Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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# Jordan-Hölder property for representations of quivers over exact categories

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Supervisor: Sondre Kvamme
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# Contents

# Abstract

We will look at what an exact category is, and show some generall results for exact categories. Afterward we will establish what Jordan-Hölder property(JHP) for exact categories is, and look at some exact categories that does have JHP, and some that don't have JHP. We will also define homomorphism category and monomorphism category of an exact category, and we show that these has JHP, if the original exact category has JHP. We will also define homomorphism/monomorphism category over a quiver and an exact category and show some results about when these has JHP.

# Sammendrag

Vi vil se på hva en eksakt kategori er, og vise noen resultater som gjelder generelt for eksakte kategorier. Etterpå vil vi ettablere hva Jordan-Hölder egenskap(JHP) for eksakte kategorier er, og se på noen eksakte kategorier som har JHP, og noen som ikke har JHP. Vi vil også definere homomorfikategori og monomorfikategori av en eksakt kategori, og vi viser at disse kategoriene har JHP dersom den opprinnelige eksakte kategorien har JHP. Vi definere også hva homomorfikategori/monomorfikategori over både en kogger og eksakt kategori er, og vi viser noen resultater for når disse har JHP.

# 1 Introduction

Exact Categories were first introduced by Quillen in [5]. This was to be able to generalize some results that abelian categories had, without the need for every morphism to kernel and cokernel. An exact category is a additive category with an exact structure, containing short exact sequences, that satisfies certain axioms. We call an short exact sequence in the exact structure a conflation, and the monomorphisms and epimorphisms that is in a conflation, are called inflations and deflations. We can then show that many results that is true for abelian categories, also is true for exact categories in a way. Afterward we show what the Jordan-Hölder property(JHP) is for exact categories. To do this we establish what subobjects, inflation series and composition series is. We then show some connection between the structure of the Grothendick group of an exact category and when the exact category has JHP, and show some types of exact categories that has JHP, and some that does not have JHP.

Later in the thesis we introduce the concept of homomorphism and monomorphism category, we then prove the following result:

**Theorem 1.1** (lemma 8.6 and lemma 8.10)**.** *Let $\mathcal{C}$ be an exact category, then the homomorphy category and monomorphy category of $\mathcal{C}$ has JHP if and only if $\mathcal{C}$.*

This makes it possible to construct new exact categories with JHP, if we got an exact category with JHP. It is also possible to repeat creating monomorphism categories of monomorphism categories and these will also have JHP if the original category had JHP.

Then we generalize monomorphism and homomorphism categories with quivers, where the ordinary monomorphism and homomorphism categories was over the quiver $1 \to 2$. When investigating when these had JHP, we found that acyclic quivers the homomorphism category had JHP if the original category had JHP (theorem 9.5). For monomorphism categories over quivers there where some requirement for the quiver.

**Theorem 1.2** (theorem 9.13)**.** *For an exact category $\mathcal{C}$ and a acyclic quiver $\Gamma$ such that between any two nodes there are at maximum 1 arrow. Then we have that the the monomorphism category of $\mathcal{C}$ over $\Gamma$ has JHP, if and only if $\Gamma$ is a tree quiver and $\mathcal{C}$ has JHP.*

# 2 Exact categories

An exact category is an additive category $\mathcal{C}$ together with a collection of kernel-cokernel pairs, a collection of short exact sequences in $\mathcal{C}$. This collection is called the exact structure, and satisfies certain properties. Exact categories is a generalization of abelian categories, and abelian categories are also exact categories if one choses the collection to contain all kernel-cokernel pairs. Monomorphism in the collection are called essential monomorphism or inflation, and the epimorhisms are called essential epimorphism or deflation. First we can refresh what a kernel and a cokernel is in an additive category.

**Definition 2.1.** *Given a morphism $f\colon A \to B$ in an additive category $\mathcal{C}$, the kernel, if it exist is a morphism $i\colon \ker(f) \to A$ such that $fi = 0$ and for any other morphism $g\colon C \to A$ such that $fg = 0$ there exist a unique morphism $d\colon C \to \ker(f)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] & A \\
C \arrow[u, "d"] \arrow[ur, "g"] &
\end{tikzcd}$$

*Similiary a cokernel of $f$ is a morphism $\pi\colon B \to \operatorname{coker}(f)$ such that for any morphism $g\colon B \to D$, there exist a unique morphism $h\colon \operatorname{coker}(f) \to D$ such that the following diagram commutes:*

$$\begin{tikzcd}
B \arrow[r, "\pi"] \arrow[dr, "g"] & \operatorname{coker}(f) \arrow[d, "h"] \\
 & D
\end{tikzcd}$$

Even if it is technically the morphism $i$ and $\pi$ that is the kernel and cokernel of $f$, we often also call $\ker(f)$ and $\operatorname{coker}(f)$ the kernel and cokernel of $f$. This can be justified in a way, since it is well defined up to isomorphism.

Then we can define what we mean by a kernel-cokernel pair. This is in a way the same as a short-exact sequence.

**Definition 2.2.** *A kernel-cokernel pair in an additive category $\mathcal{C}$ is a pair of morphisms $f\colon A \to B$ and $g\colon B \to C$, for some objects $A, B$ and $C$, such that $f$ is the kernel of $g$, and $g$ is the cokernel of $f$. This is equivalent to that the following sequence is short exact:*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Then we are ready to define what an exact category is.

**Definition 2.3.** *[2, Definition 2.1] An additive category $\mathcal{C}$ together with the exact structure $\Delta$, consisting of kernel-cokernel pairs, closed under isomorphisms and which satisfying the follow properties is an exact category:*

*(1): Both* $A \xrightarrow{id_A} A \longrightarrow 0$ *and* $0 \longrightarrow A \xrightarrow{id_A} A$ *are in* $\Delta$ *for any object in* $\mathcal{C}$ *and therefore* $id_A$ *is both an inflation and a deflation.*

*(2a): If* $i$ *is an inflation from* $A$ *to* $B$*, and a arbitrary morphism* $f$ *starting in* $A$*, we have that the pushout of* $i$ *and* $f$ *exists, and that it preserve inflations. This means that if the following is the pushout of* $i$ *and* $f$ *, then* $j$ *is an inflation.*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d] \\
C \arrow[r, "j"] & D
\end{tikzcd} \tag{1}$$

*(2b): If* $i$ *is an deflation from* $B$ *to* $A$*, and let* $f$ *be an arbitrary morphism ending in* $A$*, we have that the pullback of* $i$ *and* $f$ *exists, and that it preserve deflations. This means that if the following is the pullback of* $i$ *and* $f$ *, then* $j$ *is a deflation.*

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d] & C \arrow[d, "f"] \\
B \arrow[r, "i"] & A
\end{tikzcd} \tag{2}$$

*(3): Composition of two inflations is an inflation, and composition of two deflations is a deflation.*

If $i$ is an inflation from $A$ to $B$ i will use a tail on the arrow like this to show that it is an inflation: $A \overset{i}{\rightarrowtail} B$ and if $p$ is an deflation from $C$ to $D$ I will use an arrow with two spearheads, like this: $C \overset{p}{\twoheadrightarrow} D$
To distinguish the kernel-cokernal pairs in the exact structure from a arbitrary pair, we call the kernel-cokernel pairs in the exact structure, conflations. Note that a kernel-cokernel pair "is the same" as a short exact sequence, so a conflation is a short exact sequence containing an inflation and a deflation. To check if a short exact sequence is a conflation, it is sufficient to show that the monomorphism is an inflation, or that the epimorphism is a deflation.

It is not only the identity map on an object that is both an inflation and a deflation. Any isomorphism is also both. We show this by looking at an isomorphism $f \colon A \to B$. Then the short exact sequence

$$A \xrightarrow{f} B \longrightarrow 0$$

has an isomorphism to the conflation in (1) in the definition of an exact category in the following way:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id_A"] & B \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[d] \\
A \arrow[r, "id_A"] & A \arrow[r] & 0
\end{tikzcd}$$

So $f$ is an inflation, and it is obvious that the dual argument gives us that $f$ is a deflation.

In most cases, the collection of short exact sequences do not need to contain every kernel-cokernel pairs, but there is some short exact sequences that always haves to be in the collection, namely the split exact sequences.

**Definition 2.4.** *An short exact sequence* $A \longrightarrow B \longrightarrow C$ *is split exact if it isomorphic to a short exact sequence on the form*

$$A' \xrightarrow{(1\,0)^T} A' \oplus C' \xrightarrow{(0\,1)} C'$$

*for some objects* $A'$ *and* $C'$

**Lemma 2.5.** *[2, Lemma 2.7] If* $\mathcal{C}$ *is an exact category, then every split exact sequence in* $\mathcal{C}$ *is also a conflation*

*Proof.* From the definition of split exact sequence, it is sufficient to show that

$$A \xrightarrow{(1\,0)^T} B \xrightarrow{(0\,1)} C$$

is a conflation for any $A$ and $B$. But this is a conflation if

$$A \xrightarrow{(0\,1)^T} A \oplus B$$

is an inflation.
So look at the following diagram:

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & B \arrow[d, "(0\,1)^T"] \\
A \arrow[r, "(1\,0)^T"] & A \oplus B
\end{tikzcd}$$

Since the upper arrow is the kernel of the deflation $id_B$, it follows that the upper arrow is an inflation. Also since the diagram is a pushout, Axiom (2a) gives us that lower row is also an inflation. □

This result show us that any exact structure has to contain the split exact sequences. In fact the collection of all split exact sequences is an exact structure and it is the minimal exact structure possible. We will show this, but first we will show a result which is neccesary.

**Lemma 2.6.** *If we got a short exact sequence on the form*

$$A \xrightarrow{(f.g)^T} B \oplus C \xrightarrow{(g'.f')} D$$

*then the following commutative diagram is bicartesian(both pushout and pullback).*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "-g"] & B \arrow[d, "g'"] \\
C \arrow[r, "f'"] & D
\end{tikzcd}$$

*Proof.* We show that it is a pushout. Assume we have another commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "-g"] & B \arrow[d, "h_B"] \\
C \arrow[r, "h_C"] & E
\end{tikzcd}$$

Then using the fact that $h_B f + h_C g = 0$ we get an induced morphism $h$ from $D$ to $E$ from the cokernel property of $D$

$$\begin{tikzcd}
A \arrow[r, "(f.g)^T"] & B \oplus C \arrow[r, "(g'.f')"] \arrow[dr, "(h_B.h_C)"'] & D \arrow[d, "h"] \\
& & E
\end{tikzcd}$$

□

**Theorem 2.7.** *Let $\mathcal{C}$ be an additive category and let $\Delta$ be the collection of all split exact sequences. Then $\mathcal{C}$ is an exact category with the exact structure $\Delta$.*

*Proof.* We have to show that the axioms holds. Axiom (1) holds since both short exact sequences containing $id_A$ are split exact sequences. Now we show that axiom (2a). Look at $(1\,0)^T\colon A \to A \oplus B$, and an arbitrary morphism $f\colon A \to C$ then we construct the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "(1\,0)^T"] \arrow[d, "f"] & A \oplus B \arrow[d, "(f\, id_B)"] \\
C \arrow[r, "(0\,1)^T"] & C \oplus B
\end{tikzcd}$$

Using lemma 2.6 we get that this diagram is a pushout since the following is an short exact sequence.

$$\begin{tikzcd}
A \arrow[r, "(1.0.-f)"] & A \oplus B \oplus C \arrow[r, "\begin{pmatrix} 010 \\ f01 \end{pmatrix}"] & B \oplus C
\end{tikzcd}$$

Dually we get that axiom (2b) is true. Axiom (3) is true from the fact that composition of two split monomorphisms is a split monomorphism, and same for split epimorphisms. □

On the other side, it is not the case that the collection of all kernel-cokernel pairs is an exact structure in general. The problem is that this collection of all kernel-cokernel pair does not neccesarily satisfy all the axioms to be an exact structure. But a result from [4] shows that there exist an maximal exact structure. For a particular additive category there may also be several other potential exact structures.

As exact categories are some sort of generalization of abelian categories, it is interesting to try to find an exact category which is not abelian. An example of a non-abelian exact category is the monomorphism category of an exact category, which we define in lemma 8.6. It follows from theorem 8.9 that the monomorphism category is not abelian.

# 3 Some diagram results

Working with exact categories it is handy to know some general result. Many results for abelian categories can also be generalized for exact categories.

**Proposition 3.1.** *[2, Proposition 2.12] Assume we have the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\
C \arrow[r, tail, "i'"] & D
\end{tikzcd} \tag{3}$$

*then the following are equivalent:*
*(1) The diagram is a pushout.*
*(2) The diagram is bicartesian (Pushout and pullback)*
*(3) The following diagram commutes where the rows are exact:*

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "q"] \arrow[d, "f'"] & E \arrow[d, equal] \\
C \arrow[r, tail, "i'"] & D \arrow[r, two heads, "q'"] & E
\end{tikzcd} \tag{4}$$

*(4) The following is an exact sequence:*

$$A \xrightarrow{(i.-f)^T} B \oplus C \xrightarrow{(f'.i')} D \tag{5}$$

*Proof.* See [2] ☐

The dual of this result is also true.

**Theorem 3.2.** *[2, Lemma 3.5] Given the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & B \arrow[r, two heads, "p"] \arrow[d, tail, "j"] & C \\
A \arrow[r, tail, "ji"] & D \arrow[r, two heads, "p'"] \arrow[d, two heads, "q"] & E \\
 & F &
\end{tikzcd} \tag{6}$$

*where the two rows and the middle column is conflations. Then there exist morphisms $j'$ and $s$ such that the following diagram commutes, and both rows* *and columns are conflations:*

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & B \arrow[r, two heads, "p"] \arrow[d, tail, "j"] & C \arrow[d, tail, "j'"] \\
A \arrow[r, tail, "ji"] & D \arrow[r, two heads, "p'"] \arrow[d, two heads, "q"] & E \arrow[d, two heads, "s"] \\
& F \arrow[r, equal] & F
\end{tikzcd} \tag{7}$$

*Proof.* We attain $j'$ using the cokernel property of $p$. We then have the following commuting diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & B \arrow[r, two heads, "p"] \arrow[d, tail, "j"] & C \arrow[d, "p"] \\
A \arrow[r, tail, "ji"] & D \arrow[r, two heads, "p'"] & E
\end{tikzcd} \tag{8}$$

Then using the dual version of proposition 3.1, we get that the rightmost square is bicartesian, and therefore is a pushout. Since $j$ is an inflation, $p$ is then also an inflation. Now using proposition 3.1 on this square we get that the following diagram commutes with exact columns:

$$\begin{tikzcd}
B \arrow[r, two heads, "p"] \arrow[d, tail, "j"] & C \arrow[d, tail, "j'"] \\
D \arrow[r, two heads, "p'"] \arrow[d, two heads, "q"] & E \arrow[d, two heads, "s"] \\
F \arrow[r, equal] & F
\end{tikzcd} \tag{9}$$

Combining the two we get the desired diagram. □

**Theorem 3.3.** *[2, Corollary 3.2] Let $\mathcal{C}$ be an exact category and assume we have the following diagram where the two rows are conflations:*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f"] & B \arrow[r, two heads] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

*Then if two of $f, g$ and $h$ are inflations/deflations/isomorphism then the last one is also an inflation/deflation/isomorphism.*

*Proof.* See [2]. □

Now we want to define what is a *projective* object in an exact category is.

**Definition 3.4.** *A Projective object $P$ is defined to have the following property: For any deflation $f$ from $A$ to $B$ and morphism $g$ from $P$ to $A$, there exist morphism $h$ from $P$ to $A$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & P \arrow[ld, dashed, "h"'] \arrow[d, "g"] \\
A \arrow[r, two heads, "f"] & B
\end{tikzcd} \tag{10}$$

This definition of projective is very similiar to the definition of projective objects in abelian categories, the only difference is that the morphism $f$ must be a deflation, and not only an epimorphism. But for an abelian category with the "normal" exact structure, then this definition is the same as the other definition.

# 4 Extension closed subcategories

For certain types of subcategories of an exact category, the subcategory inherits the exact structure. This is the case for subcategories closed under extension.

**Definition 4.1.** *A subcategory $\mathcal{D}$ of an exact category $\mathcal{C}$ is closed under extension if for any conflation in $\mathcal{C}$:*

$$\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

*if both $A$ and $C$ are objects in $\mathcal{D}$ then $B$ must also be in $\mathcal{D}$*

**Proposition 4.2.** *Let $\mathcal{C}$ be an exact category and $\mathcal{D}$ a full additive subcategory of $\mathcal{C}$ such that for any conflation in $\mathcal{C}$: $0 \longrightarrow A \rightarrowtail B \twoheadrightarrow C \longrightarrow 0$ if $A$ and $C$ are objects in $\mathcal{D}$ then $B$ is also an object in $\mathcal{D}$. If this is the case then $\mathcal{D}$ is an exact category where the exact structure is the structure "inherited" from $\mathcal{C}$ ie. $A \rightarrowtail B \twoheadrightarrow C$ is exact in $\mathcal{D}$, if it is exact in $\mathcal{C}$ and $A$, $B$ and $C$ are objects in $\mathcal{D}$.*

*Proof.* The goal is to show that $\mathcal{D}$ is an exact category, so it must be shown that axioms holds It is trivial that the identity morphism on an object $A$ in $\mathcal{D}$ is both an inflation and deflation from the fact that both $A$ and 0 are objects in $\mathcal{D}$

Next step is to prove that pushout of an inflation together with a morphism from an object $A$ exists and conserves inflation, and that a pullback of a deflation and a morphism to an object $A$ exists and conserves deflations.

Let

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\
C \arrow[r, tail, "i'"] & D
\end{tikzcd}$$

be a pushout of the inflation $i$ and the morphism $f$ (which are inflation and morphism in $\mathcal{D}$) in $\mathcal{C}$, we want to show that $D$ is in $\mathcal{D}$ and that $i'$ is an inflation in $\mathcal{D}$ looking at the diagram in $\mathcal{C}$, proposition 3.1 implies that

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & E \arrow[d, equal] \\
C \arrow[r, tail, "i'"] & D \arrow[r, "p'"] & E
\end{tikzcd} \tag{11}$$

is a commutative diagram where both rows are exact in $\mathcal{C}$, and since $i$ is an inflation in $\mathcal{D}$, then $p$ is a deflation in $\mathcal{D}$ and $E$ is therefore also an object in $\mathcal{D}$. From this we have that the row on the bottom is an exact sequence where $C$ and $E$ is in $\mathcal{D}$ and therefore $D$ is also in $\mathcal{D}$. So the lower row is an exact sequence in $\mathcal{D}$ so $i'$ is an inflation
The argument for pullback is the dual of the argument for pushout.
Lastly it must be shown that a composition of two inflations is an inflation, and the composition of two deflation is a deflation.
Proof: Let $A \overset{i}{\rightarrowtail} B$ and $B \overset{j}{\rightarrowtail} C$ be two inflations in $\mathcal{D}$, then consider the diagram in $\mathcal{C}$,

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & B \arrow[r, two heads, "p"] \arrow[d, tail, "j"] & D \arrow[d, "k"] \\
A \arrow[r, tail, "ji"] & C \arrow[r, two heads, "q"] & E
\end{tikzcd}$$

where $p$ is the deflation of $i$, so D is an object in $\mathcal{D}$, and $q$ is the deflation of $ji$ in $\mathcal{C}$, as $ji$ is an inflation in $\mathcal{C}$. Since $qji = 0 = (qj)i$ we get a morphism $k$ from $D$ to $E$ such that $kp = qj$. From proposition 3.1 the $BCDE$ square is bicartesian and a pushout of the inflation $j$ and morphism $p$. Since this pushout is in $\mathcal{D}$, $E$ is an object in $\mathcal{D}$ and the lower row is therefore an exact sequence in $\mathcal{D}$ so $ji$ is an inflation in $\mathcal{D}$. □

We call the exact structure from proposition 4.2 for a extension closed subcategory $\mathcal{D}$ of $\mathcal{C}$ as the subcategory exact structure of $\mathcal{D}$. It is important to note that this exact structure does not imply that any inflation $f\colon A \to B$ in $\mathcal{C}$, where $A$ and $B$ are in $\mathcal{D}$, is also an inflation in $\mathcal{D}$. We need also that the cokernel of $f$ in $\mathcal{C}$ is also in $\mathcal{D}$.

# 5 Subobjects and Inflations Series

We want to see for which exact categories JHP is satisfied, but we have not defined what a composition series is for exact categories. In this section the goal is to show that there is a natural way to define composition series for exact categories.
The reason we specify that the categories are skelletaly small, is to not get any collections that are not sets. This could be a problem both when we talk about the poset containing subobjects of a object, but also when we start working with the grothendick monoid/group of an exact category.

First of all we want to understand when $A$ is a subobject of $B$ This is the case when there exist an inflation $i\colon A \to B$. Technically we call $i$ a subobject of $B$, but we will often say that $A$ is a subobject of $B$. So when we say that $A$ is a subobject of $B$, it is connected to an inflation from $A$ to $B$.

**Definition 5.1.** *Let $A, B$ be objects in an exact category, then $A$ is a subobject of $B$ if there exist an inflation $i$ from $A$ to $B$.*

We also define when two subobjects are equivalent:

**Definition 5.2.** *Let $A, B$ and $X$ be objects in an exact category, then $A$ and $B$ are equivalent as subobjects of $X$ if there exist an isomorphism $f$ between $A$ and $B$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f"] & X \\
B \arrow[ur, tail] &
\end{tikzcd}$$

So we can then define the poset of an object $A$ as the collection of isomorphism classes of subobjects of $A$ with a partially order relation $\subset$.

**Definition 5.3.** *For an object $X$ in a skeletally small exact category $\mathcal{C}$, we define the poset of $X$ to be the set of equivalent classes of subobjects of $X$ together with the relation $\subset$. We denote this set by $P(X)$. The relation $\subset$ on $P(X)$ is such that $A \subset B$ if there exist an inflation $f$ such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail, "f"] & X \\
B \arrow[ur, tail] &
\end{tikzcd}$$

Now we want to study the relation $\subset$ in a $P(X)$. It is clear that $\subset$ is reflexive since if $A$ and $B$ are equivalent subobjects of $X$, then obviously $A \subset B$ and $B \subset A$. $\subset$ is also transitive since if $A \subset B \subset C$ we get the following diagram:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail, "f"] & X \\
B \arrow[ur, tail] \arrow[d, tail, "g"] & \\
C \arrow[uur, tail] &
\end{tikzcd}$$

So $gf$ is the inflation that gives us that $A \subset C$. At last it is antisymetric, if $A \neq B$ but $A \subset B$ we can show that $B \not\subset A$. We show this by constructing a contradiction. Assume that $A \subset B$ and $B \subset A$, then we can construct the following diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, tail, bend left, "f"] & X \\
B \arrow[u, tail, bend left, "g"] \arrow[ur, tail, "j"]
\end{tikzcd}$$

where $i = jf$ and $j = ig$. By inserting the first equation into the second one, we get that $j = jfg$. Since $j$ is a monomorphism this forces $fg = id_B$, and we get that $gf = id_A$ when we insert the second equation into the first. This means that $f$ and $g$ are both isomorphism, and therefore $A$ and $B$ are equivalent as subobjects of $X$. An important thing to note is that sometimes the inflation matters. Let us assume we have an object $A$ and the object $\bigoplus_{i=1}^{\infty} A$. When looking at $P(\bigoplus_{i=1}^{\infty} A)$, we can create two inflations from $\bigoplus_{i=1}^{\infty} A$ to itself that are not equivalent. The first one is the identity morphism, and the other inflation is by sending the $i$ -th $A$ to the $i+1$ - th $A$ with the identity morphism on $A$. With the notation we have used, we would say that $\bigoplus_{i=1}^{\infty} A$ is not equivalent to $\bigoplus_{i=1}^{\infty} A$, but technicaly it is the inflations that are not equivalent. It is obbvious that a poset of any object contains at least itself, since the identity morphism is an inflation, and it must also contain 0. So if $X$ is a nonzero object, then $P(X)$ must at least contain the two elements $X$ and 0. We define the simple objects to be those that the poset contains exactly two elements.

**Definition 5.4.** *An object $A$ is simple if $P(A)$ contains exactly 2 elements. In other words $A$ is nonzero, and $A$ and 0 are the only subobjects of $A$.*

Often we denote a simple object with the letter $S$.
Then we move on to show some results. To do this we have to define another set.

**Definition 5.5.** *If we have a pair of objects $B, C$ in $P(X)$ such that $B \subset C$ we define $P(B, C)$ to be the set of isomorphism classes of objects $Y$ such that $B \subset Y \subset C$*

By definition of $\subset$ it follows that $P(B, C)$ is a subset of $P(X)$. Observe that in $P(X)$ we have that $P(0, A)$ is the same as $P(A)$ and $P(A, A)$ is the set containing only $A$ We can show that $P(A, B)$ for some $A \subset B$ is isomorphic to $P(B/A)$ where $B/A$ is the cokernel of the inflation from $A$ to $B$ This will help us to show that if $P(A, B)$ contains exactly 2 objects, then $B/A$ is a simple object.

**Proposition 5.6.** *[3, Proposition 2.5] Let $A$ and $B$ be objects of a poset $P(Z)$ in an exact category such that $A \subset B$. Then there exist a poset-isomorphism between $P(A, B)$ and $P(B/A)$*

*Proof.* Let $X$ be an object in $P(A, B)$. Using theorem 3.2 we get the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & X \arrow[r, two heads] \arrow[d, tail, "j"] & X/A \arrow[d, tail, "j'"] \\
A \arrow[r, tail, "ji"] & B \arrow[r, two heads, "p"] \arrow[d, two heads] & B/A \arrow[d, two heads] \\
 & B/X \arrow[r, equal] & B/X
\end{tikzcd} \tag{12}$$

So we want to show that $\sigma : P(A, B) \to P(B/A), \sigma(X) = X/A$ is an poset-isomorphism. If both $X/A = Y/A$ then from the diagram we get that $B/X = B/Y$ and from this that $X = Y$, so sigma is injective. So let $Y \subset B/A$. By taking the pullback of the inflation of Y into $B/A$ and the deflation $p$ from the diagram, and using the dual of proposition 3.1 we get the following diagram:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, tail, "j"] & Y \arrow[d, tail] \\
A \arrow[r, tail] & B \arrow[r, two heads, "p"] & B/A
\end{tikzcd} \tag{13}$$

Then $D$ is an object in $P(A, B)$ such that $\sigma(D) = Y$. Therefore $\sigma$ is at least a set-isomorphism. If we have $X \subset Y$ both in $P(A, B)$, then by constructing a diagram similiar to eq. (12), where we replace $B$ with $Y$ we get that $\sigma(X) = X/A \subset Y/A = \sigma(Y)$. So $\sigma$ is a poset isomorphism. ☐

As a consequence of this theorem we have that if $A \subset B$ such that $A$ is not isomorphic to $B$ and if $X$ is such that $A \subset X \subset B$ then $X$ is either isomorphic to $B$ or $A$, then $B/A$ is a simple object. This will be useful when we define inflation series.

**Definition 5.7.** *If $\mathcal{C}$ is an exact category and $X$ an object in $\mathcal{C}$, we define an inflation series on $X$ as a sequence of inflations and objects:*
$A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail ... \rightarrowtail A_n = X$

Note that if we have an inflation series such that each inflation in the series is not an isomorphism, then we call the inflation series a proper inflation series. We define now what an composition series is.

**Definition 5.8.** *If $\mathcal{C}$ is an exact category and $X$ an object in $\mathcal{E}$, an composition series on $X$ such that $A_0 = 0$ and $A_{i+1}/A_i$ is simple.*

It is the same as that there is no objects $X \neq A_i, A_{i+1}$ such that $A_i \subset X \subset A_{i+1}$. This follows from proposition 5.6. Most of the times there are multiple composition series, but we will define when two composition series are *similar*. This is when the cokernels given by the inflations in the compositions series are the same.

**Definition 5.9.** *If we two series $Z$ and $Z'$ of an object $A$ in an exact category, we say that $Z$ and $Z'$ are isomorphic if both the length of $Z$ and $Z'$ are the same, and there exist a permutation $\sigma$ such that if $Z_i/Z_{i-1} \cong Z'_{\sigma(i)}/Z'_{\sigma(i)-1}$*

Now we want to define what a length exact category is, but we must first decide what we mean with length.

**Definition 5.10.** *Let $\mathcal{C}$ be an exact category and let $S$ be a proper inflation series of $A$ as following: $S_0 \subset S_1 \subset ... \subset S_n = A$. Then the length of $S$ is $n$.*

**Definition 5.11.** *A exact category $\mathcal{C}$ is length exact if for every object $A$, there exist an upper bound $N_A < \infty$ such that any proper inflation series of $A$ has length less than or equal to $N_A$*

If $\mathcal{C}$ is a length exact category it is easy to show that any object $A$ has an composition series. In the following theorem we will show that any proper inflations series can be extended to composition series in length exact categories. This is shown as a part of [3, Proposition 2.7]

**Lemma 5.12.** *If $\mathcal{C}$ is a length exact category and $S$ is an proper inflation series of $A \in \mathcal{C}$, then $S$ can be extended to an composition series $S'$, which contains every object in $S$.*

*Proof.* Let $\mathcal{C}$ be a length exact category and $S$ be a proper inflation series of $A \in \mathcal{C}$ where $S_0 = 0$. If we got proper inflation series $S'$ such that $S'$ is not 0, then we define $S$ such that $S_0 = 0$ and $S_i = S'_{i-1}$ for $i \geq 1$.
Then $S$ is on the form: $0 = S_0 \subset S_1 \subset S_2 ... \subset S_n = A$. $S$ has then length $n$. For the smallest $i$ such that $S_i/S_{i-1}$ is not simple, we can find atleast one object $X$ such that $S_{i-1} \subset X \subset S_i$. We then define a new proper inflation series $Z$ where $Z_j = S_j$ for $j < i$ and $Z_{i+1} = S_i$ for $j \geq i$, this new series has length $n+1$. Then we repeat this process until we got an composition series $S'$. First the reason we know that this cant go on forever is the fact that $\mathcal{C}$ is length exact. This means that for the object $A$, there exist a constant $N_A$ such that any proper inflation series of $A$ has length at maximum $N_A$. Since the length increases by one each time we repeat expanding the series, we can at worst repeat this process $N_A - n$ times before we get a series that can't be expanded. Secondly if we have proper inflation series $S$(where the object in node 0 is 0) which can't be expanded this way, it must be a composition series, this follows from the fact that as long as there is at least one $i$ such that $S_i/S_{i-1}$ is not simple, the set $P(S_{i-1}, S_i)$ has to contain at least one element $X$ that is not $S_i$ or $S_{i-1}$, then we can expand $S$ with $X$ as described earlier in the proof. ☐

Now we define a property for exact category which is central for the thesis, namely Jordan-Hölder property.

**Definition 5.13.** *An exact category $\mathcal{C}$ satisfies the Jordan-Hölder property, abbreviated JHP, if $\mathcal{C}$ is length exact, and if for every pair of composition series $S$ and $Z$ for the same object $A$ there exist a permutation $\sigma$ such that $S_i/S_{i-1} \sim Z_{\sigma(i)}/Z_{\sigma(i-1)}$*

A natural consequence of an exact category having JHP is that any composition series for a fixed object $A$ has to have the same length. This property is called unique length property. In general an exact category with the unique length property does not neccesarily have JHP.

Earlier we observed that for an arbitrary exact category, we could have an object $X$ such that there exist an inflation $i\colon X \to X$ such that $i$ is not equivalent to $id_X$. But for length exact categories, and therefore also exact categories with JHP, this cannot occur.

**Theorem 5.14.** *Let $\mathcal{C}$ be a length exact category, and let $X$ be an object in $\mathcal{C}$. Then if $i\colon X \to X$ is an inflation, then $i$ is an isomorphism and therefore equivalent to $id_X$*

*Proof.* Assume $i$ is not an isomorphism, then we can construct a proper inflation series of $X$ with any length. The proper inflation series with length $n$ is the following:

$$0 \rightarrowtail X_1 \overset{i}{\rightarrowtail} X_2 \overset{i}{\rightarrowtail} ... \overset{i}{\rightarrowtail} X_{n-1} \overset{i}{\rightarrowtail} X_n$$

Where $X_1 = X_2 = ... = X_n = X$. But this contradicts that $\mathcal{C}$ is a length exact category, so $i$ must be an isomorphism. □

**Theorem 5.15.** *If $\mathcal{C}$ is a lenght exact category, then for any pair of object $A \subset B$ such that $A \neq B$, we have that there exist no inflation from $B$ to $A$. ($B \not\subset A$)*

*Proof.* Assume there exist two object $A$ and $B$ and two inflations $i\colon A \to B$ and $j\colon B \to A$, we can construct proper inflation series of $B$, of length $2n$ for $n \in \mathbb{N}$. The inflation series of lenght $2n$ is

$$0 \rightarrowtail A_1 \overset{i}{\rightarrowtail} B_1 \overset{j}{\rightarrowtail} ... \overset{j}{\rightarrowtail} A_n \overset{i}{\rightarrowtail} B_n$$

where $B_i = B$ and $A_i = A$ for $i = 1, 2, ..., n$. This contradicts that $\mathcal{C}$ is length exact. □

Now we start looking at length functions.

**Definition 5.16.** *Let $\mathcal{C}$ be a skelletaly small exact category, then a weakly length-like function $f\colon \mathrm{iso}(\mathcal{C}) \to \mathbf{N}$ is a map with these properties: - For a conflation $A \to B \to C$ we have that $f(C) + f(A) \leq f(B)$, and $f(A) = 0$ if and only if $A = 0$*

In general, there may not exist a weakly length-like fuction for a particular $\mathcal{C}$. Typically this is the case if we have some object that is not finitely generated. By looking at composition series of objects in $\mathcal{C}$, we can find a connection between length of composition series of $A$ and $f(A)$ if $f$ is a weakly length-like function.

**Lemma 5.17.** *[3, Lemma 4.3] Let $A$ be an object in a skelletaly small exact category $\mathcal{C}$ with a composition series $(A_i)_{i=0}^{i=n}$ of length $n$. If $f$ is a weakly length-like function of $\mathcal{C}$, we then have that $f(A) \geq n$*

*Proof.* We use induction on the length of the composition series to show this. So first assume that $A$ is a object with a composition series of length 1. This gives us that $A$ cant be 0, so from the second requirement of $f$ to be a weakly length-like function gives us that $f(A) \geq 1$. Then we assume we have shown it $n-1$, so let $A$ be a object with a composition series $(A_i)_{i=0}^{i=n}$ of length $n$. Using the first requirement for $f$ to be a weakly length-like function, we get from the conflation $A_{n-1} \to A \to A/A_{n-1}$ that $f(A) \geq f(A_{n-1}) + f(A/A_{n-1}) \geq f(A_{n-1}) + 1$. Since $A_{n-1}$ has composition series $(A_i)_{i=0}^{i=n-1}$ it follows that $f(A_{n-1}) \geq n-1$ so we get that $f(A) \geq n$ ☐

As a direct consequence of this result, we get that the existence of a weakly length-like function implies that the category is length exact, we can show that the converse also holds.

**Proposition 5.18.** *[3, Theorem 4.4] Let $\mathcal{C}$ be a skelletaly small exact category, then there exist a weakly length-like function $f$ for $\mathcal{C}$, if and only if $\mathcal{C}$ is a length exact category.*

*Proof.* If there exist a weakly length-like function $f$, we have from lemma 5.17 that for any object $A$ and any composition series of $A$ we have that the length of the composition series has to be less than $f(A)$. So $\mathcal{C}$ is length exact. In the case where $\mathcal{C}$ is a length exact category, we define the map $f\colon \operatorname{iso}(\mathcal{C}) \to \mathbf{N}$ to be the map such that $f(A)$ is the lowest upper bound on the length of any composition series of $A$. We then show that $f$ is a weakly length-like function. It is obvious that $f(0) = 0$ and if $A \neq 0$ it follows that $0 \subset A$ can be extended to a composition series with length more than or equal to 1, so it follows that $f(A) \geq 1$. Now assume we have a conflation $A \to B \to C$, it follows that for any composition series $(A_i)_{i=0}^{i=n}$, where $n = f(A)$, and we can extend the inflation series $A_0 \subset ...A_{n-1} \subset A \subset B$ to a new composition series $(A_i)_{i=0}^{i=n+m}$ for $B$ such that $(A_i/A)_{i=n}^{i=n+m}$ is a composition series of $C$ where $m = f(C)$. This gives us that $f(B) \geq f(A) + f(B)$ (may have to prove this existence) ☐

# 6 Grothendick monoid and group

Let $\mathcal{C}$ be an exact category, then we want to define the grothendick monoid of $\mathcal{C}$. But first we introduce what a monoid is. A monoid is like a generalization of groups, where one does not require that inverses exist. More formally a monoid is a set $M$ and an binary operation $+ : M \times M \to M$ such that $+$ is associative, and $M$ has to contain an identity element 0 such that $0 + x = x + 0 = x$ for all $x$ in $M$.
The classical example of a monoid is $\mathbb{N}_0$, and any group is also a monoid.
Now we are ready to define grotendick monoid of $\mathcal{C}$:

**Definition 6.1.** *Let $\mathcal{C}$ be a skelletaly small exact category then the grothendick monoid of $\mathcal{C}$, denoted $\mathrm{M}(\mathcal{C})$ is the monoid where we denote the elements as $[A]$ for any object $[A]$ in $\operatorname{Iso}(\mathcal{C})$ up to a relation $\sim$. And we define the binary operation $+$ to be like this: $[A] + [B] = [A \oplus B]$. The relation $\sim$ is the smallest* *monoid congruence such that for any conflation $A \to C \to B$, we have that $[A] + [B] \sim [C]$*

**Definition 6.2.** *A monoid congruence $\sim$ on a monoid $M$ is an equivalence relation on $M$ such that for any elements $a$ and $b$ in $M$ such that $a \sim b$, we have that $a + c \sim b + c$ for any $c$ in $M$.*

As of now, it is unclear when $[A] \sim [B]$, but we can make it more clear. We begin by presenting a result.

**Lemma 6.3.** *Let $\mathcal{C}$ be a skeletaly small category. Then if for two objects $A$ and $B$ in $\mathcal{C}$ there exist two objects $D'$ and $D''$ and two conflations $D' \to A \to D''$ and $D' \to B \to D''$ we have that $[A] = [B]$ in $\mathrm{M}(\mathcal{C})$.*

*Proof.* Since we have the two conflations $D' \to A \to D''$ and $D' \to B \to D''$ it follows directly that $[A] = [D'] + [D''] = [B]$ ☐

For a pair of objects $A$ and $B$, we say that $A$ and $B$ share *conflation partners* if it is the case that there exist $D'$ and $D''$ such that we have the two conflations in lemma 6.3 . We know can show that this has some connection to when $[A] = [B]$.

**Theorem 6.4.** *Let $\mathcal{C}$ be a skeletaly small category. Then for two objects $A$ and $B$ in $\mathcal{C}$, we have that $[A] = [B]$ if and only if there exist a sequence $A = Y_0, Y_1, ..., Y_n = B$ such that $Y_i$ and $Y_{i+1}$ share conflation partners.*

*Proof.* The proof is very similar to the proof of [3, Proposition 3.4]. For simplicity we say that $[A] \approx [B]$ when there exist such a sequence. We first start by showing that $\approx$ infact is a monoid congruence. We show first that $\approx$ is an equivalence relation. It is clear that $\approx$ is both reflexive and symmetric. For transitivity assume $A \approx B$ and $B \approx C$. We can then construct a sequence of objects $Y_i$ between $A$ and $C$ such that $Y_i$ and $Y_{i+1}$ share conflation partners. This we do by combining the sequences between $A$ and $B$ and $B$ and $C$. We then show that if $[A] \approx [B]$ then $[A] + [C] = [A \oplus C] \approx [B \oplus C] = [B] + [C]$ for any object $C \in \mathcal{C}$. Since we have that $[A] \approx [B]$, there exist a sequence $A = Y_0, Y_1, ..., Y_n = B$ where $Y_i$ and $Y_{i+1}$ share conflation partners. Since $Y_i$ and $Y_{i+1}$ share conflation partners, there exist $D'$ and $D''$ such that we got the following conflations $D' \to Y_i \to D''$ and $D' \to Y_{i+1} \to D''$, but then we can also construct the two conflations $D' \oplus C \to Y_1 \oplus C \to D''$ and $D' \oplus C \to Y_1 \oplus C \to D''$, so we then get that $Y_i \oplus C$ and $Y_{i+1} \oplus C$ share conflation partners. Then the sequence $Y_0 \oplus C, Y_1 \oplus C, ..., Y_n \oplus C$ gives us that $[A \oplus C] \approx [B \oplus C]$, so $\approx$ is a monoid congruence. We are then left to show that $\approx$ is the same as $\sim$. From lemma 6.3 we got that if $[A] \approx [B]$ then $[A] \sim [B]$ so we have that $\approx$ is a "smaller" or equal monoid congruence than $\sim$, so we just need to show that $\approx$ respects conflations. Let $A \to C \to B$ be a conflation in $\mathcal{C}$. By definition we have that $[A] + [B] = [A \oplus B]$ but it is obvious that $A \oplus B$ and $C$ share conflation partners, so $[A] + [B] = [A \oplus B] \approx [C]$. So $\approx$ is the smallest monoid congruence respecting conflations and is by definition the same as $\sim$. ☐

**Definition 6.5.** *Let* $f\colon \mathrm{Iso}(\mathcal{C}) \to M$ *be a map to a monoid* $M$*. Then* $f$ *respects conflations if for any conflation* $A \to B \to C$ *in* $\mathcal{C}$*,* $f([A]) + f([C]) = f([B])$*.*

Note that the map $\pi\colon \mathrm{Iso}(\mathcal{C}) \to \mathrm{M}(\mathcal{C})$ such that $\pi([A]) = [A]$ is a map that respects conflations. This follows from the definition of $\mathrm{M}(\mathcal{C})$. Now we show that any such map $f$ can be factorized through $\pi$.

**Theorem 6.6.** *Let* $f\colon \mathrm{Iso}(\mathcal{C}) \to M$ *be a map that respects conflations, then there exist a monoid homomorphism* $g\colon \mathrm{M}(\mathcal{C}) \to M$ *such that the following commutes:*

$$\begin{tikzcd}
\mathrm{Iso}(\mathcal{C}) \arrow[r, "f"] \arrow[d, "\pi"] & M \\
\mathrm{M}(\mathcal{C}) \arrow[ur, "g"] &
\end{tikzcd}$$

*Proof.* Assume $f$ is a map that respects conflations. We define $g\colon \mathrm{M}(\mathcal{C}) \to M$ to be such that $g([A]) = f([A])$. We need to show that $g$ is well defined. Assume we have $A$ and $B$ such that $[A] = [B]$ in $\mathrm{M}(\mathcal{C})$. From this there exist a sequence $A = Y_0, Y_1, ..., Y_n = B$ such that $Y_i$ and $Y_{i+1}$ share conflation partners. So we show that $f([Y_i]) = f([Y_{i+1}])$. Since $Y_i$ and $Y_{i+1}$ share conflation partners, we know that there exist $C', C''$ such that we can construct two conflations $C' \to Y_i \to C''$ and $C' \to Y_{i+1} \to C''$. Since $f$ respects conflations, $f([Y_i]) = f([C']) + f([C'']) = f([Y_{i+1}])$. Using this we get that $f([A]) = f([Y_1]) = ... = f([Y_{n-1}]) = f(B)$, so $g$ is well defined. It is also true that $g$ is a monoid homomorphism, since $g([A] + [B]) = g([A \oplus B]) = f([A \oplus B]) = f([A]) + f([B]) = g([A]) + g([B])$ . From this it is also clear that $g\pi = f$. ☐

This shows that the grothendick monoid we defined in definition 6.1 is the same as it is defined in [3, Definition 3.2]. We are now ready to show some properties $\mathrm{M}(\mathcal{C})$ has.

**Proposition 6.7.** *[3, Proposition 3.5] [1, Lemma 2.9] Let* $\mathrm{M}(\mathcal{C})$ *be the grothendick monoid of a scelletaly small exact category* $\mathcal{C}$ *then it has the following two properties: (1):* $[A] = 0$ *if and only if* $A = 0$ *and (2):* $[A] + [B] = 0$ *if and only if both* $A$ *and* $B$ *are* $0$*.*

*Proof.* We first show that the first property holds. If $A = 0$ it is obvious that $[A] = 0$, so we are left to show the other way. If $[A] = 0$, we have that $[A] = [0]$. So there must exist a sequence $0 = Y_0, Y_1, ..., Y_n = A$ where $Y_i$ and $Y_{i+1}$ share conflation partners, but as the only conflation with 0 in the middle is $0 \to 0 \to 0$, and if $0 \to X \to 0$ is a conflation then $X = 0$. So we get that $Y_i = 0$ for all $i$, and therefore $A = 0$. For the second property it is again obvious that if both $A$ and $B$ are 0 then $[A] + [B] = 0$. In the case that $[A] + [B] = 0$, we have that $[A] + [B] = [A \oplus B] = 0$. Using the first property, we then get that $A \oplus B = 0$ which implies both $A$ and $B$ are 0. ☐

The second property in the previous proposition is the requirement for a monoid to be reduced. We will then show that a result for the elements of the grothendick monoid that it is represented by simple objects.

**Theorem 6.8.** *[3, Lemma 3.6] If $S$ is a simple object in a skelletaly small category $\mathcal{C}$, and $A$ is an object in $\mathcal{C}$. Then $[A] = [S]$ in $\mathrm{M}(\mathcal{C})$ if and only if $A \cong S$.*

*Proof.* If $[A] = [S]$, then there exist a sequence $S = Y_0, Y_1, ..., Y_n = A$, but since the only conflations with $S$ in the middle are $S \to S \to 0$ and $0 \to S \to S$, it forces all of $Y_i \cong S$ which also means that $A \cong S$. □

We then are ready to define the *grothendick group* of an exact category $\mathcal{C}$. The grothendick group of an exact category is the group completion of the grothendick monoid. There are several ways to define this group, as can be seen in [6], but we will define it the following way.

**Definition 6.9.** *Let $\mathrm{M}(\mathcal{C})$ be the grothendick monoid of a skeletaly small exact category $\mathcal{C}$. Then we define the grothendick group of $\mathcal{C}$, denoted $\mathrm{K}_0(\mathcal{C})$, to be the free abelian group on the objects in $\mathrm{M}(\mathcal{C})$ with the relation that $[A] + [B] = [A \oplus B]$, and that for a conflation $A \to C \to B$ we have that $[A] + [B] = [C]$*

This group has a very similar property to the property shown to be true for the grothendick monoid in theorem 6.6, but where $M$ is an abelian group. This is the way [3] defines the grothendick group.

**Proposition 6.10.** *For the grothendick group $\mathrm{K}_0(\mathcal{C})$ of a skeletaly small exact category $\mathcal{C}$, if we have a inflation series of any object $A$*

$$0 = Z_0 \rightarrowtail Z_1 \rightarrowtail ... \rightarrowtail Z_{n-1} \rightarrowtail Z_n = A$$

*then $[A] = \sum_{i=1}^{n} [Z_i / Z_{i-1}]$*

*Proof.* We prove this by induction on the length of an inflation series. So assume we have an object $A$ and a inflation series of $A$ with length 1. Then it looks like this: $0 \subset A$. It is obvious that $[A] = [A/0]$ since $A/0 = A$. Then assume we know that it is true for inflation series of length $\leq k$, and assume we have a compositon series $Z$ of an object $A$ with length $k+1$. From the inflation from $Z_k$ into $Z_{k+1} = A$ we get the following conflation: $Z_k \rightarrowtail A \twoheadrightarrow Z_{k+1}/Z_k$ And from this we get that $[A] = [Z_k] + [Z_{k+1}/Z_k]$. Then the inflation series of $Z_k$ we get by removing $A$ from $Z$, has length $k$ so $[Z_k] = \sum_{i=1}^{k} [Z_i / Z_{i-1}]$. By combining the two equations we get that $[A] = [Z_k] = \sum_{i=1}^{k+1} [Z_i / Z_{i-1}]$ □

Finally by studying the structure of $\mathrm{K}_0(\mathcal{C})$ we can say something about when $\mathcal{C}$ has JHP.

**Theorem 6.11.** *[3, Theorem 4.10] Let $\mathcal{C}$ be a skelletaly small exact structure. Then $\mathcal{C}$ has JHP if and only if $\mathrm{K}_0(\mathcal{C})$ is a free abelian group with $\mathrm{Rank}(\mathrm{K}_0(\mathcal{C})) = \# \operatorname{sim}(\mathcal{C})$*

*Proof.* We prove one direction of the equivalence by showing that if $\mathcal{C}$ does not have JHP, then $\mathrm{K}_0(\mathcal{C})$ is not free abelian group generated by the simple objects in iso($\mathcal{C}$) If $\mathcal{C}$ does not have JHP, then there must exist an object $A$, and two non-similar composition series $Z$ and $Z'$ of $A$. Then we get that $[A] = \sum_{i=1}^{n} Z_i/Z_{i-1} = \sum_{i=1}^{m} Z'_i/Z'_{i-1}$. But these two sums are not similiar, so it follows that $\mathrm{K}_0(\mathcal{C})$ is not a free abelian group generated by the simples object. See [3] for the full proof. ☐

# 7 $^{\perp}U$ subcategories

Now when we have defined what Jordan-Hölder property is, we will look at a particular type of categories, namely representations over finite quivers.

**Definition 7.1.** *For a field $\mathcal{F}$ and a acyclic quiver $\Gamma$ with finite number of nodes, we define* $\mathrm{Rep}(\mathcal{F}, \Gamma)$ *to be the category of representations of $\mathcal{F}$ over $\Gamma$.*

These categories are abelian, so they have the maximum exact structure. It is also know that JHP holds for these categories. But we are mostly interested in looking at extension closed subcategories of these representation categories. So for an representation category $\mathrm{Rep}(\mathcal{F}, \Gamma)$, where $\mathcal{F}$ is a field, and $\Gamma$ an acyclic quiver with finite number of node, we want to define the subcategory $^{\perp}U$ for any object $U \in \mathrm{Rep}(\mathcal{F}, \Gamma)$ . The definition involves the Ext functor, so we first remind ourself what this is.

**Definition 7.2.** *For an abelian category $\mathcal{C}$ with enough projective objects, if we study two objects $A$ and $U$, and assume the following is a projective resolution of $A$:*

$$... \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A$$

*Then we get the complex chain:*

$$0 \longrightarrow \mathrm{Hom}(P_0, U) \xrightarrow{\pi_0} \mathrm{Hom}(P_1, U) \xrightarrow{\pi_1} ...$$

*Then* $\mathrm{Ext}^i(A, U)$ *is the homology of this complex chain in the position with* $\mathrm{Hom}(P_i, U)$.

**Definition 7.3.** *Let $U$ be an object in* $\mathrm{Rep}(\mathcal{F}, \Gamma)$*, then the subcategory $^{\perp}U$ is the full subcategory such that $X \in^{\perp} U$ if* $\mathrm{Ext}_i(U, X) = 0$ *for $i \geq 1$.*

This is the kind of subcategory we are going to look at in this section. We will show that there is a relation between when $^{\perp}U$ and the number of indecomposable projectives and simple objects. But first we have to argue that this is infact an exact category.

**Theorem 7.4.** *Let $^{\perp}U$ be the subcategory of* $\mathrm{Rep}(\mathcal{F}, \Gamma)$. *Then $^{\perp}U$ is closed under exact sequences, and is therefore an exact subcategory of* $\mathrm{Rep}(\mathcal{F}, \Gamma)$.

*Proof.* So assume we have an short exact sequence: $A \rightarrowtail B \twoheadrightarrow C$ where $A$ and $C$ is in $^{\perp}U$ and $B$ is in $\mathrm{Rep}(\mathcal{F}, \Gamma)$. Since this is a short exact sequence then from homological algebra there is an result that gives us the following exact sequence:

$$... \longrightarrow \mathrm{Ext}_i(C,U) \longrightarrow \mathrm{Ext}_i(B,U) \longrightarrow \mathrm{Ext}_i(U,A) \longrightarrow \mathrm{Ext}_{i+1}(C,U) \longrightarrow ... \tag{14}$$

Since $\mathrm{Ext}_i(A,U)$ and $\mathrm{Ext}_i(C,U)$ is 0 for $i \geq 1$ and $\mathrm{Ext}_i(B,U)$ lays between these two, then $\mathrm{Ext}_i(B,U)$ must also be zero, so $B$ is in ${}^{\perp}U$. So as ${}^{\perp}U$ is extension closed subcategory, proposition 4.2 implies that ${}^{\perp}U$ is an exact category. □

Using the same argument, since $\mathrm{Ext}_i(U,A)$ lies between $\mathrm{Ext}_i(U,B)$ and $\mathrm{Ext}_{i+1}(U,C)$, if $B$ and $C$ is in ${}^{\perp}U$, then so is $A$. We can also show that any projective objects are in ${}^{\perp}U$.

**Lemma 7.5.** *If $P$ is projective object in* $\mathrm{Rep}(\Gamma,\mathcal{F})$. *Then for any $U$ we have that $P \in^{\perp} U$*

*Proof.* Since the projective resolution of $P$ is $0 \to P \to P$, and that $\mathrm{Hom}(0,U) = 0$, we get the following complex chain:

$$0 \longrightarrow \mathrm{Hom}(P,U) \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

From this it is obvious that $\mathrm{Ext}^i(P,U) = 0$ when $i > 0$. Therefore $P \in^{\perp} U$ □

Another property of this subcategory is that it is closed under summands. This will be usefult to know, since we can find which objects are in ${}^{\perp}U$, by finding out which indecomposable objects are in ${}^{\perp}U$.

**Lemma 7.6.** *If $A \oplus B$ is in ${}^{\perp}U$ then so is both $A$ and $B$.*

*Proof.* If $A \oplus B$ is in ${}^{\perp}U$, then for $i > 0$ it follows from that the Ext-functor preserves direct sums that $0 = \mathrm{Ext}^i(A \oplus B, U) = \mathrm{Ext}^i(A,U) \oplus \mathrm{Ext}^i(B,U)$. Then both $\mathrm{Ext}^i(A,U)$ and $\mathrm{Ext}^i(B,U)$ has to be zero, and therefore both $A$ and $B$ are in ${}^{\perp}U$ □

From this result, we can see that ${}^{\perp}U$ is "generated" by the indecomposable objects in $\mathrm{Rep}(\Gamma,\mathcal{F})$ that is also in ${}^{\perp}U$. We then show some properties of the grothendick group of ${}^{\perp}U$.

**Proposition 7.7.** *[3, Proposition 5.8, (2) and (3)] For the subcategory ${}^{\perp}U$ we have that* $\mathrm{K}_0({}^{\perp}U)$ *is a free abelian group with* $\mathrm{Rank}(\mathrm{K}_0({}^{\perp}U)) = \#ind.proj({}^{\perp}U)$

**Theorem 7.8.** *[3, Theorem 5.10] Let $U$ be an object of* $\mathrm{Rep}(\mathcal{F},\Gamma)$ *for a field $\mathcal{F}$ and quiver $\Gamma$. If $\mathcal{E} =^{\perp} U$ then JHP holds for $\mathcal{E}$ if and only if the number of indecomposable projective in $\mathcal{E}$ are equal to the number of simples, up to isomorphism.*

*Proof.* We get from proposition 7.7 that $\mathrm{K}_0({}^{\perp}U)$ is free, with rank equal to the number of nonisomorphic indecomposable projective objects in ${}^{\perp}U$. Using theorem 6.11 we get that ${}^{\perp}U$ has JHP if and only if $\mathrm{K}_0({}^{\perp}U)$ is free, with rank equal to the number of nonisomorphic simple in ${}^{\perp}U$. Since we already know that $\mathrm{K}_0({}^{\perp}U)$ is free, we get that ${}^{\perp}U$ has JHP if and only if $\#ind.proj({}^{\perp}U) = \#sim({}^{\perp}U)$ $\square$

Now we can look some example of ${}^{\perp}U$ subcategories. Example: lets look at the quiver $\Gamma = 1 \xrightarrow{\alpha} 2$ and $\mathrm{Rep}(\mathcal{F}, \Gamma)$ where $\mathcal{F}$ is a field. Then since $\mathrm{Rep}(\mathcal{F}, \Gamma)$ is an artinian algebra so the result above holds, observe that the indecomposables of $\mathrm{Rep}(\mathcal{C}, \Gamma)$ are $I_1 = \mathcal{F} \longrightarrow 0$

$P_2 = 0 \longrightarrow \mathcal{F}$ and $P_1 = I_2 = \mathcal{F} \xrightarrow{id} \mathcal{F}$ where the indecomposable projectives are $P_2$ and $P_1$ and the simples are $I_1$ and $P_2$, Let $U = P_2$ and look at ${}^{\perp}U$. By "calculation" one can show that ${}^{\perp}U$ is objects of the form $(P_1)^n \oplus (P_2)^m$ where $n$ and $m$ are non-negative integers. $P_2$ is simple in $\mathrm{Rep}(\mathcal{F}, \Gamma)$ so it is simple in any subcategory, but now $P_1$ is also simple in ${}^{\perp}U$ from the fact that the inflation from $P_2$ into $P_1$ in $\mathrm{Rep}(\mathcal{F}, \Gamma)$ is not a inflation in ${}^{\perp}U$, since the cokernel is not in ${}^{\perp}U$. Since this was the only proper inflation of $P_1$, $P_1$ is simple. $P_1$ and $P_2$ are also the only indecomposable objects and are both projective so in this case we have the same number of simples and ind. projectives so this subcategory has JHP

But there exist also subcategories of this form that does not have equally many indecomposable projectives as simple objects. We will show one example of such a subcategory. Let us define $\Gamma$ as the following quiver:

$$\begin{tikzcd}
 & 1 & \\
2 \arrow[ur] & 3 \arrow[u] & 4 \arrow[ul]
\end{tikzcd} \tag{15}$$

and consider the subcategory ${}^{\perp}U$ of $\mathrm{Rep}(\Gamma, \mathcal{F})$, where where $U$ is the following object:

$$\begin{tikzcd}
 & \mathcal{F}^2 & \\
\mathcal{F} \arrow[ur, "(1\,0)^T"] & \mathcal{F} \arrow[u, "(0\,1)^T"] & \mathcal{F} \arrow[ul, "(1\,1)^T"']
\end{tikzcd}$$

It can be shown that $\mathrm{Rep}(\Gamma, \mathcal{F})$ has 12 non-isomorphic indecomposable objects. 8 of these are the injectives and projective objects generated by each node. The remaining indecomposable, are $D_i$, for $i = 2, 3$ and 4, where $D_2$ is

$$\begin{tikzcd}
 & \mathcal{F} & \\
0 \arrow[ur] & \mathcal{F} \arrow[u, "id"] & \mathcal{F} \arrow[ul, "id"]
\end{tikzcd}$$

and $D_3$ and $D_4$ is on the same form, jut where representation in node 3 and 4 is 0 instead of in node 2. The last indecomposable is $U$. Then we find out which indecomposable objects that are in ${}^{\perp}U$. From lemma 7.5 we have that $P_1, P_2, P_3$ and $P_4$ are in ${}^{\perp}U$. The rest we will have to calculate to find out wheter or not they are in ${}^{\perp}U$ So we start with showing that $I_2$, $I_3$ and $I_4$ are not in ${}^{\perp}U$. So for $I_2$ we have the following projective resolution:

$$0 \longrightarrow P_1 \xrightarrow{\pi_1} P_2 \xrightarrow{\pi_0} I_2$$

And we therefore get the complex chain:

$$0 \longrightarrow \mathrm{Hom}(P_2, U) \xrightarrow{\pi_1} \mathrm{Hom}(P_1, U) \longrightarrow 0$$

but since $\mathrm{Hom}(P_2, U) \cong \mathcal{F}$ and $\mathrm{Hom}(P_1, U) \cong \mathcal{F}^2$, we get that $\mathrm{Ext}^1(I_2, U) \neq 0$. This means that $I_2$ is not in ${}^{\perp}U$, and in the same way, we can show that $I_3$ and $I_4$ is not in ${}^{\perp}U$. Then we want to determine if $I_1$ is in ${}^{\perp}U$. The projective resolution of $I_1$ is

$$0 \longrightarrow P_1 \oplus P_1 \longrightarrow P_2 \oplus P_3 \oplus P_4 \longrightarrow I_1$$

By using a similiar argument as for the other injective objects, we observe that $\hom(P_1 \oplus P_1, U) \cong \mathcal{F}^4$, while $\mathrm{Hom}(P_2 \oplus P_3 \oplus P_4, U) \cong \mathcal{F}^3$, which forces $\mathrm{Ext}^1(I_1, U) \neq 0$. Therefore $I_1$ is not in ${}^{\perp}U$. To show that also $D_2, D_3$ and $D_4$ are not in ${}^{\perp}U$, we need to calculate $\mathrm{Ext}^1(D_i, U)$. For $D_4$ we have the projective resolution

$$0 \longrightarrow P_1 \xrightarrow{\pi_1} P_2 \oplus P_3 \longrightarrow D_4$$

Note that the map $\pi_1 \colon P_1 \to P_2 \oplus P_3$ is the zero map for every node except 1, and the map $(id, -id)$ for node 1. This gives us the following complex chain:

$$0 \longrightarrow \mathrm{Hom}(P_2 \oplus P_3, U) \xrightarrow{f} \mathrm{Hom}(P_1, U) \longrightarrow 0$$

where the morphism $f$ is such that $f(h) = h\pi_1$. It can be observed that $\ker(f) \neq 0$ , so the image, $\mathrm{Im}(f)$ is at most $\mathcal{F}$. This forces $\mathrm{Ext}^1(D_4, U) \neq 0$. Therefore $D_4$ is not in ${}^{\perp}U$ This argument can be used to also show that $D_3$ and $D_2$ are also not in ${}^{\perp}U$. At last we check if $U$ is in ${}^{\perp}U$. The projective resolution of $U$ is

$$0 \longrightarrow P_1 \xrightarrow{\pi_1} P_2 \oplus P_3 \oplus P_4 \longrightarrow U$$

We then get the following complex chain:

$$0 \longrightarrow \mathrm{Hom}(P_2 \oplus P_3 \oplus P_4, U) \xrightarrow{f} \mathrm{Hom}(P_1, U) \longrightarrow 0$$

We will then show that $\mathrm{Im}(f)$ is two-dimensional, and therefore $\mathrm{Ext}^1(U, U) = 0$. Since $f$ is a map essentially from $\mathcal{F}^3$ to $\mathcal{F}^2$ it is equivalent to show that the kernel is one-dimensional We know that $\ker(f) = \mathrm{Ext}^0(U, U) = \mathrm{Hom}(U, U)$, and it is possible to show that $\mathrm{Hom}(U, U) \cong \mathcal{F}$. Therefore $\mathrm{Ext}^1(U, U) = 0$ and it is obvious that $\mathrm{Ext}^i(U, U) = 0$ when $i > 1$. So $U \in^{\perp} U$.

From this we know that ${}^{\perp}U$ is generated by $P_1, P_2, P_3, P_4$ and $U$. The first four are the indecomposable projectives in ${}^{\perp}U$, while all five indecomposable are simple objects in ${}^{\perp}U$. From this we get that $\#\operatorname{Sim}({}^{\perp}U) = 5 \neq 4 = \#\operatorname{Ind.Proj}({}^{\perp}U)$. So from theorem 7.8 we get that ${}^{\perp}U$ does not have JHP. We can show this directly by looking at $P_2 \oplus P_3 \oplus P_4$. There then exist a inflation $i\colon P_1 \to P_2 \oplus P_3 \oplus P_4$ such that coker(i) = U. This proves directly that ${}^{\perp}U$ does not have JHP.

# 8 Homomorphy category and monomorphy category

Given an exact category it is possible to construct a new category where the objects are the morphisms of the category, and the morphisms of this new category are pairs of morphism with a certain properties. This new category is called the *homomorphism category*, and we are going to show that this category inherits some properties from the original category, including JHP.

**Definition 8.1.** *Let $\mathcal{C}$ be a category. Then let $\mathrm{H}(\mathcal{C})$ be the category such that the objects in $\mathrm{H}(\mathcal{C})$ are all morphisms in $\mathcal{C}$. If $f$ and $g$ are morphisms in $\mathcal{C}$, and therefore objects in $\mathrm{H}(\mathcal{C})$, then the morphism between $f$ and $g$ is $(h_1, h_2)$, where $h_1$ and $h_2$ are morphisms in $\mathcal{C}$ and $gh_1 = h_2 f$, ie. the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "h_2"] & D \end{tikzcd} \tag{16}$$

The composition of the morphisms in $\mathrm{H}(\mathcal{C})$ are defined as followed $f$, $g$ and $v$ are objects in $\mathrm{H}(\mathcal{C})$, $(h_1, h_2)$ is a morphism beetween $f$, and $g$, and $(h_1', h_2')$ is a morphism between $g$ and $v$ then $(h_1', h_2') \circ (h_1, h_2) = (h_1' h_1, h_2' h_2)$ It is easy to show that the properties of a category are fulfiled, so it is truly a category. To make it easier to explain I will call $\mathcal{C}$ the parent category of $\mathrm{H}(\mathcal{C})$. It is clear to see that $\mathcal{C}$ if additive, then so is $\mathrm{H}(\mathcal{C})$.

It is also true that if $\mathcal{C}$ is an exact category, we get a natural exact structure for $\mathrm{H}(\mathcal{C})$. This structure is such that $(f_1, f_2)$ is an inflation in $\mathrm{H}(\mathcal{C})$ if $f_1$ and $f_2$ both are inflations in $\mathcal{C}$.

**Proposition 8.2.** *If $\mathcal{C}$ is an exact category, then $\mathrm{H}(\mathcal{C})$ is an exact category with the exact structure that $(f_1, f_2)$ is an inflation if $f_1$ and $f_2$ both are inflations in $\mathcal{C}$, and $(f_1, f_2)$ is a deflation if $f_1$ and $f_2$ are deflations in $\mathcal{C}$*

*Proof.* We want to show that this is an exact category. So first we show that $(f_1, f_2)$ is a part of a kernel-cokernel pair if both $f_1$ and $f_2$ are inflations in the parent category. If $f_1$ and $f_2$ is inflations in $\mathcal{C}$ and $(f_1, f_2)$ is a morphism in $\mathrm{H}(\mathcal{C})$, then $(g_1, g_2)$ is also a morphism in $\mathrm{H}(\mathcal{C})$, where $g_1$ and $g_2$ are the cokernels of $f_1$ and $f_2$ respectivly. As $(g_1, g_2) \circ (f_1, f_2) = (g_1 f_1, g_2 f_2) = (0, 0)$ and if there exist another morphism $(h_1, h_2)$ such that $(g_1, g_2) \circ (h_1, h_2) = (0, 0)$, then $g_i h_i = 0$ so since $f_i$ is a kernel of $g_i$ we have $v_i$ such that $h_i = f_i v_i$. it is easy to check that $(v_1, v_2)$ is a morphism in $\mathrm{H}(\mathcal{C})$ and $(h_1, h_2) = (f_1, f_2) \circ (v_1, v_2)$, therefore $(f_1, f_2)$ is the kernel of $(g_1, g_2)$ and the dual argument gives that $(g_1, g_2)$ is the cokernel of $(f_1, f_2)$. Then we show that this collection of kernel-cokernel pairs in fact follows the requirment to be an exact structure. We will only show that the argument is true for inflations as the argument for the deflations will be the dual argument.

First we show that the identity is an inflation. For a object $f$ in $\mathrm{H}(\mathcal{C})$, which is a morphism between $A$ and $B$ in $\mathcal{C}$, then $(Id_A, Id_B)$ is the identity on $f$. As $Id_A$ and $Id_B$ are both inflations in $\mathcal{C}$, $(Id_A, Id_B)$ is an inflation in $\mathrm{H}(\mathcal{C})$.

Then we show that the push-out of a inflation together with an arbitrary morphism preserves inflation. So if $f, g$ and $h$ be objects in $\mathrm{H}(\mathcal{C})$ and let $i = (i_1, i_2)$ be a inflation from $f$ to $g$, and $v = (v_1, v_2)$ is an arbitrary morphism from $f$ to $h$, and let the following diagram be the pushout of $i$ and $v$. Then $i'$ is also an inflation:

$$\begin{tikzcd}
f \arrow[r, tail, "i"] \arrow[d, "v"] & g \arrow[d, "v'"] \\
h \arrow[r, "i'"] & p
\end{tikzcd}$$

For simplicity let $D_f$ be the domain of $f$, and $C_f$ be the codomain of $f$, and the same for $g$ and $h$. We begin at looking at the two pushouts of $i_1, v_1$ and $i_2, v_2$ in the parent category:

$$\begin{tikzcd}
D_f \arrow[r, "i_1"] \arrow[d, "v_1"] & D_g \arrow[d, "v_1'"] & C_f \arrow[r, "i_2"] \arrow[d, "v_2"] & C_g \arrow[d, "v_2'"] \\
D_h \arrow[r, "i_1'"] & E_1 & C_h \arrow[r, "i_2'"] & E_2
\end{tikzcd} \tag{17}$$

We want to show that the following diagram is commutative:

$$\begin{tikzcd}
D_f \arrow[r, "i_1"] \arrow[d, "v_1"] & D_g \arrow[d, "v_2' g"] \\
D_h \arrow[r, "i_2' h"] & E_2
\end{tikzcd} \tag{18}$$

ie. that $v_2' g i_1 = i_2' h v_1$. First from the diagram given by $i$, $v_2' g i_1 = v_2' i_2 f$. From the second pushout we get that $v_2' i_2 f = i_2' v_2 f$ and from the diagram given by v, we finally get that $i_2' v_2 f = i_2' h v_1$, so $v_2' g i_1 = i_2' h v_1$ and the square above is commutative. From the properties of the first pushout we then get a unique morphism $p$ from $E_1$ to $E_2$ such that $i_2' h = i_1' p$ and $v_2' g = v_1' p$. This $p$ is the pushout of $i$ and $v$ and the morphism from $h$ to $p$ is $(i_1', i_2')$, which is an inflation since both $i_1'$ and $i_2'$ are inflations.

The last property that must be proven is that composition of inflations are also inflations, so let $f, g$ and $h$ and let $i = (i_1, i_2)$ and $j = (j_1, j_2)$ be inflations:

$f \xrightarrow{i} g \xrightarrow{j} h$ Since $ji = (j_1 i_1, j_2 i_2)$ and $j_1 i_1$ and $j_2 i_2$ are both inflations, so $ji$ is also an inflation. ☐

**Proposition 8.3.** *Let* $\mathrm{H}(\mathcal{C})$ *be the homomorphism category of an exact category* $\mathcal{C}$*. If we have the morphism* $(f_1, f_2)$ *between objects* $g$ *and* $h$ *in* $\mathrm{H}(\mathcal{C})$ *and both* $f_1$ *and* $f_2$ *have kernels in* $\mathcal{C}$ *we get the following diagram:*

$$\begin{tikzcd}
\ker(f_1) \arrow[r, "i_1"] \arrow[d, "p"] & A_1 \arrow[r, "f_1"] \arrow[d, "g"] & B_1 \arrow[d, "h"] \\
\ker(f_2) \arrow[r, "i_2"] & A_2 \arrow[r, "f_2"] & B_2
\end{tikzcd}$$

*then* $(i_1, i_2)$ *is the kernel of* $(f_1, f_2)$*, and* $p$ *is the kernel object of* $(f_1, f_2)$*.*

The dual result is also true. Note that if $\mathcal{C}$ is preabelian, i.e. has all kernel and cokernels, then $\mathrm{H}(\mathcal{C})$ is also preabelian. This follows directly from proposition 8.3 and its dual version.

**Theorem 8.4.** *Let* $\mathcal{C}$ *be an abelian category. Then* $\mathrm{H}(\mathcal{C})$ *is also abelian.*

*Proof.* It follows, from proposition 8.3, that since $\mathcal{C}$ preabelian, then so is $\mathrm{H}(\mathcal{C})$. Let $(f_1, f_2)$ be a morphism in $\mathrm{H}(\mathcal{C})$ from $u$ to $v$ as described below:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "u"] & B_1 \arrow[d, "v"] \\
A_2 \arrow[r, "f_2"] & B_2
\end{tikzcd} \tag{19}$$

And by finding the kernels and cokernels of $(f_1, f_2)$ we get the following commuting diagram:

$$\begin{tikzcd}
\ker(f_1) \arrow[r, tail, "i_1"] \arrow[d, "k"] & A_1 \arrow[r, "f_1"] \arrow[d, "u"] & B_1 \arrow[r, two heads, "\pi_1"] \arrow[d, "v"] & \operatorname{coker}(f_1) \arrow[d, "c"] \\
\ker(f_2) \arrow[r, tail, "i_2"] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r, two heads, "\pi_2"] & \operatorname{coker}(f_2)
\end{tikzcd} \tag{20}$$

Finding the coimage and image of $(f_1, f_2)$ we get the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, two heads, "\sigma_1"] \arrow[d, "u"] & \operatorname{coim}(f_1) \arrow[r, "\psi_1"] \arrow[d, "p"] & \operatorname{im}(f_1) \arrow[r, tail, "\omega_1"] \arrow[d, "q"] & B_1 \arrow[d, "v"] \\
A_2 \arrow[r, two heads, "\sigma_2"] & \operatorname{coim}(f_2) \arrow[r, "\psi_2"] & \operatorname{im}(f_2) \arrow[r, tail, "\omega_2"] & B_2
\end{tikzcd} \tag{21}$$

Here $p$ is the coimage of $(f_1, f_2)$ while q is the image of $(f_1, f_2)$. As both $\psi_1$ and $\psi_2$ are isomorphism, it is enough to show that $(\psi_1, \psi_2)$ is a morphism in $\mathrm{H}(\mathcal{C})$. In other words showing that the middle square of the diagram commutes. As $\omega_1 \psi_1 \sigma_1 = f_1$ and $\omega_2 \psi_2 \sigma_2 = f_2$, we have that the outer square commutes, and the leftmost and rightmost squares commutes. Using the commutativity of big square we get the following:

$$v\omega_1\psi_1\sigma_1 = \omega_2\psi_2\sigma_2 u$$

And then using the fact that the rightmost and lefmost square commutes, $v\omega_1 = \omega_2 q$ and $\sigma_2 u = p\sigma_1$ we get that $\omega_2 q\psi_1\sigma_1 = \omega_2\psi_2 p\sigma_1$. Then as $\sigma_1$ is an epimorphism then it follows that $\omega_2 q\psi_1 = \omega_2\psi_2 p$. Then as $\omega_2$ is an monomorphism: $q\psi_1 = \psi_2 p$ so the middle square commutes. Therefore $\mathrm{H}(\mathcal{C})$ is abelian. □

Now we look at a subcategory of the homomorphism category. The *monomorphism category* of a category $\mathcal{C}$ is a subcategory of $\mathrm{H}(\mathcal{C})$ where the objects are inflations in $\mathcal{C}$ and the morphisms between two objects are the morphism in $\mathrm{H}(\mathcal{C})$.

**Definition 8.5.** *Let $\mathcal{C}$ be an exact category, then the monomorphy category of $\mathcal{C}$, $\mathrm{MM}(\mathcal{C})$ is a full subcategory of $\mathrm{H}(\mathcal{C})$, where the objects are inflations in $\mathcal{C}$*

We will show that this subcategory infact is closed under extensions, and therefore it follows from proposition 4.2 that the subcategory is exact with the inherited exact structure.

**Lemma 8.6.** *Let $\mathcal{G} = \mathrm{H}(\mathcal{C})$ and let $\mathcal{H} = \mathrm{MM}(\mathcal{C}) \subset \mathrm{H}(\mathcal{C})$ be the monomorphism category. $\mathcal{H}$ is an extension closed subcategory of $\mathcal{G}$. i.e. if we have the conflation $f \rightarrowtail g \twoheadrightarrow h$ where $f$ and $h$ are objects in $\mathcal{H}$ and $B$ is an object in $\mathcal{G}$, then $g$ is in $\mathcal{H}$.*

*Proof.* Assume we have an conflation as above, then we draw the diagram:

$$\begin{tikzcd}
A_1 \arrow[r, tail] \arrow[d, tail, "f"] & B_1 \arrow[r, two heads] \arrow[d, "g"] & C_1 \arrow[d, tail, "h"] \\
A_2 \arrow[r, tail] & B_2 \arrow[r, two heads] & C_2
\end{tikzcd} \tag{22}$$

As both $f$ and $h$ are inflations, and both rows are conflations, we have from theorem 3.3 that $g$ also has to be an inflation. Therefore $g$ is also an object in the monomorphism category. □

**Proposition 8.7.** *If $\mathcal{C}$ is an exact category, then so is $\mathrm{MM}(\mathcal{C})$ with the exact structure such that the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, tail] \arrow[d, tail, "f"] & B_1 \arrow[r, two heads] \arrow[d, "g"] & C_1 \arrow[d, tail, "h"] \\
A_2 \arrow[r, tail] & B_2 \arrow[r, two heads] & C_2
\end{tikzcd}$$

*is an conflation in $\mathrm{MM}(\mathcal{C})$ if it is a conflation in $\mathrm{H}(\mathcal{C})$.*

*Proof.* If $\mathcal{C}$ is an exact category, then it follows from proposition 8.2 that $\mathrm{H}(\mathcal{C})$ is also an exact category. Then lemma 8.6 gives us that $\mathrm{MM}(\mathcal{C})$ is a extension closed subcategory of $\mathrm{H}(\mathcal{C})$. Using proposition 4.2 we get that $\mathrm{MM}(\mathcal{C})$ is an exact category with subcategory exact structure from $\mathrm{H}(\mathcal{C})$, as defined above. □

This is the exact structure we give MM($\mathcal{C}$) if nothing else is specified. One could also wonder if the monomorphism category also preserves abelian properties. This is not the case, with the exception of the trivial abelian category. To help us show this, we first state a lemma.

**Lemma 8.8.** *For an exact category* $\mathcal{C}$ *and an object* $A \neq 0$ *in* $\mathcal{C}$*, it follows that the cokernel of the morphism* $(0, id_A)$ *in* MM($\mathcal{C}$) *from* $0 \to A$ *to* $A \to A$ *is* $0 \to 0$ *as in the diagram below:*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "id_A"] & 0 \arrow[d] \\
A \arrow[r, "id_A"] & A \arrow[r] & 0
\end{tikzcd}$$

Note that even though this morphism contains two inflations, it is not an inflation in MM($\mathcal{C}$) since the cokernel in H($\mathcal{C}$) is $A \longrightarrow 0$ which is not in MM($\mathcal{C}$).

*Proof.* Assume we have an object $f$ in MM($\mathcal{C}$) such that

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r, "f_B"] \arrow[d, "id_A"] & B \arrow[d, "f"] \\
A \arrow[r, "id_A"] & A \arrow[r, "f_C"] & C
\end{tikzcd}$$

the diagram commutes and such that $(f_B, f_C)(0, id_A) = (0, f_C) = (0, 0)$. This implies that $f_C = 0$, and since the diagram commutes we have that $f_C id_A = 0 = f f_B$ but since $f$ is inflation and therefore a monomorphism it follows that $f_B$ also is 0. It is then obvious that $(0, 0)$ factors through $0 \to 0$ □

**Theorem 8.9.** *If* $\mathcal{C}$ *is an exact category with at least one object* $A \neq 0$ *, then* MM($\mathcal{C}$) *is not an abelian category.*

*Proof.* Since $A \neq 0$ then we know that $A \xrightarrow{id_A} A$ and $0 \longrightarrow A$ are objects in MM($\mathcal{C}$). We also have the morphism $(0, id_A)$ in MM($\mathcal{C}$):

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[d, "id_A"] \\
A \arrow[r, "id_A"] & A
\end{tikzcd}$$

From lemma 8.8 we have that $(0, 0)$ is the cokernel of $(0, id_A)$ while the kernel is $(0, 0)$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "id_A"] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r, "id_A"] & A \arrow[r] & 0
\end{tikzcd}$$

but it then follows that $\text{Coim}((0, id_A)) = 0 \to A$ while $\text{im}((0, id_A)) = A \to A$. It is obvious that there exist no isomorphism between these two objects, therefore MM($\mathcal{C}$) cannot be abelian. □

To understand better the structure of the homomorphism category and the monomorphism category it is useful to know what objects are simple objects in these new categories, and also what are the projective objects. This will be used later, when proving that these categories conserves JHP.

**Lemma 8.10.** *For a homomorphy category $\mathcal{G} = \mathrm{H}(\mathcal{C})$ of an exact category $\mathcal{C}$, the simple objects are the morphism on the form* $S \longrightarrow 0$ *or* $0 \longrightarrow S$ *where $S$ is a simple object in $\mathcal{C}$*

*Proof.* Assume that $f$ is a simple object: $A_1 \xrightarrow{f} A_2$ Then we can construct the following diagram:

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & A_1 \arrow[d, "f"] \\
A_2 \arrow[r, tail, "id_{A_2}"] & A_2
\end{tikzcd} \tag{23}$$

But since $f$ is simple, either $A_1 = 0$ or if $A_1 \neq 0$ then $A_2 = 0$. In the case $A_1 = 0$, this forces $A_2$ to be simple, because if not then there exist an object $0 \subset C \subset A_2$ where $C \neq 0, A_2$ and $0 \longrightarrow C$ is therefore a proper subobject of $f$. In the case of $A_1 \neq 0$, the same argument forces $A_1$ to be simple. So these objects are the only objects that can possibly be simple, and it is obvious that these objects are simple. $\square$

As the monomorphism category is a exact subcategory of the homomorphism category, $0 \longrightarrow S$ which is in the monomorphism category, is also simple objects there. The other simples in the monomorphism category are of the form $S \overset{id_S}{\rightarrowtail} S$

**Lemma 8.11.** *For a monomorphism category $\mathcal{M} = \mathrm{MM}(\mathcal{C})$ of an exact category $\mathcal{C}$, the simple objects are on the form* $0 \longrightarrow S$ *or* $S \rightarrowtail S$ *for any simple object $S$ in $\mathcal{C}$*

*Proof.* For an object $f$ in $\mathcal{M}$, for a morphism $(j_1, j_2)$ into $f$ to be an inflation, it has to not only consist of two inflations, but also the cokernel in $\mathrm{H}(\mathcal{C})$ must also be in $\mathcal{M}$.

$$\begin{tikzcd}
A_1 \arrow[r, tail, "j_1"] \arrow[d, tail, "g"] & B_1 \arrow[r, two heads, "\pi_1"] \arrow[d, tail, "f"] & B_1/A_1 \arrow[d, "h"] \\
A_2 \arrow[r, tail, "j_2"] & B_2 \arrow[r, two heads, "\pi_2"] & B_2/A_2
\end{tikzcd} \tag{24}$$

In the case of the diagram above, if all squares commute, $(j_1, j_2)$ is a morphism in the monomorphism category, but not generally an inflation. For it to be an inflation, we need that the cokernel $h$ is in the monomorphism category. In other words $h$ must also be an inflation.
So let $f\colon A \to B$ be a simple object in $\mathcal{M}$, then it is obvious that both $id_A$ and $id_C$, for any proper subobject $C$ of $A$, are subobjects of $f$. Therefore if $A \neq 0$ then $id_A$ and $id_0$ are two different subobjects of $f$ and therefore $f$ has to be

isomorphic to $id_A$.
In the case that $A = 0$, it follows that the inflation $0_C \colon 0 \to C$ for a subobject $C$ of $B$, is a subobject of $f$. Therefore $B$ has to be simple, else there exist a subobject $C$ of $B$ such that $C \neq B$ or 0 and leads to $f$ having $0_0, 0_C$ and $0_B$ as subobjects. ☐

Since we know what the simples are in the homomorphism categories, we can now show that homomorphism categories inherits JHP from its parent category.

**Theorem 8.12.** *If $\mathcal{C}$ is an exact category that have JHP, then $\mathrm{H}(\mathcal{C})$ does also have JHP.*

*Proof.* Let $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$ be an object in $\mathrm{H}(\mathcal{C})$, and consider the following two composition series of $f$:

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & A_n \arrow[r, tail] \arrow[d] & A_{n-1} \arrow[r, tail] \arrow[d] & ... \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[d] & A \arrow[d] \\
0 \arrow[r, tail] & B_n \arrow[r, tail] & B_{n-1} \arrow[r, tail] & ... \arrow[r, tail] & B_1 \arrow[r, tail] & B
\end{tikzcd} \tag{25}$$

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & A'_m \arrow[r, tail] \arrow[d] & A'_{m-1} \arrow[r, tail] \arrow[d] & ... \arrow[r, tail] & A'_1 \arrow[r, tail] \arrow[d] & A \arrow[d] \\
0 \arrow[r, tail] & B'_m \arrow[r, tail] & B'_{m-1} \arrow[r, tail] & ... \arrow[r, tail] & B'_1 \arrow[r, tail] & B
\end{tikzcd} \tag{26}$$

Since these are composition series of $f$ it follows that the gives us the simples

$$\begin{tikzcd}
A_i/A_{i+1} \arrow[d, tail] & , & A'_i/A'_{i+1} \arrow[d, tail] \\
B_i/B_{i+1} & & B'_i/B'_{i+1}
\end{tikzcd} \tag{27}$$

for $0 \leq i \leq n$. We also know that the simples are on the form $S \to 0$ or $0 \to S$ from lemma 8.10, so it follows that either $A_i/A_{i+1}$ or $B_i/B_{i+1}$ are simple for each $i$ and the other one is 0. The same is true for the other composition series. We define the set of integers $i$ such that $A_i/A_{i+1}$ is simple as $I$, and $I'$ is the set containing $i$ such that $A'_i/A'_{i+1}$ is simple. From the fact that $\mathcal{C}$ has JHP, we have then that there exist permutations $\sigma$ and $\gamma$ such that $A_i/A_{i+1} = A'_{\sigma(i)}/A'_{\sigma(i)+1}$ for $i \in I$ and $B_i/B_{i+1} = B'_{\gamma(i)}/B'_{\gamma(i)+1}$ for $i$ not in $I$. Here $\sigma$ is a permutation between $I$ and $I'$ and $\gamma$ ia a permutation between $\mathbf{Z}_{n+1} - I$ and $\mathbf{Z}_{n+1} - I'$. It follows from this that we can define a new permutation $\omega$ such that $\omega(i) = \sigma(i)$

if $i \in I$ and $\omega(i) = \gamma(i)$ if $i \notin I$ . This permutation satisfies that

$$\begin{tikzcd}
A_i/A_{i+1} \arrow[dd, tail] & & A'_{\omega(i)}/A'_{\omega(i)+1} \arrow[dd, tail] \\
& = & \\
B_i/B_{i+1} & & B'_{\omega(i)}/B'_{\omega(i)+1}
\end{tikzcd} \tag{28}$$

Therefore JHP is also satisfied by $\mathrm{H}(\mathcal{C})$ ☐

We also want to show that the monomorphism category has JHP. The way to show this is quite similiar, but it is not so obvious how to construct the permutation. Therefore we introduce a result about permutations.

**Lemma 8.13.** *Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be sets of objects from a collection $\mathcal{C}$ with a equivalence relation "$\sim$" and a permutation $\sigma$ such that $a_i \sim b_{\sigma(i)}$. Then if there exist a bijection $\gamma\colon I \to \gamma(I)$ for a subset $I$ of $\{1, 2, ..., n\}$ such that for $i \in I$ we have that $a_i \sim b_{\gamma(i)}$. Then there exist a permutation $\omega$ on $\{1, 2, ..., n\}$ such that $a_i \sim b_{\omega(i)}$ for all $i$, and $\omega(i) = \gamma(i)$ when $i \in I$.*

*Proof.* We construct $\gamma$: If $i \in I$, then $\omega(i) = \gamma(i)$. And if $i \notin I$ we use the following algorithm to decide what to map $i$ to: First we look at the candidate $\sigma(i)$, if this is not in the image of $\gamma$ then we choose $\omega(i) = \sigma(i)$. If $\sigma(i)$ is in the image of $\gamma$, we then look at the new candidate $\sigma(j)$ where $j = \gamma^{-1}\sigma(i)$, if this is also in the image of $\gamma$, then we repeat this process until we get something not in the image of $\gamma$. This means that we define $\omega(i) = \sigma(\gamma^{-1}\sigma)^k(i)$, where $k$ is the smallest integer such that $\sigma(\gamma^{-1}\sigma)^k(i)$ is not in the image of $\sigma$.
First we show that this in fact a map. We do this by shoving that this method does not repeat itself. Assume for $i \notin I$, there exist $k_1 < k_2$ such that $\sigma(\gamma^{-1}\sigma)^{k_1}(i) = \sigma(\gamma^{-1}\sigma)^{k_2}(i)$ and that $\sigma(\gamma^{-1}\sigma)^j(i)$ is in the image of $\gamma$ for $j \leq k_2$. Then it follows that $\sigma(\gamma^{-1}\sigma)^{k_2-k_1}(i) = \sigma(i)$ and by applying $\sigma^{-1}$ that $(\gamma^{-1}\sigma)^{k_2-k_1}(i) = i$ . But since $k_2 - k_1 > 0$, it follows that $(\gamma^{-1}\sigma)^{k_2-k_1}(i)$ is in $I$, but we assumed $i \notin I$, therefore it does not repeat. Then it follows from the fact that $\mathrm{im}(\gamma)$ is a finite set, that we will end up outside at one point.
Now we have shown that this is a well defined map, but we also want to show that it is a permutation. To do this we show that it is injective and surjective. Injective: Assume $\omega(i) = \omega(j)$, both $i$ and $j$ are in $I$, then it is obvious that $i = j$,. Also if $i \in I$ and $j \notin I$ then we know that $\omega(j) \notin \mathrm{im}(\gamma)$ then $\omega(i) \neq \omega(j)$. So we need to check what happens if both $i$ and $j$ is not in $I$. Then we know that $\sigma(\gamma^{-1}\sigma)^{k_1}(i) = \sigma(\gamma^{-1}\sigma)^{k_2}(j)$ for some $k_1$ and $k_2$, if $k_1 = k_2$ it follows that $i = j$ and if $k_1 < k_2$ we get the same contradiction as when we showed that it was well defined. This shows that $\sigma$ is injective.
To show that $\sigma$ is surjective, we first observe that if $i \in \mathrm{im}(\gamma)$, i=$\sigma(\gamma^{-1}(i))$. If $i$ is not in the image of $\gamma$, we do the opposite of when we defined the map. We look at $(\sigma^{-1}\gamma)^k\sigma^{-1}(i)$ and this is also non repeating, for some $k$ this is not in $I$. Then $\omega((\sigma^{-1}\gamma)^k\sigma^{-1}(i)) = i$. Hence $\omega$ is both injective and surjective, and is therefore a permutation. ☐

**Theorem 8.14.** *Let* $\mathcal{C}$ *be a exact category which have JHP, and let* $\mathcal{D} = \mathrm{MM}(\mathcal{C})$ *be the monomorphy category of* $\mathcal{C}$. *Then* $\mathcal{D}$ *does also have JHP.*

*Proof.* Let $$\begin{tikzcd} A \arrow[d, tail, "i"] \\ B \end{tikzcd}$$ be an object in $\mathcal{D}$, and consider the following two composition series of $i$:

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & A_n \arrow[r, tail] \arrow[d, tail] & A_{n-1} \arrow[r, tail] \arrow[d, tail] & ... \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail, "i"] \\ 0 \arrow[r, tail] & B_n \arrow[r, tail] & B_{n-1} \arrow[r, tail] & ... \arrow[r, tail] & B_1 \arrow[r, tail] & B \end{tikzcd} \tag{29}$$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & A'_m \arrow[r, tail] \arrow[d, tail] & A'_{m-1} \arrow[r, tail] \arrow[d, tail] & ... \arrow[r, tail] & A'_1 \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail, "i"] \\ 0 \arrow[r, tail] & B'_m \arrow[r, tail] & B'_{m-1} \arrow[r, tail] & ... \arrow[r, tail] & B'_1 \arrow[r, tail] & B \end{tikzcd} \tag{30}$$

the simples given by these composition series are on the form:

$$\begin{tikzcd} A_i/A_{i+1} \arrow[dd, tail] & & A'_i/A'_{i+1} \arrow[dd, tail] \\ & , & \\ B_i/B_{i+1} & & B'_i/B'_{i+1} \end{tikzcd} \tag{31}$$

From knowing how the simples in $\mathcal{D}$ looks like from lemma 8.11 it follows that the lower rows in both composition series are itself composition series of $B$ in $\mathcal{C}$, and since $\mathcal{C}$ has JHP, $n = m$. Also there must exist an permutation $\sigma$ such that $B_i/B_{i+1} \sim B'_{\sigma(i)}/B'_{\sigma(i)+1}$ where $B_{n+1} = B'_{n+1} = 0$ The upper row is not neccesarily a composition series, but as $A_i/A_{i+1}$ is either 0 or a simple, and the same for $A'_i/A'_{i+1}$, in $\mathcal{C}$, there must exist a permutation $\omega$ such that $A_i/A_{i+1} \sim A'_{\omega(i)}/A'_{\omega(i)+1}$. But for $i$ such that $A_i/A_{i+1} \neq 0$ it follows that:

$$B_i/B_{i+1} \sim A_i/A_{i+1} \sim A'_{\omega(i)}/A'_{\omega(i)+1} \sim B'_{\omega(i)}/B'_{\omega(i)+1} \tag{32}$$

so $B_i/B_{i+1} \sim B'_{\omega(i)}/B'_{\omega(i)+1}$ when $i$ such that $A_i/A_{i+1} \neq 0$. Using lemma 8.13 we get that there must exist a permutation $\sigma'$ such that $B_i/B_{i+1} \sim B'_{\sigma'(i)}/B'_{\sigma'(i)+1}$, and $\sigma'(i) = \omega(i)$ when $A_i/A_{i+1} \neq 0$. Then we have that

$$\begin{tikzcd} A_i/A_{i+1} \arrow[dd, tail] & & A_{\sigma'(i)}/A_{\sigma'(i)+1} \arrow[dd, tail] \\ & \sim & \\ B_i/B_{i+1} & & B_{\sigma'(i)}/B_{\sigma'(i)+1} \end{tikzcd} \tag{33}$$

As the object and the two composition series where arbitrary, this shows that $\mathcal{D}$ has JHP. □

**Proposition 8.15.** *If $\mathcal{C}$ is an exact category with enough projectives, then the projective objects in* $\mathrm{MM}(\mathcal{C})$ *is of the form* $P_1 \overset{i}{\rightarrowtail} P_2$ *where* $P_1$ *and* $P_2$ *are projectives, and* $i$ *is a split monomorphism.*

*Proof.* We first show that a projective object in $\mathrm{MM}(\mathcal{C})$ has to be an inflation into a projective object, i.e. if $A_1 \overset{i}{\rightarrowtail} A_2$ is projective, then $A_2$ has to be projective in $\mathcal{C}$. If $p$ is an deflation from $A$ to $B$, and $f$ a morphism from $A_2$ to $B$, it follows from that $i$ that there exist a morphism $d$ from $A_2$ to $A$ such that the following diagram commutes:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, tail, "i"] \arrow[rr, bend left] & 0 \arrow[d] & 0 \arrow[l, two heads] \arrow[d] \\
A_2 \arrow[r, "g"] \arrow[rr, bend right, "d"'] & B & A \arrow[l, two heads, "p"]
\end{tikzcd} \tag{34}$$

this follows from the fact that $A_1 \rightarrowtail A_2$ is projective, but this also shows that $A_2$ must be projective in $\mathcal{C}$. We then show that any projective object is a split monomorphism. $P_1 \overset{i}{\rightarrowtail} P_2$ be a projective object in the monomorphism category. The goal is to show that $\mathrm{coker}(i)$ is projective, which implies that the exact sequence $P_1 \overset{i}{\rightarrowtail} P_2 \overset{\pi}{\twoheadrightarrow} \mathrm{coker}(i)$ is split exact, and therefore $i$ is a split monomorphism. So assume we have a morphism from $\mathrm{coker}(i)$ to some object $B$ and a deflasion from $A$ to $B$: $\mathrm{coker}(i) \xrightarrow{g} B \underset{p}{\twoheadleftarrow} A$ From this we can construct the following diagram:

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d, tail, "i"] & 0 \arrow[d] & 0 \arrow[l, two heads] \arrow[d] \\
P_2 \arrow[r, "g\pi"] \arrow[d, two heads, "\pi"] & B & A \arrow[l, two heads, "p"] \\
\mathrm{coker}(i) \arrow[ur, "g"] & &
\end{tikzcd} \tag{35}$$

Then from the projective property of $i$, we get a map $d$ from $P_2$ to $A$ such that $g\pi = pd$ and $di = 0$. Since $di = 0$ the cokernel property of $\pi$ gives a morphism $h$ such that $h\pi = d$. This implies that $g\pi = pd = ph\pi$, but since $\pi$ is an deflation, and therefore an epimorphism, we have that $g = ph$, so $\mathrm{coker}(i)$ is projective and $i$ is therefore split.

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[rr, bend left] \arrow[d, tail, "i"] & 0 \arrow[d, "d"'] & 0 \arrow[l, two heads] \arrow[d] \\
P_2 \arrow[r, "f"] \arrow[rr, bend left] \arrow[d, two heads, "\pi"] & B & A \arrow[l, two heads, "p"] \\
\text{coker}(i) \arrow[ur, "g"] \arrow[urr, bend right, "h"] & &
\end{tikzcd} \tag{36}$$

We are then left to show any split monomorphism $i$ between two projecive objects in $\mathcal{C}$ is a projecive object in $\text{MM}(\mathcal{C})$. $P_1 \overset{i}{\rightarrowtail} P_2$ is projective.

$\square$

# 9 Homomorphism and monomorphism categories over quivers

In this secton, when we talk about quiver, we assume they have a finite number of nodes. We observe that the monomorphism and homomorphism category look quite similar to representations over the quiver $\Gamma$: $1 \longrightarrow 2$ , but where the objects in 1 and 2 are objects in a general category, and not finite dimensional vector spaces over a field. In fact if $\mathcal{C}$ is the homomorphsim category for finite dimensional vector spaces over a field $\mathcal{F}$, then $\mathcal{C}$ is equivalent to $\text{Rep}(\mathcal{F}, \Gamma)$. We can generalize homomorphism and monomorphism category, by looking at such representation over quivers. Note that in this section we assume that the quivers has finite number of nodes, and we look at quivers such that there exist at most 1 arrow going from a node $i$ to a node $j$, for any $i$ and $j$. We will first define some concepts for quivers.

**Definition 9.1.** *If $\gamma$ is a quiver and $\alpha$ an arrow in $\Gamma$ then we denote $s(\alpha)$ as the source of $\alpha$, i.e. where the arrow begins. Also we denote $e(\alpha)$ as the target of $\alpha$., the end of the arrow.*

Likewise we want to introduce some terminology for when a node is downstream/upstream to another node,

**Definition 9.2.** *If $\Gamma$ is quiver, then we say that $i$ is downstream of $j$ if there exist a path from $j$ to $i$ in $\Gamma$. In this case we also say that $j$ is upstream of $i$*

Note that if we have a connected acyclic quiver, then for $i$ and $j$ are different nodes and $i$ is downstream of $j$, we know that $j$ is not downstream of $i$.

**Definition 9.3.** *If $\mathcal{C}$ is a category, and $\Gamma$ is a quiver, then the homomorphism category of $\mathcal{C}$ over $\Gamma$ denoted $\text{H}(\mathcal{C}, \Gamma)$ is the category where an object $(A, f)$ is the collection of objects $A(i) \in \mathcal{C}$ for each node $i$ in $\Gamma$ and morphisms $f_\alpha \colon A(s(\alpha)) \to A(e(\alpha))$ for each arrow $\alpha \in \Gamma$. A morphism $g$ between two objects $(A, f)$ and*

*$(B, f')$ in $\mathrm{H}(\mathcal{C}, \Gamma)$ is then a collection of morphisms $g_i \colon A(i) \to B(i)$ for each node $i$ such that for any arrow $\alpha \in \Gamma$, where $i = s(\alpha)$ and $j = e(\alpha)$ we have that the following diagram commutes:*

$$\begin{tikzcd}
A(i) \arrow[r, "g_i"] \arrow[d, "f_\alpha"] & B(i) \arrow[d, "f'_\alpha"] \\
A(j) \arrow[r, "g_j"] & B(j)
\end{tikzcd}$$

Note that $\mathrm{H}(\mathcal{C})$ is on this form, namely if $\Gamma = 1 \to 2$ then $\mathrm{H}(\mathcal{C}) = \mathrm{H}(\mathcal{C}, \Gamma)$. Another example is when $\Gamma$ is the quiver: $1 \xrightarrow{\alpha} 2 \xleftarrow[\beta]{} 3$ then the homomorphism category of $\mathcal{C}$ over $\Gamma$ is the category where the objects are on the form: $A \xrightarrow{f_\alpha} B \xleftarrow{f_\beta} C$ and the morphisms are on the form $(h_1, h_2, h_3)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[d, "h_1"] & B \arrow[d, "h_2"] & C \arrow[l, "f_\beta"'] \arrow[d, "h_3"] \\
D \arrow[r, "f'"] & E & F \arrow[l, "g'"']
\end{tikzcd} \tag{37}$$

i.e. $h_2 f = f' h_1$ and $h_2 g = g' h_3$.
It can be shown that if $\mathcal{C}$ is an exact category, and $\Gamma$ a quiver with finite nodes, then $\mathrm{H}(\mathcal{C}, \Gamma)$ is an exact category where an inflation is a morphism $h \colon (A, f) \to (B, f')$ between two object $(A, f)$ and $(B, f')$ in $\mathrm{H}(\mathcal{C}, \Gamma)$ is such that for any node $i$ in $\Gamma$ the morphism $h_i \colon A(i) \to B(i)$ is an inflation. Equally $h$ is a deflation if for every $i \in \Gamma$, the morphism $h_i$ is a deflation.

The monomorphism category of $\mathcal{C}$ over $\Gamma$, called $\mathrm{MM}(\mathcal{C}, \Gamma)$ is defined similiar to $\mathrm{MM}(\mathcal{C})$ as the full subcategory where the objects are the objects $(A, f)$ in $\mathrm{H}(\mathcal{C}, \Gamma)$ such that for any arrow $\alpha$ in $\Gamma$, we have that $f_\alpha$ is an inflation. It can be shown that $\mathrm{MM}(\mathcal{C}, \Gamma)$ is extension closed subcategory of $\mathrm{H}(\mathcal{C}, \Gamma)$, and is therefore also an exact category with the subcategory structure.

When does these categories have JHP. We can show with show is that JHP does not generally holds for all of these types of quiver. We use a example to show this.

Example: Let $\mathcal{F}$ be a field and let $\mathcal{C} = \mathrm{vec}(\mathcal{F})$ be the category of any finite dimensional vectorspaces over the field. Now let $\Gamma$ be the following quiver:

$$\begin{tikzcd}
2 \arrow[r] & 1 & 4 \arrow[l] \\
& 3 \arrow[u] &
\end{tikzcd} \tag{38}$$

and let us look at the monomorphism category of $\mathcal{C}$ over $\Gamma$. Then the following object $A$ in the monomorphism category does not have unique composition factors:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "(1\,0)^T"] & \mathcal{F}^2 & \mathcal{F} \arrow[l, "(1\,1)^T"'] \\
& \mathcal{F} \arrow[u, "(0\,1)^T"] &
\end{tikzcd} \tag{39}$$

this is the case, $A$ has the composition series $0 \subset S_2 \subset A$, where $S_2$ is on the form:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "id"] & \mathcal{F} & 0 \arrow[l] \\
& 0 \arrow[u] &
\end{tikzcd}$$

and the inflation from $S_2$ to $A$ is on the form $((1,0)^T, id, 0, 0)$ Then this composition series gives us the composition factors $S_2$ and $A/S_2$, where $A/S_2$ is isomorphic to

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} & \mathcal{F} \arrow[l, "id"'] \\
& \mathcal{F} \arrow[u, "id"] &
\end{tikzcd}$$

But one can construct similar composition series with $S_2$ and $S_4$ which gives other simples composition factors, and none of the compostion factors for the composition series $0 \subset S_3 \subset A$ is isomorphic to the compostion factors for $0 \subset S_2 \subset A$. Therefore the category does not have JHP.
But there also exist quivers that does inherit JHP. The simplest example is the monomorphism category of $\mathcal{C}$. This is as stated earlier the monomorphism category of $\mathcal{C}$ over the quiver: $1 \longrightarrow 2$ , and from theorem 8.14 we have that it has JHP, if $\mathcal{C}$ has JHP.

So we start by looking at homomorphism categories over acyclic quiver, we can then show the following result.

**Proposition 9.4.** *Let $\mathcal{C}$ be an exact category, and $\Gamma$ be an acyclic quiver. Then any simple object in $\mathrm{H}(\mathcal{C}, \Gamma)$ is of the form $S^i$ for a simple object $S$ in $\mathcal{C}$ and node $i$ in $\Gamma$, where $S^i = (S^i, f)$ such that $S^i(j) = 0$ if $i \neq j$ and $S^i(i) = S$. Naturally this means that $f_\alpha = 0$ for any arrow $\alpha$ in $\Gamma$.*

*Proof.* It is obvious that $S^i$ is a simple object in $\mathrm{H}(\mathcal{C}, \Gamma)$, so we then have to show that any other object is not simple. So let $(A, f)$ be an object in $\mathrm{H}(\mathcal{C}, \Gamma)$ such that for only one node $i$ in $\Gamma$, we have that $A(i) \neq 0$. If $A(i)$ is simple, then $(A, f) = A(i)^i$. In the case that $A(i)$ is not simple, there exist an object $B \subset A(i)$ such that $B \neq A(i)$. Then $(A', f')$ is a proper subobject of $(A, f)$, where $A'(i) = B$ and $A'(j) = 0$ for any other node. So in this case, $(A, f)$ is not simple. In the case that there is at least two nodes $i$ and $j$ such that $A(i)$ and $A(j)$ are nonzero, we know from that $\Gamma$ is acyclic, that there exist a node $k$ such that $A(k) \neq 0$ but for any arrow $\alpha$ ending in $k$, we have that $A(s(\alpha)) = 0$. So we can construct a subobject $(A', f')$ such that $A'(l) = A(l)$ if $l \neq k$, and $A'(k) = 0$. This object is a proper subobject of $(A, f)$, and therefore $(A, f)$ is not a simple object. $\square$

‘This leads up the following result.

**Theorem 9.5.** *Let $\mathcal{C}$ be an exact category and $\Gamma$ be an acyclic quiver. Then* $\mathrm{H}(\mathcal{C},\Gamma)$ *has JHP, if and only if $\mathcal{C}$ has JHP.*

*Proof.* Assume $\mathcal{C}$ has JHP, and let $(A, f)$ be an object in $\mathrm{H}(\mathcal{C},\Gamma)$. Then for a composition sequence $Z$ of $(A, f)$ were the length we call $n$. Then the get composition factors $Z_k$ for $0 < k \leq n$ from $Z$ is of the form $S^i$ for some simple $S \in \mathcal{C}$ and node $i \in \Gamma$. For each node $j \in \Gamma$, we can look at the composition factors $Z_k$ such that $Z_k = S_k^j$ for some simple $S_k$ in $\mathcal{C}$. Then the collection of all $S_k$ for $k$ such that $Z_k = S_k^j$ are the composition factors of a composition series of $A(j)$ and since $\mathcal{C}$ has JHP, they are unique. This forces the composition factors of the $S^j$ to be unique, but this argument works for any node, so we get that $\mathrm{H}(\mathcal{C},\Gamma)$ has JHP. ☐

Then we turn our attention to monomorphism categories. We can begin by defining a type of simple objects.

**Definition 9.6.** *Let $\mathcal{C}$ be an exact category, and $\Gamma$ a acyclic quiver. Then for any node $i$ and simple object $S$ in $\mathcal{C}$, we define $S^i$ in* $\mathrm{MM}(\mathcal{C},\Gamma)$ *to be the object such that if $S^i = (A, f)$, then $A(j) = S$ for any node $j$ such that there exist a path from $i$ to $j$ in $\Gamma$, $A(i) = S$, and in any other node $S^i$ is* $0$. *$f_\alpha$ is then identity morphism if there is a path from $i$ to both $s(\alpha)$ and $e(\alpha)$, and* $0$ *for any other arrow.*

**Proposition 9.7.** *If $S$ is a simple object in $\mathcal{C}$ and $i$ a node in $\Gamma$, then $S^i$ is a simple object in* $\mathrm{MM}(\mathcal{C},\Gamma)$.

*Proof.* Assume $A = (A, f) \neq 0$ is a subobject of $S^i$. Let $j$ be a node such that there is a path from $i$ to $j$ and such that $A(j) = 0$. From this we get that also $A(i) = 0$. From this we get that $S^i/A$ is such that $(S^i/A)(i) = S/A(i) = S$, but this forces any node with path from $i$ to also be $S$, but then $(A, f)(j) = 0$ for any node $j \in \Gamma$, which is a contradiction. Therefore $(A, f)(j) = S$ for any node $j$ with a path from $i$, but this is the object $S^i$. ☐

Let us look at an example of a object on this form. Example: Let

$$\begin{tikzcd}
2 & 3 & 4 \\
 & 1 \arrow[ul] \arrow[u] \arrow[ur] & 5 \arrow[u]
\end{tikzcd}$$

be the quiver, and $S$ be a simple object in the parent category, then $S^1$ is on the form

$$\begin{tikzcd}
S & S & S \\
 & S \arrow[ul, tail, "id"] \arrow[u, tail, "id"] \arrow[ur, tail, "id"] & 0 \arrow[u]
\end{tikzcd}$$

The reason we use the notation $S^i$ for these simple objects, are because they are generated from a simple $S$ in node $i$. This we say because if we have $S$ in node $i$, then from the requirement that it is in $\mathrm{MM}(\mathcal{C}, \Gamma)$ we have to have inflations to any node $j$ with a path from $i$, so it follows that the object in node $j$ has to have $S$ as a subobject. These may not be the only simple objects there are, but we can show that if these are the only simple objects, then it has JHP if its parent category has JHP.

**Theorem 9.8.** *Let $\mathcal{C}$ be an exact category and $\Gamma$ be an acyclic quiver. Then $\mathrm{MM}(\mathcal{C}, \Gamma)$ has JHP if $\mathcal{C}$ has JHP and any simple object in $\mathrm{MM}(\mathcal{C}, \Gamma)$ is on the form $S^i$ for a simple $S \in \mathcal{C}$ and node $i$ in $\Gamma$*

*Proof.* The way to prove this is not too different from the proof of theorem 9.5, but adjusted slightly. It also has some similarities to the proof of theorem 8.14. Let $\mathcal{C}$ have JHP, and assume any simple object in $\mathrm{MM}(\mathcal{C}, \Gamma)$ is on the form $S^i$. Assume $(A, f)$ is a object with a composition series $Z$, where $n$ is the length of $Z$. Let $Z_j$ be the composition factors of $Z$, so $Z_j = S^i$ for some simple $S$ in $\mathcal{C}$ and node $i$ in $\Gamma$. So we begin by looking at a node $i$ such that there is no other node with path to $i$. It follows that the composition factors $Z_k = S_k^i$ is such that $S_k$ is the composition factors of $A(i)$, and therefore are unique since $\mathcal{C}$. After we have considered every node $i$ such that there is no other node with paths to $i$, we look at nodes $i$ such that the nodes with paths to $i$ is already been looked at, then the $Z_k = S^i$ are the composition factors of $A(i)$ together with any $Z_k = S^j$ from any node $j$ with path to $i$. Since these last simple objects are unique, we can use a variant of lemma 8.13 to show that these $Z_k = S^i$ are also unique. Therefore $\mathrm{MM}(\mathcal{C}, \Gamma)$ has JHP ☐

In fact using a similar argument as in theorem 8.14, one can show that for a quiver on the form: $1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n-1 \longrightarrow n$ the monomorphism category over this quiver has JHP if the parent category has JHP. It is also possible to show this by showing that every simple is of the form $S^i$. We show the following result.

**Theorem 9.9.** *Let $\Gamma$ be a quiver with finite nodes of the following form:*

$$\begin{tikzcd}
2 & 3 & \ldots & n-1 & n \\
 & & 1 \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

*Then the monomorphism category $\mathrm{MM}(\mathcal{C}, \Gamma)$ has JHP, if and only if $\mathcal{C}$ has JHP.*

*Proof.* We will show this for the quiver with four nodes, but the same arguments work in the general case. So we have this quiver:

$$\begin{tikzcd}
2 & 3 & 4 \\
 & 1 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We want to show that any simple is of the form $S^i$ for some simple $S$ and node $i$. Then theorem 9.8 gives us that JHP is inherited from the parent category. Assume we have a object $Q$ in the monomorphism category which is the following:

$$\begin{tikzcd}
B & C & D \\
& A \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

When is this object simple? First we assume that $A = 0$. If atleast two of either $B$, $C$ or $D$ is nonzero, let say $B$ and $C$, then the object that is zero in every node execpt in node 2 where it is $B$

$$\begin{tikzcd}
B & 0 & 0 \\
& 0 \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

is a subobject of $Q$. So only one of $B$, $C$ and $D$ can be nonzero. Assume without loss of generality that this is $B$. Then if $B' \subset B$ is a subobject of $B$, then

$$\begin{tikzcd}
B' & 0 & 0 \\
& 0 \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

is a subobject of $Q$. So if $Q$ is simple, then $B$ has to be a simple $S$. But then $Q = S^2$.

But what if $A \neq 0$. Then if $A$ is not simple and $A'$ is a proper subobject of $A$, the object

$$\begin{tikzcd}
B & C & D \\
& A' \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

is then a subobject of $Q$. If $A$ is simple, we know that $A$ is a subobject of $B$, $C$ and $D$. If any of these are not simple, assume without loss of generality $B$ is not simple, then there exist a subobject $B'$ of $B$ such that

$$\begin{tikzcd}
B' & C & D \\
& A \arrow[ul, tail] \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

is subobject of $Q$. So for $Q$ to be simple $A$, $B$, $C$ and $D$ has to all be isomorphic to some simple $S$ and therefore $Q = S^1$. This proves that all simples are on the form $S^i$. □

**Theorem 9.10.** *Let $\mathcal{C}$ be an exact category and let $\Gamma$ be quiver on the following form:*

$$\begin{tikzcd}
 & & 1 & & \\
2 \arrow[urr, tail, "\alpha_2"] & 3 \arrow[ur, tail, "\alpha_3"'] & ... \arrow[u, tail] & n-1 \arrow[ul, tail, "\alpha_{n-1}"] & n \arrow[ull, tail, "\alpha_n"']
\end{tikzcd}$$

*where $n \geq 3$. Then* $\mathrm{MM}(\mathcal{C}, \Gamma)$ *does not have JHP.*

*Proof.* If $\mathcal{C}$ has no simple objects then the monomorphism category will also have no simple objects so it does not make sense to talk about JHP. So assume $\mathcal{C}$ has at least one simple $S$. Then we look at $Q = (Q, f)$ where $Q(1) = S^2$ and $Q(2) = Q(3) = S$ and $Q(j) = 0$ for any remaining nodes $j$. Then the inflation $f_{\alpha_2} = (1, 0)^T$ and the inflation $f_{\alpha_3} = (0, 1)^T$. Firstly we observe that there exist the composition series: $0 \rightarrowtail S^2 \overset{i}{\rightarrowtail} Q$ where $i$ is the inflation with the identity map between $S^2(2)$ and $Q(2)$, and the inflation $(1, 0)^T$ between $S^2(1)$ and $Q(1)$. This composition series gives us the composition factors $S^2$ and $S^3$. But there is also another composition series: $0 \rightarrowtail S^1 \overset{j}{\rightarrowtail} Q$ here $j$ is the inflation were the inflation between $S^1(1)$ and $Q(1)$ is $(1, 1)^T$. This composition series gives us the compositon factors $S^1$ and but also another simple $Z$, where $Z(i) = S$ for the nodes $1, 2$ and $3$ and zero elsewhere. Since these are not the same simples as the first composition series, it does not have JHP. ☐

We see from the last proof, that the existence of a simple object which is not on the form $S^i$ stops the monomorphism category from having JHP. This depends on the quiver we have, and we will now define a type of quivers that is such that any monomorphism category over the quiver has only simple of the form $S^i$

**Definition 9.11.** *We call a quiver $\Gamma$ a tree quiver if there exist a node $i$ in $\Gamma$ such that for every other node $j \in \Gamma$ there exist exactly $1$ path from $i$ to $j$. We call the node $i$ the root of the tree quiver $\Gamma$*

Note that a tree quiver must be acyclic, because a cycle would imply multiple paths to any node in the cycle. also we can show that for any node $j$ in a tree quiver there is at most 1 arrow ending in $j$.

**Lemma 9.12.** *If $\Gamma$ is a tree quiver and $i$ is a node in $\Gamma$, then there is at most one arrow ending in $j$.*

*Proof.* Assume that there exist a node $i \in \Gamma$ such that both arrows $\alpha$ and $\beta$ ends in $j$. We call the startnode of $\alpha$ and $\beta$ for $i_\alpha$ and $i_\beta$. Then from the definition of a tree quiver we have that there exist a path from the root $i$ to $i_\alpha$ and $i_beta$, denoted $\omega_\alpha$ and $\omega_\beta$. It then follows that both $\alpha\omega_\alpha$ and $\beta\omega_\beta$ are differents paths to $j$ but cannot be true, so we have a contradiction. Therefore there exist no $j$ with two (or more) arrows ending in $j$. ☐

This shows us that for a tree quiver, there is no subsection that "looks like" the quivers in theorem 9.10.

**Theorem 9.13.** *Let $\mathcal{C}$ be an exact category that have JHP and $\Gamma$ be acyclic quiver such that between two nodes there is at maximum 1 arrow. Then the monomorphism category of $\mathcal{C}$ over $\Gamma$ has JHP if and only $\Gamma$ is a tree quiver.*

*Proof.* We first show that if every node in $\Gamma$ has at most 1 incoming arrow, then all simples are on the form $S^i$ for some simple object $S$ in $\mathcal{C}$ and a node $i$. This we show by an induction argument. $\Gamma$ has to have a node that is upstream to every other node. Lets call this node 1. Every simple that is nonzero in node 1 must then also be nonzero in every node downstream of 1. This forces any node to be nonzero. It is also easy to show that the only simples on this form is the simples with the same simple object $S \in \mathcal{C}$ in every node. This is by definition $S^1$.
Then assume we know that this is true for any upstream node of $k$, and we want to show that it is also true for $k$. So assume we have a simple object $Q$ such that $Q(k) \neq 0$. If for any upstream node $i$ of $k$, we have that $Q(k) \neq 0$, we know that $Q = S^j$ for some $j$. In the case that $Q(i) = 0$ for any upstream node $i$ we study the following cases: We know that the simples that are nonzero only in the nodes downstream of $k$ must be of the form $S^k$ for some simple $S$. So assume we have a simple $Z$ which is nonzero both in $k$ and in some other node $j$ not downstream of $k$. If $j$ is upstream of $k$, we know that this simple has to be on the form of $S_i$ for some simple $S$ and node $i$. If $j$ is not upstream of $i$, then there exist a node $j'$ such that $j'$ is upstream of both $i$ and $j$ and any other node upstream of both $i$ and $j$ is also upstream of $j'$. If the object in node $j'$ is nonzero, then we again have that this simple is on the form $S_i$. In the case where the object in node $j'$ is zero, then we get that $S^k$ is an subobject of $Z$, $S^k$ is even a direct sum of $Z$. But $Z$ is not isomorphic to $S^k$ so $Z$ is not simple. This proves that any simple is of the form $S^i$. Using**??** this again implies that the monomorphism category over $\Gamma$ has JHP.
We are then left with proving that if there exist a node with at least two incoming arrow, then the monomorphism category over $\Gamma$ does not have JHP. So assume node $i$ has two arrows into it from node $j$ and node $k$ and choose a simple $S$ in $\mathcal{C}$. Let $Q$ be the object that is $S$ in $j$ and $k$ and $S^2$ in $i$. It is $S$ in every node downstream of $j$ or $k$ but not $i$, and $S^2$ in every node downstream of $i$. The inflation from $j$ to $i$ is $(0,1)^T$ and the inflation from $k$ to $i$ is $(1,0)^T$. Then by using the same argument as in theorem 9.10 we get that JHP is not satisfied for $Q$. ☐

Before we have only looked at acyclic quivers, but we may also consider cyclic quivers. If we look at homomorphism categories over quivers with cycles, we get more simple objects than the simples on the form $S^i$. Note that these now are the simples with is only nonzero in node $i$. If we call a cycle in the quiver $\gamma$, we can define a simple object $S^\gamma$ as the simple object with $S$ in the cycle, 0 elsewhere and identity map for arrows in $\gamma$. But there may also be more and potentially more complex simple objects as a consequence of the cycles. An object with simple objects in each node in $\gamma$ and 0 elsewhere is also a simple object if any morphism in the cycle is not the zero-morphism.

**Theorem 9.14.** *Let* $\mathrm{H}(\mathcal{C},\Gamma)$ *be the homomorphism category of a exact category* $\mathcal{C}$*, over a finite quiver* $\Gamma$*. Then for any cycle* $\gamma$ *in* $\Gamma$ *and simple object* $S$ *in* $\mathcal{C}$ *them the object* $S^{\gamma}$ *as defined above is a simple object in* $\mathrm{H}(\mathcal{C},\Gamma)$*.*

*Proof.* Since $\gamma$ is a cycle in a finite quiver, it has a finite number of nodes. Let us number the nodes in the cycle from 1 to $n$ such that 1 has arrow into 2, 2 has arrow into 3 and so fourth until $n$ which has arrow into 1. Let $Z = S^{\gamma}$ be the object such that the object in node $i$ denoted $Z(i) = S$ and that the morphism from $Z(i)$ to $Z(i+1)$, and $Z(n)$ to $Z(1)$ is the identity. Then Assume $A$ is a subobject of $Z$ which is not 0. This implies that there exist a node $k$ such that $A(k) \neq 0$, so $A(k)$ is isomorphic to $Z(k) = S$. Then we get the following commutative diagram:

$$\begin{tikzcd}
A(k) \arrow[r, "a_k"] \arrow[d, tail, "f_i"] & A(k+1) \arrow[d, tail, "f_{k+1}"] \\
Z(k) \arrow[r, "id_S"] & Z(k+1)
\end{tikzcd} \tag{40}$$

Since $f_k$ is an isomorphism and $z_k$ is the identity morphism, the composition $z_k f_k = f_{k+1} a_k$ is also an isomorphism. But this forces $A(k+1)$ to be nonzero, so $A(k+1)$ must be isomorphic to $Z(k+1)$. Repeating this argument again gives us that $A(k+1)$ is isomorphic to $Z(k+2)$ and so on. This in total gives us that $A(i)$ is isomorphic to $Z(i)$ for every $i$ in the cycle. But this object is clearly isomorphic to $Z$ as each $f_i$ is an isomorphism. Therefore the only subobject of $Z$ is itself and 0, so it is a simple object. ☐

For an arbitrary exact category there may exist a morphism $f\colon S \to S'$ between two nonisomorphic simple objects such that $f$ is not the zero morphism. It is not hard to find a such a category which does not satisfy this.
An example is looking at the subcategory generated by direct sums of $P_1$ and $P_2$ for the representation over the quiver $1 \to 2$ . From an earlier section we found that this was an ${}^{\perp}U$ category, with the simples $P_1$ and $P_2$. What is clear though is that there exist a nonzero morphism from $P_2$ to $P_1$ although they are not isomorphic.
These types of morphisms opens up the possibility for other simple objects for homomorphism categories over quivers with cycles. We present the following result.

**Proposition 9.15.** *Let* $\mathcal{C}$ *be a homomorphism category of a exact category, over a finite quiver* $\Gamma$*. If there exists simple objects* $S_i$ *and nonzero morphisms* $\beta_i\colon S_i \to S_{i+1}$*, for* $i = 1, 2, ..., n$ *where* $S_{n+1} = S_1$ *then for any cycle* $\gamma$ *in* $\Gamma$ *with length equal to n,(and bigger) we can create a simple object with this sequence.*

*Proof.* If we number the nodes in $\gamma$ as in the proof for theorem 9.14, and define the object $Z$ such that $Z(i) = S_i$, and the arrow from $Z(i)$ to $Z(i+1)$ is $\beta_i$. Any other node or arrow is 0. Then $Z$ is an simple object in $\mathcal{C}$. This is shown in the same way as in theorem 9.14. (It may not be neccesary that $S_i$ is simple) ☐

Note that if we have collection of $n$ simples, with the nonzero maps, we can always construct a collection with $n+1$ simples, where the last object $S_{n+1} = S_n$ with the identity morphism between them.
Observe that for a cycle $\gamma$ in $\Gamma$ and a $S$ in the parent category, we can always construct the simple $S^\gamma$ where we use the identity morphism. It is more interesting to find categories such that not every morphism in the cycle-simple is an isomorphism. From our earlier example we have shown that there is a non-isomorphic, nonzero morphism between the simples $P_2$ and $P_1$ for the subcategory ${}^\perp U$. But there is no nonzero morphisms from $P_1$ to $P_2$ so we have to look at another category to find such a cyclesimple.
If we look at the quiver $\Gamma$ looking like this:

$$\begin{tikzcd}
1 \arrow[dr] & \\
3 \arrow[u] & 2 \arrow[l]
\end{tikzcd}$$

Then by looking at the category $\mathcal{C}$ containing representations of a field $\mathcal{F}$ over $\Gamma$ modding out any path of length 2 or more and define the exact structure to be the split exact sequences. The indecomposable projective modules $P_i$ are then such that the object in node $i$ and $i + 1$ is $\mathcal{F}$ with the identity map between them, and the last node is 0. Since these are indecomposable objects in $\mathcal{C}$, they are simple with split exact structure. Then for $\mathrm{H}(\mathcal{C}, \Gamma)$ we can construct the following simple object:

$$\begin{tikzcd}
P_1 \arrow[dr, "f_1"] & \\
P_2 \arrow[u, "f_2"] & P_3 \arrow[l, "f_3"]
\end{tikzcd}$$

where the morphism $f_1$ from $P_1$ to $P_3$ is like this:

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, "id"] \arrow[rrr, bend left, "id"] & & & \mathcal{F} \arrow[dr] & \\
0 \arrow[u] \arrow[rrr, bend right] & \mathcal{F} \arrow[l] \arrow[rrr, bend right] & & \mathcal{F} \arrow[u, "id"] & 0 \arrow[l]
\end{tikzcd}$$

and $f_2$ and $f_3$ act in a similar way. Then it follows from proposition 9.15 that this is a simple object in $\mathrm{H}(\mathcal{C}, \Gamma)$.

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NTNU

Norwegian University of
Science and Technology