Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Tord Skiftestad

Enhanced Triangulated Categories

Master's thesis in Mathematical Sciences
Supervisor: Petter Andreas Bergh
June 2023

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Abstract

In this thesis we study triangulated categories which appear as the cohomology of certain differential graded categories, and those which appear as the stable category of Frobenius categories. The classes of triangulated categories arising from either construction coincide, and are known as *algebraic* triangulated categories. Given such a triangulated category, a natural question to ask is whether the corresponding differential graded category is unique.

Sammendrag

I denne oppgaven studerer vi triangulerte kategorier som oppstår som kohomologien til visse differensialgraderte kategorier, og de som oppstår som den stabile kategorien til Frobeniuskategorier. Klassene av triangulerte kategorier som oppstår fra hver av disse konstruksjonene sammenfaller, og omtales som *algebraiske* triangulerte kategorier. Gitt en slik triangulert kategori, er det naturlig å spørre om den tilsvarende differensialgraderte kategorien er unik.

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Contents

Introduction

Triangulated categories appear in multiple branches of mathematics. In an attempt to better understand them, several different situations where they arise have been characterized. After going over some preliminaries, we will study two such situations.

To understand the first case, we will define *differential graded categories*, categories where morphism spaces have the structure of chain complexes. By studying modules over such categories, we establish a variant of the *Yoneda embedding* for such categories. We then prove that the cohomology of a *pretriangulated* differential graded category is triangulated. The core idea is an adaptation of the proof that the *homotopy category of chain complexes* is triangulated, by using the equivalence established by the Yoneda embedding.

Next we study *exact categories*, a generalization of abelian categories. We will study *Frobenius categories*, exact categories where injectives and projectives coincide, and see how the *stable category* of such categories are triangulated.

Finally we see how the classes of triangulated categories arising from either of these constructions essentially coincide. We say the triangulated categories constructed in this way are *algebraic*. Afterwards we will do a brief detour looking at a broader class of triangulated categories which are called *topological*.

Given an algebraic triangulated category, a natural question to ask is whether the corresponding differential graded category is unique. We will look at an example showing that this is not necessarily the case.

1 Preliminaries

We start out this chapter by recalling some definitions from category theory which we will be using. Afterwards we have a quick recap of some basic concepts from homological algebra, and finally we recall the definition of a triangulated category. Throughout this thesis, let k be a commutative ring. All tensor product are assumed to be over k.

1.1 Categories

We will assume familiarity with basic category theory. We will however still do a quick recap of the concepts we use the most in this thesis.

For a category $\mathcal{C}$, we denote the collection of objects by $\mathrm{Obj}(\mathcal{C})$, or simply $\mathcal{C}$ when it is clear from the context. For two objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, we denote the collection of morphisms from X to Y by $\mathcal{C}(X, Y)$. If all these collections of morphisms are sets, we say that the category is *locally small*. If in addition the collection of objects $\mathrm{Obj}(\mathcal{C})$ forms a set, we say the category is *small*.

For morphisms $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, Z)$ we denote the composition by $g \circ f \in \mathcal{C}(X, Z)$. We denote identity morphisms by $1_X \in \mathcal{C}(X, X)$. For a morphism $f \in \mathcal{C}(X, Y)$ and any object $Z \in \mathcal{C}$, we can create two different maps between morphism sets given by compostion. We denote the map given by *postcomposition* with f by

$$\begin{aligned} f_* : \mathcal{C}(Z, X) &\to \mathcal{C}(Z, Y) \\ g &\mapsto f \circ g \end{aligned}$$

and the map given by *precomposition* with f by

$$\begin{aligned} f^* : \mathcal{C}(Y, Z) &\to \mathcal{C}(X, Z) \\ g &\mapsto g \circ f \end{aligned}$$

Definition. For two morphisms $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(X, Z)$, we define the *pushout* of f, g as an object P together with two morphisms $u \in \mathcal{C}(Y, P), v \in \mathcal{C}(Z, P)$ such that $u \circ f = v \circ g$. Moreover for any other object W and morphisms $s \in \mathcal{C}(Y, W), t \in \mathcal{C}(Z, W)$ such that $s \circ f = t \circ g$, we require the existence of a *unique* morphism $\phi \in \mathcal{C}(P, W)$ such that $s = \phi \circ u$ and $t = \phi \circ v$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "u"] \arrow[ddr, bend left, "s"] & \\
Z \arrow[r, "v"] \arrow[drr, bend right, "t"] & P \arrow[dr, dotted, "\exists!\phi"] & \\
& & W
\end{tikzcd}$$

Remark 1.1. The concept of a *pullback* is defined dually, though we will be most interested in pushouts in this thesis. Note that pushouts and pullbacks do not necessarily exist. If they do exist however, they can be shown to be unique up to unique isomorphism.

We state some useful properties when dealing with pushouts.

Proposition 1.2. *Consider the following diagram where P is the pushout of u and v. Let W be any object, and let $f, g : P \to W$ be morphisms such that $f \circ s = g \circ s$ and $f \circ t = g \circ t$.*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "v"] & Y \arrow[d, "t"] & \\
Z \arrow[r, "s"] & P \arrow[dr, bend left, "g"] \arrow[dr, bend right, "f"'] & \\
& & W
\end{tikzcd}$$

Then $f = g$.

Proof. First we note that we have the morphisms $f \circ s : Z \to W$, $f \circ t : Y \to W$, satisfying $(f \circ s) \circ v = (f \circ t) \circ u$. Hence the pushout property of P gives a *unique* morphism $\phi : P \to W$, satisfying $\phi \circ s = f \circ s$ and $\phi \circ t = f \circ t$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "v"] & Y \arrow[d, "t"] \arrow[ddr, bend left, "f \circ t"] & \\
Z \arrow[r, "s"] \arrow[drr, bend right, "f \circ s"'] & P \arrow[dr, dotted, "\exists ! \phi"] & \\
& & W
\end{tikzcd}$$

By assumtion both f and g satisfy both the relations required by ϕ, and since this morphisms is unique, we conclude that $f = g$. $\square$

Proposition 1.3. *Consider the following commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "s"] \\
Z \arrow[r, "g"] \arrow[d, "v"] & P \arrow[d, "t"] \\
W \arrow[r, "h"] & Q
\end{tikzcd}$$

where P is the pushout of f and u, and Q is the pushout of f and $v \circ u$. Then Q is also the pushout of g and v.

Proof. Let V be any object, and assume we have morphisms $i : P \to V$, $j : W \to V$ such that $i \circ g = j \circ v$. Considering the composition $i \circ s$, the pushout property of Q gives a unique morphism $\phi : Q \to V$ satisfying

$\phi \circ t \circ s = i \circ s$ and $\phi \circ h = j$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "s"] \arrow[ddr, bend left, "i \circ s"] & \\
Z \arrow[r, "g"] \arrow[d, "v"] & P \arrow[d, "t"] \arrow[dr, bend left, "i"] & \\
W \arrow[r, "h"] \arrow[rr, bend right, "j"'] & Q \arrow[r, dotted, "\phi"'] & V
\end{tikzcd}$$

We immediately have that $(\phi \circ t) \circ s = i \circ s$. We also have that $i \circ g = j \circ v = \phi \circ h \circ v = (\phi \circ t) \circ g$. By Proposition 1.2 we get that $i = \phi \circ t$. Thus the morphism ϕ satisfies what is needed for the induced morphism from Q as a pushout of g and v. Uniqueness follows from ϕ being unique even when required to satisfy "weaker" relations. Hence Q is the pushout of g and v. $\square$

Definition. A category $\mathcal{C}$ is called *additive* if all the following properties are satisfied.

(1) All morphism spaces $\mathcal{C}(X, Y)$ are abelian groups, and composition respects this structure. In particular this means that for $f, g \in \mathcal{C}(Y, Z)$, $u, v \in \mathcal{C}(X, Y)$, the composition must satisfy

$$(f + g) \circ u = f \circ u + g \circ u$$
$$f \circ (u + v) = f \circ u + f \circ v$$

(2) There exists a *zero object* $0 \in \mathcal{C}$ such that $\mathcal{C}(X, 0) = \mathcal{C}(0, X) = \{0\}$ for all $X \in \mathcal{C}$, where $\{0\}$ is the abelian group with one object.

(3) For every pair of objects $X, Y \in C$, the biproduct $X \oplus Y$ exists, that is, an object equiped with morphisms

$$\begin{tikzcd}
X \arrow[r, bend left, "i_X"] & X \oplus Y \arrow[l, bend left, "\pi_X"] \arrow[r, bend left, "\pi_Y"] & Y \arrow[l, bend left, "i_Y"]
\end{tikzcd}$$

satisfying $\pi_X \circ i_X = 1_X$, $\pi_Y \circ i_Y = 1_Y$ and $i_X \circ \pi_X + i_Y \circ \pi_Y = 1_{X \oplus Y}$.

Remark 1.4. We will often use *matrix notation* when dealing with biproducts. For example if $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(X,Z)$, then we may draw the following diagram.

$$X \xrightarrow{\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)} Y \oplus Z$$

Keep in mind that this is just shorthand notation for the morphism $i_Y \circ f + i_Z \circ g$.

$$\begin{tikzcd}
 & Y \arrow[dr, "i_Y"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & Y \oplus Z \\
 & Z \arrow[ur, "i_Z"'] &
\end{tikzcd}$$

It can be shown that $\pi_Y \circ i_X = 0$ if $X \neq Y$. This means that composition of morphisms written in matrix notation ends up behaving exactly like matrix multiplication. Thus we will use matrix notation for biproducts whenever this is convenient.

Definition. Let $\mathcal{C}$ be an additive category, and let $f \in \mathcal{C}(X,Y)$. A *cokernel* of f is an object Q and a morphism $q \in \mathcal{C}(Y,Q)$ such that $q \circ f = 0$. Moreover for any other object P and morphism $p \in \mathcal{C}(Y,P)$ satisfying $p \circ f = 0$, we require p to factor through q *uniquely*. In particular we require the existence of a *unique* morphism $g \in \mathcal{C}(Q,P)$ such that $p = g \circ q$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "q"] \arrow[dr, "p"'] & Q \arrow[d, dashed, "\exists! g"] \\
 & & P
\end{tikzcd}$$

Remark 1.5. The concept of *kernels* are defined dually. It can be shown that the cokernel morphism q is always an epimorphisms, and dually that all kernels are monomorphisms. Note that kernels and cokernels do not necessarily exist, but because of the uniqueness they guarantee, it can be shown that if they exist, they are unique up to unique isomorphism.

Definition. Given a category $\mathcal{C}$, and some collection of morphisms W, the localization of $\mathcal{C}$ with respect to W is the category $\mathcal{C}[W^{-1}]$, which is obtained by inverting all morphisms in W. Specifically we require that we have a functor $Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that all morphisms in W are sent to isomorphisms in $\mathcal{C}[W^{-1}]$, and so that for any other functor $P : \mathcal{C} \to \mathcal{D}$ sending all morphisms in W to isomorphisms, there exists a unique functor $S : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that $P = S \circ Q$.

Remark 1.6. We note that from this definition alone, the category $\mathcal{C}[W^{-1}]$ does not necessarily exist. If it exists however, it can be shown to be unique up to unique equivalence of categories. For more details on the construction of the category $\mathcal{C}[W^{-1}]$, see [5].

As we will not need the technical details, for our purposes it is enough to understand $\mathcal{C}[W^{-1}]$ as the most natural way to alter the category $\mathcal{C}$ such that all morphisms in W become isomorphisms.

1.2 Graded modules

A k-category $\mathcal{C}$ is a category in which all morphism spaces $\mathcal{C}(X, Y)$ are k-modules, and where the composition of morphisms respects this structure. Specifically we require that the composition

$$\rho : \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z), \quad \rho(f \otimes g) = f \circ g$$

is a k-module homomorphism. Note that all k-categories are locally small, since k-modules are sets.

A functor between k-categories $F : \mathcal{C} \to \mathcal{D}$, is a map on objects $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, together with a k-module homomorphism for every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, $F_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$, which preserves identities and composition.

A graded k-module A is a k-module which can be decomposed as a direct sum indexed over the integers.

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

For two graded k-modules A and B, a graded morphism of degree n, $f : A \to B$ is a collection of k-module morphisms

$$f = \{f^p : A^p \to B^{p+n} \mid p \in \mathbb{Z}\}$$

We denote the collection of all graded morphisms from A to B of degree n by $\mathrm{Hom}(A,B)^n$. This forms a k-module with componentwise addition; for morphisms $f,g \in \mathrm{Hom}(A,B)^n$, the sum is given by the collection

$$f+g = \{(f+g)^p := f^p + g^p : A^p \to B^{p+n} \mid p \in \mathbb{Z}\}$$

We define the composition of graded morphisms in the natural way: If $f \in \mathrm{Hom}(A,B)^n$, and $g \in \mathrm{Hom}(B,C)^m$, then the composition $g \circ f \in \mathrm{Hom}(A,C)^{n+m}$ is given by the collection

$$g \circ f = \{(g \circ f)^p := g^{p+n} \circ f^p : A^p \to C^{p+n+m} \mid p \in \mathbb{Z}\}$$

A differential graded k-module is a graded k-module A, together with a graded morphism of degree 1 from A to itself

$$d_A = \{d_A^p : A^p \to A^{p+1} \mid p \in \mathbb{Z}\}$$

such that the composition $d_A \circ d_A = 0_A$, where $0_A : A \to A$ is the graded morphism of degree 2 which is 0 everywhere. This is called the differential of the graded k-module A. Sometimes we abbreviate $d_A = d$ if there is no ambiguity which graded k-module the differential refers to. Throughout this thesis we will refer to differential graded k-modules simply as *chain complexes*, and assume that all such chain complexes are modules over the ring k unless stated otherwise.

Let A and B be chain complexes, and $f : A \to B$ a graded morphism of degree 0. If $d_B \circ f = f \circ d_A$ we say that f commutes with the differential. We call such graded morphisms of degree 0 which commute with the differential *chain maps*.

Definition. The category $\mathcal{C}(k)$ is the category where objects are chain complexes over k, and morphisms are chain maps.

For a chain complex A, we denote by Id_A the identity chain map from A to itself.

Given a chain complex A, we define the homological constructions of cocycles and cohomology. The n-th cocycle of A is $Z^n(A) := \mathrm{Ker}(d_A^n)$, and the n-th cohomology of A is $H^n(A) := \mathrm{Ker}(d_A^n)/\mathrm{Im}(d_A^{n-1})$.

Let $f \in \mathcal{C}(k)(A,B)$ be a chain map. Then $f(Z^n(A)) \subset Z^n(B)$ for all $n \in \mathbb{Z}$. We see this because for any $a \in Z^n(A)$, we have that $d_A(a) = 0$. Since f is a chain map, we know that $d_B \circ f = f \circ d_A$, and hence

$$d_B(f(a)) = f(d_A(a)) = f(0) = 0$$

Thus $f(a) \in Z^n(B)$.

The tensor product $A \otimes B$ of two graded k-modules, is a new graded k-module with components

$$(A \otimes B)^n = \bigoplus_{p+q=n} A^p \otimes B^q$$

Given graded morphisms $f : A \to A'$ and $g : B \to B'$, the tensor product $f \otimes g : A \otimes B \to A' \otimes B'$ is defined using the Koszul sign rule

$$(f \otimes g)(a \otimes b) = (-1)^{pq} f(a) \otimes g(b)$$

where p is the degree of g and $a \in A^q$. If A and B are chain complexes, then the graded module

$$A \otimes B := \bigoplus_{n \in \mathbb{Z}} (A \otimes B)^n$$

becomes a chain complex when given the differential

$$d_{A \otimes B} := d_A \otimes \mathrm{Id}_B + \mathrm{Id}_A \otimes d_B$$

For a chain complex A, the shifted chain complex $A[n]$ has components $(A[n])^p = A^{n+p}$, and differential with components $d^p_{A[n]} = (-1)^n d^{n+p}_A$. For a chain map $f : A \to B$, the shifted map $f[n] : A[n] \to B[n]$, is the collection

$$f[n] = \{f[n]^p := f^{n+p} : A^{n+p} \to B^{n+p} \mid p \in \mathbb{Z}\}$$

For a chain map $f : A \to B$, we denote the *cone*[1] of f by $C(f) := B \oplus A[1]$, that is, the chain complex with components $C(f)^p = B^p \oplus A^{p+1}$, and differential given by

$$d_{C(f)} = \begin{pmatrix} d_B & f[1] \\ 0 & d_{A[1]} \end{pmatrix}$$

1.3 Triangulated categories

We recall the definition of a triangulated category [7, Definition 3.1].

[1]This is often written as $A[1] \oplus B$ instead, with a lower triangular matrix as differential.

Definition. A *triangulated category* is an additive category $\mathcal{C}$, together with an additive autoequivalence $[1] : \mathcal{C} \to \mathcal{C}$ called *shift* (or *translation*), and a collection of *distinguished triangles* Δ, which are sequences on the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying the following.

(TR1) (a) For every object $X \in \mathcal{C}$, the triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow X[1]$$

is in Δ.

(b) For every morphism $f : X \to Y$, there exists a triangle in Δ

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

The object Z is called a *cone* of the morphism f.

(c) The collection Δ is closed under isomorphisms. That is, if the top row is in the following diagram is in Δ, and u, v, w are isomorphisms making the diagram commute, then the bottom row is also in Δ.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & X[1] \arrow[d, "u[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

(TR2) For any triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in Δ, the *rotated triangles*

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are also in Δ.

(TR3) Given the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists\, w"] & X[1] \arrow[d, "u[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

where both rows are in Δ and the leftmost square commutes, there exists a morphism w making the whole diagram commute.

(TR4) Given the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "v"] & Z' \arrow[r, "b"] & X[1] \arrow[d, equal] \\
X \arrow[r, "vu"] \arrow[d, "u"] & Z \arrow[r, "c"] \arrow[d, equal] & Y' \arrow[r, "d"] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r, "v"] \arrow[d, "a"] & Z \arrow[r, "e"] \arrow[d, "c"] & X' \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[d, "a[1]"] \\
Z' & Y' & X' & Z'[1]
\end{tikzcd}$$

where the three rows are in Δ, there exist morphisms g, h, i completing the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "v"] & Z' \arrow[r, "b"] \arrow[d, "g"] & X[1] \arrow[d, equal] \\
X \arrow[r, "vu"] \arrow[d, "u"] & Z \arrow[r, "c"] \arrow[d, equal] & Y' \arrow[r, "d"] \arrow[d, "h"] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r, "v"] \arrow[d, "a"] & Z \arrow[r, "e"] \arrow[d, "c"] & X' \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[d, "a[1]"] \\
Z' \arrow[r, "g"] & Y' \arrow[r, "h"] & X' \arrow[r, "i"] & Z'[1]
\end{tikzcd}$$

such that the bottom row is in Δ and everything commutes.

Definition. A *triangulated functor* is an additive functor $F : \mathcal{T} \to \mathcal{S}$ between triangulated categories which commutes with shifts and preserves distinguished triangles. Specifically we require a natual isomorphism ϕ with components

$$\phi_X : F(X[1]_{\mathcal{T}}) \to F(X)[1]_{\mathcal{S}}$$

such that for any triangle in $\Delta_{\mathcal{T}}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]_{\mathcal{T}}$$

the resulting triangle under F

$$FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{\phi_X \circ F(h)} F(X)[1]_{\mathcal{S}}$$

is in $\Delta_{\mathcal{S}}$. A triangulated functor which is also an equivalence of categories is called a *triangulated equivalence*.

2 Differential graded categories

In this chapter we define *differential graded categories*, or simply *dg-categories*. Our goal is to see that given a so called *pretriangulated* dg-category, we can construct a triangulated category. A good reference for many of the results and concepts in this chapter is [10].

2.1 The basics

Definition. A *dg-category* $\mathcal{A}$ is a k-category in which all morphism spaces $\mathcal{A}(X,Y)$ are chain complexes over k, and where the composition of morphisms respects this structure. Specifically we require that the composition

$$\rho : \mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z), \quad \rho(g \otimes f) = g \circ f$$

is a chain map. We will use $\mathcal{A}$ throughout this thesis to denote such a dg-category. For brevity we denote the differential of the chain complex $\mathcal{A}(X,Y)$ by $d_{X,Y}$.

Remark 2.1. We take a moment to note some consequences of this composition law. First we see that if $f \in \mathcal{A}(X,Y)^n, g \in \mathcal{A}(Y,Z)^m$, then their composition $g \circ f \in \mathcal{A}(X,Z)^{n+m}$. This is because the element $g \otimes f \in \mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y)$ is by definition in degree $n+m$, and the chain map $g \otimes f \mapsto g \circ f$ preserves this degree.

If $f \in \mathcal{A}(X,Y)^n$ and $g \in \mathcal{A}(Y,Z)^m$, what is $d_{X,Z}(g \circ f)$? By applying the definition of the differential in $\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y)$, and using the Kozul sign rule, we get

$$\begin{aligned}
&d_{Y,Z \otimes X,Y}(g \otimes f) &= \\
&(d_{Y,Z} \otimes \mathrm{Id}_{\mathcal{A}(X,Y)} + \mathrm{Id}_{\mathcal{A}(Y,Z)} \otimes d_{X,Y})(g \otimes f) &= \\
&(d_{Y,Z} \otimes \mathrm{Id}_{\mathcal{A}(X,Y)})(g \otimes f) + (\mathrm{Id}_{\mathcal{A}(Y,Z)} \otimes d_{X,Y})(g \otimes f) &= \\
&(-1)^{0 \cdot m} d_{Y,Z}(g) \otimes \mathrm{Id}_{\mathcal{A}(X,Y)}(f) + (-1)^{1 \cdot m} \mathrm{Id}_{\mathcal{A}(Y,Z)}(g) \otimes d_{X,Y}(f) &= \\
&d_{Y,Z}(g) \otimes f + (-1)^m g \otimes d_{X,Y}(f)
\end{aligned}$$

Here we use the fact that the identity maps and differentials are graded morphisms of degree 0 and 1 respectively. Since the composition map ρ is a chain map, it commutes with the differentials. Hence we get

$$\begin{aligned}
&d_{X,Z}(g \circ f) = d_{X,Z}(\rho(g \otimes f)) && = \\
&\rho(d_{Y,Z \otimes X,Y}(g \otimes f)) && = \\
&\rho(d_{Y,Z}(g) \otimes f + (-1)^m g \otimes d_{X,Y}(f)) && = \\
&d_{Y,Z}(g) \circ f + (-1)^m g \circ d_{X,Y}(f)
\end{aligned}$$

This is called the *Leibniz rule* [10, p.154], a generalization of the product rule.

As we have seen, composition preserves sum of degrees. Since composition with 1_X should leave the element unchanged (and hence degree unchanged), we conclude that $1_X \in \mathcal{A}(X,X)^0$. Using the Leibniz rule, we also get

$$\begin{aligned}
&d(1_X) && = \\
&d(1_X \circ 1_X) && = \\
&d(1_X) \circ 1_X + 1_X \circ d(1_X) && = \\
&d(1_X) + d(1_X) \\
&\implies d(1_X) = 0
\end{aligned}$$

Hence $1_X \in Z^0(\mathcal{A}(X,X))$.

Let $f \in Z^0(\mathcal{A}(X,Y)), g \in Z^0(\mathcal{A}(Y,Z))$, then the composition satisfies

$$\begin{aligned}
&d_{X,Z}(g \circ f) && = \\
&d_{Y,Z}(g) \circ f + g \circ d_{X,Y}(f) && = \\
&0 \circ f + g \circ 0 && = \\
&0
\end{aligned}$$

Hence $g \circ f \in Z^0(\mathcal{A}(X,Z))$.

Now let $f \in H^0(\mathcal{A}(X,Y)), g \in H^0(\mathcal{A}(Y,Z))$. Is there a well defined composition $g \circ f \in H^0(X,Z)$? Assume we have two different representatives for each of f and g, satisfying $f_1 - f_2 = d^{-1}_{X,Y}(s_1)$ and $g_1 - g_2 = d^{-1}_{Y,Z}(s_2)$, for

$s_1 \in \mathcal{A}(X,Y)^{-1}$ and $s_2 \in \mathcal{A}(Y,Z)^{-1}$. We see that

$$\begin{aligned}
& d^{-1}_{X,Z}(g_2 \circ s_1 + s_2 \circ f_1) && = \\
& d^{-1}_{X,Z}(g_2 \circ s_1) + d^{-1}_{X,Z}(s_2 \circ f_1) && = \\
& d^{0}_{Y,Z}(g_2) \circ s_1 + g_2 \circ d^{-1}_{X,Y}(s_1) + d^{-1}_{Y,Z}(s_2) \circ f_1 - s_2 \circ d^{0}_{X,Y}(f_1) && = \\
& g_2 \circ d^{-1}_{X,Y}(s_1) + d^{-1}_{Y,Z}(s_2) \circ f_1 && = \\
& g_2 \circ (f_1 - f_2) + (g_1 - g_2) \circ f_1 && = \\
& g_2 \circ f_1 - g_2 \circ f_2 + g_1 \circ f_1 - g_2 \circ f_1 && = \\
& - g_2 \circ f_2 + g_1 \circ f_1 && = \\
& (g_1 \circ f_1) - (g_2 \circ f_2)
\end{aligned}$$

Since $(g_1 \circ f_1) - (g_2 \circ f_2)$ is in the image of the differential $d^{-1}_{X,Z}$, we conclude that the compositions $g_1 \circ f_1$ and $g_2 \circ f_2$ represent the same element in $H^0(X,Z)$.

Definition. We define $\mathcal{C}_{dg}(k)$, the *dg-category of chain complexes* over k. The objects in this category are chain complexes. For two chain complexes A and B, the morphism space is given by the graded module of graded morphisms from A to B, $\mathcal{C}_{dg}(k)(A,B) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(A,B)^n$. This graded module can be given the differential $d_{A,B} : \operatorname{Hom}(A,B)^n \to \operatorname{Hom}(A,B)^{n+1}$ defined by $d_{A,B}(f) = d_B \circ f + (-1)^{n+1} f \circ d_A$. This makes $\mathcal{C}_{dg}(k)$ into a dg-category.

Since the morphism spaces $\mathcal{A}(X,Y)$ in a dg-category are chain complexes, we can construct new categories where the morphism spaces are the 0-th cocycles and 0-th cohomology.

Definition. Let a $\mathcal{A}$ be a dg-category. Then we define the following two categories.

(1) The category $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$, with morphism spaces given by $Z^0(\mathcal{A})(X,Y) := Z^0(\mathcal{A}(X,Y))$. The composition is induced by the composition in $\mathcal{A}$. This is well defined, because as we saw in Remark 2.1, the composition of two morphisms in $Z^0(\mathcal{A})$ is again in $Z^0(\mathcal{A})$, and $Z^0(\mathcal{A})$ contains the identity morphisms. Note that this is not a dg-category, but a k-category.

(2) The category $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$, with morphism spaces given by $H^0(\mathcal{A})(X,Y) := H^0(\mathcal{A}(X,Y))$. The composition is induced by the composition in $\mathcal{A}$, and is well defined by Remark 2.1. Note that this is not a dg-category, but a k-category.

Remark 2.2. For any dg-category $\mathcal{A}$, we get a natural projection functor which we denote by $\pi_{\mathcal{A}} : Z^0(\mathcal{A}) \to H^0(\mathcal{A})$. This functor is the identity on objects, and the natural projection $\pi_{X,Y} : Z^0(\mathcal{A}(X,Y)) \to H^0(\mathcal{A}(X,Y))$ on morphisms.

In the following example, we will study the dg-category $\mathcal{C}_{dg}(k)$, and determine what $Z^0(\mathcal{C}_{dg}(k))$, and $H^0(\mathcal{C}_{dg}(k))$ are.

Example. First we take a look at $Z^0(\mathcal{C}_{dg}(k))$. Let $A, B \in \mathcal{C}_{dg}(k)$. We know that $\mathcal{C}_{dg}(k)(A,B)$ is a chain complex, where the n-th component is given by the graded morphisms of degree n from A to B. To study $Z^0(\mathcal{C}_{dg}(k))(A,B) := \operatorname{Ker}(d^0_{A,B})$, we look at the differential in degree zero: $d^0_{A,B} : \mathcal{C}_{dg}(k)(A,B)^0 \to \mathcal{C}_{dg}(k)(A,B)^1$, which is given by $d^0_{A,B}(f) = d_B \circ f - f \circ d_A$. We get

$$\begin{aligned}
&d^0_{A,B}(f) = 0 && \iff \\
&d_B \circ f - f \circ d_A = 0 && \iff \\
&d_B \circ f = f \circ d_A
\end{aligned}$$

That is, $f \in Z^0(\mathcal{C}_{dg}(k))(A,B) \iff f$ is a chain map. Hence we can conclude that $Z^0(\mathcal{C}_{dg}(k)) = \mathcal{C}(k)$, as this is exactly the category where all morphisms are required to be chain maps.

What about $H^0(\mathcal{C}_{dg}(k))$? Let $A, B \in \mathcal{C}_{dg}(k)$, and recall that

$$H^0(\mathcal{C}_{dg}(k)(A,B)) := \operatorname{Ker}(d^0_{A,B}) / \operatorname{Im}(d^{-1}_{A,B}) = C(k)(A,B) / \operatorname{Im}(d^{-1}_{A,B})$$

What is $\operatorname{Im}(d^{-1}_{A,B})$? If we take some $s \in \mathcal{C}_{dg}(k)(A,B)^{-1}$, then s is what is usually described as a *chain homotopy* in homological algebra.

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{-2}"] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[dl, "s^{-1}"'] & A^0 \arrow[r, "d_A^0"] \arrow[dl, "s^0"'] & A^1 \arrow[r, "d_A^1"] \arrow[dl, "s^1"'] & \cdots \arrow[dl, "s^2"'] \\
\cdots \arrow[r, "d_B^{-2}"] & B^{-1} \arrow[r, "d_B^{-1}"] & B^0 \arrow[r, "d_B^0"] & B^1 \arrow[r, "d_B^1"] & \cdots
\end{tikzcd}$$

Applying the diffential $d^{-1}_{A,B}$, we get the chain map $d^{-1}_{A,B}(s) = d_B \circ s + s \circ d_A$. Thus the image of the differential $d^{-1}_{A,B}$ are exactly the chain maps f which can be written as $f = d_B \circ s + s \circ d_A$ for some chain homotopy s. We say that such maps f are *homotopic to zero*. We say that two chain maps $g, h \in \mathcal{C}(k)(A, B)$ are *homotopic* if $g - h$ is homotopic to zero. We get

$$H^0(\mathcal{C}_{dg}(k))(A, B) := \mathrm{Ker}(d^0_{A,B}) / \mathrm{Im}(d^{-1}_{A,B}) = \mathcal{C}(k)(A, B)/\sim$$

where $\sim$ is the equivalence relation given by homotopy. The resulting category $K(k) := \mathcal{C}(k)/\sim$ is usually called the *homotopy category of chain complexes* over k, with chain complexes as objects, and where the morphisms are chain maps, up to homotopy.

Definition. For two dg-categories $\mathcal{A}$ and $\mathcal{B}$, a dg-functor $F : \mathcal{A} \to \mathcal{B}$ consist of a map on objects $F : \mathrm{Obj}(\mathcal{A}) \to \mathrm{Obj}(\mathcal{B})$, together with a chain map for each pair of objects $X, Y \in \mathcal{A}, F_{X,Y} : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$. These maps should also preserve identities $F_{X,X}(1_X) = 1_{F_X}$, and composition $F(g \circ_{\mathcal{A}} f) = F(g) \circ_{\mathcal{B}} F(f)$. We can picture this last property with the following commutative diagram.

$$\begin{tikzcd}
\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \arrow[r, "\circ_{\mathcal{A}}"] \arrow[d, "F_{Y,Z} \otimes F_{X,Y}"'] & \mathcal{A}(X, Z) \arrow[d, "F_{X,Z}"] \\
\mathcal{B}(FY, FZ) \otimes \mathcal{B}(FX, FY) \arrow[r, "\circ_{\mathcal{B}}"] & \mathcal{B}(FX, FZ)
\end{tikzcd}$$

In order to study a dg-category $\mathcal{A}$, it will be helpful to define the concept of a *dg $\mathcal{A}$-module*. This is a special type of dg-functor, namely one mapping into the category $\mathcal{C}_{dg}(k)$. Because we understand this category well, studying such modules will be helpful in understand the category $\mathcal{A}$. Specifically we make the following definition

Definition. A (right)[2] dg $\mathcal{A}$-module M is a contravariant dg-functor

$$M : \mathcal{A} \to \mathcal{C}_{dg}(k)$$

In order to define what morphisms between dg $\mathcal{A}$-modules are, we first define transformations of dg-functors.

[2]Because of the contravariance, these are often referred to as right dg $\mathcal{A}$-modules (with the covariant case referred to as left dg $\mathcal{A}$-modules). We will in this thesis simply refer to right dg $\mathcal{A}$-modules as dg $\mathcal{A}$-modules for brevity.

Definition. For two contravariant dg-functors $F, G : \mathcal{A} \to \mathcal{B}$, a graded transformation ϕ of degree n from F to G is a collection of elements

$$\phi = \{\phi_X \in \mathcal{B}(FX, GX)^n \mid X \in \mathcal{A}\}$$

such that for any two objects $X, Y \in \mathcal{A}$ and any morphism $f \in \mathcal{A}(Y, X)$, the following diagram commutes.

$$\begin{array}{ccc} FX & \xrightarrow{\phi_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\phi_Y} & GY \end{array}$$

We will refer to ϕ_X as the *component* of ϕ on X. The k-module $\mathrm{Hom}(F, G)^n$ is defined as the collection of all graded transformations of degree n from F to G. We define the graded k-module $\mathrm{Hom}(F, G)$ to have $\mathrm{Hom}(F, G)^n$ as its n-th component. We refer to elements in $\mathrm{Hom}(F, G)$ as *transformations* from F to G. The graded k-module $\mathrm{Hom}(F, G)$ can be made into a chain complex with a differential induced by the differential in the dg-category $\mathcal{B}$, component-wise. Specifically we construct a differential $d_{F,G} : \mathrm{Hom}(F, G)^n \to \mathrm{Hom}(F, G)^{n+1}$ as follows. Let $\phi \in \mathrm{Hom}(F, G)^n$. For each object $X \in \mathcal{A}$, we have the element $d_{FX,GX}(\phi_X) \in \mathcal{B}(FX, GX)^{n+1}$, where $d_{FX,GX}$ is the differential on the chain complex $\mathcal{B}(FX, GX)$. Then we define $d_{F,G}(\phi)$ to be given by the collection

$$d_{F,G}(\phi) = \{d_{FX,GX}(\phi_X) \mid X \in \mathcal{A}\}$$

If $\phi : F \to G$ is a transformation such that $\phi_X \in Z^0(\mathcal{B}(FX, GX))$ for all $X \in \mathcal{A}$, then we call ϕ a cycle transformation.[3]

Using these definitions we can define what morphisms between dg $\mathcal{A}$-modules are. Specifically we have that for two dg $\mathcal{A}$-modules $M, N : \mathcal{A} \to \mathcal{C}_{dg}(k)$, the morphisms from M to N are given by transformations from M to N.

Definition. The *dg-category of dg $\mathcal{A}$-modules $\mathcal{C}_{dg}(\mathcal{A})$*, has dg $\mathcal{A}$-modules as objects, and transformations of dg-functors as morphisms.

[3]Not to be confused with other potential meanings of the words *cycle/cyclic*, this simply refers to how for a chain complex A, the elements in $Z^0(A)$ are often called *(zero-th)-cycles.*

Remark 2.3. Given functors $F, G : \mathcal{A} \to \mathcal{B}$, the differential structure on $\mathrm{Hom}(F, G)$ is entirely determined by the differentials in the category $\mathcal{B}$. Since all dg $\mathcal{A}$-modules are functors into $\mathcal{C}_{dg}(k)$, this means that for dg $\mathcal{A}$-modules M, N, the differential structure on $\mathcal{C}_{dg}(\mathcal{A})(M, N)$ is induced by that of $\mathcal{C}_{dg}(k)$, which we understand well. For brevity, we often denote the differential on $\mathcal{C}_{dg}(\mathcal{A})(M, N)$ by $d_{\mathcal{C}_{dg}(\mathcal{A})}$, when it is clear from context which dg $\mathcal{A}$-modules we are studying.

In particular we have the following. Let $\phi \in \mathcal{C}_{dg}(\mathcal{A})(M, N)$ be a graded transformation of degree n. This means all components $\phi_X \in \mathcal{C}_{dg}(k)(MX, NX)^n$, that is, all ϕ_X are graded morphisms of degree n between the chain complexes MX and NX. We also have that if $\phi \in \mathcal{C}_{dg}(\mathcal{A})(M, N)$ is a cycle transformation, then all components $\phi_X \in Z^0(\mathcal{C}_{dg}(k)(MX, NX)) = \mathcal{C}(k)(MX, NX)$, in other words, all components are chain maps.

Definition. The *category of dg $\mathcal{A}$-modules* $\mathcal{C}(\mathcal{A})$, has dg $\mathcal{A}$-modules as objects and cycle transformations as morphisms. By the above remark we can equivalently define $\mathcal{C}(\mathcal{A}) := Z^0(\mathcal{C}_{dg}(\mathcal{A}))$. Similarly we define the *category up to homotopy of dg $\mathcal{A}$-modules* $\mathcal{H}(\mathcal{A}) := H^0(\mathcal{C}_{dg}(\mathcal{A}))$

We define shifts and cones in $\mathcal{C}(\mathcal{A})$ as follows. For an object $M \in \mathcal{C}(\mathcal{A})$, $M[n]$ is the map which takes an object $X \in \mathcal{A}$ to the chain complex $M(X)[n]$. For a morphism $F \in \mathcal{C}(\mathcal{A})(M, N)$, the shifted morphism $F[n] \in \mathcal{C}(\mathcal{A})(M[n], N[n])$ has components given by

$$F[n]_X := F_X[n] : M(X)[n] \to N(X)[n]$$

where $F_X[n]$ is the usual shift of the chain map F_X. Together these shifts on objects and morphisms constitute an autoequivalence $\Sigma : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$. For a morphism $F \in \mathcal{C}(\mathcal{A})(M, N)$, the cone of F, denoted $C(F)$, is the map which takes an object $X \in \mathcal{A}$ to the usual cone of the chain map $F_X : M(X) \to N(X)$. Specifically we have $C(F)(X) := C(F_X)$ for all $X \in \mathcal{A}$.

We define a particular type of dg $\mathcal{A}$-module, which will be central in a lot of the future discussions.

Definition. Let $X \in \mathcal{A}$. We define the *dg $\mathcal{A}$-module represented by X* as the contravariant dg-functor $X^\wedge := \mathcal{A}(\,\cdot\,, X)$.

Given some morphism $f \in \mathcal{A}(X, Y)$, we get an induced morphism on the represented dg $\mathcal{A}$-modules. We call this the *morphism represented by*

f, which we denote by $f^\wedge$. This morphism has all components given by postcomposition, $f^\wedge\colon \mathcal{A}(\,\cdot\,, X) \xrightarrow{f_*} \mathcal{A}(\,\cdot\,, Y)$.

This induced morphism has some nice relations to f itself. If $f \in \mathcal{A}(X,Y)^n$, then we know that for any $Z \in \mathcal{A}$, the morphism $f^\wedge_Z\colon \mathcal{A}(Z,X) \xrightarrow{f_*} \mathcal{A}(Z,Y)$ has degree n. This is because composition in $\mathcal{A}$ preserves sum of degrees. Thus all components of $f^\wedge$ have degree n, and hence it is a graded transformation of degree n.

If $f \in Z^0(\mathcal{A})(X,Y)$, we show that the induced morphism $f^\wedge$ is a cycle transformation. We know $f^\wedge$ is of degree 0, so we need to show that $d_{\mathcal{C}_{dg}(\mathcal{A})}(f^\wedge) = 0$. The transformation $d_{\mathcal{C}_{dg}(\mathcal{A})}(f^\wedge)$ consists of components on the form $d_{\mathcal{A}(Z,X),\mathcal{A}(Z,Y)}(f_*)$ for all $Z \in \mathcal{A}$, where $d_{\mathcal{A}(Z,X),\mathcal{A}(Z,Y)}$ is the differential on the chain complex $\mathcal{C}_{dg}(k)(\mathcal{A}(Z,X), \mathcal{A}(Z,Y))$. We have that $d_{\mathcal{A}(Z,X),\mathcal{A}(Z,Y)}(f_*) = d_{Z,Y} \circ f_* - f_* \circ d_{Z,X}$. This being zero is the same as the following diagram commuting.

$$\begin{array}{ccc}
\mathcal{A}(Z,X) & \xrightarrow{f_*} & \mathcal{A}(Z,Y) \\
\downarrow{\scriptstyle d_{Z,X}} & & \downarrow{\scriptstyle d_{Z,Y}} \\
\mathcal{A}(Z,X) & \xrightarrow{f_*} & \mathcal{A}(Z,Y)
\end{array}$$

Take any $g \in \mathcal{A}(Z,X)$. Going along the top we get

$$d_{Z,Y} \circ f_*(g) = d_{Z,Y}(f \circ g) = d_{X,Y}(f) \circ g + f \circ d_{Z,X}(g)$$

by the Leibniz rule. Now since $f \in Z^0(\mathcal{A})(X,Y)$, we have

$$d_{X,Y}(f) \circ g + f \circ d_{Z,X}(g) = f \circ d_{Z,X}(g) = f_* \circ d_{Z,X}(g)$$

This is exactly what going along the bottom path in the diagram means. Hence the diagram commutes, which means $d_{\mathcal{A}(Z,X),\mathcal{A}(Z,Y)}(f_*) = 0$ for all $Z \in \mathcal{A}$, and hence $f^\wedge$ is a cycle transformation.

Proposition 2.4. *Let $s \in \mathcal{A}(X,Y)^{-1}$. Then $d^{-1}_{X,Y}(s)^\wedge = d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)$.*

Proof. Because $s \in \mathcal{A}(X,Y)^{-1}$, this gives us the morphism $s^\wedge \in \mathcal{C}_{dg}(\mathcal{A})(X^\wedge, Y^\wedge)^{-1}$, where all components are given by postcomposition with s. Recall that the differential of the chain complex $\mathcal{C}_{dg}(\mathcal{A})(X^\wedge, Y^\wedge)$ is given by taking the differential at each component. Each component is the chain complex $\mathcal{C}_{dg}(k)(\mathcal{A}(Z,X),\mathcal{A}(Z,Y))$, with the differential

$$d_{\mathcal{A}(Z,X),\mathcal{A}(Z,Y)}(s_*) = d_{Z,Y} \circ s_* + s_* \circ d_{Z,X}$$

Hence we get that for any $t \in \mathcal{A}(Z,X)$

$$\begin{aligned}
&d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)_Z(t) &&= \\
&d_{Z,Y} \circ s_*(t) + s_* \circ d_{Z,X}(t) &&= \\
&d_{Z,Y}(s \circ t) + s \circ d_{Z,X}(t) &&= \\
&d^{-1}_{X,Y}(s) \circ t - s \circ d_{Z,X}(t) + s \circ d_{Z,X}(t) &&= \\
&d^{-1}_{X,Y}(s) \circ t &&= \\
&(d^{-1}_{X,Y}(s))_*(t) &&= \\
&d^{-1}_{X,Y}(s)^\wedge_Z(t) &&=
\end{aligned}$$

Hence $d^{-1}_{X,Y}(s)^\wedge_Z = d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)_Z$ for all $Z \in \mathcal{A}$, and thus $d^{-1}_{X,Y}(s)^\wedge = d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)$. □

In order to study the represented dg $\mathcal{A}$-modules, we might make the following definition.

Definition. The category of *strictly representable* dg $\mathcal{A}$-modules $\mathcal{R}_s(\mathcal{A})$, is a subcategory of $\mathcal{C}(\mathcal{A})$. The objects are the strictly representable dg $\mathcal{A}$-modules. That is, it contains all objects on the form $X^\wedge \in \mathcal{C}(\mathcal{A})$ for all $X \in Z^0(\mathcal{A})$. Morphisms are the strictly representable morphisms. A morphism $F \in \mathcal{C}(\mathcal{A})(X^\wedge, Y^\wedge)$ is strictly representable if there exists $f \in Z^0(\mathcal{A})(X,Y)$, such that $f^\wedge = F$.

One problem with this definition however, is that this category only contains the dg $\mathcal{A}$-modules which are directly represented by some object in $Z^0(\mathcal{A})$. This means there can exist a dg $\mathcal{A}$-module $M \in \mathcal{C}(\mathcal{A})$, such that $M \simeq X^\wedge$, while $M \notin \mathcal{R}_s(\mathcal{A})$. That is, $\mathcal{R}_s(\mathcal{A})$ is not necessarily closed under isomorphisms in $\mathcal{C}(\mathcal{A})$. A more natural category to study is the following.

Definition. The category of *representable* dg $\mathcal{A}$-modules $\mathcal{R}(\mathcal{A})$, is a subcategory of $\mathcal{C}(\mathcal{A})$. Its objects are the representable dg $\mathcal{A}$-modules, i.e. objects $M \in \mathcal{C}(\mathcal{A})$ such that there exists $X \in Z^0(\mathcal{A})$ with $M \simeq X^\wedge$ in $\mathcal{C}(\mathcal{A})$. Morphisms are the representable morphisms. For two representable objects

$M \simeq X^\wedge$ and $N \simeq Y^\wedge$, a morphism $F \in \mathcal{C}(\mathcal{A})(M,N)$ is said to be representable if there exists $f \in Z^0(\mathcal{A})(X,Y)$, such that $F = v \circ f^\wedge \circ u$, for isomorphisms $u : M \to X^\wedge$ and $v : Y^\wedge \to N$.

With this definition, we get a category that is closed under isomorphisms in the supercategory $\mathcal{C}(\mathcal{A})$.

2.2 Yoneda embedding

To better understand a dg-category $\mathcal{A}$, it will be useful to study the category of dg $\mathcal{A}$-modules, $\mathcal{C}(\mathcal{A})$. To do this, we will look at the *Yoneda functor*, which will allow us to identify $Z^0(\mathcal{A})$ with a full subcategory of $\mathcal{C}(\mathcal{A})$.

Definition. Let $\mathcal{A}$ be a dg-category. The *Yoneda functor*, which we will denote by Γ, maps objects and morphisms to their representatives in the module category $\mathcal{C}(\mathcal{A})$. Specifically we define $\Gamma : Z^0(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ by

$$X \mapsto X^\wedge := \mathcal{A}(\,\cdot\,, X) \qquad\qquad f \mapsto f^\wedge : \mathcal{A}(\,\cdot\,, X) \xrightarrow{f_*} \mathcal{A}(\,\cdot\,, Y)$$

Remark 2.5. Notice how the essential image of Γ are the objects in $\mathcal{R}(\mathcal{A})$. Thus we can also restrict Γ and consider it as a dense functor $\Gamma : Z^0(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$.

We take a moment to discuss the following construction, which will be very useful in the coming arguments. For a morphism between strictly represented dg $\mathcal{A}$-modules $F \in \mathcal{C}(\mathcal{A})(X^\wedge, Y^\wedge)$, we can obtain a morphism $f \in Z^0(\mathcal{A})(X,Y)$. We do this by looking at the component of F on the object X, $F_X : \mathcal{A}(X,X) \to \mathcal{A}(X,Y)$, and evaluating this at the element $1_X \in \mathcal{A}(X,X)$. This gives us some morphism $f := F_X(1_X) \in \mathcal{A}(X,Y)$. Since $1_X \in Z^0(\mathcal{A}(X,X))$, it follows that $f \in Z_0(\mathcal{A}(X,Y))$, since F_X is a chain map.

Proposition 2.6. *The functor* Γ *is full and faithful.*

Proof. Let $X, Y \in Z^0(\mathcal{A})$, and let $F : X^\wedge \to Y^\wedge$. Define $f := F_X(1_X) \in Z^0(\mathcal{A})(X,Y)$. We show that $f^\wedge = F$.

Let $Z \in \mathcal{A}$. We have both $F_Z, f^\wedge_Z : \mathcal{A}(Z,X) \to \mathcal{A}(Z,Y)$. Take any $g \in \mathcal{A}(Z,X)$. Since F is a cycle transformation, we get that the following commutative diagram holds.

$$\begin{array}{ccc} \mathcal{A}(X,X) & \xrightarrow{F_X} & \mathcal{A}(X,Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{A}(Z,X) & \xrightarrow{F_Z} & \mathcal{A}(Z,Y) \end{array}$$

This gives us that

$$f_Z^\wedge(g) = f \circ g = g^*(f) = g^*(F_X(1_X)) = F_Z(g^*(1_X)) = F_Z(g)$$

Hence $f_Z^\wedge(g) = F_Z(g)$ for any $g \in \mathcal{A}(Z, X)$. Since the choice of Z was arbitrary, this shows that $f^\wedge = F$. Hence Γ is full.

To show that Γ is faithful, assume we have $f, g \in Z^0(\mathcal{A})(X, Y)$, such that $f^\wedge = g^\wedge$. Looking at the component on X and evaluating at 1_X we get that $f_X^\wedge(1_X) = g_X^\wedge(1_X)$. Writing out what this means, we get that $f \circ 1_X = g \circ 1_X \implies f = g$. Hence Γ is faithful. $\square$

Corollary 2.7. *The restricted Yoneda functor $\Gamma : Z^0(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$ as a functor onto its essential image, is full, faithful and dense. Hence $Z^0(\mathcal{A}) \simeq \mathcal{R}(\mathcal{A})$ are equivalent k-categories.*

Remark 2.8. What we have shown here is a variant of the *Yoneda embedding*. Essentially this states that the equivalence above gives us a way to embed $Z^0(\mathcal{A})$ as a full subcategory of $\mathcal{C}(\mathcal{A})$.

Proposition 2.9. *The Yoneda functor induces a well defined functor $\Gamma_H : H^0(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ making the following diagram commute.*

$$\begin{tikzcd}
Z^0(\mathcal{A}) \arrow[r, "\Gamma"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{C}(\mathcal{A}) \arrow[d, "\pi_{\mathcal{C}_{dg}(\mathcal{A})}"] \\
H^0(\mathcal{A}) \arrow[r, dotted, "\Gamma_H"] & \mathcal{H}(\mathcal{A})
\end{tikzcd}$$

Proof. Since the functors $\pi_A, \pi_{\mathcal{C}_{dg}(\mathcal{A})}$ both are the identy on objects, it is natural for Γ_H to be the same on objects as Γ, hence we define $\Gamma_H(X) := \Gamma(X)$. Now let $X, Y \in \mathcal{A}$, and take some representative of $f \in H^0(\mathcal{A})(X, Y)$. We define the image of $\Gamma_H(f)$ in $\mathcal{H}(\mathcal{A})(\Gamma(X), \Gamma(Y))$ to be represented by $\Gamma(f)$. We show that this is well defined and independent of representative for the morphism f.

Let $f, g \in Z^0(\mathcal{A})(X, Y)$, such that $f - g \in \mathrm{Im}(d_{X,Y}^{-1})$. Then we know there exists a morphism $s \in \mathcal{A}(X, Y)^{-1}$ such that $d_{X,Y}^{-1}(s) = f - g$. Applying Γ to this equation we get that $\Gamma(d_{X,Y}^{-1}(s)) = \Gamma(f - g) = \Gamma(f) - \Gamma(g)$. By Proposition 2.4, we know that $\Gamma(d_{X,Y}^{-1}(s)) = d_{\mathcal{C}_{dg}(\mathcal{A})}^{-1}(\Gamma(s))$, and hence $\Gamma(f) - \Gamma(g) \in \mathrm{Im}(d_{\mathcal{C}_{dg}(\mathcal{A})}^{-1})$. $\square$

Proposition 2.10. *The functor Γ_H is full and faithful.*

Proof. Let $X, Y \in \mathcal{A}$, and choose some representative for the morphism $f \in \mathcal{H}(\mathcal{A})(X^\wedge, Y^\wedge)$. Since Γ is full, there exists some $g \in Z^0(\mathcal{A})$ such that $\Gamma(g) = f$. Then

$$\Gamma_H(\pi_{\mathcal{A}}(g)) = \pi_{\mathcal{C}_{dg}(\mathcal{A})}(\Gamma(g)) = \pi_{\mathcal{C}_{dg}(\mathcal{A})}(f) = f$$

Hence Γ_H is full. Now assume $\Gamma_H(f) = 0$. This means there exists $s \in \mathcal{C}_{dg}(\mathcal{A})(X^\wedge, Y^\wedge)^{-1}$, such that $d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s) = \Gamma(f)$. By taking the component of these transformations on the object X, and evaluating at 1_X, we get

$$\begin{aligned}
&d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s) = \Gamma(f) && \Longrightarrow\\
&d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s)_X(1_X) = \Gamma(f)_X(1_X) && \Longrightarrow\\
&(d^{-1}_{X,Y} \circ s_X + s_X \circ d_X)(1_X) = f_*(1_X) && \Longrightarrow\\
&d^{-1}_{X,Y} \circ s_X(1_X) = f \circ 1_X && \Longrightarrow\\
&d^{-1}_{X,Y}(s_X(1_X)) = f
\end{aligned}$$

That is, $f \in \mathrm{Im}(d^{-1}_{X,Y})$, and hence $f = 0$ in $H^0(\mathcal{A})$. Hence Γ_H is faithful. $\square$

2.3 Triangulated categories from dg-categories

In this section we will prove that for a so-called *pretriangulated* dg-category $\mathcal{A}$, the category $H^0(\mathcal{A})$ has a canonical triangulation. We will first see how we can use the functors discussed in the previous section to better understand the category $H^0(\mathcal{A})$, and to define what a *pretriangulated* dg-category is. Finally we will prove that $H^0(\mathcal{A})$ is triangulated.

Recall that we could consider $\Gamma : Z^0(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$ an equivalence onto the full subcategory with objects in the essential image of Γ. Similarly we denote $\mathcal{R}(\mathcal{A})_H$ for the full subcategory of $\mathcal{H}(\mathcal{A})$ with objects in the essential image of Γ_H. Then we can consider the equivalence $\Gamma_H : H^0(\mathcal{A}) \to \mathcal{R}(\mathcal{A})_H$. But what is $\mathcal{R}(\mathcal{A})_H$? Let $M \in \mathcal{R}(\mathcal{A})$, then there exists some $X \in Z^0(\mathcal{A})$ such that $\Gamma(X) \simeq M$ in $\mathcal{C}_{dg}(\mathcal{A})$. Since functors preserve isomorphisms, this means that $\pi_{\mathcal{C}_{dg}(\mathcal{A})}(\Gamma(X)) \simeq \pi_{\mathcal{C}_{dg}(\mathcal{A})}(M)$, which implies that $\Gamma_H(X) \simeq M$ in $\mathcal{H}(\mathcal{A})$. Thus we have that $\mathrm{Obj}(\mathcal{R}(\mathcal{A})) \subseteq \mathrm{Obj}(\mathcal{R}(\mathcal{A})_H)$. The result is that the restricted functor $\pi_{\mathcal{C}_{dg}(\mathcal{A})} : \mathcal{R}(\mathcal{A}) \to \mathcal{R}(\mathcal{A})_H$ is well defined.

Because $\Gamma_H : H^0(\mathcal{A}) \to \mathcal{R}(\mathcal{A})_H$ is an equivalence, we know there exists a functor $\beta : \mathcal{R}(\mathcal{A})_H \to H^0(\mathcal{A})$, satisfying $\beta \circ \Gamma_H \simeq 1_{H^0(\mathcal{A})}$, and $\Gamma_H \circ \beta \simeq 1_{\mathcal{R}(\mathcal{A})_H}$. This allows us to make the following diagram.

$$\begin{tikzcd}
Z^0(\mathcal{A}) \arrow[r, "\Gamma"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{R}(\mathcal{A}) \arrow[d, "\pi_{\mathcal{C}_{dg}(\mathcal{A})}"] \\
H^0(\mathcal{A}) \arrow[r, "\Gamma_H"] & \mathcal{R}(\mathcal{A})_H \arrow[l, bend left, "\beta"]
\end{tikzcd}$$

We have that

$$\begin{aligned}
&\pi_{\mathcal{C}_{dg}(\mathcal{A})} \circ \Gamma = \Gamma_H \circ \pi_A && \Longrightarrow \\
&\beta \circ \pi_{\mathcal{C}_{dg}(\mathcal{A})} \circ \Gamma = \beta \circ \Gamma_H \circ \pi_A && \Longrightarrow \\
&\beta \circ \pi_{\mathcal{C}_{dg}(\mathcal{A})} \circ \Gamma \simeq 1_{H^0(\mathcal{A})} \circ \pi_A && \Longrightarrow \\
&\beta \circ \pi_{\mathcal{C}_{dg}(\mathcal{A})} \circ \Gamma \simeq \pi_A
\end{aligned}$$

This factorization of π_A allows us to "visit" $\mathcal{R}(\mathcal{A})$ on the way to $H^0(\mathcal{A})$. This category has some structure we are interrested in, in particular since it is a subcategory of $\mathcal{C}(\mathcal{A})$ there are well defined shifts and cones. In general however, $\mathcal{R}(\mathcal{A})$ is not closed under these operations. Namely for some representable dg $\mathcal{A}$-module M, we cannot be sure that the shifted module $M[n]$ is representable. The same is true for cones, in general, the cone of a representable morphism might not be a representable object.

This is where the property of $\mathcal{A}$ being *pretriangulated* comes in, as it ensures that the category of representable dg $\mathcal{A}$-modules is closed under both shifts and cones. In particular, following Keller [10, p.172] we have the following definition.

Definition. A dg-category $\mathcal{A}$ is *pretriangulated*, if both of the following hold.

1. For every object $X \in Z^0(\mathcal{A})$, and for all $n \in \mathbb{Z}$, there exists a unique object (up to isomorphism) $X[n] \in Z^0(\mathcal{A})$, such that $X[n]^\wedge \simeq X^\wedge[n]$ in $\mathcal{C}(\mathcal{A})$.

2. For every morphism $f \in Z^0(\mathcal{A})(X, Y)$, there exists a unique object (up to isomorphism) $C(f) \in Z^0(\mathcal{A})$, such that $C(f)^\wedge \simeq C(f^\wedge)$ in $\mathcal{C}(\mathcal{A})$.

Remark 2.11. The first property gives us a well defined shift $\Sigma : Z^0(\mathcal{A}) \to Z^0(\mathcal{A})$, induced by the shift in $\mathcal{C}(\mathcal{A})$. The second property gives us a way to construct cones in $Z^0(\mathcal{A})$. We write out explicitly what this construction looks like. We start with some morphism $f \in Z^0(\mathcal{A})(X, Y)$. This gives us a morphism $f^\wedge \in \mathcal{C}(\mathcal{A})(X^\wedge, Y^\wedge)$, with components $f_* : \mathcal{A}(Z, X) \to \mathcal{A}(Z, Y)$. The cone is defined componentwise, so $C(f^\wedge)_Z = \mathcal{A}(Z, Y) \oplus \mathcal{A}(Z, X)[1]$, where the right hand side has the differential

$$d^n = \begin{pmatrix} d^n_{Z,Y} & f_* \\ 0 & -d^{n+1}_{Z,X} \end{pmatrix}$$

Thus we have that

$$\mathcal{A}(Z, C(f)) = C(f)^\wedge_Z \simeq C(f^\wedge)_Z = \mathcal{A}(Z, Y) \oplus \mathcal{A}(Z, X)[1] \simeq \mathcal{A}(Z, Y) \oplus \mathcal{A}(Z, X[1])$$

To show that $H^0(\mathcal{A})$ is triangulated, we first need to know that it is additive. We do this by showing that $Z^0(\mathcal{A})$ is additive. For a general dg-category $\mathcal{A}$, this is not guaranteed. All morphisms spaces are abelian groups (since they are k-modules), but the existence of a biproduct is not guaranteed by our definitions. However when $\mathcal{A}$ is pretriangulated, then it follows that all biproducts exist in $Z^0(\mathcal{A})$.

Proposition 2.12. *Let $\mathcal{A}$ be a pretriangulated dg-category. Then $Z^0(\mathcal{A})$ is additive.*

Proof. Let $X, Y \in \mathcal{A}$, and take the zero morphism $0 \in \mathcal{A}(Y[-1], X)$. Define the cone of this morphism $B := C(0)$. We show that B is the biproduct of X and Y in $Z^0(\mathcal{A})$.

By the definition of the cone, we know that $B^\wedge \simeq C(0^\wedge)$ are isomorphic in $\mathcal{C}(\mathcal{A})$. The morphism $0^\wedge : Y[-1]^\wedge \to X^\wedge$ has cone isomorphic to

$$X^\wedge \oplus Y[-1]^\wedge[1] \simeq X^\wedge \oplus Y^\wedge[-1][1] = X^\wedge \oplus Y^\wedge$$

where $X^\wedge \oplus Y^\wedge$ is equipped with the appropriate differential. Thus we know there exists an isomorphism in $\mathcal{C}(\mathcal{A})$

$$\phi : B^\wedge \to X^\wedge \oplus Y^\wedge$$

This means that for all $Z \in \mathcal{A}$, we have an isomorphism of chain complexes

$$\phi_Z : \mathcal{A}(Z, B) \to \mathcal{A}(Z, X) \oplus \mathcal{A}(Z, Y)$$

natural with respect to changes in Z. Note that since we are taking the cone of the zero morphism, the differential on $\mathcal{A}(Z,X) \oplus \mathcal{A}(Z,Y)$ is a diagonal matrix, and thus $Z^0(\mathcal{A}(Z,X) \oplus \mathcal{A}(Z,Y)) \simeq Z^0(\mathcal{A})(Z,X) \oplus Z^0(\mathcal{A})(Z,Y)$. Looking specifically at the components $Z \in \{X, Y, B\}$, we define the following morphisms.

$$\begin{aligned} i_X &:= \phi_X^{-1}(1_X, 0) \in Z^0(\mathcal{A})(X,B) \\ i_Y &:= \phi_Y^{-1}(0, 1_Y) \in Z^0(\mathcal{A})(Y,B) \\ (\pi_X, \pi_Y) &:= \phi_B(1_B) \in Z^0(\mathcal{A})(B,X) \oplus Z^0(\mathcal{A})(B,Y) \end{aligned}$$

All these morphisms are in $Z^0(\mathcal{A})$, since all components ϕ_Z are chain maps. We now show that these morphisms satisfy the properties we expect from a biproduct. For any $f \in \mathcal{A}(X,B)$, we get the following commutative diagram

$$\begin{tikzcd} \mathcal{A}(B,B) \arrow[r, "\phi_B"] \arrow[d, "f^*"] & \mathcal{A}(B,X) \oplus \mathcal{A}(B,Y) \arrow[d, "f^*"] \\ \mathcal{A}(X,B) \arrow[r, "\phi_X"] & \mathcal{A}(X,X) \oplus \mathcal{A}(X,Y) \end{tikzcd}$$

If we specifically let $f = i_X$ and take $1_B \in \mathcal{A}(B,B)$, we get

$$\begin{aligned} &(1_X, 0) && = \\ &\phi_X(i_X) && = \\ &\phi_X(i_X^*(1_B)) && = \\ &i_X^*(\phi_B(1_B)) && = \\ &i_X^*(\pi_X, \pi_Y) && = \\ &(\pi_X \circ i_X, \pi_Y \circ i_X) \end{aligned}$$

Hence $\pi_X \circ i_X = 1_X$. Similarly we can show that $\pi_Y \circ i_Y = 1_Y$. Now for any $g \in \mathcal{A}(B,X)$, we get the commutative diagram

$$\begin{tikzcd} \mathcal{A}(X,X) \oplus \mathcal{A}(X,Y) \arrow[r, "\phi_X^{-1}"] \arrow[d, "g^*"] & \mathcal{A}(X,B) \arrow[d, "g^*"] \\ \mathcal{A}(B,X) \oplus \mathcal{A}(B,Y) \arrow[r, "\phi_B^{-1}"] & \mathcal{A}(B,B) \end{tikzcd}$$

Choose $g = \pi_X$, and look at $(1_X, 0) \in \mathcal{A}(X,X) \oplus \mathcal{A}(X,Y)$ to get that

$$\begin{aligned}
i_X \circ \pi_X &= \\
\pi_X^*(i_X) &= \\
\pi_X^*(\phi_X^{-1}(1_X, 0)) &= \\
\phi_B^{-1}(\pi_X^*(1_X, 0)) &= \\
\phi_B^{-1}(\pi_X, 0)
\end{aligned}$$

Similarly we can show that $i_Y \circ \pi_Y = \phi_B^{-1}(0, \pi_Y)$, and thus we get

$$\begin{aligned}
i_X \circ \pi_X + i_Y \circ \pi_Y &= \\
\phi_B^{-1}(\pi_X, 0) + \phi_B^{-1}(0, \pi_Y) &= \\
\phi_B^{-1}(\pi_X, \pi_Y) &= \\
1_B
\end{aligned}$$

Hence B is the biproduct of X and Y in $Z^0(\mathcal{A})$. $\square$

We take some time to discuss some useful notation which will use from here on.

Notation. For an object $M \in \mathcal{C}_{dg}(\mathcal{A})$, M_n denotes map $X \mapsto M(X)_n$, for any $X \in \mathcal{A}$, where $M(X)_n$ is the n-th component of the chain complex $M(X)$. Similarly for $f \in \mathcal{C}_{dg}(\mathcal{A})(M,N)^k$ a graded morphism of degree k, we denote by f_n the map $X \mapsto f(X)_n$, where $f(X)_n$ is the n-th component of the graded morphism of chain complexes $f(X) \in \mathcal{C}_{dg}(k)(MX, NX)^k$. Similarly for an object $M \in \mathcal{C}_{dg}(\mathcal{A})$, we denote by d_M^n the map $X \mapsto d_{MX}^n$, where d_{MX}^n is the n-th component of the differential on the chain complex MX. Composition of morphisms with this notation is defined in the natural way. Namely if $f \in \mathcal{C}_{dg}(\mathcal{A})(M,N)^j$, $g \in \mathcal{C}_{dg}(\mathcal{A})(N,V)^k$, then $(g \circ f)_n$ represents the map $X \mapsto (g \circ f)(X)_n = g(X)_{n+j} \circ f(X)_n$ for any $X \in \mathcal{A}$. Thus composition is defined by $(g \circ f)_n = g_{n+j} \circ f_n$.

Using this notation allows us to simplify arguments. For example we have the following proposition which makes it easier to discuss homotopic maps in $\mathcal{C}(\mathcal{A})$.

Proposition 2.13. *Let $f \in \mathcal{C}(\mathcal{A})(M,N)$, and let $s \in \mathcal{C}_{dg}(\mathcal{A})(M,N)^{-1}$. If $f_n = d_N^{n-1} \circ s_n + s_{n+1} \circ d_M^n$ for all $n \in \mathbb{Z}$, then $d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s) = f$.*

Proof. Let $X \in \mathcal{A}$, then we know that

$$\begin{aligned} f_n(X) &= d_N^{n-1}(X) \circ s_n(X) + s_{n+1}(X) \circ d_M^n(X) \qquad \Longrightarrow \\ f(X)_n &= d_{NX}^{n-1} \circ s(X)_n + s(X)_{n+1} \circ d_{MX}^n \end{aligned}$$

Since this is true for all $n \in \mathbb{Z}$, we consider the chain maps with all these morphisms as components. We get

$$f(X) = d_{NX} \circ s(X) + s(X) \circ d_{MX} = d_{MX,NX}(s(X))$$

The last equality we get by the definition of the differential in $\mathcal{C}_{dg}(k)$. Considering the definiton of the differential in $\mathcal{C}_{dg}(\mathcal{A})$, we get

$$d_{\mathcal{C}_{dg}(\mathcal{A})}(s) = \{d_{MX,NX}(s(X)) \mid X \in \mathcal{A}\} = \{f(X) \mid X \in \mathcal{A}\} = f$$

□

Since $\mathcal{A}$ is pretriangulated, we have well-defined shifts and cones in $Z^0(\mathcal{A})$. Using this we define them similarly in $H^0(\mathcal{A})$.

Definition. The shift $\Sigma : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ is the functor induced by the shift in $Z^0(\mathcal{A})$. Given a morphism $f \in H^0(\mathcal{A})(X, Y)$, we choose some representative for f, and construct the object $C(f) \in Z^0(\mathcal{A})$. Then we define the cone of f as the object $C(f) \in H^0(\mathcal{A})$.

Remark 2.14. It can be shown that this definition of the cone in $H^0(\mathcal{A})$ is independent of the representative for the morphism f.

Definition. A standard triangle in $Z^0(\mathcal{A})$ is a sequence on the form

$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1]$$

for some morphism $f \in Z^0(\mathcal{A})$. Here u and v are the morphisms making the following diagram commute in $\mathcal{C}(\mathcal{A})$

$$\begin{tikzcd} X^\wedge \arrow[r, "f^\wedge"] \arrow[d, equal] & Y^\wedge \arrow[r, "u^\wedge"] \arrow[d, equal] & C(f)^\wedge \arrow[r, "v^\wedge"] \arrow[d, "\phi"] & X^\wedge[1] \arrow[d, equal] \\ X^\wedge \arrow[r, "f^\wedge"] & Y^\wedge \arrow[r, "\binom{1}{0}"] & Y^\wedge \oplus X^\wedge[1] \arrow[r, "(0\ 1)"] & X^\wedge[1] \end{tikzcd}$$

where ϕ is an isomorphism granted by knowing that $C(f)^\wedge \simeq C(f^\wedge) = Y^\wedge \oplus X^\wedge[1]$. We define the collection Δ of *distinguished triangles* in $H^0(\mathcal{A})$ as all triangles isomorphic to the image of a standard triangle under the projection functor $\pi_{\mathcal{A}}$.

We will show that $(H^0(\mathcal{A}), \Sigma, \Delta)$ is a triangulated category, but first we discuss a pattern that will repeat throughout the proof. We will start with some collection of objects and morphisms in $H^0(\mathcal{A})$, say for example

$$X \xrightarrow{f} Y$$

$$U \xrightarrow{g} V$$

We then apply Γ_H to pass to the category $\mathcal{R}(\mathcal{A})_H$. Here we will use some of the extra structure to obtain some new morphisms, making the following diagram commute in $\mathcal{R}(\mathcal{A})_H$.

$$\begin{tikzcd} X^\wedge \arrow[r, "f^\wedge"] \arrow[d, dotted, "s"] & Y^\wedge \arrow[d, dotted, "t"] \\ U^\wedge \arrow[r, "g^\wedge"] & V^\wedge \end{tikzcd}$$

By applying β, we come back to $H^0(\mathcal{A})$. Since $\beta \circ \Gamma_H \simeq 1_{H^0(\mathcal{A})}$, we have a natural isomorphism $\phi : 1_{H^0(\mathcal{A})} \to \beta \circ \Gamma_H$, giving us the following commutative diagram in $H^0(\mathcal{A})$.

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi_X} & & \downarrow{\phi_Y} \\
\beta(X^\wedge) & \xrightarrow{\beta(f^\wedge)} & \beta(Y^\wedge) \\
\downarrow{\beta(s)} & & \downarrow{\beta(t)} \\
\beta(U^\wedge) & \xrightarrow{\beta(g^\wedge)} & \beta(V^\wedge) \\
\phi_U\uparrow & & \phi_V\uparrow \\
U & \xrightarrow{g} & V
\end{array}$$

Note that the middle square commutes since β preserves compositions. The top and bottom squares commute because ϕ is a natural transformation. Using this diagram we can fill in the missing morphisms in the original diagram, by defining the *corresponding morphisms* to s and t in $H^0(\mathcal{A})$.

$$s' := \phi_U^{-1} \circ \beta(s) \circ \phi_X \in H^0(\mathcal{A})(X, U)$$
$$t' := \phi_V^{-1} \circ \beta(t) \circ \phi_Y \in H^0(\mathcal{A})(Y, V)$$

Since all squares in this diagram commute, these morphisms make the original diagram commute as well. Note that if for example s is an isomorphism in $\mathcal{R}(\mathcal{A})_H$, then the corresponding morphism s' is an isomorphism in $H^0(\mathcal{A})$.

The following proof is an adaptation of the proof given in [7, Theorem 6.7], to work when dealing with dg-categories.

Theorem 2.15. *Let $\mathcal{A}$ be a pretriangulated dg-category, and let $\Sigma : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ and Δ be defined as above. Then $(H^0(\mathcal{A}), \Sigma, \Delta)$ becomes a triangulated category.*

Proof. (TR1) By construction, Δ is closed under isomorphisms and all morphisms fit into a distinguished triangle. Let $X \in \mathcal{A}$, and look at the triangle

$$X \xrightarrow{1_X} X \xrightarrow{u} C(1_X) \xrightarrow{v} X[1]$$

We want to prove that this is isomorphic to the triangle

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1]$$

In particular, we need to show that $C(1_X) \simeq 0$ in $H^0(\mathcal{A})$. We first show that $C(1_X)^\wedge \simeq X^\wedge \oplus X^\wedge[1] \simeq 0$ in $\mathcal{H}(\mathcal{A})$. Denote the differential on $X^\wedge \oplus X^\wedge[1]$ by

$$d^n_{C(1_X)^\wedge} = \begin{pmatrix} d^n_{X^\wedge} & 1_{X^\wedge_{n+1}} \\ 0 & -d^{n+1}_{X^\wedge} \end{pmatrix}$$

There is only one choice of morphism in either direction between the objects 0 and $C(1_X)$, so we need to show that these are inverses in $\mathcal{H}(\mathcal{A})$. The composition $0 \to C(1_X) \to 0$ is clearly equal to 1_0. We make the following diagram for the morphism $C(1_X) \to 0 \to C(1_X)$.

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & X^\wedge_{-1} \oplus X^\wedge_0 \arrow[r, "d^{-1}"] \arrow[d, "0"] & X^\wedge_0 \oplus X^\wedge_1 \arrow[r, "d^0"] \arrow[d, "0"] \arrow[dl, "s_0"] & X^\wedge_1 \oplus X^\wedge_2 \arrow[r, "d^1"] \arrow[d, "0"] \arrow[dl, "s_1"] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & X^\wedge_{-1} \oplus X^\wedge_0 \arrow[r, "d^{-1}"] & X^\wedge_0 \oplus X^\wedge_1 \arrow[r, "d^0"] & X^\wedge_1 \oplus X^\wedge_2 \arrow[r, "d^1"] & \cdots
\end{tikzcd}$$

Here s is the graded transformation of degree -1 with components given by

$$s_n = \begin{pmatrix} 0 & 0 \\ 1_{X^\wedge_n} & 0 \end{pmatrix}$$

We get

$$\begin{aligned}
& d^{n-1}_{C(1_X)^\wedge} \circ s_n + s_{n+1} \circ d^n_{C(1_X)^\wedge} & = \\
& \begin{pmatrix} d^{n-1}_{X^\wedge} & 1_{X^\wedge_n} \\ 0 & -d^n_{X^\wedge} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_{X^\wedge_n} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1_{X^\wedge_{n+1}} & 0 \end{pmatrix} \begin{pmatrix} d^n_{X^\wedge} & 1_{X^\wedge_{n+1}} \\ 0 & -d^{n+1}_{X^\wedge} \end{pmatrix} & = \\
& \begin{pmatrix} 1_{X^\wedge_n} & 0 \\ d^n_{X^\wedge} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -d^n_{X^\wedge} & 1_{X^\wedge_{n+1}} \end{pmatrix} & = \\
& \begin{pmatrix} 1_{X^\wedge_n} & 0 \\ 0 & 1_{X^\wedge_{n+1}} \end{pmatrix} = \\
& 1_{C(1_X)^\wedge_n}
\end{aligned}$$

By Proposition 2.13, we get that $1_{C(1_X)^\wedge} = d^{-1}_{\mathcal{C}_{dg}(\mathcal{A})}(s)$, which means $C(1_X)^\wedge \simeq 0$ in $\mathcal{H}(\mathcal{A})$. Applying β preserves isomorphisms, hence we get that $0 \simeq \beta(0) \simeq \beta(C(1_X)^\wedge) \simeq C(1_X)$ in $H^0(\mathcal{A})$.

(TR2) Let $X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1]$ be a standard triangle in $H^0(\mathcal{A})$. The left rotation of this triangle is $Y \xrightarrow{u} C(f) \xrightarrow{v} X[1] \xrightarrow{-f[1]} Y[1]$. We need to show that we have an isomorphism $\phi' : C(u) \to X[1]$ in $H^0(\mathcal{A})$, such that the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, equal] & C(f) \arrow[r, "w"] \arrow[d, equal] & C(u) \arrow[r, "t"] \arrow[d, dotted, "\phi'"] & Y[1] \arrow[d, equal] \\
Y \arrow[r, "u"] & C(f) \arrow[r, "v"] & X[1] \arrow[r, "-f[1]"] & Y[1]
\end{tikzcd}$$

We have that $C(u)^\wedge \simeq C(f)^\wedge \oplus Y^\wedge[1] \simeq Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1]$, with the differential

$$d^n_{C(u)^\wedge} = \begin{pmatrix} d^n_{Y^\wedge} & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \\ 0 & -d^{n+1}_{X^\wedge} & 0 \\ 0 & 0 & -d^{n+1}_{Y^\wedge} \end{pmatrix}$$

We define the morphisms $\phi : Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \to X^\wedge[1]$ given by components

$$\phi_n = \begin{pmatrix} 0 & 1_{X^\wedge_{n+1}} & 0 \end{pmatrix}$$

and $\theta : X[1]^\wedge \to Y^\wedge \oplus X[1]^\wedge \oplus Y[1]^\wedge$ given by components

$$\theta_n = \begin{pmatrix} 0 \\ 1_{X^\wedge_{n+1}} \\ -f^\wedge_{n+1} \end{pmatrix}$$

One can check that these commute with the respective differentials. We consider the following diagram

$$\begin{tikzcd}
Y^\wedge \arrow[r, "u^\wedge"] \arrow[d, equal] & Y^\wedge \oplus X^\wedge[1] \arrow[r, "w^\wedge"] \arrow[d, equal] & Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \arrow[r, "t^\wedge"] \arrow[d, bend left, "\phi"] & Y^\wedge[1] \arrow[d, equal] \\
Y^\wedge \arrow[r, "u^\wedge"] & Y^\wedge \oplus X^\wedge[1] \arrow[r, "v^\wedge"] & X^\wedge[1] \arrow[r, "-f^\wedge[1]"] \arrow[u, bend left, "\theta"] & Y^\wedge[1]
\end{tikzcd}$$

where

$$u^\wedge = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v^\wedge = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$w^\wedge = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad t^\wedge = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

We check that ϕ and θ make the diagram commute in $\mathcal{H}(\mathcal{A})$. We see that

$$(\phi \circ w^\wedge)_n = \begin{pmatrix} 0 & 1_{X^\wedge_{n+1}} & 0 \end{pmatrix} \begin{pmatrix} 1_{Y^\wedge_n} & 0 \\ 0 & 1_{X^\wedge_{n+1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_{X^\wedge_{n+1}} \end{pmatrix} = v^\wedge_n$$

and

$$(t^\wedge \circ \theta)_n = \begin{pmatrix} 0 & 0 & 1_{Y^\wedge_{n+1}} \end{pmatrix} \begin{pmatrix} 0 \\ 1_{X^\wedge_{n+1}} \\ -f^\wedge_{n+1} \end{pmatrix} = -f^\wedge_{n+1}$$

Hence ϕ makes the left square commute, and θ makes the right square commute already in $\mathcal{C}(\mathcal{A})$. For the other compositions to commute, we need to pass to $\mathcal{H}(\mathcal{A})$. First observe that

$$\begin{aligned}
&t^\wedge_n - (-f^\wedge[1] \circ \phi)_n && = \\
&\begin{pmatrix} 0 & 0 & 1_{Y^\wedge_{n+1}} \end{pmatrix} - \left(-f^\wedge_{n+1} \begin{pmatrix} 0 & 1_{X^\wedge_{n+1}} & 0 \end{pmatrix}\right) && = \\
&\begin{pmatrix} 0 & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix}
\end{aligned}$$

Let $s : Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \to Y^\wedge[1]$ be the graded transformation of degree -1 given by components

$$s_n = \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \end{pmatrix}$$

This gives us that

$$\begin{aligned}
&- d^n_{Y^\wedge} \circ s_n + s_{n+1} \circ d^n_{C(u)^\wedge} && = \\
&- d^n_{Y^\wedge} \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1_{Y^\wedge_{n+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^n_{Y^\wedge} & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \\ 0 & -d^{n+1}_{X^\wedge} & 0 \\ 0 & 0 & -d^{n+1}_{Y^\wedge} \end{pmatrix} && = \\
&\begin{pmatrix} -d^n_{Y^\wedge} & 0 & 0 \end{pmatrix} + \begin{pmatrix} d^n_{Y^\wedge} & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix} && = \\
&\begin{pmatrix} 0 & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix}
\end{aligned}$$

Hence by Proposition 2.13, ϕ also makes the right square commute in $\mathcal{H}(\mathcal{A})$. Similarly, first we calculate that

$$\begin{aligned}
&w_n^\wedge - (\theta \circ v^\wedge)_n &&= \\
&\begin{pmatrix} 1_{Y_n^\wedge} & 0 \\ 0 & 1_{X_{n+1}^\wedge} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1_{X_{n+1}^\wedge} \\ -f_{n+1}^\wedge \end{pmatrix} \begin{pmatrix} 0 & 1_{X_{n+1}^\wedge} \end{pmatrix} &&= \\
&\begin{pmatrix} 1_{Y_n^\wedge} & 0 \\ 0 & 1_{X_{n+1}^\wedge} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1_{X_{n+1}^\wedge} \\ 0 & -f_{n+1}^\wedge \end{pmatrix} &&= \\
&\begin{pmatrix} 1_{Y_n^\wedge} & 0 \\ 0 & 0 \\ 0 & f_{n+1}^\wedge \end{pmatrix}
\end{aligned}$$

Let $s : Y^\wedge \oplus X^\wedge[1] \to Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1]$ be the graded transformation of degree -1 given by components

$$s_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1_{Y_n^\wedge} & 0 \end{pmatrix}$$

We get

$$\begin{aligned}
&d_{C(u)^\wedge}^{n-1} \circ s_n + s_{n+1} \circ d_{C(f)^\wedge}^n &&= \\
&\begin{pmatrix} d_{Y^\wedge}^{n-1} & f_n^\wedge & 1_{Y_n^\wedge} \\ 0 & -d_{X^\wedge}^n & 0 \\ 0 & 0 & -d_{Y^\wedge}^n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1_{Y_n^\wedge} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1_{Y_{n+1}^\wedge} & 0 \end{pmatrix} \begin{pmatrix} d_{Y^\wedge}^n & f_{n+1}^\wedge \\ 0 & -d_{X^\wedge}^{n+1} \end{pmatrix} &&= \\
&\begin{pmatrix} 1_{Y_n^\wedge} & 0 \\ 0 & 0 \\ -d_{Y^\wedge}^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ d_{Y^\wedge}^n & f_{n+1}^\wedge \end{pmatrix} &&= \\
&\begin{pmatrix} 1_{Y_n^\wedge} & 0 \\ 0 & 0 \\ 0 & f_{n+1}^\wedge \end{pmatrix}
\end{aligned}$$

Hence by Proposition 2.13 θ also makes the left square commute in $\mathcal{H}(\mathcal{A})$.

Now we show that ϕ and θ are inverses in $\mathcal{H}(\mathcal{A})$. Composing them we get

$$\phi_n \circ \theta_n = 1_{X_{n+1}^\wedge}$$

This shows they are already one-sided inverses of oneanother in $\mathcal{C}(\mathcal{A})$. For the other composition order we get

$$\theta_n \circ \phi_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} & 0 \\ 0 & -f^\wedge_{n+1} & 0 \end{pmatrix}$$

Now let $s : Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \to Y^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1]$ be the graded transformation of degree -1 given by components

$$s_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1_{Y^\wedge_n} & 0 & 0 \end{pmatrix}$$

We get

$$\begin{aligned}
& d^{n-1}_{C(u)^\wedge} \circ s_n + s_{n+1} \circ d^n_{C(u)^\wedge} & = \\
& \begin{pmatrix} d^{n-1}_{Y^\wedge} & f^\wedge_n & 1_{Y^\wedge_n} \\ 0 & -d^n_{X^\wedge} & 0 \\ 0 & 0 & -d^n_{Y^\wedge} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1_{Y^\wedge_n} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1_{Y^\wedge_{n+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^n_{Y^\wedge} & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \\ 0 & -d^{n+1}_{X^\wedge} & 0 \\ 0 & 0 & -d^{n+1}_{Y^\wedge} \end{pmatrix} & = \\
& \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \\ 0 & 0 & 0 \\ -d^n_{Y^\wedge} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d^n_{Y^\wedge} & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix} & = \\
& \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix}
\end{aligned}$$

Now notice that

$$\begin{aligned}
& d^{n-1}_{C(u)^\wedge} \circ s_n + s_{n+1} \circ d^n_{C(u)^\wedge} + \theta_n \circ \phi_n & = \\
& \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} & 0 \\ 0 & -f^\wedge_{n+1} & 0 \end{pmatrix} & = \\
& \begin{pmatrix} 1_{Y^\wedge_n} & 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} & 0 \\ 0 & 0 & 1_{Y^\wedge_{n+1}} \end{pmatrix} & = \\
& 1_{C(u)^\wedge_n}
\end{aligned}$$

Hence by Proposition 2.13, $\theta \circ \phi = 1_{C(u)^\wedge}$. This means ϕ and θ are inverses and $X^\wedge[1] \simeq C(u)^\wedge$ in $\mathcal{H}(\mathcal{A})$. By applying β, we get an isomorphism $\phi' \in H^0(\mathcal{A})(C(u), X[1])$ corresponding to ϕ which makes the original diagram commute.

(TR3) Assume we have the folowing diagram in $H^0(\mathcal{A})$, where the rows are triangles and the leftmost square commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "u"] \arrow[d, "b"] & C(f) \arrow[r, "v"] \arrow[d, dotted, "\exists ? \phi'"] & X[1] \arrow[d, "a[1]"] \\ V \arrow[r, "g"] & W \arrow[r, "w"] & C(g) \arrow[r, "t"] & V[1] \end{tikzcd}$$

We need to show that there exists a morphism $\phi' \in H^0(\mathcal{A})(C(f), C(g))$ such that the two other squares also commute in $H^0(\mathcal{A})$.

The commutativity of the leftmost square in $Z^0(\mathcal{A})$ implies that there exists $s \in \mathcal{A}(X, W)^{-1}$, such that $b \circ f - g \circ a = d_{X,W}^{-1}(s)$ in $Z^0(\mathcal{A})$. This gives us the graded transformation $s^\wedge \in \mathcal{C}_{dg}(\mathcal{A})(X^\wedge, W^\wedge)^{-1}$. We construct the following diagram in $\mathcal{C}(\mathcal{A})$, with appropriate differentials on the cones.

$$\begin{tikzcd} Y^\wedge \arrow[r, "\binom{1}{0}"] \arrow[d, "b^\wedge"] & Y^\wedge \oplus X^\wedge[1] \arrow[r, "(0\ 1)"] \arrow[d, dotted, "\phi"] & X^\wedge[1] \arrow[d, "a^\wedge[1]"] \\ W^\wedge \arrow[r, "\binom{1}{0}"] & W^\wedge \oplus V^\wedge[1] \arrow[r, "(0\ 1)"] & V^\wedge[1] \end{tikzcd}$$

Where ϕ is the morphism with components

$$\phi_n = \begin{pmatrix} b_n^\wedge & s_{n+1}^\wedge \\ 0 & a_{n+1}^\wedge \end{pmatrix}$$

This clearly makes the diagram commute, as the term $s_{n+1}^\wedge$ vanishes in both compositions. We check that ϕ commutes with the differentials.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{-2}_{C(f)^\wedge}} & Y^\wedge_{-1}\oplus X^\wedge_0 & \xrightarrow{d^{-1}_{C(f)^\wedge}} & Y^\wedge_0\oplus X^\wedge_1 & \xrightarrow{d^{0}_{C(f)^\wedge}} & Y^\wedge_1\oplus X^\wedge_2 & \xrightarrow{d^{1}_{C(f)}} & \cdots \\
 & & \Big\downarrow{\phi_{-1}} & & \Big\downarrow{\phi_0} & & \Big\downarrow{\phi_1} & & \\
\cdots & \xrightarrow{d^{-2}_{C(g)^\wedge}} & W^\wedge_{-1}\oplus V^\wedge_0 & \xrightarrow{d^{-1}_{C(g)^\wedge}} & W^\wedge_0\oplus V^\wedge_1 & \xrightarrow{d^{0}_{C(g)^\wedge}} & W^\wedge_1\oplus V^\wedge_2 & \xrightarrow{d^{1}_{C(g)^\wedge}} & \cdots
\end{array}$$

We have the differentials

$$d^n_{C(f)^\wedge}=\begin{pmatrix} d^n_{Y^\wedge} & f^\wedge_{n+1}\\ 0 & -d^{n+1}_{X^\wedge}\end{pmatrix}\qquad\qquad d^n_{C(g)^\wedge}=\begin{pmatrix} d^n_{W^\wedge} & g^\wedge_{n+1}\\ 0 & -d^{n+1}_{V^\wedge}\end{pmatrix}$$

We get

$$\begin{aligned}
&\phi_n\circ d^{n-1}_{C(f)^\wedge}-d^{n-1}_{C(g)^\wedge}\circ\phi_{n-1} &&=\\
&\begin{pmatrix} b^\wedge_n & s^\wedge_{n+1}\\ 0 & a^\wedge_{n+1}\end{pmatrix}\begin{pmatrix} d^{n-1}_{Y^\wedge} & f^\wedge_n\\ 0 & -d^n_{X^\wedge}\end{pmatrix}-\begin{pmatrix} d^{n-1}_{W^\wedge} & g^\wedge_n\\ 0 & -d^n_{V^\wedge}\end{pmatrix}\begin{pmatrix} b^\wedge_{n-1} & s^\wedge_n\\ 0 & a^\wedge_n\end{pmatrix} &&=\\
&\begin{pmatrix} b^\wedge_n\circ d^{n-1}_{Y^\wedge} & b^\wedge_n\circ f^\wedge_n-s^\wedge_{n+1}\circ d^n_{X^\wedge}\\ 0 & -a^\wedge_{n+1}\circ d^n_{X^\wedge}\end{pmatrix}-\begin{pmatrix} d^{n-1}_{W^\wedge}\circ b^\wedge_{n-1} & d^{n-1}_{W^\wedge}\circ s^\wedge_n+g^\wedge_n\circ a^\wedge_n\\ 0 & -d^n_{V^\wedge}\circ a^\wedge_n\end{pmatrix} &&=\\
&\begin{pmatrix} b^\wedge_n\circ d^{n-1}_{Y^\wedge}-d^{n-1}_{W^\wedge}\circ b^\wedge_{n-1} & (b^\wedge_n\circ f^\wedge_n-g^\wedge_n\circ a^\wedge_n)-(d^{n-1}_{W^\wedge}\circ s^\wedge_n+s^\wedge_{n+1}\circ d^n_{X^\wedge})\\ 0 & -(a^\wedge_{n+1}\circ d^n_{X^\wedge}-d^n_{V^\wedge}\circ a^\wedge_n)\end{pmatrix} &&=\\
&\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}
\end{aligned}$$

In the last equality everything vanishes because of the following. All components of $a^\wedge$ and $b^\wedge$ are chain maps, hence we know that $b^\wedge_n\circ d^{n-1}_{Y^\wedge}-d^{n-1}_{W^\wedge}\circ b^\wedge_{n-1}=0$ and $a^\wedge_{n+1}\circ d^n_{X^\wedge}-d^n_{V^\wedge}\circ a^\wedge_n=0$. Since $b\circ f-g\circ a=d^{-1}_{X,W}(s)$, we get the following by Proposition 2.4.

$$b^\wedge_n\circ f^\wedge_n-g^\wedge_n\circ a^\wedge_n=d^{-1}_{X,W}(s)^\wedge_n=d_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)_n$$

On the other hand we also have

$$d_{\mathcal{C}_{dg}(\mathcal{A})}(s^\wedge)_n=d^{n-1}_{W^\wedge}\circ s^\wedge_n+s^\wedge_{n+1}\circ d^n_{X^\wedge}$$

from the definition of the differential in $\mathcal{C}_{dg}(\mathcal{A})$. Hence the top right term in the matrix also vanishes, and thus ϕ commutes with the differentials.

Applying β we get a morphism $\phi' \in H^0(\mathcal{A})(C(f), C(g))$ corresponding to ϕ which makes the original diagram commute.

(TR4) Assume we have the following diagram in $H^0(\mathcal{A})$, where the three first rows are triangles

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "v"] & C(u) \arrow[r, "b"] & X[1] \arrow[d, equal] \\
X \arrow[r, "vu"] \arrow[d, "u"] & Z \arrow[r, "c"] \arrow[d, equal] & C(vu) \arrow[r, "d"] & X[1] \arrow[d, "{u[1]}"] \\
Y \arrow[r, "v"] \arrow[d, "a"] & Z \arrow[r, "e"] \arrow[d, "c"] & C(v) \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[d, "{a[1]}"] \\
C(u) & C(vu) & C(v) & C(u)[1]
\end{tikzcd}
$$

Then we need to show that the bottom row also forms a triangle in $H^0(\mathcal{A})$. Transfering this diagram over to $\mathcal{C}(\mathcal{A})$ and decomposing the cones with appropriate differentials, we get the following diagram.

$$
\begin{tikzcd}
X^{\wedge} \arrow[r, "u^{\wedge}"] \arrow[d, equal] & Y^{\wedge} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "v^{\wedge}"] & Y^{\wedge} \oplus X^{\wedge}[1] \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "g^{\wedge}"] & X^{\wedge}[1] \arrow[d, equal] \\
X^{\wedge} \arrow[r, "(vu)^{\wedge}"] \arrow[d, "u^{\wedge}"] & Z^{\wedge} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, equal] & Z^{\wedge} \oplus X^{\wedge}[1] \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "h^{\wedge}"] & X^{\wedge}[1] \arrow[d, "{u^{\wedge}[1]}"] \\
Y^{\wedge} \arrow[r, "v^{\wedge}"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & Z^{\wedge} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & Z^{\wedge} \oplus Y^{\wedge}[1] \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, equal] & Y^{\wedge}[1] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\
Y^{\wedge} \oplus X^{\wedge}[1] \arrow[r, "g"] & Z^{\wedge} \oplus X^{\wedge}[1] \arrow[r, "h"] & Z^{\wedge} \oplus Y^{\wedge}[1] \arrow[r, "i"] & Y^{\wedge}[1] \oplus X^{\wedge}[2]
\end{tikzcd}
$$

We have filled in the missing parts with morphisms $g^\wedge, h^\wedge$ and $i^\wedge$ with components

$$g_n = \begin{pmatrix} v_n^\wedge & 0 \\ 0 & 1_{X_{n+1}^\wedge} \end{pmatrix} \qquad h_n = \begin{pmatrix} 1_{Z_n^\wedge} & 0 \\ 0 & u_{n+1}^\wedge \end{pmatrix} \qquad i_n = \begin{pmatrix} 0 & 1_{Y_{n+1}^\wedge} \\ 0 & 0 \end{pmatrix}$$

One can check that these commute with the respective differentials, and makes the entire diagram commute. To show that the bottom row gives us a triangle, we need to consider the object $C(g)^\wedge \simeq Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2]$, with the differential

$$d^n_{C(g^\wedge)} = \begin{pmatrix} d^n_{Z^\wedge} & (vu)^\wedge_{n+1} & v^\wedge_{n+1} & 0 \\ 0 & -d^{n+1}_{X^\wedge} & 0 & 1_{X^\wedge_{n+2}} \\ 0 & 0 & -d^{n+1}_{Y^\wedge} & -u^\wedge_{n+2} \\ 0 & 0 & 0 & d^{n+2}_{X^\wedge} \end{pmatrix}$$

We have to show that we have an isomorphism of triangles in $\mathcal{H}(\mathcal{A})$.

$$\begin{tikzcd}
Y^\wedge \oplus X^\wedge[1] \arrow[r, "g"] \arrow[d, equal] & Z^\wedge \oplus X^\wedge[1] \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[d, equal] & Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2] \arrow[r, "\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, dotted, bend left, "\tau"] & Y^\wedge[1] \oplus X^\wedge[2] \arrow[d, equal] \\
Y^\wedge \oplus X^\wedge[1] \arrow[r, "g"] & Z^\wedge \oplus X^\wedge[1] \arrow[r, "h"] & Z^\wedge \oplus Y^\wedge[1] \arrow[r, "i"] \arrow[u, dotted, bend left, "\sigma"] & Y^\wedge[1] \oplus X^\wedge[2]
\end{tikzcd}$$

We construct the following two morphisms.

$$\sigma : Z^\wedge \oplus Y^\wedge[1] \to Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2],$$
$$\sigma_n = \begin{pmatrix} 1_{Z_n^\wedge} & 0 \\ 0 & 0 \\ 0 & 1_{Y_{n+1}^\wedge} \\ 0 & 0 \end{pmatrix}$$
$$\tau : Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2] \to Z^\wedge \oplus Y^\wedge[1],$$
$$\tau_n = \begin{pmatrix} 1_{Z_n^\wedge} & 0 & 0 & 0 \\ 0 & u_{n+1}^\wedge & 1_{Y_{n+1}^\wedge} & 0 \end{pmatrix}$$

One can check that τ makes the left square commute, and σ makes the right square commute in $\mathcal{C}(\mathcal{A})$. One can also check that σ makes the left square commute in $\mathcal{H}(\mathcal{A})$, by using $s : Z^\wedge \oplus X^\wedge[1] \to Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2]$, the graded transformation of degree -1 with components

$$s_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} \end{pmatrix}$$

Similarly we can check that τ makes the right square commute by using $s : Z^\wedge \oplus X^\wedge[1] \oplus Y^\wedge[1] \oplus X^\wedge[2] \to Y^\wedge[1] \oplus X^\wedge[2]$ given by components

$$s^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} & 0 & 0 \end{pmatrix}$$

Now we check that σ and τ are inverses in $\mathcal{H}(\mathcal{A})$.

$$\tau_n \circ \sigma_n = \begin{pmatrix} 1_{Z^\wedge_n} & 0 \\ 0 & 1_{Y^\wedge_{n+1}} \end{pmatrix} \qquad \sigma_n \circ \tau_n = \begin{pmatrix} 1_{Z^\wedge_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & u^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $s : C(g)^\wedge \to C(g)^\wedge$ be the graded transformation of degree -1 with components

$$s_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1_{X^\wedge_{n+1}} & 0 & 0 \end{pmatrix}$$

We get

$$d^{n-1}_{C(g)^\wedge} \circ s_n + s_{n+1} \circ d^n_{C(g)^\wedge} =$$

$$\begin{pmatrix} d^n_{Z^\wedge} & (vu)^\wedge_{n+1} & v^\wedge_{n+1} & 0 \\ 0 & -d^{n+1}_{X^\wedge} & 0 & 1_{X^\wedge_{n+2}} \\ 0 & 0 & -d^{n+1}_{Y^\wedge} & -u^\wedge_{n+2} \\ 0 & 0 & 0 & d^{n+2}_{X^\wedge} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1^{n+1}_{X^\wedge} & 0 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1^{n+2}_{X^\wedge} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^n_{Z^\wedge} & (vu)^\wedge_{n+1} & v^\wedge_{n+1} & 0 \\ 0 & -d^{n+1}_{X^\wedge} & 0 & 1_{X^\wedge_{n+2}} \\ 0 & 0 & -d^{n+1}_{Y^\wedge} & -u^\wedge_{n+2} \\ 0 & 0 & 0 & d^{n+2}_{X^\wedge} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_{X^\wedge_{n+1}} & 0 & 0 \\ 0 & u^\wedge_{n+1} & 0 & 0 \\ 0 & -d^{n+1}_{X^\wedge} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d^{n+1}_{X^\wedge} & 0 & -1_{X^\wedge_{n+2}} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_{X^\wedge_{n+1}} & 0 & 0 \\ 0 & u^\wedge_{n+1} & 0 & 0 \\ 0 & 0 & 0 & -1_{X^\wedge_{n+2}} \end{pmatrix}$$

Now notice that

$$d^{n-1} \circ s_n + s_{n+1} \circ d^n + 1_{C(g)^\wedge_n} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_{X^\wedge_{n+1}} & 0 & 0 \\ 0 & u^\wedge_{n+1} & 0 & 0 \\ 0 & 0 & 0 & -1_{X^\wedge_{n+2}} \end{pmatrix} + \begin{pmatrix} 1_{Z^\wedge_n} & 0 & 0 & 0 \\ 0 & 1_{X^\wedge_{n+1}} & 0 & 0 \\ 0 & 0 & 1_{Y^\wedge_{n+1}} & 0 \\ 0 & 0 & 0 & 1_{X^\wedge_{n+2}} \end{pmatrix} =$$

$$\begin{pmatrix} 1_{Z^\wedge_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & u^\wedge_{n+1} & 1_{Y^\wedge_{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sigma_n \circ \tau_n$$

Hence by Proposition 2.13 we get that $\sigma \circ \tau = 1_{C(g)}$ in $\mathcal{H}(\mathcal{A})$. For the other composition order, we already have that $\tau \circ \sigma = 1_{C(v)^\wedge}$ in $\mathcal{C}(\mathcal{A})$, hence it is also true in $\mathcal{H}(\mathcal{A})$. Hence $C(v)^\wedge \simeq C(g)^\wedge$ in $\mathcal{H}(\mathcal{A})$. By applying β we conclude that the original diagram commutes and the bottom row is a triangle in $H^0(\mathcal{A})$. $\square$

3 Exact categories

In abelian categories, we can form exact sequences by using the concepts of kernels and cokernels. The goal in this section is to study *exact categories*, a generalization of abelian categories. These categories allow us to define *exact sequences* in categories where not all morphisms necessarily have kernels and/or cokernels. After defining the first concepts, we define the *stable category* of a special type of exact category, and see how this category is triangulated.

3.1 The basics

There are several equivalent formulations of the properties an exact category should satisfy, here we will use the definition given by Keller [9, p.405].

Definition. An exact category is an additive category $\mathcal{C}$, together with a family E of sequences on the form

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{d} Z \longrightarrow 0$$

where i is a kernel of d, and d is a cokernel of i. These sequences are called *exact sequences*. The morphisms i are called *inflations*, and the morphisms d are called *deflations*[4]. Furthermore, we require the following to hold

(1) The family E is closed under isomorphisms. That is, given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "\wr"'] & Y \arrow[r, "d"] \arrow[d, "\wr"'] & Z \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "i'"] & Y' \arrow[r, "d'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

where the top row is in E and the vertical morphisms are all isomorphisms, then the bottom row is also in E.

(2) 1_0 is a deflation.

[4]These are sometimes instead called *admissible monomorphisms/epimorphisms* respectively

(3) The composition of two deflations is a deflation.

(4) Given a diagram

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[r, "d"] & Z \end{tikzcd}$$

where d is a deflation and f is any morphism, then the pullback exists

$$\begin{tikzcd} P \arrow[r, "\hat{d}"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ X \arrow[r, "d"] & Z \end{tikzcd}$$

and the morphism $\hat{d}$ is also a deflation.

(5) Dually, given a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \\ Z & \end{tikzcd}$$

where i is an inflation and f is any morphism, the pushout exists

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ Z \arrow[r, "\hat{i}"] & P \end{tikzcd}$$

and the morphism $\hat{i}$ is also an inflation.

Remark 3.1. It can be shown that there is an equivalent characterization of exact categories. Namely an additive category $\mathcal{C}$ is exact if and only if it can be embedded as a full, *extension-closed* subcategory of some abelian category $\mathcal{D}$. Here extension-closed refers to the property that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence in $\mathcal{D}$, and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. Some authors use this as the definition of an exact category, see for example [6, p.10].

Example. A simple example of an exact category is **Ab**, the category of abelian groups. The exact structure is given by the collection $E_{\mathbf{Ab}}$ containing all short exact sequences in the usual sense.

Let $X \in \mathcal{C}$. Since $\mathcal{C}$ is additive, we can consider the following additive functors.

$$\mathcal{C}(\,\cdot\,, X) : \mathcal{C}^{op} \to \mathbf{Ab}$$
$$\mathcal{C}(X, \,\cdot\,) : \mathcal{C} \to \mathbf{Ab}$$

We establish a useful fact about these functors.

Proposition 3.2. *The functor $\mathcal{C}(\,\cdot\,, X)$ is left-exact. That is, given any exact sequence in $\mathcal{C}$*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

and any $V \in \mathcal{C}$, then the sequence

$$0 \longrightarrow \mathcal{C}(Z, V) \xrightarrow{g^*} \mathcal{C}(Y, V) \xrightarrow{f^*} \mathcal{C}(X, V)$$

is exact in ***Ab****.*

Proof. Let $g^*(a) = 0$ for some $a \in \mathcal{C}(Z, V)$. This means that $a \circ g = 0$, and since g is an epimorphism, this implies that $a = 0$. Hence g^* is injective. Now consider $c := g^*(b) = b \circ g$ for some $b \in \mathcal{C}(Z, V)$. We get that $f^*(c) = c \circ f = b \circ g \circ f = 0$. Hence $\mathrm{Im}(g^*) \subseteq \mathrm{Ker}(f^*)$. Now let $d \in \mathcal{C}(Y, V)$ such that $f^*(d) = 0$. This means $d \circ f = 0$, and since Z is a cokernel of f, we get the existence of a unique morphism h completing the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[dr, "d"'] & Z \arrow[d, dashed, "\exists! h"] \\ & & V \end{tikzcd}$$

Specifically we have that $d = h \circ g = g^*(h)$, and hence $\mathrm{Ker}(f^*) \subseteq \mathrm{Im}(g^*)$. $\square$

Remark 3.3. Note that a similar statement can be proven for the functor $\mathcal{C}(\,\cdot\,, X)$.

Definition. An exact functor is an additive functor $F : \mathcal{C} \to \mathcal{D}$ between exact categories such that the image of any exact sequence in $\mathcal{C}$ is an exact sequence in $\mathcal{D}$.

As we have seen, the functors $\mathcal{C}(\,\cdot\,, X)$ and $\mathcal{C}(X,\,\cdot\,)$ are *almost* exact. We denote the objects which make these exact in the following way.

Definition. Let $\mathcal{C}$ be an exact category. An object $X \in \mathcal{C}$ is called

(1) *injective* if the functor $\mathcal{C}(\,\cdot\,, X)$ is exact.

(2) *projective* if the functor $\mathcal{C}(X,\,\cdot\,)$ is exact.

We now give an equivalent characterization of injective objects.

Proposition 3.4. *Let $I \in \mathcal{C}$ be an object in an exact category. Then I is injective if and only if the following holds. For any morphism $i \in \mathcal{C}(X, I)$ and any inflation $f \in \mathcal{C}(X, Y)$, there exists a morphism $h \in \mathcal{C}(Y, I)$ making the following diagram commute.*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

Proof. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. Applying the functor $\mathcal{C}(\,\cdot\,, I)$ we get the following sequence in **Ab**.

$$0 \longrightarrow \mathcal{C}(Z, I) \xrightarrow{g^*} \mathcal{C}(Y, I) \xrightarrow{f^*} \mathcal{C}(X, I) \longrightarrow 0$$

By definition, we know this sequence is exact if and only if I is injective. Since $\mathcal{C}(\,\cdot\,, I)$ is always left-exact, this sequence being exact is equivalent to f^* being surjective. The morphism f^* being surjective is equivalent to the following. For any $i \in \mathcal{C}(X, I)$, there exists some $h \in \mathcal{C}(Y, I)$ such that $i = f^*(h)$. But since $f^*(h) = h \circ f$, this is equivalent to $i = h \circ f$ for some $h \in \mathcal{C}(Y, I)$. This is exactly what we wanted to show, and hence this is an equivalent characterization of injective objects. $\square$

Remark 3.5. There is a similar characterization of projective objects, using the dual of the property described here.

The following is a useful fact about exact sequences starting with an injective.

Proposition 3.6. *Let*

$$0 \longrightarrow I \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence where I is injective. Then the sequence splits, and $Y \simeq I \oplus Z$.

Proof. Because I is injective, there exists $h : Z \to I$ such that $h \circ f = 1_I$.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "f"] \arrow[d, "1_i"] & Z \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

Consider the idempotents $s = f \circ h : Y \to Y$ and $t = 1_Y - s$. Notice that

$$\begin{aligned} & t \circ f && = \\ & (1_Y - s) \circ f && = \\ & 1_Y \circ f - s \circ f && = \\ & f - f \circ h \circ f && = \\ & f - f \circ 1_I && = \\ & 0 && \end{aligned}$$

hence by the cokernel property of Z we get a unique morphism $\phi : Z \to Y$ such that $t = \phi \circ g$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[dr, "t"'] & Z \arrow[d, dashed, "\phi"] \\ & & Y \end{tikzcd}$$

Putting it all together, we get the diagram

$$
\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "f"'] \arrow[d, equal] & Y \arrow[l, bend right, "h"'] \arrow[r, "g"'] \arrow[d, "\left(\begin{smallmatrix} h \\ g \end{smallmatrix}\right)"] & Z \arrow[r] \arrow[d, equal] \arrow[l, bend right, "\phi"'] & 0 \\
0 \arrow[r] & I \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & I \oplus Z \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"'] & Z \arrow[r] & 0
\end{tikzcd}
$$

We will show that $\begin{pmatrix} h \\ g \end{pmatrix}$ is an isomorphism, with $\begin{pmatrix} f & \phi \end{pmatrix}$ as inverse.

$$
\begin{aligned}
&\begin{pmatrix} f & \phi \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} && = \\
&f \circ h + \phi \circ g && = \\
&s + t && = \\
&s + (1_Y - s) && = \\
&1_Y
\end{aligned}
$$

Checking the other composition order we get

$$
\begin{aligned}
&\begin{pmatrix} h \\ g \end{pmatrix} \begin{pmatrix} f & \phi \end{pmatrix} && = \\
&\begin{pmatrix} h \circ f & h \circ \phi \\ g \circ f & g \circ \phi \end{pmatrix} && = \\
&\begin{pmatrix} 1_X & h \circ \phi \\ 0 & g \circ \phi \end{pmatrix}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\phi \circ g = 1_Y - e && \Longrightarrow \\
&g \circ \phi \circ g = g \circ (1_Y - e) && \Longrightarrow \\
&g \circ \phi \circ g = g - g \circ f \circ h && \Longrightarrow \\
&g \circ \phi \circ g = g && \Longrightarrow \\
&g \circ \phi = 1_Z && \text{Since } g \text{ epimorphism}
\end{aligned}
$$

We also see that

$$
\begin{aligned}
&(f \circ h) \circ (\phi \circ g) = (s) \circ (t) && \Longrightarrow \\
&f \circ h \circ \phi \circ g = s \circ (1_Y - s) && \Longrightarrow \\
&f \circ h \circ \phi \circ g = s - s && \Longrightarrow \\
&f \circ h \circ \phi \circ g = 0 && \Longrightarrow \\
&h \circ \phi = 0 && \text{Since } g \text{ epimorphism, } f \text{ monomorphism}
\end{aligned}
$$

Hence we get

$$\begin{pmatrix} h \\ g \end{pmatrix} \begin{pmatrix} f & \phi \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}$$

□

Remark 3.7. Note that the dual statement is also true; an exact sequence ending with a projective is split, with the middle object isomorphic to the biproduct of the outer objects.

Definition. An exact category $\mathcal{C}$ has *enough projectives* (resp. *enough injectives*), if for every object $X \in \mathcal{C}$, there exists a deflation $P \to X$, P projective (exists an inflation $X \to I$, I injective).

Definition. An exact category $\mathcal{F}$ is called a *Frobenius category*, if it has enough projectives and injectives, and if the collection of projective objects coincides with the collection of injective objects. That is for all $X \in \mathcal{F}$, X projective $\iff$ X injective.

We will look at an example of some Frobenius categories. In order to see that the categories in question indeed are Frobenius categories, the following proposition will be useful.

Proposition 3.8. *Let S be a PID, (p) a maximal ideal and denote $R := S/(p^2)$. Then there are only two finitely generated indecomposable R-modules, R and $R/(p)$.*

Proof. We know that R-modules are in 1-1 correspondence with S-modules annihilated by (p^2). By *the structure theorem for finitely generated modules over a PID*, we know any finitely generated S-module can be decomposed into indecomposables S or $S/(q)$, with $(q) \neq 0$ a power of a prime ideal. Since

S is a PID, $(p^2) \neq 0$, and hence S is not annihilated by (p^2). Assume now that an indecomposable S-module $S/(q)$ is annihilated by (p^2). We get that

$$(p^2) \cdot S/(q) = 0 \iff (p^2) \subseteq (q) \implies p^2 \in (q) \implies p^2 = q \cdot n$$

for some $n \in S$. Since every PID is a UFD and p is irreducible, this implies that $q = p$ or $q = p^2$ up to multiplication by a unit. Hence $S/(p^2) = R$ and $S/(p) \simeq R/(p)$ are the only indecomposable R-modules. $\square$

Remark 3.9. Note that the ring R has an ideal (p) corresponding to the ideal (p) in S, since $(p^2) \subseteq (p)$. By the third isomorphism theorem we get the last isomorphism $S/(p) \simeq R/(p)$. Considering this ideal (p) in R, we get the two R-modules (p) and $R/(p)$. We have a surjective morphism of R-modules $f : R \to (p)$ given by multiplication by p. $\mathrm{Ker}(f) = (p)$, so $(p) \simeq R/(p)$. In particular this means we have an exact sequence

$$0 \longrightarrow (p) \xrightarrow{\ i\ } R_i \xrightarrow{\ \pi\ } R/(p) \longrightarrow 0$$

with isomorphic endpoints.

Example. Let $p \in \mathbb{N}$ be a prime number, and let $\mathbb{F}_p$ be the field with p elements. We define the two rings $R_1 := \mathbb{Z}/p^2\mathbb{Z}$, and $R_2 := \mathbb{F}_p[\epsilon] \simeq \mathbb{F}_p[x]/(x^2)$. Since these rings are finite (and hence artinian), and of finite representation type (we will show this), when classifying the modules over these rings it will be enough to study the finitely generated ones. This is because any module over these rings is isomorphic to a (potentially infinite) direct sum of finitely generated indecomposable modules, see [1].

First we look at some properties of these rings. Denote the elements $p_1 = p \in R_1$ and $p_2 = \epsilon \in R_2$. The elements p_i generate a maximal ideal in both rings which we denote by $I_1 = (p), I_2 = (\epsilon)$ respectively. We have that $R_1/I_1 \simeq \mathbb{F}_p \simeq R_2/I_2$. Thus $\mathbb{F}_p$ is naturally a module over both rings. Are there other indecomposable R_i-modules than $\mathbb{F}_p$ and the ring itself? Using Proposition 3.8, we get that R_i and $R_i/I_i \simeq \mathbb{F}_p$ are indeed the only indecomposables. To see this, for $i = 1$, set $S = \mathbb{Z}, (p) = p\mathbb{Z}$. For $i = 2$, set $S = \mathbb{F}_p[x], (p) = (x)$.

Next, we want to see that $\mathbb{F}_p$ is neither a projective nor injective R_i-module. Consider the exact sequence of R_i-modules

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\ i\ } R_i \xrightarrow{\ \pi\ } \mathbb{F}_p \longrightarrow 0$$

Where i is given by multiplication with p_i, and $\pi : R_i \to R_i/I_i \simeq \mathbb{F}_p$ is the natural projection. If we assume $\mathbb{F}_p$ is injective, by Proposition 3.6, we would have that $R_i \simeq \mathbb{F}_p \oplus \mathbb{F}_p$. But R_i is indecomposable, hence $\mathbb{F}_p$ cannot be injective. Dually if we assume $\mathbb{F}_p$ is projective, the dual statement to Proposition 3.6, implies the same decomposition, and hence $\mathbb{F}_p$ cannot be projective either. Since these rings have at least one indecomposable projective and at least one indecomposable injective module, we can conclude that R_i must be both a projective and injective R_i-module. Thus the categories $\mathrm{Mod}(R_i)$ are both Frobenius categories, where M is projective $\iff$ M is injective $\iff$ M is free.

3.2 Triangulated categories from exact categories

In this section we will define the *stable category* of a Frobenius category. After establishing some useful results, we show that this is a triangulated category.

Definition. Let $\mathcal{C}$ be an exact category. A morphism $f \in \mathcal{C}(X,Y)$ is said to *factor through an injective* if there exists an injective object $I \in \mathcal{C}$, and morphisms $g \in \mathcal{C}(X,I), h \in \mathcal{C}(I,Y)$ such that $f = h \circ g$. We define the subset $I(X,Y) \subseteq \mathcal{C}(X,Y)$ to be set of all morphisms from X to Y which factor through an injective.

Proposition 3.10. *The set $I(X,Y)$ is a subgroup of $\mathcal{C}(X,Y)$.*

Proof. Let $f, g \in I(X,Y)$. If f factors through I as $f = f_2 \circ f_1$, then $(-f_1) \in \mathcal{C}(X,I)$, which means $(-f) = f_2 \circ (-f_1)$, hence $I(X,Y)$ is closed under inverses. Now assume g factors through J as $g = g_2 \circ g_1$. Then we get the following diagram

$$
\begin{tikzcd}
 & X \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
I \arrow[r, shift left, "i_I"] \arrow[dr, "f_2"'] & I \oplus J \arrow[l, shift left, "\pi_I"] \arrow[r, shift left, "\pi_J"] & J \arrow[l, shift left, "1_J"] \arrow[dl, "g_2"] \\
 & Y &
\end{tikzcd}
$$

We look at the morphisms $i_I \circ f_1 + i_J \circ g_1 \in \mathcal{C}(X, I \oplus J)$ and $f_2 \circ \pi_I + g_2 \circ \pi_J \in \mathcal{C}(I \oplus J, Y)$ and observe that their composition

$$
\begin{aligned}
&(f_2 \circ \pi_I + g_2 \circ \pi_J) \circ (i_I \circ f_1 + i_J \circ g_1) &&= \\
&f_2 \circ \pi_I \circ i_I \circ f_1 + f_2 \circ \pi_I \circ i_J \circ g_1 + g_2 \circ \pi_J \circ i_I \circ f_1 + g_2 \circ \pi_J \circ i_J \circ g_1 &&= \\
&f_2 \circ 1_I \circ f_1 + f_2 \circ 0 \circ g_1 + g_2 \circ 0 \circ f_1 + g_2 \circ 1_J \circ g_1 &&= \\
&f_2 \circ f_1 + g_2 \circ g_1 &&= \\
&f + g
\end{aligned}
$$

Hence $f + g$ factors through $I \oplus J$, and by noting that the direct sum of injectives again is injective, this concludes the proof. $\square$

Definition. Let $\mathcal{F}$ be a Frobenius category. We define the *stable category of* $\mathcal{F}$, $S(\mathcal{F})$, having the same objects as $\mathcal{F}$, and morphisms are stable equivalence classes of morphisms, $S(\mathcal{F})(X, Y) := \mathcal{F}(X, Y)/I(X, Y)$. That is, we identify morphisms $f, g \in \mathcal{F}(X, Y)$ if $f - g$ factors through an injective. It can be checked that this category has well defined composition induced by the composition in $\mathcal{F}$.

Remark 3.11. One immediate consequence of this construction, is that any injective object in $\mathcal{F}$ is identified with the zero object in $S(\mathcal{F})$. This is because any morphism into or out of an injective object, factors through said injective. Thus for an injective I, all morphisms $X \xrightarrow{f} I$ are identified with eachother, and similarly all morphisms $I \xrightarrow{g} Y$ are identified with eachother. Hence there is only one morphism into, and one morphism out of I, which means I is isomorphic to the zero object in $S(\mathcal{F})$.

The following proposition will be one of the key ingredients in constructing triangles.

Proposition 3.12. *Given an exact sequence*

$$0 \longrightarrow X \xrightarrow{x} X'' \xrightarrow{\bar{x}} X' \longrightarrow 0$$

and a morphism $f : X \to Y$, there is a unique *(up to isomorphism) way to construct the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & X' \arrow[r] \arrow[d, equal] & 0 \\ & Y \arrow[r, "v"] & P \arrow[r, "w"] & X' & \end{tikzcd}$$

*such that P is the pushout of x and f and $w \circ v = 0$. Moreover the bottom row is always exact. We call this the **induced exact sequence** of the top row and f.*

Proof. We start with the following diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] & X' \arrow[r] \arrow[d, equal] & 0 \\ & Y & & X' & \end{tikzcd}$$

Since the morphism x is an inflation, we can construct the pushout P, which is unique up to isomorphism. This gives the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "v"] & P & X' & \end{tikzcd}$$

where v is an inflation. Now we use the pushout property of P, by noting that both $0 \circ f = \bar{x} \circ x = 0$. This gives a unique morphism $w : P \to X'$, satisfying $w \circ v = 0$ and $w \circ \bar{f} = \bar{x}$. This completes the diagram.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] \arrow[rr, bend right, "0"'] & P \arrow[r, dotted, "\exists w"] & X' &
\end{tikzcd}$$

Now we show that the bottom row is also exact. Since v is an inflation, there exists a corresponding deflation $u : P \to Z$. Since this is a cokernel of v, and $w \circ v = 0$, we also get a unique morphism $t : Z \to X'$ satisfying $t \circ u = w$. We propose that this is in fact an isomorphism. Using the cokernel property of X' together with the fact that $0 = u \circ v \circ f = (u \circ \bar{f}) \circ x$, we get a unique morphism $s : X' \to Z$ satisfying $s \circ \bar{x} = u \circ \bar{f}$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] & P \arrow[r, "w"] \arrow[dr, "u"'] & X' \arrow[d, dotted, shift left, "\exists s"] & \\
 & & & Z \arrow[u, dotted, shift left, "\exists t"] \arrow[dr] & \\
 & & & & 0
\end{tikzcd}$$

We want to show that s and t are inverses of eachother. We get

$$\begin{aligned}
& s \circ \bar{x} = u \circ \bar{f} && \Longrightarrow \\
& t \circ s \circ \bar{x} = t \circ u \circ \bar{f} && \Longrightarrow \\
& t \circ s \circ \bar{x} = w \circ \bar{f} && \Longrightarrow \\
& t \circ s \circ \bar{x} = \bar{x} && \Longrightarrow \\
& t \circ s = 1_{X'} && \text{Since } \bar{x} \text{ epimorphism}
\end{aligned}$$

For the other composition order, we first need to show that $u = s \circ w$. We have that $(s \circ w) \circ v = 0 = u \circ v$ and $(s \circ w) \circ \bar{f} = s \circ \bar{x} = u \circ \bar{f}$.

By Proposition 1.2, we conclude that $u = s \circ w$. Now we get

$$\begin{aligned}
&t \circ u = w && \Longrightarrow \\
&s \circ t \circ u = s \circ w && \Longrightarrow \\
&s \circ t \circ u = u && \Longrightarrow \\
&s \circ t = 1_Z && \text{Since } u \text{ epimorphism}
\end{aligned}$$

Thus $X' \simeq Z$, and since exact sequences are closed under isomorphisms, the bottom row is also exact. □

When studying $S(\mathcal{F})$, we will be interrested in a specific kind of exact sequence, namely those where the middle term is injective.

Proposition 3.13. *Consider the following two exact sequences in $\mathcal{F}$,*

$$0 \longrightarrow X \xrightarrow{\mu_1} I \xrightarrow{\pi_1} Y \longrightarrow 0$$

$$0 \longrightarrow X \xrightarrow{\mu_2} J \xrightarrow{\pi_2} Z \longrightarrow 0$$

where the middle objects I and J are injective. Then Y and Z are isomorphic in $S(\mathcal{F})$.

Proof. Because of the injective property of I and J, there exist morphisms f_1, f_2 making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\mu_1"] \arrow[d, equal] & I \arrow[r, "\pi_1"] \arrow[d, "f_1"] & Y \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "\mu_2"] \arrow[d, equal] & J \arrow[r, "\pi_2"] \arrow[d, "f_2"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "\mu_1"] & I \arrow[r, "\pi_1"] & Y \arrow[r] & 0
\end{tikzcd}$$

Since Y and Z are cokernels of μ_1 and μ_2 respectively, and since $0 = \pi_1 \circ \mu_1 = (\pi_1 \circ f_2) \circ \mu_2$ (similarly $0 = \pi_2 \circ \mu_2 = (\pi_2 \circ f_1) \circ \mu_1$), we get that there also exist morphisms g_1, g_2 making the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu_1} & I & \xrightarrow{\pi_1} & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \\
0 & \longrightarrow & X & \xrightarrow{\mu_2} & J & \xrightarrow{\pi_2} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_2} & & \\
0 & \longrightarrow & X & \xrightarrow{\mu_1} & I & \xrightarrow{\pi_1} & Y & \longrightarrow & 0
\end{array}$$

Now notice two different paths from X to I gives us that $f_2 \circ f_1 \circ \mu_1 = \mu_1 \iff (f_2 \circ f_1 - 1_I) \circ \mu_1 = 0$. Using the fact that Y is a cokernel for μ_1, there exists a unique morphism $h : Y \to I$ making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "\mu_1"] & I \arrow[r, "\pi_1"] \arrow[dr, "(f_2 \circ f_1 - 1_I)"'] & Y \arrow[d, dashed, "h"] \\
 & & I
\end{tikzcd}$$

In other words, we have $h \circ \pi_1 = f_2 \circ f_1 - 1_I$. Postcomposition with π_1 gives us that

$$\begin{aligned}
&\pi_1 \circ h \circ \pi_1 &&= \\
&\pi_1 \circ (f_2 \circ f_1 - 1_I) &&= \\
&\pi_1 \circ f_2 \circ f_1 - \pi_1 &&= \\
&g_2 \circ g_1 \circ \pi_1 - \pi_1 &&= \\
&(g_2 \circ g_1 - 1_Y) \circ \pi_1 &&
\end{aligned}$$

Since π_1 is an epimorphism, this implies that $\pi_1 \circ h = (g_2 \circ g_1 - 1_Y)$. Since the morphism $\pi_1 \circ h$ factors through the projective I, we get that $0 = (g_2 \circ g_1 - 1_Y) \implies g_2 \circ g_1 = 1_Y$ in the category $S(\mathcal{F})$. Similarly we can argue that $g_1 \circ g_2 = 1_Z$ in $S(\mathcal{F})$, by swapping the roles of the two exact sequences in the argument. Thus $Y \simeq Z$ in $S(\mathcal{F})$. □

The above proposition allows us to construct a well defined map $T : \mathrm{Obj}(S(\mathcal{F})) \to \mathrm{Obj}(S(\mathcal{F}))$ by doing as follows. For every object $X \in \mathcal{F}$, we choose some exact sequence starting with X, and where the middle object is injective.

$$0 \longrightarrow X \xrightarrow{x} I(X) \xrightarrow{\bar{x}} Y \longrightarrow 0$$

We can do this for every object, since $\mathcal{F}$ has enough injectives. Then we define $T(X) := Y$ in $S(\mathcal{F})$. By Proposition 3.13, we know that this construction is well defined up to isomorphism in $S(\mathcal{F})$.

When studying $S(\mathcal{F})$, we come across morphisms which factor through *some* injective. The following proposition will give us more control over exactly which injective morphisms may factor through.

Proposition 3.14. *Let $X, Y \in \mathcal{F}$, and let*

$$0 \longrightarrow X \xrightarrow{x} I(X) \xrightarrow{\bar{x}} T(X) \longrightarrow 0$$

be a sequence like constructed previously. Let $f \in \mathcal{F}(X, Y)$ be a morphism factoring through an injective, in other words $f = 0$ in $S(\mathcal{F})$. Then f factors through $I(X)$ as $f = \alpha \circ x$, for some $\alpha \in \mathcal{F}(I(X), Y)$.

Proof. Since f factors through an injective, there exists some $J \in \mathcal{F}$, $f_1 \in \mathcal{F}(X, J)$, $f_2 \in \mathcal{F}(J, Y)$ such that $f = f_2 \circ f_1$. We get the following diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, equal] & I(X) \arrow[r, "\bar{x}"] \arrow[d, dotted, "\exists h"] & T(X) \arrow[r] & 0 \\
 & X \arrow[r, "f_1"] & J \arrow[r, "f_2"] & Y &
\end{tikzcd}$$

now since J is injective, and x is an inflation, we get that there exists a morphism $h : I(X) \to J$ such that $h \circ x = f_1$. Thus the morphism $f = f_2 \circ f_1 = (f_2 \circ h) \circ x$ factors through $I(X)$. $\square$

We want to extend the map $T : \mathrm{Obj}(S(\mathcal{F})) \to \mathrm{Obj}(S(\mathcal{F}))$ to a functor. For a morphism $f \in \mathcal{F}(X, Y)$, we want to construct the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & I(X) \arrow[r, "\bar{x}"] \arrow[d, "I(f)"] & T(X) \arrow[r] \arrow[d, "T(f)"] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\bar{y}"] & T(Y) \arrow[r] & 0
\end{tikzcd}$$

Since $I(Y)$ is injective, there exists some $I(f)$ making the leftmost square commute. This is not necessarily unique, so we choose one. After this choice is made, we get a unique morphism $T(f)$ making the rightmost square commute, because $T(X)$ is a cokernel of x. It can be shown that that when passing to $S(\mathcal{F})$, the morphism $T(f)$ is well defined and independent of $I(f)$ and representative of the morphisms f. This means we can consider T as a functor $T : S(\mathcal{F}) \to S(\mathcal{F})$. It can be shown that this indeed is an automorphism. For more details, see [6, p.12].

Proposition 3.15. *Consider the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & X'' \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r, "v"] & Z \arrow[r, "w"] & X' \arrow[r] & 0 \end{tikzcd}$$

where both rows are exact and I is injective. Then the bottom row is the induced exact sequence of the top row and f. In particular, Z is the pushout of x and f.

Proof. Let

$$0 \longrightarrow I \xrightarrow{s} P \xrightarrow{t} X' \longrightarrow 0$$

be the induced exact sequence of top row and the morphism f. Since both this and the sequence

$$0 \longrightarrow I \xrightarrow{v} Z \xrightarrow{w} X' \longrightarrow 0$$

are exact sequences starting with an injective, we get the following commutative diagram by Proposition 3.6

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "\phi"] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & I \oplus X' \arrow[r, "(0\ 1)"] \arrow[d, "\psi"] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r, "s"] & P \arrow[r, "t"] & X' \arrow[r] & 0 \end{tikzcd}$$

where ϕ and ψ are isomorphisms. Hence $Z \simeq P$. □

Putting the above constructions together we can define what the triangles in $S(\mathcal{F})$ are.

Definition. Let $f \in S(\mathcal{F})(X,Y)$. We choose some representative $f \in \mathcal{F}(X,Y)$, and construct the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"] & I(X) \arrow[r, "\bar{x}"] \arrow[d, "\bar{f}"] & T(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] & C(f) \arrow[r, "w"] & T(X) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the induced exact sequence of the top row and f. The following sequence in $S(\mathcal{F})$

$$X \xrightarrow{f} Y \xrightarrow{v} C(f) \xrightarrow{w} T(X)$$

is called a *standard triangle*, and the object $C(f)$ is called the *cone* of the morphism f.

We show that this is a well defined construction when passing to $S(\mathcal{F})$. In particular we show that the object $C(f) \in S(\mathcal{F})$ is independent of both the choice of injective $I(X)$, and representative of the morphism $f \in S(\mathcal{F})$.

Proposition 3.16. *The cone of a morphism $f \in S(\mathcal{F})(X,Y)$ is independent of the choice of representative.*

Proof. Let f, g be two different representatives for some morphism in $S(\mathcal{F})(X,Y)$. Then we know by Proposition 3.14 that $g - f = \alpha \circ x$ for some $\alpha \in \mathcal{F}(I(X),Y)$. By the pushout properties, we get the unique morphisms ϕ and ψ between the objects $C(f)$ and $C(g)$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"] & I(X) \arrow[d, "\bar{f}"] \arrow[ddr, bend left, "\bar{g} - u \circ \alpha"] & \\
Y \arrow[r, "v"] \arrow[drr, bend right, "u"'] & C(f) \arrow[dr, dotted, "\exists! \phi"] & \\
 & & C(g)
\end{tikzcd}$$

$$
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "g"] & I(X) \arrow[d, "\bar{g}"] \arrow[ddr, bend left, "\bar{f} + v \circ \alpha"] & \\
Y \arrow[r, "u"] \arrow[drr, bend right, "v"'] & C(g) \arrow[dr, dotted, "\exists ! \psi"] & \\
& & C(f)
\end{tikzcd}
$$

since

$$
\begin{aligned}
&u \circ f &&= \\
&u \circ (g - \alpha \circ x) &&= \\
&u \circ g - u \circ \alpha \circ x &&= \\
&\bar{g} \circ x - (u \circ \alpha) \circ x &&= \\
&(\bar{g} - u \circ \alpha) \circ x &&
\end{aligned}
$$

and

$$
\begin{aligned}
&v \circ g &&= \\
&v \circ (f + \alpha \circ x) &&= \\
&v \circ f + v \circ \alpha \circ x &&= \\
&\bar{f} \circ x + (v \circ \alpha) \circ x &&= \\
&(\bar{f} + v \circ \alpha) \circ x &&
\end{aligned}
$$

These morphisms satisfy

$$
\begin{aligned}
&\phi \circ v = u \\
&\phi \circ \bar{f} = \bar{g} - u \circ \alpha \\
&\psi \circ u = v \\
&\psi \circ \bar{g} = \bar{f} + v \circ \alpha
\end{aligned}
$$

We confirm that ϕ and ψ are actually inverses of each other. We use Proposition 1.2 for this.

$$\begin{aligned}
&\psi \circ \phi \circ v &&= \\
&\psi \circ u &&= \\
&1_{C(f)} \circ v
\end{aligned}$$

$$\begin{aligned}
&\psi \circ \phi \circ \bar{f} &&= \\
&\psi \circ (\bar{g} - u \circ \alpha) &&= \\
&\psi \circ \bar{g} - \psi \circ u \circ \alpha &&= \\
&(\bar{f} + v \circ \alpha) - v \circ \alpha &&= \\
&1_{C(f)} \circ \bar{f}
\end{aligned}$$

Hence $\psi \circ \phi = 1_{C(f)}$. Checking the other composition order is similar. Hence $C(f) \simeq C(g)$ in $\mathcal{F}$, which means they are also isomorphic in $S(\mathcal{F})$. $\square$

Proposition 3.17. *The cone of a morphism $f \in S(\mathcal{F})(X, Y)$ is independent of the choice of injective $I(X)$.*

Proof. We start with two different injective objects I, J, and get the following diagrams for the construction of the two cones P and Q.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i} & I & \xrightarrow{i'} & V & \longrightarrow & 0 \\
& & \Big\downarrow f & & \Big\downarrow s & & \Big\| & & \\
0 & \longrightarrow & Y & \xrightarrow{v} & P & \xrightarrow{\bar{v}} & V & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{j} & J & \xrightarrow{j'} & W & \longrightarrow & 0 \\
& & \Big\downarrow f & & \Big\downarrow t & & \Big\| & & \\
0 & \longrightarrow & Y & \xrightarrow{u} & Q & \xrightarrow{\bar{u}} & W & \longrightarrow & 0
\end{array}$$

Since I, J are injective, and i, j inflations, we get morphisms $h : J \to I$ and $g : I \to J$, satisfying $i = h \circ j$ and $j = g \circ i$. We obtain morphisms ϕ and ψ

between P and Q from the following pushout diagrams

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & I \arrow[d, "s"] \arrow[ddr, bend left, "t \circ g"] & \\
Y \arrow[r, "v"] \arrow[drr, bend right, "u"'] & P \arrow[dr, dotted, "\exists! \phi"] & \\
 & & Q
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "f"] & J \arrow[d, "t"] \arrow[ddr, bend left, "s \circ h"] & \\
Y \arrow[r, "u"] \arrow[drr, bend right, "v"'] & Q \arrow[dr, dotted, "\exists! \psi"] & \\
 & & P
\end{tikzcd}$$

satisfying $\phi \circ v = u, \phi \circ s = t \circ g, \psi \circ u = v$ and $\psi \circ t = s \circ h$. Now we notice that we have $(h \circ g - 1_I) \circ i = h \circ g \circ i - i = i - i = 0$. The cokernel property of V gives a unique morphism $\alpha : V \to I$

$$\begin{tikzcd}
X \arrow[r, "i"] & I \arrow[r, "i'"] \arrow[dr, "(h \circ g - 1_I)"'] & V \arrow[d, dashed, "\alpha"] \\
 & & I
\end{tikzcd}$$

satisfying $\alpha \circ i' = h \circ g - 1_I$. Postcomposition with s gives us

$$\begin{aligned}
&s \circ (h \circ g - 1_I) = s \circ \alpha \circ i' && \Longrightarrow \\
&s \circ h \circ g - s = s \circ \alpha \circ \bar{v} \circ s && \Longrightarrow \\
&\psi \circ t \circ g = s \circ \alpha \circ \bar{v} \circ s + s && \Longrightarrow \\
&(\psi \circ \phi) \circ s = (s \circ \alpha \circ \bar{v}) \circ s + s && \Longrightarrow \\
&(\psi \circ \phi) \circ s = (s \circ \alpha \circ \bar{v} + 1_P) \circ s &&
\end{aligned}$$

Thus we have one of the requirements to use Proposition 1.2. We only need to check that

$$
\begin{aligned}
(s \circ \alpha \circ \bar{v} + 1_P) \circ v & = \\
s \circ \alpha \circ \bar{v} \circ v + 1_P \circ v & = \\
v & = \\
\psi \circ u & = \\
(\psi \circ \phi) \circ v &
\end{aligned}
$$

Hence we conclude that $\psi \circ \phi = s \circ \alpha \circ \bar{v} + 1_P$, and since $s \circ \alpha \circ \bar{v}$ factors through I, we get that $\psi \circ \phi = 1_P$ in $S(\mathcal{F})$. The proof for the other composition order is completely analogous, by just swapping the roles of I and J. Thus $P \simeq Q$ in $S(\mathcal{F})$. $\square$

We define the triangulation of $S(\mathcal{F})$ to be given by the autoequivalence T, and the collection Δ containing all triangles which are isomorphic to standard triangles in $S(\mathcal{F})$.

Theorem 3.18. *Let $\mathcal{F}$ be a Frobenius category, and let $T : S(\mathcal{F}) \to S(\mathcal{F})$ and Δ be defined as above. Then $(S(\mathcal{F}), T, \Delta)$ becomes a triangulated category.*

Proof. It is enough to only consider the cases for standard triangles. For full details, see [6, Theorem 2.6].

(TR1) By construction Δ is closed under isomorphisms and all morphisms fit into a distinguished triangle. Let $X \in \mathcal{F}$, and consider the following pushout diagram.

$$
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "1_X"] & I(X) \arrow[d, "u"] \arrow[ddr, bend left, "1_{I(X)}"] & \\
X \arrow[r, "v"] \arrow[drr, bend right, "x"'] & C(1_X) \arrow[dr, dotted, "\exists! \phi"] & \\
 & & I(X)
\end{tikzcd}
$$

We immediately get that $\phi \circ u = 1_{I(X)}$. On the other hand, we have

$$u \circ \phi \circ u = u \circ 1_{I(X)} = 1_{C(1_X)} \circ u$$

and

$$u \circ \phi \circ v = u \circ x = v \circ 1_X = 1_{C(1_X)} \circ v$$

by Proposition 1.2 we get that $u \circ \phi = 1_{C(1_X)}$, and hence $C(1_X) \simeq I(X)$. Thus $C(1_X) = 0$ in $S(\mathcal{F})$, and the triangle

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow T(X)$$

is in Δ.

(TR2) Let $X \xrightarrow{u} Y \xrightarrow{v} C(f) \xrightarrow{w} T(X)$ be a standard triangle. Consider the following commutative diagram.

$$\begin{tikzcd}
Y \arrow[r, "v"] & C(u) \arrow[r, "w"] & T(X) \\
X \arrow[u, "u"] \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[u, "\bar{u}"] \arrow[r, "\bar{x}"] \arrow[d, "I(u)"] & T(X) \arrow[u, equal] \arrow[d, "T(u)"] \\
Y \arrow[r, "y"] & I(Y) \arrow[r, "\bar{y}"] & T(Y)
\end{tikzcd}$$

Since $y \circ u = I(u) \circ x$, the pushout property of $C(u)$ gives a unique morphism $f : C(u) \to I(Y)$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[d, "\bar{u}"] \arrow[ddr, bend left, "I(u)"] & \\
Y \arrow[r, "v"] \arrow[drr, bend right, "y"] & C(u) \arrow[dr, dotted, "\exists! f"] & \\
& & I(Y)
\end{tikzcd}$$

We have the two compositions $T(u) \circ w, \bar{y} \circ f : C(u) \to T(Y)$. By noting that

$$(\bar{y} \circ f) \circ v = \bar{y} \circ y = 0 = (T(u) \circ w) \circ v$$
$$(\bar{y} \circ f) \circ \bar{u} = \bar{y} \circ I(u) = T(u) \circ \bar{x} = (T(u) \circ w) \circ \bar{u}$$

we get by Proposition 1.2 that $\bar{y} \circ f = T(u) \circ w$. Thus the following composition is zero.

$$C(u) \xrightarrow{\binom{f}{w}} I(Y) \oplus T(X) \xrightarrow{(\bar{y}\ -T(u))} T(Y)$$

We construct the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r, "v"] \arrow[d, "y"] & C(u) \arrow[r, "w"] \arrow[d, "\left(\begin{smallmatrix} f \\ w \end{smallmatrix}\right)"] & T(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I(Y) \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\bar{y}"] & I(Y) \oplus T(X) \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \bar{y} & -T(u) \end{smallmatrix}\right)"] & T(X) \arrow[r] & 0 \\
 & T(Y) \arrow[r, equal] \arrow[d] & T(Y) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since the upper two rows are exact and $I(Y)$ is injective, by Proposition 3.15, we get that the middle row is the induced exact sequence by the upper row and y. In particular we get that $I(Y) \oplus T(X)$ is the pushout of v and y.

Now we consider the leftmost and middle columns. Since $I(Y) \oplus T(X)$ is the pushout of y and v, and $\left(\begin{smallmatrix} \bar{y} & -T(u) \end{smallmatrix}\right)\left(\begin{smallmatrix} f \\ w \end{smallmatrix}\right) = 0$, we get by Proposition 3.12 that the middle column is the induced exact sequence of the left column and v. Thus we get that the sequence

$$Y \xrightarrow{v} C(u) \xrightarrow{\left(\begin{smallmatrix} f \\ w \end{smallmatrix}\right)} I(Y) \oplus T(X) \xrightarrow{\left(\begin{smallmatrix} \bar{y} & -T(u) \end{smallmatrix}\right)} T(Y)$$

is a standard triangle, which is isomorphic to

$$Y \xrightarrow{v} C(u) \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

in $S(\mathcal{F})$.

(TR3) Let $X \xrightarrow{u} Y \xrightarrow{v} C(u) \xrightarrow{w} T(X)$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} C(u') \xrightarrow{w'} T(X')$ be two standard triangles. Let $f : X \to X'$, $g : Y \to Y'$ be morphisms such that $u' \circ f = g \circ u$ in $S(\mathcal{F})$. We also have the morphisms $I(f) : I(X) \to I(X')$ and $T(f) : T(X) \to T(X')$. We get the following diagram with exact rows and where all squares commute in $\mathcal{F}$.

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[ddd, bend right=60, "g"'] & C(u) \arrow[r, "w"] & T(X) \\
X \arrow[u, "u"] \arrow[r, "x"] \arrow[d, "f"] & I(X) \arrow[u, "\bar{u}"] \arrow[r, "\bar{x}"] \arrow[d, "I(f)"] & T(X) \arrow[u, equal] \arrow[d, "T(f)"] \\
X' \arrow[r, "x'"] \arrow[d, "u'"] & I(X') \arrow[r, "\bar{x}'"] \arrow[d, "\bar{u}'"] & T(X') \arrow[d, equal] \\
Y' \arrow[r, "v'"] & C(u') \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

Since the morphisms g, f commute in $S(\mathcal{F})$, we know because of Proposition 3.2 that there exists some $\alpha : I(X) \to Y'$ such that $g \circ u = u' \circ f + \alpha \circ x$. Now we see that

$$\begin{aligned}
&(v' \circ g) \circ u && = \\
&v' \circ (u' \circ f + \alpha \circ x) && = \\
&v' \circ u' \circ f + v' \circ \alpha \circ x && = \\
&\bar{u}' \circ I(f) \circ x + v' \circ \alpha \circ x && = \\
&(\bar{u}' \circ I(f) + v' \circ \alpha) \circ x
\end{aligned}$$

Using the pushout property of $C(u)$ we get a unique morphism $h : C(u) \to C(u')$ satisfying $h \circ v = v' \circ g$, and $h \circ \bar{u} = \bar{u}' \circ I(f) + v' \circ \alpha$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & C(u) \arrow[r, "w"] \arrow[d, dotted, "h"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & C(u') \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

From this definition of h, we immediately get that the second square commutes. Now we just need to check that the rightmost square commutes, that is $w' \circ h = T(f) \circ w$. To see this, we first note that $(w' \circ h) \circ v = w' \circ v' \circ g = 0 = (T(f) \circ w) \circ v$. We also have that

$$
\begin{aligned}
&(T(f) \circ w) \circ \bar{u} && = \\
&w' \circ \bar{u}' \circ I(f) + 0 && = \\
&w' \circ \bar{u}' \circ I(f) + w' \circ v' \circ \alpha && = \\
&w' \circ (\bar{u}' \circ I(f) + v' \circ \alpha) && = \\
&(w' \circ h) \circ \bar{u}
\end{aligned}
$$

By Proposition 1.2, we get that $w' \circ h = T(f) \circ w$.

(TR4) We start out with the morphisms

$$
\begin{aligned}
&u : X \to Y \\
&v : Y \to Z \\
&w := v \circ u : X \to V
\end{aligned}
$$

and want to construct three standard triangles. We do this in a particular way so that everything works out nicely. For the morphism u we construct the following standard triangle.

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[r, "\bar{x}"] \arrow[d, "\bar{u}"] & T(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & Z' \arrow[r, "i'"] & T(X) \arrow[r] & 0
\end{tikzcd}
$$

Now we choose some exact sequence for the object Z'

$$
0 \longrightarrow Z' \xrightarrow{z} I(Z') \xrightarrow{\bar{z}} T(Z') \longrightarrow 0
$$

with $I(Z')$ injective. Now notice that we have the inflations i and z. The composition is an inflation into an injective $z \circ i : Y \to I(Z')$. By choosing this inflation for constructing the standard triangle of the morphism v, we get

$$
\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "z \circ i"] \arrow[d, "v"] & I(Z') \arrow[r, "\bar{y}"] \arrow[d, "\bar{v}"] & T(Y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & X' \arrow[r, "j'"] & T(Y) \arrow[r] & 0
\end{tikzcd}
$$

For the morphism $w = v \circ u$ we construct the following standard triangle.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "w"] & I(X) \arrow[r, "\bar{x}"] \arrow[d, "\bar{w}"] & T(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "k"] & Y' \arrow[r, "k'"] & T(X) \arrow[r] & 0
\end{tikzcd}$$

We want to find the morphisms to complete the following diagram.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "i'"] & T(X) \arrow[d, equal] \\
X \arrow[r, "w"] \arrow[d, "u"] & Z \arrow[r, "k"] \arrow[d, equal] & Y' \arrow[r, "k'"] & T(X) \arrow[d, "T(u)"] \\
Y \arrow[r, "v"] \arrow[d, "i"] & Z \arrow[r, "j"] \arrow[d, "k"] & X' \arrow[r, "j'"] \arrow[d, equal] & T(Y) \arrow[d, "T(i)"] \\
Z' & Y' & X' & T(Z')
\end{tikzcd}$$

Since Z' is a pushout and $\bar{w} \circ x = k \circ w = k \circ v \circ u$, we get a unique morphism $f : Z' \to Y'$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[d, "\bar{u}"] \arrow[ddr, bend left, "\bar{w}"] & \\
Y \arrow[r, "i"] \arrow[drr, bend right, "k \circ v"'] & Z' \arrow[dr, dotted, "\exists! f"] & \\
& & Y'
\end{tikzcd}$$

such that $f \circ i = k \circ v$ and $f \circ \bar{u} = \bar{w}$. Similarly, since Y' is a pushout and $j \circ w = j \circ v \circ u = \bar{v} \circ (z \circ i) \circ u = \bar{v} \circ z \circ \bar{u} \circ x$, we get a unique morphism

$g : Y' \to X'$,

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "w"] & I(X) \arrow[d, "\bar{w}"] \arrow[ddr, bend left, "\bar{v} \circ z \circ \bar{u}"] & \\
Z \arrow[r, "k"] \arrow[drr, bend right, "j"'] & Y' \arrow[dr, dotted, "\exists ! g"] & \\
& & X'
\end{tikzcd}$$

such that $g \circ k = j$ and $g \circ \bar{w} = \bar{v} \circ z \circ \bar{u}$. Recall how the morphisms $T(u)$ and $T(i)$ are constructed. In the following diagram, we choose morphisms $I(u), I(i)$ which make the leftmost squares commute. Then there exist unique morphisms $T(u)$ and $T(i)$ such that the whole diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[r, "\bar{x}"] \arrow[d, "I(u)"] & T(X) \arrow[r] \arrow[d, "T(u)"] & 0 \\
0 \arrow[r] & Y \arrow[r, "z \circ i"] \arrow[d, "i"] & I(Z') \arrow[r, "\bar{y}"] \arrow[d, "I(i)"] & T(X) \arrow[r] \arrow[d, "T(i)"] & 0 \\
0 \arrow[r] & Z' \arrow[r, "z"] & I(Z') \arrow[r, "\bar{z}"] & T(Z') \arrow[r] & 0
\end{tikzcd}$$

We see that we can choose $I(i) := 1_{I(Z')}$. We also have that $z \circ i \circ u = z \circ \bar{u} \circ x$, hence we can choose $I(u) := z \circ \bar{u}$. Thus we obtain the relations $T(u) \circ \bar{x} = \bar{y} \circ z \circ \bar{u}$ and $T(i) \circ \bar{y} = \bar{z}$. We put everything together in the following diagram.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & T(X) \\
\| & & \downarrow{v} & & \downarrow{f} & & \| \\
X & \xrightarrow{w} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & T(X) \\
\downarrow{u} & & \| & & \downarrow{g} & & \downarrow{T(u)} \\
Y & \xrightarrow{v} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & T(Y) \\
\downarrow{i} & & \downarrow{k} & & \| & & \downarrow{T(i)} \\
Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{T(i)\circ j'} & T(Z')
\end{array}$$

What remains is to see that this entire diagram commutes, and that the bottom row is a triangle. By construction, we already have that $f \circ i = k \circ v$ and $g \circ k = j$. To see that $i' = k' \circ f$, we use Proposition 1.2, and note that

$$\begin{aligned}
&i' \circ i = 0 \\
&k' \circ f \circ i = k' \circ k \circ v = 0 \\
&i' \circ \bar{u} = \bar{x} = k' \circ \bar{w} = k' \circ f \circ \bar{u}
\end{aligned}$$

Again using the same proposition, we show that $T(u) \circ k' = j' \circ g$. We have that

$$\begin{aligned}
&T(u) \circ k' \circ k = 0 \\
&j' \circ g \circ k = j' \circ j = 0 \\
&T(u) \circ k' \circ \bar{w} = T(u) \circ \bar{x} = \bar{y} \circ z \circ \bar{u} = j' \circ \bar{v} \circ z \circ \bar{u} = j' \circ g \circ \bar{w}
\end{aligned}$$

Finally, to show that the bottom row is a triangle in Δ, consider the following diagram.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"] & I(X) \arrow[d, "\bar{u}"] & \\
Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "z"] \arrow[d, "f"] & I(Z') \arrow[d, "\bar{v}"] \\
Z \arrow[r, "k"] & Y' \arrow[r, "g"] & X'
\end{tikzcd}$$

We know the two leftmost squares commute. Since Z' is a pushout, and

$$\begin{aligned}
&\bar{v} \circ z \circ i = j \circ v = g \circ k \circ v = g \circ f \circ i \\
&\bar{v} \circ z \circ \bar{u} = g \circ \bar{w} = g \circ f \circ \bar{u}
\end{aligned}$$

we conclude by Proposition 1.2 that the bottom right square commutes as well. Now we want to show that the bottom right square is a pushout. Since Z' is the pushout of x and u and Y' is the pushout of x and $v \circ u$, Proposition 1.3 gives us that Y' is also the pushout of i and v. Using the same proposition again, together with the fact that X' is the pushout of v and $z \circ i$, we get that X' is also the pushout of z and f. Finally we check that

$$\begin{aligned}
&T(i) \circ j' \circ g \circ k = T(i) \circ j' \circ j = 0 \\
&T(i) \circ j' \circ g \circ \bar{w} = T(i) \circ j' \circ \bar{v} \circ z \circ \bar{u} = T(i) \circ \bar{y} \circ z \circ \bar{u} = \bar{z} \circ z \circ \bar{u} = 0
\end{aligned}$$

Hence we get by Proposition 1.2 that $T(i) \circ j' \circ g = 0$. We constuct the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r, "z"] \arrow[d, "f"] & I(Z') \arrow[r, "\bar{z}"] \arrow[d, "\bar{v}"] & T(Z') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r, "g"] & X' \arrow[r, "T(i) \circ j'"] & T(Z') \arrow[r] & 0
\end{tikzcd}$$

Since $T(i) \circ j' \circ \bar{v} = T(i) \circ \bar{y} = \bar{z}$, the rightmost square commutes as well. Putting everything together we get that the bottom row is the induced exact sequence of the top row and f, and hence

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(i) \circ j} T(Z')$$

is a standard triangle in $S(\mathcal{F})$. $\square$

Example. Recall the rings $R_1 := \mathbb{Z}/p^2\mathbb{Z}$ and $R_2 := \mathbb{F}_p[\epsilon] \simeq \mathbb{F}_p[x]/(x^2)$ that we studied earlier. We have seen that the module categories $\mathrm{Mod}(R_i)$ are Frobenius categories, so then we may ask, what is $S(\mathrm{Mod}(R_i))$? Let M be a finitely generated R_i-module with decomposition $M \simeq \mathbb{F}_p^s \oplus R_i^t$ into indecomposables. When passing to $S(\mathrm{Mod}(R_i))$, all the injective summands vanish, and we are left with $M \simeq \mathbb{F}_p^s$ in $S(\mathrm{Mod}(R_i))$. By the previous discussion about classifying modules over these rings [p.54], this fact generalizes to all R_i-modules, which implies that $S(\mathrm{Mod}(R_i)) \simeq \mathrm{Mod}(\mathbb{F}_p)$.

Since $\mathbb{F}_p$ is a field, it is well known that the resulting triangulation on $\mathrm{Mod}(\mathbb{F}_p)$ is given as follows. The shift is the identity, and distinguished triangles are the triangles which induce long exact sequences.

4 Enhancements

A triangulated category can be said to have an *enhancement*, which contains additional information about it. This allows for a better understanding of the category in question [10, 1.1]. In this chapter we will first define what a *dg-enhancement* is, and then state a result regarding equivalent characterizations of the triangulated categories which have such enhancements. Afterwards we will consider the question regarding whether such enhancements are unique.

4.1 Algebraic triangulated categories

We have seen that for a pretriangulated dg-category $\mathcal{A}$, we can construct the triangulated category $H^0(\mathcal{A})$. Does every triangulated category arise from this construction? In general the answer is no, but when a triangulated category arises from this construction, we call it *algebraic*. Specifically we have the following definition.

Definition. Let $\mathcal{T}$ be a triangulated category. We say that $\mathcal{T}$ is *algebraic*, if there exists a pretriangulated dg-category $\mathcal{A}$, together with an equivalence of triangulated categories $H^0(\mathcal{A}) \simeq \mathcal{T}$. We then say that $\mathcal{A}$ is a *dg-enhancement* of the category $\mathcal{T}$.

In Chapter 2 we defined the category $\mathcal{C}_{dg}(k)$, and looked at how $H^0(\mathcal{C}_{dg}(k)) \simeq K(k)$ is the *homotopy category of chain complexes over k*. We can generalize these constructions; instead of considering complexes of modules over k, we consider complexes of objects in some additive category $\mathbf{A}$.

Definition. Let $\mathbf{A}$ be an additive category. The *dg-category of chain complexes over* $\boldsymbol{A}$, denoted $\mathcal{C}_{dg}(\mathbf{A})$, has objects which are chain complexes of objects in $\mathbf{A}$. For two chain complexes A, B, the set of morphisms forms a chain complex, with the n-th component consisting of graded morphisms of degree n from A to B.

We define the *homotopy category of chain complexes over* $\boldsymbol{A}$, denoted $K(\mathbf{A})$. Obects in this category are chain complexes with components in $\mathbf{A}$, and morphisms are *chain maps up to chain homotopy.* Similar to what we saw in Chapter 2, we have that $K(\mathbf{A}) = H^0(\mathcal{C}_{dg}(\mathbf{A}))$.

It is well known that the category $K(\mathbf{A})$ is triangulated, see [7, Theorem 6.7] for more details.

The following definition concerns how we define the concept of long exact sequences in exact categories.

Definition. A sequence in an exact category

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

is called an *acyclic* (or *long exact*), if each morphism d^n can be decomposed as

$$A^n \xrightarrow{\pi^n} Z^{n+1}(A) \xrightarrow{i^{n+1}} A^{n+1}$$

where $Z^{n+1}(A) := \mathrm{Ker}(d^{n+1})$, such that each sequence

$$0 \longrightarrow Z^n(A) \xrightarrow{i^n} A^n \xrightarrow{\pi^n} Z^{n+1}(A) \longrightarrow 0$$

is exact.

Definition. Let $\mathcal{F}$ be a Frobenius category. We denote by $\mathcal{C}_{dg}^{ai}(\mathcal{F})$ the full subcategory of $\mathcal{C}_{dg}(\mathcal{F})$, with objects the acyclic complexes with all components injective.

Similarly we denote $K^{ai}(\mathcal{F}) := H^0(\mathcal{C}_{dg}^{ai}(\mathcal{F}))$, which can equivalently be described as the full subcategory of $K(\mathcal{F})$, with objects the acyclic complexes with all components injective.

Remark 4.1. The category $K^{ai}(\mathcal{F})$ is clearly closed under shift of chain complexes. It can also be shown that the cone of a morphism between acyclic complexes is again an acyclic complex, see for example [2, Lemma 10.3]. The cone clearly also has all components injective. Hence $K^{ai}(\mathcal{F})$ is a triangulated subcategory of $K(\mathcal{F})$, with $\mathcal{C}_{dg}^{ai}(\mathcal{F})$ as a dg-enhancement.

Proposition 4.2. *For a Frobenius category $\mathcal{F}$, there is a triangulated equivalence $G : K^{ai}(\mathcal{F}) \to S(\mathcal{F})$, given by $X \mapsto Z^0(X)$ on objects.*

Proof. First we define what G does to morphisms. For a morphism $f \in K^{ai}(\mathcal{F})(X, Y)$, we see that the image in $S(\mathcal{F})$ is independent of representative, since for any chain homotopy s, each morphism $s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$ factors through $X^{n+1} \oplus Y^{n-1}$, which is injective. Thus we simply choose some chain map to represent f, and make the following diagram.

$$\begin{tikzcd}
X^{-1} \arrow[r, "\pi_X^{-1}"] \arrow[d, "f^{-1}"] & Z^0(X) \arrow[r, "i_X^0"] \arrow[d, dotted, "\exists! \bar{f}"] & X^0 \arrow[d, "f^0"] & \\
Y^{-1} \arrow[r, "\pi_Y^{-1}"] & Z^0(Y) \arrow[r, "i_Y^0"] & Y^0 \arrow[r, "d_Y^0"] & Y^1
\end{tikzcd}$$

Since we have

$$\begin{aligned}
d_Y^0 \circ f^0 \circ i_X^0 \circ \pi_X^{-1} &= \\
d_Y^0 \circ i_Y^0 \circ \pi_Y^{-1} \circ f^{-1} &= \\
0
\end{aligned}$$

and since π_X^{-1} is an epimorphism, by the kernel property of $Z^0(Y)$ we get the morphism $\bar{f}$ making the right square in the diagram commute. Moreover, since f is a chain map, we have that

$$\begin{aligned}
f^0 \circ i_X^0 \circ \pi_X^{-1} = i_Y^0 \circ \pi_Y^{-1} \circ f^{-1} & \quad \Longrightarrow \\
i_Y^0 \circ \bar{f} \circ \pi_X^{-1} = i_Y^0 \circ \pi_Y^{-1} \circ f^{-1} & \quad \Longrightarrow \\
\bar{f} \circ \pi_X^{-1} = \pi_Y^{-1} \circ f^{-1} & \quad \text{Since } i_Y^0 \text{ monomorphism}
\end{aligned}$$

hence the leftmost square commutes as well. We define $G(f)$ to be the morphism represented by $\bar{f}$ in $S(\mathcal{F})$, and call this the *induced morphism on kernels*.

To show that G is faithful, let $f \in K^{ai}(\mathcal{F})(A, B)$ such that $G(f) := \bar{f} = 0$ in $S(\mathcal{F})$. This means that $\bar{f}$ factors through some injective. We make the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "\pi_A^{-1}"] \arrow[d, "f^{-1}"] & Z^0(A) \arrow[r, "i_A^0"] \arrow[d, "\bar{f}"] & A^0 \arrow[r, "\pi_A^0"] \arrow[d, "f^0"] & Z^1(A) \arrow[r, "i_A^1"] \arrow[d, "\bar{f}^1"] & A^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "\pi_B^{-1}"] & Z^0(B) \arrow[r, "i_B^0"] & B^0 \arrow[r, "\pi_B^0"] & Z^1(B) \arrow[r, "i_B^1"] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

By Proposition 3.14, and since i_A^0 is an inflation, we get that $\bar{f}$ factors through A^0 as $\bar{f} = \alpha \circ i_A^0$ for some $\alpha : A^0 \to Z^0(B)$. Note that the morphism α constructed in that proof also factors through an injective. By using the dual statement of Proposition 3.14, and since π_B^{-1} is a deflation, we get that α factors through B^{-1} as $\alpha = \pi_B^{-1} \circ s^0$, which put together gives the factorization $\bar{f} = \pi_B^{-1} \circ s^0 \circ i_A^0$. Now notice that we have

$$f^0 \circ i_A^0 = i_B^0 \circ \bar{f} = (i_B^0 \circ \pi_B^{-1} \circ s) \circ i_A^0$$

which means that

$$(f^0 - i_B^0 \circ \pi_B^{-1} \circ s^0) \circ i_A^0 = 0$$

Since $Z^1(A)$ is the cokernel of i_A^0, this gives a unique morphism $t : Z^1(A) \to B^0$ satisfying $t \circ \pi_A^0 = f^0 - i_B^0 \circ \pi_B^{-1} \circ s^0$. Rearranging this gives us that $f^0 = t \circ \pi_A^0 + d_B^{-1} \circ s^0$. Since i_A^1 is an inflation and B^0 is injective, we can factor $t = s^1 \circ i_A^1$, thus we can write $t \circ \pi_A^0 = s^1 \circ i_A^1 \circ \pi_A^0 = s^1 \circ d_A^0$, and hence $f^0 = s^1 \circ d_A^0 + d_B^{-1} \circ s^0$. This gives us the start of a chain homotopy. To extend it to the right, notice that

$$
\begin{aligned}
& f^0 = s^1 \circ d_A^0 + d_B^{-1} \circ s^0 && \Longrightarrow \\
& \pi_B^0 \circ f^0 = \pi_B^0 \circ s^1 \circ d_A^0 + \pi_B^0 \circ d_B^{-1} \circ s^0 && \Longrightarrow \\
& \bar{f}^1 \circ \pi_A^0 = \pi_B^0 \circ s^1 \circ i_A^1 \circ \pi_A^0 + 0 && \Longrightarrow \\
& \bar{f}^1 = \pi_B^0 \circ s^1 \circ i_A^1 && \text{Since } \pi_A^0 \text{ epimorphism}
\end{aligned}
$$

Hence s^1 satisfies the same starting conditions as s^0, only one "step" to the right, and we can repeat the argument to complete extending the chain homotopy to the right. A dual argument can be used to extend the chain homotopy to the left. Specifically it starts out by using $Z^0(B)$ as a kernel of π_B^0, and A^0 being projective, and then allows us to continue extending the chain homotopy to the left. Hence f is chain homotopic to 0, and G is faithful.

To show that G is full, start with some $\bar{f} \in S(\mathcal{F})(A, B)$. Since i_A^0 is an inflation and B^0 is injective, we get a morphism $f^0 : A^0 \to B^0$ such that $f^0 \circ i_A^0 = i_B^0 \circ \bar{f}$. Now since $\pi_B^0 \circ f^0 \circ i_A^0 = \pi_B^0 \circ i_B^0 \circ \bar{f} = 0$, we get by the cokernel property of $Z^1(A)$, a unique morphism $\bar{f}^1 : Z^1(A) \to Z^1(B)$ satisfying $\bar{f} \circ \pi_A^0 = \pi_B^0 \circ f^0$.

$$
\begin{tikzcd}
Z^0(A) \arrow[r, "i_A^0"] \arrow[d, "\bar{f}"] & A^0 \arrow[r, "\pi_A^0"] \arrow[d, dotted, "\exists f^0"] & Z^1(A) \arrow[d, dotted, "\exists! \bar{f}^1"] \\
Z^0(B) \arrow[r, "i_B^0"] & B^0 \arrow[r, "\pi_B^0"] & Z^1(B)
\end{tikzcd}
$$

Now we can repeat the argument over and over, extending f to the right making all squares commute. By a dual argument we can extend f to the left. Since all squares commute, this means $f = \{f^n \mid n \in \mathbb{Z}\}$ is a chain map, which also satisfies $Z^0(f) = \bar{f}$. Hence G is full.

To show that G is dense, start with any $X \in S(\mathcal{F})$. Since $\mathcal{F}$ has enough injectives, we can construct a sequence

$$
X \xrightarrow{i^0} I^0 \xrightarrow{\pi^0} Z^1 \xrightarrow{i^1} I^1 \xrightarrow{\pi^1} Z^2 \longrightarrow \cdots
$$

where all I^n are injective, all i^n are inflations and where each $\xrightarrow{\pi^n} Z^{n+1}$ is the cokernel of i^n. Dually, since $\mathcal{F}$ has enough projectives, we can construct a sequence

$$\cdots \longrightarrow Z^{-2} \xrightarrow{i^{-2}} P^{-2} \xrightarrow{\pi^{-2}} Z^{-1} \xrightarrow{i^{-1}} P^{-1} \xrightarrow{\pi^{-1}} X$$

such that all all P^n are projective, all π^n are deflations, and each $Z^n \xrightarrow{i^n}$ is the kernel of π^n. Putting the sequences together we make the following complex which we denote by C.

$$\cdots \longrightarrow P^{-2} \xrightarrow{i^{-1}\circ\pi^{-2}} P^{-1} \xrightarrow{i^{0}\circ\pi^{-1}} I^0 \xrightarrow{i^{1}\circ\pi^{0}} I^1 \longrightarrow \cdots$$

This is acyclic by construction, and $Z^0(C) = \mathrm{Ker}(i^1 \circ \pi^0) = \mathrm{Ker}(\pi^0) = X$. Hence G is dense.

Finally we show that triangles in $K^{ai}(\mathcal{F})$ are transformed to triangles in $S(\mathcal{F})$ under G. Let

$$A \xrightarrow{f} B \xrightarrow{v} B \oplus A[1] \xrightarrow{w} A[1]$$

be a standard triangle in $K^{ai}(\mathcal{F})$, where $v = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$, and we denote the differential on the chain complex $C := B \oplus A[1]$ by

$$d_C^n = \begin{pmatrix} d_B^n & f^{n+1} \\ 0 & -d_A^{n+1} \end{pmatrix}$$

We want to construct the corresponding standard triangle in $S(\mathcal{F})$, thus we make the following diagram

$$\begin{tikzcd}
Z^0(A) \arrow[r, "i_A^0"] \arrow[d, "\bar{f}"] & A^0 \arrow[r, "\pi_A^0"] \arrow[d, "u"] & Z^1(A) \arrow[d, equal] \\
Z^0(B) \arrow[r, "\bar{v}"] & Z^0(C) \arrow[r, "\bar{w}"] & Z^1(A)
\end{tikzcd}$$

The morphism denoted u is the composition

$$A^0 \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} B^{-1} \oplus A^0 \xrightarrow{\pi_C^{-1}} Z^0(C)$$

The morphism $\bar{v}$ is the induced morphism making the following diagram commute

$$\begin{tikzcd}
B^{-1} \arrow[r, "\pi_B^{-1}"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & Z^0(B) \arrow[r, "i_B^0"] \arrow[d, dotted, "\exists!\bar{v}"] & B^0 \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\
B^{-1} \oplus A^0 \arrow[r, "\pi_C^{-1}"] & Z^0(C) \arrow[r, "i_C^0"] & B^0 \oplus A^1
\end{tikzcd}$$

and similarly for $\bar{w}$

$$\begin{tikzcd}
B^{-1} \oplus A^0 \arrow[r, "\pi_C^{-1}"] \arrow[d, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] & Z^0(C) \arrow[r, "i_C^0"] \arrow[d, dotted, "\exists!\bar{w}"] & B^0 \oplus A^1 \arrow[d, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \\
A^0 \arrow[r, "\pi_A^0"] & Z^1(A) \arrow[r, "-i_A^1"] & A^1
\end{tikzcd}$$

First we check that the morphism u makes both squares commute. Starting with the right square, we have

$$\bar{w} \circ u = \bar{w} \circ \pi_C^{-1} \circ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \pi_A^0 \circ \left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right) \circ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \pi_A^0$$

For the left square, we first calculate that

$$\begin{aligned}
& i_C^0 \circ u \circ i_A^0 \circ \pi_A^{-1} && = \\
& i_C^0 \circ \pi_C^{-1} \circ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \circ d_A^{-1} && = \\
& d_C^{-1} \circ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \circ d_A^{-1} && = \\
& \begin{pmatrix} d_B^{-1} & f^0 \\ 0 & -d_A^0 \end{pmatrix} \circ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \circ d_A^{-1} && = \\
& \begin{pmatrix} f^0 \\ -d_A^0 \end{pmatrix} \circ d_A^{-1} && = \\
& \begin{pmatrix} f^0 \circ d_A^{-1} \\ -d_A^0 \circ d_A^{-1} \end{pmatrix} && = \\
& \begin{pmatrix} f^0 \circ d_A^{-1} \\ 0 \end{pmatrix}
\end{aligned}$$

While on the other hand

$$
\begin{aligned}
&i_C^0 \circ \bar{v} \circ \bar{f} \circ \pi_A^{-1} && = \\
&i_C^0 \circ \bar{v} \circ \pi_B^{-1} \circ f^{-1} && = \\
&i_C^0 \circ \pi_C^{-1} \circ v^{-1} \circ f^{-1} && = \\
&d_C^{-1} \circ v^{-1} \circ f^{-1} && = \\
&\begin{pmatrix} d_B^{-1} & f^0 \\ 0 & -d_A^0 \end{pmatrix} \circ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \circ f^{-1} && = \\
&\begin{pmatrix} d_B^{-1} \\ 0 \end{pmatrix} \circ f^{-1} && = \\
&\begin{pmatrix} d_B^{-1} \circ f^{-1} \\ 0 \end{pmatrix} && = \\
&\begin{pmatrix} f^0 \circ d_A^{-1} \\ 0 \end{pmatrix} && \text{Since } f \text{ is a chain map}
\end{aligned}
$$

This shows that

$$i_C^0 \circ \bar{v} \circ \bar{f} \circ \pi_A^{-1} = i_C^0 \circ u \circ i_A^0 \circ \pi_A^{-1}$$

and since i_C^0 is a monomorphism and π_A^{-1} is an epimorphism, we conclude that $\bar{v} \circ \bar{f} = u \circ i_A^0$.

To show that this diagram represents a standard triangle, we first need that $\bar{w} \circ \bar{v} = 0$. Since $w \circ v = 0$, we get that the same is true for the induced maps. Next we need to show that $Z^0(C)$ is the pushout of i_X^0 and $\bar{f}$. Take any $Z \in \mathcal{F}$ and two morphisms s, t such that $s \circ i_A^0 = t \circ \bar{f}$.

$$
\begin{tikzcd}
Z^0(A) \arrow[r, "i_A^0"] \arrow[d, "\bar{f}"] & A^0 \arrow[d, "u"] \arrow[rdd, bend left, "s"] & \\
Z^0(B) \arrow[r, "\bar{v}"] \arrow[rrd, bend right, "t"'] & Z^0(C) \arrow[rd, dotted, "\exists ? \phi"] & \\
& & Z
\end{tikzcd}
$$

We want to get a morphism out of $Z^0(C)$ by using it as a cokernel of the morphism i_C^{-1} in the following diagram.

$$\begin{tikzcd}
Z^{-1}(C) \arrow[r, "i_C^{-1}"] & B^{-1} \oplus A^0 \arrow[r, "\pi_C^{-1}"] \arrow[dr, "\left(t\circ\pi_B^{-1} \;\; s \right)"'] & Z^0(C) \arrow[d, dashed, "\exists!\phi"] \\
& & Z
\end{tikzcd}$$

The need to show that $\left(t\circ\pi_B^{-1} \;\; s \right)\circ i_C^{-1} = 0$. Precomposing with the epimorphism π_C^{-2} we get

$$\begin{aligned}
&\left(t \circ \pi_B^{-1} \quad s\right) \circ i_C^{-1} \circ \pi_C^{-2} && = \\
&\left(t \circ \pi_B^{-1} \quad s\right) \circ d_C^{-2} && = \\
&\left(t \circ \pi_B^{-1} \quad s\right) \circ \begin{pmatrix} d_B^{-2} & f^{-1} \\ 0 & -d_A^{-1} \end{pmatrix} && = \\
&\left(t \circ \pi_B^{-1} \circ d_B^{-2} \quad t \circ \pi_B^{-1} \circ f^{-1} - s \circ d_A^{-1}\right) && = \\
&\left(t \circ \pi_B^{-1} \circ i_B^{-1} \circ \pi_B^{-2} \quad t \circ \pi_B^{-1} \circ f^{-1} - s \circ d_A^{-1}\right) &&
\end{aligned}$$

We know $\pi_B^{-1} \circ i_B^{-1} = 0$, so we look at what remains in the second component.

$$\begin{aligned}
&t \circ \pi_B^{-1} \circ f^{-1} - s \circ d_A^{-1} && = \\
&t \circ \bar{f} \circ \pi_A^{-1} - s \circ d_A^{-1} && = \\
&s \circ i_A^0 \circ \pi_A^{-1} - s \circ d_A^{-1} && = \\
&s \circ d_A^{-1} - s \circ d_A^{-1} && = \\
&0 &&
\end{aligned}$$

Since π_C^{-2} is an epimorphism, this means the original composition is zero, and thus the cokernel property of $Z^0(C)$ gives the unique morphism ϕ satisfying

$$\phi \circ \pi_C^{-1} = \left(t \circ \pi_B^{-1} \quad s\right)$$

Now we only need to check that ϕ actually makes the original pushout diagram commute. We have that

$$\begin{aligned}
&\phi \circ u && = \\
&\phi \circ \pi_C^{-1} \circ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) && = \\
&\left(t \circ \pi_B^{-1} \quad s\right) \circ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) && = \\
&s &&
\end{aligned}$$

We also have

$$
\begin{aligned}
&\phi \circ \bar{v} \circ \pi_B^{-1} &&= \\
&\phi \circ \pi_C^{-1} \circ v^{-1} &&= \\
&\left(t \circ \pi_B^{-1} \quad s\right) \circ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &&= \\
&t \circ \pi_B^{-1}
\end{aligned}
$$

Which implies that $\phi \circ \bar{v} = t$, since π_B^{-1} is an epimorphism. The uniqueness of ϕ follows from the cokernel which gave us ϕ, and hence $Z^0(C)$ is indeed a pushout. $\square$

We have the following result regarding how the triangulated categories we have seen thus far are related, from [3, Proposition 3.1].

Theorem 4.3. *The following are equivalent for a triangulated category $\mathcal{T}$.*

(1) $\mathcal{T}$ is algebraic, that is, there is a triangulated equivalence $\mathcal{T} \to H^0(\mathcal{A})$ for some pretriangulated dg-category $\mathcal{A}$.

(2) There is a fully faithful triangulated functor $\mathcal{T} \to S(\mathcal{F})$ for some Frobenius category $\mathcal{F}$.

(3) There is a triangulated equivalence $\mathcal{T} \to S(\mathcal{F})$ for some Frobenius category $\mathcal{F}$.

(4) There is a fully faithful triangulated functor $\mathcal{T} \to K(\mathbf{A})$ for some additive category $\mathbf{A}$.

Proof. It can be shown that the category $\mathcal{C}(\mathcal{A})$ is Frobenius category and that $S(\mathcal{C}(\mathcal{A})) \simeq \mathcal{H}(\mathcal{A})$ [10, Lemma 3.3a)]. It can also be shown that the Yoneda functor $\Gamma_H : H^0(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ is triangulated, as the triangulation in $H^0(\mathcal{A})$ is essentially inherrited from the triangulation in $\mathcal{H}(\mathcal{A})$, as we saw in the proof in Chapter 2. As we have already seen, Γ_H is full and faithful, and hence (1) $\implies$ (2).

Now let $\phi : \mathcal{T} \to S(\mathcal{F})$ be a fully faithful triangulated functor. We denote by $\mathcal{F}'$ the full subcategory of $\mathcal{F}$ whose objects are those in the essential image of ϕ. It can be checked that $\mathcal{F}'$ is also a Frobenius category. Then the restricted functor $\phi : \mathcal{T} \to S(\mathcal{F}')$ is still triangulated, and hence a triangulated equivalence. Hence (2) $\implies$ (3).

By Proposition 4.2, we know that there is a triangulated equivalence $\phi : S(\mathcal{F}) \to K^{ai}(\mathcal{F})$. Considering the inclusion of the full triangulated subcategory $i : K^{ai}(\mathcal{F}) \to K(\mathcal{F})$, we get a fully faithful triangulated functor $i \circ \phi : S(\mathcal{F}) \to K(\mathcal{F})$. Hence (3) $\implies$ (4).

Finally, we know that $\mathcal{C}_{dg}(\mathbf{A})$ is an enhancement of $K(\mathbf{A})$, which means $K(\mathbf{A}) \simeq H^0(\mathcal{C}_{dg}(\mathbf{A}))$. Hence we have a fully faithful triangulated functor $\phi : \mathcal{T} \to H^0(\mathcal{C}_{dg}(\mathbf{A}))$. Denote the full subcategory of $\mathcal{C}_{dg}(\mathbf{A})$ with objects in the essential image of ϕ by $\mathcal{B}$. We need to show that $\mathcal{B}$ is a pretriangulated dg-category. Let $f \in H^0(\mathcal{B}(X,Y))$. Since X, Y are in the essential image of ϕ, and ϕ is full and faithful, we know there exists a triangle in $\mathcal{T}$

$$U \xrightarrow{g} V \longrightarrow C(g) \longrightarrow U[1]$$

such that the following diagram commutes in $H^0(\mathcal{B})$, and the vertical arrows are isomorphisms.

$$\begin{tikzcd}
\phi(U) \arrow[r, "\phi(g)"] \arrow[d] & \phi(V) \arrow[r] \arrow[d] & \phi(C(g)) \arrow[r] & \phi(U[1]) \\
X \arrow[r, "f"] & Y & &
\end{tikzcd}$$

Since ϕ is triangulated, we get that $\phi(U[1]) \simeq \phi(U)[1] \simeq X[1]$ in $H^0(\mathcal{C}_{dg}(\mathbf{A}))$, which means $X[1] \in \mathcal{B}$. Since ϕ is triangulated, the top row is a triangle in $H^0(\mathcal{C}_{dg}(\mathbf{A}))$. Thus we know that $\phi(C(g)) \simeq C(\phi(g)) \simeq C(f)$ in $H^0(\mathcal{C}_{dg}(\mathbf{A}))$. Hence $C(f) \in \mathcal{B}$. This means $\mathcal{B}$ is pretriangulated, and clearly $\phi : \mathcal{T} \to H^0(\mathcal{B})$ is then a triangulated equivalence. Hence (4) $\implies$ (1). $\square$

4.2 Topological triangulated categories

We take a moment to briefly discuss *model categories*. Getting familiar with these concepts will help us in Section 4.3, where we will use them to study an example of a triangulated category admitting two *different* dg-enhancements. We will not go into too much detail, but rather try to get a rough idea of some core concepts.

A *model category* is a category $\mathcal{C}$, equiped with a *model structure*. This structure is given by three distiguished classes of morphisms, called *weak equivalences*, *fibrations* and *cofibrations*, satisfying certain properties. For more details, see [8, 1.1].

These definitions allow us to define a notion of shifts and cones in $\mathcal{C}$, as well as the *homotopy category* $\mathrm{Ho}(\mathcal{C}) := \mathcal{C}[W^{-1}]$, the localization of $\mathcal{C}$ with respect to the class of weak equivalences. If the shift functor is an autoequivalence on $\mathrm{Ho}(\mathcal{C})$, then we call $\mathcal{C}$ a *stable model category* [14, 2.1.1]. For our purposes these are the model categories we are interrested in, as this is when the category $\mathrm{Ho}(\mathcal{C})$ is triangulated, see [8, Proposition 7.1.6]. We call the triangulated categories arising from such a construction *toplogical*. It can be shown that any algebraic triangulated category is also topological [13, p.872]. The converse however, is not true in general, see for example [13, p.873]. In fact there are triangulated categories which are not even topological, see [11].

Example. Let R be a ring such that $\mathrm{Mod}(R)$ is a Frobenius category. Then we can equip the category $\mathrm{Mod}(R)$ with a model structrue, which we will refer to as *standard* [8, Theorem 2.2.12]. It is given by cofibrations as the injections, fibrations as the surjections, and weak equivalences as the morphisms which become isomorphisms when passing to $S(\mathrm{Mod}(R))$. This construction results in the fact that $\mathrm{Ho}(\mathrm{Mod}(R)) \simeq S(\mathrm{Mod}(R))$.

The notion of equivalence between model categories, called *Quillen equivalence*, is given by certain kinds of adjoint functors which induce equivalences on the homotopy categories [14, 2.5]. We look at an example giving us such a Quillen equivalence.

Example. Recall the rings $R_1 := \mathbb{Z}/p^2\mathbb{Z}$ and $R_2 := \mathbb{F}_p[\epsilon] \simeq \mathbb{F}_p[x]/(x^2)$ which we studied previously. We have seen that $\mathrm{Mod}(R_i)$ is a Frobenius category, so we consider it as a model category with the standard model structure as described in the previous example. We want to apply the statement given in [4, Theorem 3.5]. It can be checked that $\mathbb{F}_p$ is a so called *compact, weak generator* of $\mathrm{Mod}(R_i)$. Let B_i be the chain complex

$$\cdots \xrightarrow{p_i} R_i \xrightarrow{p_i} R_i \xrightarrow{p_i} R_i \xrightarrow{p_i} \cdots$$

where p_i is given by multiplication with the element p and ϵ in B_1 and B_2 respectively. This is a *complete resolution* of $\mathbb{F}_p$, that is, an acyclic chain complex of injectives such that $Z^0(B_i) \simeq \mathbb{F}_p$. We denote by D_i the full dg-subcategory of $C^{ai}_{dg}(\mathrm{Mod}(R_i))$ with B_i as its only object. Let $\mathcal{C}(D_i)$ be the category of dg D_i-modules. This category has a model stucture where the weak equivalences are the quasi-isomorphisms, and the fibrations are the surjections [4, p.146]. Then [4, Theorem 3.5] states that there is a Quillen equivalence between the model categories $\mathcal{C}(D_i)$ and $\mathrm{Mod}(R_i)$.

4.3 Uniqeness of enhancements

Given an algebraic triangulated category $\mathcal{T}$, we have seen that it has a *dg-enhancement*. A natural question to ask is whether such dg-enhancements are unique. To answer this, we first need to define the appropriate notion of uniqueness.

Definition. Let $\mathcal{A}, \mathcal{B}$ be dg-categories, and $F : \mathcal{A} \to \mathcal{B}$ be a dg-functor. We say that F is a *quasi-equivalence*, if it satisfies both the following:

(1) The chain map $F_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{B}(FX, FY)$ is a quasi-isomorphism for all $X, Y \in \mathcal{A}$. That is, the induced morphisms on homology $H^n(F_{X,Y}) : H^n(\mathcal{A}(X,Y)) \to H^n(\mathcal{B}(FX, FY))$ are isomorphisms for all $n \in \mathbb{Z}$.

(2) The induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is dense.

Remark 4.4. Note that given a quasi-equivalence $F : \mathcal{A} \to \mathcal{B}$ between pre-triangulated dg-categories, we immediately get that $H^0(\mathcal{A}) \simeq H^0(\mathcal{B})$ as triangulated categories. In other words $\mathcal{A}$ and $\mathcal{B}$ are dg-enhancements of the same triangulated category. To see this, observe that taking $n = 0$, we get morphisms $H^0(F_{X,Y}) : H^0(\mathcal{A})(X,Y) \to H^0(\mathcal{B})(FX, FY)$ which are all isomorphisms. Since the induced functor $H^0(F)$ is also dense, we conclude that F is an equivalence of triangulated categories.

Because of the above remark, if we should have any chance of dg-enhancements being unique, they have to be *unique up to quasi-equivalence*. Thus we make the following definition.

Definition. Let $\mathbf{DGCat_k}$ denote the category of all small dg-categories over k. We define the category $\mathbf{Hqe}$ as the localization of $\mathbf{DGCat_k}$ with respect to the collection of quasi-equivalences.

With this definition in place, we can state what it means for a dg-enhancement of a triangulated category to be unique [3, Definition 1.15].

Definition. Let $\mathcal{T}$ be an algebraic triangulated category. We say that $\mathcal{T}$ has a *unique* dg-enhancement, if when given any two dg-enhancements $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{T}$, then $\mathcal{A}_1 \simeq \mathcal{A}_2$ in $\mathbf{Hqe}$.

Here we will combine several of the results we have looked at to understand an example given in [3, 3.3]. In this example we construct two *different* dg-enhancements $\mathcal{A}_1, \mathcal{A}_2$ of the triangulated category $\mathcal{T}_p := \mathrm{Mod}(\mathbb{F}_p)$. We will show that they are indeed different by contradiction; if $\mathcal{A}_1 \simeq \mathcal{A}_2$ in **Hqe**, then that would contradict the following theorem, given in [3, Theorem 3.9], see also [12, Proposition 1.7] and [4, Theorem 4.5] for details.

Theorem 4.5. *Let p be a prime number and let $R_1 := \mathbb{Z}/p^2\mathbb{Z}$ and $R_2 := \mathbb{F}_p[\epsilon] \simeq \mathbb{F}_p[x]/(x^2)$. Then the categories $\mathrm{Mod}(R_1)$ and $\mathrm{Mod}(R_2)$ with their standard model structures are not Quillen equivalent.*

Example. Let the rings R_1, R_2 be as above, and let $\mathcal{T}_p := \mathrm{Mod}(\mathbb{F}_p)$ be the triangulated category discussed in the example at the end of Chapter 3. We have seen that we have equivalences of triangulated categories $S(\mathrm{Mod}(R_i)) \simeq \mathcal{T}_p$. By Proposition 4.2, we also have triangulated equivalences $K^{ai}(\mathrm{Mod}(R_i)) \simeq S(\mathrm{Mod}(R_i))$. Since $C^{ai}_{dg}(\mathrm{Mod}(R_i))$ is a natural enhancement of $K^{ai}(\mathrm{Mod}(R_i))$, we thus have that $C^{ai}_{dg}(\mathrm{Mod}(R_1))$ and $C^{ai}_{dg}(\mathrm{Mod}(R_2))$ are both enhancements of $\mathcal{T}_p$.

Now assume that $C^{ai}_{dg}(\mathrm{Mod}(R_1)) \simeq C^{ai}_{dg}(\mathrm{Mod}(R_2))$ in **Hqe**. This means that we have a zig-zag of quasi-equivalences between $C^{ai}_{dg}(\mathrm{Mod}(R_1))$ and $C^{ai}_{dg}(\mathrm{Mod}(R_2))$, possibly involving other dg-categories. This induces an equivalence of triangulated categories $\phi : K^{ai}(\mathrm{Mod}(R_1)) \to K^{ai}(\mathrm{Mod}(R_2))$. Let $B_i \in C^{ai}_{dg}(\mathrm{Mod}(R_i))$ be the chain complex

$$\cdots \xrightarrow{p_i} R_i \xrightarrow{p_i} R_i \xrightarrow{p_i} R_i \xrightarrow{p_i} \cdots$$

where p_i is given by multiplication with the element p and ϵ in B_1 and B_2 respectively. The equivalence $K^{ai}(\mathrm{Mod}(R_i)) \to S(\mathrm{Mod}(R_i)) \simeq \mathcal{T}_p$, maps the object B_i to $Z^0(B_i) = \mathrm{Ker}(p_i) \simeq \mathbb{F}_p \in \mathcal{T}_p$ for both $i = 1, 2$. Hence B_1 must correspond to B_2 up to isomorphism under ϕ. Let D_i be the full dg-subcategory of $C^{ai}_{dg}(\mathrm{Mod}(R_i))$, which has B_i as the only object. Because B_1 corresponds to B_2, the zig-zag of quasi-equivalences between $C^{ai}_{dg}(\mathrm{Mod}(R_1))$ and $C^{ai}_{dg}(\mathrm{Mod}(R_2))$ induces a zig-zag of quasi-equivalences between D_1 and D_2. Now we use [15, Proposition 3.2]. For our purposes, it states that our zig-zag of quasi-equivalences between D_1 and D_2 induces a chain of Quillen equivalences between the categories of dg modules over the respective dg-categories [15, Corollary 3.4]. In particular we get that the categories $\mathcal{C}(D_1)$, $\mathcal{C}(D_2)$, with model structure as described in the last example in the previous section, are Quillen equivalent. As we also saw in this example, using [4, Theorem 3.5] we concluded that $\mathcal{C}(D_i)$ and $\mathrm{Mod}(R_i)$ are Quillen equivalent. Thus we have the following chain of Quillen equivalences.

$$\mathrm{Mod}(R_1) \longrightarrow \mathcal{C}(D_1) \longrightarrow \mathcal{C}(D_2) \longrightarrow \mathrm{Mod}(R_2)$$

This shows that $\mathrm{Mod}(R_1)$ and $\mathrm{Mod}(R_2)$ are Quillen equivalent. This contradicts Theorem 4.5, and hence the assumtion that $C^{ai}_{dg}(\mathrm{Mod}(R_1)) \simeq C^{ai}_{dg}(\mathrm{Mod}(R_2))$ in **Hqe** is wrong. Thus $C^{ai}_{dg}(\mathrm{Mod}(R_1))$ and $C^{ai}_{dg}(\mathrm{Mod}(R_2)$ are two *different* dg-enhancements of the triangulated category $\mathcal{T}_p$.

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