Doctoral theses at NTNU, 2023:270

Vegard Topphol

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An Exploratory Inquiry

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
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ISBN 978-82-326-7240-0 (printed ver.)
ISBN 978-82-326-7239-4 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2023:270

Printed by NTNU Grafisk senter

Preface

This doctoral work was financed by the Norwegian University of Science and Technology (NTNU), partly by funding from NTNU Teaching Excellence and partly by funding from the Department of Mathematical Sciences. I am grateful for all the resources provided by the university and for all the help and support by the staff at both NTNU and the Department of Mathematical Sciences while being employed here at the department. The willingness to always help me when I needed it has been invaluable.

First and foremost, I will thank my two supervisors, Heidi Strømskag and Frode Rønning, for all the help and support in this work. You have shared with me of your knowledge and insights, you have been sparring partners and critical voices when necessary, and you have supported me and believed in me when I may not have had the same faith in this project. Overall, it has been a pleasure and honour to get to know you, and I will thank you for all the conversations we have had the last five years. I deeply appreciate you both as my supervisors and as persons!

I will also thank everyone who have contributed by providing data. The data provided by the students who gave of their time for the interviews have been indispensable, and I greatly appreciate this. Not only has it provided valuable insights, but the conversations have been interesting, and I feel fortunate to have been able to meet you all. I wish you all the best in your further studies and careers. I also feel a great gratitude towards everyone who have been involved in the writing and publishing of the textbooks that have been analysed in this work. Thank you for allowing me to use these books in my research. I have great respect for the work you do as authors, editors, and illustrators, and I hope I have been able to demonstrate this in my treatment of your works.

My gratitude goes out to the scientific community that I have been introduced to. I will thank all who have been critical voices, and you have provided valuable advice and insights. These include fellow PhD students and lecturers in the courses I attended at the University of Agder, Carl Winsløw who received me as a guest during my stay at the University of Copenhagen, and everyone who has shown interest in my work at the conferences I have attended: INDRUM 2020, NORMA 20, and CITAD 7.

I also want to thank everyone at the Department of Mathematical Sciences for providing an amazing inclusive environment. I want to thank the "lunch group", who met every day at the 13th floor, and for the weird, interesting, and always engaging conversations, and for the excellent social life. In particular, I will thank my office mate, Kristian Aga, for all support, critical questions and discussions, and distractions whenever needed.

At last, I will thank my family. To my siblings, Guro, Dagny Karin, Birger and Rolf Arild I owe a great thank you for always being there for me. You have always shown your support and interest for what I have been doing, and you have always shown that you had faith in me through the whole PhD work. I would have been a lesser man without you!

To my parents, Kjersti and Arne Kåre, I owe everything! Thank you for all support, love, and care in times when I have needed it the most. Thank you for showing me the advertisement for the PhD position and thank you for encouraging me to apply for the position. Thank you both for always believing in me and for your unconditional love. I love you deeply!

The heart of the discerning acquires knowledge, for the ears of the wise seek it out.

Proverbs 18:15

Vegard Topphol

Trondheim, Norway

April, 2023

Abstract

All teaching and learning are governed by conditions and constraints that encourage or discourage certain modes of study, inquiry, teaching, and learning. These conditions and constraints affect institutional positions in different ways (Chevallard, 2020a). In this dissertation, I report on research, presented in four papers, that identifies conditions and constraints that govern mainly the student position. But it also touches upon conditions and constraints for the positions of teacher and for textbook author.

Two different theoretical frameworks are applied in the dissertation. The initial study was designed and analysed informed by the Instrumental Approach (Rabardel & Samurçay, 2001; Trouche, 2004). But to expand the scope of the study to allow for an analysis that accounts for the transition of knowledge between different institutions, the switch was made to applying the Anthropological Theory of the Didactic (Chevallard, 2019).

The study uses several sources of data and methodical approaches, all aimed at identifying *different* sets of conditions and constraints that govern the teaching and learning of calculus themes. The mathematical theme in three of the papers in the study is centred on the Fundamental Theorem of Calculus (FTC) and integration, and one paper is mainly centred on the theme of mathematical modelling. Of the three papers where the FTC is central, two of them were mainly based on data from a series of task-based interviews with six first-year university calculus students, and one paper is based on a didactic transposition analysis (Chevallard & Bosch, 2014) relating to a chapter on integration in a Grade 13 mathematics textbook. The last of the four papers in the dissertation is a presentation and analysis of a Study and Research Path (SRP) (Chevallard, 2020a).

Four distinct sets of conditions and constraints were identified in the analyses. The first set of conditions relates to the use of tasks by students as instruments for developing mathematical competence. The research indicates that an awareness of how mathematical tasks can be modified and created, and that the students themselves get the opportunity to practice both modifying and posing of mathematical tasks are both conditions for the development of mathematical competence. The second set of conditions and constraints relates to the FTC. The lack of a concept of an *area function*, algebraically expressed as $A(x) = \int_a^x f(t)dt$, hinders the development of techniques and theory that are necessary for solving one of the interview questions. Similarly, the lack of the idea of using *general functions* rather than *specific functions* in investigating mathematical problems made two of the other interview tasks impossible to solve. The third set of conditions and constraints appears in the analysis of the

textbook *Matematikk R2* (Borge et al., 2022). The lack of the notion of *boundedness of functions*, and only a superficial treatment of *integrability* seem to constrain the expansion of the praxeology of integration in the textbook. Particularly, a set of three tasks found in the textbook, which could otherwise have served as the foundation for an expansion of the theme of integration, remain as separate, seemingly unrelated phenomena. The last set of conditions and constraints relate to the posing of a generating question in an SRP. In particular, the findings indicates that the set of preconditions, and expectations about the nature of a finally accepted answer to the generating question, both form important conditions and constraints that govern the course and outcome of an SRP.

In general, this dissertation contributes to a better understanding of the conditions and constraints that govern the dissemination of integral calculus, in particular the FTC. It also provides a foundation for asking further questions about the responsibilities of the institutional positions of students and teachers, and about the role of textbooks under the new paradigm of questioning the world.

Samandrag

All undervisning og læring er styrt av vilkår og avgrensingar som motiverer eller hindrar visse former for studie, undersøking, undervisning og læring. Desse vilkåra og avgrensingane påverkar institusjonelle posisjonar på ulike måtar (Chevallard, 2020a). I denne avhandlinga rapporterer eg frå forsking, presentert i fire artiklar, som identifiserer vilkår og avgrensingar som styrer hovudsakleg studentposisjonen. Men den rører også ved vilkår og avgrensingar for posisjonane som lærar og som lærebokforfattar.

To ulike teoretiske rammeverk er nytta i avhandlinga. Den fyrste studien var designa og analysert ut i frå den Instrumentelle Tilnærminga (Rabardel & Samurçay, 2001; Trouche, 2004). Men for å utvide omfanget av studien til også å tillate ei analyse som inkluderer transposisjon av kunnskap mellom institusjonar, vart det teoretiske rammeverket endra til den Antropologiske Teorien for det Didaktiske (Chevallard, 2019).

Studien nyttar fleire datakjelder og metodar, som alle siktar på å identifisere *ulike* grupper av vilkår og avgrensingar som styrer undervising og læring av tema innanfor kalkulus. Det matematiske temaet i tre av artiklane i studien er knytt til Analysens Fundamentalteorem (AFT) og integrasjon, og ein artikkel er hovudsakleg knytt til temaet matematisk modellering. Av dei tre artiklane der AFT er sentralt, så er to av artiklane hovudsakeleg basert på data frå ein serie med oppgåvebaserte intervju med seks fyrsteårsstudentar i analyse, og ein artikkel er basert på ein didaktisk transposisjonsanalyse (Chevallard, 1989; Chevallard & Bosch, 2014) av eit kapittel i ei VG3 lærebok i matematikk. Den siste av dei fire artiklane i avhandlinga er ein presentasjon og analyse av ei Studie og Forskingsløype (SFL) (Chevallard, 2020a).

Fire distinkte sett av vilkår og avgrensingar vart identifiserte i analysane. Det fyrste settet med vilkår og avgrensingar relaterer seg til studentars bruk av oppgåver som instrument for å utvikle matematisk kompetanse. Forskinga indikerer at medvit om korleis matematiske oppgåver kan modifiserast og lagast, og det at studentane får moglegheit til å øve på å modifisere og lage matematiske oppgåver sjølve er begge vilkår for å utvikle matematisk kompetanse. Det andre settet av vilkår og avgrensingar er relatert til AFT. Mangelen på eit konsept om ein *arealfunksjon*, uttrykt algebraisk som $A(x) = \int_a^x f(t)dt$, hindrar utviklinga av teknikkar og teori som er nødvendige for å løyse eitt av intervjuspørsmåla. Ein liknande mangel på idéen om å bruke ein *generell* funksjon i staden for ein *spesifikk funksjon* i utforskinga av matematiske problem gjorde to av dei andre intervjuspørsmåla umoglege å løyse. Det tredje settet av vilkår og avgrensingar kjem fram i analysen av læreboka *Matematikk R2* (Borge et al., 2022). Mangelen på omgrepet *avgrensing av funksjonar*, og berre ei overflatisk behandling av

integrerbarheit, ser ut til å avgrense ei utviding av prakseologien for integrasjon i læreboka. Heilt konkret, så gjer dette at tre oppgåver, som elles kunne ha danna grunnlaget for ei utviding av temaet integrasjon, står som separate, tilsynelatande urelaterte fenomen. Det siste settet av vilkår og avgrensingar er relatert til det å stille eit genererande spørsmål i ei SFL. Funna indikerer særskild at settet av forutsetningar, og forventingar om karakteren av eit endeleg akseptert svar på det genererande spørsmålet begge dannar viktige villkår og avgrensingar som styrer gangen i og resultatet av ei SFL.

Generelt bidreg denne avhandlinga til ei betre forståing av vilkår og avgrensingar som styrer formidlinga av integralrekning, særleg AFT. Den dannar også eit grunnlag for å stille vidare spørsmål om ansvaret som ligg i dei institusjonelle posisjonane som studentar og lærarar, og om rolla til lærebøker under det nye paradigmet om å stille spørsmål til verda.

Abbreviations

ATD – Anthropological Theory of the Didactic

FTC – Fundamental Theorem of Calculus

IA – Instrumental Approach

PQW – Paradigm of Questioning the World

PVW – Paradigm of Visiting Works

SRP – Study and Research Path

Publications

Throughout the dissertation, the following numbering will be used to refer to the papers which is the foundation for the dissertation. This ordering is not strictly temporal, but rather thematic, in the order which they appear in the main argument of the dissertation.

P_1: Topphol, V. (2021). A novel application of the instrumental approach in research on mathematical tasks. In G. A. Nordtvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankova-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Preceedings of NORMA 20: The Ninth Nordic Conference on Mathematics Education* (pp. 265–272). NCM & SMDF.

P_2: Topphol, V., & Strømskag, H. (2022). An analysis of a first-year university student's construction of a praxeology of integration tasks. Manuscript submitted to the journal *Teaching Mathematics and Its Applications*. Department of Mathematical Sciences, Norwegian University of Science and Technology.

P_3: Topphol, V. (2023). Didactic transposition of the Fundamental Theorem of Calculus. Manuscript submitted to the journal *REDIMAT, Journal of Research in Mathematics Education*. Department of Mathematical Sciences, Norwegian University of Science and Technology.

P_4: Topphol, V. (accepted). A study and research path on hyperthermia in children left in parked cars. In *Proceedings of the 7th International Conference of the Anthropological Theory of the Didactic.* Birkhäuser.

Table of Contents

1 Introduction 3
1.1 Didactic Themes Discussed in the Papers 3
1.2 Questions and a New Didactic Paradigm 4
1.2.1 The Notions of Mathematical Questions and Tasks 4
1.2.2 The Paradigm of Visiting Works 5
1.2.3 A New Counter Paradigm 5
1.3 The Fundamental Theorem of Calculus 6
1.3.1 Statement of the FTC 7
1.3.2 Notes on the Conditions for the FTC 8
2 Theoretical Tools Used in the Dissertation 9
2.1 The Anthropological Theory of the Didactic 9
2.1.1 Persons, Institutions and Institutional Positions 10
2.1.2 Conditions and Constraints 10
2.1.3 Praxeologies: Models of Knowledge 10
2.1.4 Model of Didactic Moments 12
2.1.5 Didactic Transposition: Institutional Relativity of Knowledge 13
2.1.6 Study and Research Paths: A Methodology for Questioning the World . 15
2.2 The Instrumental Approach 17
2.3 Networking of Theories 18
2.3.1 Networking Methodologies Using the ATD 18
2.3.2 A Change of Theoretical Framework From the IA to the ATD 19
2.3.3 Contributions From the Instrumental Approach 21
3 Research Questions 23
4 Methodology 25
4.1 Emergence of the Published Papers 25
4.2 Data Material and Data Collection 26

4.3 Research Participants 29
4.4 Data Analysis 29
4.4.1 Analysis of a Process of Instrumental Genesis 29
4.4.2 Analysis Using the Model of Didactic Moments 30
4.4.3 Didactic Transposition Analysis 32
4.4.4 Analysis of the Construction of an SRP 33
4.5 Strengths and Limitations of the Reported Research 34
4.6 Ethical Considerations 35
5 Results and Discussion 37
5.1 Summary of Results From Individual Papers 37
5.1.1 Discussion and Conclusion From P_1 37
5.1.2 Discussion and Conclusion From P_2 38
5.1.3 Discussion and Conclusion From P_3 39
5.1.4 Discussion and Conclusion From P_4 40
5.2 General Discussion 41
5.2.1 Conditions and Constraints Related to the Student Position 42
5.2.2 Conditions and Constraints Related to the Teacher and Textbook Author Positions 43
5.3 Conclusions and Open Questions 44
5.3.1 General Conclusions 44
5.3.2 Open Questions 45
References 47
Papers 55

1 Introduction

All human activities are governed by *conditions* and *constraints* that either permit, facilitate, hinder, or even prevent certain actions (Chevallard, 2020b; Chevallard & Bosch, 2020), and so also with teaching and learning situations, no matter the scale of the situation. Classroom situations, where students are solving tasks individually or in groups, self-study situations with only the student present, or tutoring situations with a single student and a single teacher are all governed by conditions and constraints unique to the specific situation. Even the production of teaching materials, like tasks, lesson plans, and textbooks are governed by their own conditions and constraints.

This dissertation builds on four different papers, where a common theme is the identification of conditions and constraints that govern different aspects of both study processes and the production of study materials. The work presented is multifaceted, with multiple sources of data, including student interviews, mathematics textbooks, and a report from a small-scale Study and Research Path (SRP). The Instrumental Approach (IA) (Rabardel & Samurçay, 2001; Trouche 2004) was used during the initial design of the research, and in the first paper (P_1), but for reasons that will be discussed in Section 2.5, a switch was made to the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019, 2020).

1.1 Didactic Themes Discussed in the Papers

The common strand that runs through all four papers are the notions of *conditions and constraints* (Chevallard, 2020b; Chevallard & Bosch, 2020). The precise definitions of these terms, and their relations to other notions will be described in Section 2.1.1. But in short, a condition is any aspect of a system that influences something within the system, while a constraint is a condition that seems unchangeable from within the frame, or the system considered.

Three different types of systems are under consideration here. In papers P_1 and P_2, student interviews, where individual students are given a set of four mathematical tasks, are investigated. The system can therefore be described as the individual asking and solving mathematical tasks. In P_2 the focus is on the student and the conditions and constraints that govern the solving process. While that is also an aspect of P_1, an equally prominent aspect is the production of the interview tasks and what conditions and constraints governed the design choices. In P_3 the chapter on integration, found in a Grade 13 textbook (Borge et al., 2022) has been analysed with a specific focus on the Fundamental Theorem of Calculus (FTC). There, conditions and constraints which govern the design outcomes, as well as conditions and constraints imposed by the design choices themselves, are identified. These are conditions and constraints that influence what types of tasks can be found in the textbook, and likely also how these tasks can be used and connected in a teaching and learning situation. In P_4, the conditions and constraints that govern the solving and evaluation of an SRP are identified, and in particular conditions and constraints related to preconditions and a-priori expectations about the nature of the accepted answer to a generating question.

In three of the papers, P_1, P_2 and P_3, the mathematical theme of integration, and in particular, the FTC is an important aspect. A description of this is given in Section 1.3. All of the papers do also rely on some form of mathematical questions, either as the foundation for interview tasks or for textbook tasks, or as generating questions of an SRP. Therefore, a description of the role of questions in Chapters 1 to 5 will follow in Section 1.2.1.

In P_1, the notion of *competence*, described by Niss and Højgaard as a "readiness to act" (Niss & Højgaard, 2011, p. 49) was also a guiding principle. Being *ready* to *act* stresses the two aspects of competence of both having the *ability* to carry out procedures, as well as being confident in this ability due to knowledge about why, when and how the procedures work. This notion is not used in paper P_2, P_3 and P_4, however, as the notion of *praxeologies* (Chevallard, 2019) replaces this as a model of mathematical knowledge and ability.

1.2 Questions and a New Didactic Paradigm

1.2.1 The Notions of Mathematical Questions and Tasks

In the ATD the words *tasks* and *types of tasks* have quite a specific meaning (Chevallard, 2019). Any activity, according to the ATD, consists of a set of basic actions, called *tasks*. Each task can be analysed as being of a specific type, having in common that all tasks of the same type can be solved using the same technique. A more thorough description of this will be presented in Section 2.1.3. Colloquially, the word *task* is often used to denote typical *textbook tasks*, or more generally, written tasks (as in exam tasks, test tasks, etc.), that are still so prevalent in mathematics education (Watson & Ohtani, 2015). In this dissertation, the word *task* will be used generally for *basic actions*, while in cases where it is necessary to distinguish, the notions *textbook tasks* or *interview tasks* will be used.

Related to, but distinct from the notion of mathematical tasks is the notion of *mathematical questions*. A task can often be *formulated* as a question, and a mathematical question is often the *foundation* of a task. For example, the task "Calculate the antiderivative of $f(x) = 3x^2 - 4x$" is a task, as it can be carried out as an action. At the basis of this task is the question "what is the antiderivative of $f(x) = 3x^2 - 4x$? In cases like this, the task can be seen as answering a specific mathematical question. This relation between the mathematical tasks and mathematical questions is an essential one, and when I use the term *task*, there is always a mathematical question (explicitly or implicitly) at the core of the task.

1.2.2 The Paradigm of Visiting Works

To give students the opportunity to practice studying and answering mathematical questions, textbook tasks have long been utilised. Solving tasks has therefore been, and still is a staple in all levels of mathematics education as a primary method of mathematical study, both in general (Watson & Ohtani, 2015) and in calculus specifically (Broley & Hardy, 2022). In Norway and the Nordic countries in general, it is no different (Bergqvist, 2007; Lithner, 2017; Opsal & Topphol, 2015). In the Nordic countries, the situation where students are given task after tas by the teacher and solve them individually is identified by Stieg Mellin-Olsen (1996) as the *task discourse*[1].

A related term used in the ATD, with a somewhat broader meaning, is the *Paradigm of Visiting Works* (PVW) (Chevallard, 2015). This paradigm is characterised by a fragmentation of the knowledge base into smaller bits and pieces, where the problems and questions that generated this knowledge in the first place is largely absent. Moreover, the "students are reduced to almost mere spectators" (Chevallard, 2015, p. 175), that is, they do not participate significantly in the exploration of the mathematical themes but are given lectures and demonstrations by the teacher from the front of the classroom. And then, using the techniques presented by the teachers, students are given tasks to solve. One can thus say that the *task discourse* is one of the major symptoms of the PVW.

1.2.3 A New Counter Paradigm

There is no shortage of critical voices to the current situation. In the Nordic countries both Niss (2007), and Mellin-Olsen (1996) are two of them, voicing much of the same critique as Chevallard (2015). Mellin-Olsen points out the impact a strong focus on tasks and individual task solving has on education and teaching. By referring to an underlying *counter discourse* prevalent among teachers, he shows how teachers see the need for a new way of teaching that promotes understanding and includes a practical and complete mathematics.

[1] "Oppgavediskursen" in Norwegian. The word "oppgave" can mean both *task* and *exercise* depending on context.

There has been much effort to overcome this challenge. Both by communalizing and expanding upon the problem-solving process, as seen for example in Japanese lesson studies (Fernandez & Yoshida, 2004; Winsløw et al., 2018), to implement more carefully selected principles for task design (e.g., Coles & Brown, 2016; Lithner, 2017), or to introduce new inquiry-based methodologies (e.g., Artigue & Blomhøj, 2013; Skovsmose, 2003) and contextualised learning activities (Heckman & Weissglass, 1994).

The solution to the challenge, suggested by the ATD is the counter paradigm called the *Paradigm of Questioning the World* (PQW) (Chevallard, 2015, 2022), a paradigm characterised by the study of questions instead of monolithic works. This process of study and research is done and modelled by what is called *study and research paths* (SRPs), which have been studied and applied extensively in the ATD (e.g., Barquero & Bosch, 2015; Barquero et al., 2018; Florensa, 2018; Jessen, 2017; Rodrígues-Quintana et al., 2008).

In this dissertation, both study situations that have characteristics of the PVW, and study situations that are designed with the PQW in mind are prominent. In P_1 and P_2, study situations where students are solving mathematical tasks are presented and investigated, while P_4 is based on a small-scale SRP. The paper P_3, being an investigation of a chapter in the Grade 13 mathematics textbook "Matematikk R2" (Borge et al., 2022), also shows evidence of the PVW.

1.3 The Fundamental Theorem of Calculus

The mathematical theme that is under question in the three first papers is the Fundamental Theorem of Calculus (FTC) (see e.g., Adams & Essex, 2018; Lindstrøm, 2016). The FTC is selected as a theme because of the significance it has for other fields of mathematics, like differential equations and Fourier analysis, and because of the difficulties students have in understanding integration calculus (see e.g., Thompson & Harel, 2021). In Norway, the theme of integration and the FTC is a part both of the Grade 13 curriculum and of first year university calculus and is therefore a theme that spans the transition from upper secondary education to tertiary education.

The FTC is sometimes presented as one theorem with two parts (e.g., Adams & Essex, 2018, pp. 313–314), or as a theorem and a corollary (e.g., Lindstrøm, 2016). In this dissertation, I follow a combination of these two approaches, with a Part 1, a corollary to the FTC Part 1, and a Part II, which is a generalisation of the corollary. Note, the FTC Part II presented here is not exactly the same as in Adams and Essex (2018, pp. 313–314), but based on Botsko (1991), which presents a more general form of the FTC Part II, which does not demand a continuous integrand. Since the student interviews focus purely on the *use* of the FTC, and since the textbook analysis also mainly focuses on the consequences of the conditions for the applicability of the FTC, only the statement of the FTC is given here without the proofs.

1.3.1 Statement of the FTC

FTC Part I:

Consider a real valued function $f(x)$ which is continuous on an interval I on the real number line, and a number $a \in I$. Let $F(x) = \int_a^x f(t)dt$ for $x \in I$. Then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x),$$

which means that $F(x)$ is an antiderivative of $f(x)$ on I.

Corollary:

Consider a real valued function $f(x)$ which is continuous on an interval I on the real number line, and two numbers $a, b \in I$. Let $G(x)$ be any antiderivative of $f(x)$ on I. Then

$$\int_a^b f(x)dx = G(b) - G(a).$$

The formula arrived at in the corollary is called the *Newton-Leibniz formula*. The conditions for the applicability of the corollary can be improved[2] by considering boundedness as a criterion for the integrand instead of the criterion of continuity on *I* (Botsko, 1991). The reformulation of the corollary into what is called the *FTC Part II* can thus be described.

FTC Part II:

Consider a real valued function $f(x)$ which is bounded on an interval I containing the points $a, b \in \mathbb{R}$. Let $G(x)$ be a continuous function defined on I and assume that $G(x)$ is the antiderivative of $f(x)$ *almost everywhere* on I. Then

$$\int_a^b f(x)dx = G(b) - G(a).$$

The term antiderivative *almost everywhere* means for the function $G(x)$ that it is the antiderivative of $f(x)$ for all points in *I*, except for a set of points with measure 0.

[2] *Improved* in the sense of allowing for a wider range of integrands to be acceptable.

The corollary is what is typically included in textbooks, but Part II, following Botsko (1991), is included for a fuller understanding of the role of boundedness. This is particularly relevant for P_3. The role of boundedness is explained in what follows.

1.3.2 Notes on the Conditions for the FTC

One can see three different sets of conditions here, one for the FTC Part I, one for the corollary, and one for the FTC Part II. For the FTC Part I, note first that the function f is assumed to be continuous on an interval which *contains* the point a. This places some clear restrictions on what type of interval I can be. Any interval of the types (a, c), $(a, c]$, (c, a) or $[c, a)$ cannot function as *I*, since these do not include a. The intervals of the type $[a, c]$, and even $[a, c)$ will hold, so long as $f(x)$ is defined and continuous on these intervals.

For the corollary, the point *b* is an element of the interval *I*, and thus, assuming $a < b$, the smallest interval *I*, for which the corollary holds, is the interval $[a, b]$. A consequence is that, since $f(x)$ is continuous on *I*, $f(x)$ must also be continuous on $[a, b]$. Since any function which is continuous on a closed interval, is also bounded on that interval, $f(x)$ is bounded on $[a, b]$.

This leads to the conditions for the *FTC Part II*. Here, the condition of continuity for $f(x)$ is replaced by boundedness, which makes the FTC applicable to all Riemann-integrable functions[3]. This change of the conditions provides the foundation for applying existing techniques to integrands that are not continuous, but still Riemann-integrable. And importantly, it provides the foundation for understanding why some functions are *not* Riemann-integrable, and what one can do in those cases. For example, functions for which $\lim_{x \to p} f(x) \to \pm\infty$ for some $p \in \mathbb{R}$ are not Riemann-integrable on intervals that include p.

[3] The Lebesgue condition for Riemann-integrability states that a bounded function on a compact interval is Riemann-integrable if and only if the set of points of discontinuity has measure zero (Brown, 1936).

2 Theoretical Tools Used in the Dissertation

The two theoretical approaches applied in this dissertation, the IA (Rabardel & Samurçay, 2001; Trouche 2004) and the ATD (Chevallard, 2019, 2020), will in this section be described in more detail. Most attention is given to the ATD, since this is the approach under which this document is framed, and since three out of the four papers are written under this approach. The IA will be given a shorter treatment, and only aspects that are directly relevant to the paper P_1 will be presented.

Any theoretical approach of didactic needs to contend with what it means to know something. The view on the nature of knowledge and learning has wide ranging consequences for what we as researchers in the didactic of mathematics are able to say with any degree of certainty what it means to learn and to teach. In the IA and in the ATD, the knowledge of an object[4] is modelled in different ways. The details will be described in the following sections. But the major difference is in what parts of knowledge is modelled. The ATD models knowledge in general as a relation between the knowing subject and the object which the subject knows something about (Chevallard, 2019). The IA models how an acting subject develops a relation to an artifact[5], which can be used as an instrument in the subject's activity (Rabardel & Samurçay, 2001; Vérillon & Rabardel, 1995), and the knowledge modelled is therefore that of knowing how and when a certain instrument can be used within an activity.

2.1 The Anthropological Theory of the Didactic

Being part of the French didactical tradition, the ATD is both an epistemological approach (Chevallard, 2006, 2007, 2022; Gascón, 2003) and an institutional approach (Chevallard, 1989, 2019). As an epistemological approach, the ATD has a *content specific* focus, meaning that the nature of the knowledge at hand imposes conditions and constraints on the analytical approach.

Knowledge is modelled through the concept of *praxeologies*, which accounts for both the *active* part of knowledge (*know-how*) and the *explanatory* part (*know-why*) (Chevallard, 2019). Commonly in ATD research, an a-priori analysis of the content to be taught is carried out, called a *praxeological analysis* with respect to both the active and the explanatory part. Second, all knowledge is *institutionally relative*, that is, how knowledge of a given object appears is dependent upon the institution in which the knowledge is active (Chevallard, 1989).

[4] Any sort of *object* that "something can be known about". These can be both concrete, like an animal or a city, or abstract, like mathematical or philosophical concepts.
[5] An *artifact* (from Latin *arte factum* – artificially made) is a thing made by "human workmanship or modification as distinguished from a natural object" (Merriam-Webster, n.d.).

And third, because of institutional relativity, when knowledge is adapted from one institution and implemented in another institution, the knowledge necessarily changes, dependent on the differences in conditions and constraints present in each of the two institutions (Chevallard, 1989).

The ATD goes beyond the institutional and introduces *anthropological* aspects. The ATD came about partly as a response to a challenge in describing how knowledge respond to transition between institutions, and how a person's relation to an object of knowledge is changed when the institution in which they are operating changes. Under the name of the *Theory of Didactic Transposition*, Chevallard introduced this aspect on the French scene in 1985 (Bosch & Gascón, 2006; Chevallard & Sensevy, 2014).

2.1.1 Persons, Institutions and Institutional Positions

The three concepts of *persons*, *institutions* and *institutional positions* are closely related (Chevallard, 2020b; Chevallard & Bosch, 2020). A *person, x,* is simply a human being, like a child or an adult of any capacity. An *institution, I*, is something which is *instituted*, that is, it is something which consists of persons joined for some reason, like a school, a class, a family, or an interview. Institutions are not simply a collection of persons, though, and in all institutions, there are institutional *positions, p*. In a class, there are the positions of teacher and of student; in an interview there are the positions of interviewer and interviewee.

2.1.2 Conditions and Constraints

A central notion in the ATD specifically, and in didactics in general, is the notion of *conditions and constraints* (Chevallard, 2020b; Chevallard & Bosch, 2020). A condition is, in general, "anything purported to have influence over at least something" (Chevallard, 2020b, p. xx). In a classroom situation this can be the different elements of the curriculum, the textbook used in a class, the explicit and implicit rules governing the interaction between the different persons in the class, or generally anything that influences the activity of the class. If the condition seems unchangeable for a person within the institution, this condition is now what is called a *constraint*. Indeed, "a constraint is any condition which appears to be unmodifiable by occupants – acting as such – of a given institutional *position*" (Chevallard, 2020b, pp. xx–xxi).

2.1.3 Praxeologies: Models of Knowledge

A fundamental tenet of the ATD is that all knowledge can be modelled by a *praxeology* (from Greek: πράξις – *praxis, deed, action*, and λογία – *reason, account, explanation*) (Chevallard, 2019). The origin of the term is old, and at least one instance of its use goes back to 1608

(Timpler, 1608, p. 387): "Fuit aretologia: Sequitur praxiologia: quae est altera pars ethicae, tractans generaliter de actionibus moralibus" – "There was aretology: and following that, praxiology: which is a second part of ethics, which in general discusses moral actions". *Aretology* (from Greek: ἀρετή – *virtue*) is the study of *virtue* or *moral being*. The origin of the term praxeology is therefore that of describing *moral actions*, or the *praxis* of a virtuous person, in contrast to *being* virtuous. In its modern form, often written with an *e* instead of an *i*, it is generalized to all motivated action, making praxeology "the formal science (*logos*) of praxis, or human actions" (Prychitko, 1994, p.77). It is the latter meaning of the word, that of a science of human actions, which is relevant to the ATD.

The following description of the term praxeology is based on Chevallard (2019). A praxeology, $\wp$, consists of four components, T, τ, θ and Θ. Here, T is a type of tasks, or a set of types of tasks of which a numbered T_i is one certain type. The ATD also distinguishes between a certain *type* of task and a *task t* being of the type T. For example, a task t_1 can be "calculate the definite integral $\int_{-\pi/4}^{\pi/4} \sin x \, dx$", while its type T_1 is "calculate the definite integral $\int_{-a}^{a} f(x) dx$ for an odd, continuous, bounded function f, and a real number a". This relation is denoted $t \in T$.

The second component of a praxeology is the *technique*, τ. A particular technique is one specific way of solving a task $t \in T$. For a given type of task there might also be more than one technique. So, for the example task t_1, of solving $\int_{-\pi/4}^{\pi/4} \sin x \, dx$, one technique τ_{1a} could be to "calculate using the Newton-Leibniz formula", which yields the calculation $\int_{-\pi/4}^{\pi/4} \sin x \, dx = [-\cos x]_{-\pi/4}^{\pi/4} = 0$. But since t_1 is of type T_1, where f is said to be *odd*, another technique, τ_{1b}, is also possible, exploiting odd symmetry, by simply stating that "$\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$, due to odd symmetry". Thus, for the case of odd symmetry, a quicker technique than using the Newton-Leibniz formula exists.

The pair, $\Pi = [T / \tau]$, is called the *praxis* block of the praxeology, and describes the active part of knowledge. That is, it describes *what* is being done by x, in relation to the object o, and *how* it is done. The two other components of the praxeology belong to what is called the *logos* block, $\Lambda = [\theta / \Theta]$. The logos deals with giving reasons for the praxeology's working and existence. The first of these two elements, the *technology*, θ, provides the justification for why the technique, τ, works. Additionally, in an optimal technology, it should also aid the understanding of *why* the technique works, that is, it should not only justify, but also explain or clarify the technique.

A technology can consist of a formal proof, which is the most precise sort of justification, based on definitions, previously proved theorems, and axioms. In cases where a type of task has several correct techniques, each of the techniques has their own justification, and therefore their own corresponding technology. Less rigorous forms of justifications, like the use of paradigmatic examples or demonstrations relying on “common sense”, can also make up the core of a technology. For the FTC and the Newton-Leibnitz formula this can be a proof or demonstration that accounts for a subset of the functions for which the FTC is applicable to, like continuous, positive, and increasing functions. Or it can be a justification based mainly on informal argumentation supported by, for example a graph.

The second component of the logos block, and the last component of a praxeology, is called a *theory*. In the ATD, a theory, Θ, denotes a *discourse*, that can “generate, control, justify and make intelligible a given technology” (Chevallard, 2019, p. 91). In broad terms, it is a justification for why one should even be interested in the given praxis in the first place, a *raison d’être* for the praxeology. For the FTC, such a raison d’être can be seen in the historical roots of calculus (Kline, 1972), by Newton and Leibniz independently, in the study of planetary motion, and physics more generally. It also lies in the fact that it connects the two important fields of calculus: differential calculus, which deals with tangents, slope, and rate of change; and integral calculus, which deals with accumulation, area, and volume calculations, among other things.

The pairs $\Lambda = [\theta / \Theta]$, the logos block, and $\Pi = [T / \tau]$, the praxis block, together make up a praxeology. This is commonly written as $\wp = [T / \tau / \theta / \Theta]$. Using these notions, the ATD claims to be able to model knowledge in action, as a dialectic relationship between the four components of the praxeology. More precisely, it models knowledge by describing how a set of types of tasks is solved by the careful utilisation of a set of techniques connected to the types of tasks, and these techniques are in turn justified and clarified by a set of technologies, which again is justified, clarified, constructed, and organised by a theory. Thus, it can be said that the praxeological model of knowledge found in the ATD considers both the “know-how” with the praxis block, and the “know-why” with the logos block. These two blocks are connected in a complete knowledge structure, the praxeology, which models the dialectical relationship between these two modes of knowledge.

2.1.4 Model of Didactic Moments

In the ATD, the process of studying a question is modelled in terms of six *didactic moments* (Chevallard, 1999, 2022). The six didactic moments are: 1) the *moment of the first encounter*

– this moment is characterised by the student of $\wp$'s first encounter with the praxeology through the introduction to tasks of the type *T*; 2) the *moment of the exploration of the type of tasks T and of the emergence of the technique* τ; 3) the *moment of the constitution of the technological-theoretical environment* $[\theta/\Theta]$ – this moment is characterized by a first emergence of the logos block, where a justification of the technique τ in relation to the type of tasks, *T*, is developed; 4) the *moment of working on the praxeological organization* $\wp$, in which the praxeology is further developed and elaborated – this moment is highly dependent on the availability of appropriate tasks, dependent on both the precise type of task at stake, and the scope of the praxeology (whether it is a point praxeology, local praxeology or a global praxeology); 5) the *moment of institutionalisation of* $\wp$ – here, the elements relevant to the given institution are selected and integrated into $\wp$; 6) the *moment of evaluation*.

The term didactic moments should not be understood in a temporary sense, where for example moment 1 necessarily precedes moment 2, and is never revisited after it has been visited once. Instead, the moments are to be understood functionally. They are working modes in the development of a praxeology, $\wp$, based on a type of task, *T*, in which a certain part or relation within the praxeology is being elaborated. Granted, a first encounter with a type of task often does precede the other moments. After all, to develop a technique for solving *T* does imply the existence of at least one task of type *T*. But a situation is completely conceivable, where a *raison d'être* for the existence of a task of the type *T* exists already at the inception of *T*. Thus, a justification in some form might already exist, making the technological-theoretical environment already a reality.

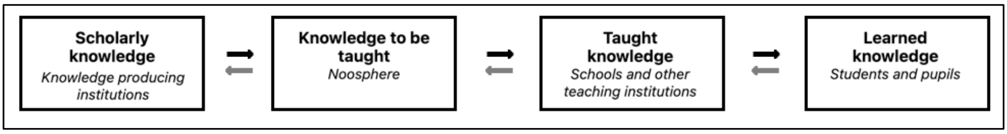
2.1.5 Didactic Transposition: Institutional Relativity of Knowledge

Praxeologies, didactic systems, and institutions do not live in isolation. Two main models, the theory of *didactic transposition* (Bosch & Gascón, 2006; Chevallard & Bosch, 2014) and the *scale of levels of didactic co-determinacy* (Bosch & Gascón, 2006; Chevallard, 2020a), have been developed within the ATD to describe how institutions interrelate, and how knowledge plays a role in this.

The theory of *didactic transposition* (see Figure 1) (Bosch & Gascón, 2006; Chevallard, 1989; Chevallard & Bosch, 2014) models how a knowledge object, created in one institution, is transformed when it passes to another institution. Typically, knowledge is created in a scholarly institution, as *scholarly knowledge*, it is then transformed in three stages, as it is adapted for teaching in a didactic institution. It is first prepared by the *noosphere*, as knowledge to be taught, before it is taught in the teaching institutions. The knowledge that is acquired by the students is then a function of this teaching, as well as the conditions and constraints that govern both the teaching institution and the students, both as a group and as individuals.

Figure 1

The Model of Didactic Transposition (adapted from Chevallard & Bosch, 2014, p. 171)

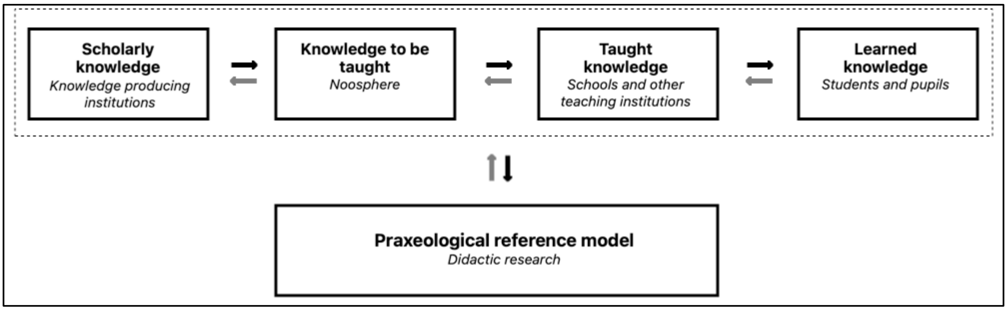


All these transpositions also entail a transformation of the knowledge object. For example, the Fundamental Theorem of Calculus (FTC) can be selected to be taught at a given secondary school. The conception of the FTC is often a very different one as scholarly knowledge, where it exists in a very formalised and rigorous shape, than what is taught in secondary school calculus classes, in which the goal often is to provide students with a *method* for calculating areas under the graph of a function. What students are left with is often something different from that again, maybe even lacking some of the technical details. A student might remember how to calculate the area under the graph of a positive valued function, but once it is negative, or even changes between positive and negative, the technique has been forgotten, or was not even properly learned at all.

A characteristic of the epistemological approaches to didactic research is the priority of and significance of the subject matter to the whole process of research. Both the research questions, methods of analysis and the answers to the research questions are conditioned by the nature of the mathematical theme under study. To be able to study the whole process of didactic transposition, the researchers therefore need to be able to position themselves external to the institutions, with no priority to any of the internal positions. The researchers therefore need to continually develop and update a model of the body of knowledge under consideration which, although necessarily a product of the same processes that shape knowledge in general, is a representation of the researchers own perspective on the body of knowledge studied. This model, a *reference praxeological model* (see Figure 2) (Bosch & Gascón, 2006; Chevallard & Bosch, 2014; Wijayanti & Winsløw, 2017), used as the reference against which the praxeological organisation at a given institution or didactic system is analysed. This model, as it is the researchers' attempt at distancing themselves from the institutional knowledges under study, should therefore be constantly updated and challenged.

Figure 2

The Researcher's Point of Reference (adapted from Chevallard & Bosch, 2014, p. 171)



The *scale of levels of didactic co-determinacy* (Bosch & Gascón, 2006; Chevallard, 2020a) models how conditions and constraints that affect learning in an institutional setting is not decided only by the given institution itself but at a higher level of organisation. For example, a school might decide methods used for teaching a certain theme, which imposes constraints on how classes can organise the teaching of that given theme. If a relatively large school, with a high number of students are following the same course, the school might decide to have two or more parallel classes following a common course. Then, it is typically not up to the teacher in one of the parallel classes to decide how much time is to be used on that one particular theme. Similarly, the school cannot typically decide completely freely what themes to teach. A given society, like a country, typically decides what themes are to be taught, which puts constraints on the teaching in the schools within the country. This phenomenon is described in terms of *levels of didactic co-determinacy* (Figure 3).

Figure 3

The Scale of Levels of Co-Determinacy

Civilization ⇄ *Society* ⇄ *School* ⇄ *Pedagogy* ⇄

Discipline ⇄ *Domain* ⇄ *Sector* ⇄ *Theme* ⇄ *Question*

2.1.6 Study and Research Paths: A Methodology for Questioning the World

In the ATD, a study process, called *Study and Research Paths* (SRPs), can also be designed around a given generating question, *Q*, rather than a specific type of tasks (Chevallard, 2019, 2022). This is a part of the ATD's answer to the challenge of the PVW, and it is the ATD's model of inquiry. A more detailed description of such a process is found in P_4. But a characteristic of SRPs is that they aim at letting the questions, and the answers to these questions, arise naturally during the process of study.

In an SRP, a generating question Q_0 is studied, and used as the starting point for an inquiry (Chevallard, 2019). The SRP is conducted through a process of searching for answers, in a collection of works, and through this process, arriving at a final question agreed upon by the participants of the didactic system in which Q_0 is studied. As a consequence, in an SRP aimed at studying mathematical themes, the generating question would need to either be a mathematical question itself, or a question that is likely to lead to the discovery of works that are mathematically themed.

The process can be modelled symbolically by the *Herbartian* schema (Chevallard, 2019). The Herbartian schema describes how a didactic system S, through the interaction with a *didactic milieu* M, produces an answer $A^{\heartsuit}$, to the question Q_0.

The didactic system (Chevallard, 2019, 2022), S, consists of the participants together with the *didactic stake*, or the *thing to learn or study*. A didactic stake can be a question, as in an SRP, or more generally, any object o that can be studied. The participants in the didactic system are the students, denoted as a group with a capital X, of which there are individual students $x \in X$, and the different "helpers", Y, whose jobs are to help the student to study in different ways. An individual "helper", $y \in Y$, can for example be a teacher, a librarian, or any person assigned with the task of assisting in a study process. Algebraically, a didactic system can be described as $S(X, Y, o)$, or $S(X, Y, Q_0)$, in the case of the didactic system in an SRP.

The *semi-developed Herbartian schema*, written as $[S(X, Y, Q_0)$➦$M]$➥$A^{\heartsuit}$, describes the process of inquiry (Chevallard, 2019). The didactic system both interacts with and produces the didactic milieu, and through this interaction, the answer $A^{\heartsuit}$ is built up. The superscript $\heartsuit$ denotes that $A^{\heartsuit}$ is an answer, that under the given conditions and constraints, is the optimal answer that S can produce.

The Herbartian schema can be further developed into the *developed Herbartian schema* (Chevallard, 2019) by considering the components of the milieu. A didactic milieu consists in general of the four components $A_i^{\lozenge}$, W_j, Q_k, and D_l. $A_i^{\lozenge}$ denotes the pre-existing institutional answers to Q_0 that the group of students X discover in the institutions they interact with. W_j are *additional works* in which the $A_i^{\lozenge}$ are found, and which are used to make sense of the $A_i^{\lozenge}$. Through the study of these pre-existing answers, works, and generating question, *derived questions*, Q_k, also arise, and will need to be answered to produce $A^{\heartsuit}$. The last element, the D_l, are sets of data gathered during the inquiry process, and which functions as a foundation for the answers of the didactic system. Thus, in the *developed Herbartian schema*, the letter *M* is replaced with the expression

$$M = \{A_1^{\Diamond}, \dots, A_m^{\Diamond}, W_{m+1}, \dots W_n, Q_{n+1}, \dots Q_p, D_{p+1}, \dots D_q\}.$$

2.2 The Instrumental Approach

The instrumental approach (IA) is a theory developed originally by Pierre Rabardel (Rabardel & Samurçay, 2001; Vérillon & Rabardel, 1995) to describe the human-instrument interaction and the processes through which human activity is mediated by the use of tools. After the introduction to mathematics, this approach has been connected strongly with digital tools (Trouche, 2004), but not exclusively (e.g., Mesa et al., 2021; Shinno & Mizoguchi, 2021). The main idea behind the IA lies in the distinction between a *tool* and an *instrument*. The tool is described as "something which is available for sustaining human activity" (Trouche, 2004, p. 282). A tool is simply an *artifact*, meaning that it is a *man-made object*, and even if it has been specifically created for the purpose of being used as an instrument, it is not yet an instrument, before it has become appropriated by an acting subject, and incorporated into the subject's activity (Vérillon & Rabardel, 1995). A tool is also not part of the human body itself, but it can be said to *extend* it, by adding functionality that otherwise would either be difficult or even impossible, like the way snowshoes make walking on snow easier, or a hammer makes it possible to hammer in a nail. An activity, mediated by the usage of an instrument, is called an *instrumented activity*. For the tool to become an instrument, and the activity of the subject to become instrumented, *utilisation schemes* for the utilisation of the tool needs to be developed. A utilisation scheme can be defined as the "structured set of the generalizable characteristics of artifact utilization activities" which "form a stable basis for [their] activity" (Vérillon & Rabardel, 1995, p. 86). The utilisation schemes can be *private*, relating an individual subject's activity to the artifact. They also have a *social* dimension, both because they emerge from a collective process of usage and design, and because of the social transmission processes involved in teaching and learning how a tool can be used. Without these schemes, the tool will not become an instrument.

The process of learning to use a tool as an instrument is in the IA called an *instrumental genesis* (Rabardel & Samurçay, 2001), and is characterised by the two subprocesses of *instrumentalisation* and *instrumentation.* Instrumentalisation is the process of turning the tool into an instrument through modifications done to the tool (selection of functionalities, physical modifications, grouping it with other artifacts, etc.). Instrumentation is the process in which the acting subject becomes a tool user by incorporating social utilisation schemes into their private set of schemes. Instrumentation is governed by the instrument's characteristics (Drijvers & Trouche, 2008). These include *constraints*[6], or the characteristics of an instrument which discourage or hinder certain actions, and *possibilities*, or characteristics that enables or encourages actions. These characteristics shape the thinking of the subject.

Instrumentalisation consists of three stages, namely *discovery and selection of relevant functions*, *personalisation*, and *transformation of the artifact* (Trouche, 2004). Discovery and selection of relevant functions is exactly what the term describes, and exploration of the different operations possible to carry out with a tool, like the different operations programmed into a calculator, and the selection of what functions are relevant to the activities of the subject. Personalisation is different from discovery and selection of relevant functions in that it is a process of figuring out how the functionality of the tool best fits the personal activity of the subject. Trouche uses the metaphor that "one fits the artifact to one's hand" (Trouche, 2004, p. 293). The stage of transformation of the artifact is the direct modification of the tool, like making custom keyboard shortcuts in a digital tool, or adding physical modifications to a tool for it to best fit the subject's activity.

As the instrument refers mainly to the mental construction of the tool user, the tool does not need to be a physical, concrete object (Lagrange et al., 2001). Indeed, a tool can be material, like a hammer or a computer, but it can also be immaterial, like a procedure, technique, or even a task (Trouche, 2004).

2.3 Networking of Theories

2.3.1 Networking Methodologies Using the ATD

As a means of connecting the results of the papers in this dissertation, it is necessary to know how the two theoretical frameworks can communicate. The link between P_1 and P_2 is particularly important, because these two papers both use the interviews as the main data material. In this section, the aim is therefore mainly to make a connection between these two papers.

Several ways of *networking theories* have been described (Bikner-Ahsbahs & Prediger, 2014), ranging from making theories understandable to other approaches, applying different theoretical lenses to the same problem in a process of comparison and contrasting, to different degrees of combinations and integrations of either parts of theories, or whole theoretical approaches (Bikner-Ahsbahs & Prediger, 2010). Within the ATD, some work has been done to

[6] Note the difference in usages of the notion of a *constraint* between the IA and the ATD.

explore the possibility of networking ATD with other theoretical approaches (Artigue & Bosch, 2014; Bosch et al., 2017; Rodrígues-Quintana et al., 2008).

In Rodrígues-Quintana et al. (2008), the challenge of using one framework to answer a research question formulated in another framework was explored. It was pointed out that a research question formulated within one framework might not always be meaningful in another framework. This is due to differences in both terminology, epistemological and ontological assumptions, and research priorities. They do, however, demonstrate a method based on identifying a *problematic question* which lies at the origin of the research project, and use that to *compare and contrast* the findings from the two theories. A *problematic question* is to be understood as a naturally occurring question, which is not yet formulated in scientific terms. In the method devised, the problematic question is seen as the "common denominator of the theoretical developments that are to be compared" (Rodrígues-Quintana et al., 2008).

2.3.2 A Change of Theoretical Framework From the IA to the ATD

A consideration specific to this dissertation is that a change of theoretical frameworks is prompted by a necessity of expanding the didactic frame, from a local focus on tasks, to a wider focus including a description of people and knowledge undergoing institutional transitions. The method used for combining results from the two theoretical approaches therefore needs to account for both the work that is done using the two theoretical lenses, and the reasons for the switch from one theory to the other. What made it beneficial to switch from an IA approach to that of the ATD?

The aim of both P_1 and P_2 was, partially, to examine how first-year university students work with a set of unfamiliar tasks and learn from them. Formulated as a question, this would be: "How do first-year university students work with unfamiliar tasks?" In P_1, the goal was to demonstrate *that* tasks could be analysed as instruments, and that an instrumental genesis could be identified in a student's work with a set of tasks, while in P_2 the main goal was to examine the praxeology that a student demonstrated and developed through working with the same set of tasks (see Section 4.2.1 for the description of the tasks). The problematic question is therefore the "common denominator" between P_1 and P_2 that allows a connection between these two papers, and through this, a connection to the rest of the dissertation is possible. In handing this problematic question, the ATD and the IA provide two different approaches.

The IA, which was applied in the initial work on the dissertation, provides a way to view tasks as a human made tool, which can be used for learning. A definition of tasks[7], conductive to this sort of approach, was devised for the analysis. In this definition, the *purpose* of a given task was central. In order to examine how a task could be used as a tool, it was assumed to be important to know *why* this task existed, and what sort of mathematical theory and techniques can be relayed by giving this specific task to a student.

The ATD, on the other hand, has tasks, or more precisely *types of tasks* as an integral part of the notion of a praxeology (Chevallard, 2019). Thus, there is a difference in the use of terminology and focus that makes the two theories not directly compatible. A notion of *constraints* (Chevallard, 2020b; Chevallard & Bosch, 2020) found in the ATD, is also seemingly paralleled in the IA. In the IA, a constraint is a characteristic of an instrument (Drijvers & Trouche, 2008) which hinders or discourages a certain action using the given instrument (it *constrains* the action). This function of *discouraging* an action is not an essential part of a constraint in the ATD usage of the term, but rather a constraint is a condition which cannot be changed by the members of a given institution in which the constraint is acting (Chevallard & Bosch, 2020). This again adds to the incommensurability of the terminology, and a choice therefore needs to be made about which set of terms to use. Since most of the dissertation is framed as an ATD study, the ATD notions of conditions and constraints are used generally in the dissertation.

There is, nevertheless, some correspondence between the ATD notion of types of tasks, and the notion of tasks presented in P_1, which makes it possible to recontextualise some of the notions developed for that paper. First, it is clear that tasks are only one part of knowledge, evident from the definition of knowledge as praxeologies. The dialectic between tasks and other components of knowledge, represented in the IA as a *tool-utilisation scheme* dialectic, is therefore paralleled in the ATD in the form of a dialectic between *types of tasks* and *techniques*. Asking about the role of tasks therefore still makes sense in the ATD. Second, the ATD provides a way of describing how the work on a given unfamiliar task develops into a praxeological organisation, through the theory of the didactic moments. Thus, some sort of *usage* of these tasks can be seen in this description, and the developmental aspect is also preserved. Third, and last, the description of a technique, τ_1, being divisible into smaller tasks needed to be performed to solve the type of task, T_1, that τ_1 is created to solve, corresponds well with the observation

[7] In P_1 called *formal tasks*. This definition is not used in the rest of the dissertation, but the notion of a *purpose* of a task, and how a task can potentially condition an activity is important also for the rest of the dissertation.

of students identifying subtasks when working on the tasks they were given. Thus, a recontextualisation of the study under the ATD is possible, by reframing the question of using tasks as instruments, to asking a question about what roles they play in study situations, and thus, how they can condition the formation of the technological-theoretical block of a praxeology.

What proved difficult to do with the tools of the IA was to describe the transition of knowledge between institutions. This prompted the change of theoretical approach. Three options existed for the expansion of the theoretical framework. The tools could be developed from notions already existing in the activity theoretical tradition (Engeström, 1978; Leont'ev, 1978; Vygotsky, 1978) within which the instrumental approach lives. Alternatively, they could be incorporated into the IA from another theory. Or lastly, the project could be reframed within another theoretical approach better suited for the task.

These realisations, that the notion of types of tasks within the notion of praxeologies, and the language for describing the institutional relativity of knowledge were well suited both to describe the data material and to handle the question of transition between educational institutions made the switch of theoretical approaches well justified. What is gained from the work carried out under the IA should nevertheless not be bypassed, and in the following, the main contributions from the work done using the IA will be described.

2.3.3 Contributions From the Instrumental Approach

There were two main contributions from the work done in P_1, and the IA. First, the description of tasks as instruments used for the development of mathematical competence, proved to be a guiding principle in the question asking within the further research and analysis, carried out using the ATD. Concretely, it allowed me to realise a particular set of conditions and constraints surrounding the posing of tasks. By considering that mathematical tasks are not and should not be the ends of the praxeologies of teaching and learning a mathematical theme, but rather the tasks are used as means for teaching and learning mathematical practices and theory, it allowed me to question the role of tasks within a study situation. This idea then guided the work presented in both P_2 and P_3.

The second contribution is a methodological development originating from this same idea. Resulting from the work presented in P_1 is the use of the notion of subtasks as an analytical tool for the data from student interviews. By observing and identifying how students divided the tasks they were given into subtasks in the study process, it was possible to map out the argumentative nature of the students' explorations in flowcharts, resembling tree structures, or argumentation trees. An example of these flowcharts and how they are constructed can be found in Section 4.4.2.

3 Research Questions

With this background, a statement of the goal of this research project can be presented. The general goal is to examine conditions and constraints that govern the dissemination, teaching, and learning of calculus, particularly integral calculus. Related to this are three subgoals. The first goal is the examination of the work done by a group of students on a set of unfamiliar tasks based on the FTC, and identification of conditions and constraints that govern the development of praxeologies related to these tasks. The second goal, related to this, is to investigate how the conditions and constraints originating from the praxeological organisation of the FTC in a Grade 13 mathematics textbook affects what types of tasks can be posed, how they are answerable, and possible consequences this might have for the further development of a praxeology of integration tasks. The third goal, related to mathematical questions in general, is to examine an SRP, and through it, also explore a problematique relating to the formulation of the generating question, Q_0, and the explicit and implicit preconditions, and expectations about how the question will be answered.

The claim that the set of tasks given to the students is unfamiliar is a bold claim and should be treated carefully. It is of course not possible to guarantee in advance that a set of tasks is unknown to a group of students. The claim will, nevertheless, be substantiated, both by the nature of the tasks, and by reference to statements made by the students during interview sessions.

Table 1

Research Questions in the Papers of the Dissertation

Paper	Research Questions
P_1	RQ_1: Based on a series of task-based interviews in early university calculus, and using the instrumental approach, what sort of evidence is there for saying that tasks can be used as instruments for developing mathematical competence?
P_2	RQ_2: Given a set of four tasks, t_1 to t_4, what is the praxeology constructed by a first-year university mathematics student to solve these tasks, and what are the tools used in this construction?
P_3	$RQ_{3,1}$: What sort of changes have been made during the transposition of the theme of integration, and particularly the FTC, from scholarly knowledge to knowledge to be taught in upper secondary school? $RQ_{3,2}$: In case of any unused potential in the presentation of the FTC, with regard to strengthening its logos, what does this potential consist of?
P_4	RQ_4: What sorts of mathematical models are used to answer Q_0, and how are they interconnected? [Q_0 = “Why do babies die of heat stroke in cars parked in the sun”]

4 Methodology

The central place of the object of knowledge in the ATD has consequences for the methodology, and how the methods are applied. This is particularly true for how the analyses are conducted. The main types of analyses from the ATD used in this dissertation are *praxeological analyses* (Chevallard & Bosch, 2020) and *didactic transposition analyses* (Chevallard, 1989; Chevallard & Bosch, 2014)), both of which are based on a careful description of the epistemological foundation of the praxeologies under study by creating a *praxeological reference model*.

The focus on the object of knowledge under study also governs what types of data will be needed, specifically, data that are rich enough to identify significant praxeological elements. This can include interview data and questionnaires (e.g., González-Martín & Camacho, 2004; Ladage et al., 2020), classroom observations (e.g., Artaud, 2020), and written or printed materials (e.g., Strømskag & Chevallard, 2022; Wijayanti & Winsløw, 2017).

4.1 Emergence of the Published Papers

The Instrumental Approach (IA) (Rabardel & Samurçay, 2001; Trouche, 2004) was used in the beginning both to guide the initial design of the project and as the analytic tool in the first paper. But as the research also included an interest in the effects of how knowledge and people move from one institution to another, the Anthropological Theory of the Didactic (ATD) (Chevallard, 2007, 2019) seemed like a better fit for the study. In addition, due to the Covid-19 lockdown, the intended form of data collection, a longitudinal interview intervention, was not possible. Only the first part of the data collection was possible to carry out. A result of this broadening of the data collection and the change of theoretical approaches is a collection of four very different papers.

The ordering of the papers in the dissertation is mainly thematic. The papers P_1, P_2 and P_3 all have some themes in common which are not present in P_4. The papers P_1–P_3 have the common themes of mathematics tasks and the FTC, while the themes of P_4, of SRPs and modelling, are not so directly connected to the three other papers. P_1, P_2, and P_3 are in their temporal ordering, due to how the writing of one paper leads to questions that were later examined in the later paper.

P_1 was a consequence of an interest in tasks, and how tasks can be used by students, in different ways, as instruments for learning integral calculus. P_2 is the paper where the data from the student interviews are most deeply examined and is also the first paper where the ATD is used as an analytical tool of the interview data. P_3, about the praxeological analysis of an upper secondary mathematics textbook, was a result of questions arising in the work with P_2 regarding the connection between the praxeologies observed in the student group, and indications of some of the causes of shortcomings of these praxeologies.

P_4, which is ordered last in the dissertation, did not arise from a question prompted by any of the other three papers. It is included in the dissertation because it exemplifies the alternative way that the ATD proposes for the teaching and study of open questions. Under the PQW, as opposed to PVW, the methodology of *study and research paths* (SRPs) is utilised to focus on the exploration of a question, or set of questions, as a driver of exploration and learning. This is contrasted with learning defined through a pre-defined set of particular *knowledges*, typically described through curriculum bullet points.

The studies in this dissertation were designed (with the exception of the SRP), conducted, transcribed, and analysed by me. I have a master's degree in mathematics, with focus on numerical integration, and pedagogical education. Before starting the PhD work, I had six years of experience in teaching mathematics in upper secondary school. I am also the main author of all the papers that the dissertation builds on, the sole author of the papers P_1, P_3 and P_4. The SRP was designed by the lecturer of the course it was part of.

4.2 Data Material and Data Collection

Two of the papers (P_1 and P_2) are based on interviews. The interviews were conducted in two periods during the autumn of 2019, in a first-year calculus course at NTNU. The interviews were task-based, and semi-structured, and video recordings were made and analysed. The students were all enrolled in the course "MA1101 – Grunnkurs i Analyse 1" (MA1101 – Basic Calculus 1) (MA1101, n.d.) at the Norwegian University of Science and Technology, during the autumn of 2019. A third interview was planned for the end of the spring semester of 2020, after they had finished the course "MA1102 – Grunnkurs i Analyse 2" (MA1102 – Basic Calculus 2) (MA1102, n.d.), and possibly also "MA1103 – Flerdimensjonal Analyse" (MA1103 – Vector Calculus) (MA1103, n.d.). But the last interview round was cancelled due to lockdown.

Four tasks were given, and the students were told to solve them while explaining their thought process during the solving of the tasks. Video recordings, with the camera focused on the table between the student and the interviewer, were used for documentation, and the interviews were later transcribed with focus on spoken words. Other modes of communications were also included in the transcriptions when they were found relevant (e.g., pointing, hand gestures, drawings). All the interviews were conducted and transcribed in Norwegian, but translations into English of relevant sections for inclusion in published work. The interviews were with individual students, and in each interview only the student and the interviewer were present. The didactic system can thus be represented as $S(x, y, k)$, where x is the student interviewed, y is the interviewer, and k is the knowledge at stake. A part of the explicit contract was that y should not give any hint about the solution of the task while x was solving it. The only permissible gestures for y were to present the task, by handing a printed paper with the task written on it, receiving the task when x had decided to finish it, and to prompt x to speak more or louder if deemed necessary. "Finishing the task" was x's own privilege, and x could decide to stop at any point, whether the task was solved, in x's opinion, or not. This was all made clear to the students beforehand.

The tasks were designed with the specific goal that they should be of a type that the students had most likely not seen before. This is naturally difficult to be certain about, but comments by the students during the interviews did indicate that the types of tasks were unfamiliar to the students, except for the first task. The second task was also of a form that would resemble a type of task likely to be familiar to the students, but with some information missing, making it impossible to solve without making some interpretations of the task itself. The two last tasks were selected to be tasks of proving propositions, and to be more similar to tasks they would likely meet in university mathematics, and at the same time more advanced than what would be expected from upper secondary integration tasks.

In P_2 and P_3, mathematics textbooks are also used as data material. In P_2, a textbook was used as additional support and warrant for the results emerging from the analysis of the interview data, while in P_3 textbooks were the main source of data. Two different editions of the textbooks entitled *Matematikk R2* were used (Borge et al., 2022; Heir et al., 2016). In P_2, the edition from 2016 was used, as this is the textbook that the interviewed student had used during his Grade 13 mathematics course, and in P_3 the edition from 2022 was analysed. Between the publishing of the two editions of *Matematikk R2*, a new curriculum reform was implemented (Directorate for Education and Training, 2020b). By selecting the edition of *Matematikk R2* published after 2020 (Borge et al., 2022), an insight into how the new curriculum has been implemented in a textbook was possible.

The last paper, P_4, is based on materials collected from an SRP. This includes handouts that were given before the start of the SRP, works (scientific articles, news articles, webpages, etc.) collected during the performance of the SRP, and a report written by the participant of the SRP.

Interview Tasks Used in P_1 and P_2

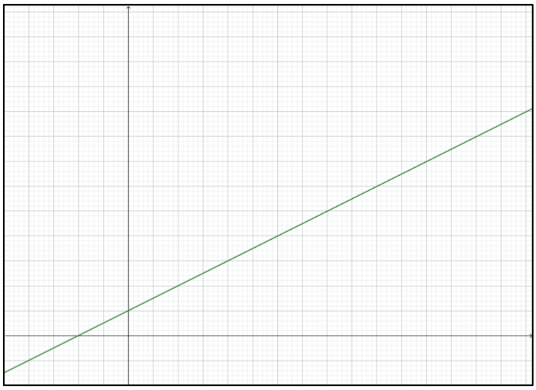
Four tasks were used in the student interviews referred to in P_1 and P_2. These tasks are:

t_1: Integrate the function $f(t) = t^2 + 2t$.

t_2: Integrate the following function. (“function” refers to the graph in Figure 4)

Figure 4

Graph of the Function Found in Task t_2



t_3: What can you say about $G'(x)$ in Figure 5?

Figure 5

General Function

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

What can you say about $G'(x)$?

t_4: What can you say about $G'(x)$ in Figure 6?

Figure 6

Periodic Function

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

where $f(t)$ is a periodic function with period 2, so $f(t) = f(t+2)$ for all $t \in \mathbb{R}$
What can you say about $G'(x)$?

Suggested solutions of the tasks, and reasons for the design of the tasks, can be seen in Topphol & Strømskag (2022).

4.3 Research Participants

The studies in this dissertation were designed (with the exception of the SRP), conducted, transcribed, and analysed by me. I have a master's degree in mathematics, with focus on numerical integration, and pedagogical education. Before starting the PhD work, I had six years of experience in teaching mathematics in upper secondary school. I am also the main author of all the papers that the dissertation builds on, the sole author of the papers P_1, P_3 and P_4.

The main supervisor of the PhD participates as the co-author on P_2 and is the designer of the SRP presented in P_4.

In the interview, six first-year mathematics students volunteered for being interviewed. All the students took part in a first-year university calculus course (MA1101, n.d.) in the autumn of 2019, and in lectures and task sessions during the beginning of the course the students were informed about the study and recruited in person by me. Five of the students participated in two interviews each, while one student participated in one interview. Each interview was with one student, making each interview a didactical system with two persons, the student and the interviewer, participating. All interviews were conducted by me.

4.4 Data Analysis

4.4.1 Analysis of a Process of Instrumental Genesis

In P_1, where the IA was used, the analysis was based on the two components of the *instrumental genesis*, the *instrumentation* and *instrumentalisation* (Rabardel & Samurçay, 2001), the focus being on instrumentation. The paper was mainly written as a theoretical paper, and the analysis presented was aimed at being a support for the claim that tasks *can* be seen as instruments. To do this, an *instrumental genesis* of mathematics tasks had to be identified, and essential to this analysis was therefore the identification of an *instrumentation process*. The method is based on observing statements and gestures consistent with the three phases of instrumentation, the *discovery and selection of relevant functions*, *personalisation*, and *transformation of the artifact* (Trouche, 2004). Instrumentalisation was assumed to occur whenever a student learns something from a task, since the task would then *act upon* the student by allowing the acquisition of new knowledge, and thus the task "*prints its mark* on the subject" (Trouche, 2004, p 290).

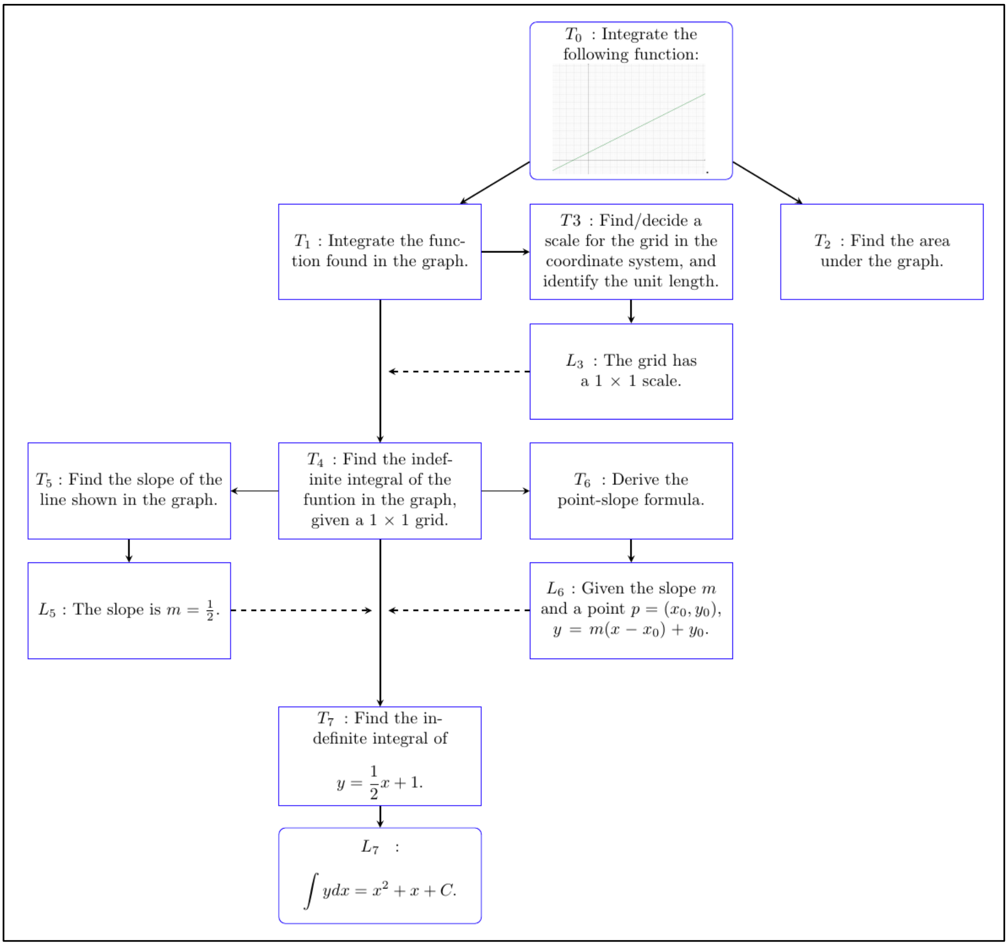
4.4.2 Analysis Using the Model of Didactic Moments

The interview data were first transcribed, with a focus on utterances and on what was written and drawn by the students. Gestures were transcribed when deemed relevant to the communication about the tasks. These were typically pointing or hand gestures indicating specific areas of interest on a graph, formula, or equation, or "drawing in the air" suggesting overall shape of functions or curves.

The analysis can be divided in two steps. First, a general step, where information about the argumentation built up by and communicated by the students was organised in flowcharts (see an example in Figure 7). The subtasks that the students derived from the tasks they were given were identified by examining the choices that the students made during the solving process and by examining utterances and gestures. Each time a student mentioned a question about the task, a challenge found in the task, made a choice, or expressed the necessity to do *something* to solve the task, it was interpreted in the analysis as a subtask. Thus, the solution presented in the flowcharts are often not the optimal solutions, as they include investigations or tasks that do not lead to the solution presented in the end.

Figure 7

Flowchart Mapping out the Exploration Process of a Task



In the graphic representation, each T_n denotes a subtask or subproblem, and each L_n a *corresponding* solution or answer. The numbering of the subtasks is done according to when they first appeared in the interview data, while the numbering of the answers indicates which subtask the given answer belongs to. Further, two types of connections between subtasks and solutions are illustrated. A solid arrow indicates a *direct derivation*, *generalisation,* or *interpretation* of a task. This is either a new subtask, possibly necessary to solve the task it is derived from, or an answer to the given task. The dotted arrows indicate *warrant*, that is, it shows how the answer to one subtask lends support to another connection. For example, in the example above (Figure 7), the student had forgotten the point-slope formula, and could therefore not formulate the general function, as described in T_4, as a specific function, as in T_7, without working out this formula. Therefore, the answer L_6, to the task T_6, enables the student to make the connection (a reformulation) between T_4 and T_7, and is thus interpreted as supporting this connection. In the example above (Figure 7), one can see how the student generalises the task of integrating a function found in a graph (T_1), to the task of integrating the function $y = \frac{1}{2}x + 1$. The student also explicitly identified the need to find the slope of the line, and by mentioning this, it gave reason for analysing this as a subtask. Note, however, that the possible subtask of finding the y-intercept of the function is not represented as a subtask. That is because it was not mentioned in any way by the student as a specific "thing to do". The technique was just applied without any mention.

By identifying the subtasks that the students communicated, and by identifying connections between them, the flowchart, resembling a directed graph, is constructed. This is then used in the second step of the analysis, a *praxeological analysis* (Chevallard, 1989; Chevallard & Bosch, 2014). In the analysis presented in P_2, the praxeological analysis of the work of the observed student is centred on the notion of didactic moments. As part of the praxeological analysis of the student work, a *praxeological reference model* is constructed, against which the student work is analysed.

4.4.3 Didactic Transposition Analysis

In both P_2 and P_3 sections of Grade 13 mathematics textbooks are analysed. P_2 contains a short analysis of themes relevant to the analysis of the interview data. In the paper, one student was selected as a particular case, and the section of the textbook that he had used in Grade 13, containing a presentation and treatment of the FTC was therefore analysed. Using this analysis of the textbook, causal relations between the mathematical organisation found in the textbook and the praxeological development of the student could be identified.

The paper P_3 presents a praxeological analysis of the textbook chapter detailing the FTC in the textbook *Matematikk R2* (Borge et al., 2022). The analysis is modelled partially after the analysis found in Wijayanti and Winsløw (2017). An addition to Wiyajanti and Winsløw's method is a structuring of the analysis around the terms *structure, functioning*, and *utility* (Chevallard, 2022) of the mathematical object under study.

The analysis follows Wijayanti and Winsløw (2017) by being a *didactic transposition analysis*. The analysis consists of three steps. First, a reference praxeological model of the theme of integration is constructed. The goal of the reference model is to present a general statement of the FTC, a description of which concepts are connected by the theorem, and what sort of tasks and techniques can be derived from the theorem. This serves as a general model of the theme, and as a model of scholarly knowledge, as well as being a reference towards which knowledge to be taught, knowledge taught, and learned knowledge can be analysed. Specifically, to P_3, it serves as a general model of knowledge of the FTC, towards which knowledge to be taught, as represented by the Grade 13 textbook *Matematikk R2* (Borge et al., 2022), can be analysed. The second step consists of analysing the textbook by identifying the praxeological elements present in the textbook. The third step is to compare the praxeological organisation found in the textbook to the reference model. In that way, the results of the *didactic transposition* that has generated the organisation in the textbook can be identified, together with what consequences this might have for teaching and learning using the textbook.

Where the analysis in P_3 diverges from Wiyajanti and Winsløw is in the explicit organisation of each of the three steps of the analysis around the notions of structure, functioning and utility of the mathematical object. By doing this, the analysis enables an identification of characteristics of the mathematical object under study, the FTC. These characteristics again imposes certain conditions and constraints for the teaching of the FTC and integral calculus, and on what sort of insights students are able to get from the study of the FTC using the textbook which has been analysed in P_3.

4.4.4 Analysis of the Construction of an SRP

The last numbered paper in the dissertation, P_4, presents an analysis of an SRP based on the generating question Q_0 = "Why do babies die of heat stroke in cars parked in the sun?". The question was accompanied by a handout describing the SRP, including the question Q_0, a guideline that pointed out that the answer should be mathematically oriented.

The analysis was carried out in two phases. The first phase consisted of conducting the SRP. This was done by first investigating the generating question and the literature provided in the handout. From this, more questions are derived, the *derived questions*. The derived questions are categorised according to what sorts of answers are likely to emerge from answering them. As the guideline pointed out a mathematical orientation, and in particular mathematics focused on modelling a physical system, this was also a categorisation criterion. A literature search was then conducted, both by following the trail of references found in the handed-out literature, and by general search using keywords related to hyperthermia, heat death, babies, parked cars, allometry, biological scaling, surface area, and heat transfer.

The second phase consists of an analysis of the didactic potential of the generating question. This includes both a description of how the answers came about as a result of the generating question, and what other potential avenues of investigations could be derived from Q_0, that was not investigated in this SRP. A particular focus is on how the SRP is not only conditioned and constrained by Q_0, but also by the implicit and explicit preconditions and expectations surrounding the answers to Q_0.

4.5 Strengths and Limitations of the Reported Research

One structural characteristic of this research project gives rise to both strengths and limitations, namely the differences in both theoretical, methodological and thematical foci in the four papers. By basing the dissertation on four so diverse papers, a deep examination of one single didactical theme is difficult. Consequently, the conclusions arrived at in the papers are to some degree four separate conclusions. Care must therefore be taken when trying to generalise from the findings.

What the study lacks in deep focus on one single and specific didactic phenomenon, however, it gains in being able to identify multiple facets relating to more general didactical themes. An important part of these multiple facets is the fact that the reported studies focus on different *institutional positions*. P_1 is seen partly from the position of task designer. Here considerations necessary for creating tasks that allow students to use tasks as tools in a study process are identified. The position as student is also seen in P_1, which identifies some possibilities under the conditions and constraints imposed by a study situation with a single student solving exercises designed to be unfamiliar. P_2 is focused on the position as student, facilitated by the tool of flowchart, which visualises the choices that students make in a solving process, and therefore also help in indicating conditions and constraints that govern the solving process. P_3 focuses on the position of textbook designer and can therefore identify conditions and constraints that govern the writing of a textbook chapter. It also identifies how the choices, shaped by those conditions and constraints, in turn creates conditions for the classroom setting. P_4 focuses on the position as student in an SRP, and both serves to contrast the study in P_2, and to highlight unique conditions and constraints that govern the study of a more open question. It affords a study of the expectations, both implicit and explicit, about the nature of the answer to the studied question

The studies report on small-scale[8] study processes (P_1, P_2, and P_4 specifically), which stands in contrast to what is common in research in the ATD (e.g., Bourgade, 2016; Florensa, 2018; Jessen, 2017). The studies therefore occupy a place which is relatively unique in the ATD tradition. The didactical systems seen in the studies are closer to the ones seen in a tutoring situation or autonomous self-study processes. In the light of promoting the PQW to all types of

[8] Small-scale in terms of numbers of participants, with each of the study situations containing only one student, and either one or zero "helpers".

study processes (not only large scale and cooperative study processes) the studies in the dissertation can therefore give valuable insights to the conditions and constraints that govern studies with more limited resources and workforce.

4.6 Ethical Considerations

All participation in the interviews was voluntary, and the selection was done by actively opting in. All students who opted in were part of the study. As required by Norwegian regulations, the project has been registered in Sikt (formerly NSD, the Norwegian Agency for Shared Services in Education and Research, https://sikt.no/en/home), and the collection, storing and handling of personal information has been approved, with project reference number 806467. All participants in the interviews have been informed in writing about their rights and have signed a consent form. The interview data have been transcribed in anonymous form, and references to the individuals participating in the interviews are by codenames, to secure the participants' anonymity.

The authors of the textbooks analysed in P_2 and P_3 have been noticed in writing about the project. High resolution versions of illustrations and digital versions of the referenced chapters of the textbooks have been made available to the research project upon request. Permissions to use the texts and the original illustrations have been given, provided that the illustrators, as well as the authors are given due credit in the papers. This has been followed in both P_2 and P_3. In addition, the editorial director of the textbooks has requested to be notified of the publishing of P_2 and P_3, and to receive copies of the finally published papers. This request will be followed as soon as the papers have been published.

5 Results and Discussion

5.1 Summary of Results From Individual Papers

In this part of the dissertation, I will summarise the individual contributions of each of the papers, before I explain the relationship between the published works, and how they all relate to the overarching project. In the papers P_1 and P_2, two cases from the student interviews are given particular focus. Episodes from one student interview are presented in P_1, while episodes from another student interview is presented in P_2. The student in P_1 was simply called “the student” in the paper, while the student in the case in P_2 was given the pseudonym “John”. Both students were first-year students following a calculus course at NTNU took part as volunteers in the interviews. The student in P_1 had already attended one university mathematics course in algebra while still being a student in upper secondary, and the conditions for his solving of the tasks in the interview was therefore likely different from the conditions and constraints governing the five other students’ solving process.

5.1.1 Discussion and Conclusion From P_1

This paper presents the first analysis of the student interviews. Specifically, the paper presents an analysis of one student interview, as part of an examination of the question "Based on a series of task-based interviews in early university calculus, and using the instrumental approach, what sort of evidence is there for saying that tasks can be used as instruments for developing mathematical competence?".

To conclude that an instrumental genesis is taking place, both elements of both the instrumentation and instrumentalisation processes were identified. Specifically, it was observed that the student in the case presented was able to change the task t_4 (described in Section 4.2.1), from a task asking about periodic functions, to a less general task, which he called an “extreme example”. The change of the task is interpreted as evidence of an instrumentalisation, as the student creates an instrument, the new, more specific example, suitable for exploring the original task t_4.

Two consequences were noted which places some constraints on the design of tasks with a goal of enabling the development of mathematical competence. First, the tasks and examples that are given to students need to be carefully designed in such a way that not only the mathematical themes are evident, but also the nature of the underlying mathematical questions, and how the tasks can be changed to allow for the investigation of different aspects of the mathematical objects. And second, the students need to be given the opportunity to take part in this exploration of how tasks can be changed and designed purposefully to allow for a more careful investigation of a mathematical theme.

5.1.2 Discussion and Conclusion From P_2

This paper presents an examination of a praxeology built up by John working on the four tasks t_1 to t_4 (see Section 4.2.1). In the paper, a praxeological reference model for these tasks is presented. The observations made in the analysis of the praxeology built up by John are corroborated and substantiated by the analyses of the five other students' work on t_1 to t_4. This includes the claim that the tasks seem to be new to the students and can be seen both by statements that the students give and by the nature of their work with the tasks. The last point can easily be seen in the flowcharts used to illustrate the solving process.

Through the analysis of the interview data, and through a comparison to the textbook (Heir et al., 2016) that John had used during Grade 13, a causal relationship between the mathematical organisation he had been exposed to and difficulties he later had in the praxeological development was revealed. Particularly clear was a lack in the conceptualisation of the area in terms of a function, essentially Part I of the FTC. This posed an important constraint on the solving of tasks of the same type as t_2, which made the task unsolvable.

Two other phenomena were identified. Related to the first observation, a difficulty in generalising is seen. t_3 and t_4 could only be solvable by defining a new general function, $F(t)$ for which $F'(t) = f(t)$, but as John expressed in the second interview, the idea of using this as a technique to solve a task had never occurred to him before university calculus. And the second observation is a relative evaluation of algebraic and symbolic technique as more useful than graphical techniques. This is likely attributed to the algorithmic focus in upper secondary teaching, and probably strengthened by the focus on rigor in university mathematics courses. This is seen even in the exploratory phase of the praxeological development, where techniques of calculation have not yet been established. Using graphs and drawings seems to be an underdeveloped technique and is used mainly for illustrative purposes.

In sum, two important constraints on the solving process were observed, namely the missing idea of area in terms of a function, which is the essentially the result of the FTC Part I, and the missing idea of generalisation of the function concept. Both these missing ideas were common difficulties among all six students, with a partial exception for the student in P_1, which managed to solve t_4. Moreover, John managed to solve t_3 in the second interview, referring precisely to the idea of defining a general function as the solving idea. Both these two observations indicate that the lack of these two ideas are constraints relating to the institutional position as a Grade 13 student, and not only specific to the individual students interviewed. A third condition is the nature of the formulation of the tasks themselves. This was particularly evident in t_2, which was purposefully formulated vaguely. This made the precise interpretation of this task difficult, and several students, including John, noted a difficulty in understanding what the task was asking the students to do.

Two suggestions were proposed for further avenues of action. First, a deeper look at textbooks and their praxeological organisation of integration and the FTC is warranted. This led to the research proposal for P_3. And last, it adds to the suggestions of diversifying teaching methods, that includes giving students strategies for exploring unfamiliar problems. These strategies will need to include both different modes of exploration, not just algebraic and symbolic, but also graphic. Combining these two suggestions, the paper raises the question of textbook design, and what the role of textbooks could be in the new paradigm.

5.1.3 Discussion and Conclusion From P_3

In P_3, a didactic transposition analysis of Chapter 2 of the textbook, *Matematikk R2* (Borge et al., 2022). *Matematikk R2* is a new textbook written for Grade 13 advanced mathematics, and Chapter 2 is the introduction to integral calculus. The textbook was published after a recent curriculum reform (Directorate for Education and Training, 2020b) and is therefore an example of a written implementation of the content of this new reform. The publisher of *Matematikk R2*, is also the publisher of the textbook that John from P_2 had used during Grade 13 mathematics.

The analysis presented reveals that, although the FTC was presented in a more rigorous way after the introduction of the new curriculum, and justified by other previously treated theorems and definitions, a new problem was introduced. The presentation of the definitions upon which the FTC is based, continuity and the definition of the integral, is missing the notion of boundedness.

Without boundedness, three tasks found in the textbook cannot be connected or justified properly and appear more as mathematical curiosities. With the addition of boundedness, the three tasks could have functioned as a foundation for both a strengthening of the logos with regards to the theme of integration and the theme of continuity of functions in general. The tasks could also have functioned as justification for generalising the definite integrals to also include partially continuous functions and improper integrals. All the technical components to do so are present, but the missing notion of boundedness makes the connection impossible.

Thus, in the textbook, the conditions for expanding and modifying the praxeology of the FTC, beyond the introduction of examples that can be described as *mathematical curiosities*, are not present. To be able to expand the praxeology systematically, a notion of boundedness and of integrability would be necessary. It is not clear, however that the conditions for *actually* including these notions are present. Both boundedness, and criteria for integrability in general, are concepts that students have been shown to struggle with (e.g., González-Martín & Camacho, 2004; González-Martín & Correira de Sá, 2007; Rúbio & Gómez-Chacón, 2011). To give these themes a proper treatment, either more time is needed on the study of mathematics in general, or these themes would need to replace another theme. Thus, the analyses of the praxeological organisation of integration in Matematikk R2 described in P_3 illustrates well the effects of the constraints that the textbook authors are under. The change in the curriculum introduces a requirement to include a more thorough treatment of the FTC, but a similar imperative to treat the underlying concepts of integrability and boundedness is not present.

5.1.4 Discussion and Conclusion From P_4

P_4, which was presented at "the 7th Conference of the Anthropological Theory of Didactic", was the second paper to be accepted for publication. The SRP presented in the paper was special in three ways. First, it was a small-scale SRP, both in terms of the number of participants and in terms of time limitations. Although two students took part in the course which the SRP was a part of, each student performed an SRP individually, both answering the same generating question Q_0. There was, however, a seminar about halfway during the course of the SRP, where the two students and the two course supervisors met and discussed the progression so far and the preliminary answers to the generating question. The SRP was also performed over the course of only five weeks. Since I was both student performing the SRP and the author of the paper P_4, the insights gained from occupying the student position gives the paper a unique perspective. The experiences relayed are therefore first-hand experiences. And third, expectations about how Q_0 should be answered have been made explicit in a written handout before the start of the SRP. Particularly, the handout states that the answer should be mathematical, and suggests an investigation of physical and physiological factors that makes babies extra vulnerable to heat stroke. They also suggest an approach of modelling the human body.

The direct result from the SRP was a mathematical model, connecting the geometric characteristics of a human body, its heat capacity and heat transfer potential, and the effect of the ambient temperature. This model was again used to answer Q_0. But from the investigation of the generated SRP, it became clear that there were many avenues that were not investigated, and the sole reason not to investigate them was the expectations of a mathematical-physiological answer. Other derived questions could possibly also be answered using mathematics, like statistics and probability, but were not necessarily connected to physiological factors.

The explicit expectations in the handout therefore made it possible to discuss the role and nature of the generating question Q_0. The preconditions and expectations that are communicated about the answer quite clearly act as conditions and constraints on the course of the SRP, both with regards to what the accepted answer is, and with regards to what derived questions are further investigated. Especially due to the constraints put on the formation of the finally accepted answer, a question of whether the preconditions should be regarded as a part of the generating question itself or as something different and external to the generating question emerges from this.

An argument is made that much of the preconditions are represented in the pre-existing answers, $A^{\lozenge}_i$. However, it is noted that pre-existing works do not suffice to fully explain the effect the preconditions have, and particularly the expectations about the *nature* of the answer. Nor is it obvious that one could analyse the preconditions as parts of Q_0. The argument for this view is that expectations about both what Q_0 is about, and what would be an acceptable answer might differ between different people. The conclusion on this part of the discussion is therefore that the locus of the expectations about the nature of an answer to Q_0 in an SRP is not easily analysable in the current framework. The expectations do have a strong guiding force on how Q_0 is answered, something which could points towards them being an integral part of the question Q_0. But that seems to obscure the fact that different persons might have different expectations about the answer $A^{\heartsuit}$. Thus, neither saying that the preconditions and expectations are all part of the pre-existing answers $A^{\lozenge}$, nor saying that they are part of the generating question Q_0 is entirely satisfactory. A strong conclusion to this last question is not reached and left as an open question. The main result with regards to conditions and constraints governing the formation of an answer to Q_0 is in pointing to the *importance* of the a-priori expectations about the nature of the answer, both when analysing an SRP and when designing and implementing SRPs.

5.2 General Discussion

In the three papers P_1, P_2 and P_3, the mathematical themes of the FTC and integral calculus are at stake, either as the theme of tasks designed and implemented in student interviews, or as the theme for textbook chapters. In P_4, the theme is not directly related to the FTC, but similar to P_2 and to some extent P_1, a special focus on the student position in a study situation can be seen.

Common to all papers are a focus on conditions and constraints governing the study of mathematical themes, either directly observed in study situations (P_1, P_2, and P_4), or identified in textbook analyses (P_3 and P_2). Conditions governing other positions, like the teacher position, can also be seen, but not to the same extent. Importantly, the constraints identified for the student position might be conditions for the teacher position, but not *necessarily* constraints.

5.2.1 Conditions and Constraints Related to the Student Position

In both P_1 and P_2, study situations involving the tasks t_1 to t_4 are presented, and conditions and constraints governing the role these tasks have in the study situations are identified. In P_1, for tasks to function as instruments for the development of mathematical competence, two conditions were suggested, both relating to the forming and posing of mathematical questions. The first is the design of the tasks and examples in such a way that the nature of the tasks and how tasks can be changed to explore mathematical themes. The second relates to the study situation itself, when it is suggested that students themselves need to take part in the posing and changing of tasks. Although the conclusion from the research in P_1 is a result of applying the IA, the conclusion does seem to echo the ATD notion of the PQW and the importance of studying *questions*. This is particularly evident when considering that a task can both be formulated as a question, and even when it is not formulated as a question, a mathematical question is still the foundation of the task. An ability to consider and formulate mathematical questions is therefore necessary for changing and creating tasks.

In P_2 it was demonstrated how the lack of an idea resulting from the FTC Part I, that of an area function, the definite integral $A(x) = \int_a^x f(t)dt$, made solving t_2 impossible. It was also demonstrated how the lack of the idea of defining general functions as a technique constrained the development of the techniques necessary to solve t_3 and t_4. This reflects the findings presented in Winsløw (2008), demonstrating students' struggles with abstraction. The fact that all the students showed these difficulties supports the claim that these ideas are constraints on the institutional position as Grade 13 students. Moreover, the only exceptions also serve to support this conclusion. The first exception is the student in P_1, who was the only student who was able to solve t_4. As noted in Section 5.1, he had already followed a university course in algebra, which likely allowed him to overcome the constraints that would otherwise have governed the position he was occupying. And the second exception is seen in John's performance in the second interview, where he did manage to solve t_3 by defining a general function. This also indicates that the constraints belong mainly to the position of Grade 13 students rather than university calculus students.

Important conditions and constraints are introduced by the study material available to the students, and particularly by the textbooks that the students have available. After all, the textbook is still one of the most important resources for students (Rezat, 2010). In P_2, a connection is identified between the difficulties the students have when attempting to solve the tasks t_1 to t_4, and missing elements of the praxeological organisation of the textbooks they had used. Additionally, although the missing praxeological elements identified in P_2 were to a large extent remedied in the new edition of Matematikk R2 (Borge et al., 2022), new constraints were introduced by the lack of the notion of boundedness and the lack of a proper treatment of integrability. This likely introduces strong restrictions on possible expansions on the praxeology surrounding the FTC found in the textbook.

This is contrasted with the conditions and constraints governing the student position in a study situation focused on an SRP. In particular, the conditions imposed by a textbook will naturally not be as important, as the students here are encouraged to do the information search themselves. Other conditions and constraints are, however, more important, and in particular conditions and constraints related to the formulation and context of the generating question. This is seen clearly in P_4. Students in a situation centred on an SRP are now given the conditions for both changing and posing questions, and for searching for literature themselves, rather than relying purely on the tasks and literature provided by the teacher. Communication about preconditions and expectations about the nature of the finally accepted answer is, however, more important, as these have a strong guiding force on what course the SRP takes. This adds to the literature showing the importance of the nature, and even the formulation of the question itself (e.g., Bourgade, 2016) and of the effects of the expectations and preconceptions that students have about a theme (e.g., Strømskag, 2022).

5.2.2 Conditions and Constraints Related to the Teacher and Textbook Author Positions

The constraints described for the student position is, however, not necessarily clearly identified as constraints for the teacher position and for the textbook author position. The teacher is, after all, in the position to change the conditions by introducing new teaching materials and select tasks and examples that the teaching is focused on, while the textbook authors are in position to *produce* these materials.

The conditions imposed by the textbook are still significant for the teacher position. The importance of the textbook for students (Rezat, 2010) is also true for teachers (Lepik et al., 2015), something which stresses the importance of the quality of the textbooks. Although the praxeological organisation of a textbook does not *constrain* the teachers' organisation and presentation of the subject matter, it does strongly condition it. It does so in concert with the constraints imposed by the available time for instruction (Leong & Chick, 2011; Teig et al., 2019) and pressure from high-stakes testing regime (Chichekian & Shole, 2016).

The evidence presented in P_3 does also illuminate some conditions for textbook authorship. Both the time constraint that affects the teachers, and the constraints put on the school system by the curriculum have a strong effect on the design of textbooks. Together with the other papers, the findings in P_3 illustrate well the causal links between the change in curriculum and the design of textbooks. And importantly, it does also indicate that the changes made in the curriculum do not suffice to move away from the PWV. For that to occur, it likely requires a more overall reorganisation of the school system in general, which addresses the time constraints (Leong & Chick, 2011; Teig et al., 2019) and high-stakes testing (Chichekian & Shole, 2016) that still govern teaching and learning.

5.3 Conclusions and Open Questions

5.3.1 General Conclusions

The general observations about the conditions and constraints affecting both students, teachers and textbook authors do highlight the interconnectivity between the conditions and constraints that govern the different institutional positions in the school system, and especially the conditions and constraints imposed by curriculum contents and curricular materials. A causal link was established in P_2 between textbooks and the praxeologies students develop in mathematics, and a similar causal link between the curriculum reform and the praxeological organisation of textbooks is established in P_3. Although the intension behind the curriculum reform is to encourage deep learning (Directorate for Education and Training, 2019) and increasing autonomy in inquiries about mathematical themes (Directorate for Education and Training, 2020a), there are conditions and constraints that work against these principles, like the time constraints and the high-stakes testing regimes that students and teachers are under. Even under the new curriculum, the textbook analysed seems to still be affected by the PVW, despite efforts to introduce new types of tasks aimed at inquiry. P_1 also suggests that if the goal is for students to become autonomous in their study of mathematics, they need to be given the opportunity to create and modify tasks, and therefore also ask mathematical questions themselves, an essential element of the PQW. A shift from focusing on established works to a focus on questions entails a different set of conditions and constraints, in particular conditions and constraints surrounding the posing and answering of a generating question and the preconditions and expectations about the finally accepted answer.

5.3.2 Open Questions

As a result of the multifaceted nature of the dissertation, many different avenues for further questions and investigations can be derived. These relate to themes ranging from the role of textbooks and other curricular resources for teaching, the nature and context of mathematical questions and their relation to the inquiry processes, and the changing roles and responsibilities of students and teachers in the PQW. Stemming from the conditions and constraints identified, three major avenues seem particularly fruitful.

In the research presented in the dissertation, all the study situations analysed in P_1, P_2, and P_4 have been "small-scale" study situations, meaning all the didactical systems have been of the type $S(x, y, o)$ and $S(x, Ø, o)$, with a single student. Moreover, the mathematical questions investigated by the students in P_1 and P_2, when solving the tasks that they were given, have also been "small-scale questions", meaning questions that are closed-ended, asking for a specific answer, within a single mathematical context. This then raises the question of the place of these types of study situations in the PQW. How could the principle of focusing on *questions* as it appears in the PQW be applied to smaller scale study situations? After all, questions with different scope do exist also outside of the classroom, and the study of these "small-scale questions" in "small-scale study situations" might also benefit from the focus promoted by the PQW.

A second avenue is the question of textbooks and their potential role. What role could mathematics textbooks play under the PQW? Are they obsolete relics of the past educational systems, or can they still play a significant role? One thing seems to be clear. In the new paradigm, they cannot play the only role. If exploration and study processes of different scales, different modalities, and with different purposes are to become the core of education, teaching would have to expand its scope to include all sorts of sources of knowledge, and textbooks would need to find a new place in this landscape. If textbooks still do play a role in the PQW, how should they be designed? What design principles should be implemented to accommodate for an expansion of the knowledge base?

And third, the roles and responsibilities of students and teachers still need to be investigated in relation to the PQW. The study in P_4 demonstrates the significance of the preconditions and expectations about a finally accepted answer to a generating question in shaping the course and outcome of an SRP. Two questions arise from this. First, the question of the burden of responsibilities, and how these preconditions and expectations are to be handled in a classroom setting. How do one decide the criteria for accepting an answer, who gets to decide, and what role should the explicit preconditions and expectations take in this? And the second question is about predictability. To what extent, and in which way can the course and outcome of an SRP be predicted by the preconditions and expectations?

References

Adams, R., & Essex, C. (2018). *Calculus: A complete course* (9th ed.). Pearson.

Artaud, M. (2020). Praxeologies to be taught and praxeologies for teaching. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in mathematics education: A comprehensive casebook* (pp. 33–40). Routledge.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM – International Journal of Mathematics Education*, *45*(6), 797–810. https://doi.org/10.1007/s11858-013-0506-6

Artigue, M., & Bosch, M. (2014). Reflection on networking through the praxeological lens. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 249–265). Springer.

Barquero, B., Bosch, M. (2015). Didactic engineeesing as a research methodology – From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education* (pp. 249–272). Springer. https://doi.org/10.1007/978-3-319-09629-2_8

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education – Dealing with institutional constraints. *ZDM – International Journal of Mathematics Education*, *50*, 31–43. https://doi.org/10.1007/s11858-017-0907-z

Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *Journal of Mathematical Behavior*, *26*(4), 348–370. https://doi.org/10.1016/j.jmathb.2007.11.001

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education* (pp. 483–506). Springer. http://dx.doi.org/10.1007/978-3-642-00742-2_46

Bikner-Ahsbahs, A., & Prediger, S. (Eds.) (2014). *Networking of theories as a research practice in mathematics education*. Springer.

Borge, I., Engeseth, J., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2022). *Matematikk R2* [Mathematics R2] (4th ed.). Aschehoug.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51–65.

Bosch, M., Gascón, J., & Trigueros, M. (2017). Dialogue between theories interpreted as research praxeologies – The case of APOS and the ATD. *Educational Studies in Mathematics*, *95*(1), 39–52. https://doi.org/10.1007/s10649-016-9734-3

Botsko, M. W. (1991). A fundamental theorem of calculus that applies to all Riemann integrable functions. *Mathematics Magazine*, *64*(5), 347–348. https://doi.org/10.1080/0025570X.1991.11977632

Bourgade, J. P. (2016). Difficulties of the profession and dissemination as catalysers for the construction of professional praxeologies. In E. Nardi, C. Winsløw, & T. Hausberger (Eds.), *Proceedings of INDRUM 2016 – First conference of the International Network for Didactic Research in University Mathematics* (pp. 350 – 359). University of Montpellier and INDRUM.

Broley, L., & Hardy, N. (2022). The evolution of students' learning from calculus to analysis: How students solve analysis tasks that look like calculus tasks. *International Journal of Research in Undergraduate Mathematics Education*, *8*(2), 269–292. https://doi.org/10.1007/s40753-022-00171-2

Brown, A. B. (1936). A proof of the Lebesgue condition for Riemann integrability. *The American Mathematical Monthly*, *43*(7), 396–398. https://doi.org/10.1080/00029890.1936.11987865

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In H. G. Steiner & M. Hejny (Eds.), *Proceedings of the International Symposium on Selected Domains of Research and Development in Mathematics Education* (pp. 51–62). University of Bielefeld and University of Bratislava. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/On_Didactic_Transposition_Theory.pdf

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique [The analysis of teaching practices in the anthropological theory of the didactic]. *Recherches en Didactique des Mathématiques*, *19*(2), 221–266. https://revue-rdm.com/1999/l-analyse-des-pratiques/

Chevallard, Y. (2006). Steps towards a new epistemology. In M. Bosch (Ed.), *Proceedings of the 4th Congress of the European society for Research in Mathematics Education* (pp. 21–30). FUNDEMI-IQS and ERME.

Chevallard, Y. (2007). Readjusting didactics to a changing epistemology. *European Educational Research Journal*, *6*(2), 131–134. https://doi.org/10.2304/eerj.2007.6.2.131

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: a case for an oncoming counter paradigm. In S. J. Cho (Ed.), *The proceedings of the 12th International Congress on Mathematical Education: Intellectual and attitudinal challenges* (pp. 173–187). Springer International Publishing. https://doi.org/10.1007/978-3-319-12688-3

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114. https://doi.org/10.24529/hjme.1205

Chevallard, Y. (2020a). Some sensitive issues in the use and development of the anthropological theory of the didactic. *Educação Matemática Pesquisa*, *22*(4), 13–53. https://doi.org/10.23925/1983-3156.2020v22i4p013-053

Chevallard, Y. (with Bosch, M.). (2020b). A short (and somewhat subjective) glossary of the ATD. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in mathematics education – A comprehensive casebook* (pp. xviii–xxxvii). Routledge.

Chevallard, Y. (2022). *What is questioning the world? Towards an epistemological and curricular break* [Keynote address]. 7th International Conference on the Anthropological Theory of the Didactic (CITAD7). Barcelona, Spain.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Chevallard, Y., & Bosch, M. (2020). Anthropological theory of the didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 53–61). Springer. https://doi.org/10.1007/978-3-030-15789-0_100034

Chevallard, Y., & Sensevy, G. (2014). Anthropological approaches in mathematics education, French perspectives. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 38–43). Springer. https://doi.org/10.1007/978-94-007-4978-8_9

Chichekian, T., & Shole, B. M. (2016). Preservice and practicing teachers' self-efficacy for inquiry-based instruction. *Cogent Education*, *3*(1), Article 1236872. https://doi.org/10.1080/2331186X.2016.1236872

Coles, A., & Brown, L. (2016). Task design for ways of working: making distinctions in teaching and learning mathematics. *Journal of Mathematics Teacher Education, 19*(2), 149–168. https://doi.org/10.1007/s10857-015-9337-4

Directorate for Education and Training. (2019). *Hvorfor har vi fått nye læreplaner?* [Why have we got new curricula?]. https://www.udir.no/laring-og-trivsel/lareplanverket/stotte/hvorfor-nye-lareplaner/

Directorate for Education and Training. (2020a). *Core curriculum – learning to learn.* https://www.udir.no/lk20/overordnet-del/prinsipper-for-laring-utvikling-og-danning/2.4-a-lare-a-lare/?kode=mat03-02&lang=eng

Directorate for Education and Training. (2020b). *Læreplan i matematikk for realfag (matematikk R) (MAT3-02)* [Curriculum for mathematics in natural science and technology studies (mathematics R) (MAT03-02)]. https://www.udir.no/lk20/mat03-02

Drijvers, P., & Trouche, L. (2008). From artifact to instrument: A theoretical framework behind the orchestra metaphor. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives* (pp. 363–392). Information Age. https://doi.org/10.1007/978-1-4419-0146-0_7

Engeström, Y. (1978). *Learning by expanding: An activity-theoretical approach to developmental research.* Orienta Konsultit.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese way to improving mathematics teaching and learning.* Routledge.

Florensa, I. (2018). *Contributions of the epistemological and didactic analysis: question-answer maps in engineering and in teacher education* [Doctoral dissertation, Universitat Ramon Llull].

Gascón, J. (2003). From the cognitive to the epistemological programme in the didactics of mathematics; Two incommensurable scientific research programmes? *For the Learning of Mathematics*, *23*(2), 44–55. https://www.jstor.org/stable/40248420

González-Martín, A. S., & Camacho, M. (2004). What is first-year mathematics students' actual knowledge about improper integrals? *International Journal of Mathematics Education*, *35*(1), 73–89. https://doi.org/10.1080/00207390310001615615

González-Martín. A. S., & Correia de Sá, C. (2007). Historical-epistemological dimension of the improper integral as a guide for new teaching practices. In E. Barbin, N. Stehlíková, & C. Tzanakis (Eds.), *History and epistemology in mathematics education: Proceedings of the 5th European Summer University* (pp. 211–223). Vydavatelský servis. https://publimath.univ-irem.fr/numerisation/ACF/ACF08022/ACF08022.pdf

Heckman, P. E., & Weissglass, J. (1994). Contextualized mathematics instruction: Moving beyond recent proposals. *For the Learning of Mathematics*, *14*(1), 29–33. https://www.jstor.org/stable/40248100

Heir, O., Borge, I., Engeseth, J., Haug, H., & Moe, H. (2016). *Matematikk R2* [Mathematics R2] (3rd ed.). Aschehoug.

Jessen, B. E. (2017). *Study and research paths at upper secondary mathematics education: A praxeological and explorative study* [Doctoral dissertation, University of Copenhagen, Denmark]. IND's skriftserie.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. Oxford University Press.

Ladage, C., Achiam, M., & Marandino, M. (2020). Research on ATD outside mathematics. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in mathematics education: A comprehensive casebook* (pp. 12–29). Routledge.

Lagrange, J. B., Artigue, M., Laborde, C., & Trouche, L. (2001). A meta study on IC technologies in education: Towards a multidimensional framework to tackle their integration. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 111–122). PME.

Leong, Y. H., & Chick, H. L. (2011). Time pressure and instructional choices when teaching mathematics. *Mathematics Educational Research Journal*, *23*(3), Article 347. https://doi.org/10.1007/s13394-011-0019-y

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Prentice-Hall.

Lepik, M., Grevholm, B., & Viholainen, A. (2015). Using textbooks in the mathematics classroom – The teacher's view. *Nordic Studies in Mathematics Education*, *20*(3–4), 129–156. https://ncm.gu.se/wp-content/uploads/2020/06/20_34_129156_lepik.pdf

Lindstrøm, T. (2016). *Kalkulus* [Calculus] (4th ed.). Oslo: Universitetsforlaget.

Lithner, J. (2017) Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM – Mathematics Education*, *49*(6), 937–949. https://doi.org/10.1007/s11858-017-0867-3

MA1101. (n.d.). *Basic Calculus 1.* Retrieved 23 February 2023 from https://www.ntnu.edu/studies/courses/MA1101/2019#tab=omEmnet

MA1102. (n.d.). *Basic Calculus 2.* Retrieved 23 February 2023 from https://www.ntnu.edu/studies/courses/MA1102/2019#tab=omEmnet

MA1103. (n.d.). *Vector Calculus.* Retrieved 23 February 2023 from *https://www.ntnu.edu/studies/courses/MA1103/2019#tab=omEmnet*

Mellin-Olsen, S. (1996). Oppgavediskursen i matematikk. Rekonstruksjon av en diskurs [The exercise-discourse in mathematics. Reconstruction of a discourse]. *Tangenten, 7*(2), 9–15.

Merriam-Webster. (n.d.). Artifact. In *Merriam-Webster.com dictionary.* Retrieved 4 April 2023 from https://www.merriam-webster.com/dictionary/artifact

Mesa, V., Ma, Y., Quiroz, C., Gerami, S., Liakos, Y., Judson, T., & Chamberlain, L. (2021). University instructors' use of questioning devices in mathematics textbooks – An instrumental approach. *ZDM – Mathematics Education*, *53*(6), 1299–1311. https://doi.org/10.1007/s11858-021-01296-5

Niss, M. (2007). Opgavediskursen i matematikundervisningen [The exercise-discourse in mathematics teaching]. *MONA,* (1) 7–17.

Niss, M., & Højgaard, T. (Eds). (2011). *Competencies and mathematical learning*. Roskilde University. https://pure.au.dk/ws/files/41669781/THJ11_M

Opsal, H., & Topphol, A. (2015) Matematikkoppgåver og læringspotensial [Mathematics tasks and learning potential]. In P. Haug (Ed.), *Elev- og lærarrolla* (pp. 149–171). Det Norske Samlaget.

Prychitko, D. L., (1994). Praxeology. In P. J. Boettke (Ed.), *The Elgar companion to Austrian economics.* Edward Elgar.

Rabardel, P., & Samurçay, R. (2001, March). From artifact to instrument-mediated learning. In *New challenges to research on learning*. International symposium organized by the Center for Activity Theory and Developmental Work Research, University of Helsinki.

Rezat, S. (2010). The utilization of mathematics textbooks as instruments for learning. In V. Durant-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixt Congress of the European Society for Research in Mathematics Education* (pp. 1260–1269). INRP and ERME.

Rodrígues-Quintana, E., Bosch, M., & Gascón J. (2008). A networking method to compare theories: Metacognition in problem solving reformulated within the Anthropological Theory of the Didactic. *ZDM – Mathematics Education*, *40*(2), 287–301. https://doi.org/10.1007/s11858-008-0094-z

Rúbio, B. S., & Gómez-Chacón, I. M. (2011). Challenges with visualization: the concept of integral with undergraduate students. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2073–2081). University of Rzeszów and ERME. http://erme.site/wp-content/uploads/2021/06/CERME7.pdf

Shinno, Y., & Mizoguchi, T. (2021). Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics in textbooks: two cases from *kyouzai kenkyuu* in Japan. *ZDM – Mathematics Education*, *53*(6), 1387–1402. https://doi.org/10.1007/s11858-021-01269-8

Skovsmose, O. (2003). Undersøgelseslandskaber [Landscapes of inquiry]. In M. Blomhøj & O. Skovsmose (Eds.) *Kan det virkelig passe?* (pp. 143–157). L&R Uddannelse.

Strømskag, H. (accepted). *Study and research paths in teacher* education. In *Proceedings of the 7th International Conference of the Anthropological Theory of the Didactic.* Birkhäuser.

Strømskag, H., & Chevallard, Y. (2023). Didactic transposition of concavity of functions: From scholarly knowledge to mathematical knowledge to be taught in school. *International Network for Didactic Research in University Mathematics: INDRUM 2022*, 120–129.

Teig, N., Scherer, R., & Nilsen T. (2019). I know I can, but do I have the time? The role of teachers' self-efficacy and perceived time constraints in implementing cognitive-activation strategies in science. *Frontiers in Psychology*, *10*, Article 1697. https://doi.org/10.3389/fpsyg.2019.01697

Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM – Mathematics Education*, *53*(3), 507–519. https://doi.org/10.1007/s11858-021-01270-1

Timpler, C. (1612). *Philosophiae practicae systema methodicum; In tres partes digestum: In quo universa probe honesteque vivendi ratio tam generatim, quam speciatim per praecepta & quaestiones breviter ac perspicuè explicatur & probatur, pars prima; Complectens ethicam generalem, libris IV pertractatam* [A methodical system of practical philosophy; A treatise in three parts: In which the whole system of proper and honest living, both in general and specifically through instructions and questions, is briefly and clearly explained and demonstrated, the first part; About general ethics, treated in 4 books]. Apud haeredes Guilielmi Antonii. https://archive.org/details/bub_gb_IaPUCZDkdQYC (Accessed 24 February 2023)

Topphol, V. (2021). A novel application of the instrumental approach in research on mathematical tasks. In G. A., Nordtvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankova-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Preceedings of NORMA 20: The Ninth Nordic Conference on Mathematics Education* (pp. 265–272). NCM & SMDF.

Topphol, V. (accepted). A study and research path on hyperthermia in children left in parked cars. In *Proceedings of the 7th International Conference of the Anthropological Theory of the Didactic.* Birkhäuser.

Topphol, V. (2023). Didactic transposition of the Fundamental Theorem of Calculus. Manuscript submitted for publication. Department of Mathematical Sciences, Norwegian University of Science and Technology.

Topphol, V., & Strømskag, H. (2022). An analysis of a first-year university student's construction of a praxeology of integration tasks. Manuscript submitted for publication. Department of Mathematical Sciences, Norwegian University of Science and Technology.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, *9*, 281–307. https://doi.org/10.1007/s10758-004-3468-5

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education*, *10*(1), 77–101. https://doi.org/10.1007/BF03172796

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Harvard University Press.

Watson, A., & Ohtani, M. (Eds.) (2015). *Task design in mathematics education*. Springer.

Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *REDIMAT*, *6*(3), 307–330. https://doi.org/10.17583/redimat.2017.2078

Winsløw, C. (2008). Transformer la theorie en taches: La transition du concret a l'abstrait en analyse reelle [Turning theory into tasks: The transition from concrete to abstract in real analysis]. In R. Rouchier & I. Bloch (Eds.), *Actes de la XIIIème Ecole d'Eté de Didactique des Mathématiques* (cd-rom). La Pensée Sauvage.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018). Theorizing lesson study: Two related frameworks and two Danish case studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. Ní Shúilleabháin & A. Takahashi (Eds.), *Mathematics lesson study around the world. ICME-13 monographs* (pp. 123–142). Springer. https://doi.org/10.1007/978-3-319-75696-7_7

Papers

This section contains all the papers that the dissertation is built upon. The papers are presented here as they appear in print, with only minor corrections with regards to spelling or grammatical errors.

Paper 1

A Novel Application of the Instrumental Approach in Research on Mathematical Tasks

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Published in Preceedings of Norma 20

The Ninth Conference on Mathematics Education

A novel application of the instrumental approach in research on mathematical tasks

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In this paper I explore a new approach to analysing tasks in mathematics education. By seeing tasks as an instrument in the activity of learning mathematics, I propose to use the instrumental approach and the notion of instrumental genesis to describe how a student could be able to internalise mathematical knowledge and methods through working with tasks.

Keywords: student practices at university level, tasks, competencies, the instrumental approach.

Introduction

In the Nordic countries, a focus on the nature of mathematical tasks and their use in education has been a central theme (e.g. Bergqvist, 2007; Lithner, 2017), together with the notion of competence (Haavold, 2011; Lithner, 2017). This has in Norway over the last decades informed both policy-making and the discussion of students' mathematical achievements (Botten-Verboven et al., 2010).

The focus in this paper will be on tasks. I will present a, to my knowledge, new way of applying the instrumental approach (Rabardel, 2000) by describing mathematical tasks as instruments for developing mathematical competence. This is part of a PhD project focusing on first year university calculus and the secondary-tertiary transition. The link between tasks and transition can be seen in, for example (Bergqvist, 2007), where she examines tasks in early calculus courses. Roh and Lee also talk about tasks designed to "bridge a gap between students' intuition and mathematical rigor" (Roh & Lee, 2016, p. 34), pointing towards a connection between how tasks are formulated and presented in upper secondary and in university, and the issue of transition. One of the main research questions in my PhD project, is "How can tasks be used as instruments in developing competences?" I will however, not be able to answer this question fully in this paper. Instead, I will focus on the narrower question "Based on a series of task-based interviews in early university calculus, and using the instrumental approach, what sort of evidence is there for saying that tasks *can* be used as instruments for developing mathematical competence?"

I will divide the argument into two parts. First, I present my theoretical framework. Then I make my case for why tasks can be seen as instruments according to the instrumental approach. The use of the theory will be exemplified through a short case study, selected from one of the interviews.

Theoretical framework

There have been several ways of describing tasks and describing ways of implementing and working with tasks (Watson & Ohtani, 2015). Tasks have been described as mediating artefacts in teaching and learning mathematics by Clarke, Strømskag, Johnson, Bikner-Ahsbahs and Gardner (2014) and by Johnson, Coles and Clarke (2017). The idea of tasks as artefacts can also be identified in an article by Watson and Mason where they talk about "seeing an exercise as a single mathematical object" (Watson & Mason, 2006, p.91). Tasks and their role in mathematics education is a matter, not only of being the things that one does in class to practice doing mathematics, but they may also act as an aid in developing deeper mathematical insight. It is not only the content of the task that matters for what is being learned, but also how the task is designed and embedded in the teaching context, and what sort of guidance is given before, during and after solving the task. The last point can be evidenced by the findings of Haavold (2011), where he shows that even high achieving students tend to rely on imitative reasoning rather than creative reasoning, when proper guidance is not given.

In this article I use Activity Theory (AT) (Leont'ev, 1978) to describe the context in which tasks are solved. Through activity, humans try to achieve some *objective*, and this activity is made possible by and mediated through artefacts. Such an activity is said to be *object-oriented*. The artefact plays an important role, as the activity that is conditioned by the artefact could not even be possible without the artefact. In Leont'ev's description, human activity is divided into processes of three different levels, where the activity itself is at the top most level, and is driven by some *motive*, meaning there is a need that must be fulfilled, which *motivates* the activity. The difference between objective and motive is a subtle one. I use the word objective when talking about the concrete end towards which the activity is directed, and motive when talking about the drive towards this objective. Further, each activity consists of *actions*, which have different *goals*. A goal is the concrete end towards which an action is directed. An important distinction between motives and goals is that the subject needs not be conscious about the motive at all times during the activity, whereas the goal is always consciously present during the action. At the bottom level, an individual action is performed through a number of *operations*. The operations are determined by the *conditions*, that is, the material and immaterial resources available to and constraints imposed on the acting subject, both by the environment, but also by the prior knowledge and abilities of the subject itself.

For describing what it means to become competent, one idea that has been guiding me is the competence framework of Niss and Højgaard (2011). In particular, the description of a competency as "a readiness to act" (Niss & Højgaard, 2011, p. 49), stresses that a person is not only *able* to carry out a mathematical procedure, but is also *ready* in the sense of knowing why, when and how the procedure works, as well as having the confidence to be able to perform the procedure.

Definition of tasks

I will now use the theoretical approach described above to define what I understand by the word task. First, I describe a task in general, and then I will describe what I define as a formal mathematics task. The description by Watson and Ohtani as the "things to do" (2015, p. 3) highlights the active nature of working with a task. In addition, there should be some sort of obligation connected to a task. At least the person performing the task should have the belief that this is something that he or she should do. And thus, the task should be given by someone, possibly the same person that performs it. Five roles can then be identified in how a person can relate to the task: The roles of designing the task; presenting the task; performing the task; presenting the solution; and evaluating a given solution. In order to distinguish between these roles, I use the words *designer*, *task presenter*, *performer*, *solution presenter* and *evaluator*. I distinguish between these as roles, but it may well be that the same person could play more than one role, or that one role is played by more than one person. The designer could also be the task presenter, and in some cases also the performer, solution presenter and evaluator. I will mainly focus on the three first roles in this text, but for learning, the two last are also important.

Drawing on ideas from AT and the general description of tasks and of competence, I define a *formal task* as a task that fulfils four criteria, to be presented below. A task that fails to fulfil at least one of these criteria will be called an *informal task*.

First, since the motive of the activity is to become more competent, the task should have a *purpose* in achieving this. It is however not necessary that the performer of the task has been informed about this purpose. There might be good reasons for not disclosing the full reasoning behind a particular choice of tasks. For example, if the purpose of a task is to check whether the performer recognises a particular mathematical pattern, informing the performer beforehand about this might void the task of its purpose. The purpose is neither equal to the motive of the activity nor the goal of the action, but is related to answering the question of why this task in particular is chosen. In fact, if the purpose of the task is changed, the task itself changes, as I see the purpose being an integral part of the task itself.

The second criterion is that the purpose should be known to the designer of the task securing that the designer can state the reasons for, and therefore also argue for why someone should solve such a task. A task fulfilling this criterion is called *formulated*. This criterion is similar to the first one, but by securing that the designer knows the purpose of the task, it is possible for someone else to find this purpose without having to solve the task themselves. Thus, a way of selecting tasks is possible informed only by the objective of the activity and the intended purpose of the tasks.

The third criterion is connected to whether the task has an endpoint attainable within a predictable time limit. Such a task is called solvable. The stricter case, where the task has a clear and single answer, as for instance calculating a sum, will be called answerable. The criterion of solvability will exclude many tasks that are in some sense open ended. For example, one can imagine the task of finding an *exhaustive* answer to a *why* question, where one can continually probe deeper into the explanations, without being able to know whether a solution exists. Nevertheless, such a task can still often be divided into solvable subtasks. Still, many tasks considered open can possess the criterion of being solvable, as long as there is a possibility for the performer to be satisfied that the task has reached a solution. If for example the task is *not* to continually probe deeper into a why question, but rather to arrive at an explanation, based on some finite number of assumptions and preconditions, it can be possible to find a solution to the task. It is not, however, necessary for the task itself to provide a systematic way of validating the solution. The solution can still be invalid, but it must be clearly distinguishable as a plausible solution. A trivial example might be the solving of a simple equation, where the answer is a number. Any number could be *plausible*, depending on the knowledge of the performer, but something which is not a number, will not be a solution.

The fourth criterion is inspired by Niss and Højgaard's statement that an answer "must be produced by calculations, that is by a mathematical procedure and not by measurement" (2011, p. 94). A procedure is a way of arriving at a solution through logical inference, possibly as simple as counting on fingers for adding numbers. A task where there exists a way of arriving at the solution in this way, will be called proceduriseable. This does not mean, however, that guessing or recollecting has no place, but there should at least be some way of "sifting" the solutions by means of inference.

In addition to these criteria, tasks can also be composed. A highly composed task will have many subtasks that are more or less necessary for the whole task to be solved.

The instrumental approach

I will here present a short description of the instrumental approach.

The instrumental approach (Rabardel, 2000) has, from the introduction to mathematics education, been connected to digital tools (Trouche, 2004). A tool is seen as "something which is available for sustaining human activity" (Trouche, 2004, p. 282). The main idea of the theory is that an instrument is an object consisting of the tool, together with usage patterns and mental schemes connected to the tool. Without the usage patterns and mental schemes, the tool will not yet be useful to the tool user, and will need to go through a process, called an instrumental genesis to become an instrument (Rabardel, 2000). This process consists of two parts: *instrumentalisation* and *instrumentation*.

Through the process of instrumentalisation, the artefact becomes an instrument. That is, the subject personalises the artefact, and creates an instrument by appropriating it into the subject's activity. This might happen through three phases: discovery and selection of relevant functions of the artefact; personalisation, where the user finds the preferred way to apply the functions; and transformation of the artefact, where modifications are made to the artefact to fit the user's usage patterns (Trouche, 2004). Through instrumentation, the subject also changes, to become a tool user. The usage patterns are internalised and the activity of using the tool is conditioned by the artefact. This instrumental genesis is dependent upon the properties of the artefact, or its constraints and potentialities, and it is also dependent upon the subject, its activity, prior knowledge and working methods (Trouche, 2004).

The instrumental approach is also applicable to tools other than digital. A non-physical tool can become an instrument when the tool changes from being a mere artefact into something that in the mind of the tool-user has a purpose and can help in achieving some goal or objective. Indeed, according to Lagrange et al. (2001, p. 6), "While the artefact refers to the objective tool, the instrument refers to a mental construction of the tool by the user". Moreover, a tool "can be material or cultural" (Trouche, 2004, p. 282). Tools need not be understood as physical entities, but can also be abstract, such as formulas, algorithms, and as I will argue, tasks.

Tasks as instruments

For many students, the most immediate goal while working on a task is to get the task done. This, however might not be the most effective way of achieving competence in mathematics. In my view, using Leont'ev's three levels of activity, it makes more sense to see the task itself as a *tool* in the activity, with the objective of becoming competent in mathematics. Solving tasks are then actions in this activity, and the different operations done to solve the task corresponds to usage patterns.

Since I cannot provide an exhaustive account of the different ways a task can be used as an instrument within the confines of this paper, I will in the rest of the text argue that the idea of tasks as instruments is viable and observable. The core of the argument will be the short case study, but a note about the process of instrumental genesis is needed to argue observability.

In order for a task to become an instrument, it needs to go through an instrumental genesis. As the performer solves a task, I assume that the task "acts upon" the person solving it, when learning happens. The subject is changed by the artefact, through instrumentation. Whether this has taken place can be seen for instance through tests or exams, as the observation that a student becomes more secure and their success rate increases in solving a particular type of task is a sign of this. But in addition, an instrumentalisation process is also necessary. The task needs to be appropriated by the task performer in order for it to become an instrument. A key to the observation of this is in the three different stages of instrumentalisation, found in (Trouche, 2004). In order to demonstrate how one could observe these different stages, I present an example from an interview. This is selected from a series of video recorded interviews conducted over the course of one semester, in their first university level calculus course. The purpose was to track the development of how six different students would reason during solution of tasks and how they might use tasks for learning.

Example tasks

In the first interviews, the students were given tasks on integration, and then told to freely describe their reasoning while solving the tasks. They were themselves responsible for stating when they considered the task solved, and the interviewer would only intervene when the students were silent, by asking them to continue talking. Two of the tasks given are shown in Figures 1 and 2. They will be analysed below using the four criteria of a formal task, and then a case study of one interview where these tasks were used, will be presented, together with observations from the other interviews.

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

What can you say about $G'(x)$?

Figure 1: Task 1

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

where $f(t)$ is a periodic function with period 2, so $f(t) = f(t+2)$ for all $t \in \mathbb{R}$
What can you say about $G'(x)$?

Figure 2: Task 2

The criteria for formal tasks can be applied to these two tasks individually. The tasks have a potential to demonstrate properties of the definite integral, and in the second task, also of periodic functions. This leads me to say they are both purpose oriented, as the purpose of giving such tasks to a student could be to highlight these properties. In addition, taken together as two parts of a composed task, they could serve as a way of highlighting for the student some possible false perceptions about the fundamental theorem of calculus (FTC).

They are both solvable. In Task 1, a possible solution could be to find the algebraic expression $G'(x) = f(x+1) - f(x-1)$, and be satisfied with this as a solution, but because of the vagueness of the question, it is not strictly answerable. Another plausible solution could be to state that $G'(x)$ represents a change in area. In contrast, these solutions would probably not be considered a satisfactory solution to task 2, since it does not take the periodicity of the function into account. Here the intended conclusion would be the observation that $G'(x) = 0$. They are both proceduriseable as well, since applying the FTC could be one such procedure.

For the students, these tasks were difficult, and not all managed to give answers that they themselves were satisfied with. It is worth noticing that although the question of saying something about $G'(x)$ was identical in both of the tasks, the answers the students gave to each of the two tasks varied considerably between the tasks, which correspond well with the pre-analysis of the tasks.

Case study

One student had a particularly interesting approach to solving in particular Task 2, using what he himself calls "an extreme example". Moreover, he was the only one who solved Task 2 correctly, although not entirely rigorously. This student had already followed an algebra course at another university while he was still in secondary school, after having finished the highest level of secondary school mathematics a year early. Task 1, he solved relatively quickly, stating that $G'(x)$ relates to the area of the graph between $x-1$ and $x+1$. On Task 2 he spent more time. The excerpt below shows his description of his own thinking at the moment when he arrived at his conclusion:

Student: I want to say that $G'(x)$ is zero … but that is for the most part a gut feeling … [partially inaudible] … Okay. Then I'll go for an extreme example. That usually works. [draws a graph]. 2 in the centre … that area [points to the graph] is equal to that area. Yeah, I just want to … I'm going to think about this when I get home …

Interviewer: Just say what you are thinking.

Student: I think $G'(x)$ is … zero … that there is no change in area. Since the period is 2, that is it repeats itself, so we know at least that $f(t)$ on both ends are of equal height. That is given. And then we know also that it will be relatively symmetric. We can always do a translation... or a reflection. For example, I could take that part, move it over there [draws an arrow from the centre of the graph over to the right]. If we had moved that one a little bit to the left, then we would have gotten more of that one and less of that one [pointing at the two maxima on the graph], then we could move that one over there. Yes! $G'(x) = 0$. That is my final answer [drops the pen].

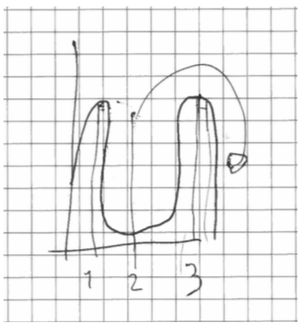


Figure 3: Graph of the "extreme example" drawn by the student.

Several parts of the instrumental genesis process can be observed here. From the triumphant drop of the pen, and declaration that he had found the answer, it seems reasonable to suggest that he has discovered something new by solving the task, and that he has learned something. It is worth noticing that the solution he arrived at is not a rigorous description of the general principle, but it is strong enough to convince him of the truth of his conclusion, and thus induce learning. Steps towards an internalisation of the usage pattern are taken, as the student might have discovered some relation between a periodic function, and the change of area under its graph.

Different stages of the instrumentalisation can also be observed. The discovery phase can be seen, as he tries different operations on the task. In this excerpt, he tests two operations. He constructs an "extreme example", as he refers to it, in the form of the graph of a function, and he operates on this graph by performing an imagined moving of one part of the graph to another part. Personalisation can also be seen in the statement that an extreme example "usually works". This shows a personal preference to certain operations, and the fact that the example made it possible to convince himself of the solution likely strengthens this preference. Transformation can also be seen in two ways. First, as the two tasks are similar, this constitutes an example of such a change, from the most general case to the periodic, albeit not performed by the student. But the student makes a similar change by further limiting the scope of the task, from finding the solution of the periodic case, into solving the extreme example, and thereby using the now changed task as a tool for exploring the more general case.

Other students also tried similar strategies. Especially the selection of an example function, and then exploring the implications that the question would have to this case. The difference would be that the other students would rather choose a *typical* periodic function, like a trigonometric function, and often explicitly defined, rather than an unspecified periodic function that lies at a perceived extreme. They also in many cases did not make the connection between $G'(x)$ and the change of area. Nevertheless, attempts at exploring and changing of the scope of the task can still be observed also in these cases.

Final remarks

Some consequences for how we use task seems to emerge from this. Since task are central to learning mathematic, and since tasks can be seen as instruments, as I have argued, this suggests that we need to take the different phases of the instrumental genesis into account when designing and using tasks. One such way may be to give students the opportunity to explore and discover the different ways a task can be used, and what sort of changes can be done to the task. This might be done simply by providing the time and opportunity for such discovery, but also by providing good examples that show how tasks can be used for different purposes, and how to change tasks in order to achieve this.

There are however still more questions to be answered. First, in this paper I have only demonstrated that tasks *can* be seen as instruments, but there is the need to examine *different ways* tasks can be used as instruments. The question of transition seems also likely to be connected to this point, as diversification of the way tasks are used would constitute one way in which the complexity increases. Finally, these observations might also have consequences on the design of tasks, and possibly give rise to some design principles. Different types of tasks might very well lend themselves to use as instruments in different way, possibly also dependent upon and conditioned by the sort of guidance provided by the presenter of the task.

References

Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *Journal of Mathematical Behavior, 26*(4), 348–370.

Botten-Verboven, C., Maugesten, M., Bendiksen, V., Nilsen, G., Dalvang, T. et al. (2010). *Matematikk for alle, ... men alle behøver ikke å kunne alt.* Oslo/Trondheim: Utdanningsdirektoratet.

Clarke, D., Strømskag, H., Johnson, H. L., Bikner-Ahsbahs, A., & Gardner, K. (2014). Mathematical tasks and the student. In P. Liljedahl, S. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group of Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 1, pp. 117–143). Vancouver: PME.

Johnson, H. L., Coles, A., & Clarke. D. (2017). Mathematical tasks and the student: Navigating "tensions of interest" between designers, teachers and students. *ZDM Mathematics Education, 49*(6), 813–822.

Haavold, P. (2011) What characterises high achieving students' mathematical reasoning? In B. Sriraman, K. H. Lee (Eds.), *The Elementselements of Creativitycreativity and Giftednessgiftedness in Mathematicsmathematics*, (pp. 193–215). Rotterdam: Sense Publishers.

Lagrange, J. B., Artigue, M., Laborde, C., & Trouche, L. (2001). A meta study on IC technologies in education: Towards a multidimensional framework to tackle their integration. In M. Van den Heuvel-Panhuizen (Ed.) *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 111–122). Utrecht: PME

Lithner, J. (2017) Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM Mathematics Education, 49*(6), 937–949.

Leont'ev, A. N. (1978). *Activity, consciousness and personality*. Englewood Cliffs: Prentice-Hall.

Niss, M., & Højgaard, T. (Eds). (2011). *Competencies and mathematical learning*. Roskilde: Roskilde University.

Roh, K. H., & Lee, Y. H. (2016). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education, 3*(1), 34–68.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*, 281–307.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, 8(2), 91–111.

Watson, A., & Ohtani, M. (Eds.) (2015). *Task design in mathematics education*. Heidelberg: Springer.

Paper 2

An Analysis of a First-Year University Student's Construction of a Praxeology of Integration Tasks

Vegard Topphol and Heidi Strømskag

Paper Submitted to the Journal Teaching Mathematics and its Applications

December 2022

This paper is awaiting publication and is not included

Paper 3

Didactic Transposition of the Fundamental Theorem of Calculus

Vegard Topphol

Paper Submitted to REDIMAT, Journal of Research in Mathematics Education

March 2023

Didactic Transposition of the Fundamental Theorem of Calculus

Abstract

Using the tools of praxeological analysis and didactical transposition analysis, the treatments of the Fundamental Theorem of Calculus in one Norwegian, Grade 13 textbook is analysed, with a particular focus on the development of the logos block of the FTC. The terms *structure*, *functioning* and *utility*, first introduced by Chevallard in 2022, is further to describe different dimensions of the mathematical object at stake. Through the analysis, a lack in the logos relating to the concept of integrability is identified in the textbook, and consequences of this is explored in relation to a set of tasks found in the book.

Keywords: didactic transposition, Grade 13 textbooks, praxeological analysis, the Fundamental Theorem of Calculus

Introduction

The movement, transformation, and incorporation of knowledge from one institution, where it is created, into the activity of other institutions (typically educational institutions) has been studied in mathematics and stem education in general (Freudenthal, 1983/2002; Bosch & Gascón, 2006) and in calculus specifically (e.g., Petropoulou et al., 2016; Strømskag & Chevallard, in press.) In the Anthropological Theory of the Didactic (ATD), this process of transposing an object of knowledge from one institution of knowledge to another institution is modelled by the concept of a *didactic transposition* (Bosch & Gascón, 2006). The transposition of the scholarly concept of integral analysis to the techniques and concepts of integrals found in upper secondary mathematics courses is an example of this.

In this paper, a study and analysis of a Norwegian textbook, *Matematikk R2* (Borge et al., 2022), for upper secondary school Grade 13 (hereafter simply Grade 13) mathematics is presented, focusing on the Fundamental Theorem of Calculus (FTC) and the didactic transposition of this theme.

The textbook selected is a part of the resources produced for the recent curriculum reform in Norway, *Kunnskapsløftet 20* (Directorate of Education and Training, 2020). The reform was implemented for Grade 13 in 2022, after Grade 12 in 2021 and Grade 11 in 2020, and consists of a substantial reorganisation of the curricula and their contents. The reform introduced more specific goals for learning integral calculus. Students is now expected to be able to “account for the fundamental theorem of calculus, and account for consequences of the theorem”. The previous reform, *Kunnskapsløftet 06*, did not mention the FTC (Directorate of Education and Training, 2006).

Much weight is put on integral calculus in higher mathematics education, and students has been shown to have numerous difficulties in understanding the concept (e.g., Orton, 1983; Thompson & Harel, 2021; Burgos et al., 2021). A previous study (Topphol & Strømskag, 2020), identified a difficulty in relating the *indefinite integral* (which will be defined later) and the antiderivative, namely the definite integral, $\int_a^x f(t)dt$ (essentially the first part of the FTC) and showed that this difficulty could be traced back to the textbooks they had used in upper secondary. One textbook (Heir et al., 2016), the previous edition of *Matematikk R2*, written for *Kunnskapsløftet 06*, was examined specifically. *Matematikk R2* (Borge et al., 2022), was also the first textbook written for *Kunnskapsløftet 20* that was available to the author of this article.

With the advent of a new curriculum in mathematics, it is therefore of interest to investigate how the theme of integration is treated under the new curriculum. More concretely, I investigate the question of how integral calculus is presented in *Matematikk R2* (Borge et al., 2022), and what consequences there might be. Specifically, I seek to answer the questions:

1. What sort of changes have been made during the transposition of the theme of integration, and particularly the FTC, from scholarly knowledge to knowledge to be taught at Grade 13, as presented in *Matematikk R2*?
2. In case of any unused potential in the presentation of the FTC, with regard to strengthening its logos in Matematikk R2, what does this potential consist of?

Theoretical tools

This study is conducted with theoretical tools from the Anthropological Theory of the Didactic (ATD; Chevallard, 2019).

Knowledge is within the ATD modelled in terms of a praxeology, $\wp$, consisting of four components: type(s) of tasks, T, a technique, τ (or set of techniques), used to solve the tasks, a technology, θ, used to describe and explain the techniques, and a theory, Θ, used to justify the technology. The types of tasks and the techniques make up the praxis block of the praxeology, and the technology and the theory make up the logos block (Chevallard, 2019). Schematically, it is commonly written as $\wp = [T / \tau / \theta / \Theta]$.

Subscripts *u* (for *university*) and *s* (for *school* or *secondary*) respectively, are used to distinguish the praxeological elements. Thus, T_u is the types of tasks found in university mathematics textbooks, while T_s are types of tasks in the Grade 13 textbook.

The concept of a didactical transposition refers to a process, where an object of knowledge is transformed, from scholarly knowledge, through its selection by the noosphere to become knowledge to be taught, until it is actually taught, and becomes available to the students, in the teaching institutions (Chevallard & Bosch, 2014) (Figure 1).

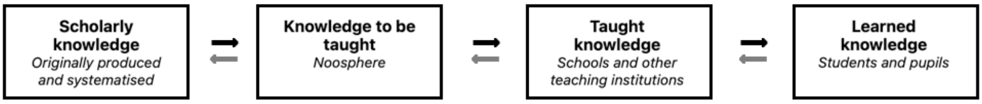


Figure 1. The Didactic Transposition Process (Adapted from Chevallard & Bosch, 2014, p. 171)

As explained by Strømskag and Chevallard:

> A praxeology $\mathscr{p}$ is usually the product of the activity of an institution or a collective of institutions *I*. It is often a result of an *institutional transposition* of a praxeology $\mathscr{p}^*$ living in a collective of institutions I^* to a praxeology $\mathscr{p}$ that has to live within *I* and thus has to satisfy a set of conditions and constraints specific to *I* (Chevallard, 2020). This is the case when *I* is a collective of "didactic" institutions, that is, institutions declaring to teach some bodies of knowledge, such as secondary school for example. This is referred to as *didactic transposition* of I^* into *I.* (Strømskag & Chevallard, in press)

In the study of a mathematical object, $\mathscr{o}$, here the FTC, one can talk about the object's *structure*, *functioning*, and *utility* (Chevallard, 2022). Structure refers to what $\mathscr{o}$ consists of, or what elements the object ties together. Functioning refers to how $\mathscr{o}$ works to tie the elements together. Utility refers to what $\mathscr{o}$ can be used for. I distinguish between *intra mathematical utility*, or utility to mathematics itself, and *extra mathematical utility*, or utility to fields outside of mathematics.

Methodology

The methodological approach is a *didactic transposition analysis* (Chevallard, 1989; Chevallard & Bosc, 2014), where a reference praxeological model is constructed, and used to analyse the Grade 13 textbook (see e.g., Wijayanti & Winsløw, 2017). A *reference praxeological model* of the theme of integration is first created, a model where the researchers expose their own perspectives on the body of knowledge at hand. Then, an analysis of the Grade 13 textbook is conducted where praxeological elements are identified. At last, the reference model and the Grade 13 textbooks are compared. In all three steps I will structure the descriptions around the notions of *structure*, *functioning*, and *utility* of the mathematical object, adding to the method of Wijayanti and Winsløw (2017). I focus mainly on the *intra mathematical utility*, in addition to structure and functioning of the mathematical object.

The reference praxeological model is partially based on *Calculus: A Complete Course* (Adams & Essex, 2018), from here on referenced as *Calculus*. This book was chosen because of its use in many of the early mathematics courses in my own home university, and the widespread international audience and the authors' long experience in writing calculus textbooks. An article by Botsko (1991), presenting a more general form of the FTC than is found in *Calculus*, and the Norwegian calculus book *Kalkulus* (Lindstrøm, 2016), are used as supplementary sources. As a single textbook is itself a result of a didactic transposition (Winsløw, 2022), it does not in general suffice alone as a description of scholarly knowledge.

A reference praxeological model for the FTC

The FTC connects the concepts of *antiderivatives*, the *indefinite integral*, and the *definite integral*, defined as *Riemann integrals* (see e.g., Adams & Essex, 2018, pp. 302–307). By FTC establishing the *Newton-Leibniz formula*,

$$\int_a^b f(x)dx = F(b) - F(a),$$

and what I will call the *derivative-integral formula*,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

the FTC provides results that allows for calculations of areas that are not easily measured through simpler geometric means, and for doing calculations on accumulation. These results have numerous applications in other fields (for examples, see any university level calculus textbook, e.g., Adams & Essex, 2018, pp. 393–458; Lindstrøm, 2016, pp. 439–459). Through extensions and generalisations, like Fourier analysis and differential equations, based on improper integrals, it has proved indispensable in our technology-driven world.

Conditions for Riemann integrability

Integrability and continuity, are the main conditions for the FTC to work. For a function to be Riemann integrable, the integrand function must be bounded, and the upper and lower Riemann sums must exist. For an integral of a function over a closed interval, continuity is a sufficient condition, but not necessary. A detailed discussion of Riemann sums, integrability, and boundedness, can be found in *Calculus'* Appendix Sections III and IV (Adams & Essex, 2018, A-21 – A-31).

Definitions

An *antiderivative* of $f(x)$ on an interval *I*, is defined as a function, $F(x)$, such that $F'(x) = f(x)$ for all $x \in I$.

An *indefinite integral* of *f* on an interval *I*, defined as

$$\int f(x)dx = F(x) + C \quad \text{on } I,$$

where *F* is an antiderivative of *f* for all x $\in I$, and *C* is a real valued constant. The addition of the constant *C* makes it possible to use the indefinite integral to represent all antiderivatives in one expression.

A definition of the definite integral can now be stated (Adams & Essex, 2018, p. 304):

> Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have
>
> $$L(f, P) \leq I \leq U(f, P).$$
>
> Then we say that the function f is **integrable** on $[a, b]$, and we call I the **definite integral** of f on $[a, b]$. The definite integral is denoted by the symbol
>
> $$I = \int_a^b f(x)dx.$$

L and U are the lower and upper Riemann sums for a partition of the interval $[a, b]$. Boundedness plays a role in the existence of lower and upper Riemann sums. If the function f is not bounded on the interval, then either a lower or an upper Riemann sum cannot exist (details can be found in Adams & Essex, 2018, A-28–A-29).

Theorems which the FTC builds on

Three theorems will be used in proving the FTC. The derivative of a constant function is zero (Theorem 13, Adams & Essex, 2018, p 142). A zero-width integral has result zero, and integrals have the *additivity property* (Theorem 3, Adams & Essex, 2018, p 308). The Mean-Value Theorem for Integrals (Theorem 4, Adams & Essex, 2018, p 310). Additivity will also prove significant as it provides a basis for a common technique used for, for example, area calculations.

The statement of the FTC

A statement of the FTC is seen in *Calculus* (Adams & Essex, 2018, pp. 313–314):

> Suppose that the function f is continuous on an interval I containing the point a.
>
> **PART I.** Let the function F be defined on I by
>
> $$F(x) = \int_a^x f(t)dt.$$

> Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:
>
> $$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$
>
> **PART II.** If $G(x)$ is *any* antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I, we have
>
> $$\int_a^b f(x)dx = G(b) - G(a).$$

A similar statement can be found in *Kalkulus* (Lindstrøm, 2016, p. 416). Part II is there referred to as a corollary.

In both treatments, continuity of the integrand is assumed both in Part I and Part II. This is also a necessary condition for the conclusion in Part I of the FTC. However, there is a version of the FTC Part II, which is rather based on an integrand bounded on the interval of integration, allowing a countable (possibly countably infinite and possibly zero) number of discontinuities (i.e. the conditions for Riemann integrability). Such a function is called *continuous almost everywhere*. Similarly, a function *G* which is the derivative of another function *f* everywhere, except for a countable number of points is said to be *derivative of f almost everywhere*.

In other words, there exists a version of the FTC Part II which can be applied to all Riemann integrable functions (Botsko, 1991). The FTC Part II can be restated:

> **PART II.** If $f(x)$ is a Riemann integrable function, and if $G(x)$ is a continuous function for which $G'(x) = f(x)$ almost everywhere on *I*, then for any *a* and *b* in *I*, we have
>
> $$\int_a^b f(x)dx = G(b) - G(a).$$

The condition that $G(x)$ is continuous is important, and the lack of this condition would have some consequences (see e.g., Pavlyk, 2008).

Now, it is not obvious why this is relevant for an upper secondary calculus textbook. I do also not expect secondary students to learn this version of the FTC. But the existence of this form of the theorem illustrates two important points. First, the difference between the two formulations of the definite integral, the definition based on Riemann sums, and the calculational formulation based on antiderivatives, often do have different conditions for their validity, in their forms expressed in typical textbooks. This difference is not always clearly communicated. And second, it illustrates one effect of the condition of boundedness. This, as will be demonstrated, is another crucial point that is not communicated in the textbook examined in this article.

Proving the FTC

A proof of Part I can be found in *Calculus* (Adams & Essex, 2018):

> Using the definition of the derivative, we calculate
>
> $$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt \quad \text{by Theorem 3(d)} \\ &= \lim_{h\to 0}\frac{1}{h}hf(c) \qquad \begin{array}{l}\text{for some } c = c(h) (\text{depending on } h) \\ \text{between } x \text{ and } x+h \text{ (Theorem 4)}\end{array} \\ &= \lim_{c\to x} f(c) \qquad \text{since } c\to x \text{ as } h\to 0 \\ &= f(x) \qquad \text{since } f \text{ is continuous.} \end{aligned}$$
>
> Also, if $G'(x) = f(x)$, then $F(x) = G(x) + C$ on I for some constant C (by Theorem 13 of section 2.8). Hence,
>
> $$\int_a^x f(t)dt = F(x) = G(x) + C.$$

A proof of the FTC Part II, applying to all Riemann integrable functions, can be found in Botsko (1991).

Types of tasks and techniques from university textbooks

The intra-mathematical utility of the FTC can be seen in the types of tasks it provides the foundation for. Based on tasks found in the two textbooks *Calculus* by Adams and Essex (2018) and *Kalkulus* by Lindstrøm (2016), seven types of tasks can be identified, and they make up the bulk of T_u:

Table 1.
Types of tasks relating to the FTC

Type of tasks.	**Technique required (τ)**
t_1: Evaluate a definite integral	τ_1: - Find an antiderivative and apply Newton-Leibniz formula.
t_2: Find the area of a bounded region	τ_2: - Find all zeros of the integrand on the interval of integration. - Evaluate the definite integral over each subinterval. Negate value if the area lies below the abscissa. - Add the resulting integrals.
t_3: Derivative of functions defined by an integral with variable integration limit.	τ_3: - Apply the derivative-integral formula.
t_4: Find the average value of a function	τ_4: - Find the area of a bounded region (τ_2). - Divide by length of integration interval.
t_5: Integral equation	τ_5: - Apply the derivative-integral formula (τ_3). - Solve resulting algebraic equation.
t_6: Approximating a sum using an integral	τ_6: - Recognize the sum as a Riemann sum.

	- Find a non-discrete real valued function, *f*, corresponding to the expression in the sum. - Find an antiderivative of *f* and apply Newton-Leibniz formula.
t_7: Approximating/calculating an integral using a Riemann sum	τ_7: - Calculating function values for the integrand in each subinterval. - Approximate area over each subinterval using rectangles. - Approximate integral by summation of rectangles.

With the possibility of applying the FTC to discontinuous integrands, all these types of tasks can be extended. In several cases, solving the discontinuous versions do require extra techniques.

Reference Example 1

Calculate the integral $\int_{-1}^{2} f(x)dx$ for

$$f(x) = \text{sign(x)} = \begin{cases} -1 \;\forall\; x < 0 \\ 0 \;\forall\; x = 0 \\ 1 \;\forall\; x > 0. \end{cases}$$

Two techniques can be used. For the first technique, observe that $F(x) = |x|$, is an antiderivative of $f(x)$ everywhere except $x = 0$, and F is continuous. Using this antiderivative,

$$\int_{-1}^{2} f(x)dx = |x|\Big|_{-1}^{2} = 2 - 1 = 1.$$

Note that this technique is the same as for tasks of type t_1. This is, however, not general, and only works for certain cases of discontinuous, bounded functions.

The second technique is more general. The interval of integration is subdivided, such that *f* is continuous on each of the subintervals. The integral is calculated over each subinterval separately, and then added, which is possible due to the additivity of integrals. This technique yields

$$\int_{-1}^{2} f(x)dx = \int_{-1}^{0} -xdx + \int_{0}^{2} xdx = -1 + 2 = 1.$$

Note the similarity between this technique and τ_2. The only difference is that the areas are not considered negative when they lie below the abscissa.

Reference Example 2: Examples of improper integrals

One example and one task from *Calculus*, provide an interesting case. Example 6 (Adams & Essex, 2018, p. 316) starts with the function $f(x) = \frac{1}{x}$, and explains that the integral of $f(x)$ from -1 to 1 diverges. The book does not present the full argument at this point and instead refers to this integral being of a type called and *improper* integral. But the key reason for why $\int_{-1}^{1} \frac{1}{x} dx = 0$ is false, is that the function is not defined and has no limit in $x = 0$, and is not integrable on neither $[-1,0]$ nor $[0,1]$. The FTC does therefore not apply. Note the significance of the criterion of *integrability*.

The consequence of the lack of integrability is clearer in Task 49 (Adams & Essex, 2018, p. 319), a classical example (see e.g. Orton, 1983; Rubio & Gómez-Chacón, 2011). Here, the erroneous calculation

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1 + \frac{1}{-1} = -2$$

is to be criticized.

To see the solution, note first that since the function is strictly positive, the integral should also be positive, and the answer -2 is clearly wrong. Also, since the function $1/x^2$ is not defined at $x = 0$, and is in fact unbounded on any interval including $x = 0$ and therefore not integrable, the FTC does not apply to the interval [-1,1].

Praxeological analysis of the FTC in *Matematikk R2*

I here describe the treatment of the FTC found in the textbook *Matematikk R2* (Borge et al., 2022)[1, 2]. In doing so, I examine the mathematical organization of integration and the FTC. This will then be used as a foundation for the praxeological analysis. The examination therefore has a focus on how the techniques are developed and justified and then applied to tasks.

Organisation of integration and the FCT in *Matematikk R2*

Aschehoug's Matematikk R2 divides the treatment of the integral into six chapters. They deal with the definite integral (Chapter 2A, pp. 90–103), numerical integration and Riemann sums (Chapter 2B, pp. 104–112), different uses of the definite integral (Chapter 2C, pp. 113–127), the FTC (Chapter 2D, pp. 128–142), methods of integration (Chapter 2E, pp. 1143–153), and some volume and surface integrals (Chapter 2F, pp. 154–169). I will mainly focus on Chapter 2A, 2B and 2D, but one example is also taken from Chapter 2F.

One feature of the organisation of the book, are the activities called *explore*[3] and *talk*[4]. These are activities intended to help students explore and talk about these themes collectively and are often placed strategically as part of the theoretical treatment of the themes. These tasks are not uniquely named, and therefore, I will give them reference names here, which do not correspond to any naming found in the textbook itself.

The logos elements of *Matematikk R2*

The notions of *limits, continuity* and *existence* of functions are discussed in the Grade 12 mathematics textbook *Matematikk R1*, providing a foundation for integral and differential calculus (Borgan et al., 2021). I cannot provide any detailed account of this here, but it suffices to say that the treatment is based on intuitive notions of what it means for a function to *tend to* a limit, and what it means for a function or a value to *tend to infinity*. The distinction of bounded and unbounded functions is not made, but different sorts of discontinuity are discussed and demonstrated.

The definition of the definite integral

Chapter 2A starts with an *explore* task (Borge et al., 2022, p. 90), named *Explore-Task 1* henceforth. The students are tasked with examining the area under two graphs by making a lower and upper approximation of the area under $f(x) = x^2$ and $g(x) = 5x$, using rectangles with equal width. The terms *upper* and *lower staircase sums*, a simplification of Riemann sums, not to be confused with the step function, are then defined. "The collected area of the rectangles below and above the graph we call a *lower staircase sum, N,* and an *upper staircase sum, Ø*, respectively" (Borge et al., 2022, p. 91). The true area under the graph lies between these two staircase sums.

Then, a definition of the integral based on staircase sums is presented. Starting with an area, *A,* between the values $x = a$ and $x = b$, and under the graph of a continuous function *f*, defined on the interval $[a, b]$, where $f(x) \geq 0$ for all $x \in [a, b]$ (see Figure 2). The interval is divided in *n* equal subintervals. Points from $x_0 = a$ to $x_n = b$, with distance $\Delta x = \frac{b-a}{n}$, such that $x_i - x_{i-1} = \Delta x$, are marked on the *x*-axis. The i-th subinterval is $[x_{i-1}, x_i]$ (see Figure 3). For one subinterval, a pair of rectangles are defined, one with height equal to the lowest function value, and one with height equal to the highest function value on the interval. This process is repeated for all *n* subintervals. To get the lower staircase sum, N_n, the smallest rectangles for each subinterval are selected, and correspondingly, the largest rectangles for the upper staircase sum, $Ø_n$.

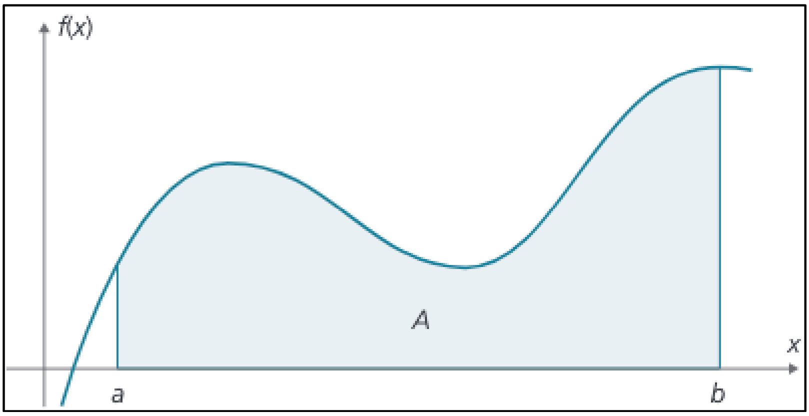


Figure 2. Area under the graph of *f*(*x*) (taken from Borge et al., 2022, p. 95)

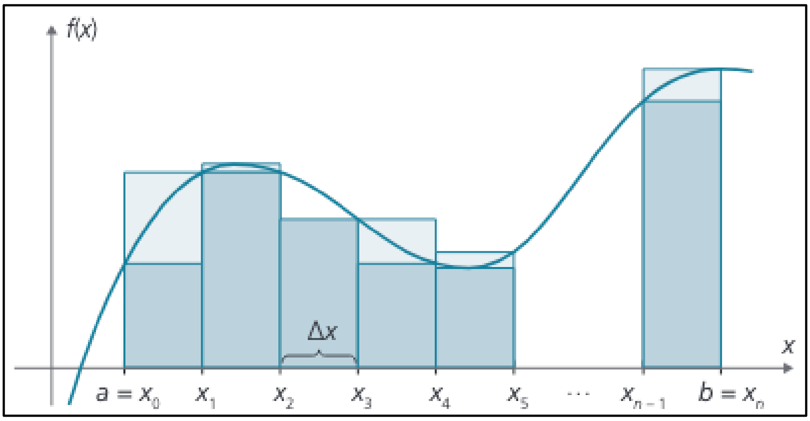


Figure 3. Subdivision of the area under the graph of *f*(*x*) (taken from Borge et al., 2022, p. 95)

Then the book explains the convergence of staircase sum:

> We let $n \to \infty$, so that $\Delta x \to 0$. We say that the sequence of staircase sums, $\{N_n\}$ and $\{Ø_n\}$, converge towards a limit value if N_n and $Ø_n$ gets closer and closer to that value when $n \to \infty$. When the two sequences converge toward the same limit, we call this limit *the definite integral* of *f* on the interval [a,b], and write $\int_a^b f(x)dx$. We

> read this as “the definite integral of f from a to b”. (Borge et al., 2022, p. 95)

After presenting the definition, a note about integrability is given.

> If the two limits are equal, the definite integral is equal to the area A. This is always the case for continuous functions, and we say that f is integrable on the interval $[a, b]$.
> If the limits are different, f is not integrable.
> All the functions you will meet in R2 are integrable. (Borge et al., 2022, p. 96)

The note about integrability does not seem to serve much purpose, besides reassuring the students that they will not need to deal with this topic in depth, since all functions in the following sections are promised to be integrable. However, as will be seen, this is not true, for at least one case, and can potentially lead to false justifications.

Riemann sums

In Chapter 2B, the concept of staircase sums is expanded upon to define Riemann sums. First, the selection of height of the rectangles in the interval $[x_{i-1}, x_i]$, is changed from the strictly highest and strictly lowest in the interval, to an arbitrary value. A value, $x_i^* \in [x_{i-1}, x_i]$, in each subinterval is selected, and the function value for each x_i^* is calculated. The book then notes that both the upper and lower staircase sums are on the form

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x,$$

called a Riemann sum.

It is noted that the Riemann sums require that f is defined on a closed interval $[a, b]$, and that the n subintervals can have varying width, but in Grade 13 mathematics they will consider only cases with constant width. The fact that continuous functions are integrable is reiterated, but again without mentioning why. The definite integral is then defined as a sequence of Riemann sums:

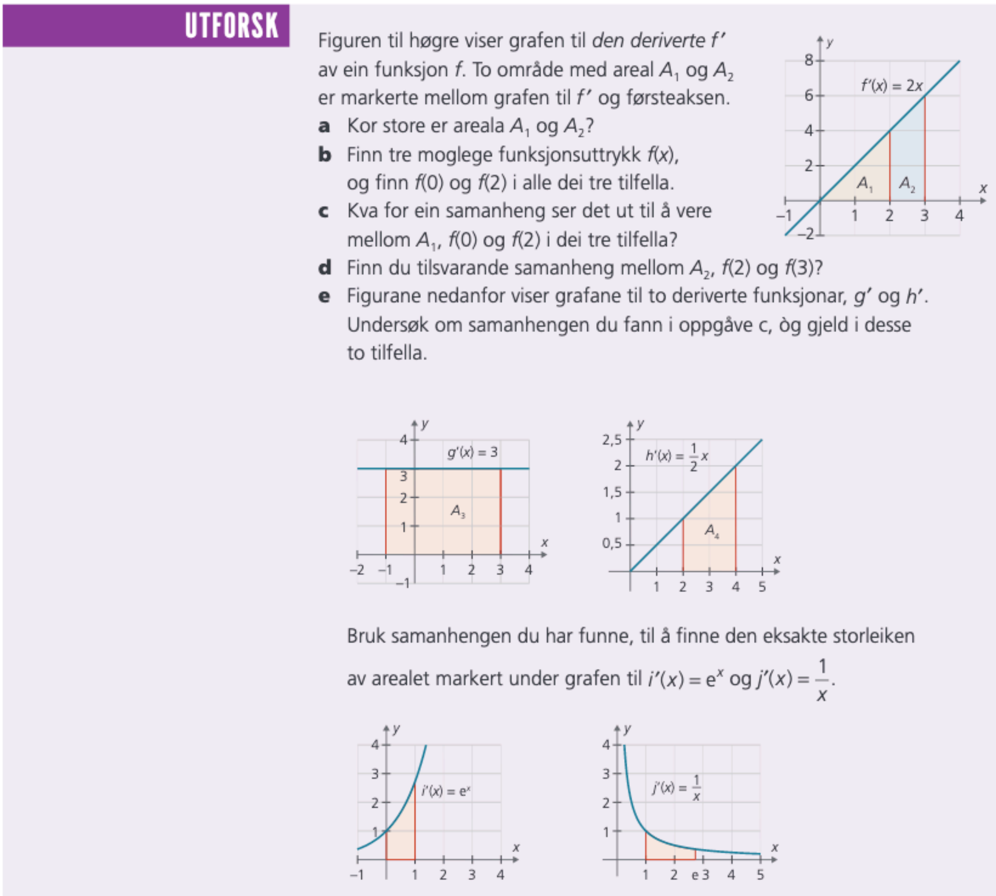
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}.$$

The following pages of Chapter 2B present different types of numerical integration, and the following Chapter 2C illustrates different uses of the integral, with a focus on techniques for area calculations.

Antiderivatives and indefinite integrals

Chapter 2D begins by presenting antiderivatives and indefinite integrals. First, an *explore-task* is presented (*Explore-Task 2*), where the derivative of a function $f'(x) = 2x$ is given, together with its graph (see Figure 4). Two areas under the graph, A_1 between $x = 0$ and $x = 2$, and A_2 between $x = 2$ and $x = 3$, are shown in the graph, and five tasks are given (Borge et al., 2022, p 128):

> *a*) How big are the two areas A_1 and A_2?
> *b*) Find three possible $f(x)$, and calculate $f(0)$ and $f(2)$ in all three cases.
> *c*) What connection does it appear to be between A_1, $f(0)$ and $f(2)$ in the three cases?
> *d*) Can you find a corresponding connection between A_2, $f(2)$ and $f(3)$ in the three cases?
> *e*) The figures below (see Figure 4) show the graphs of two derivatives $g'(x)$ and $h'(x)$. Examine whether the connection you found in task *c*) also holds for these two cases.
> Use the same technique to calculate the exact areas under the function $i'(x) = e^x$ and under the function $j'(x) = \frac{1}{x}$.



UTFORSK

Figuren til høgre viser grafen til *den deriverte* f' av ein funksjon f. To område med areal A_1 og A_2 er markerte mellom grafen til f' og førsteaksen.

a Kor store er areala A_1 og A_2?

b Finn tre moglege funksjonsuttrykk $f(x)$, og finn $f(0)$ og $f(2)$ i alle dei tre tilfella.

c Kva for ein samanheng ser det ut til å vere mellom A_1, $f(0)$ og $f(2)$ i dei tre tilfella?

d Finn du tilsvarande samanheng mellom A_2, $f(2)$ og $f(3)$?

e Figurane nedanfor viser grafane til to deriverte funksjonar, g' og h'. Undersøk om samanhengen du fann i oppgåve c, òg gjeld i desse to tilfella.

Bruk samanhengen du har funne, til å finne den eksakte storleiken av arealet markert under grafen til $i'(x) = e^x$ og $j'(x) = \frac{1}{x}$.

Figure 4. *Explore* task about the indefinite integral (Borge et al., 2022, p. 128)

The technique of finding an $f(x)$ when you know $f'(x)$ is named *to find the antiderivative*. The book observes that the only difference between the three functions found in Question *b*) is a constant, justifying the introduction of a general constant, *C*. Antiderivatives are then defined: "If $K'(x) = f(x)$, we say that *K* is one antiderivative of *f*. All antiderivatives of *f* are then given as $K(x) + C$, where $C \in \mathbb{R}$." (Borge et al., 2022, p. 129). This is called an *indefinite integral* and defined as $\int f(x)dx = K(x) + C$, where $K'(x) = f(x)$ and $C \in \mathbb{R}$. The process of finding an indefinite integral is called *to integrate*.

Note that a connection between the antiderivative, the indefinite integral, and the area under a graph is communicated, constituting an attempt at sharing the burden of the work done by the later proof of the FTC.

The Fundamental Theorem of Calculus

Another *explore-task* immediately precedes the FTC (*Explore-Task 3*). The students are given the function $f(x) = 2x + 3$ and a graph of *f*, and they are asked to use the formula of a trapezoid to explain why $F(x) = x^2 + 3x$ describes the area under the graph, from $x = 0$ to an arbitrary *x*-value greater than 0. Continuing, the students are asked to use the area function to explain why the area under the graph from $x = 2$ to $x = 5$, becomes $A = F(5) - F(2) = 40 - 10 = 30$, and why this implies

$$\int_2^5 f(x)dx = F(5) - F(2).$$

The proof, or rather a demonstration, is presented. The book does not call it a proof, but claims to be demonstrating the *carrying idea* of what could become a proof:

> We shall show the carrying idea in the proof for the Fundamental Theorem of Calculus, using the figure below (see Figure 5).

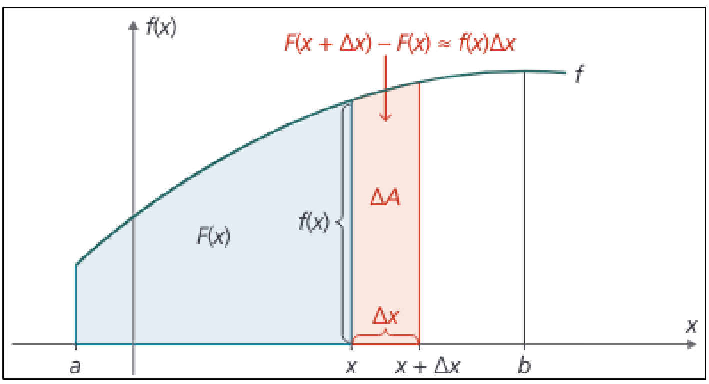


Figure 5. Definite integral of *f*(*x*) from *a* to *b* (taken from Borge et al., 2022, p. 136)

We call the area of the blue region $F(x)$. This area corresponds to the definite integral that we defined using Riemann sums

$$F(x) = \int_a^x f(t)dt.$$

The area of the pink region, ΔA, is a small additional area. The sum of the two area corresponds to the definite integral

$$F(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt.$$

The area of only the pink region is therefore the difference between the two area above.

$$\Delta A = F(x + \Delta x) - F(x).$$

An approximation for ΔA is a rectangle of width Δx and height $f(x)$

$$\Delta A \approx \Delta x \cdot f(x).$$

As with the Riemann sums, the approximation is more accurate the narrower the rectangle is, that is, the smaller Δx is.
We set the two expressions for ΔA equal to each other, and recalculate

$$F(x + \Delta x) - F(x) \approx \Delta x \cdot f(x)$$

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} \approx f(x)$$

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$$

$$F'(x) = f(x).$$

So F is therefore the antiderivative of f.
The step from the approximation value on line 2 to the limit on line 3 of the calculation on the previous page demands a formal proof, which we will not enter in R2, but this step is the carrying idea in the proof. (Borge et al., 2022, p. 136)

After this, the book provides an example and a few tasks where the FTC is used to differentiate functions defined by definite integrals. The Newton-Leibniz formula is then proved:

Starting with the FTC, we can now develop a useful result.
$\int_a^a f(x)dx = F(a) = 0$ because we do not have a region with area when the upper and the lower limits of the integral are equal.
Now, let K be an arbitrary antiderivative of f. Then $F(x) = K(x) + C$. Thus

$$\begin{aligned}\int_a^b f(x)dx &= F(b) = F(b) - F(a) \\ &= (K(b) + C) - (K(a) + C) \\ &= K(b) - K(a) \\ &= [K(x)]_a^b\end{aligned}$$

Here $F(b) = F(b) - F(a)$ since $F(a) = 0$.
$[K(x)]_a^b$ is a shorthand for $K(b) - K(a)$. (Borge et al., 2022, p 138)

Tasks and techniques in *Matematikk R2*

The theory is then used as foundation for what types of tasks can be given. In the textbook, tasks of all the seven types defined in the reference model were found. The significance of this is that the technology, θ_s, seems to be relatively similar to θ_u. One significant difference will, however, be seen in three specific tasks.

Two of the tasks are *examine* tasks and *talk* tasks. As they have no solutions provided, students and teachers are left with the option to either argue well enough to be convinced, or to seek answers from external sources. In some cases, answers are implied in the following text, but not in all.

Towards the end of the section, two *examine*-tasks are given. Both are motivated simply by stating the mathematical problem, and the book does not provide any reason for the utility of the techniques demonstrated in these tasks. The second of these tasks demonstrates a technique relevant to the discussion. I call this task *Explore-Task 4.*

Explore-Task 4 shows a calculation,

$$\int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} (x - x^3) dx + \int_{1}^{2} (x^3 - x) dx = \frac{11}{4},$$

and asks why this calculation holds. Note the similarity between the technique used to solve *Explore-Task 4*, to the second technique used in *Reference Example 1*. Dividing the area of integration, as a technique, is well within the scope of the textbook, and not restricted to examples with calculations of areas. There are, however, no similar tasks later, and the task seems therefore to serve a purpose as a mathematical curiosity. The utility of the FTC to this task, and possibly similar types of tasks, is not examined.

The *talk* task, from now on called *Talk-Task 1*, that comes after the introduction of the Newton-Leibniz formula is also worth some attention (Borge et al., 2022, p. 140). Here the students are asked to discuss an erroneous result. “Discuss what is wrong with this calculation:”

$$\int_{-1}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$$

Note the similarity between this task, and Task 49 from *Reference Example 2*. *Matematikk R2* does not present any solution. However, the intended solution is likely to be related to the area interpretation, given θ_s. Boundedness as a condition for integrability is not part of θ_s, and only the area analogy is present in detail in the preceding theoretical discussion. Since the task does not have a solution presented in the textbook, it is therefore unlikely that students would discover the significance of the criteria of integrability and boundedness.

Task 2.121 is a third task relevant to the discussion (Borge et al., 2022, p. 168). This task presents the famous *Gabriel's Horn*. The function $f(x) = 1/x$ is given, and the students are asked to define the integrals of the volume and surface of revolution, $V(a)$ and $A(a)$ respectively, about the abscissa from $x = 1$ to $x = a$, where $a > 1$. Then, by letting $a \to \infty$, they are tasked with examining whether the limits $\lim_{a\to\infty} V(a)$ and $\lim_{a\to\infty} A(a)$ exist. To solve the task, the limits

$$\lim_{a\to\infty} V(a) = \lim_{a\to\infty} \pi \int_1^a \frac{dx}{x^2} = \pi,$$

and

$$\lim_{a\to\infty} A(a) = \lim_{a\to\infty} 2\pi \int_1^a \frac{1}{x}\sqrt{1 + (\ln x)^2}\,dx \to \infty,$$

are calculated.

This task demonstrates a type of improper integral with integration limits that tend to infinity, that is, it breaks the criterion of a closed interval of integration. It also demonstrates that some integrals of this type can be calculated to a concrete value, while others cannot. *Task* 2.121 is the only instance of such a task and seems to be another case of a mathematical curiosity. The technique used in this task is not used for anything else, nor are any later uses for the techniques mentioned.

A common theme of these three tasks is that of examining the very limits of the FTC. More specifically, they illustrate what sort of functions are permissible as integrands in a definite integral. And with the addition of *Task* 2.121, it illustrates how one can handle cases where the FTC cannot be applied directly, but where it needs to be modified in certain ways. The connection between them is, however, not explicitly made.

Elements of the didactic transposition

In this section I will compare the praxeological organisation found in the Grade 13 book *Matematikk R2* with the reference model. In this way I will be able to describe the didactic transposition from scholarly knowledge to the Grade 13 noosphere. I do it element by element first, and then, in the following section I discuss implications and answer the research questions concretely.

Didactical transposition – Elements of the logos

Comparing the demonstration of the FTC in *Matematikk R2* with the reference proof for the FTC Part I, we first see some similarities. The premises are the same, that of a continuous integrand, and they therefore have the same applicability. As the reference proof, it also starts by defining the function $F(x)$ using an integral, and both have the goal of proving that $F'(x) = f(x)$.

But we do see some major differences. Whereas the reference model states the theorem formally first, *Matematikk R2* presents the proof *before* the formal statement of the theorem. As a result, it is less clear in the beginning of the proof what to expect as the end goal. By stating the goal in the beginning, the reference proof start by using the definition of the derivative, and directly show that by rewriting $F'(x)$, we will end up with $F'(x) = f(x)$.

The reference proof also bases its argument on previously proven results, which in turn are based on formal definitions, making the proofs rigorous. The argument in *Matematikk R2*, is instead based on graphical representations, and justifies the algebraic expressions it later manipulates using this graph. What it does reference, and therefore lends its legitimacy to, is the definition of the definite integral and Riemann sums. It is therefore crucial that these are defined properly for the FTC to be properly justified.

In *Matematikk R2*, there is also one major step within the proof that is not explained. For the argument to become rigorous, the step from

$$\frac{F(x+\Delta x)-F(x)}{\Delta x} \approx f(x)$$

to the equality

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x} = f(x),$$

needs to be argued. This is not done in the proof, nor does it reference any previously proven theorems. The book does, however, not claim to present a formal proof. They instead call this step the *carrying idea* and foreshadows a more complete proof to be found later in the students' journey towards knowledge.

While a rigorous approach would base the argument on previously proven theorems, founded on the formal definition of limits, *Matematikk R2* bases its argument on intuitions and algebraic manipulations. The *importance* of rigor can, however, be seen. *Matematikk R2* shows that by referring to the fact that a more rigorous proof exists, that the students will possibly encounter somewhere later along their trajectory of learning.

One structural change which is consequential, is the dependence on a correct definition of the integral. *Matematikk R2* does have a definition that is useable in most cases encountered in the textbook, but not, as claimed, in all cases. By basing the definition of a definite integral on continuous functions, and not contending with what it means for a function to be integrable, more than in a passing note, it leaves out a crucial piece of information.

Didactic transposition – Tasks

The *intra-mathematical utility* of the FTC presented in *Matematikk R2* seems to be quite similar to that of the university textbooks. Much of the same types of tasks available in the university textbooks are also available using θ_s.

The exception is when integrability is at stake. The case of Task 49 from *Calculus* and *Talk-Task 1* in *Matematikk R2* exemplifies this. Although *Matematikk R2* has a logos that can provide support for justifying *that* the calculation fails, through arguing that the area cannot be negative, it does not have a means of explaining *why* the calculation fails. The fact that boundedness is a criterion for integrability, is not discussed, and neither is the fact that it is precisely because of boundedness that a function which is continuous on a closed interval is also always integrable. A pertinent question relating to this, would be “what would the students have made of the task if the interval of integration were $[0,1]$ rather than $[-1,1]$?” Certainly, the function would *look* continuous on the whole interval.

Furthermore, *Examine Task 2*, and *Task 2.121* show the use of techniques and themes that could have been useful in a more thorough treatment of integrability. The technique of dividing the area of integration into subintervals, seen in *Examine Task 2*, which can also be used for piecewise continuous functions, as seen in *Reference Example 1*, could be instrumental in providing examples of integrable non-continuous functions. In that way, the importance of boundedness could be illustrated.

Task 2.121 is an example of an improper integral. The fact that this task is included, does show the willingness of the textbook to include integrals that are not proper definite integrals, but which are nevertheless extensions of the concept of definite integrals. With relatively few modifications, a discussion about other types of improper integrals, for example of the type where the integrand itself tends to infinity rather than the independent variable, could be included.

Thus, in these three examples, one can see a potential for a deepening of the understanding of the FTC, and particularly for the premises for its application. For that to be possible, a more precise notion of integrability is needed, also including the distinction of boundedness. The connection is, however, not made clear, and the three tasks stand as separate examples of *mathematic curiosities* rather than providing justification for further theoretical developments. The lack of this distinction in some form is therefore a major constraint.

Concluding remarks

On this background, the didactic transposition can be summarised. It is first important to note the clear similarities between the organisation seen in *Matematikk R2* and the one identified in the referenced university textbooks. The treatment of the FTC, and not only the *Newton-Leibniz* formula, allows for a broader range of tasks and techniques. In particular, the inclusion of and focus on Part I of the FTC presents conditions that allow for a closer connection between the analogies of accumulation and area, well known to be a difficulty for students (Thompson & Harel, 2021; Burgos et al., 2021).

But the main concern is that the notion of integrability is undeveloped. Although the term is used once, it is never defined properly, and the condition of boundedness is never mentioned or described. Thus, the lack of a structural element, the notion of boundedness, has consequences for the utility of the FTC.

The importance of boundedness is apparent when the integrand is either *not* bounded, or the function is not continuous but still integrable. Didactic implications of boundedness, and of closure of the interval of integration, in relation to improper integrals has been examined in several publications (e.g., Gonzáles-Martín & Camacho, 2004; Gonzáles-Martín & Correira de Sá, 2007; Rúbio & Gómez-Chacón, 2011), showing both that first-year university students have great difficulty in comprehending the importance and significance of these two criteria and even seem to be generally unaware of this importance.

It is therefore, in my opinion, a disservice to the Grade 13 students to not discuss what significance boundedness has, while at the same time include tasks that could clearly benefit from such a discussion. It is also likely that a discussion about boundedness and integrability could strengthen the conceptions of continuity of functions in general, another area of calculus that has proved difficult conceptually for students (Hanke, 2018; Lankeit & Biehler, 2020). Thus, by not including boundedness, an important part of the FTC's utility is left out, reducing the scope of both the set of available techniques, τ_s, and types of tasks, T_s.

The observations in this study and research of the organisation of the FTC in *Matematikk R2* illustrates well the challenge of including new material in a textbook. The praxis block has clearly been strengthened by an explicit inclusion of the FTC and not only the Newton-Leibniz formula. But with this, new challenges arrive. Because of an undeveloped notion of integrability, the students do not get the resources to know the conditions for when the FTC can be applied, and why the conditions are as they are. The consequences can be seen in three tasks, which without a concept of integrability which includes boundedness, cannot be connected, and therefore remain as mathematical curiosities, instead of contributing to the FTC's intra-mathematical and extra-mathematical utility.

However, the choices of the textbook authors are, just as the activities of students and teachers, formed by their conditions and constraints. In this case, through a new curriculum reform, the requirement of introducing a more concrete treatment of the FTC, a constraint, was introduced, but the underlying concepts of integrability and boundedness has not been given the same attention. And with the time constraint put on the school system (Leong & Chick, 2011; Teig et al., 2019) and pressure added from high-stakes testing (Chichekian & Shole, 2016), balancing the size and content of the curriculum, and consequently also textbooks' contents, is not an easy task. If one adds something, another thing must often go. In this case, I claim it is sensible to include boundedness as a criterion for integrability, since it provides both a more solid foundation for the FTC, and because of the insight it might provide into details about the concept of continuity.

Since the analysis here focuses on a textbook, two immediate questions remain. How is the concept of the FTC treated in other Grade 13 textbooks in Norway? What impact does this change in curriculum, and the consequent change in the textbooks affect students' learning and readiness for further mathematics studies? The last of these questions may only be answered in a few years, when the first-year students that have been taught using this textbook, under the new curriculum, arrives at the universities.

Notes

[1] Figures taken from the textbook by Borge et al. (2022) are reproduced with permission from the publisher, Aschehoug. All figures are designed by Eirek Engmark at "Framnes Tekst & Bilde AS".

[2] All translations from Norwegian to English are made by the author of this article.

[3] *Utforsk* in Norwegian.

[4] *Snakk* in Norwegian.

References

Adams, R., & Essex, C. (2018). *Calculus: A complete course* (9th ed.). Pearson.

Borgan, Ø., Borge, I. C., Engeseth, J., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2021). *Matematikk R1* (4th ed.). Aschehoug.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51–63.

Borge, I., Engeseth, J., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2022). *Matematikk R2* (4th ed.). Aschehoug.

Botsko, M. W. (1991). A fundamental theorem of calculus that applies to all Riemann integrable functions. *Mathematics Magazine*, *64*(5), 347–348. https://doi.org/10.1080/0025570X.1991.11977632

Burgos, M., Buenos, S., Godino, J. D., & Pérez, O. (2021). Onto-semiotic complexity of the definite integral. Implications for teaching and learning calculus. *REDIMAT – Journal of Research in Mathematics Education, 10*(1), 4–40. http://dx.doi.org/10.17583/redimat.2021.6778

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In H. G. Steiner & M. Hejny (Eds.), *Proceedings of the International Symposium on Selected Domains of Research and Development in Mathematics Education* (pp. 51–62). University of Bielefeld, and University of Bratislava. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/On_Didactic_Transposition_Theory.pdf

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima J. Math. Educ.*, *12*, 71–114.

Chevallard, Y. (2022). *What is questioning the world? Towards an epistemological and curricular break* [Keynote address]. 7th International Conference on the Anthropological Theory of the Didactic (CITAD7). Barcelona, Spain.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Chichekian, T., & Shole, B. M. (2016). Preservice and practicing teachers' self-efficacy for inquiry-based instruction. *Cogent Education*, *3*(1), 1236872. https://doi.org/10.1080/2331186X.2016.1236872

Directorate for Education and Training. (2006). *Læreplan i matematikk for realfag: Programfag i utdanningsprogram for studiespesialisering. (MAT3-01)* [Curriculum for mathematics in natural science and technology studies: Programme subject in educational programmes for specialisation in general studies (MAT3-01)] http://www.udir.no/kl06/MAT3-01

Directorate for Education and Training. (2020). *Læreplan i matematikk for realfag (matematikk R) (MAT3-02)* [Curriculum for mathematics in natural science and technology studies (mathematics R) (MAT03-02)]. https://www.udir.no/lk20/mat03-02

Freudenthal, H. (2002). *Didactic phenomenology of mathematical structures.* Springer. (Original work published 1983)

González-Martín, A. S., & Camacho, M. (2004). What is first-year mathematics students' actual knowledge about improper integrals? *International Journal of Mathematics Education 35*(1), 73–89. https://doi.org/10.1080/00207390310001615615

Gonzáles-Martín. A. S., & Correia de Sá, C. (2007). Historical-epistemological dimension of the improper integral as a guide for new teaching practices. In E. Barbin, N. Stehlíková, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 5th European Summer University* (pp. 211–223). Vydavatelský servis. https://publimath.univ-irem.fr/numerisation/ACF/ACF08022/ACF08022.pdf

Hanke, E. (2018). "A function is continuous if an only if you can draw its graph without lifting the pen from the paper" – Concept usage in proofs by students in a topology course. In V. Durand-Guerrier, R.

Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of INDRUM 2018. Second conference of the International Network for Didactic Research in University Mathematics* (pp. 44–53), University of Agder and INDRUM. https://hal.archives-ouvertes.fr/hal-01849969

Heir, O., Borge, I., Engeseth, J., Haug, H., & Moe, H. (2016). *Matematikk R2* (3rd ed.). Aschehoug.

Lankeit, E., & Biehler, R. (2020). "I only know the absolute value function" – About students' concept images and example spaces concerning continuity and differentiability. In T. Hausberger, M. Bosch, & F. Chellougui (pp. 133–142). University of Carthage and INDRUM. https://hal.archives-ouvertes.fr/hal-03113859

Leong, Y. H., & Chick, H. L. (2011). Time pressure and instructional choices when teaching mathematics. *Mathematics Educational Research Journal, 23*(3), Article 347. https://doi.org/10.1007/s13394-011-0019-y

Lindstrøm, T. (2016). *Kalkulus* (4th ed.). Oslo: Universitetsforlaget.

Orton, A. (1983) Students' understanding of integration. *Educational Studies in Mathematics, 14*, 1–18. https://doi.org/10.1007/BF00704699

Petropoulou, G., Jaworski, B., Potari, D., & Zachariades, T. (2016). How do research mathematicians teach calculus? In Krainer, K., & Vondrová, N. (Eds.), *Proceedings of the ninth congress of the European Society for Research in Mathematics Education.* (pp. 2221-2227) Charles University in Prague, Faculty of Education and ERME. https://hal.archives-ouvertes.fr/hal-01288620

Rúbio, B. S., & Gómez-Chacón, I. M. (2011). Challenges with visualization: the concept of integral with undergraduate students. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2073–2081). University of Rzeszów and ERME. http://erme.site/wp-content/uploads/2021/06/CERME7.pdf

Strømskag, H., & Chevallard, Y. (in press). Didactic transposition of concavity of functions: From scholarly knowledge to mathematical knowledge to be taught in school. In *Proceedings of the Fourth Conference of the International Network for Didactic Research in University Mathematics.* Leibniz University Hanover and INDRUM.

Teig, N., Scherer, R. & Nilsen T. (2019). I know I can, but do I have the time? The role of teachers' self-efficacy and perceived time constraints in implementing cognitive-activation strategies in science. *Frontiers in Psychology, 10*, Article 1697. https://doi.org/10.3389/fpsyg.2019.01697

Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM – Mathematics Education, 53*(3), 507–519. https://doi.org/10.1007/s11858-021-01270-1

Topphol, V. (2021). A novel application of the instrumental approach in research on mathematical tasks. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, J., & A. Wernberg, A. (Eds.), *Preceedings of NORMA 20 The Ninth Nordic Conference on Mathematics Education* (pp. 265–272). Svensk Förening för Matematikdidaktisk Forskning. https://www.uv.uio.no/ils/english/about/events/2021/norma/proceedings/norma_20_preceedings.pdf

Topphol, V., & Strømskag, H. (2022). *A first-year mathematics student's construction of a praxeology of integration tasks* [Manuscript submitted for publication]. Department of Mathematical Sciences, Norwegian University of Science and Technology.

Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *REDIMAT, 6*(3), 307–330. https://doi.org/10.17583/redimat.2017.2078

Winsløw, C. (2022). Mathematical analysis at university. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & Ruiz-Munzón (Eds.), *Advances in the anthropological theory of the didactic* (pp. 295–305). Springer, Cham.

Paper 4

A Study and Research Path on Hyperthermia in Children Left in Parked Cars

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Paper Accepted for Publication in the Proceedings of the 7th International Conference of the Anthropological Theory of the Didactic

A study and research path on hyperthermia in children left in parked cars

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This paper presents a study and research path based on a question about hyperthermia deaths in children placed in cars parked in the sun. The text demonstrates the potential of the study and research path for generating questions about modelling, and what sort of mathematics and physics are involved in modelling the human body. It also suggests a broader interdisciplinary potential, and in particular avenues for investigations that include sociological and statistical themes.

Keywords: Study and research paths, hyperthermia, modelling, allometry.

Introduction

Two PhD students (the author and another one), took part in a PhD course about study and research paths, and conducted an SRP individually. The generating question Q_0 = "Why do babies die of heat stroke in cars parked in the sun?" was used as the starting point of the SRP. In a handout describing the SRP, Q_0 was succeeded by the following guidelines:

> Are the possible causes of these deaths studied in the scientific literature? If so, what are the physical and physiological or other factors identified by the researchers? Does the fact that children have a greater ratio of the body's surface area to its volume than adults play a role according to these research studies? Which one? More broadly, what mathematics is useful or even indispensable to model the relevant factors and their interactions?

Three references were also provided as starting points for the SRP, a fact sheet about heat death (The European Child Safety Alliance, 2013), and two scientific papers (McLaren et al., 2005; Booth et al., 2010). A mid-way seminar was held, where the work was presented, and tips on further investigations were shared. A focus on modelling the human body was suggested at this point. Both the guidelines, the provided references, and the suggested focus made up a concrete set of preconditions for the SRP, and communicated expectations about how the question Q_0 was to be answered.

In this paper, I take as a starting point the report I wrote from the conducted SRP, and use it to answer the following research question: "What sort of mathematical models are used to answer Q_0, and how are they interconnected?" In answering this, I will also include thoughts about how the preconditions and expectations about the nature of an answer to the question Q_0 affect the outcome, and how it affects the way we might analyse the SRP.

Theoretical tools

In this study, tools from the ATD are used. These tools will be explained in the following section.

Praxeology

The ATD proposes a *praxeology* as a general model of human activity, including activities related to producing, diffusing and appropriating knowledge (Chevallard, 2020). A praxeology is described in terms of four constituents; type(s) of task(s) (T); a technique or set of techniques (τ) used to solve the given types of tasks; a technology (or discourse, θ) used to describe and explain the techniques; and a theory (Θ) that justifies the technology. These constituents are grouped in the two blocks, called the *praxis* block, consisting of T and τ, and the *logos* block, consisting of θ and Θ. A praxeology $\wp$, can then be described algebraically as $\wp = [T / \tau / \theta / \Theta]$.

Study and research path and the Herbartian schema

In this paper the notion of a study and research path (SRP) and the Herbartian schema are also used. A good description of both can be found in (Chevallard, 2020). In short, a Herbartian schema describes the institutional setting, using the notion of a didactical system S. The didactical system can be described algebraically as $S(X;Y;\heartsuit)$, where X is the group of students, Y are the study assistants (e.g. teacher, a librarian…), and ♥ the didactic stake, or the something that the X are intended to learn.

The stake ♥ can be a question Q_0, the starting point of an SRP. To describe an SRP, the developed Herbartian schema is used, and can be written symbolically as

$[S(X;Y;Q_0) \Rightarrow \{A^{\Diamond}_i, W_j, Q_k\}] \Rightarrow A^{\heartsuit}$.

Here the $A^{\Diamond}_i$ are the pre-existing answers, found in the literature and other sources of information, from which are also extracted the works W_j. These answers and works give rise to derived questions Q_k, and the final answer $A^{\heartsuit}$ produced by the didactical system. The final answer is seen collectively by the participants of the SRP as answering Q_0 to a satisfactory level. The final answer, $A^{\heartsuit}$, is an aggregate of intermediate answers A_k to the derived questions Q_k.

Description of the SRP

The SRP, based on the question Q_0, about heat stroke, with the extra premise that answers should be backed mathematically, and the focus should be on physiological and physical reasons, was conducted over approximately one month in 2019. The background for Q_0 is in the American statistics about heat death and babies left in cars. In 2019, 51 heatstroke deaths among children in parked cars were reported in the U.S., a slight reduction from the 2018 record high of 53 deaths (National Safety Council, n.d.). This issue has gained regular attention from media (e.g. Paybarah, 2019; Kalaichandra, 2019), and is therefore of public interest.

Note here, that the question Q_0, does not stand alone in this SRP. Both regarding the preconditions for the question, and the expectations of how the question will be answered. The statistical report, which is a crucial part of the preconditions, lends legitimacy to the claim that children are more vulnerable to hyperthermia, which in turn demonstrates both the relevance and legitimacy of the question itself. And the expectations of how the question is to be answered might have implications both on how the SRP is conducted, but also on how it is analysed afterwards. This further highlights the question about the nature of Q_0. In what way could the preconditions and expectations be considered an integral part of the question Q_0 itself?

Methodology

The methodology can be divided in two phases. First, the SRP is conducted by the author himself, occupying the role as a student exploring the generating question. And then, from this, a didactical analysis of the resulting data will be done, with the aim of uncovering the didactical potential of this SRP, particularly what sort of mathematical models emerge from the SRP. In the following two sections, these two phases are described in more detail. Note, however, that these phases are not meant to be strictly consecutive. Analysing the SRP is after all part of conducting it.

Method of analysis

The resulting questions that emerge through the SRP are first categorized thematically, according to what sort of answers that are likely to emerge from a careful study of the questions. This includes both partial questions that are directly extracted from the generating question, and questions that are either asked by, or answered by the literature initially provided in the handout. These questions are then categorized according to the nature of the answers an examination of these questions might provide. Most importantly, what sort of questions are likely to give rise to answers with a clear mathematical content, and what sort of mathematics?

Following this a more thorough literature search is conducted, using these initially extracted questions. The literature search is conducted in two steps. First, the references in the literature found so far are examined, to find out how the answers they provide are argued for. This includes examining what theoretical foundations lie behind the answers, which will be an important selection criterion when deciding which branches to follow in the SRP. Due to the particular interest in applying mathematics to answer the questions, after the first step in the literature search, only the branches that seem to harbour a particular potential for mathematically centred analyses will be pursued.

Initial questions and literature search

The starting point of the SRP is the question Q_0, in addition to a fact sheet from the European Child Safety Alliance (The European Child Safety Alliance, 2013), providing the first pre-existing answer ($A_1^\lozenge$) to Q_0.

A reason for starting with this text is that it gives short, easy to understand answers to 5 questions related to Q_0:

1. Why are babies left behind in cars?
2. What is special about cars?
3. What does it mean to be too hot, and how does this affect babies differently from adults?
4. How common is this phenomenon?
5. How can it be prevented?

Q_5, about prevention, is not presented directly as a question, but as tips directed at parents, about how to prevent hyperthermia in the first place. A related issue, which the text does *not* address directly, is the question about how death can be prevented when a baby is already experiencing hyperthermia, with the possible exception “Dial 112 immediately if you see a child alone in a car”. Therefore, Q_5 is divided in two related questions:

a. How can hyperthermia be prevented?
b. How can death be prevented for babies experiencing hyperthermia?

Q_{5a} is directed at parents and society at large, about how we can prevent hyperthermia in the first place. Q_{5b}, on the other hand, addresses the medical question of saving someone who is already experiencing hyperthermia.

These questions can be addressed in different ways, according to how we expect them to be answered. Q_1 is expected to be answered by sociological or psychological reasons. The fact sheet points at both intentionally leaving the baby behind, and unintentionally due to forgetting the baby, or not being aware of the dangers of overheating in a car. Q_2 has to do with the physical properties of a car, and how this affects heat absorption and heat transfer.

In the rest of the paper, I only consider physical and physiological factors. That means, the questions Q_2 and Q_3 mainly. In particular, the differences between children and adults are interesting, since it seems to lend itself to a challenge of modelling, and they relate to the core issue of why this is specific to small children.

These questions have also been answered to greater or lesser degree in the literature which the European Child Safety Alliance base their fact sheet on. I will here present some of the papers and webpages that the fact sheet refers to, what sort of answers they give, and how they arrive at these answers. I will also include some papers and texts found by a limited literature search on the keywords “heat death”, “babies” and “parked cars”.

In the papers referenced by the fact sheet, several different physical and physiological factors are mentioned, and in the next section, I present one example of a physical variable that is discussed.

An example of a physical variable: The colour of the interior of the car

The fact sheet refers to a number of articles and webpages which answers Q_2 and Q_3 experimentally (McLaren et al., 2005; Null 2010) (W_1 and W_2), and by examining 231 lethal hyperthermia cases (Booth et al., 2010) (W_3). In these, the question of interior colour of the car (Q_6), in addition to whether cracking open a window would help in regulating temperature (Q_7). Here, only a lighter interior colour of the car seemed to have any significant effect.

Other papers and webpages also deal with the same question, but the overall trend seems to follow the above-mentioned texts. An experiment from 1995 on two differently coloured cars (Gibbs et al., 1995) (W_4) had the same conclusion as W_1 and W_2. The 1995 paper differed only in adding that the *exterior* colour of the car did not matter significantly. They also mention more clothes, cushioned seats, and a position below window level as reasons for small children being particularly vulnerable.

About allometry

None of the articles and webpages presented so far have dealt directly with modelling the human body to determine the heat response of children, although they do mention its results. Both W_1 and W_3 mentions the surface area to volume ratio as having an effect, and W_3 refers to another paper (Tsuzuki-Hayakawa et al., 1995) (W_5), where they showed that small children had a higher and faster heat increase than their mothers when exposed to moderate heating. They suggested two possible explaining factors. The difference in surface area to mass ratio would result in the bodies of young children being easier to heat, and the thermoregulatory systems, sweating and blood circulation among others, might be less developed in children.

Both these themes, the surface area to volume ratio and the effectiveness of the thermoregulatory system, are important factors in the study of scaling in biology, also called allometry. Therefore, “allometry” and “biological scaling” are also included as keywords in the literature search, in trying to answer the question of what role biological scaling has in answering why children are more vulnerable to overheating (Q_8). In addition, to explain overheating, the question of how heat transfer works also needs to be answered (Q_9).

The first text found by this extension of the search, is the text “Allometry: the study of biological scaling” (Shingleton, 2010) (W_6). This is an educational article presenting the concept of allometry. It introduces the knowledge that different parts of the body scales at different rates in relation to the overall size of the body. Examples from the human body are the heart, which grows at approximately the same rate as the body itself, and the brain, which grows slower than the rest of the body. Of specific mathematical interest, the article also presents the fact that many of the observed scaling relationships turned out to be linear, when plotted on a log-log plot, and follow the equation

$$\log y = \alpha \log x + \log b,$$

which can be written as

$$y = bx^{\alpha}.$$

Here x is body size, y is the organ size, and $\log b$ is the y intercept. The factor α is called the allometric coefficient. Each organ has its specific allometric coefficient, and the size of the coefficient describes how fast a certain body part grows relative to the growth of the rest of the body. A body part having a higher growth rate than the rest of the body, then $\alpha > 1$, and conversely, when $\alpha < 1$ the growth rate of the body part is lower than the rest of the body. Further, the text expands the concept of allometry to include other aspects, such as running speed and metabolic rate.

The last of these two was mentioned in W_5 as one of two dimensions providing explanations to why children are more vulnerable to overheating. The text W_6 does, however, not describe the surface area to volume ratio as one of the allometric variables. By expanding the search to also include the word “surface area” together with allometry, some new texts were found.

The first article showing up was from Britannica (Glitterman, n.d.) (W_7), describing both these dimensions as important measures that displays allometric scaling. Area and body mass are related by area growing by a 2/3 power of the body mass, and metabolic rate grows by a ¾ power of the body mass. And the second was chapter 4 in an online textbook in biology (Sam Houston State University, n.d.) (W_8), where the relation between surface area and volume is explained through geometrical examples. The first scholarly article showing up in this search (W_9), was an article on how surface area scaling on both microscopic and macroscopic levels are related (Okie, 2013). It explores the different strategies organisms have for dealing with the challenges related to how surface area and volume scales at different rates, and it develops a theory for modelling the effects of these different strategies. The details in this last article go far beyond the scope of this SRP, but the ubiquity of the surface area to volume scaling problem as an explanatory factor in biology which W_9 refers to lends weight to the importance of this dimension.

Heat transfer

The last piece in this puzzle is the question Q_9 about how heat transfer works on bodies. Here the search term "heat transfer", in addition to surface area to volume ratio, was used. The first promising article that was found using this search term was an article describing interdisciplinary teaching in physics and biology, where the heating problem is a major theme (Planinšič & Vollmer, 2008) (W_{10}). Here an example teaching unit is described where they experiment with melting cubes of cheese, and use Newton's law of cooling, in addition to the surface area to volume ratio. Then they apply the physics learned from this to explain the differing metabolic rates of different sized animals. For a more in depth description of Newton's law of cooling, W_{10} references another paper (O'Sullivan, 1990) (W_{11}). Here the law is described in its differential form, which will be used in this SRP. A similar description can also be found at a mathematics teaching web resource (math24.ner, n.d.) (W_{12}). This site also describes the role of heat capacity on the system.

Modelling the human body

From the answers provided by literature, a simple model can be described, based on four assumptions. First, the shape of an ordinary person is largely consistent, and thus a "large" person is just a scaled-up version of a "small" person. Thus, the only dimension important for determining how well a person can stand up to heat exposure is the height. Secondly, all tissues of a person have the same heat conductivity. Thirdly, transfer of energy between a body and the environment is mainly dependent on and proportional to the surface area of the body, while the total temperature is proportional to volume.

In the following calculations, the relations and formulas found in literature are used, particularly W_8 and W_{11}.

A consequence of assuming the growth of a person as purely geometric scaling is that we can use some general geometry true for all bodies in three-dimensional space, following the argumentation shown in W_8. As the dimensions (length, width, and height), scales linearly, the surface area scales quadratically, and volume cubically. Moreover, I will assume that density, heat conductivity and heat capacity is relatively similar for all human bodies. From W_{11}, we also get Newton's law of cooling:

$\frac{dQ}{dt} = h \cdot A \cdot (T_a - T(t)).$

Here Q is the thermal energy of a body, A is the surface area, T_a and $T(t)$ are the ambient and body temperatures respectively, and h is a heat transfer coefficient. The energy transfer and the body temperature are both time dependent, meaning that without more information, we are not able to solve this equation. But we can suggest another equation, by the observation that the total thermal energy contained in a body is proportional to the temperature, and dependent on the total heat capacity of the body:

$$\frac{dQ}{dt} = C\frac{dT}{dt}.$$

Here C is the total heat capacity, which is the product of the mass (m) and the mass-specific heat capacity of the material (s) (Chang, 2008, p. 186). Since the mass is proportional to volume (V), we can write this relation as $C = m \cdot s = V \cdot c$, where c is a constant factor. Combined, these two equations give us the equation

$$\frac{dT}{dt} + \frac{A \cdot h}{V \cdot c} T = \frac{A \cdot h}{V \cdot c} T_a,$$

which has the general solution

$$T(t) = T_a + De^{-\frac{A \cdot h}{V \cdot c} t}.$$

Assuming $t = 0$ at the beginning of the heating we can see that $T_0 = T(0) = T_a + D$ implies $D = T_0 - T_a$. D is therefore the temperature difference between the body and the surroundings.

We can interpret the solution as temperature difference decreasing according to $e^{-\frac{A \cdot h}{V \cdot c} t}$. When the body temperature is lower than the ambient temperature, this results in the increase in body temperature. The speed of the temperature change is then dependent upon the coefficient $\frac{A}{V} \cdot \frac{k}{c}$. If both k and c are constants, the only variable parameter is the fraction A/V which is proportional to $L^2/L^3 = 1/L$, where L is height. This fraction decreases as height increases, and consequently, a shorter person, such as a baby, is more prone to heating.

Modelling

Constructed models and answers to Q_0

From the above argument and calculations, a potential $A^\heartsuit$ can be described. Since the body of a child is smaller than the body of an adult, the child has a larger surface area to volume ratio than the adult, and the effect of heating is then much greater on the child than the adult (A_3). In concert with other factors, such as positioning in and thermal characteristics of the car (A_2), and a less developed system for heat regulation (A_8), this makes for a deadly combination. This provides an answer to why children that are left in cars on sunny days are prone to die of heat stroke. The path of the study and research is modelled by a directed graph in Figure 1. The elements of the milieu and intermediate answers are displayed in Table 1.

Note that Figure 1 is itself a model of the SRP conducted. It does not indicate any ordering of the process itself and is only a still image of the result. It does not show which branches were followed in which order. And moments already treated at one point during the process were revisited several times more during the process. In that regard the figure should be seen more as a model of the structuring of the items of knowledge at the time of the writing of this paper, rather than a precise map of the process itself. Moreover, from the figure, it seems like the question Q_0 is the very start of the SRP. This is however more complicated since the question also arises from somewhere. The generating question might itself be a response to the established answers, and in particular the ones found in the pre-existing answers. It is also not the case that the derived questions stem from the pre-existing answers alone, but from the conjunction of the question Q_0, together with the pre-existing answers and preconditions of the SRP.

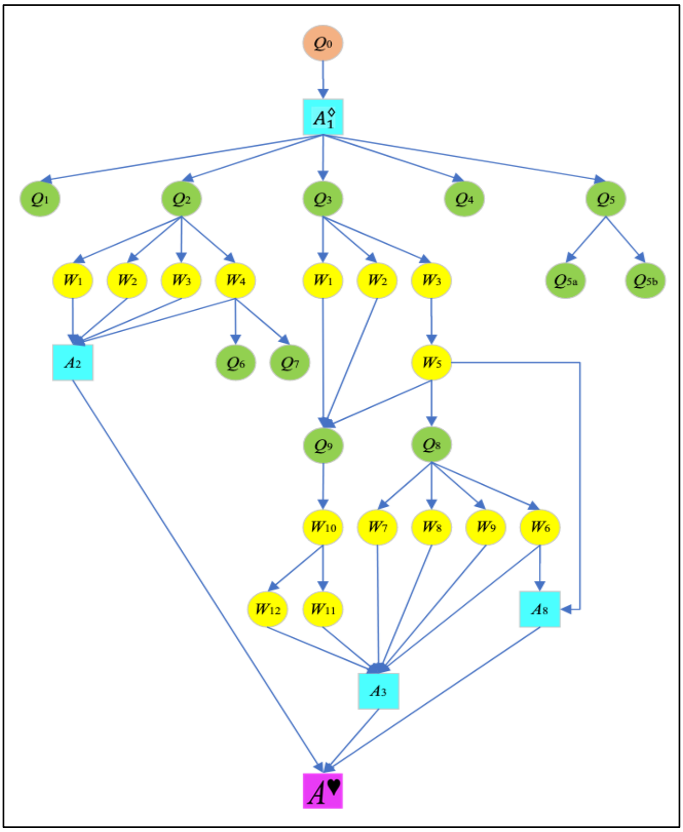

Figure 1: Schematic representation of the SRP

Table 1: Elements of the SRP

Questions	
Q_0	Why do babies die of heart stroke in cars parked in the sun?
Q_1	Why are babies left behind in cars?
Q_2	What is special about cars?
Q_3	What does it mean to be too hot, and how does it affect babies differently from adults?
Q_4	How common is this phenomenon?
Q_{5a}	How can hyperthermia be prevented?
Q_{5b}	How can death be prevented for babies experiencing hyperthermia?

Q_6	How does the colour (interior/exterior) of the car affect heating?
Q_7	How does opening a window affect heating?
Q_8	Why are children more vulnerable to overheating?
Q_9	How does heat transfer work on a body?
Existing answers to Q_0	
$A_1^\Diamond$	European Child Safety Alliance, 2013
Additional works	
W_1	McLaren et al., 2005
W_2	Null, 2010
W_3	Booth et al., 2010
W_4	Gibbs et al., 1995
W_5	Tsuzuki-Hayakawa et al., 1995
W_6	Shingleton, 2010
W_7	Glitterman, n.d.
W_8	Sam Houston State University, n.d.
W_9	Okie, 2013
W_{10}	Planinšič & Vollmer, 2008
W_{11}	O'Sullivan, 1990
W_{12}	math24.ner, n.d.
Intermediate answers	
A_2	Positioning and thermal characteristics of the car
A_3	Effect of the surface area to volume ratio
A_8	Development level of thermo-regulatory system

In this SRP, the choice of focusing on physical and physiological factors lead to the development of the model described in the last section, where the role of the surface area of different sized bodies is used to explain differences in how heating affects the bodies. And models explaining the heating of cars have been referenced, but not fully described.

However, other choices could have been made. Several of the questions proposed in the initial exploration can be answered following methods that are different from the ones used to answer Q_3 and Q_2. Notably, several of them seem to lend themselves more to investigations of sociological factors than to physical or physiological. This highlights the interdisciplinary potential of Q_0. Although the focus here is on physical and physiological factors, using geometry and differential equations to model the heating problem, there is also potential for investigating social and sociological factors, in addition to other disciplines within mathematics, such as statistical modelling.

Handling preconditions to Q_0

In the end, I return to the issue of the preconditions and expectations for the generating question Q_0. As we see, the question, as it is formulated, is quite an open question, with several available avenues of investigation, both physical and mathematical, and others, like sociological and psychological. But if we include the specified preconditions and expectations about the answer, it seems clear that this has had a strong guiding force on how the SRP was conducted, and on the answer $A^\heartsuit$ resulting from it. Then the question arises, whether the Q_0 is just the formulation presented in the beginning of this paper, or should the preconditions be “included” in the question in some way. A part of the answer is that the preconditions are represented in the pre-existing answers $A^\Diamond{}_i$. In particular, this is true for the handed-out texts that were to be used as a starting point.

But there are more parts of the preconditions than the pre-existing answers and the handed-out texts. The explicit and implicit expectations about permissible answers, are not so easy to position within the current framework. The preconditions clearly condition the way in which Q_0 is answered, which could point towards them being an integral part of the question itself. But I do think by doing this, we lose some important distinctions. It might obscure the fact that different people have different understandings and expectations about a given question. Will they then not be answering the same Q_0?

From the small SRP presented in this paper, it is not however possible to give any clear general answer to, nor a fully worked through example demonstrating the question of how we should treat the parts of the preconditions not representable as pre-existing answers $A^\Diamond{}_i$, and only hint at the issue. I leave it open as a question for further discussions and investigations.

References

Booth, J., Davis, G., Waterbor, J., & McGwinJr., G. (2010). Hyperthermia deaths among children in parked vehicles: anA analysis of 231 fatalities in the United States, 1999 –2007. *Forensic Science, Medicine and Pathology*, 6, 99–105.

Chang, R. (2008). *General chemistry. The essential concepts* (5th ed.). Mc Graw-Hill Higher Education.

Chevallard, Y. (2020). Some sensitive issues in the use and development of the anthropological theory of the didactic. *Educação Matemática Pesquisa, 22*(4), 13–53.

Guard, A., Gallagher, S.S. (2005). Heat related deaths to young children in parked cars: analysis of 171 fatalities in the United States, 1995–2002. *Injury Prevention, 11*, 33–37.

Gibbs, L., Lawrence, D., & Kohn, M. (1995). Heat exposure in an enclosed automobile. *J La State Med Soc. 6*, 545-546

Glitterman, J. L. (n.d). Allometry. *Encyclopedia Britannica*. https://www.britannica.com/science/allometry

Kalaichandran, A. (2019, August 14). How we can help prevent children from dying in hot cars. *The Washington Post*. https://www.washingtonpost.com/opinions/how-we-can-help-prevent-children-from-dying-in-hot-cars/2019/08/13/b49a016c-be0d-11e9-a5c6-1e74f7ec4a93_story.html

Math24 (n.d.). Newton's law of cooling. *Math24.* https://math24.net/newtons-law-cooling.html

McLaren, C., Null, J., & Quinn, J. (2005). Heat stress from enclosed vehicles, moderate ambient temperatures cause significant temperature rise in enclosed vehicles. *Pediatrics*, *116*(1), 109–112.

National Safety Council (n.d.) Hot car deaths. https://injuryfacts.nsc.org/motor-vehicle/motor-vehicle-safety-issues/hotcars/

Null, J. (2010, June 30). Hyperthermia death of children in vehicles. https://web.archive.org/web/20100701004956/http://ggweather.com/heat/

Okie, J. G. (2012). General models for the spectra of surface area scaling strategies of cells and organisms: Fractality, geometric dissimilitude and internalization. *The American Naturalist, 181*(3), 421–439. https://doi.org/10.1086/669150

O'Sullivan, C. T. (1990). Newton's law of cooling – A critical assessment. *American Journal of Physics 58*(10), 956–960. https://doi.org/10.1119/1.16309

Paybarah, A. (2019, August 1). Painfully common: 52 children died last year in hot cars. *New York Times.* https://www.nytimes.com/2019/08/01/nyregion/newyorktoday/hot-car-deaths-nytoday.html

Planinšič, G., & Vollmer, M. (2008). The surface-to-volume ratio in thermal physics: from cheese cube physics to animal metabolism. *European Journal of Physics, 29*, 369–384.

Sam Houston State University. (n.d.) *Biological design*. https://www.shsu.edu/~bio_mlt/Chapter4.html

Shingleton, A. (2010). Allometry: The study of biological scaling. *Nature Educational Knowledge, 3*(10). https://www.nature.com/scitable/knowledge/library/allometry-the-study-of-biological-scaling-13228439/

The European Child Safety Alliance. (2013, September 26). Children in Hot Cars: Hyperthermia. https://www.safecommunitiesportugal.com/wp-content/uploads/2021/02/Children-in-Hot-Cars-Hyperthermia.pdf

Tsuzuki-Hayakawa, K., Tochihara, Y., & Ohnaka, T. (1995). Thermoregulation during heat exposure of young children compared to their mothers. *European Journal of Applied Physiology 72*, 12-17.

ISBN 978-82-326-7240-0 (printed ver.)
ISBN 978-82-326-7239-4 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

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