrals, simplify to the final two terms on the right-hand side of (6.10). Following a similar procedure, one finds that the differentiation of all the other integrals is also justified.

Step 3. Next, we show the improved estimate.

Lemma 6.3. *The improved estimate*

$$|x^{1/2}(x-h)^2(u''(x+h)-u''(x-h))| \lesssim h$$

holds uniformly on $[0,\nu]$*, for some* $\nu > 0$.

Proof. Consider $0 < h \le x \le \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. For $h \in [x/2, x]$, we have

$$|(x-h)^2(u''(x+h)-u''(x-h))| \lesssim h^{1/2} = \frac{h}{h^{1/2}} \le \frac{h}{(x/2)^{1/2}} \lesssim \frac{h}{x^{1/2}},$$

where the first inequality is simply Lemma 6.2. We now show that it also holds for $h < x/2$, and consider the same central difference equation as in the proof of Lemma 6.2.

Note from the proof Lemma 6.2 that all the right-hand side integrals, except for the one on $(-h/3, 7h/3)$ and the one on $(4h/3, 2x/3)$, were in-fact bounded by h. Therefore, we only need to show that these two integrals are also bounded by h to arrive at the improved estimate.

First we consider the integral over $(-h/3, 7h/3)$, where we use Lemma 6.2 to bound the integral by

$$
\begin{aligned}
&\left|\int_{-h/3}^{7h/3} - K_{\text{sing}}(y)\delta_{2(y-h)}u''(x)(x-h)^2 \, \mathrm{d}y\right| \\
&\quad \lesssim x^2 \int_{-h/3}^{7h/3} |K_{\text{sing}}(y)| \frac{|y-h|^{1/2}}{|x-(y-h)|^2} \, \mathrm{d}y \\
&\quad \leq x^2 \int_{-h/3}^{7h/3} |K_{\text{sing}}(y)| \frac{|y-h|^{1/2}}{|x-\frac{4}{3}h|^2} \, \mathrm{d}y \\
&\quad \lesssim \int_{-1/3}^{7/3} |K_{\text{sing}}(\tau h)||\tau h - h|^{1/2} h \, \mathrm{d}\tau \\
&\quad = h \int_{-1/3}^{7/3} |K_{\text{sing}}(\tau)||\tau - 1|^{1/2} \, \mathrm{d}\tau \simeq h,
\end{aligned}
$$

since the final integral converges.

Using Lemma 6.2, the integral on $(4h/3, 2x/3)$ can be bounded by

$$
\begin{aligned}
&\left|\int_{4h/3}^{2x/3} - \delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)(x-h)^2 \, \mathrm{d}y\right| \\
&\quad \lesssim x^2 \int_{4h/3}^{2x/3} |\delta_{2h}K_{\text{sing}}(y)| \frac{y^{1/2}}{(x-y)^2} \, \mathrm{d}y \\
&\quad \lesssim \int_{4h/3}^{2x/3} |\delta_{2h}K_{\text{sing}}(y)| y^{1/2} \, \mathrm{d}y \\
&\quad \lesssim h\Big(1 + \Big|\ln\Big(\frac{h}{x}\Big)\Big|\Big),
\end{aligned}
$$

where the final inequality follows from the same calculation as in (4.30). As was the case in the proofs for the two previous improved estimates, this bound is not quite good enough yet. Combining the bounds we have so far, however, gives us the estimate

$$
|x^{1/2}(x-h)^2(u''(x+h) - u''(x-h))| \lesssim h\Big(1 + \Big|\ln\Big(\frac{h}{x}\Big)\Big|\Big),
$$

which we insert back into the calculations above to find that

$$
\left|\int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)(x-h)^2 \, \mathrm{d}y\right| \lesssim h,
$$

by the same calculation as in (4.31).

As we have managed to show that the two integrals were also bounded by h, we arrive at the improved estimate. □

Step 4. With the first and the improved estimate for the central difference of u'' at hand, we are ready for the main result of this section.

Proposition 6.4. *The third derivative of the solution u enjoys the limit*

$$
\lim_{x \to 0} \frac{u'''(x)}{x^{-5/2}} = \frac{3}{8}\sqrt{\frac{\pi}{8}}.
$$

Proof. With $0 < 2h < x \leq \nu$, we consider the central difference equation (6.10) satisfied by u'', divided by $2h$ and multiplied by x^2:

$$
\frac{1}{2h}(2x^2u'(x+h)^2 + 2x^2u(x+h)u''(x+h)
$$

$$
\begin{aligned}
&\qquad - 2x^2u'x^2 - 2x^2u(x-h)u''(x-h)) \\
&= \frac{1}{2h}\int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u''(x)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\text{sing}}(x+y) + \delta_{2h}K''_{\text{sing}}(x-y)\big)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{-x/3}^{5x/3} \delta_{2h}K''_{\text{sing}}(x+y)u(y)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K''_{\text{reg}}(x+y) + \delta_{2h}K''_{\text{reg}}(x-y)\big)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{-x}^{x} \delta_{2h}K''_{\text{reg}}(x+y)u(y)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K''(x+y) + \delta_{2h}K''(x-y)\big)x^2 \,\mathrm{d}y \\
&\quad + \frac{1}{2h}\delta_{2h}K'_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big(-u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big)\Big)x^2 \\
&\quad + \frac{1}{2h}\delta_{2h}K_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big)\Big)x^2.
\end{aligned}
$$

For the left-hand side, we know from (6.11) and (6.12) that

$$
\begin{aligned}
&\lim_{x\to 0}\lim_{h\to 0}\frac{2x^2u'(x+h)^2 + 2x^2u(x+h)u''(x+h) - 2x^2u'x^2 - 2x^2u(x-h)u''(x-h))}{2h} \\
&\qquad = -\frac{3}{4}\frac{\pi}{8} + 2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u'''(x)}{x^{-5/2}}.
\end{aligned}
$$

Next we show that the h- and x-limit of the right-hand side evaluates to zero. We will again do so by showing that each of the integrals are dominated by integrable functions on their respective domains of integration, whence we can interchange limits and integrals.

The first integral can be written as

$$
\frac{1}{2h}\int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u''(x)x^2 \,\mathrm{d}y = \int_{-1/3}^{7/3} \frac{-1}{\sqrt{8\pi}h^{1/2}}\frac{1}{|\tau|^{1/2}}\delta_{2(\tau h-h)}u''(x)x^2 \,\mathrm{d}\tau,
$$

and the integrand bounded by

$$
\left|\frac{-1}{\sqrt{8\pi}h^{1/2}}\frac{1}{|\tau|^{1/2}}\delta_{2(\tau h-h)}u''(x)x^2\right| \lesssim \frac{x^2}{h^{1/2}}\frac{1}{|\tau|^{1/2}}\frac{|\tau h-h|^{1/2}}{|x-(\tau h-h)|^2} \lesssim \frac{|\tau-1|^{1/2}}{|\tau|^{1/2}},
$$

where we have used Lemma (6.2). As the right-hand side is integrable on $(-1/3, 7/3)$, we can take the limit inside the integral to find that

$$
\lim_{h\to 0}\frac{-1}{\sqrt{8\pi}h^{1/2}}\frac{1}{|\tau|^{1/2}}\delta_{2(\tau h-h)}u''(x)x^2 = \lim_{h\to 0}\frac{-h^{1/2}}{\sqrt{8\pi}}\frac{1}{|\tau|^{1/2}}2(\tau-1)\frac{\delta_{2(\tau h-h)}u''(x)}{2(\tau h-h)}x^2 = 0,
$$

since $u''(x)$ is differentiable on $0 < x < P$.

The second integral can be written as

$$
\frac{1}{2h}\int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u''(x)x^2 \,\mathrm{d}y = \int_{\frac{4h}{3x}}^{2/3} -\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u''(x)x^3 \,\mathrm{d}\tau,
$$

where the integrand can be expressed as

$$
-\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u''(x)x^3 = \frac{1}{\sqrt{2\pi}}\frac{1}{\Big(\tau^2-(\frac{h}{x})^2\Big)^{1/2}\Big((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\Big)}x^{3/2}\delta_{2\tau x}u''(x),
$$

by the same calculation as in (4.34). Using the improved estimate of Lemma (6.3) and a calculation similar to (4.35), the integrand is bounded by

$$\left|\frac{1}{\sqrt{2\pi}}\frac{1}{\left(\tau^2-(\frac{h}{x})^2\right)^{1/2}\left((\tau+\frac{h}{x})^{1/2}+(\tau-\frac{h}{x})^{1/2}\right)}x^{3/2}\delta_{2\tau x}u''(x)\right|\lesssim\frac{1}{\tau^{1/2}(1-\tau)^2},$$

for all $0<2h<x\le\nu$ and $\frac{4h}{3x}<\tau<2/3$. Since the right-hand side is integrable on $(0,2/3)$, we can move the limits inside the integral. We then get

$$\lim_{h\to 0}-\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u''(x)x^3=\frac{1}{\sqrt{2\pi}}\frac{x^{3/2}\delta_{2\tau x}u''(x)}{2\tau^{3/2}},$$

for each $x\in(0,\nu]$ and $\tau\in(0,2/3)$, and

$$\lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\frac{x^{3/2}\delta_{2\tau x}u''(x)}{2\tau^{3/2}}=-\frac{1}{32\tau^{3/2}}\left(\frac{1}{(1+\tau)^{3/2}}-\frac{1}{(1-\tau)^{3/2}}\right),$$

for each $\tau\in(0,2/3)$ by calculations analogous to (4.36) and (4.37), respectively. In summary, we have that

$$\lim_{x\to 0}\lim_{h\to 0}\int_{\frac{4h}{3x}}^{2/3}-\frac{\delta_{2h}K_{\text{sing}}(\tau x)}{2h}\delta_{2\tau x}u''(x)x^3\,\mathrm{d}\tau=\int_0^{2/3}-\frac{1}{32\tau^{3/2}}\left(\frac{1}{(1+\tau)^{3/2}}-\frac{1}{(1-\tau)^{3/2}}\right)\mathrm{d}\tau.$$

The third integral can be written as

$$\begin{aligned}
&\frac{1}{2h}\int_{5x/3}^{2\nu+x}u(y)\big(\delta_{2h}K''_{\text{sing}}(x+y)+\delta_{2h}K''_{\text{sing}}(x-y)\big)x^2\,\mathrm{d}y\\
&=\int_{5/3}^{\frac{2\nu}{x}+1}\frac{1}{2h}u(\tau x)\big(\delta_{2h}K''_{\text{sing}}(x+\tau x)+\delta_{2h}K''_{\text{sing}}(x-\tau x)\big)x^3\,\mathrm{d}\tau\\
&=\int_{5/3}^{\frac{2\nu}{x}+1}\underbrace{u(\tau x)\frac{\delta_{2h}K''_{\text{sing}}(x+\tau x)}{2h}x^3}_{(A)}\,\mathrm{d}\tau+\int_{5/3}^{\frac{2\nu}{x}+1}\underbrace{u(\tau x)\frac{\delta_{2h}K''_{\text{sing}}(x-\tau x)}{2h}x^3}_{(B)}\,\mathrm{d}\tau.
\end{aligned}$$

We can rewrite (A) as

$$\begin{aligned}
&\frac{\delta_{2h}K''_{\text{sing}}(x+\tau x)}{2h}u(\tau x)x^3=\frac{1}{2h}\left(\frac{3}{\sqrt{\pi}|2(x+\tau x+h)|^{5/2}}-\frac{3}{\sqrt{\pi}|2(x+\tau x-h)|^{5/2}}\right)u(\tau x)x^3\\
&=\frac{3}{\sqrt{2^7}\pi h}\frac{\left((\tau+1-\frac{h}{x})^{5/2}-(\tau+1+\frac{h}{x})^{5/2}\right)}{(\tau+1+\frac{h}{x})^{5/2}(\tau+1-\frac{h}{x})^{5/2}}u(\tau x)x^{1/2}\\
&=\frac{3}{\sqrt{2^7}\pi h}\frac{\left((\tau+1-\frac{h}{x})^{5/2}-(\tau+1+\frac{h}{x})^{5/2}\right)}{(\tau+1+\frac{h}{x})^{5/2}(\tau+1-\frac{h}{x})^{5/2}}\frac{\left(2\sum_{k=1,3,5}\binom{5}{k}(\tau+1)^{5-k}\left(\frac{h}{x}\right)^k\right)}{\left((\tau+1+\frac{h}{x})^5-(\tau+1-\frac{h}{x})^5\right)}u(\tau x)x^{1/2}\\
&=\frac{3}{\sqrt{2^7}\pi h}\frac{\left((\tau+1-\frac{h}{x})^{5/2}-(\tau+1+\frac{h}{x})^{5/2}\right)}{(\tau+1+\frac{h}{x})^{5/2}(\tau+1-\frac{h}{x})^{5/2}}\\
&\qquad\cdot\frac{\left(2\sum_{k=1,3,5}\binom{5}{k}(\tau+1)^{5-k}\left(\frac{h}{x}\right)^k\right)}{\left((\tau+1+\frac{h}{x})^{5/2}+(\tau+1-\frac{h}{x})^{5/2}\right)\left((\tau+1+\frac{h}{x})^{5/2}-(\tau+1-\frac{h}{x})^{5/2}\right)}u(\tau x)x^{1/2}\\
&=\frac{3}{\sqrt{2^7}\pi h}\frac{-1}{(\tau+1+\frac{h}{x})^{5/2}(\tau+1-\frac{h}{x})^{5/2}}\frac{\left(2\frac{h}{x}\sum_{k=1,3,5}\binom{5}{k}(\tau+1)^{5-k}\left(\frac{h}{x}\right)^{k-1}\right)}{\left((\tau+1+\frac{h}{x})^{5/2}+(\tau+1-\frac{h}{x})^{5/2}\right)}u(\tau x)x^{1/2}
\end{aligned}\tag{6.14}$$

$$= \frac{-3}{\sqrt{2^5\pi}} \frac{1}{\Big((\tau+1)^2 - (\frac{h}{x})^2\Big)^{5/2}} \frac{\Big(\sum_{k=1,3,5}\binom{5}{k}(\tau+1)^{5-k}\Big(\frac{h}{x}\Big)^{k-1}\Big)}{\Big((\tau+1+\frac{h}{x})^{5/2} + (\tau+1-\frac{h}{x})^{5/2}\Big)} \frac{u(\tau x)}{x^{1/2}}.$$

While similar to the calculation in (4.34), we have included much of the calculation as we now had to include a binomial sum in the numerator for the new fraction in the third equality to. This difference in the calculations is a result of the exponent now being 5/2 instead of 1/2.

In an analogous manner, (B) can be rewritten as

$$\frac{\delta_{2h}K''_{\text{sing}}(x-\tau x)}{2h} u(\tau x)x^3$$
$$= \frac{3}{\sqrt{2^5\pi}} \frac{1}{\Big((\tau-1)^2 - (\frac{h}{x})^2\Big)^{5/2}} \frac{\Big(\sum_{k=1,3,5}\binom{5}{k}(\tau-1)^{5-k}\Big(\frac{h}{x}\Big)^{k-1}\Big)}{\Big((\tau-1+\frac{h}{x})^{5/2} + (\tau-1-\frac{h}{x})^{5/2}\Big)} \frac{u(\tau x)}{x^{1/2}}.$$

We can bound (A) by

$$\left|\frac{-3}{\sqrt{2^5\pi}} \frac{1}{\Big((\tau+1)^2 - (\frac{h}{x})^2\Big)^{5/2}} \frac{\Big(\sum_{k=1,3,5}\binom{5}{k}(\tau+1)^{5-k}\Big(\frac{h}{x}\Big)^{k-1}\Big)}{\Big((\tau+1+\frac{h}{x})^{5/2} + (\tau+1-\frac{h}{x})^{5/2}\Big)} \frac{u(\tau x)}{x^{1/2}}\right|$$
$$\lesssim \frac{1}{\Big((\tau+1)^2 - (\frac{3}{16}(\tau+1))^2\Big)^{5/2}} \frac{\Big(\sum_{k=1,3,5}(\tau+1)^{5-k}\Big)}{(\tau+1)^{5/2}} \frac{(\tau x)^{1/2}}{x^{1/2}}$$
$$\le \frac{\Big(\sum_{k=1,3,5}(\tau+1)^{5-k}\Big)}{(\tau+1)^{15/2}}(\tau+1)^{1/2}$$
$$\le \frac{\Big(\sum_{k=1,3,5}(\tau+1)^{4}\Big)}{(\tau+1)^{15/2}}(\tau+1)^{1/2}$$
$$= \frac{1}{(\tau+1)^3},$$

and (B) by

$$\left|\frac{3}{\sqrt{2^5\pi}} \frac{1}{\Big((\tau-1)^2 - (\frac{h}{x})^2\Big)^{5/2}} \frac{\Big(\sum_{k=1,3,5}\binom{5}{k}(\tau-1)^{5-k}\Big(\frac{h}{x}\Big)^{k-1}\Big)}{\Big((\tau-1+\frac{h}{x})^{5/2} + (\tau-1-\frac{h}{x})^{5/2}\Big)} \frac{u(\tau x)}{x^{1/2}}\right| \lesssim \frac{1}{(\tau-1)^3}$$

in an analogous manner, both of which are integrable on $(5/3,\infty)$. Thus, rewriting (A) and (B) as above, and noting that all terms except for the first in the binomial sum contain h in the numerator, we find that

$$\lim_{h\to 0} u(\tau x)\Big(\frac{\delta_{2h}K''_{\text{sing}}(x+\tau x)}{2h} + \frac{\delta_{2h}K''_{\text{sing}}(x-\tau x)}{2h}\Big)x^3$$
$$= \frac{15}{\sqrt{2^7\pi}} \frac{u(\tau x)}{x^{1/2}} \left(-\frac{1}{(\tau+1)^{7/2}} + \frac{1}{(\tau-1)^{7/2}}\right)$$

for the h-limit. By using the u-limit, we get

$$\lim_{x\to 0} \frac{15}{\sqrt{2^7\pi}} \frac{u(\tau x)}{x^{1/2}} \left(-\frac{1}{(\tau+1)^{7/2}} + \frac{1}{(\tau-1)^{7/2}}\right) = \frac{15}{2^5}\tau^{1/2}\left(-\frac{1}{(\tau+1)^{7/2}} + \frac{1}{(\tau-1)^{7/2}}\right),$$

for the x-limit. In summary, we have that

$$\lim_{x\to 0}\lim_{h\to 0}\int_{5/3}^{\frac{2\nu}{x}+1} \frac{1}{2h} u(\tau x)\big(\delta_{2h}K''_{\text{sing}}(x+\tau x) + \delta_{2h}K''_{\text{sing}}(x-\tau x)\big)x^3 \, \mathrm{d}\tau$$

$$= \int_{5/3}^{\infty} \frac{15}{2^5} \tau^{1/2} \bigg(- \frac{1}{(\tau+1)^{7/2}} + \frac{1}{(\tau-1)^{7/2}} \bigg) \,\mathrm{d}\tau.$$

The fourth integral can be written as

$$\frac{1}{2h} \int_{-x/3}^{5x/3} \delta_{2h} K''_{\text{sing}}(x+y) u(y) x^2 \,\mathrm{d}y = \int_{-1/3}^{5/3} \frac{\delta_{2h} K''_{\text{sing}}(x+\tau x)}{2h} u(\tau x) x^3 \,\mathrm{d}\tau.$$

This integrand is exactly the same as (A) from the third integral. There we saw that (A) is bounded by $(\tau+1)^{-3}$, which happens to also integrable on $(-1/3, 5/3)$. We can therefore move the limits inside the integral, arriving at

$$\lim_{x\to 0} \lim_{h \to 0} \int_{-1/3}^{5/3} \frac{\delta_{2h} K''_{\text{sing}}(x+\tau x)}{2h} u(\tau x) x^3 \,\mathrm{d}\tau = \int_{-1/3}^{5/3} \frac{-15}{2^5} \frac{|\tau|^{1/2}}{(\tau+1)^{7/2}} \,\mathrm{d}\tau.$$

Notice that we cannot drop the absolute value on $|\tau|^{1/2}$ here due to the domain of integration.

The fifth, sixth and final integral can all be bounded as in Lemma (6.2), and all three limits tend to zero:

$$u(y) \bigg(\frac{\delta_{2h} K''_{\text{reg}}(x+y)}{2h} + \frac{\delta_{2h} K''_{\text{reg}}(x-y)}{2h} \bigg) x^2 \xrightarrow{h \to 0} u(y) \bigg(K'''_{\text{reg}}(x+y) + K'''_{\text{reg}}(x-y) \bigg) x^2 \xrightarrow{x \to 0} 0,$$

$$u(y) \frac{\delta_{2h} K''_{\text{reg}}(x+y)}{2h} x^2 \xrightarrow{h \to 0} u(y) K'''_{\text{reg}}(x+y) x^2 \xrightarrow{x \to 0} 0,$$

$$u(y) \bigg(\frac{\delta_{2h} K''(x+y)}{2h} + \frac{\delta_{2h} K''(x-y)}{2h} \bigg) x^2 \xrightarrow{h \to 0} u(y) \bigg(K'''(x+y) + K'''(x-y) \bigg) x^2 \xrightarrow{x \to 0} 0,$$

respectively.

Now we consider the limits of the boundary terms. For the first one, the h-limit is

$$\lim_{h\to 0} \frac{1}{2h} \delta_{2h} K'_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(- u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big) \bigg) x^2 = K''_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(- u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big) \bigg) x^2,$$

followed by

$$\begin{aligned} \lim_{x\to 0} K''_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(- u\Big(\frac{5}{3}x\Big) + u\Big(\frac{x}{3}\Big) \bigg) x^2 &= \lim_{x \to 0} \frac{3^{7/2}}{2^5 \sqrt{\pi}} \frac{x^{5/2}}{x^{5/2}} \bigg(- \Big(\frac{5}{3}\Big)^{1/2} \frac{u\big(\frac{5}{3}x\big)}{\big(\frac{5x}{3}\big)^{1/2}} + \frac{1}{3^{1/2}} \frac{u\big(\frac{x}{3}\big)}{\big(\frac{x}{3}\big)^{1/2}} \bigg) \\ &= \frac{3^3(1-5^{1/2})}{2^{13/2}}, \end{aligned}$$

for the x-limit. Here we have used the u-limit in the last equality.

For the second one, we have

$$\lim_{h\to 0} \frac{1}{2h} \delta_{2h} K_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big) \bigg) x^2 = K'_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big) \bigg) x^2,$$

followed by

$$\begin{aligned} &\lim_{x\to 0} K'_{\text{sing}}\Big(\frac{2}{3}x\Big) \bigg(u'\Big(\frac{5}{3}x\Big) + u'\Big(\frac{x}{3}\Big) \bigg) x^2 \\ &= \lim_{x\to 0} \frac{-\operatorname{sgn}\big(\frac{2}{3}x\big)}{\sqrt{2^3\pi}\big|\frac{2}{3}x\big|^{3/2}} x^{3/2} \bigg(\frac{\big(\frac{5}{3}x\big)^{1/2} u'\big(\frac{5}{3}x\big)}{\big(\frac{5}{3}\big)^{1/2}} + \frac{\big(\frac{1}{3}x\big)^{1/2} u'\big(\frac{x}{3}\big)}{\big(\frac{1}{3}\big)^{1/2}} \bigg) \\ &= \frac{-3^2}{2^{11/2}} \Big(\frac{1}{5^{1/2}} + 1 \Big), \end{aligned}$$

where we have used the u'-limit.

Thus we shown that the limits of the right-hand side exists, and combining all of the above limits gives us that

$$-\frac{3}{4}\frac{\pi}{8}+2\sqrt{\frac{\pi}{8}}\lim_{x\to 0}\frac{u'''(x)}{x^{-5/2}} = \int_0^{2/3} -\frac{1}{32\tau^{3/2}}\left(\frac{1}{(1+\tau)^{3/2}}-\frac{1}{(1-\tau)^{3/2}}\right)\mathrm{d}\tau + \int_{5/3}^{\infty}\frac{15}{2^5}\tau^{1/2}\left(-\frac{1}{(\tau+1)^{7/2}}+\frac{1}{(\tau-1)^{7/2}}\right)\mathrm{d}\tau + \int_{-1/3}^{5/3}\frac{-15}{2^5}\frac{|\tau|^{1/2}}{(\tau+1)^{7/2}}\,\mathrm{d}\tau + \frac{3^3(1-5^{1/2})}{2^{13/2}}+\frac{-3^2}{2^{11/2}}\left(\frac{1}{5^{1/2}}+1\right). \tag{6.15}$$

Computing the right-hand side, one finds that it equals zero. Thus, we get

$$\lim_{x\to 0}\frac{u'''(x)}{x^{-5/2}} = \frac{3}{8}\sqrt{\frac{\pi}{8}},$$

after rearranging. □

7 The limit for the n-th derivative

Inspecting all the proofs from Section 6, we see that the approach lends itself nicely to an inductive proof for the general $u^{(n)}$-limit. As we will see in the Main Theorem 7.6, there is, however, one part of the proof of Proposition 6.4 which is more difficult in the general case of the n-th derivative, namely the calculation (7.2) which corresponds to the calculation of the right-hand side of (6.15). In (6.15), the right-hand side expression is a sum of tedious integrals and fractions, but it was still a manageable computation to do by hand. In the general n-th derivative case, the corresponding expression is an even more tedious sum of integrals and fractions, now depending on n. We expect that this expression equals zero for a general $n \geq 1$, though we have not been able to show this. Evaluating the expression for explicit values of n using the computer program Maple [13] has indeed given the result zero for all the values of n we have tried. We expect further considerations to yield an analytical solution to this step.

Thankfully, the value of the expression is only necessary for establishing the exact value of the limit. Consequently, it follows as a corollary to the Main Theorem that the $u^{(n)}$-limit exists for all $n \in \mathbb{N}$ unconditionally. This corollary, Corollary 7.7, is the other main result of this section and of the entire paper.

This section also follows the four main steps listed in the introduction, and they can be treated completely analogously to the corresponding steps in Section 6. In addition, now that we are building up to a general inductive proof, there are two more lemmas that we will also need. We begin with the first of these lemmas, which confirms Remark 6.1.

Lemma 7.1.

$$\lim_{h\to 0}\frac{\sum_{i=0}^n\binom{n}{i}u^{(i)}(x+h)u^{(n-i)}(x+h)-\sum_{i=0}^n\binom{n}{i}u^{(i)}(x-h)u^{(n-i)}(x-h)}{2h} = \frac{d}{dx}\sum_{i=0}^{n}\binom{n}{i}u^{(i)}(x)u^{(n-i)}(x) = \sum_{i=0}^{n+1}\binom{n+1}{i}u^{(i)}(x)u^{(n+1-i)}(x)$$

Proof. The first equality is the definition of the derivative. Taking the derivative, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{i=0}^{n}\binom{n}{i}u^{(i)}(x)u^{(n-i)}(x) = \sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(u^{(i)}(x)u^{(n-i)}(x)\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \binom{n}{i} \Big(u^{(i+1)}(x) u^{(n-i)}(x) + u^{(i)}(x) u^{(n+1-i)}(x) \Big) \\
&= \sum_{i=0}^{n} \binom{n}{i} u^{(i+1)}(x) u^{(n-i)}(x) + \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x) u^{(n+1-i)}(x).
\end{aligned}
$$

Making the change of variables $j = i + 1$, we find that

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} u^{(i+1)}(x) u^{(n-i)}(x) + \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x) u^{(n+1-i)}(x) \\
&\quad = \sum_{j=1}^{n+1} \binom{n}{j-1} u^{(j)}(x) u^{(n-(j-1))}(x) + \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x) u^{(n+1-i)}(x) \\
&\quad = \binom{n}{n} u^{(n+1)}(x) u^{(0)}(x) + \sum_{j=1}^{n} \binom{n}{j-1} u^{(j)}(x) u^{(n+1-j)}(x) \\
&\qquad + \binom{n}{0} u^{(0)}(x) u^{(n+1)}(x) + \sum_{i=1}^{n} \binom{n}{i} u^{(i)}(x) u^{(n+1-i)}(x) \\
&\quad = \binom{n}{n} u^{(n+1)}(x) u^{(0)}(x) + \binom{n}{0} u^{(0)}(x) u^{(n+1)}(x) \\
&\qquad + \sum_{i=1}^{n} \left(\binom{n}{i-1} + \binom{n}{i} \right) u^{(i)}(x) u^{(n+1-i)}(x) \\
&\quad = \binom{n+1}{n+1} u^{(0)}(x) u^{(n+1)}(x) + \binom{n+1}{0} u^{(0)}(x) u^{(n+1)}(x) \\
&\qquad + \sum_{i=1}^{n} \binom{n+1}{i} u^{(i)}(x) u^{(n+1-i)}(x) \\
&\quad = \sum_{i=0}^{n+1} \binom{n+1}{i} u^{(i)}(x) u^{(n+1-i)}(x),
\end{aligned}
$$

which is the expression in the statement of the lemma. Here we have used that $\binom{n}{n} = 1 = \binom{n+1}{n+1}$ and $\binom{n}{0} = 1 = \binom{n+1}{0}$, and the identity $\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$. $\square$

Step 1. Next, we show the complete general form of the central difference equation satisfied by $u^{(n)}$, given that differentiating under the integral sign is justified.

Lemma 7.2. *Given that differentiating with respect to x under the integral sign is justified, the n-th derivative of u satisfies the following equation for $n \geq 1$ and $0 < 2h < x \leq \nu$:*

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x+h) u^{(n-i)}(x+h) - \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x-h) u^{(n-i)}(x-h) \\
&\quad = \int_{-h/3}^{7h/3} -K_{\text{sing}}(y) \delta_{2(y-h)} u^{(n)}(x) \, \mathrm{d}y \\
&\qquad + \int_{4h/3}^{2x/3} -\delta_{2h} K_{\text{sing}}(y) \delta_{2y} u^{(n)}(x) \, \mathrm{d}y \\
&\qquad + \int_{5x/3}^{2\nu+x} u(y) \big(\delta_{2h} K_{\text{sing}}^{(n)}(x+y) + \delta_{2h} K_{\text{sing}}^{(n)}(x-y) \big) \, \mathrm{d}y \\
&\qquad + \int_{-x/3}^{5x/3} \delta_{2h} K_{\text{sing}}^{(n)}(x+y) u(y) \, \mathrm{d}y \qquad\qquad (7.1) \\
&\qquad + \int_{x}^{2\nu+x} u(y) \big(\delta_{2h} K_{\text{reg}}^{(n)}(x+y) + \delta_{2h} K_{\text{reg}}^{(n)}(x-y) \big) \, \mathrm{d}y \\
&\qquad + \int_{-x}^{x} \delta_{2h} K_{\text{reg}}^{(n)}(x+y) u(y) \, \mathrm{d}y
\end{aligned}
$$

$$+\int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K^{(n)}(x+y)+\delta_{2h}K^{(n)}(x-y)\big)\ \mathrm{d}y$$
$$+\sum_{i=0}^{n-1}\delta_{2h}K_{\text{sing}}^{(i)}\Big(\frac{2}{3}x\Big)\bigg((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big)+u^{(n-1-i)}\Big(\frac{x}{3}\Big)\bigg)$$

Proof. The discussion from Step 1 (Altered) in Section 6 serves as a base case. Now we need to show that the inductive steps also holds. That is, given that the statement of the lemma is true for the n-th derivative, we need to show that it also holds for the $(n+1)$-th derivative.

Differentiating the left-hand side of the equation with respect to x, we indeed get

$$\sum_{i=0}^{n+1}\binom{n+1}{i}u^{(i)}(x+h)u^{(n+1-i)}(x+h)-\sum_{i=0}^{n+1}\binom{n+1}{i}u^{(i)}(x-h)u^{(n+1-i)}(x-h),$$

by Lemma 7.1.

We are given that differentiating with respect to x under the integral sign is justified. Differentiating the right-hand side of the equation satisfied by the n-th derivative, it is clear that the integrals are of the correct form. For the boundary conditions, it is straightforward, though tedious, to show that the derivative of the sum on the right-hand side of (7.1), together with the boundary terms from differentiating the integrals, simplify to the desired sum (though one must not forget that K_{sing} is an even function, so its odd derivatives are odd and its even derivatives are even). In total, when differentiating the equation for the n-th derivative with respect to x we get

$$\begin{aligned}
&\sum_{i=0}^{n+1}\binom{n+1}{i}u^{(i)}(x+h)u^{(n+1-i)}(x+h)-\sum_{i=0}^{n+1}\binom{n+1}{i}u^{(i)}(x-h)u^{(n+1-i)}(x-h)\\
&\quad=\int_{-h/3}^{7h/3}-K_{\text{sing}}(y)\delta_{2(y-h)}u^{(n+1)}(x)\ \mathrm{d}y\\
&\qquad+\int_{4h/3}^{2x/3}-\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u^{(n+1)}(x)\ \mathrm{d}y\\
&\qquad+\int_{5x/3}^{2\nu+x}u(y)\big(\delta_{2h}K_{\text{sing}}^{(n+1)}(x+y)+\delta_{2h}K_{\text{sing}}^{(n+1)}(x-y)\big)\ \mathrm{d}y\\
&\qquad+\int_{-x/3}^{5x/3}\delta_{2h}K_{\text{sing}}^{(n+1)}(x+y)u(y)\ \mathrm{d}y\\
&\qquad+\int_{x}^{2\nu+x}u(y)\big(\delta_{2h}K_{\text{reg}}^{(n+1)}(x+y)+\delta_{2h}K_{\text{reg}}^{(n+1)}(x-y)\big)\ \mathrm{d}y\\
&\qquad+\int_{-x}^{x}\delta_{2h}K_{\text{reg}}^{(n+1)}(x+y)u(y)\ \mathrm{d}y\\
&\qquad+\int_{2\nu+x}^{\infty}u(y)\big(\delta_{2h}K^{(n+1)}(x+y)+\delta_{2h}K^{(n+1)}(x-y)\big)\ \mathrm{d}y\\
&\qquad+\sum_{i=0}^{n+1}\delta_{2h}K_{\text{sing}}^{(i)}\Big(\frac{2}{3}x\Big)\bigg((-1)^i u^{(n-i)}\Big(\frac{5}{3}x\Big)+u^{(n-i)}\Big(\frac{x}{3}\Big)\bigg),
\end{aligned}$$

which is satisfied by the $(n+1)$-th derivative. As this equation is of the same form as the equation for the n-th derivative, the inductive step holds. □

Step 2. Now we show that the first estimate holds for the central difference $u^{(n)}$, given that the $u^{(n)}$-limit exists.

Lemma 7.3. *Given that*

$$\lim_{x\to 0}\frac{u^{(n)}(x)}{x^{\frac{1}{2}-n}}$$

exists, it follows that there is some $\nu>0$ so that

$$|(x-h)^n(u^{(n)}(x+h)-u^{(n)}(x-h))|\lesssim h^{1/2}$$

holds uniformly on $[0, \nu]$.

Proof. We prove the lemma by strong induction. With the u-limit known from Proposition 4.8, we will consider $n \geq 1$ in the following and take the u'-limit as our base case. Now we need to show that the inductive step also holds. That is, given that the statement of the lemma holds for all $k \in \{1, ..., n\}$, we need to show that it also holds for $n + 1$.

Consider $0 < h \leq x \leq \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. The claim follows directly from the limit at the origin for large h, say $h \in [x/2, x]$. Indeed, for such h we have that

$$\begin{aligned}
&|(x-h)^n(u^{(n)}(x+h) - u^{(n)}(x-h))| \\
&\lesssim |x-h|^n(|x+h|^{\frac{1}{2}-n} + |x-h|^{\frac{1}{2}-n}) \lesssim |x-h|^n|x-h|^{\frac{1}{2}-n} \leq |2h-h|^{1/2} = |h|^{1/2}.
\end{aligned}$$

We now show that it also holds for $h < x/2$.

Consider the central difference equation satisfied by $u^{(n)}$ from Lemma 7.1, multiplied by $(x-h)^n$,

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (x-h)^n u^{(i)}(x+h) u^{(n-i)}(x+h) - \sum_{i=0}^{n} \binom{n}{i} (x-h)^n u^{(i)}(x-h) u^{(n-i)}(x-h) \\
&= \int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)} u^{(n)}(x)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{4h/3}^{2x/3} -\delta_{2h} K_{\text{sing}}(y) \delta_{2y} u^{(n)}(x)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h} K^{(n)}_{\text{sing}}(x+y) + \delta_{2h} K^{(n)}_{\text{sing}}(x-y)\big)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{-x/3}^{5x/3} \delta_{2h} K^{(n)}_{\text{sing}}(x+y) u(y)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{x}^{2\nu+x} u(y)\big(\delta_{2h} K^{(n)}_{\text{reg}}(x+y) + \delta_{2h} K^{(n)}_{\text{reg}}(x-y)\big)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{-x}^{x} \delta_{2h} K^{(n)}_{\text{reg}}(x+y) u(y)(x-h)^n \, \mathrm{d}y \\
&\quad + \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h} K^{(n)}(x+y) + \delta_{2h} K^{(n)}(x-y)\big)(x-h)^n \, \mathrm{d}y \\
&\quad + \sum_{i=0}^{n-1} \delta_{2h} K^{(i)}_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big) + u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)(x-h)^n
\end{aligned}$$

From the inductive hypothesis we have that $u^{(i)}(x \pm h) \simeq x^{\frac{1}{2}-i}$ for all $i \in \{0, 1, 2, ..., n-1\}$ and for $h < x/2$ (after possibly shrinking ν), so the left-hand side can be written as

$$\begin{aligned}
&\left| \sum_{i=0}^{n} \binom{n}{i} (x-h)^n u^{(i)}(x+h) u^{(n-i)}(x+h) - \sum_{i=0}^{n} \binom{n}{i} (x-h)^n u^{(i)}(x-h) u^{(n-i)}(x-h) \right| \\
&\simeq \big| x^{1/2}(x-h)^n \big(u^{(n)}(x+h) - u^{(n)}(x-h)\big) \big|.
\end{aligned}$$

Now we consider the right-hand side, beginning with the boundary terms. Expanding the derivatives of K_{sing} using the binomial theorem shows that they all tend to zero to zero on the order of h:

$$\begin{aligned}
&\left| \sum_{i=0}^{n-1} \delta_{2h} K^{(i)}_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big) + u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)(x-h)^n \right| \\
&\lesssim \sum_{i=0}^{n-1} \Big|\delta_{2h} K^{(i)}_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big| \Big(\Big|\frac{5}{3}x\Big|^{\frac{1}{2}-(n-1-i)} + \Big|\frac{1}{3}x\Big|^{\frac{1}{2}-(n-1-i)} \Big) x^n \\
&= \sum_{i=0}^{n-1} \Big|\delta_{2h} K^{(i)}_{\text{sing}}\Big(\frac{2}{3}x\Big)\Big| \Big(\Big(\frac{5}{3}\Big)^{\frac{1}{2}-n+1+i} + \Big(\frac{1}{3}\Big)^{\frac{1}{2}-n+1+i} \Big) |x|^{\frac{1}{2}+1+i}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{i=0}^{n-1} \left| \frac{(-\mathrm{sgn}(\frac{2}{3}x+h))^i}{|\frac{2}{3}x+h|^{\frac{1}{2}+i}} - \frac{(-\mathrm{sgn}(\frac{2}{3}x-h))^i}{|\frac{2}{3}x-h|^{\frac{1}{2}+i}} \right| |x|^{\frac{1}{2}+1+i} \\
&= \sum_{i=0}^{n-1} \left| \Big(\frac{2}{3}+\frac{h}{x}\Big)^{-\frac{1}{2}-i} - \Big(\frac{2}{3}-\frac{h}{x}\Big)^{-\frac{1}{2}-i} \right| x^{-\frac{1}{2}-i} |x|^{\frac{1}{2}+1+i} \\
&= x \sum_{i=0}^{n-1} \left| \sum_{k=0}^{\infty} \binom{-\frac{1}{2}-i}{k} \Big(\frac{2}{3}\Big)^{-\frac{1}{2}-i-k} \Big(\frac{h}{x}\Big)^k - \sum_{k=0}^{\infty} \binom{-\frac{1}{2}-i}{k} \Big(\frac{2}{3}\Big)^{-\frac{1}{2}-i-k} \Big(-\frac{h}{x}\Big)^k \right| \\
&= x \sum_{i=0}^{n-1} \left| 2 \sum_{\substack{k=1,\\ k \text{ odd}}}^{\infty} \binom{-\frac{1}{2}-i}{k} \Big(\frac{2}{3}\Big)^{-\frac{1}{2}-i-k} \Big(\frac{h}{x}\Big)^k \right| \\
&= h \sum_{i=0}^{n-1} \left| 2 \sum_{\substack{k=1,\\ k \text{ odd}}}^{\infty} \binom{-\frac{1}{2}-i}{k} \Big(\frac{2}{3}\Big)^{-\frac{1}{2}-i-k} \Big(\frac{h}{x}\Big)^{k-1} \right| \\
&\simeq h \le (xh)^{1/2},
\end{aligned}
$$

where we recall that the binomial coefficient is defined as $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$ for an arbitrary number r.

Using the estimate at the origin for $u^{(n)}$ and u, the boundedness of u, the mean value theorem on $\delta_{2h}K_{\mathrm{sing}}(x \pm y)$, and the smoothness and decay of K_{reg} and K away from its singularity, the seven integrals can be bounded as

$$\left| \int_{-h/3}^{7h/3} -K_{\mathrm{sing}}(y)\delta_{2(y-h)}u^{(n)}(x)(x-h)^n \,\mathrm{d}y \right| \lesssim (xh)^{1/2},$$

$$\left| \int_{4h/3}^{2x/3} -\delta_{2h}K_{\mathrm{sing}}(y)\delta_{2y}u^{(n)}(x)(x-h)^n \,\mathrm{d}y \right| \lesssim (xh)^{1/2},$$

$$\left| \int_{5x/3}^{2\nu+x} u(y)\big(\delta_{2h}K^{(n)}_{\mathrm{sing}}(x+y) + \delta_{2h}K^{(n)}_{\mathrm{sing}}(x-y)\big)(x-h)^n \,\mathrm{d}y \right| \lesssim h \le (xh)^{1/2},$$

$$\left| \int_{-x/3}^{5x/3} \delta_{2h}K^{(n)}_{\mathrm{sing}}(x+y)u(y)(x-h)^n \,\mathrm{d}y \right| \lesssim h \le (xh)^{1/2},$$

$$\left| \int_{x}^{2\nu+x} u(y)\big(\delta_{2h}K^{(n)}_{\mathrm{reg}}(x+y) + \delta_{2h}K^{(n)}_{\mathrm{reg}}(x-y)\big)(x-h)^n \,\mathrm{d}y \right| \lesssim h \le (xh)^{1/2},$$

$$\left| \int_{-x}^{x} \delta_{2h}K^{(n)}_{\mathrm{reg}}(x+y)u(y)(x-h)^n \,\mathrm{d}y \right| \lesssim h \le (xh)^{1/2},$$

$$\left| \int_{2\nu+x}^{\infty} u(y)\big(\delta_{2h}K^{(n)}(x+y) + \delta_{2h}K^{(n)}(x-y)\big)(x-h)^n \,\mathrm{d}y \right| \lesssim h \le (xh)^{1/2},$$

by calculations completely analogous to those in the proof of Lemma 6.2.

Combining all the right-hand side estimates with the left-hand side, we arrive at the desired result after dividing away the factor $x^{1/2}$. □

Step 4. With the first estimate at hand, we move on to the improved estimate.

Lemma 7.4. *Given Lemma 7.3, it follows that the improved estimate*

$$|x^{1/2}(x-h)^n(u^{(n)}(x+h) - u^{(n)}(x-h))| \lesssim h$$

holds uniformly on $[0, \nu]$*, for some* $\nu > 0$.

Proof. Consider $0 < h \le x \le \nu \ll 1$ for some $\nu > 0$ which we shrink whenever necessary. For $h \in [x/2, x]$ we have

$$|(x-h)^n(u^{(n)}(x+h) - u^{(n)}(x-h))| \lesssim |h|^{1/2} = \frac{h}{h^{1/2}} \le \frac{h}{(x/2)^{1/2}} \lesssim \frac{h}{x^{1/2}},$$

by the estimate in Lemma 7.3. We now show that it also holds for $h < x/2$, and consider the same equation as in the proof of Lemma 7.3. Notice from the proof of Lemma 7.3 that all the terms on the right-hand side are bounded by h, except for the first two integrals on $(-h/3, 7h/3)$ and $(4h/3, 2x/3)$. Therefore we only need to show that these two integrals are also bounded by h to arrive at the statement of the lemma.

The following bounds all follow from calculations analogous to those in the proof of Lemma (6.3). Applying Lemma 7.3, one can show that

$$\left| \int_{-h/3}^{7h/3} -K_{\text{sing}}(y)\delta_{2(y-h)}u^{(n)}(x)(x-h)^n \, \mathrm{d}y \right| \lesssim h,$$

and

$$\left| \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u^{(n)}(x)(x-h)^n \, \mathrm{d}y \right| \lesssim h\Big(1 + \Big|\ln\Big(\frac{h}{x}\Big)\Big|\Big).$$

Again, the final bound is not quite good enough, but combining the bounds so far gives us the estimate

$$|x^{1/2}(x-h)^n(u^{(n)}(x+h) - u^{(n)}(x-h))| \lesssim h\Big(1 + \Big|\ln\Big(\frac{h}{x}\Big)\Big|\Big),$$

which we can use to find that

$$\left| \int_{4h/3}^{2x/3} -\delta_{2h}K_{\text{sing}}(y)\delta_{2y}u^{(n)}(x)(x-h)^n \, \mathrm{d}y \right| \lesssim h.$$

Having shown that the integrals on $(-h/3, 7h/3)$ and on $(4h/3, 2x/3)$ are also bounded by h, we arrive at improved estimate. □

We have one more lemma left before we are ready for the $u^{(n)}$-limit, but first we need to recall the concept of a *double factorial*. The double factorial of a number n, denoted by $n!!$, is the product of all positive integers up to n with the same parity as n (i.e. for odd numbers, $(2n+1)!! = (2n+1)(2n-1)\cdots 5\cdot 3\cdot 1$). We will use double factorials, as it allows us to succinctly express the n-th derivative of $x^{1/2}$ as

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{1/2} = (-1)^{n-1}\frac{(2n-3)!!}{2^n}x^{\frac{1}{2}-n},$$

and the n-th derivative of K_{sing} as

$$K_{\text{sing}}^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}|2\pi x|^{-1/2} = (-\text{sgn}(x))^n \frac{(2n-1)!!}{\sqrt{\pi}|2x|^{n+\frac{1}{2}}}.$$

Note that these expressions also hold for the 0-th and 1-st derivative, as one can extend the double factorial of odd numbers to take any negative odd integer argument. Without going into the details of how this extension is done, we simply state that it gives us that $(-1)!! = 1$ and $(-3)!! = -1$.

Lemma 7.5. *For $n \in \mathbb{N}$, the identity*

$$\sum_{i=1}^{n} \binom{n+1}{i}(2i-3)!!(2n-2i-1)!! = 2\cdot(2n-1)!!$$

holds.

Proof. We proceed by induction. For $n = 1$, we find that

$$\sum_{i=1}^{1} \binom{2}{i} (2i-3)!!(2\cdot 1 - 2i - 1)!! = 2 = 2\cdot(2\cdot 1 - 1)!!,$$

which proves the base case. Now we assume the identity in the statement of the lemma holds for n, and must show that it holds for $n+1$ as well. Considering the left-hand side of the identity for $n+1$, Pascal's identity gives us that

$$\begin{aligned}
&\sum_{i=1}^{n+1} \binom{n+2}{i} (2i-3)!!(2(n+1) - 2i - 1)!! \\
&= \sum_{i=1}^{n+1} \binom{n+1}{i} (2i-3)!!(2(n+1) - 2i - 1)!! + \sum_{i=1}^{n+1} \binom{n+1}{i-1} (2i-3)!!(2(n+1) - 2i - 1)!!.
\end{aligned}$$

The right-hand side can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{n+1} \binom{n+1}{i} (2i-3)!!(2(n+1) - 2i - 1)!! \\
&\qquad + \sum_{j=0}^{n} \binom{n+1}{j} (2(j+1)-3)!!(2(n+1) - 2(j+1) - 1)!! \\
&= \sum_{i=1}^{n+1} \binom{n+1}{i} (2i-3)!!(2n - 2i - 1)!!(2n-2i+1) \\
&\qquad + \sum_{j=0}^{n} \binom{n+1}{j} (2j-3)!!(2n - 2j - 1)!!(2j-1),
\end{aligned}$$

where we first made the change of variables $j = i - 1$, then used the definition of the double factorial. Splitting of the $n+1$-term and the 0-th term from the first and second sum, respectively, allows us to rewrite the expression as

$$\begin{aligned}
&2\cdot(2n-1)!! + \sum_{i=1}^{n} \binom{n+1}{i} (2i-3)!!(2n-2i-1)!!\big((2n-2i+1) + (2i-1)\big) \\
&= 2\cdot(2n-1)!! + 2n\sum_{i=1}^{n} \binom{n+1}{i} (2i-3)!!(2n-2i-1)!! \\
&= 2\cdot(2n-1)!! + 2n\cdot 2\cdot(2n-1)!! \\
&= 2\cdot(2(n+1)-1)!!,
\end{aligned}$$

where we used the inductive hypothesis in the second-to-last equality. Thus we have shown that the inductive step holds, which concludes the proof. □

Step 4. Now we arrive at the main result of this paper.

Main Theorem 7.6. *The n-th derivative of u admits the limit*

$$\lim_{x\to 0} \frac{u^{(n)}}{x^{\frac{1}{2}-n}} = (-1)^{n-1}\frac{(2n-3)!!}{2^n}\sqrt{\frac{\pi}{8}},$$

if

$$\begin{aligned}
&(2k-3)!! \int_0^{2/3} \frac{1}{\tau^{3/2}}\Big((1+\tau)^{\frac{1}{2}-k} - (1-\tau)^{\frac{1}{2}-k}\Big)\, d\tau \\
&\quad + (2k+1)!! \int_{2/3}^{\infty} \frac{1}{\tau^{k+\frac{3}{2}}}\Big(|\tau-1|^{1/2} + (-1)^{k-1}(\tau+1)^{1/2}\Big)\, d\tau \\
&\quad + 3^k \sum_{i=0}^{k-1} \frac{(2i+1)!!(2k-2i-5)!!}{2^{i+\frac{1}{2}}}\Big((-1)^i 5^{-k+i+\frac{3}{2}} + 1\Big)
\end{aligned}$$

$$= 0,$$

for all $k \in \{2, .., n-1\}$ *when* $n \geq 3$. *This expression has been validated to equal zero for* $n \in \{1, .., 100\}$ *using a computer-aided approach.*

Proof. We will prove the statement by strong induction. We already know that the limit in the statement of the theorem is true for $n = 0$, $n = 1$, $n = 2$ and $n = 3$. When $n = 3$, the expression in the theorem statement is equal to zero for $k = 2$ (as we will see later, for $k = 2$ the expression is in-fact equivalent to the right-hand side of (6.15) which we recall was equal to zero), and thus have that a base case holds. In the following we will therefore consider $n \geq 3$. We need to show that the inductive step also holds. That is, given that the statement of the theorem is true for the k-th derivative of u for all $k \in \{3, ..., n\}$, we need to show that it is also holds for the $(n+1)$-th derivative.

Since we are given that the statement is true for the n-th derivative, we have that the estimates

$$|(x-h)^n (u^{(n)}(x+h) - u^{(n)}(x-h))| \lesssim h^{1/2}$$

and

$$|x^{1/2}(x-h)^n (u^{(n)}(x+h) - u^{(n)}(x-h))| \lesssim h$$

hold uniformly on $[0, \nu]$ for some small enough $\nu > 0$, by Lemma 7.3 and Lemma 7.4 respectively. Consequently, by an argument analogous to the one used to justify the differentiation under the integral sign in Section 6, the calculations in the proof of Lemma 7.3 justify the differentiation under the integral sign so that the assumption in Lemma 7.2 is true.

We consider the central difference equation satisfied by $u^{(n)}$ from Lemma 7.2, divided by $2h$ and multiplied by x^n:

$$\begin{aligned}
&\frac{\sum_{i=0}^{n} \binom{n}{i} x^n u^{(i)}(x+h) u^{(n-i)}(x+h) - \sum_{i=0}^{n} \binom{n}{i} x^n u^{(i)}(x-h) u^{(n-i)}(x-h)}{2h} \\
&= \frac{1}{2h} \int_{-h/3}^{7h/3} -K_{\text{sing}}(y) \delta_{2(y-h)} u^{(n)}(x) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{4h/3}^{2x/3} -\delta_{2h} K_{\text{sing}}(y) \delta_{2y} u^{(n)}(x) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{5x/3}^{2\nu+x} u(y) \big(\delta_{2h} K_{\text{sing}}^{(n)}(x+y) + \delta_{2h} K_{\text{sing}}^{(n)}(x-y)\big) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{-x/3}^{5x/3} \delta_{2h} K_{\text{sing}}^{(n)}(x+y) u(y) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{x}^{2\nu+x} u(y) \big(\delta_{2h} K_{\text{reg}}^{(n)}(x+y) + \delta_{2h} K_{\text{reg}}^{(n)}(x-y)\big) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{-x}^{x} \delta_{2h} K_{\text{reg}}^{(n)}(x+y) u(y) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \int_{2\nu+x}^{\infty} u(y) \big(\delta_{2h} K^{(n)}(x+y) + \delta_{2h} K^{(n)}(x-y)\big) x^n \, \mathrm{d}y \\
&\quad + \frac{1}{2h} \sum_{i=0}^{n-1} \delta_{2h} K_{\text{sing}}^{(i)}\Big(\frac{2}{3}x\Big) \bigg((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big) + u^{(n-1-i)}\Big(\frac{x}{3}\Big) \bigg) x^n.
\end{aligned}$$

Recall that we are working with $0 < 2h < x \leq \nu$.

Consider the left-hand side of the equation. By Lemma (7.1), we get that

$$\begin{aligned}
&\lim_{h \to 0} \frac{\sum_{i=0}^{n} \binom{n}{i} x^n u^{(i)}(x+h) u^{(n-i)}(x+h) - \sum_{i=0}^{n} \binom{n}{i} x^n u^{(i)}(x-h) u^{(n-i)}(x-h)}{2h} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} x^n u^{(i)}(x) u^{(n+1-i)}(x)
\end{aligned}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} \frac{u^{(i)}(x)}{x^{\frac{1}{2}-i}} \frac{u^{(n+1-i)}(x)}{x^{\frac{1}{2}-(n+1-i)}},$$

where we use that $x^n = \frac{1}{x^{\frac{1}{2}-i}} \frac{1}{x^{\frac{1}{2}-(n+1-i)}}$. Using the inductive hypothesis, we take the x-limit of this expression:

$$\begin{aligned}
&\lim_{x\to 0} \sum_{i=0}^{n+1} \binom{n+1}{i} \frac{u^{(i)}(x)}{x^{\frac{1}{2}-i}} \frac{u^{(n+1-i)}(x)}{x^{\frac{1}{2}-(n+1-i)}} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} \lim_{x\to 0} \frac{u^{(i)}(x)}{x^{\frac{1}{2}-i}} \lim_{x\to 0} \frac{u^{(n+1-i)}(x)}{x^{\frac{1}{2}-(n+1-i)}} \\
&= 2\sqrt{\frac{\pi}{8}} \lim_{x\to 0} \frac{u^{(n+1)}(x)}{x^{\frac{1}{2}-(n+1)}} + \sum_{i=1}^{n} \binom{n+1}{i} \lim_{x\to 0} \frac{u^{(i)}(x)}{x^{\frac{1}{2}-i}} \lim_{x\to 0} \frac{u^{(n+1-i)}(x)}{x^{\frac{1}{2}-(n+1-i)}} \\
&= 2\sqrt{\frac{\pi}{8}} \lim_{x\to 0} \frac{u^{(n+1)}(x)}{x^{\frac{1}{2}-(n+1)}} + \frac{(-1)^{n-1}}{2^{n+1}} \frac{\pi}{8} \sum_{i=1}^{n} \binom{n+1}{i} (2i-3)!!(2n-2i-1)!! \\
&= 2\sqrt{\frac{\pi}{8}} \lim_{x\to 0} \frac{u^{(n+1)}(x)}{x^{\frac{1}{2}-(n+1)}} + \frac{(-1)^{n-1}}{2^{n}} \frac{\pi}{8} (2(n+1)-3)!!.
\end{aligned}$$

The final equality follows from Lemma 7.5. Thus, it is clear that we arrive at the desired result if we are able to show that the limits of the right-hand side equals zero.

For the right-hand side, all seven integrals can be rewritten and bounded in a completely analogous manner to the seven integrals in the proof of Proposition 6.4, only now with the estimates from Lemma 7.3 and Lemma 7.4 instead. Then, as each integrand is dominated by an integrable function on their respective domains of integration, we can move the h-limit and x-limit inside of the integrals, which also are completely analogous to the ones in the proof of Proposition 6.4. The only change that we would like to explicitly point out, is that in the calculation corresponding to (6.14), the fraction with the binomial sum in the numerator will here be

$$\frac{\left(2 \sum_{\substack{k=1,\\ k \text{ odd}}}^{2n+1} \binom{2n+1}{k} (\tau+1)^{2n+1-k} \left(\frac{h}{x}\right)^{k}\right)}{\left((\tau+1+\frac{h}{x})^{2n+1} - (\tau+1-\frac{h}{x})^{2n+1}\right)}.$$

That is, we have

$$\lim_{x\to 0} \lim_{h\to 0} \int_{-h/3}^{7h/3} -\frac{1}{2h} K_{\text{sing}}(y) \delta_{2(y-h)} u^{(n)}(x) x^n \, \mathrm{d}y = 0,$$

$$\lim_{x\to 0} \lim_{h\to 0} \int_{4h/3}^{2x/3} -\frac{1}{2h} \delta_{2h} K_{\text{sing}}(y) \delta_{2y} u^{(n)}(x) x^n \, \mathrm{d}y = \int_0^{2/3} (-1)^{n-1} \frac{(2n-3)!!}{2^{n+3}} \frac{1}{\tau^{3/2}} \left(\frac{1}{(1+\tau)^{n-\frac{1}{2}}} - \frac{1}{(1-\tau)^{n-\frac{1}{2}}} \right) \mathrm{d}\tau,$$

$$\lim_{x\to 0} \lim_{h\to 0} \int_{5x/3}^{2\nu+x} \frac{1}{2h} u(y) \big(\delta_{2h} K_{\text{sing}}^{(n)}(x+y) + \delta_{2h} K_{\text{sing}}^{(n)}(x-y)\big) x^n \, \mathrm{d}y = \int_{5/3}^{\infty} \frac{(2n+1)!!}{2^{n+3}} \tau^{1/2} \left((-1)^{n+1} \frac{1}{(\tau+1)^{n+\frac{3}{2}}} + \frac{1}{(\tau-1)^{n+\frac{3}{2}}} \right) \mathrm{d}\tau,$$

$$\lim_{x\to 0} \lim_{h\to 0} \int_{-x/3}^{5x/3} \frac{1}{2h} \delta_{2h} K_{\text{sing}}^{(n)}(x+y) u(y) x^n \, \mathrm{d}y = \int_{-1/3}^{5/3} (-1)^{n+1} \frac{(2n+1)!!}{2^{n+3}} \frac{|\tau|^{1/2}}{(\tau+1)^{n+\frac{3}{2}}} \, \mathrm{d}\tau,$$

$$\lim_{x\to 0} \lim_{h\to 0} \int_{x}^{2\nu+x} \frac{1}{2h} u(y) \big(\delta_{2h} K_{\text{reg}}^{(n)}(x+y) + \delta_{2h} K_{\text{reg}}^{(n)}(x-y)\big) x^n \, \mathrm{d}y = 0,$$

$$\lim_{x\to 0}\lim_{h\to 0}\int_{-x}^{x}\frac{1}{2h}\delta_{2h}K_{\mathrm{reg}}^{(n)}(x+y)u(y)x^n\ \mathrm{d}y=0,$$

$$\lim_{x\to 0}\lim_{h\to 0}\int_{2\nu+x}^{\infty}\frac{1}{2h}u(y)\big(\delta_{2h}K^{(n)}(x+y)+\delta_{2h}K^{(n)}(x-y)\big)x^n\ \mathrm{d}y=0,$$

for the limits of the seven integrals.

All that is remaining on the right-hand side is the limits of the boundary terms. For the h-limit, we get

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{2h}\sum_{i=0}^{n-1}\delta_{2h}K_{\mathrm{sing}}^{(i)}\Big(\frac{2}{3}x\Big)\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big)+u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)x^n\\
&=\sum_{i=0}^{n-1}K_{\mathrm{sing}}^{(i+1)}\Big(\frac{2}{3}x\Big)\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big)+u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)x^n\\
&=\sum_{i=0}^{n-1}(-1)^{i+1}\frac{(2i+1)!!}{\sqrt{\pi}\left|\frac{4}{3}x\right|^{i+\frac{3}{2}}}\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big)+u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)x^n.
\end{aligned}$$

For the x-limit, we use the inductive hypothesis to find that

$$\begin{aligned}
&\lim_{x\to 0}\sum_{i=0}^{n-1}(-1)^{i+1}\frac{(2i+1)!!}{\sqrt{\pi}\left|\frac{4}{3}x\right|^{i+\frac{3}{2}}}\Big((-1)^i u^{(n-1-i)}\Big(\frac{5}{3}x\Big)+u^{(n-1-i)}\Big(\frac{x}{3}\Big)\Big)x^n\\
&=\lim_{x\to 0}\sum_{i=0}^{n-1}\frac{(2i+1)!!}{\sqrt{\pi}\left|\frac{4}{3}x\right|^{i+\frac{3}{2}}}x^{i+\frac{3}{2}}\Big(-\Big(\frac{5}{3}\Big)^{\frac{1}{2}-(n-1-i)}\frac{u^{(n-1-i)}\big(\frac{5}{3}x\big)}{\big(\frac{5}{3}x\big)^{\frac{1}{2}-(n-1-i)}}\\
&\qquad\qquad+(-1)^{i+1}\Big(\frac{1}{3}\Big)^{\frac{1}{2}-(n-1-i)}\frac{u^{(n-1-i)}\big(\frac{x}{3}\big)}{\big(\frac{1}{3}x\big)^{\frac{1}{2}-(n-1-i)}}\Big)\\
&=\sum_{i=0}^{n-1}\frac{(2i+1)!!}{\sqrt{\pi}\big(\frac{4}{3}\big)^{i+\frac{3}{2}}}\Big(-\Big(\frac{5}{3}\Big)^{\frac{1}{2}-(n-1-i)}(-1)^{n-1-i-1}\frac{(2(n-1-i)-3)!!}{2^{n-1-i}}\sqrt{\frac{\pi}{8}}\\
&\qquad\qquad+(-1)^{i+1}\Big(\frac{1}{3}\Big)^{\frac{1}{2}-(n-1-i)}(-1)^{n-1-i-1}\frac{(2(n-1-i)-3)!!}{2^{n-1-i}}\sqrt{\frac{\pi}{8}}\Big)\\
&=\sum_{i=0}^{n-1}(-1)^{n-1}\frac{(2i+1)!!(2n-2i-5)!!}{2^{n+i+\frac{7}{2}}}3^n\Big((-1)^i5^{-n+i+\frac{3}{2}}+1\Big).
\end{aligned}$$

Combining all of the terms, we find that the right-hand side equals

$$\begin{aligned}
&\int_0^{2/3}(-1)^{n-1}\frac{(2n-3)!!}{2^{n+3}}\frac{1}{\tau^{3/2}}\Big(\frac{1}{(1+\tau)^{n-\frac{1}{2}}}-\frac{1}{(1-\tau)^{n-\frac{1}{2}}}\Big)\ \mathrm{d}\tau\\
&+\int_{5/3}^{\infty}\frac{(2n+1)!!}{2^{n+3}}\tau^{1/2}\Big((-1)^{n+1}\frac{1}{(\tau+1)^{n+\frac{3}{2}}}+\frac{1}{(\tau-1)^{n+\frac{3}{2}}}\Big)\ \mathrm{d}\tau\\
&+\int_{-1/3}^{5/3}(-1)^{n+1}\frac{(2n+1)!!}{2^{n+3}}\frac{|\tau|^{1/2}}{(\tau+1)^{n+\frac{3}{2}}}\ \mathrm{d}\tau\\
&+\sum_{i=0}^{n-1}(-1)^{n-1}\frac{(2i+1)!!(2n-2i-5)!!}{2^{n+i+\frac{7}{2}}}3^n\Big((-1)^i5^{-n+i+\frac{3}{2}}+1\Big),
\end{aligned}$$

after taking the h- and x-limits. Note that this expression corresponds to the right-hand side of (6.15) from the proof of the u'''-limit. Through changes of variables one finds that the two middle integrals can be combined, allowing us to rewrite this as

$$\begin{aligned}\frac{(-1)^{n-1}}{2^{n+3}}\Bigg((2n-3)!!\int_0^{2/3}\frac{1}{\tau^{3/2}}\Big((1+\tau)^{\frac{1}{2}-n}-(1-\tau)^{\frac{1}{2}-n}\Big)\,\mathrm{d}\tau \\ +(2n+1)!!\int_{2/3}^{\infty}\frac{1}{\tau^{n+\frac{3}{2}}}\Big(|\tau-1|^{1/2}+(-1)^{n-1}(\tau+1)^{1/2}\Big)\,\mathrm{d}\tau \\ +3^n\sum_{i=0}^{n-1}\frac{(2i+1)!!(2n-2i-5)!!}{2^{i+\frac{1}{2}}}\Big((-1)^i5^{-n+i+\frac{3}{2}}+1\Big)\Bigg),\end{aligned} \tag{7.2}$$

after pulling out common factors.

Assume for a moment that (7.2) equals zero for all integers $n \geq 3$ (we know from Proposition (6.4) that it equals zero when $n = 2$). Combining everything we have so far would then give us that

$$\lim_{x\to 0}\frac{u^{(n+1)}(x)}{x^{\frac{1}{2}-(n+1)}}=\frac{(-1)^{n-1}}{2^{n+1}}\sqrt{\frac{\pi}{8}}(2(n+1)-3)!!,$$

after rearranging, which would prove that the inductive step holds. Unfortunately, we have not been able to show that (7.2) equals zero in the case of a general positive integer n, which is why the value of the $u^{(n)}$-limit in the theorem statement is conditional on the value of (7.2). Using the computer program Maple, (7.2) has been evaluated to zero for the specific cases of $n \in \{1, ..., 100\}$, but we have not managed to prove the analytical value of (7.2) in the general case. We expect further considerations will result in an analytical proof for the value of (7.2) in the case of a general n. □

Let us take a closer look at (7.2) from the proof of Theorem 7.6, which we did not manage to determine the exact value of. One might wish to try integration by parts n times on the middle integral, as the integrand would then be very similar to the integrand in the first integral, but this would introduce non-integrable singularities. In-fact, note that the integrals in this expression are exactly those from (6.8), which we got by formally integrating by parts while trying to get away from the non-integrable singularities in (6.7). Thus, (7.2) is, in a sense, what we would get if we *could* integrate (6.7) by parts, which we could not do due to the boundary terms exploding (and is why we had to alter our approach for the u'''-limit and beyond in the first place).

Doing integration by parts only one time on each of the integrals in (7.2) does, however, not introduce any singularities:

$$\begin{aligned}\frac{(-1)^n}{2^{n+3}}(2n-1)!!\Bigg(\int_0^{2/3}\frac{1}{\tau^{1/2}}\Big((1+\tau)^{-\frac{1}{2}-n}+(1-\tau)^{-\frac{1}{2}-n}\Big)\,\mathrm{d}\tau \\ +\int_{2/3}^{\infty}\frac{1}{\tau^{n+\frac{1}{2}}}\Big(\operatorname{sgn}(1-\tau)|\tau-1|^{-\frac{1}{2}}+(-1)^n(\tau+1)^{-\frac{1}{2}}\Big)\,\mathrm{d}\tau \\ -\frac{3^n}{(2n-1)!!}\sum_{i=0}^{n-1}\frac{(2i+1)!!(2n-2i-5)!!}{2^{i+\frac{1}{2}}}\Big((-1)^i5^{-n+i+\frac{3}{2}}+1\Big) \\ -3^n2^{\frac{1}{2}}\Big(\frac{-5^{\frac{1}{2}-n}+1}{2n-1}+\frac{(-1)^{n-1}5^{\frac{1}{2}}+1}{2^n}\Big)\Bigg),\end{aligned}$$

This integration by parts gives us a common factor of $(2n-1)!!$ in front of the integrals, but also some new boundary terms, so the new expression is not really any simpler than (7.2), but we decided to at least mention it.

We have considered the possibility of an inductive proof to show that (7.2) equals zero, but this seems very difficult as we cannot see a clear way to express the $n+1$-th case of the integrals in terms of the n-th case. Still, we believe that further considerations of (7.2) will result in an analytical proof of its exact value.

Anyways, despite the troubles with evaluating (7.2), the proof of Theorem 7.6 at least shows that $u^{(n)}$-limit actually exists, as the following corollary shows.

Corollary 7.7. *The limit of the n-th derivative of u,*

$$\lim_{x \to 0} \frac{u^{(n)}}{x^{\frac{1}{2}-n}},$$

exists.

Proof. Switch out the statement of Theorem 7.6 with the statement of this corollary, and proceed with the proof exactly as before. As we know that (7.2) exists (despite not knowing its exact value) we get that the $u^{(n+1)}$-limit exists, which shows that the inductive step holds. $\square$

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