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Matthew Tandy

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Doctoral thesis

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Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
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Preface

This thesis is submitted as a required part of the fulfillment of the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology (NTNU). Funding for the PhD has been provided by the project *Wave Phenomena and Stability – a Shocking Combination* (WaPheS) from the Research Council of Norway, for which I am grateful. I would also like to express my thanks towards the Department of Mathematical Sciences (IMF) at NTNU for their support and the working environment they have provided.

First, and most importantly, I would like to thank my main supervisor Katrin Grunert for her supervision, advice during the PhD, providing answers to my many questions, and her valuable collaboration and feedback on my work. I would also like to thank my co–supervisor Helge Holden for his support during my PhD.

I am grateful for the supervision and invitation to stay at the University of Brescia from my other co–supervisor Rinaldo M. Colombo. Particularly he provided valuable insights and exceptional ideas that have served as great inspiration, as well as knowing the best places to have a bite to eat. He and Mauro Garavello were fantastic to work with on the article we created together. I would also like to thank the University of Brescia for funding towards the trip from the fund of international cooperation.

The support from my parents Susan and Andrew Tandy were invaluable, and I suppose I should mention my siblings Joseph and Abigayil.

Finally, I would like to extend my gratitude toward my many friends and colleagues that have supported me throughout, both locally and across the world. In particular I mention my companions in my field Sondre and Thomas, as well as Silius, Philip and Halvard, for our many interesting dinners and conversations.

Tandy

Matthew Tandy

Trondheim, 11.04.2023

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Part I

Introduction

Chapter 1

An overview and the thesis structure

This work focuses on the stability, and some well-posedness results, for a variety of partial differential equations.

We begin by studying the Hunter–Saxton equation, in the form

$$u_t + uu_x = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(y,t)\,\mathrm{d}y - \int_{x}^{\infty} u_x^2(y,t)\,\mathrm{d}y\right).$$

This problem fits well into the realm of other non-linear hyperbolic PDEs such as conservation laws, the KDV equation, and the Camassa-Holm equation. A formal yet simple description of the Hunter–Saxton equation is that it is the (inviscid) Burgers' equation[1] with the addition of a non-local source term. The equation is of interest due to the nature of its solutions, whose behaviour is related, but also compellingly different to those of Burgers' equation. Its solutions may experience wave breaking in finite time, however rather than the development of shock singularities, i.e. spatial discontinuities, continuity is retained for all time. Multiple different closely related solution concepts emerge which pose interesting problems. A solution of Burgers' equation and the Hunter–Saxton equation can be seen in Figure 1.1.

We are interested in the study of the stability for different notions of solution. More specifically, we aim at the construction of a metric which renders the flow of solutions Lipschitz continuous with respect to initial data.

A detailed background and an introduction to the Hunter–Saxton equation is given in Chapter 2. Section 2.1 provides a brief overview

[1](Inviscid) Burgers' Equation: $u_t + \left(\frac{u^2}{2}\right)_x = 0$.

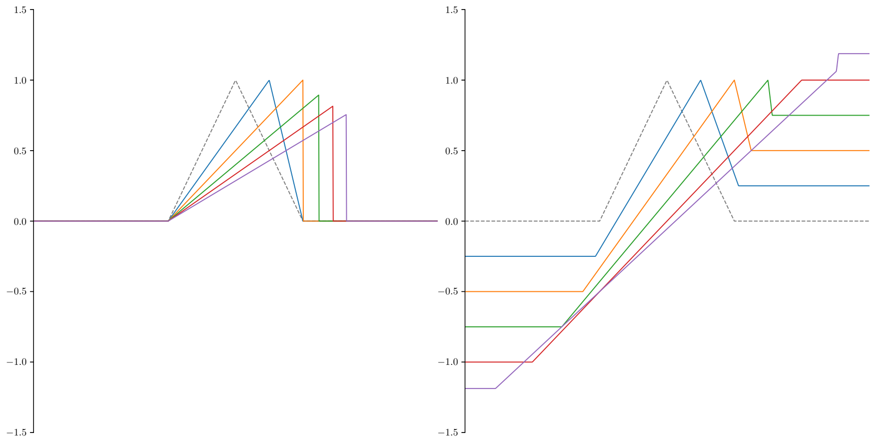


Figure 1.1: On the left: A solution to Burgers' equation. On the right: A solution to the Hunter–Saxton equation. Each curve is the solution at a different point in time. Both use the same initial data, given by the grey dashed line.

and history of the equation. In Section 2.2, the original derivation by Hunter and Saxton is detailed, as well as some of the key properties of the equation in an explanation of how one obtains classical solutions via characteristics. This is then expanded upon in Section 2.3 in a presentation of the framework used for general solutions of the equation. The origin of the scheme, its application, and the results of the first two articles [27, 28] are then shown in Sections 2.4 and 2.5. Finally, the results are contextualised in Section 2.6 by outlining two metrics that were previously constructed.

Our attention then shifts to the problem of the well–posedness in the coupling of a variety of different partial differential equations, often of hyperbolic type. The presented article [14], an extended version of the one to be published [12], was written during a research stay abroad at the University of Brescia, hosted by the co-author Rinaldo M. Colombo, and with meetings with the other co-author Mauro Garavello. Said works main results are built for the wider realm of problems in complete metric spaces, who lose the usual vector space structure in Banach spaces, but retain a well defined notion of distance. Results were constructed specifically with the future extension and application to problems in the field of epidemiological models in mind. We require that the constituent problems in the coupling have suitable stability results, and we inherit stability for the generated solution flow for the coupled problem.

An introduction and a detailing of the ideas can be found in Chapter

3, in addition to an overview of the results of the third article [14]. Further, an alternative approach to one of the models in said article is presented, which was removed due to space constraints.

Chapter 2

The Hunter–Saxton equation

2.1 A brief history

Hunter and Saxton first introduced the Hunter–Saxton equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \tag{2.1.1}$$

in their seminal paper [33]. The equation was derived by considering "unidirectional non-linear asymptotic waves" of the non-linear variational wave equation, a PDE that serves as a model for nematic liquid crystals. Early on this paper established a variety of important results and properties that make the model of particular interest to study. The local existence of classical solutions was established, constructed via the method of characteristics, and it was identified that said solutions may break down in finite time, generating singularities where the derivative of the solutions diverges to $-\infty$. The behaviour of these singularities, and their development, are denoted "wave breaking".

It was quickly noted that one cannot expect uniqueness of solutions unless additional constraints are imposed. Solutions remain continuous in space before and after wave breaking, notably different to the wave-breaking behaviour of Burgers' equation in which shocks emerge. Correspondingly, the concept of entropy conditions from the realm of conservation laws, see [4, 32], is unsuitable. Instead, different solution concepts emerge based on how one treats the energy, given by $E(t) = \|u_x^2(\cdot, t)\|_1$ for a solution u, at wave–breaking times. Energy that was initially spread over regions, less formally sets of positive measure, concentrates into singular points, i.e. sets of measure zero. The two most studied solution concepts are conservative and dissipative solutions, in which this concentrated energy is retained or fully discarded respectively.

It was also noted that solutions satisfy an infinite number of conservation laws. Finally, a wide class of solutions was identified that can be constructed explicitly, called multi-peakons, consisting of a series of piecewise affine curves connected continuously. That such solutions can be constructed is a tremendous boon to the study of the Hunter–Saxton equation, providing a wide class of examples, and serving as powerful tools for numerical methods. It was further noted that the uniqueness of solutions of the Cauchy problem requires additional constraints, which can immediately be shown via multi-peakon examples.

It is important to note that the initially derived form (2.1.1) is throughout this paper often referred to as the differentiated form of the Hunter–Saxton equation. We consider the integrated form, introduced in Section 2.2. Different notions of solution, usually with some incompatibilities, are dependent on the form and interpretation used.

Early on, the notions of dissipative and conservative solutions were defined via the limits of sequences of multi-peakons, in which said multi-peakons retained or disregarded concentrated energy at wave breaking. With these definitions in mind, existence of solutions was demonstrated on the positive half plane if the initial data has compactly supported derivatives with bounded variation [34]. Simultaneously, dissipative solutions for simple initial data on the positive half plane were constructed via a process of vanishing viscosity [35].

Zhang and Zheng then used the theory of Young measures to establish global existence and uniqueness of solutions on the half-line for the wider class of compactly supported initial data [46, 47, 48]. The notion of solution they used was no longer defined by the limit of multi-peakons, and instead used the distributional form of the Hunter–Saxton equation. In the dissipative case they used an Oleĭnik type entropy condition.

Later, Bressan and Constantin focused on the dissipative case, using the method of characteristics to establish the existence of solutions for initial data without the assumption of compact support [7]. Furthermore, they defined a Kantorovich-Rubinstein type distance establishing Lipschitz stability of dissipative solutions with respect to time and initial data.

Temporarily shifting focus, the Hunter–Saxton equation is closely related to the Camassa-Holm equation, for which similar techniques are employed in the analysis. Thus, often tools are developed for one equation and then applied or adapted to the other. A direct connection between the two equations is seen in the fact that the extended Hunter–

Saxton equation [1] is the high frequency limit of the Camassa-Holm equation, see [19].

In [6, 31], generalised characteristics were employed for the Camassa-Holm equation, and global existence of conservative solutions was established. In this method, the authors transform the equation into an equivalent semi-linear ordinary differential equation system, denoted the Lagrangian system and whose variables are denoted Lagrangian variables. Equivalently, the original equation variables are referred to as the Eulerian variables. They furthermore introduced an additional Radon measure variable μ that forms a part of the solution, corresponding to the current energy in the system. This additional variable allows for the separation of data in which energy is concentrated on a set of measure zero, as said energy inhabits the singular part of μ. They showed the ODE system can be solved, and the respective solution can be transformed back to form the solution of the Camassa-Holm equation. This method has proven to be adaptable and extremely useful when tackling problems.

Subsequently the method was applied to the Hunter–Saxton equation to demonstrate the existence of conservative solutions for non compactly-supported initial data in [8]. The Lagrangian ODE system in this case has the benefit of being linear. Note also that characteristics for the Camassa–Holm equation may exchange energy upon contact. This is not a feature for the Hunter–Saxton equation, and thus often schemes require a less technical approach for this equation. Additionally, a metric inspired by methods in Riemannian geometry was constructed in the Lagrangian setting, and used to define a metric in the Eulerian setting.

The behaviour of these generalised characteristics for the Hunter–Saxton equation differs greatly from those for Burgers' equation and other scalar conservation laws in two notable respects. Rather than characteristics colliding at wave-breaking times, they focus, as depicted in Figure 2.1. Furthermore, for Burgers' equation individual characteristics may experience multiple collisions, while for Hunter–Saxton they can focus at most once.

Uniqueness of dissipative solutions was then established by Dafermos [18] for initial data defined on the real line, via generalised characteristics, by demonstrating that the emanating characteristics are unique. See also [11, 17], in which it is demonstrated that the notion of dissipative solution used is indeed the weak solution with maximal energy loss at wave-breaking. Uniqueness in the conservative case presented a particular challenge in comparison to the dissipative case, as the ODEs for the characteristics may not have unique solutions. Uniqueness was

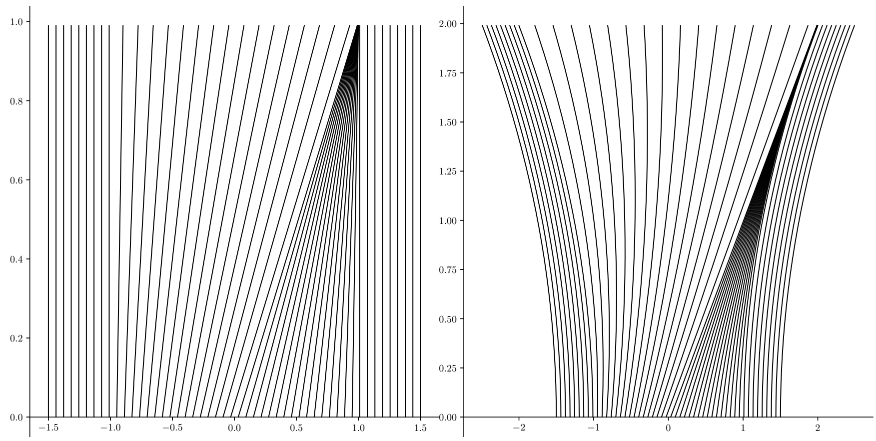


Figure 2.1: On the left: Characteristics for a solution to Burgers' equation. On the right: Characteristics for a solution to the Hunter–Saxton equation. Notice how the characteristics on the right collide, while those on the left focus. Also note that the vertical time axes are different.

established by Grunert and Holden [22], by demonstrating that any solution is equivalent to the one given by construction via the Lagrangian ODE system.

In [38], Nordli establishes via the Lagrangian system existence of conservative solutions for the two component Hunter–Saxton system. The system consists of the Hunter–Saxton equation coupled with an additional conservation law. He also constructs a metric via the Lagrangian variables establishing a Lipschitz continuous dependence on initial data.

Extending the notion of conservative and dissipative solutions, one introduces the concept of α-dissipative solutions, in which an α proportion of energy is lost after energy concentrates. While such an idea was remarked upon in earlier works, such solutions were concretely introduced for the two component Camassa-Holm system in [23]. In said work, existence of solutions was established using the Lagrangian system. The ideas were then used to construct α-dissipative solutions to the Hunter–Saxton equation in [25]. In said work, it was assumed that wave-breaking does not occur in the initial data, and what was constructed was a time-dependent Lipschitz metric. It is from this work that we continue.

The goal of the first two articles [27, 28] presented in this thesis is to establish a metric rendering α-dissipative solutions to the Hunter–Saxton equation Lipschitz continuous. In comparison to the previously discussed metric, we will allow wave breaking at time zero and said metric will not be dependent on the time variable. We will split into cases, establishing 3 different metrics. In the first, we consider if α is constant. In such a case one can take advantage of the property that the energy lost at wave breaking time is known from the initial data. In the remaining two cases, we wish to construct metrics in which we can compare solutions dependent on different α. In the second case, α is assumed to be a constant, and in the final a function. Each subsequent scenario will require a more technical approach. We leave the details and complications in the works to Section 2.5.

A wide variety of other results and analyses of the Hunter–Saxton equation have been conducted. Numerical methods are still an area of active development. Convergence of some finite difference upwind schemes on the positive half plane for dissipative solutions was established in [29]. The ideas were generalised to a discontinuous Galerkin method in [45]. More recently [26], a numerical scheme was developed and shown to converge to conservative solutions on the whole half plane. This made use of piecewise linear interpolation and the ability to explicitly construct the multi-peakon solutions.

In [44] a generalised Hunter-Saxton system is studied. Local existence of solutions with periodic boundary conditions, and global existence of weak solutions is established via a modified characteristics approach.

A stochastic version of the Hunter–Saxton equation was studied in [30], in which existence results for the stochastic analogue of dissipative and conservative solutions are shown by a method of characteristics approach.

2.2 Derivation and classical solutions

We begin with a brief overview of the derivation used by Hunter and Saxton [33] to obtain the Hunter–Saxton equation,

$$u_t + uu_x = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(y,t)\,\mathrm{d}y - \int_{x}^{\infty} u_x^2(y,t)\,\mathrm{d}y\right). \tag{HS}$$

In [43] Saxton derived a simplified model for nematic liquid crystals[1] in which the orientation field effects are emphasised in comparison to the velocity field. In the general case, this is referred to as the Non-Linear

[1]Nematic liquid crystals are liquid crystals whose director field $\mathbf{n}$ is invariant under the transformation $\mathbf{n} \mapsto -\mathbf{n}$.

Variational Wave Equation, and is given by

$$\phi_{tt} - c(\phi)(c(\phi)\phi_x)_x = 0, \tag{NLVW}$$

which is dependent on some given positive function $c : \mathbb{R} \to \mathbb{R}_+$.

Hunter and Saxton looked for weak solutions consisting of small perturbations to constant solutions by unidirectional waves over a large time scale. That is, solutions of the form

$$\phi_\epsilon(x,t) = \phi_0 + \epsilon\phi_1(\theta,\tau) + \mathcal{O}(\epsilon^2), \tag{2.2.1}$$

with ϕ_0 some real number, and

$$\tau = \epsilon t, \text{ and } \theta = x - c_0 t,$$

where $c_0 = c(\phi_0)$.

Substitution of (2.2.1) into (NLVW), equating coefficients of ϵ^2, and the transformation of variables $u = c_0'(\phi_0)$, $x = \theta$ and $t = \tau$ leads to the differentiated form of (HS),

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2. \tag{DHS}$$

Thus the space variable x in the Hunter–Saxton equation represents the position in a frame of reference moving with constant speed equal to the unperturbed wave speed, and t is a slow time variable.

There are, of course, other forms of the Hunter–Saxton equation. For instance, it could take the form

$$u_t + uu_x = \frac{1}{2}\int_{-\infty}^{x} u_x^2(y,t)\,\mathrm{d}y\,,$$

but in this work the symmetric form (HS) is preferred.

To begin finding solutions to (HS), a natural technique to use is the method of characteristics, applied early on in the seminal paper [33]. The following exploration takes inspiration from [7]. Assume formally that we have a smooth classical solution $u \in C^2(\mathbb{R}\times\mathbb{R}^+)$ to (HS) with $u_x(\cdot,t) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}^+$. Define the characteristics y of particles $\xi \in \mathbb{R}$ by

$$y_t(\xi,t) = u(y(\xi,t),t), \qquad y(\xi,0) = \xi, \tag{2.2.2}$$

and velocity along the path of a particle by

$$U(\xi,t) = u(y(\xi,t),t).$$

From (DHS), we have

$$(u_{tx} + uu_{xx})(y(\xi,t),t) = -\frac{1}{2}u_x^2(y(\xi,t),t),$$

or, after rearranging,

$$\frac{\partial}{\partial t}(u_x(y(\xi,t),t)) = -\frac{1}{2}u_x^2(y(\xi,t),t).$$

For each $\xi \in \mathbb{R}$ this defines an ODE for the function $z(t) := u_x(y(\xi,t),t)$ with initial data, from (2.2.2), $z(0) = u_{0,x}(\xi)$. The solution of this ODE is given by

$$z(t) = \frac{2u_{0,x}(\xi)}{2 + tu_{0,x}(\xi)}. \tag{2.2.3}$$

We can make an important conclusion from this. If the initial data satisfies $u_{0,x}(\xi) > 0$, then this can be solved for all time. Otherwise, there is a blow up in the derivative at time

$$\tau(\xi) := -\frac{2}{u_{0,x}(\xi)}.$$

Thus, classical solutions to (HS) are only guaranteed to exist on the time interval $[0, \mathcal{T})$, with $\mathcal{T} = \inf\{\tau(\xi) \mid \xi \in \mathbb{R}, u_{0,x}(\xi) < 0\}$.

Consider $t \in [0, \mathcal{T})$. One finds, differentiating (2.2.2) w.r.t. ξ,

$$\frac{\partial}{\partial t}y_\xi(\xi,t) = u_x(y(\xi,t),t)y_\xi(\xi,t) = \frac{2u_{0,x}(\xi)}{2 + tu_{0,x}(\xi)}y_\xi(\xi,t),$$

which, for each $\xi \in \mathbb{R}$, defines an ODE for $y_\xi(\xi,t)$, with initial data $y_\xi(\xi,0) = 1$. This ODE has as its solution

$$y_\xi(\xi,t) = \left(1 + \frac{1}{2}tu_{0,x}(\xi)\right)^2,$$

and hence, on the interval $[0, \mathcal{T})$, y_ξ is positive for all $\xi \in \mathbb{R}$. Thus, for each time $t \in [0, \mathcal{T})$, y is a differentiable homeomorphism on the real line.

Note also that classical solutions have a conservation of energy equation,

$$(u_x^2)_t = 2u_x u_{xt} = 2u_x\left(-uu_{xx} - \frac{1}{2}u_x^2\right) = -(uu_x^2)_x.$$

Introduce now the cumulative energy function V, given by

$$V(\xi,t) = \int_{-\infty}^{y(\xi,t)} u_x^2(x,t)\,\mathrm{d}x\,.$$

Differentiating V with respect to time, we see

$$\begin{aligned}
V_t(\xi,t) &= U(\xi,t)u_x^2(y(\xi,t),t) + \int_{-\infty}^{y(\xi,t)} (u_x^2)_t(x,t)\,\mathrm{d}x \\
&= U(\xi,t)u_x^2(y(\xi,t),t) + \int_{-\infty}^{y(\xi,t)} -(uu_x^2)_x(x,t)\,\mathrm{d}x \\
&= U(\xi,t)u_x^2(y(\xi,t),t) - U(\xi,t)u_x^2(y(\xi,t),t) \\
&= 0,
\end{aligned}$$

hence V is constant.

Further note that, via (HS),

$$\begin{aligned}
4U_t(\xi,t) &= 4(u_t + uu_x)(y(\xi,t),t) \\
&= V(\xi,t) - \int_{y(\xi,t)}^{\infty} u_x^2(x,t)\,\mathrm{d}x \\
&= 2V(\xi,t) - \int_{\mathbb{R}} u_x^2(x,t)\,\mathrm{d}x \\
&= 2V(\xi,t) - V_\infty(t),
\end{aligned}$$

where we have introduced the notation $V_\infty(t) = \lim_{\xi\to\infty} V(\xi,t)$.

And thus we see that one can solve the characteristic ODE's to obtain

$$y(\xi,t) = \xi + u_0(\xi)t + \frac{1}{8}\left(2V(\xi,t) - V_\infty(t)\right)t^2, \tag{2.2.4a}$$

$$U(\xi,t) = u_0(\xi) + \frac{1}{4}\left(2V(\xi,t) - V_\infty(t)\right)t, \tag{2.2.4b}$$

$$V(\xi,t) = \int_{-\infty}^{\xi} u_{0,x}^2(x)\,\mathrm{d}x\,. \tag{2.2.4c}$$

The solution of the Hunter-Saxton equation can then be obtained in this case on the interval $[0,\mathcal{T})$, as y is invertible, a consequence of it being a differentiable homeomorphism. More specifically, the solution is given by

$$u(x,t) = U(\xi,t), \qquad \text{for all } \xi \in \mathbb{R} \text{ s.t. } x = y(\xi,t),$$

for each $x \in \mathbb{R}$.

This method breaks down at time $\mathcal{T}$. At this point, the derivative u_x diverges to $-\infty$, and energy that was initially spread over a set of positive measure concentrates. Multiple solution concepts emerge, dependent on how one treats this energy.

2.3 General Solutions to the Hunter–Saxton equation

As we have seen, global existence of classical solutions cannot be guaranteed for a wide class of initial data. Thus the solution concept is expanded to that of weak, distributional solutions.

Definition 2.3.1. A continuous function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, absolutely continuous in space at each time, with $t \mapsto u_x(\cdot, t) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}))$, is called an admissible weak solution to the Cauchy problem of (HS), with initial data $u_0 \in C(\mathbb{R})$ satisfying $u_{0,x} \in L^2(\mathbb{R})$, if

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} u\varphi_t(x,t) &+ \frac{1}{2}u^2\varphi_x(x,t)\\ &+ \frac{1}{4}\left(\int_{-\infty}^x u_x^2(y,t)\,\mathrm{d}y - \int_x^\infty u_x^2(y,t)\,\mathrm{d}y\right)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x\,.\end{aligned}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, and $u(0,x) = u_0(x)$ pointwise in $\mathbb{R}$.

However, it was seen early on in [7, 33] that such solutions are not unique, as demonstrated in the following example.

Example 2.3.2. Consider as initial data

$$u_0(x) = \begin{cases} 1, & x \le 0,\\ 1-x, & 0 < x \le 1,\\ 0, & 1 < x.\end{cases}$$

Setting

$$v(x,t) := \begin{cases} 1 - \frac{1}{4}t, & x \le t - \frac{1}{8}t^2,\\ \frac{-4-t+4x}{2(t-2)}, & t - \frac{1}{8}t^2 < x \le 1 + \frac{1}{8}t^2,\\ \frac{1}{4}t, & 1 + \frac{1}{8}t^2 < x,\end{cases}$$

one finds that both

$$u_1(x,t) = \begin{cases} v(x,t), & t \neq 2,\\ \frac{1}{2}, & t = 2,\end{cases} \tag{2.3.1}$$

and

$$u_2(x,t) = \begin{cases} v(x,t), & t < 2,\\ \frac{1}{2}, & t \ge 2,\end{cases} \tag{2.3.2}$$

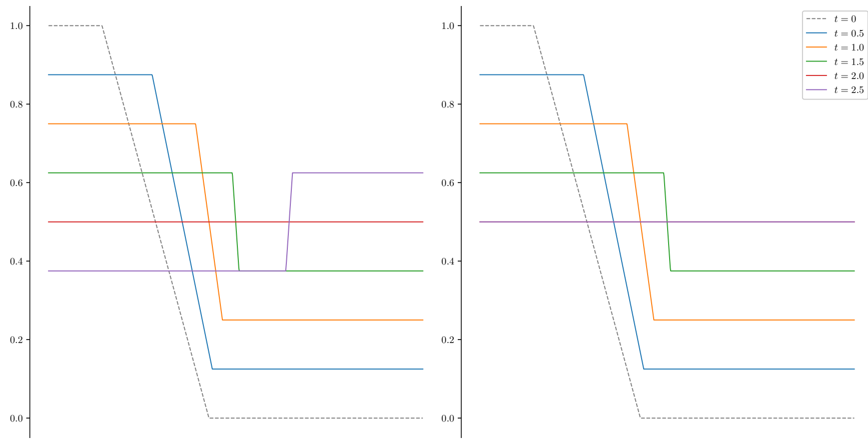


Figure 2.2: On the left: $u_1(x,t)$ at different times t. On the right: $u_2(x,t)$ at different times t.

are solutions in the sense of Definition 2.3.1. Notice that both u_1 and u_2 experience wave breaking at time $t = 2$. See Figure 2.2 for plots of u_1 and u_2 at different times.

Example 2.3.2 exemplifies two important classes of weak-solutions that are studied for the Hunter–Saxton equation.

Consider the energy $E_i(t) := \int_{\mathbb{R}} u_{i,x}^2(x,t)\,\mathrm{d}x$ for $i = 1, 2$. We see that for the first solution u_1, $E_1(t) = 1$ for almost all time. That is to say, after wave-breaking at time $t = 2$, the solution conserves its energy. On the other hand, for the second solution u_2, $E_2(t) = 0$ for $t \geq 2$, and the maximal amount of energy is dissipated at wave breaking. Thus, we refer to these solutions as conservative and dissipative solutions respectively.

Consider also the following example demonstrating a further type of solution.

Example 2.3.3. Consider the initial data $u_0 \equiv 0$. Then $u(\cdot, t) \equiv 0$ for all $t \in \mathbb{R}^+$ is a solution, as is

$$u_1(x,t) = \begin{cases} -\frac{1}{4}t, & x \leq -\frac{1}{8}t^2, \\ \frac{2x}{t}, & -\frac{1}{8}t^2 < x \leq \frac{1}{8}t^2, \\ \frac{1}{4}t, & \frac{1}{8}t^2 < x. \end{cases} \quad t > 0. \tag{2.3.3}$$

In fact, for any $\beta > 0$,

$$u_\beta(x,t) = \begin{cases} -\frac{1}{4}\beta t, & x \leq -\frac{1}{8}\beta t^2, \\ \frac{2x}{t}, & -\frac{1}{8}\beta t^2 < x \leq \frac{1}{8}\beta t^2, \\ \frac{1}{4}\beta t, & \frac{1}{8}\beta t^2 < x, \end{cases} \quad t > 0, \tag{2.3.4}$$

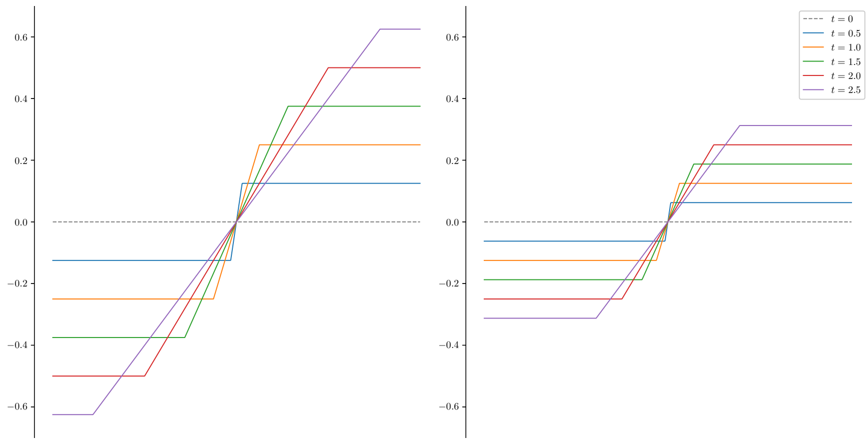


Figure 2.3: On the left: $u_1(x,t)$ from (2.3.3) at different times t. On the right: $u_{\frac{1}{2}}(x,t)$ from (2.3.4) at different times t.

is a solution. See Figure 2.3 for a plot of u_1 and $u_{\frac{1}{2}}$.

In this example we see multiple solutions exist corresponding to different amounts of energy concentrated on sets of measure zero initially. The constant 0 solution is a dissipative solution, while simultaneously they are all conservative solutions. Thus, to obtain further well-posedness results, Definition 2.3.1 must be refined into multiple concepts to distinguish the solutions introduced above.

The concept of a solution is augmented with an additional measure μ, corresponding to the current energy in the system at a given time. We begin by introducing the function spaces in which solutions will lie.

By $H^1(\mathbb{R})$ we refer to the usual Sobolev space of weakly differentiable functions in $L^2(\mathbb{R})$ with derivatives in $L^2(\mathbb{R})$. Define the Banach spaces and their associated norms

$$H_i := H^1(\mathbb{R}) \times \mathbb{R}^i, \qquad \|(f,a)\|_{H^i} = (\|f\|_{H^1}^2 + |a|^2)^{\frac{1}{2}}, \qquad i = 1, 2,$$

with the notation $|x|$ for the normal Euclidean norm for $x \in \mathbb{R}^n$. Define two functions χ^- and χ^+ in $C^\infty(\mathbb{R})$ satisfying

$$\chi^-(x) + \chi^+(x) = 1, \text{ and } 0 \le \chi^+ \le 1, \text{ for all } x \in \mathbb{R}, \tag{2.3.5a}$$

whose supports satisfy

$$\operatorname{supp}(\chi^-) \subset (-\infty, 1), \text{ and } \operatorname{supp}(\chi^+) \subset (-1, \infty). \tag{2.3.5b}$$

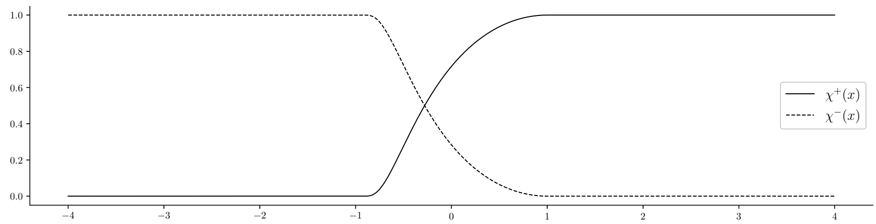


Figure 2.4: Plot of functions χ^+ and χ^- satisfying the requirements.

Consequently, these two functions have compactly supported derivatives with support in $[-1, 1]$. See Figure 2.4 for a plot of some possible functions. Further, they converge to 1 on opposite ends of the real line. Using these, we define the two linear mappings,

$$R_1 : H_1 \to E, \qquad (f, a) \mapsto f + a \cdot \chi^+, \tag{2.3.6a}$$

$$R_2 : H_2 \to E, \qquad (f, a, b) \mapsto f + a \cdot \chi^+ + b \cdot \chi^-, \tag{2.3.6b}$$

with the Banach space E, and its associated norm, given by

$$E \coloneqq \{f \in L^\infty(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}, \qquad \|f\|_E = \|f\|_\infty + \|f'\|_2. \tag{2.3.7}$$

R_1 and R_2 are continuous mappings. Indeed as

$$\|R_1((f, a))\|_E = \|f + a \cdot \chi^+\|_\infty + \|f' + a \cdot \chi'^+\|_2,$$

then, via the Sobolev embedding theorem A.1.1, $\|f\|_\infty \leq \|f\|_{H^1(\mathbb{R})}$, and we get

$$\begin{aligned}\|R_1((f, a))\|_E &\leq (2 + \max\{\|\chi^+\|_\infty, \|\partial_x \chi^+\|_2\})(\|f\|_{H^1(\mathbb{R})} + |a|) \\ &\leq C\|(f, a)\|_{H^i},\end{aligned}$$

for a constant $C > 0$, where in the final step we have used the Cauchy-Schwarz inequality.

Further, these mappings are injective. Indeed, consider $(f_i, a_i, b_i) \in H_2$, with $i = 1, 2$. Set $g_i = R_2((f_i, a_i, b_i)) = f_i + a_i \cdot \chi^+ + b_i \cdot \chi^-$, for $i = 1, 2$. Then, if $g_1 = g_2$,

$$f_1(x) + a_1 \cdot \chi^+(x) + b_1 \cdot \chi^-(x) = f_2(x) + a_2 \cdot \chi^+(x) + b_2 \cdot \chi^-(x),$$

and, as $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$, see [21], taking the limits at $+\infty$ and $-\infty$ confirms $(a_1, b_1) = (a_2, b_2)$. It is then immediate that $f_1 = f_2$, as required.

Using these functions we define the Banach subspaces of E, and their associated norms,

$$E_i = R_i(H_i), \qquad \|f\|_{E_i} = \|R_i^{-1}(f)\|_{H_i}, \qquad i = 1, 2.$$

Note that these spaces are independent of the choice of functions χ^-, χ^+ satisfying the assumptions above.

We also introduce the space $H_0 = L^2(\mathbb{R}) \times \mathbb{R}$, and define the set E_0, given by

$$E_0 \coloneqq \{f + a \cdot \chi^+ \mid (f, a) \in H_0\}, \qquad \|f + a \cdot \chi^+\|_{E_0} = (\|f\|_2^2 + |a|^2)^{\frac{1}{2}}.$$

This is well defined, i.e. each element in E_0 is uniquely identified by a pair $(f, a) \in H_0$. Indeed, suppose $f = f_1 + a_1 \cdot \chi^+ = f_2 + a_2 \cdot \chi^+$, with $f_i \in L^2(\mathbb{R})$, and $a_i \in \mathbb{R}$, $i = 1, 2$. Then, for $x \geq 1$, $\chi^+(x) = 1$, and hence

$$f_1(x) - f_2(x) = a_2 - a_1.$$

Then, as $f_1 - f_2 \in L^2((1, \infty))$, $a_2 = a_1$. Thus $f_1 = f_2$, as required.

The spaces E_i are essentially elements of E with well defined left and right asymptotes. They are a necessary choice to obtain uniqueness of conservative solutions, see [22].

With these things in place, we introduce the solution space we consider.

Definition 2.3.4 (The solution space $\mathcal{D}_{HS}$)**.** The space $\mathcal{D}_{HS}$ contains all pairs $(u, \mu) \in E_2 \times \mathcal{M}^+(\mathbb{R})$, satisfying

$$\mu((-\infty, \cdot)) \in E_0, \text{ and } d\mu_{ac} = u_x^2 \, \mathrm{d}x\,.$$

Everything is in place to introduce the solution concept we make use of.

Definition 2.3.5. A mapping $(u, \mu) : \mathbb{R}^+ \to \mathcal{D}_{HS}$ is a solution of (HS) with initial data $(u_0, \mu_0) \in \mathcal{D}_{HS}$ if

- $(u(t), \mu(t)) \in \mathcal{D}_{HS}$ for each time $t \in \mathbb{R}^+$;
- $u \in C^{0,\frac{1}{2}}(\mathbb{R} \times [0, T]; \mathbb{R})$, for all $T \in \mathbb{R}^+$;
- (u, μ) satisfy

$$\begin{aligned}\int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big[u\varphi_t(x,t) &+ \frac{1}{2}u^2\varphi_x(x,t) \\ &+ \frac{1}{4}\left(\int_{-\infty}^{x} \mathrm{d}\mu(t) - \int_{x}^{\infty} \mathrm{d}\mu(t)\right)\varphi(x,t)\Big] \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x\,,\end{aligned}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, and

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\phi_t + u\phi_x)(x,t)\,\mathrm{d}\mu(t)\,\mathrm{d}t \geq -\int_{\mathbb{R}} \phi(x,0)\,\mathrm{d}\mu(0)\,, \qquad (2.3.8)$$

for any non-negative test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^+)$;

- $(u(0), \mu(0)) = (u_0, \mu_0)$.

A solution is conservative if $\mu \in C_{weak*}(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}))$, $\mu(t) = u_x^2\,\mathrm{d}x$ for almost every time $t \in \mathbb{R}^+$, and equation (2.3.8) is satisfied as an equality.

A solution is dissipative if

$$\mu(t) = \mu_{ac}(t)^-, \text{ and } \mu(s) \overset{*}{\rightharpoonup} \mu(t), \text{ as } s \downarrow t$$

for all times $t \in \mathbb{R}^+$, with μ^- given by

$$\mu(s) \overset{*}{\rightharpoonup} \mu(t)^-, \qquad \text{as } s \uparrow t.$$

α-dissipative solutions

We concern ourselves with a broader set of classes of solutions, called α-dissipative solutions. These are solutions for which a proportion of the concentrated energy at wave breaking times is lost, given by some α. Here α could be a constant in $[0,1]$, or it could be a function lying in the set

$$\Lambda := W^{1,\infty}(\mathbb{R}; [0,1)) \cup \{1\}.$$

Definition 2.3.6. Let $\alpha \in \Lambda$. A mapping $(u,\mu) : \mathbb{R}^+ \to \mathcal{D}_{HS}$ is an α-dissipative solution if it is a solution in the sense of Definition 2.3.5, and

$$\mu(t) = \mu_{ac}(t)^- + (1-\alpha)\mu_s(t)^-, \text{ and } \mu(s) \overset{*}{\rightharpoonup} \mu(t), \text{ as } s \downarrow t$$

for all times $t \in \mathbb{R}^+$, with μ^- given by

$$\mu(s) \overset{*}{\rightharpoonup} \mu^-(t), \qquad \text{as } s \uparrow t.$$

In the following space the pairs (u,μ) are supplemented with a dummy variable, the energy measure ν, that is not part of the actual solution to the Hunter–Saxton equation. While the μ corresponds to the current energy in the system, ν corresponds to the original energy at time zero carried forwards in time. The space will be of use in the method of construction of solutions below. In particular, it enables the constructed method to be a semi-group.

Definition 2.3.7. Let $\alpha \in \Lambda$. The set $\mathcal{D}^\alpha$ contains all $Y = (u, \mu, \nu) \in \mathcal{D}_{HS} \times \mathcal{M}^+(\mathbb{R})$ such that

1. $\mu \leq \nu$ and $\mu_{ac} \leq \nu_{ac}$;
2. $\nu\left((-\infty, \cdot)\right) \in E_0$;
3. $\mu\left((-\infty, \cdot)\right) \in E_0$;
4. If $\alpha \equiv 1$, then $\nu_{ac} = \mu = u_x^2 \, \mathrm{d}x$;
5. If $\alpha \in W^{1,\infty}(\mathbb{R}; [0, 1))$, then $\frac{d\mu}{d\nu}(x) > 0$, and $\frac{d\mu_{ac}}{d\nu_{ac}}(x) = 1$ if $u_x(x) < 0$ for any $x \in \mathbb{R}$.

The set $\mathcal{D}$ is defined as

$$\mathcal{D} := \{Y^\alpha = (Y, \alpha) \mid \alpha \in \Lambda, Y \in \mathcal{D}^\alpha\} = \bigcup_{\alpha \in \Lambda} (\mathcal{D}^\alpha \times \{\alpha\}).$$

When comparing different α-dissipative solutions to the Hunter–Saxton equation in our metric we need to include the α in our coordinates. Furthermore, for each pair $(u, \mu) \in \mathcal{D}_{HS}$ and $\alpha \in \Lambda$ there is an equivalence class of possible ν such that the triple $((u, \mu, \nu), \alpha) \in \mathcal{D}$. For this reason, we introduce the following:

Definition 2.3.8 (The set of equivalence classes in $\mathcal{D}$)**.** The sets $\mathcal{D}_0^\alpha$ and $\mathcal{D}_0$ are given by

$$\mathcal{D}_0^\alpha = \{Z = (u, \mu) \in \mathcal{D}_{HS} \mid \mu = u_x^2 \, \mathrm{d}x \ \text{ if } \alpha = 1\}.$$

and

$$\mathcal{D}_0 = \{Z^\alpha = ((u, \mu), \alpha) \in \mathcal{D}_{HS} \times \Lambda \mid \mu = u_x^2 \, \mathrm{d}x \ \text{ if } \alpha = 1\}.$$

For each $Z^\alpha = ((u, \mu), \alpha) \in \mathcal{D}_0$, we define the set of ν such that acceptable Eulerian coordinates are formed,

$$\mathcal{V}(Z^\alpha) = \{\nu \in \mathcal{M}^+(\mathbb{R}) \mid ((u, \mu, \nu), \alpha) \in \mathcal{D}\}.$$

Finally, we also define the set $\mathcal{D}_{0,M}^L$ of elements whose energy is bounded by $M > 0$ and the change in α is bounded by $L > 0$, i.e.

$$\mathcal{D}_{0,M}^L = \{Z^\alpha \in \mathcal{D}_0 \mid \mu(\mathbb{R}) \leq M, \text{ and } \|\alpha'\|_\infty \leq L\}.$$

Solutions via a Generalised Method of Characteristics

As seen in Section 2.2, in the setting of smooth solutions the solution of (HS) can be constructed via the method of characteristics by solving a linear system of ODEs in Lagrangian coordinates. The existence of a solution is only guaranteed up to the first time at which wave breaking occurs. Further, this construction assumed no wave breaking at time zero.

The goal of this section is to outline how these ideas have been extended to obtain solutions for all time.

A generalised method of characteristics was applied in [31] to the Camassa-Holm equation, see also [23], and is an extension upon the ideas above. It can be employed to construct global solutions to (HS) which can experience wave breaking initially. See [25] for its use for α-dissipative solutions of the Hunter–Saxton equation.

Before introducing the ODE system that underlines this method, we introduce the quantity

$$\tau(\xi) = \begin{cases} 0, & y_{0,\xi}(\xi) = U_{0,\xi}(\xi) = 0, \\ -2\frac{y_{0,\xi}(\xi)}{U_{0,\xi}(\xi)}, & U_{0,\xi}(\xi) < 0, \\ +\infty, & \text{otherwise}, \end{cases} \qquad \xi \in \mathbb{R}, \tag{2.3.9}$$

which is the wave breaking time for the characteristic associated to particle ξ. The derivation of this quantity can be done via similar calculations used to obtain (2.2.3), with now undetermined initial data for the characteristic y.

Corresponding to the (u, μ, ν), the equivalent ODE system in Lagrangian coordinates for solutions to (HS) is given by

$$y_t(\xi, t) = U(\xi, t), \tag{2.3.10a}$$

$$U_t(\xi, t) = \frac{1}{2}V(\xi, t) - \frac{1}{4}V_\infty(t), \tag{2.3.10b}$$

$$H_t(\xi, t) = 0, \tag{2.3.10c}$$

with

$$V(\xi, t) = \int_{-\infty}^{\xi} V_{\xi,0}(\eta)(1-\alpha(y(\eta, \tau(\eta))) \mathbb{1}_{\{r\in\mathbb{R} | 0<\tau(r)<t\}}(\eta)) \, \mathrm{d}\eta \,. \tag{2.3.10d}$$

We consider the following sets for our setting in Lagrangian coordinates.

Definition 2.3.9 (The space of Lagrangian coordinates, $\mathcal{F}$)**.** Let $\alpha \in \Lambda$. The space $\mathcal{F}^\alpha$ consists of all $X = (y, U, H, V)$ such that $(y-\mathrm{id}, U, H, V) \in E_2 \times E_2 \times E_1 \times E_1$ and

1. $y - \mathrm{id}, U, H$ and V are in $W^{1,\infty}(\mathbb{R})$;
2. $y_\xi, H_\xi \geq 0$, and $0 < c < y_\xi + H_\xi$ a.e. for some constant $c > 0$;
3. $y_\xi V_\xi = U_\xi^2$;
4. If $\alpha = 1$, then $y_\xi(\xi) = 0$ implies $V_\xi(\xi) > 0$, and $y_\xi(\xi) > 0$ implies $V_\xi(\xi) = H_\xi(\xi)$ a.e.;
5. If $\alpha \in W^{1,\infty}(\mathbb{R}; [0,1))$, then there exists a function $\kappa : \mathbb{R} \to (0,1]$ such that $V_\xi(\xi) = \kappa(y(\xi))H_\xi(\xi)$ a.e., with $\kappa(y(\xi)) = 1$ for $\xi \in \mathbb{R}$ s.t. $U_\xi(\xi) < 0$.

The set $\mathcal{F}$ is given by

$$\mathcal{F} \coloneqq \{X^\alpha = (X, \alpha) \mid \alpha \in \Lambda, X \in \mathcal{F}^\alpha\} = \bigcup_{\alpha \in \Lambda} (\mathcal{F}^\alpha \times \{\alpha\}).$$

Further, for constants $M, L > 0$, representing respectively an energy bound and a bound for the change in dissipation over space, define,

$$\mathcal{F}_M^L \coloneqq \left\{X^\alpha \in \mathcal{F} \mid \|V\|_\infty \leq M, \|\alpha'\|_\infty \leq L\right\}. \tag{2.3.11}$$

Finally, we introduce the sets $\mathcal{F}_0^\alpha$ and $\mathcal{F}_0$, whose importance will be described later, given by

$$\mathcal{F}_0^\alpha = \{X \in \mathcal{F}^\alpha \mid y + H = \mathrm{id}\},$$

and

$$\mathcal{F}_0 = \{X^\alpha \in \mathcal{F} \mid y + H = \mathrm{id}\}.$$

Note. As usual with a Lagrangian coordinate representation, the values $y(\xi, t)$ and $U(\xi, t)$ denote the positions and velocity of each particle $\xi \in \mathbb{R}$. The variables H and V correspond to the μ and ν in Eulerian coordinates respectively. The H is the cumulative energy conserved forwards in time, and the V is the true current energy, with dissipation at wave breaking times.

The following result was established in [25, Lemma 2.3] via a fixed point iteration method.

Lemma 2.3.10. *Let $\alpha \in \Lambda$ and $X_0 \in \mathcal{F}^\alpha$. There exists a unique solution $X \in C(\mathbb{R}^+; \mathcal{F}^\alpha)$, satisfying $X(0) = X_0$, to the Cauchy problem for* (2.3.10).

Definition 2.3.11. Denote by $S : \mathbb{R}_+ \times \mathcal{F} \to \mathcal{F}$ the solution operator that maps a given time t and initial data $X_0^\alpha \in \mathcal{F}$ to the α-dissipative solution $S_t X_0^\alpha = X^\alpha(t)$ of (2.3.10) at time t.

As seen in Example 2.3.3, when transforming for Eulerian to Lagrangian variables one cannot assume that $y_0(\xi) = \xi$, as wave breaking may occur at time zero. Mappings developed for the Camassa-Holm equation, see [23, 31], are used to transform between the two instead. These were first employed for the Hunter-Saxton equation in [38] for conservative solutions.

Definition 2.3.12 (Transforming from Eulerian to Lagrangian coordinates)**.** The mapping $\hat{L} : \mathcal{D} \to \mathcal{F}_0$, used to transform from Eulerian to Lagrangian coordinates, is given by $\hat{L}(Y^\alpha) = X^\alpha$, with $X = (y, U, H, V)$ given by

$$y(\xi) = \sup\{x \in \mathbb{R} \mid x + \nu\left((-\infty, x)\right) < \xi\}, \tag{2.3.12a}$$

$$U(\xi) = u(y(\xi)), \tag{2.3.12b}$$

$$H(\xi) = \xi - y(\xi), \tag{2.3.12c}$$

and

$$V(\xi) = \int_{-\infty}^{\xi} H_\xi(\eta) \frac{d\mu}{d\nu} \circ y(\eta)\, \mathrm{d}\eta\,. \tag{2.3.12d}$$

Definition 2.3.13 (Transforming from Lagrangian to Eulerian coordinates)**.** The mapping $\hat{M} : \mathcal{F} \to \mathcal{D}$, used to transform from Lagrangian to Eulerian coordinates, is defined as $\hat{M}(X^\alpha) = Y^\alpha$, with $Y = (u, \mu, \nu)$ given by

$$u(x) = U(\xi), \qquad \text{for all } \xi \in \mathbb{R} \text{ such that } x = y(\xi), \tag{2.3.13a}$$

$$\mu = y_\#(V_\xi \ \mathrm{d}\xi), \tag{2.3.13b}$$

$$\nu = y_\#(H_\xi \ \mathrm{d}\xi). \tag{2.3.13c}$$

Here we have used the push forward measure for a Radon measure $\mu \in \mathcal{M}(\mathbb{R})$, μ-measurable function $f : \mathbb{R} \to \mathbb{R}$, and Borel measurable set A, given by

$$f_\#(\mu)(A) = \mu(f^{-1}(A)).$$

That $\hat{L}$ and $\hat{M}$ are well defined mappings follows from [39], whose proof is for the two component Hunter-Saxton system, and is inspired by those for the Camassa-Holm equation, see [23, 31].

An important consequence of switching from Eulerian to Lagrangian coordinates is the introduction of a redundancy; there are four Lagrangian coordinates to three Eulerian coordinates. The mapping $\hat{L}$

is non-surjective with respect to the co-domain $\mathcal{F}$, and multiple Lagrangian coordinates represent the same Eulerian coordinates. These coordinates are related via an equivalence relation, defined using what are known as relabelling functions.

Definition 2.3.14 (Relabelling functions)**.** Let $\mathcal{G}$ be the set containing all homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f - \mathrm{id} \in W^{1,\infty}(\mathbb{R}), \qquad f^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R}), \qquad f_\xi - 1 \in L^2(\mathbb{R}).$$

Define the action $\circ : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$, called "relabelling $X^\alpha \in \mathcal{F}$ by a function $f \in \mathcal{G}$", by

$$(X^\alpha, f) \mapsto X^\alpha \circ f = ((y \circ f, U \circ f, V \circ f, H \circ f), \alpha).$$

Proposition 2.3.15. *Suppose $f \in \mathcal{G}$ and*

$$\|f - \mathrm{id}\|_{W^{1,\infty}(\mathbb{R})} + \|f^{-1} - \mathrm{id}\|_{W^{1,\infty}(\mathbb{R})} \leq c \tag{2.3.14}$$

for a constant $c > 0$. Then f is strictly increasing, and in fact $\frac{1}{1+c} \leq f_\xi \leq 1 + c$ almost everywhere.

Proof. See [31, Lemma 3.2]. By Rademacher's theorem, the set of points for which the derivative of a Lipschitz function g does not exist, denoted B_g^c, has measure zero. Thus B_f and $B_{f^{-1}}$ both have full measure.

Consider $\xi \in f^{-1}(B_{f^{-1}}) \cap B_f$. Then f^{-1} is differentiable at $f(\xi)$ and f is differentiable at ξ. We have

$$|f^{-1}(f(\xi')) - f^{-1}(f(\xi))| \leq (c+1)|f(\xi') - f(\xi)|$$

for any $\xi, \xi' \in \mathbb{R}$, by assumption (2.3.14). So,

$$f_\xi(\xi) = \lim_{\xi' \to \xi} \frac{f(\xi') - f(\xi)}{\xi' - \xi} = \lim_{\xi' \to \xi} \frac{f(\xi') - f(\xi)}{f^{-1}(f(\xi')) - f^{-1}(f(\xi))} \geq \frac{1}{(c+1)}.$$

Furthermore, Lipschitz continuous bijections map measure zero sets to measure zero sets, and hence $f^{-1}(B_{f^{-1}}) \cap B_f$ is of full measure, thus this inequality is true almost everywhere, as required.

For the remaining inequality, we use $f_\xi(\xi) \leq \|f_\xi - 1\|_\infty + 1 \leq c+1$. □

Lemma 2.3.16. *The set $\mathcal{G}$ is a group, the mapping $\circ$ is a well-defined group action, and the relation*

$$X_A^{\alpha_A} \sim X_B^{\alpha_B} \text{ if there exists } f \in \mathcal{G} \text{ such that } X_A^{\alpha_A} = X_B^{\alpha_B} \circ f$$

defines an equivalence relation on the set $\mathcal{F}$.

Proof. See [31, Proposition 3.4]. $\mathcal{G}$ is a subgroup of the group of all homeomorphisms. Indeed, consider $f, g \in \mathcal{G}$, then

$$|f \circ g - \mathrm{id}| \leq |f \circ g - g| + |g - \mathrm{id}| \leq \|f - \mathrm{id}\|_\infty + \|g - \mathrm{id}\|_\infty < \infty,$$

and hence, $f \circ g - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$, inheriting the Lipschitz continuity from the composition. Via similar logic $(f \circ g)^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$, as it is bounded and Lipschitz continuous via the composition of Lipschitz functions. Furthermore, by Rademacher's theorem the sets B_f and B_g where the classical derivatives of f and g respectively exist are of full measure. Hence, as g is Lipschitz continuous and strictly increasing, the set

$$\{\xi \in \mathbb{R} \mid g \text{ is differentiable at } \xi, f \text{ is differentiable at } g(\xi)\}$$

is of full measure. In other words, $(f \circ g)_\xi(\xi) = f_\xi \circ g(\xi) g_\xi(\xi)$ almost everywhere. Further,

$$\begin{aligned}\int_{\mathbb{R}} ((f \circ g)_\xi - 1)^2 \,\mathrm{d}\xi &\leq 2 \int_{\mathbb{R}} ((f_\xi \circ g - 1)^2 g_\xi^2 + (g_\xi - 1)^2) \,\mathrm{d}\xi \\ &\leq 2\|f_\xi - 1\|_2^2 \|g_\xi\|_\infty + 2\|g_\xi - 1\|_2^2 < \infty\end{aligned} \tag{2.3.15}$$

and hence $(f \circ g)_\xi - 1 \in L^2(\mathbb{R})$, and via similar calculations $(f \circ g)_\xi^{-1} - 1 \in L^2(\mathbb{R})$, as required.

We next show that $\circ$ is well-defined. Let $f \in \mathcal{G}$. One has, via the triangle inequality, and that f is a homeomorphism,

$$|y \circ f - \mathrm{id}| \leq \|y - \mathrm{id}\|_\infty + \|f - \mathrm{id}\|_\infty < \infty,$$

and

$$\begin{aligned}&|y \circ f(\xi_2) - \xi_2 - (y \circ f(\xi_1) - \xi_1)| \\ &\leq |(y - \mathrm{id}) \circ f(\xi_2) - (y - \mathrm{id}) \circ f(\xi_1)| + |f(\xi_2) - \xi_2 - (f(\xi_1) - \xi_1)| \\ &\leq \|y_\xi - 1\|_\infty |f(\xi_2) - f(\xi_1)| + \|f_\xi - 1\|_\infty |\xi_2 - \xi_1| \\ &\leq (\|y_\xi - 1\|_\infty (\|f_\xi - 1\|_\infty + 1) + \|f_\xi - 1\|_\infty) |\xi_2 - \xi_1|\end{aligned}$$

and hence $y \circ f - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$. Similarly $U \circ f, H \circ f, V \circ f \in W^{1,\infty}(\mathbb{R})$. We have

$$(y \circ f)_\xi (V \circ f)_\xi = (y_\xi \circ f)(V_\xi \circ f) f_\xi^2 = (U_\xi \circ f)^2 f_\xi^2 = (U \circ f)_\xi^2,$$

almost everywhere, via once again Rademacher's theorem and the fact that f is a Lipschitz bijection. It can be shown, via a similar argument to (2.3.15) that $(y \circ f)_\xi - 1, (U \circ f)_\xi, (H \circ f)_\xi, (V \circ f)_\xi \in L^2(\mathbb{R})$. Further,

$$\int_{\mathbb{R}} (U \circ f - U_\infty \chi^+ - U_{-\infty} \chi^-)^2 \,\mathrm{d}\xi$$

$$
\begin{aligned}
&= \int_{-\infty}^{-1} (U\circ f - U_{-\infty})^2 \,\mathrm{d}\xi + \int_{1}^{\infty} (U\circ f - U_{\infty})^2 \,\mathrm{d}\xi \\
&\quad + \int_{-1}^{1} (U\circ f - U_{\infty}\chi^{+} - U_{-\infty}\chi^{-})^2 \,\mathrm{d}\xi \\
&= \int_{-\infty}^{f(-1)} (U - U_{-\infty})^2 \frac{1}{f_\xi \circ f^{-1}} \,\mathrm{d}\xi + \int_{f(1)}^{\infty} (U - U_{\infty})^2 \frac{1}{f_\xi \circ f^{-1}} \,\mathrm{d}\xi \\
&\quad + \int_{-1}^{1} (U\circ f - U_{\infty}\chi^{+} - U_{-\infty}\chi^{-})^2 \,\mathrm{d}\xi\,.
\end{aligned}
$$

The first two integrals here are finite as $\frac{1}{\|f_\xi\|_\infty} < \infty$ from Proposition 2.3.15, combined with the fact $U \in E_2$. The final integral is finite as the integrand is bounded. Hence $U \circ f \in E_2$. A similar argument demonstrates that $y \circ f - \mathrm{id} \in E_2$, and $H \circ f, V \circ f \in E_1$.

Properties 2, 4 and 5 of Definition 2.3.9 are an immediate consequence of the fact that f is a Lipschitz continuous bijection, and via the chain rule.

For the equivalence relation, reflexivity and symmetry follow from taking the inverse of the relabelling function for each case. Suppose $X_A^{\alpha_A} = X_B^{\alpha_B} \circ f$, $X_B^{\alpha_B} = X_C^{\alpha_C} \circ g$, for $f, g \in \mathcal{G}$. Then $g \circ f \in \mathcal{G}$ from the group properties, and

$$X_A^{\alpha_A} = X_B^{\alpha_B} \circ f = X_C^{\alpha_C} \circ g \circ f$$

hence $X_A^{\alpha_A} \sim X_C^{\alpha_C}$. Thus transitivity is satisfied. □

Thus we have equivalence classes on $\mathcal{F}$ defined by the equivalence relation $\sim$, and for each class we identify a representative by the element in $\mathcal{F}_0$, which, after a careful observation, is exactly the element of $\mathcal{F}$ Eulerian coordinates are mapped to via $\hat{L}$.

Given $X^\alpha \in \mathcal{F}$, notice that $y + H - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$, and $(y+H)_\xi - 1 \in L^2(\mathbb{R})$, directly from Definition 2.3.9. Furthermore, as there exists $c > 0$ such that $0 < c < (y+H)_\xi$, $y + H$ is a homeomorphism. Finally,

$$(y+H)_\xi^{-1} = \frac{1}{(y+H)_\xi \circ (y+H)^{-1}} \leq \frac{1}{c},$$

and hence $(y+H)^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$. These calculations thus demonstrate that $y + H \in \mathcal{G}$.

We thus define the mapping $\Pi : \mathcal{F} \to \mathcal{F}_0$ via

$$\Pi X^\alpha = X^\alpha \circ (y+H)^{-1},$$

which maps an element of an equivalence class to its representative.

The following two well established results are crucial. First, the validity of the relabelling relationship.

Lemma 2.3.17. *The output of the mapping $\hat{M}$ is independent of the relabelling. That is, for any $X^\alpha \in \mathcal{F}$ and $f \in \mathcal{G}$,*

$$\hat{M}(X^\alpha \circ f) = \hat{M}(X^\alpha).$$

Proof. Original proved for the Camassa-Holm equation in [31, Theorem 3.11], and for the Hunter–Saxton equation in [39, Proposition 2.1.10]. □

In particular, it can be shown that $\hat{L}$ and $\hat{M}$ are bijective mappings between $\mathcal{D}$ and $\mathcal{F}_0$, via an extension of the proof of [31, Theorem 3.12].

Second, we have equivariance of the solution operator.

Lemma 2.3.18. *The solution operator is equivariant under the relabelling operation. That is, for $X^\alpha \in \mathcal{F}$ and $f \in \mathcal{G}$,*

$$S_t(X^\alpha \circ f) = S_t(X^\alpha) \circ f.$$

Proof. Original proved for the Camassa-Holm equation in [31, Theorem 3.7], and for the Hunter–Saxton equation in [25, Proposition 3.7]. □

Which tells us that the equivalence classes of initial data each correspond uniquely to a respective Eulerian evolution.

Finally, we can define the mapping $T : \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$ by

$$T_t Y^\alpha = (\hat{M} \circ S_t \circ \hat{L}) Y^\alpha,$$

which inherits the semigroup property from the mapping S_t. To each initial data $Y^\alpha = (Y, \alpha) \in \mathcal{D}$, T_t associates an α-dissipative solution satisfying Definition 2.3.6, see [25, Theorem 3.14]. Henceforth, when speaking about an α-dissipative solution in Eulerian coordinates, we are referring to the solution given by this mapping.

It is for these solutions we wish to construct a metric which is Lipschitz continuous with respect to initial data, a challenge we move on to next.

2.4 The scheme - Metrics via generalised characteristics

We now begin by describing the ideas behind the construction of metrics via a generalised method of characteristics.

The aim is to begin by constructing some metric $d_{\mathcal{F}} : \mathcal{F}^2 \to \mathbb{R}^+$ rendering flows generated by the ODE system (2.3.10) locally Lipschitz continuous with respect to initial data. That is, it satisfies an inequality of the form

$$d_{\mathcal{F}}(S_t X_A^{\alpha_A}, S_t X_B^{\alpha_B}) \leq L(t) d_{\mathcal{F}}(X_A^{\alpha_A}, X_B^{\alpha_B}), \tag{2.4.1}$$

for some function $L(t) : \mathbb{R}^+ \to \mathbb{R}^+$, preferably of the form $L(t) = e^{At}$, with A some constant. Using the transformation mappings $\hat{L}$ and $\hat{M}$, given by Definitions 2.3.12 and 2.3.13 respectively, we use $d_{\mathcal{F}}$ to define a distance rendering solutions constructed via the method Lipschitz continuous in Eulerian coordinates.

However, things are not so simple. Consider a solution $Y^\alpha(t) \in \mathcal{D}$. After transforming, $X_0^\alpha = \hat{L}(Y^\alpha(0))$ and $X_1^\alpha = \hat{L}(Y^\alpha(1))$, both X_0^α and X_1^α are in $\mathcal{F}_0$. However, in general

$$S_1 X_0^\alpha \neq \Pi S_1 X_0^\alpha = X_1^\alpha.$$

Consider a second solution $\hat{Y}^{\hat{\alpha}}(t) \in \mathcal{D}$, with $\hat{X}_0^{\hat{\alpha}} = \hat{L}(\hat{Y}^{\hat{\alpha}}(0))$. One has, in general,

$$\begin{aligned}
d_{\mathcal{F}}(\hat{L}(Y^\alpha(1)), \hat{L}(\hat{Y}^{\hat{\alpha}}(1))) &= d_{\mathcal{F}}(\Pi S_1 X_0^\alpha, \Pi S_1 \hat{X}_0^{\hat{\alpha}}) \\
&\neq d_{\mathcal{F}}(S_1 X_0^\alpha, S_1 \hat{X}_0^{\hat{\alpha}}) \\
&\leq L(1) d_{\mathcal{F}}(X_0^\alpha, \hat{X}_0^{\hat{\alpha}}) \\
&= L(1) d_{\mathcal{F}}(\hat{L}(Y^\alpha(0)), \hat{L}(\hat{Y}^{\hat{\alpha}}(0))).
\end{aligned}$$

In other words, it is not sufficient that we have a metric that renders elements of $\mathcal{F}$ Lipschitz continuous with respect to initial data. Our metric needs to render elements in $\mathcal{F}$ related by relabelling, i.e. in the same equivalence class, equivalent. Or rather, we need a metric on the space of equivalence classes that is Lipschitz continuous.

In the context of the Hunter–Saxton equation, this problem was first tackled for conservative solutions in [8], with a construction inspired by ideas from Riemannian geometry. We begin by detailing the construction of this metric.

When considering conservative solutions of HS, the additional measure ν and associated Lagrangian coordinate H is unnecessary. In particular, the lack of energy loss enables the solution mapping to be a semigroup without these components. Hence, we introduce the following set.

Definition 2.4.1. Denote by $\mathcal{F}^{cons}$ (respectively $\mathcal{F}^{cons}_{\geq}$) the set of $(y - \mathrm{id}, U, V) \in E_2 \times E_2 \times E_1$ such that

- $y - \mathrm{id}, U$, and V are in $W^{1,\infty}(\mathbb{R})$;
- $y_\xi, V_\xi \geq 0$ *a.e.*;
- $0 < c < y_\xi + V_\xi$ a.e.;
- $y_\xi V_\xi = U_\xi^2$ (resp. $y_\xi V_\xi \geq U_\xi^2$),

for some constant $c > 0$.

By $\mathcal{F}^{cons}_0$ (resp. $\mathcal{F}^{cons}_{\geq,0}$) we refer to the subset of elements for whom $y + V = \mathrm{id}$.

Set

$$B^2 = E_2^2 \times E_2^2 \times E_1^2 \subset E_2 \times E_2 \times E_1 = B \tag{2.4.2}$$

to be the subset given by $E_i^2 = R_i(H^2(\mathbb{R}) \times \mathbb{R}^i)$, $i = 1, 2$, with associated product norm.

Rather than constructing a metric in the space $\mathcal{F}^{cons}$, a metric is constructed in the wider space $\mathcal{F}^{cons}_{\geq}$, taking advantage of the less stringent restrictions.

Given $X \in \mathcal{F}^{cons}_{\geq} \cap B^2$, a seminorm $|||\cdot|||_X$ on B is defined, see [8, Definition 3.5]. It is given by

$$|||\hat{X}|||_X = \|\hat{X} - g(X, \hat{X}) X_\xi\|_B,$$

where $X_\xi = (y_\xi - 1, U_\xi, V_\xi)$, and $g = g(X, \hat{X})$ is the unique element of E_2 satisfying the property

$$\|\hat{X} - g X_\xi\|_B \leq \|\hat{X} - h X_\xi\|_B, \qquad \text{for any } h \in E_2.$$

The set of solution curves C_0 is defined as the mappings $X : [0, 1] \to \mathcal{F}^{cons}_{\geq,0} \cap B^2$ satisfying

$$X \in C([0, 1]; B^2), \qquad X_s \in C_{pc}([0, 1]; B),$$

with $X_s = \partial_s X = (y_s, U_s, V_s)$, and $C_{pc}([0, 1]; B)$ denoting the set of piecewise continuous curves from $[0, 1]$ to B.

Via these curves a metric for elements $X_0, X_1 \in \mathcal{F}^{cons}_{\geq,0} \cap B^2$ is defined by

$$d(X_0, X_1) = \inf_{C_0(X_0, X_1)} \int_0^1 |||X_s(s)|||_{X(s)} \,\mathrm{d}s\,,$$

where $C_0(X_0, X_1)$ is the set of curves $X \in C_0$ satisfying $X(0) = X_0$ and $X(1) = X_1$. This metric then carries the property that elements of the same equivalence class have distance zero.

This construction relies on the non-emptiness of the set $C(X_0, X_1)$, which follows from the fact that $\mathcal{F}^{cons}_{0,\geq}$ is convex.

Finally, the metric is expanded to the entire space $\mathcal{F}^{cons}_{\geq,0}$ by considering the approximation of elements in B. In other words, given two elements $X_0, X_1 \in B$, the distance is given by

$$d(X_0, X_1) = \lim_{n\to\infty} d(X_{0,n}, X_{1,n}), \tag{2.4.3}$$

where $\{X_{0,n}\}_{n\in\mathbb{N}}$ and $\{X_{1,n}\}_{n\in\mathbb{N}}$ are two sequences in $\mathcal{F}^{cons}_{\geq,0} \cap B^2$ converging to X_0 and X_1 respectively in the B-norm.

Such a construction can be expanded to the general α-dissipative setting, via analogous set definitions. However, developing a satisfactory Lipschitz estimate of the form (2.4.1) relies heavily on the fact that in the outlined conservative setting, i.e. $\alpha = 0$, the system of ODEs (2.3.10) reduces to a Linear system of ordinary differential equations with $V_t = 0$.

The outlined construction nonetheless provides a metric d, given by (2.4.3), satisfying

$$d(X_0(t), X_1(t)) \leq e^{Ct} d(X_0(0), X_1(0)),$$

with $C > 0$ a constant, for conservative solutions $X_1(t), X_0(t)$.

The paper of Nordli [38] takes an alternative route, with an approach developed initially for the Camassa-Holm equation [24], which was itself inspired by the previous metric construction. In Nordli's paper, the two component Hunter–Saxton equation is considered, however this discussion will be constrained to our setting.

The B norm, see (2.4.2), separates elements in the same equivalence class. Using B the mapping J is defined by

$$J(X_A, X_B) = \inf_{f,g\in\mathcal{G}} \{\|X_A \circ f - X_B\|_B + \|X_A - X_B \circ g\|_B\}.$$

This does satisfy our desired property that if X_A and X_B in $\mathcal{F}^{cons}$ lie in the same equivalence class, $J(X_A, X_B) = 0$, however one cannot show this satisfies the triangle inequality.

It should also be noted that J does not satisfy invariance with respect to relabelling, i.e. it may hold that

$$J(X_A \circ f, X_B \circ g) \neq J(X_A, X_B),$$

for $X_A, X_B \in \mathcal{F}^{cons}$ and $f, g \in \mathcal{G}$.

Thankfully, a metric can be constructed using J. Define the metric $d : \mathcal{F}^{cons} \times \mathcal{F}^{cons} \to \mathbb{R}$ by

$$d(X_A, X_B) = \inf_{\hat{\mathcal{F}}^{cons}(X_A, X_B)} \sum_{n=1}^{N} J(X_n, X_{n-1}),$$

where the infimum is taken over the set $\hat{\mathcal{F}}^{cons}(X_A, X_B)$ of finite sequences $\{X_n\}_{n=0}^{N}$ of arbitrary length satisfying $X_0 = \Pi X_A$ and $X_N = \Pi X_B$. This metric is invariant under relabelling, and hence

$$d(\Pi X_A, \Pi X_B) = d(X_A, X_B).$$

It is then shown that this metric satisfies, for two conservative solutions $X_A(t), X_B(t)$ in $\mathcal{F}^{cons}$, with initial data $X_{A,0}, X_{B,0} \in \mathcal{F}_0^{cons}$,

$$d(X_A(t), X_B(t)) \leq e^{\frac{1}{2}t}(\frac{1}{2}t^2 + t + 1)d(X_{A,0}, X_{B,0}) \leq e^{\frac{3}{2}t} d(X_{A,0}, X_{B,0}),$$

exactly the relation (2.4.1) as required.

This approach is exactly the one we expand upon to construct a metric in the case of α-dissipative solutions. It was also used in its sequel [38] in the case of α-dissipative solutions. However, these solutions were assumed to not experience wave breaking at zero and the construction was a time-dependent metric. These were two properties we sought to overcome.

2.5 Paper 1 and paper 2 - The scheme for α-dissipative solutions and main results

We now shift our focus to outlining the ideas and results of the first two articles [27, 28], in which the goal was to extend the scheme outlined in Section 2.4 to the case of α-dissipative solutions.

In the first paper [28], a Lipschitz continuous metric for comparison between solutions for which α is a constant, and both solutions share the same α, was constructed.

In the second paper [27], we expand upon our ideas to consider the case of α-dissipative solutions with $\alpha \in \Lambda$. The metric constructed can be used to compare solutions with different values of α.

A natural first question to ask is "what added complications are there in the α-dissipative case compared to the conservative case?".

Immediately the additional measure ν required may seem the most significant issue. As seen in [27, Lemma 2.13], different choices of ν in the same equivalence class have no affect on the solution. So while this additional constraint means we require a metric that does not distinguish elements of said equivalence classes, the same techniques as in the Lagrangian case can be applied to overcome this issue.

The most significant challenge is the discontinuity present in V_ξ. Consider the case where two solutions X_A, X_B are compared via the E norm for the difference in the V. Then, the discontinuities can cause $|V_{A,\xi} - V_{B,\xi}|$ to grow on an interval, and hence the respective $L^2(\mathbb{R})$ norm grows in a discontinuous manner. This is demonstrated in the following example.

Example 2.5.1. Consider as initial data

$$u_{0,-}(x) = \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 < x \leq 1 \\ 0, & 1 < x, \end{cases} \qquad \text{and } u_{0,+}(x) = 1 - u_{0,-}(x),$$

with $\mu_0 = u_{0,+,x}^2 \, \mathrm{d}x = u_{0,-,x}^2 \, \mathrm{d}x$. We consider solutions when $\alpha = \frac{1}{2}$.

We end up with, constructing the solution in Lagrangian coordinates with initial data $X_\pm(0) = \hat{L}((u_{0,\pm}, \mu_0, \mu_0), \alpha)$,

$$V_-(\xi, t) = \begin{cases} \begin{cases} 0, & \xi \leq 0, \\ \frac{\xi}{2}, & 0 < \xi \leq 0, \quad t < 2, \\ 1, & 2 < \xi \end{cases} \\ \begin{cases} 0, & \xi \leq 0, \\ \frac{\xi}{4}, & 0 < \xi \leq 0, \quad 2 \leq t, \\ \frac{1}{2}, & 2 < \xi \end{cases} \end{cases} \qquad V_+(\xi, t) = V_-(\xi, 0).$$

The difference in the $L^2(\mathbb{R})$ norm grows after the wave breaking time $t = 2$, and is given by

$$\|V_{+,\xi}(t) - V_{-,\xi}(t)\|_2 = \frac{1}{2\sqrt{2}} \mathbb{1}_{[2,+\infty)}(t).$$

Explicitly, we have the issue,

$$\|V_{+,\xi}(2+) - V_{-,\xi}(2+)\|_2 \not\leq A \|V_{+,\xi}(2-) - V_{-,\xi}(2-)\|_2,$$

for any value of $A \geq 0$.

There is one particular advantage in the case where α is constant. One knows how large the discontinuity generated will be initially. In the general case $\alpha \in \Lambda$, the characteristic position at the time of wave breaking determines the size of the discontinuity, in particular it is determined by $\alpha(y(\xi,\tau(\xi)))$ as can be seen in the definition of V in (2.3.10d). Thus it is not known initially. This property is what leads to a much more approachable metric construction in the first paper compared to the second.

To resolve the issue of the discontinuities, we overestimate the norm terms involving the V_ξ, such that when comparing two solutions said difference is decreasing in time, with discontinuous drops. To do so, we split the real line into three disjoint sets corresponding to possible behaviours of characteristics for the two solutions.

Consider $X_i^{\alpha_i}, X_j^{\alpha_j} \in \mathcal{F}$, for some labels i, j. The set

$$\mathcal{A}_i = \mathcal{A}(X_i^{\alpha_i}) \coloneqq \{\xi \in \mathbb{R} \mid U_{i,\xi}(\xi) \geq 0\}, \tag{2.5.1a}$$

contains particles for which wave breaking will not happen in the future for the solution generated by $X_i^{\alpha_i}$, and using these we define

$$\mathcal{A}_{i,j} \coloneqq \mathcal{A}_i \cap \mathcal{A}_j. \tag{2.5.1b}$$

The set

$$\mathcal{B}_{i,j} = \mathcal{B}(X_i^{\alpha_i}, X_j^{\alpha_j}) \coloneqq \{\xi \in \mathbb{R} \mid 0 < \tau_i(\xi) = \tau_j(\xi) < \infty\}, \tag{2.5.1c}$$

contains the particles for which wave breaking happens for both solutions generated by X_i and X_j at the same time in the future. Finally, the remaining particles, for which wave breaking happens at different times in the future or for which wave breaking occurs only for one of the solutions, are in the compliment of the set

$$\Omega_{i,j} = \Omega(X_i^{\alpha_i}, X_j^{\alpha_j}) \coloneqq \mathcal{A}_{i,j} \cup \mathcal{B}_{i,j}. \tag{2.5.1d}$$

For an α-dissipative solution the sets change in time. Set $\mathcal{A}_i(t) = \mathcal{A}_i(X(t))$, and analogously define $\mathcal{B}_{i,j}(t)$ and $\Omega_{i,j}(t)$. Elements ξ in $\mathcal{B}_{i,j}(t)$ or $\Omega_{i,j}^c(t)$ transfer to $\mathcal{A}_{i,j}(t)$ after $t = \max(\tau_i(\xi), \tau_j(\xi))$. Hence $\mathcal{A}_{i,j}$ is growing, and the other two sets are shrinking, forwards in time.

Example 2.5.2. Consider the characteristic functions

$$y_A(\xi,t) = \begin{cases} t - \frac{1}{8}t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{8}t^2 + \frac{1}{8}(t-2)^2\xi, & 0 < \xi \leq 2, \\ -1 + \frac{1}{8}t^2 + \xi, & 2 < \xi, \end{cases} \tag{2.5.2}$$

$$y_B(\xi,t) = \begin{cases} t - \frac{1}{8}t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{8}t^2 + \frac{1}{4}(t-2)^2\xi, & 0 < \xi \leq \frac{5}{2}, \\ -\frac{3}{2}t + \frac{1}{2}t^2 + \xi, & \frac{5}{2} < \xi, \end{cases} \tag{2.5.3}$$

$$y_C(\xi,t) = \begin{cases} t - \frac{1}{4}t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{4}t^2 + \frac{1}{5}(t-1)^2\xi, & 0 < \xi \leq \frac{5}{2}, \\ -2 + \frac{1}{4}t^2 + \xi, & \frac{5}{2} < \xi. \end{cases} \tag{2.5.4}$$

The curves of $y_k(\xi_i, t)$, with $k = A, B, C$ and $i = 1, \ldots, 4$, are plotted in Figure 2.5, with $\{\xi_i\}_{i=0}^4$ some increasing sequence of particles in $\mathbb{R}$.

Comparing plots, the first particle ξ_0 will be in the set $\mathcal{A}_k(0)$ for $k = A, B, C$, as it doesn't experience wave breaking in any of the cases. If we compare the characteristics corresponding to ξ_1, ξ_2 and ξ_3, the breaking times of the curves in plots (2.5a) and (2.5b) are the same, $t = 2$, and hence $\xi_i \in \mathcal{B}_{A,B}(0)$ for $i = 1, 2, 3$. Comparing plots (2.5a) and (2.5c) we see they break at different times, hence $\xi_i \in \Omega^c_{A,C}(0)$ for $i = 1, 2, 3$. Finally, the characteristic corresponding to ξ_4 does not break in plot (2.5a), while it breaks at different times in the others, hence ξ_4 is in $\Omega^c_{A,B}(0)$, $\Omega^c_{A,C}(0)$, and $\Omega^c_{B,C}(0)$.

The act of relabelling on these sets is explored in [27]. Introducing the notation, for relabelling functions $f, h \in \mathcal{G}$,

$$\mathcal{A}^f_A = \mathcal{A}(X^{\alpha_A}_A \circ f), \qquad \mathcal{A}^{f,h}_{A,B} = \mathcal{A}(X^{\alpha_A}_A \circ f) \cap \mathcal{A}(X^{\alpha_B}_B \circ h),$$

and

$$\mathcal{B}^{f,h}_{A,B} = \mathcal{B}(X^{\alpha_A}_A \circ f, X^{\alpha_B}_B \circ h), \qquad \Omega^{f,h}_{A,B} = \Omega(X^{\alpha_A}_A \circ f, X^{\alpha_B}_B \circ h),$$

it is shown that

$$f(\mathcal{A}^f_A) = \mathcal{A}_A, \qquad f(\mathcal{A}^{f,h}_{A,B}) = \mathcal{A}^{\mathrm{id},h\circ f^{-1}}_{A,B}, \qquad f(\mathcal{B}^{f,h}_{A,B}) = \mathcal{B}^{\mathrm{id},h\circ f^{-1}}_{A,B}, \tag{2.5.5a}$$

and

$$f(\Omega^{f,h,c}_{A,B}) = \Omega^{\mathrm{id},h\circ f^{-1},c}_{A,B}. \tag{2.5.5b}$$

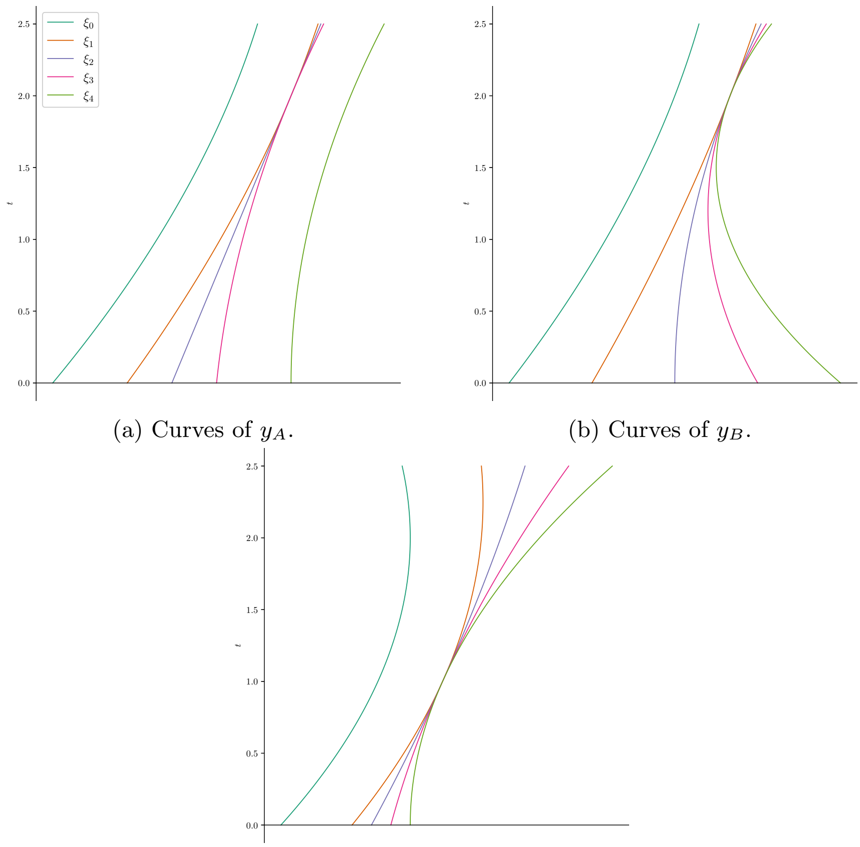
With our ingredients in place, and to outline the method, we explore the construction of a metric in a special case.

Application in a special case

To outline the ideas of said papers [27, 28], we now consider the case of comparing two solutions in which the α may be different, but are assumed to be constant. This was briefly discussed in the second paper [27], but without rigour.

We begin by defining the subset containing Lagrangian coordinates for which α is constant,

$$\mathcal{F}_c = \{X^\alpha \in \mathcal{F} \mid \alpha \in [0,1]\}. \tag{2.5.6}$$



(a) Curves of y_A.

(b) Curves of y_B.

(c) Curves of y_C.

Figure 2.5: Plots of $y_i(\xi_j, \cdot)$ for $i = A, B, C$ and $j = 0, \dots, 4$. The horizontal position of each curve is the value of $y_i(\xi_j, t)$ at time $t \in [0, 2.5]$. Note that the curves that focus experience wave breaking at the point of collision.

We introduce two functions, that make use of the sets introduced in the previous section, see (2.5.1).

Given $X^\alpha \in \mathcal{F}_c$, define

$$V_\xi^d(\xi, t) = \alpha V_\xi(\xi, t) \mathbb{1}_{\mathcal{A}^c(t)}(\xi), \qquad V_\xi^c(\xi, t) = (1 - \alpha \mathbb{1}_{\mathcal{A}^c(t)}(\xi)) V_\xi(\xi, t).$$

With this construction V_ξ^c is constant in time, thus we drop time dependence, and

$$V_\xi(t) = V_\xi^c + V_\xi^d(t). \tag{2.5.7}$$

After wave breaking time $0 < \tau(\xi) \leq t < +\infty$, we have

$$V_\xi^d(\xi, t) = 0, \qquad V_\xi(\xi, t) = V_\xi^c.$$

In other words, $V_\xi^d(\xi)$ is the part $V_\xi(\xi)$ loses after wave breaking, and $V_\xi^c(\xi)$ corresponds to the value of $V_\xi(\xi)$ after wave breaking.

We introduce $G : \mathcal{F}_c^2 \to \mathbb{R}^+$ that replaces, and overestimates, the difference in the V_ξ in our metric. First set, for $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$,

$$g_{A,B}(\xi) = g(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) = |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)|, \tag{2.5.8a}$$

$$\begin{aligned} \hat{g}_{A,B}(\xi) &= \hat{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) \\ &= |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + |V_{A,\xi}^d(\xi) - V_{B,\xi}^d(\xi)|, \end{aligned} \tag{2.5.8b}$$

$$\begin{aligned} \bar{g}_{A,B}(\xi) &= \bar{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) \\ &= |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + V_{A,\xi}^d(\xi) \vee V_{B,\xi}^d(\xi), \end{aligned} \tag{2.5.8c}$$

where $a \vee b = \max\{a, b\}$ for $a, b \in \mathbb{R}$. G is then given by

$$\begin{aligned} G_{A,B}(\xi) &= G\left(X_A^{\alpha_A}, X_B^{\alpha_B}\right)(\xi) \\ &= g_{A,B}(\xi)\mathbb{1}_{\mathcal{A}_{A,B}}(\xi) + \hat{g}_{A,B}(\xi)\mathbb{1}_{\mathcal{B}_{A,B}}(\xi) + \bar{g}_{A,B}(\xi)\mathbb{1}_{\Omega_{A,B}^c}(\xi). \end{aligned} \tag{2.5.9}$$

The fact that G is an overestimate, that is

$$|V_{A,\xi}(\xi) - V_{B,\xi}(\xi)| \leq G_{A,B}(\xi),$$

is immediate as from (2.5.7), as we have

$$\begin{aligned} |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)| &\leq |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + |V_{A,\xi}^d(\xi) - V_{B,\xi}^d(\xi)| \\ &\leq |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + V_{A,\xi}^d(\xi) \vee V_{B,\xi}^d(\xi). \end{aligned}$$

Proposition 2.5.3. *Let $X_A^{\alpha_A}, X_B^{\alpha_B} \in \mathcal{F}_c$ be two α-dissipative solutions. For any fixed $\xi \in \mathbb{R}$, $G_{A,B}(\xi, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, given by $G_{A,B}(t) = G(X_A(t), X_B(t))$, is a decreasing function in time.*

Proof. If $\xi \in \mathcal{A}_{A,B}(0)$, no wave breaking occurs. So we only need to consider the other two cases.

If $\xi \in \mathcal{B}_{A,B}(0)$, then

$$G_{A,B}(\xi, t) = |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + |V_{A,\xi}^d(\xi, t) - V_{B,\xi}^d(\xi, t)|$$

for all time. The first summand is constant, while the second is zero for $t \geq \tau_A(\xi) = \tau_B(\xi)$. Hence $G_{A,B}(\xi, t)$ is decreasing forward in time.

On the other hand, if $\xi \in \Omega_{A,B}^c(0)$, then

$$G_{A,B}(\xi, t) = |V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + V_{A,\xi}^d(\xi, t) \vee V_{B,\xi}^d(\xi, t)$$

for all time. Again, the first summand is constant, while the second is the maximum of two positive decreasing functions, and hence $G_{A,B}(\xi, t)$ is decreasing forward in time. □

Proposition 2.5.4. *G, given in* (2.5.9)*, satisfies the triangle inequality, i.e.*

$$G_{A,C}(\xi) \le G_{A,B}(\xi) + G_{B,C}(\xi), \qquad \text{for all } \xi \in \mathbb{R}, \tag{2.5.10}$$

for $X_A^{\alpha_A}, X_B^{\alpha_B}$ and $X_C^{\alpha_C}$ in $\mathcal{F}_c$.

Proof. Consider $X_A^{\alpha_A}, X_B^{\alpha_B}$ and $X_C^{\alpha_C}$ in $\mathcal{F}_c$. Once again, we split into different cases. For simplicity, we drop the ξ when writing V_ξ in this proof.

It is immediate that if $\xi \in \mathcal{A}_i$ for $i = A, B, C$ then the triangle inequality is satisfied.

Consider first the case $\xi \in \mathcal{A}_{A,C}$ and $\xi \notin \mathcal{A}_B$. Then $\xi \in \Omega^c_{A,B} \cap \Omega^c_{B,C}$, and

$$\begin{aligned}
g_{A,C}(\xi) = |V_{A,\xi} - V_{C,\xi}| &\le |V_{A,\xi} - V_{B,\xi}| + |V_{B,\xi} - V_{C,\xi}| \\
&\le |V_{A,\xi} - V^c_{B,\xi}| + |V^c_{B,\xi} - V_{C,\xi}| + 2V^d_{B,\xi} \\
&= |V^c_{A,\xi} - V^c_{B,\xi}| + V^d_{A,\xi} \vee V^d_{B,\xi} \\
&\quad + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{B,\xi} \vee V^d_{C,\xi} \\
&= \bar{g}_{A,B}(\xi) + \bar{g}_{B,C}(\xi),
\end{aligned}$$

as required.

Suppose now $\xi \in \mathcal{B}_{A,C}$. If $\xi \in \mathcal{B}_{B,C}$ then $\xi \in \mathcal{B}_{A,B}$, and once again the triangle inequality is immediate. On the other hand, if $\xi \notin \mathcal{B}_{B,C}$, then $\xi \in \Omega^c_{A,B} \cap \Omega^c_{B,C}$, and

$$\begin{aligned}
\hat{g}_{A,C}(\xi) &= |V^c_{A,\xi} - V^c_{C,\xi}| + |V^d_{A,\xi} - V^d_{C,\xi}| \\
&\le |V^c_{A,\xi} - V^c_{B,\xi}| + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{A,\xi} + V^d_{C,\xi} \\
&\le |V^c_{A,\xi} - V^c_{B,\xi}| + V^d_{A,\xi} \vee V^d_{B,\xi} \\
&\quad + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{B,\xi} \vee V^d_{C,\xi} \\
&= \bar{g}_{A,B}(\xi) + \bar{g}_{B,C}(\xi).
\end{aligned}$$

The final case to consider is $\xi \in \Omega^c_{A,C}$. If $\xi \in \Omega^c_{A,B} \cap \Omega^c_{B,C}$, then the inequality is once again immediate. In the sub case $\xi \in \mathcal{A}_A \cap \mathcal{A}^c_C$, there are two possibilities left to consider. First, if $\xi \in \mathcal{A}_B$, then $V^d_{A,\xi} = V^d_{B,\xi} = 0$, and $V^c_{A,\xi} = V_{A,\xi}$, and so

$$\begin{aligned}
\bar{g}_{A,C}(\xi) &= |V^c_{A,\xi} - V^c_{C,\xi}| + V^d_{C,\xi} \\
&\le |V_{A,\xi} - V_{B,\xi}| + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{B,\xi} \vee V^d_{C,\xi} \\
&= g_{A,B}(\xi) + \bar{g}_{A,C}(\xi).
\end{aligned}$$

Secondly, if $\xi \in \mathcal{B}_{B,C}$, then

$$\begin{aligned}\bar{g}_{A,C}(\xi) &= |V^c_{A,\xi} - V^c_{C,\xi}| + V^d_{C,\xi} \\ &\leq |V^c_{A,\xi} - V^c_{B,\xi}| + V^d_{B,\xi} + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{C,\xi} - V^d_{B,\xi} \\ &\leq \bar{g}_{A,B}(\xi) + \hat{g}_{A,C}(\xi).\end{aligned}$$

The other subcase to consider is if $\xi \in \mathcal{A}^c_A \cap \mathcal{A}^c_C \setminus \mathcal{B}_{A,C}$ and $\xi \in \mathcal{B}_{A,B}$. Then we need the inequality, for any real numbers $a, b, c \geq 0$,

$$a \vee c \leq |a - b| + b \vee c.$$

So

$$\begin{aligned}\bar{g}_{A,C}(\xi) &= |V^c_{A,\xi} - V^c_{C,\xi}| + V^d_{A,\xi} \vee V^d_{C,\xi} \\ &\leq |V^c_{A,\xi} - V^c_{B,\xi}| + |V^d_{A,\xi} - V^d_{B,\xi}| + |V^c_{B,\xi} - V^c_{C,\xi}| + V^d_{B,\xi} \vee V^d_{C,\xi} \\ &= \hat{g}_{A,B}(\xi) + \bar{g}_{B,C}(\xi).\end{aligned}$$

Combining the inequalities of each of the cases considered, we obtain the triangle inequality (2.5.10). □

With everything in place, we can define our metric $D : \mathcal{F}^2_c \to \mathbb{R}^+$ for $X^{\alpha_A}_A, X^{\alpha_B}_B \in \mathcal{F}_c$ by

$$\begin{aligned}D(X^{\alpha_A}_A, X^{\alpha_B}_B) \coloneqq{}& \|y_A - y_B\|_\infty + \|U_A - U_B\|_\infty + \|H_A - H_B\|_\infty \\ &+ \|y_{A,\xi} - y_{B,\xi}\|_2 + \|U_{A,\xi} - U_{B,\xi}\|_2 \\ &+ \frac{1}{4}\|G_{A,B}\|_1 + \frac{1}{2}\|G_{A,B}\|_2 + |\alpha_A - \alpha_B|. \qquad (2.5.11)\end{aligned}$$

Lemma 2.5.5. *Let $X^{\alpha_A}_A$ and $X^{\alpha_B}_B$ be two α-dissipative solutions to the Lagrangian ODE system* (2.3.10) *with initial data $X^{\alpha_A}_{A,0}, X^{\alpha_B}_{B,0} \in \mathcal{F}_c$ respectively. Then*

$$D(X^{\alpha_A}_A(t), X^{\alpha_B}_B(t)) \leq e^t D(X^{\alpha_A}_{A,0}, X^{\alpha_B}_{B,0}).$$

Proof. As H does not change with time $\|H_A - H_B\|_\infty$ is constant.

We also have that, for any $t \in \mathbb{R}^+$,

$$\|V_{A,\xi}(t) - V_{B,\xi}(t)\|_i \leq \|G_{A,B}(t)\|_i, \text{ for } i = 1, 2,$$

and

$$\|G_{A,B}(t)\|_i \leq \|G_{A,B}(0)\|_i, \text{ for } i = 1, 2.$$

Combining these estimates with those from [27, Corollary 2.5], we obtain

$$D(X_A(t), X_B(t)) \leq D(X_A(0), X_B(0)) + \int_0^t D(X_A(s), X_B(s))\, ds,$$

and thus the result follows from Grönwall's inequality. □

The metric D satisfies a Lipschitz estimate we desire, however it fails as a metric over equivalence classes related by relabelling. In particular, two elements of the same equivalence class may have positive distance when compared via D.

The resolution of this issue begins by introducing a mapping on the space $\mathcal{F}_c$ which is zero when measuring members of the same equivalence class. Define $J : \mathcal{F}_c^2 \to \mathbb{R}$ by

$$J(X_A^{\alpha_A}, X_B^{\alpha_B}) = \inf_{f,g\in\mathcal{G}} \left(D(X_A^{\alpha_A} \circ f, X_B^{\alpha_B}) + D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ g) \right).$$

Proposition 2.5.6. *If $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$ share the same equivalence class,*

$$J(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0.$$

Furthermore, if $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ are in $\mathcal{F}_c \cap \mathcal{F}_0$, and

$$J(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0,$$

then $X_A^{\alpha_A} = X_B^{\alpha_B}$.

Proof. Suppose that $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ share the same equivalence class. Then there exists relabelling functions $f, g \in \mathcal{G}$ such that $X_A^{\alpha_A} \circ f = X_B^{\alpha_B}$ and $X_A^{\alpha_A} = X_B^{\alpha_B} \circ g$. Thus

$$\begin{aligned} 0 \leq J(X_A^{\alpha_A}, X_B^{\alpha_B}) &\leq D(X_A^{\alpha_A} \circ f, X_B^{\alpha_B}) + D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ g) \\ &= D(X_B^{\alpha_B}, X_B^{\alpha_B}) + D(X_A^{\alpha_A}, X_A^{\alpha_A}) \\ &= 0, \end{aligned}$$

and hence $J(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0$.

The remainder of this proof inherits the ideas of [24, Lemma 3.2]. Consider the norm $\|\cdot\| : \mathcal{F}_c^2 \to \mathbb{R}$ given by

$$\|X^\alpha\| = \|y - \mathrm{id}\|_\infty + \|U\|_\infty + \|H\|_\infty + \|V\|_\infty + |\alpha|.$$

Consider $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c \cap \mathcal{F}_0$. Let $f \in \mathcal{G}$. Consider any Lipschitz continuous function h with Lipschitz constant bounded by 1. As $y_A + H_A = y_B + H_B = \mathrm{id}$,

$$\begin{aligned} |h \circ f - h| \leq |f - \mathrm{id}| &= |(y_A + H_A) \circ f - (y_B + H_B)| \\ &\leq |y_A \circ f - y_B| + |H_A \circ f - H_B|. \end{aligned} \tag{2.5.12}$$

$X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ are in $\mathcal{F}_0$, and hence y, U, H and V are Lipschitz continuous with constant bounded by 1, and result (2.5.12) is applicable. Hence

$$\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq \|X_A^{\alpha_A} - X_A^{\alpha_A} \circ f\| + \|X_A^{\alpha_A} \circ f - X_B^{\alpha_B}\|$$

$$\begin{aligned}&\leq 4(\|y_A\circ f-y_B\|_\infty+\|H_A\circ f-H_B\|_\infty)\\&\quad+\|X_A^{\alpha_A}\circ f-X_B^{\alpha_B}\|\\&\leq 5\|X_A^{\alpha_A}\circ f-X_B^{\alpha_B}\|.\end{aligned}$$

Consequently, for any $f, g \in \mathcal{G}$,

$$\begin{aligned}2\|X_A^{\alpha_A}-X_B^{\alpha_B}\| &\leq 5(\|X_A^{\alpha_A}\circ f-X_B^{\alpha_B}\|+\|X_A^{\alpha_A}-X_B^{\alpha_B}\circ g\|)\\&\leq 5(D(X_A^{\alpha_A}\circ f, X_B^{\alpha_B})+D(X_A^{\alpha_A}, X_B^{\alpha_B}\circ g)),\end{aligned} \tag{2.5.13}$$

where we have made use of the property

$$\|V_1-V_2\|_\infty \leq \|V_{1,\xi}-V_{2,\xi}\|_1 \leq \|G_{1,2}\|_1,$$

for any $X_1^{\alpha_1}$, $X_2^{\alpha_2} \in \mathcal{F}_c$, see [28, (3.16)]. Taking the infimum over all $f, g \in \mathcal{G}$ in (2.5.13), we obtain

$$2\|X_A^{\alpha_A}-X_B^{\alpha_B}\| \leq 5J(X_A^{\alpha_A}, X_B^{\alpha_B}), \tag{2.5.14}$$

and the second part of the proposition is an immediate consequence. □

Further, J satisfies the following result.

Lemma 2.5.7. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be in $\mathcal{F}_c$. Then*

$$D(X_A^{\alpha_A}\circ f, X_B^{\alpha_B}\circ h) \leq \max\{\|f_\xi\|_\infty^{\frac{1}{2}}, 1\} D(X_A^{\alpha_A}, X_B^{\alpha_B}\circ w), \tag{2.5.15}$$

for any $f, h \in \mathcal{G}$, where $w := h \circ f^{-1}$.

Hence,

$$J(X_A^{\alpha_A}\circ f, X_B^{\alpha_B}\circ h) \leq \max\{\|f_\xi\|_\infty^{\frac{1}{2}}, \|h_\xi\|_\infty^{\frac{1}{2}}, 1\} J(X_A^{\alpha_A}, X_B^{\alpha_B}). \tag{2.5.16}$$

Proof. The idea of this proof is from [38, Lemma 4.8]. We begin by obtaining estimates for the first three terms of D, i.e. those in the $L^\infty(\mathbb{R})$ norm, see (2.5.11). Consider any function $\psi \in L^\infty(\mathbb{R})$. As f and h are homeomorphisms, in particular bijective, one has

$$\|\psi\circ f-\psi\circ h\|_\infty = \|\psi\circ f\circ f^{-1}-\psi\circ h\circ f^{-1}\|_\infty = \|\psi-\psi\circ w\|_\infty. \tag{2.5.17}$$

Replacing ψ by y, U and H we obtain equalities for the first three terms.

We now consider the terms involving y_ξ and U_ξ in (2.5.11). One has, dropping the ξ in our notation for convenience,

$$\begin{aligned}|(y_A\circ f)_\xi-(y_B\circ h)_\xi|\circ f^{-1} &= |y_{A,\xi}f_\xi\circ f^{-1}-(y_{B,\xi}\circ h\circ f^{-1})h_\xi\circ f^{-1}|\\&= |y_{A,\xi}-(y_{B,\xi}\circ w)\frac{h_\xi\circ f^{-1}}{f_\xi\circ f^{-1}}|f_\xi\circ f^{-1}\end{aligned}$$

$$= |y_{A,\xi} - (y_B \circ w)_\xi| f_\xi \circ f^{-1},$$

from which we can conclude, taking the substitution $\eta = f(\xi)$,

$$\begin{aligned} \|(y_A \circ f)_\xi - (y_B \circ h)_\xi\|_2^2 &= \int_{\mathbb{R}} |(y_A \circ f)_\xi(\xi) - (y_B \circ h)_\xi(\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} |y_{A,\xi}(\eta) - (y_B \circ w)_\xi(\eta)|^2 f_\xi \circ f^{-1}(\eta) \,\mathrm{d}\eta \\ &\leq \|f_\xi\|_\infty \|y_{A,\xi} - (y_B \circ w)_\xi\|_2^2. \end{aligned} \tag{2.5.18}$$

Via similar calculations one finds

$$\|(U_A \circ f)_\xi - (U_B \circ h)_\xi\|_2^2 \leq \|f_\xi\|_\infty \|U_{A,\xi} - (U_B \circ w)_\xi\|_2^2. \tag{2.5.19}$$

It remains to find estimates for the terms involving G in (2.5.11). To begin

$$\begin{aligned} g(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} &= |(V_A \circ f)_\xi - (V_B \circ w)_\xi| \\ &= |V_{A,\xi} - (V_B \circ w)_\xi| f_\xi \circ f^{-1} \\ &= g(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1}, \end{aligned} \tag{2.5.20}$$

repeating the previous calculations.

Furthermore, from (2.5.5), we have

$$\mathbb{1}_{\mathcal{A}_A^{f,c}} \circ f^{-1} = \mathbb{1}_{\mathcal{A}_A^c}, \qquad \mathbb{1}_{\mathcal{A}_B^{h,c}} \circ f^{-1} = \mathbb{1}_{\mathcal{A}_B^{w,c}}, \qquad \mathbb{1}_{\mathcal{A}_{A,B}^{f,h}} \circ f^{-1} = \mathbb{1}_{\mathcal{A}_{A,B}^{\mathrm{id},w}},$$

and

$$\mathbb{1}_{\mathcal{B}_{A,B}^{f,h}} \circ f^{-1} = \mathbb{1}_{\mathcal{B}_{A,B}^{\mathrm{id},w}}, \qquad \mathbb{1}_{\Omega_{A,B}^{f,h,c}} \circ f^{-1} = \mathbb{1}_{\Omega_{A,B}^{\mathrm{id},w,c}}.$$

Then,

$$\begin{aligned} (V_A^c \circ f)_\xi \circ f^{-1} &= (1 - \alpha \mathbb{1}_{\mathcal{A}_A^{f,c}} \circ f^{-1})(V_A \circ f)_\xi \circ f^{-1} \\ &= (1 - \alpha \mathbb{1}_{\mathcal{A}_A^c}) V_{A,\xi} f_\xi \circ f^{-1} \\ &= V_{A,\xi}^c \, f_\xi \circ f^{-1}, \end{aligned}$$

and similarly,

$$(V_B^c \circ h)_\xi \circ f^{-1} = (V_B^c \circ w)_\xi f_\xi \circ f^{-1}.$$

Furthermore,

$$\begin{aligned} (V_A^d \circ f)_\xi \circ f^{-1} &= (V_A \circ f)_\xi \circ f^{-1} - (V_A^c \circ f)_\xi \circ f^{-1} \\ &= (V_{A,\xi} - V_{A,\xi}^c) f_\xi \circ f^{-1} \end{aligned}$$

$$= V_A^d f_\xi \circ f^{-1},$$

and similarly,

$$(V_B^d \circ h)_\xi \circ f^{-1} = (V_B^d \circ w)_\xi f_\xi \circ f^{-1}.$$

From these results, we can conclude,

$$\hat{g}(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} = \hat{g}(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1}, \qquad (2.5.21)$$

and

$$\bar{g}(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} = \bar{g}(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1}. \qquad (2.5.22)$$

and via (2.5.20), (2.5.21), and (2.5.22), we obtain

$$G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} = G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1},$$

almost everywhere, leading to

$$\|G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h)\|_1 = \|G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w)\|_1, \qquad (2.5.23)$$

and

$$\|G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h)\|_2^2 \leq \|f_\xi\|_\infty \|G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w)\|_2^2. \qquad (2.5.24)$$

A combination of (2.5.17), (2.5.18), (2.5.19), (2.5.23), and (2.5.24) leads to (2.5.15).

Considering now the second result, we have

$$\begin{aligned}
&J(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \\
&= \inf_{f_1, f_2 \in \mathcal{G}} \left(D(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h \circ f_1) + D(X_A^{\alpha_A} \circ f \circ f_2, X_B^{\alpha_B} \circ h) \right) \\
&\leq \inf_{f_1, f_2 \in \mathcal{G}} \Big(\max\{\|f_\xi\|_\infty^{\frac{1}{2}}, 1\} D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ h \circ f_1 \circ f^{-1}) \\
&\qquad\qquad + \max\{\|h_\xi\|_\infty^{\frac{1}{2}}, 1\} D(X_A^{\alpha_A} \circ f \circ f_2 \circ h^{-1}, X_B^{\alpha_B}) \Big) \\
&\leq \max\{\|f_\xi\|_\infty^{\frac{1}{2}}, \|h_\xi\|_\infty^{\frac{1}{2}}, 1\} \\
&\quad \times \inf_{f_1, f_2 \in \mathcal{G}} \left(D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ f_1) + D(X_A^{\alpha_A} \circ f_2, X_B^{\alpha_B}) \right),
\end{aligned}$$

where in the last inequality we have used that, via the group properties of $\mathcal{G}$, any element of the group can be written as $h \circ f_1 \circ f^{-1}$ for a suitable choice of $f_1 \in \mathcal{G}$, and similarly as $f \circ f_2 \circ h^{-1}$ for a suitable choice of $f_2 \in \mathcal{G}$. □

Consider $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$, and set $\hat{X}_A^{\alpha_A} = \Pi X_A^{\alpha_A}$, $\hat{X}_B^{\alpha_B} = \Pi X_B^{\alpha_B}$. Then after substituting (2.5.16) into (2.5.14), we find that

$$\|\hat{X}_A^{\alpha_A} - \hat{X}_B^{\alpha_B}\| \leq CJ(X_A^{\alpha_A}, X_B^{\alpha_B}),$$

for some constant $C > 1$. Hence we have the corollary:

Corollary 2.5.8. *J is a semi-metric on the space $\mathcal{F}_c$ in the sense of equivalence classes, i.e. elements of the same equivalence class have distance zero.*

Finally, we have a result that is useful later,

Corollary 2.5.9. *Suppose we have two solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$ of the ODE system* (2.3.10) *with initial data $X_{A,0}^{\alpha_A}$ and $X_{B,0}^{\alpha_B}$ in $\mathcal{F}_c \cap \mathcal{F}_0$. Then*

$$J(\Pi X_A^{\alpha_A}(t), \Pi X_B^{\alpha_B}(t)) \leq e^{\frac{1}{4}t} J(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)).$$

Proof. Set $f_A = (y_A + H_A)^{-1}(\cdot, t)$ and $f_B = (y_B + H_B)^{-1}(\cdot, t)$, where we are considering the spatial inverse. Referring to the calculations in [38, Theorem 3.3], we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{(y_{A,\xi} + H_{A,\xi})(\xi, t)}\right) &= -\frac{U_{A,\xi}(\xi, t)}{(y_{A,\xi} + H_{A,\xi})^2(\xi, t)} \\
&\leq \frac{1}{(y_{A,\xi} + H_{A,\xi})(\xi, t)} \frac{\frac{1}{2}(y_{A,\xi} + V_{A,\xi})(\xi, t)}{(y_{A,\xi} + H_{A,\xi})(\xi, t)} \\
&\leq \frac{1}{2} \frac{1}{(y_{A,\xi} + H_{A,\xi})(\xi, t)}
\end{aligned}$$

where we have used,

$$U_{A,\xi}(\xi, t) = \pm\sqrt{y_{A,\xi}(\xi, t) V_{A,\xi}(\xi, t)} \leq \frac{1}{2}(y_{A,\xi}(\xi, t) + V_{A,\xi}(\xi, t)).$$

As a consequence,

$$\frac{1}{(y_{A,\xi} + H_{A,\xi})(\xi, t)} \leq e^{\frac{1}{2}t},$$

using that, as $X_A^{\alpha_A} \in \mathcal{F}_0$, $(y_{A,\xi} + H_{A,\xi})(\xi, 0) = 1$.

These calculations can be repeated for $X_B^{\alpha_B}$, and we can conclude

$$f_{i,\xi}(\xi) = \frac{1}{(y_{i,\xi} + H_{i,\xi})(f_i(\xi), t)} \leq e^{\frac{1}{2}t}, \qquad \text{for } i = A, B,$$

and hence $\|f_{i,\xi}\|_\infty \leq e^{\frac{1}{2}t}$ for $i = A, B$. The result then follows from using (2.5.16). □

The mapping J does not satisfy the triangle inequality. However, it can be used to construct a metric. Define the distance $d : \mathcal{F}_c^2 \to \mathbb{R}$ by

$$d(X_A^{\alpha_A}, X_B^{\alpha_B}) = \inf_{\hat{\mathcal{F}}_c(X_A^{\alpha_A}, X_B^{\alpha_B})} \sum_{i=1}^{N} J(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}}),$$

with $\hat{\mathcal{F}}_c(X_A^{\alpha_A}, X_B^{\alpha_B})$ the set of all finite sequences $\{X_n^{\alpha_n}\}_{n=0}^N$ in $\mathcal{F}_c \cap \mathcal{F}_0$ of arbitrary length satisfying

$$X_0^{\alpha_0} = \Pi X_A^{\alpha_A} \qquad \text{and} \qquad X_N^{\alpha_N} = \Pi X_B^{\alpha_B}.$$

Proposition 2.5.10. *The mapping d is a metric, in the sense of equivalence classes.*

Furthermore, for any α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$ with initial data $X_{A,0}^{\alpha_A}$ and $X_{B,0}^{\alpha_B}$ in $\mathcal{F}_c \cap \mathcal{F}_0$,

$$d(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \leq e^{\frac{5}{4}t} d(X_{A,0}^{\alpha_A}, X_{B,0}^{\alpha_B}). \tag{2.5.25}$$

Proof. Consider $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_c$. Symmetry of d is immediate. We have that, from the definition of d,

$$d(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0 \implies J(\Pi X_A^{\alpha_A}, \Pi X_B^{\alpha_B}) = 0,$$

and hence $X_A^{\alpha_A} \sim X_B^{\alpha_B}$ by Proposition 2.5.6. That $\alpha_A = \alpha_B$ follows from the inclusion of the $|\alpha_A - \alpha_B|$ term in D, see (2.5.11). On the other hand, if $X_A^{\alpha_A} \sim X_B^{\alpha_B}$, considering the sequence containing only $\Pi X_A^{\alpha_A}$ and $\Pi X_B^{\alpha_B}$ immediately gives $d(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0$.

Setting our sights on the triangle inequality, we consider a third element $X_C^{\alpha_C} \in \mathcal{F}_c$. Let $\epsilon > 0$. As a consequence of the infimum, there exist two finite sequences, $\{X_n^{\alpha_n}\}_{n=0}^N$ in $\hat{\mathcal{F}}_c(X_A^{\alpha_A}, X_B^{\alpha_B})$ and $\{X_n^{\alpha_n}\}_{n=N}^M$ in $\hat{\mathcal{F}}_c(X_B^{\alpha_B}, X_C^{\alpha_C})$ with $1 < N < M$, such that

$$\sum_{n=1}^{N} J(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}}) \leq d(X_A^{\alpha_A}, X_B^{\alpha_B}) + \frac{\epsilon}{2},$$
$$\sum_{n=N+1}^{M} J(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}}) \leq d(X_B^{\alpha_B}, X_C^{\alpha_C}) + \frac{\epsilon}{2}.$$

Note that the last element of the first sequence is the same as the first element of the second. By this construction, the sequence $\{X_n^{\alpha_n}\}_{n=0}^M$ is in $\hat{\mathcal{F}}(X_A^{\alpha_A}, X_C^{\alpha_C})$. So

$$d(X_A^{\alpha_A}, X_C^{\alpha_C}) \leq \sum_{n=1}^{M} J(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}})$$

$$\leq d(X_A^{\alpha_A}, X_B^{\alpha_B}) + d(X_B^{\alpha_B}, X_C^{\alpha_C}) + \epsilon.$$

This inequality holds for any $\epsilon > 0$, and the triangle inequality follows.

A generalisation of the scheme presented above is given as a result in the Appendix of Paper 2.

It remains to show (2.5.25). Let $\epsilon > 0$. By the properties of the infimum, there exists a finite sequence $\{X_n^{\alpha_n}\}_{n=0}^N$ in $\hat{\mathcal{F}}_c(X_{A,0}^{\alpha_A}, X_{B,0}^{\alpha_B})$ with $X_0^{\alpha_0} = X_{A,0}^{\alpha_A}$ and $X_N^{\alpha_N} = X_{B,0}^{\alpha_B}$, and relabelling functions $\{f_n\}_{n=1}^N$, $\{h_n\}_{n=0}^{N-1}$ in $\mathcal{G}$ such that

$$\sum_{n=1}^N \big(D(X_n^{\alpha_n} \circ f_n, X_{n-1}^{\alpha_{n-1}}) + D(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}} \circ h_{n-1})\big) \leq d(X_{A,0}^{\alpha_A}, X_{B,0}^{\alpha_B}) + \epsilon.$$

Denote by $X_n^{\alpha_n}(t) = S_t X_n^{\alpha_n}$ for $n = 0, \dots, N$. Then

$$\begin{aligned}
d(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) &\leq \sum_{n=1}^N J(\Pi X_n^{\alpha_n}(t), \Pi X_{n-1}^{\alpha_{n-1}}(t)) \\
&\leq e^{\frac{1}{4}t} \sum_{n=1}^N J(X_n^{\alpha_n}(t), X_{n-1}^{\alpha_{n-1}}(t)) \\
&\leq e^{\frac{1}{4}t} \sum_{n=1}^N \big(D(X_n^{\alpha_n}(t) \circ f_n, X_{n-1}^{\alpha_{n-1}}(t)) \\
&\qquad + D(X_n^{\alpha_n}(t), X_{n-1}^{\alpha_{n-1}}(t) \circ h_{n-1})\big) \\
&\leq e^{\frac{5}{4}t} \sum_{n=1}^N \big(D(X_n^{\alpha_n} \circ f_n, X_{n-1}^{\alpha_{n-1}}) \\
&\qquad + D(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}} \circ h_{n-1})\big) \\
&\leq e^{\frac{5}{4}t}(d(X_{A,0}^{\alpha_A}, X_{B,0}^{\alpha_B}) + \epsilon).
\end{aligned}$$

Again, as this holds for any $\epsilon > 0$, equation (2.5.25) is satisfied. □

Results of Papers 1 and 2

We now define the metrics from the first two papers [27, 28], and explore their respective results.

The Lagrangian Setting

In the context of paper 1, both the solutions we compare share the same α. Hence the construction is built in the setting of $\mathcal{F}^\alpha$ for $\alpha \in [0, 1]$ fixed.

We first have the analog of G from the previous section.

Lemma 2.5.11. *Consider X_A and X_B in $\mathcal{F}^\alpha$ for some $\alpha \in [0,1]$. Then*

$$\|V_{A,\xi} - V_{B,\xi}\|_i \leq \|G_{A,B}\|_i, \qquad i = 1, 2,$$

where

$$\begin{aligned} G_{A,B}(\xi) = G(X_A, X_B)(\xi) = |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)|\mathbb{1}_{\Omega_{A,B}}(\xi) \\ + (V_{A,\xi}(\xi) \vee V_{B,\xi}(\xi))\mathbb{1}_{\Omega^c_{A,B}}(\xi). \end{aligned} \tag{2.5.26}$$

For two α-dissipative solutions X_A and X_B in $\mathcal{F}^\alpha$, we introduce the notation

$$G_{A,B}(\xi, t) = G(X_A(t), X_B(t))(\xi),$$

for any $\xi \in \mathbb{R}$. Then G is a decreasing function in time for any $\xi \in \mathbb{R}$.

We then define the metric D_1 by,

$$\begin{aligned} D_1(X_A, X_B) = \|y_A - y_B\|_\infty + \|U_A - U_B\|_\infty \\ + \|y_{A,\xi} - y_{B,\xi}\|_2 + \|U_{A,\xi} - U_{B,\xi}\|_2 \\ + \|H_A - H_B\|_\infty + \|G_{A,B}\|_1 + \|G_{A,B}\|_2, \end{aligned} \tag{2.5.27}$$

for $X_A, X_B \in \mathcal{F}^\alpha$ and $\alpha \in [0,1]$, with $G_{A,B}$ given by (2.5.26).

In the more general setting of paper 2, we begin by defining three help functions. Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be in $\mathcal{F}$. Define

$$g_{A,B}(\xi) = g(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) = |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)|, \tag{2.5.28a}$$

$$\begin{aligned} \hat{g}_{A,B}(\xi) &= \hat{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) \\ &= |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)| + \|\alpha_A - \alpha_B\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi) \\ &\quad + \|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi) \\ &\qquad \times (|y_A(\xi) - y_B(\xi)| + |U_A(\xi) - U_B(\xi)|) \end{aligned} \tag{2.5.28b}$$

and

$$\begin{aligned} \bar{g}_{A,B}(\xi) &= \bar{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi) \\ &= |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)| \\ &\quad + (V_{A,\xi} \wedge V_{B,\xi})(\xi)(\alpha_A(\xi)\mathbb{1}_{\mathcal{A}^c_A}(\xi) + \alpha_B(\xi)\mathbb{1}_{\mathcal{A}^c_B}(\xi)) \\ &\quad + \|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi) \\ &\qquad \times \Big(|y_A(\xi) - \xi|\mathbb{1}_{\mathcal{A}^c_A}(\xi) + |y_B(\xi) - \xi|\mathbb{1}_{\mathcal{A}^c_B}(\xi) \\ &\qquad\quad + (|U_A(\xi)| + |U_B(\xi)|)(\mathbb{1}_{\mathcal{A}^c_A}(\xi) + \mathbb{1}_{\mathcal{A}^c_B}(\xi))\Big), \end{aligned} \tag{2.5.28c}$$

where we use the notation $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$, and

$$\alpha'_{A,B} = \max\{\alpha'_A, \alpha'_B\}.$$

As usual, if we consider two α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}$ we set

$$g_{A,B}(\xi, t) = g(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t))(\xi)$$

and analogously define $\hat{g}_{A,B}(\xi, t)$ and $\bar{g}_{A,B}(\xi, t)$.

Lemma 2.5.12. *Consider $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}$. Then*

$$\|V_{A,\xi} - V_{B,\xi}\|_i \leq \|G_{A,B}\|_i, \qquad i = 1, 2,$$

where

$$\begin{aligned} G_{A,B}(\xi) &= G(X_A, X_B)(\xi) \\ &= g_{A,B}(\xi)\mathbb{1}_{\mathcal{A}_{A,B}}(\xi) + \hat{g}_{A,B}(\xi)\mathbb{1}_{\mathcal{B}_{A,B}}(\xi) + \bar{g}_{A,B}(\xi)\mathbb{1}_{\Omega^c_{A,B}}(\xi) \\ &\quad + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi) \\ &\quad \times (\|V_{A,\xi}\|_1 + \|V_{B,\xi}\|_1 + 1) \\ &\quad \times (\mathbb{1}_{\mathcal{A}^c_A}(\xi) + \mathbb{1}_{\mathcal{A}^c_B}(\xi))\mathbb{1}_{\mathcal{B}^c_{A,B}}(\xi). \end{aligned}$$

For two α-dissipative solutions X_A and X_B in $\mathcal{F}$, we introduce the notation

$$G_{A,B}(\xi, t) = G(X_A(t), X_B(t))(\xi),$$

for any $\xi \in \mathbb{R}$. Then G is a decreasing function over breaking times, i.e.

$$G_{A,B}(\xi, \tau(\xi)) \leq \lim_{t \uparrow \tau(\xi)} G_{A,B}(\xi, t), \qquad \text{for any } \xi \in \mathbb{R}.$$

Furthermore, we have

$$\begin{aligned} \|G_{A,B}(\cdot, t)\|_1 \leq \|G_{A,B}(\cdot, 0)\|_1 & \\ + \int_0^t \Big(\|G_{A,B}(\cdot, s)\|_1 & \\ + \frac{1}{4} M_{A,B}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(\cdot, s)\|_1\Big)\, \mathrm{d}s\,, & \end{aligned} \tag{2.5.29}$$

and

$$\begin{aligned} \|G_{A,B}(\cdot, t)\|_2 \leq \|G_{A,B}(\cdot, 0)\|_2 & \\ + \int_0^t \Big(\|G_{A,B}(\cdot, s)\|_2 & \\ + \frac{1}{4}\sqrt{M_{A,B}}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(\cdot, s)\|_1\Big)\, \mathrm{d}s\,, & \end{aligned} \tag{2.5.30}$$

where

$$M_{A,B} = \max\{\|V_A(\cdot, 0)\|_\infty, \|V_B(\cdot, 0)\|_\infty\}. \tag{2.5.31}$$

We then define the mapping $D_2 : \mathcal{F}^2 \to \mathbb{R}$, which is not a metric, by

$$\begin{aligned} D_2(X_A^{\alpha_A}, X_B^{\alpha_B}) = &\|y_A - y_B\|_\infty + \|U_A - U_B\|_\infty \\ &+ \|y_{A,\xi} - y_{B,\xi}\|_2 + \|U_{A,\xi} - U_{B,\xi}\|_2 \\ &+ \|H_A - H_B\|_\infty + \frac{1}{4}\|G_{A,B}\|_1 + \frac{1}{2}\|G_{A,B}\|_2 \\ &+ \|\alpha_A - \alpha_B\|_\infty. \end{aligned} \tag{2.5.32}$$

This fails to be a metric due to G not satisfying the triangle inequality. The other conditions of a metric are satisfied, and thus it is a semi-metric on the space $\mathcal{F}$.

As it turns out, there is no issue in the lack of the triangle inequality on this mapping, as being a semi-metric is sufficient for the construction of the metric on equivalence classes.

Via an application of Grönwall's inequality, we obtain the following results.

Lemma 2.5.13. *The metric D_1 satisfies*

$$D_1(X_A(t), X_B(t)) \leq e^t D_1(X_A(0), X_B(0)),$$

for two α-dissipative solutions X_A and X_B in $\mathcal{F}^\alpha$ with $\alpha \in [0,1]$.

The metric D_2 satisfies

$$D_2(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \leq e^{C_{A,B}t} D_2(X_A^{\alpha_A}(0), X_B^{\alpha_B}(0)),$$

for two α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ with $X_A^{\alpha_A}(0) \in \mathcal{F}_0$ and $X_B^{\alpha_B}(0) \in \mathcal{F}$, where

$$C_{A,B} = 2 + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (M_{A,B} + 2\sqrt{M_{A,B}}),$$

and $M_{A,B}$ is given by (2.5.31).

We thus see a notable difference between the simpler case of α being a constant versus $\alpha \in \Lambda$. In the second case, the estimate above depends on an initial energy bound $M_{A,B}$ and a bound on the derivatives of α in the term $\|\alpha'_{A,B}\|_\infty$. When trying to obtain an estimate for the metric we construct on equivalence classes, these dependencies can cause an issue unless we bound their values for the considered finite sequences. Hence in this circumstance we consider the set $\mathcal{F}_M^L$, see (2.3.11).

Eventually, the energy bound M is required when constructing the metric in Eulerian coordinates. The main drawback in the second case is thus the requirement that $\|\alpha'_{A,B}\|_\infty \leq L$ for some $L > 0$.

Let $\alpha \in [0,1]$. Define $J_1 : \mathcal{F}^\alpha \times \mathcal{F}^\alpha \to \mathbb{R}^+$ by

$$J_1(X_A, X_B) = \inf_{f,g \in \mathcal{G}} (D_1(X_A, X_B \circ f) + D_1(X_A \circ g, X_B)),$$

and subsequently define

$$d_1(X_A, X_B) = \inf_{\hat{\mathcal{F}}_1(X_A, X_B)} \sum_{i=1}^{N} J_1(X_n, X_{n-1}),$$

where the infimum is taken over the set $\hat{\mathcal{F}}_1(X_A, X_B)$ containing all finite sequences $\{X_n\}_{n=0}^N$ of arbitrary length in $\mathcal{F}_0^\alpha$, satisfying $X_0 = \Pi X_A$ and $X_N = \Pi X_B$. Here we have introduced the notation for X in $\mathcal{F}^\alpha$,

$$\Pi X = X \circ (y + H)^{-1}.$$

Define also $J_2 : \mathcal{F}^2 \to \mathbb{R}$ by

$$J_2(X_A^{\alpha_A}, X_B^{\alpha_B}) = \inf_{f,g \in \mathcal{G}} \big(D_2(X_A^{\alpha_A}, X_B^{\alpha_B} \circ f) + D_2(X_A^{\alpha_A} \circ g, X_B^{\alpha_B})\big)$$

and subsequently define, for $X_A^{\alpha_A}, X_B^{\alpha_B} \in \mathcal{F}_M^L$,

$$d_2(X_A^{\alpha_A}, X_B^{\alpha_B}) = \inf_{\hat{\mathcal{F}}_2(X_A^{\alpha_A}, X_B^{\alpha_B})} \sum_{i=1}^{N} J_2(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}}),$$

where the infimum is taken over the set $\hat{\mathcal{F}}_2(X_A^{\alpha_A}, X_B^{\alpha_B})$ containing all finite sequences of arbitrary length $\{X_n^{\alpha_n}\}_{n=0}^N$ in $\mathcal{F}_M^L \cap \mathcal{F}_0$ satisfying $X_0^{\alpha_0} = \Pi X_A^{\alpha_A}$ and $X_N^{\alpha_N} = \Pi X_B^{\alpha_B}$.

d_1 and d_2 are metrics, and with them we get the final Lipschitz stability results we desire.

Lemma 2.5.14. *Let $\alpha \in [0,1]$. Consider two α-dissipative solutions X_A and X_B in $\mathcal{F}^\alpha$, with initial data $X_A(0)$ and $X_B(0)$ in $\mathcal{F}_0^\alpha$. Then*

$$d_1(X_A(t), X_B(t)) \leq e^{\frac{3}{2}t} d_1(X_A(0), X_B(0)).$$

Lemma 2.5.15. *Consider two α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}_M^L$ with initial data $X_A^{\alpha_A}(0)$ and $X_B^{\alpha_B}(0)$ in $\mathcal{F}_M^L \cap \mathcal{F}_0$. Then*

$$d_2(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \leq e^{R_M^L t} d_2(X_A^{\alpha_A}(0), X_B^{\alpha_B}(0)),$$

where

$$R_M^L = 4\bar{M} + \frac{5}{2} + \frac{1}{4} L(M + 2\sqrt{M}),$$

where $\bar{M} = M \vee 1$.

The Eulerian Setting

It remains to use the construction in the Lagrangian setting to define a metric in the Eulerian setting. For simplicity we outline the ideas in the more complicated setting of the second paper [27], and we will provide only the final result of the first paper [28].

Using the transformation $\hat{L}$, we can define the distance between two Eulerian coordinates by their distance in Lagrangian coordinates, i.e. via the metric $d_{\mathcal{D}}$ given by

$$d_{\mathcal{D}}(Y_A^{\alpha_A}, Y_B^{\alpha_B}) = d_2(\hat{L}(Y_A^{\alpha_A}), \hat{L}(Y_B^{\alpha_B})),$$

for $Y_A^{\alpha_A}$ and $Y_B^{\alpha_B}$ in $\mathcal{D}_M^L$.

We immediately have the following result: For two α-dissipative solutions $Y_A^{\alpha_A}$ and $Y_B^{\alpha_B}$ in $\mathcal{D}_M^L$ with initial data $Y_A^{\alpha_A}(0)$ and $Y_B^{\alpha_B}(0)$ in $\mathcal{D}_M^L$,

$$d_{\mathcal{D}}(Y_A^{\alpha_A}(t), Y_B^{\alpha_B}(t)) \leq e^{R_M^L t} d_{\mathcal{D}}(Y_A^{\alpha_A}(0), Y_B^{\alpha_B}(0)).$$

As discussed earlier, for an α-dissipative solution $Y^\alpha = ((u, \mu, \nu), \alpha)$, the solution to the Hunter–Saxton equation consists of the pair (u, μ), and the variable α determines the type of solution. The ν is a dummy variable to enable the semigroup property for solutions. Most importantly, there are equivalence classes of ν, given by $\mathcal{V}((u, \mu), \alpha)$ in Definition 2.3.8, that generate the same Eulerian solution, see [27, Lemma 2.13]. We wish to define a metric for measuring triples $Z^\alpha = ((u, \mu), \alpha) \in \mathcal{D}_0$.

Thankfully, we can repeat the same strategy we used in the construction of our metric in Lagrangian coordinates. First, we define the mapping $\hat{J} : \mathcal{D}_0^2 \to \mathbb{R}^+$ by

$$\hat{J}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) = \inf_{(\nu_1, \nu_2) \in \mathcal{V}(Z_A^{\alpha_A}) \times \mathcal{V}(Z_B^{\alpha_B})} d_{\mathcal{D}}(((Z_A, \nu_1), \alpha_A), ((Z_B, \nu_2), \alpha_B)).$$

where $\mathcal{V}$ is given in Definition 2.3.8.

It cannot be concluded that the triangle inequality is satisfied, hence $\hat{J}$ is not a metric itself. We then define the mapping $\bar{d} : \mathcal{D}_{0,M}^L \times \mathcal{D}_{0,M}^L \to \mathbb{R}$ by

$$\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) = \inf_{\hat{\mathcal{D}}_0(Z_A^{\alpha_A}, Z_B^{\alpha_B})} \sum_{n=1}^{N} \hat{J}(Z_n^{\alpha_n}, Z_{n-1}^{\alpha_{n-1}}),$$

where $\hat{\mathcal{D}}_0(Z_A^{\alpha_A}, Z_B^{\alpha_B})$ denotes the set of finite sequences $\{Z_n^{\alpha_n}\}_{n=0}^N$ of arbitrary length in $\mathcal{D}_{0,M}^L$ satisfying $Z_0^{\alpha_0} = Z_A^{\alpha_A}$ and $Z_N^{\alpha_N} = Z_B^{\alpha_B}$.

$\bar{d}$ is indeed a metric, and with it we can obtain our final result.

Theorem 2.5.16. *Let $Z_A^{\alpha_A}$ and $Z_B^{\alpha_B}$ be two α-dissipative solutions to the Hunter–Saxton equation* (HS)*, constructed via the generalised method of characteristics, with initial data $Z_A^{\alpha_A}(0)$ and $Z_B^{\alpha_B}(0)$ in $\mathcal{D}_{0,M}^L$ respectively. Then*

$$\bar{d}(Z_A^{\alpha_A}(t), Z_B^{\alpha_B}(t)) \leq e^{R_M^L t}\bar{d}(Z_A^{\alpha_A}(0), Z_B^{\alpha_B}(0)).$$

We have a similar result for the case of $\alpha \in [0, 1]$.

Theorem 2.5.17. *Let $\alpha \in [0, 1]$. Let Z_A and Z_B be two α-dissipative solutions to* (HS)*, with initial data $Z_A(0)$ and $Z_B(0)$ in $\mathcal{D}_0^\alpha$ with*

$$\max\{\mu_A(\mathbb{R}), \mu_B(\mathbb{R})\} \leq M,$$

for some $M > 0$. Then there exists a metric $\bar{d}_1$, constructed via a similar method as outlined above but using the metric d_1 in Lagrangian coordinates, such that

$$\bar{d}_1(Z_A(t), Z_B(t)) \leq e^{\frac{3}{2}t}\bar{d}_1(Z_A(0), Z_B(0)).$$

2.6 Two alternative previously constructed metrics

To provide context in the field, we provide a short overview of two previous metrics that have been constructed for solutions to the Hunter–Saxton equation making use of different techniques.

Bressan and Constantin - A metric for dissipative solutions

The following section details the metric created in the paper of Bressan and Constantin [7] for dissipative solutions of the Hunter-Saxton equation. To avoid confusion during this discussion, we attempt to emulate the notation introduced in [7].

In said work, a different definition of solution is used. We begin by introducing its definition.

Definition 2.6.1. Let $T > 0$. A function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a solution of the Cauchy problem for (HS), if

1. $u \in C([0, T] \times \mathbb{R}; \mathbb{R})$, and $u(0, x) = u_0(x)$;
2. For any $t \in [0, T]$ the map $x \mapsto u(t, x)$ is an absolutely continuous function, and $u_x(t, \cdot) \in L^2(\mathbb{R})$. Furthermore, the map $t \mapsto u_x(t, \cdot)$ lies in the space $L^\infty([0, T]; L^2(\mathbb{R}))$;

3. The map $t \mapsto u(t,\cdot)$ lies in $L^2_{loc}(\mathbb{R})$, is absolutely continuous when restricted to bounded intervals $[a,b] \subset [0,T]$ with respect to the $L^2([a,b])$ metric, and satisfies (HS) for almost every $t \in [0,T]$.

The solution is said to be dissipative if

$$\int_{\mathbb{R}} u_x^2(t_2,x)\,\mathrm{d}x - \int_{\mathbb{R}} u_x^2(t_1,x)\,\mathrm{d}x \\ \leq \int_{t_1}^{t_2}\int_{\mathbb{R}} (u_x^2\varphi_t(t,x) + uu_x^2\varphi_x(t,x))\,\mathrm{d}x\,\mathrm{d}t \quad (2.6.1)$$

for any $t_2 > t_1 > 0$ and any test function $\varphi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$.

The solution concept here is fundamentally different to Definition 2.3.5. Most notably, solutions here may be unbounded in space at any point in time, and the existence of the almost everywhere derivative in time is required for u.

Nonetheless, existence of dissipative solutions was demonstrated in this paper for absolutely continuous initial data with $L^2(\mathbb{R})$ first derivatives, via a method of characteristics.

Shifting focus to the metric, an additional constraint is required. Specifically, the initial data, and hence the solution, must be bounded in space, and lie in the space E as defined in (2.3.7). To begin, the metric space $X = (\mathbb{R}^2 \times (-\frac{\pi}{2}, \frac{\pi}{2}]) \cup \{\infty\}$ is introduced. This space is endowed with a metric,

$$d_X((x,u,w),(\tilde{x},\tilde{u},\tilde{w})) \coloneqq \min\{|x-\tilde{x}| + |u-\tilde{u}| + k_o|w-\tilde{w}|, \\ k_o|\frac{\pi}{2} + w| + k_o|\frac{\pi}{2} + \tilde{w}|\},$$
$$d_X((x,u,w),\infty) \coloneqq k_o|\frac{\pi}{2} + w|,$$

with k_o a sufficiently large constant.

Given a function $u \in H^1_{loc}(\mathbb{R})$, with $L^2(\mathbb{R})$ first derivative, a measure $\mu^u \in \mathcal{M}^+(\mathbb{R})$ is associated, given by

$$\mu^u(\{\infty\}) = 0, \qquad \mu^u(A) = \int_{\{x\in\mathbb{R} | (x,u(x),\arctan u_x(x))\in A\}} u_x^2(x)\,\mathrm{d}x\,,$$

for any Borel set $A \subseteq \mathbb{R}^2 \times (-\frac{\pi}{2}, \frac{\pi}{2}]$.

The metric constructed is described in the context of the Kantorovich–Rubinstein distance. Given u, v in $H^1_{loc}(\mathbb{R})$ with $L^2(\mathbb{R})$ first derivatives, the set of transport plans $\mathcal{F}$ consists of triplets (ψ, ϕ_1, ϕ_2) with

- ψ a strictly increasing and absolutely continuous surjection;

- $\phi_i : \mathbb{R} \to [0,1]$ are simple and Borel measurable functions, for $i = 1, 2$;
- $\phi_1(x)u_x^2(x) = \psi'(x)\phi_2 \circ \psi(x) v_x^2 \circ \psi(x)$ almost everywhere.

From the associated measure μ^u and μ^v the "transportation cost" under a plan (ψ, ϕ_1, ϕ_2) is defined as

$$\begin{aligned} J^{\psi,\phi_1,\phi_2}(u,v) := \int_{\mathbb{R}} & d_X((x, u(x), \arctan \circ u_x(x)), \\ & \qquad (x, v, \arctan \circ v_x) \circ \psi(x))\phi_1(x)u_x^2(x)\,\mathrm{d}x \\ & \int_{\mathbb{R}} d_X((x, u(x), \arctan \circ u_x(x)), \infty)(1-\phi_1(x))u_x^2(x)\,\mathrm{d}x \\ & + \int_{\mathbb{R}} d_X(\infty, (x, v, \arctan \circ v_x) \circ \psi(x)) \\ & \qquad \times (1 - \phi_2 \circ \psi(x))(v_x^2 \circ \psi(x))\psi'(x)\,\mathrm{d}x\,, \end{aligned}$$

and, as usual, the distance is given by optimising (minimising) the cost over all transport plans, i.e.

$$d(u,v) = \inf_{(\psi,\phi_1,\phi_2)\in\mathcal{F}} J^{\psi,\phi_1,\phi_2}(u,v).$$

Comparing solutions using this metric, two key properties are shown. Solutions to (HS) in the sense of Definition 2.6.1 are Lipschitz continuous in time with respect to this metric, i.e. for $t_1, t_2 \in [0, T]$,

$$d(u(t_1), u(t_2)) \le C|t_1 - t_2|,$$

with the constant C dependent on $T, u(0)$ and k_o. Observe that this property is not shown to be satisfied by the metrics constructed in papers 1 and 2.

Additionally, solutions are Lipschitz continuous with respect to initial data, i.e. for u_1, u_2 two solutions in the sense of Definition 2.6.1 with initial data in E,

$$d(u_1(t), u_2(t)) \le e^{2t} d(u_1(0), u_2(0)).$$

A key advantage in the construction of dissipative solutions via a method of characteristics in this way is the lack of the requirement for the additional energy measure μ. This is precisely because in this case the concentrated energy at wave breaking is completely destroyed, or in other words the energy is always given by the density $u_x^2\,\mathrm{d}x$. Furthermore, dissipative solutions are unique [18].

The approach outlined in this paper does not seem well suited for the non-dissipative case of the Hunter–Saxton equation, and furthermore its construction relies on a deep understanding of Wasserstein/Kantorovich-Rubinstein type distances.

Carrillo, Grunert and Holden - A metric via generalised inverses

The following section focuses on the metric constructed in the paper [10]. In said work, a metric is constructed for conservative solutions to the Hunter–Saxton equation via the use of pseudo-inverses, taking inspiration from Wasserstein/Kantorovich-Rubinstein type distances again.

For an element $(u, \mu) \in \mathcal{D}_{HS}$, define the pseudo-inverse $\chi : [0, M] \to \mathbb{R}$ of $F(x) = \mu((-\infty, x))$, with $M = \mu(\mathbb{R})$,

$$\chi(\eta) = \sup\{x \in \mathbb{R} \mid \bar{F}(x) < \eta\},$$

and then define

$$\mathcal{U}(\eta) = u(\chi(\eta)).$$

This alternative change of variables present another way of constructing conservative solutions to the Hunter–Saxton equation. For a conservative solution (u, μ) of the Hunter–Saxton equation the associated time dependent quantities satisfy the ODE system

$$\chi_t(t, \eta) = \mathcal{U}(t, \eta), \quad \mathcal{U}_t(t, \eta) = \frac{1}{2}\eta - \frac{1}{4}M,$$

on the interval $(0, M)$. In the case $\chi(\mathbb{R}) \neq \mathbb{R}$ these are extended continuously at the endpoints.

This system is equivalent to the one given by equations (32) and (33) in [9], which focuses on a generalised version of the Hunter–Saxton equation.

Consider two conservative solutions to the Hunter–Saxton equation, (u_A, μ_A) and (u_B, μ_B), with energies $\mu_A(\mathbb{R}) = M_A$, and $\mu_B(\mathbb{R}) = M_B$. It is shown that the metric

$$\begin{aligned} d((u_A, \mu_A), (u_B, \mu_B)) = {} & \|\chi_A(M_A \cdot) - \chi_B(M_B \cdot)\|_{L^1([0,1])} \\ & + \|\mathcal{U}_A(M_A \cdot) - \mathcal{U}_B(M_B \cdot)\|_{L^\infty([0,1])} + |C_1 - C_2|, \end{aligned}$$

satisfies

$$d((u_A(t), \mu_A(t)), (u_B(t), \mu_B(t)))$$

$$\leq \left(1 + t + \frac{1}{8}t^2\right) d((u_A(0), \mu_A(0)), (u_B(0), \mu_B(0))),$$

whose Lipschitz constant is quadratic rather than exponential, under the constraint

$$\mu_i((-\infty, \cdot)) - M_i \chi^+(\cdot) \in L^1(\mathbb{R}), \qquad \text{for } i = A, B.$$

with χ^+ given in (2.3.5).

Chapter 3

Well-posedness for coupled differential models

Susceptible, infectious and recovered (SIR) models are epidemiological models used to describe the transmission of infectious diseases. The simplest model consists of a coupled system of ODEs, describing three populations at points in time. Susceptible individuals, $S = S(t)$, transfer into the infected population, $I = I(t)$, upon infection, whom then transfer into a recovered population, $R = R(t)$, after said infection has passed. Various extensions and generalisations of said models have been constructed, adapted to specific scenarios. The literature for such models is extremely broad, see [42, Section 1.5.1] and references therein for an overview.

With this motivation in mind, a classical question arises. If the individual component parts of the model are well-posed, under what conditions is the system well-posed? This question can be extended beyond the bounds of SIR models, to cover a wide range of different systems of differential equations.

Article 3 [14] establishes a result that can be used to obtain well-posedness results for systems of problems defined on metric spaces. Metric spaces maintain the concept of distance but lose the structure and operations that are a boon for problems set in vector spaces. The established result is then shown applicable to systems of equations that can consist of some particular ordinary, partial, and measure valued differential equations.

The framework and results established follow from the precursor work [15], whose focus is establishing well-posedness for differential equations in metric spaces, which itself was based on the framework detailed in [5, 40], see also [41]. Said framework differs from, but offers similar results to, the approach of mutational equations in metric spaces established by Aubin [2, 3], and extended by Lorenz [37].

We begin with some formal motivation. Let $I \subseteq \mathbb{R}$ be some interval containing 0, and X a Banach space. A semigroup operator on X is an operator $S : I \times X \to X$ satisfying

$$S_0 = Id, \qquad S_{t+s} = S_t S_s,$$

for t, s, and $t+s$ in I, with Id the identity operator on X. Consider a general autonomous evolution equation defined on the Banach space X, with a well defined solution concept. The existence of a semigroup operator, Lipschitz in both variables, mapping some initial data in X to the solution of the Cauchy problem at time $t \in I$, is often a powerful tool in showing the well-posedness of the problem.

A global process is an extension of the semigroup operator for the non-autonomous setting. It is a mapping dependent on time, the initial time and initial data. Henceforth fix $I \subset \mathbb{R}$ to be some sub interval of the real line, and X some metric space.

Definition 3.0.1. Let $\mathcal{D}$ be a closed subset of X. Fix a family of sets $\{\mathcal{D}_{t_0}\}_{t_0 \in I}$ with $\mathcal{D}_{t_0} \subseteq \mathcal{D}$ for all $t_0 \in I$. Then define the set

$$\mathcal{A} = \{(t, t_0, u) \mid t_0 \leq t, (t_0, t) \subseteq I, \text{ and } u \in \mathcal{D}_{t_0}\}. \tag{3.0.1}$$

A global process on X is a map $P : \mathcal{A} \to X$ satisfying

$$P(t_0, t_0)u = u, \tag{3.0.2a}$$
$$P(t_1, t_0)u \in \mathcal{D}_{t_1}, \tag{3.0.2b}$$
$$P(t_2, t_1) \circ P(t_1, t_0)u = P(t_2, t_0)u, \tag{3.0.2c}$$

for $t_0, t_1, t_2 \in I$ with $t_0 \leq t_1 \leq t_2$, and $u \in \mathcal{D}_{t_0}$.

We will often drop the symbol $\circ$ when performing the composition in a global process for notational simplicity.

We will sometimes use the notation $\mathcal{D}_t^X$ when referring to the associated family of subsets for the process defined on the space X.

In the case of a non-autonomous evolution equation defined on X, with some suitable concept of solution, the existence of a Lipschitz global process is again tied to the well-posedness of the problem.

[15, Theorem 2.6] establishes that a global process can be generated by a local flow under suitable conditions. A local flow can be thought of as a tangent vector field to X.

Definition 3.0.2. Let $\delta > 0$ and $\mathcal{D} \subseteq X$ be closed. A local flow is a continuous map $F : [0, \delta] \times I \times \mathcal{D} \to X$ satisfying, for some $L > 0$,

$$F(0, t_0)u = u, \tag{3.0.3a}$$
$$d(F(\tau, t_0)u, F(\tau', t_0)u') \leq L\left(|\tau - \tau'| + d(u, u')\right), \tag{3.0.3b}$$

for all $\tau, \tau' \in [0, \delta]$, $u, u' \in \mathcal{D}$ and $t_0 \in I$.

Global processes generated in such a way are Lipschitz continuous and have a tangency condition with respect to the local flow. Hence, a schema for obtaining solutions to evolution equations is established. If one is able to define a local flow from a given non-autonomous evolution equation, under suitable conditions a global process may be generated via said result. One then needs to show that this global process is a solution operator, with the boon that one can make use of the tangency condition.

Consider now a system of coupled differential equations. The simplest example is a system of ODEs,

$$\dot{u} = f(t, u, w), \qquad \dot{w} = g(t, u, w),$$

with $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Of course, under suitable assumptions on f and g, the well-posedness of such a problem is established in a variety of literature. Nonetheless, this serves as a good establishing example. We first assume that well-posedness of the individual equations, when w or u are fixed respectively, have been established. In other words a Lipschitz continuous global process has been constructed. Said processes, for the general case, must satisfy the following definition.

Definition 3.0.3. Consider two metric spaces $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$. A Lipschitz process on $\mathcal{U}$, parameterised by $w \in \mathcal{W}$, is a family of maps $P^w : \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$, with

$$\mathcal{I} = \{(t, t_0) \in I^2 \mid t_0 \leq t\}, \tag{3.0.4a}$$
$$\mathcal{A}_{\mathcal{U}} = \{(t, t_0, u) \mid (t, t_0) \in \mathcal{I}, u \in \mathcal{D}^{\mathcal{U}}_{t_0}\} \tag{3.0.4b}$$
$$\mathcal{D}^{\mathcal{U}}_{t_0} \subseteq \mathcal{U}, \tag{3.0.4c}$$

such that for all $w \in \mathcal{W}$, P^w is a global process in the sense of Definition 3.0.1, and said family is Lipschitz continuous with respect to the initial data, time and parameters. That is,

$$d_{\mathcal{U}}(P^w(t, t_0)u_1, P^w(t, t_0)u_2) \leq e^{C_u(t-t_0)} d_{\mathcal{U}}(u_1, u_2), \tag{3.0.5a}$$

$$d_{\mathcal{U}}(P^w(t_1,t_0)u, P^w(t_2,t_0)u) \leq C_t|t_1 - t_2|, \tag{3.0.5b}$$
$$d_{\mathcal{U}}(P^{w_1}(t,t_0)u, P^{w_2}(t,t_0)u) \leq C_w(t-t_0)d_{\mathcal{W}}(w_1,w_2), \tag{3.0.5c}$$

for some constants C_u, C_t and C_w, possible dependent on the sets $\mathcal{D}^{\mathcal{U}}$, $\mathcal{W}$ and I, for all $t_0, t_1, t_2, t \in I$ such that $t_0 \leq \min\{t, t_1, t_2\}$, $w, w_1, w_2 \in \mathcal{W}$, and $u, u_1, u_2 \in \mathcal{D}^{\mathcal{U}}_{t_0}$.

We can couple the two respective processes, and said coupling is a local flow. Supposing that the considered metric spaces are complete, such a local flow can itself generate a global process. This is the central result in the third article, see [14, Theorem 2.6], in which a more detailed version of the following theorem is presented.

Theorem 3.0.4. *Consider two complete metric spaces $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$. Define the product metric $d : (\mathcal{U} \times \mathcal{W})^2 \to \mathbb{R}_+$ by*

$$d((u,w),(u',w')) = d_{\mathcal{U}}(u,u') + d_{\mathcal{W}}(w,w').$$

Let $P^w : \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$ and $P^u : \mathcal{A}_{\mathcal{W}} \to \mathcal{W}$ be two Lipschitz processes parameterised by $\mathcal{W}$ and $\mathcal{U}$ respectively. Then, there exists a local flow $F : \mathcal{A}_F \to \mathcal{U} \times \mathcal{W}$, satisfying

$$F(\tau,t_0)(u,w) = (P^w(t_0+\tau,t_0)u, P^u(t_0+\tau,t_0)w),$$

for all $(\tau, t_0, (u,w)) \in \mathcal{A}_F$, with $\mathcal{A}_F$ given by

$$\mathcal{A}_F = \left\{(\tau, t_0, (u,w)) \mid \tau \geq 0, t_0, t_0+\tau \in I, (u,w) \in \mathcal{D}^{\mathcal{U}}_{t_0} \times \mathcal{D}^{\mathcal{W}}_{t_0}\right\}.$$

This local flow generates a unique global process $P : \mathcal{A} \to \mathcal{U} \times \mathcal{W}$, Lipschitz continuous with respect to $(t, t_0, (u,w)) \in \mathcal{A}$, and satisfying a tangency condition

$$\frac{1}{\tau} d(P(t_0+\tau,t_0)(u,w), F(\tau,t_0)(u,w)) \leq \mathcal{O}(\tau), \tag{3.0.6}$$

for all $(t_0+\tau, t_0, (u,w)) \in \mathcal{A}$, with $\tau \in (0, \delta]$.

The consequences of this result are inherited from [15, Theorem 2.6].

Note. In the cases we consider, the set $\mathcal{A}$ is ensured to be non-empty.

The generated process P is not automatically a solution of the considered problem. This theorem however serves as a starting point to establish existence of solutions via P, and furthermore ensures that solutions constructed by P are unique. The majority of article 3 is then establishing such results for different problems. In fact, we show such a process generates a solution in the coupling of a system of equations consisting of particular

- Ordinary differential equations;
- Semilinear initial value problems;
- Semilinear initial boundary value problems;
- Measure valued balance laws;
- Scalar conservation laws.

We then contextualise the results by applying them to some proposed applications.

It should be noted that the original context of SIR models and their extensions poses an additional challenge. Often, in said models, the initial data for one unknown function is dependent on another unknown function, see for example [16, 36]. Either the scheme currently is not well suited to such situations, or future work could be done to incorporate them.

3.1 An approach excluded from the paper

During the writing of article 3 [14], an approach was constructed for the section "The Boundary Value Problem for a Linear Balance Law", but not included. In the following, we include the details of this approach, which may provide ideas for tackling similar problems in the future. We remark that the content of this chapter are the product of work with the co-authors of article 3.

We begin by introducing the model considered. Fix a subinterval $\hat{I} \subset \mathbb{R}$ containing 0. We consider the boundary value problem

$$\partial_t r(t,x) + \partial_x (v(t,x) r(t,x)) = m(t,x,w) r(t,x) + q(t,x,w), \qquad (3.1.1)$$

for $(t,x) \in [t_0,T] \times \mathbb{R}_+$, under the boundary and initial value conditions

$$\begin{aligned} r(t,0) &= b(t), && \text{for } t \in [t_0,T], \\ r(t_0,x) &= r_0(x), && \text{for } x \in \mathbb{R}_+, \end{aligned}$$

with $0 \leq t_0$, $[t_0,T] \subset \hat{I}$, $r_0 \in (L^1 \cap BV)(\mathbb{R}_+;\mathbb{R})$ and $b \in BV(\hat{I};\mathbb{R})$. Here the PDE is parameterised by the variable w lying in the complete metric space $\mathcal{W}$. The setup and assumptions of this model are developed from [13].

Throughout this section we consider only left-continuous functions in $BV(\mathbb{R})$.

We use the following notion of solution.

Definition 3.1.1. For a fixed $w \in \mathcal{W}$, a function

$$r \in C^0([t_0, T]; L^1(\mathbb{R}_+; \mathbb{R})), \qquad \text{with } r(t) \in BV(\mathbb{R}_+; \mathbb{R}),$$

for *a.e.* $t \in [t_0, T]$ is a solution to the boundary value problem of (3.1.1) if

- The PDE is satisfied in a distributional sense, i.e.

$$\int_{t_0}^{T} \int_{\mathbb{R}_+} \big(r(t,x)\partial_t \varphi(t,x) + v(t,x) r(t,x) \partial_x \varphi(t,x) \\ + (m(t,x,w) r(t,x) + q(t,x,w))\varphi(t,x)\big) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

  for every test function $\varphi \in C_c^\infty((t_0, T) \times (0, +\infty); \mathbb{R})$;

- $r(t_0, x) = r_0(x)$ for *a.e.* $x \in \mathbb{R}_+$;

- $\lim_{x \to 0+} r(t,x) = b(t)$ for *a.e.* $t \in [t_0, T]$.

The first step of the scheme is the construction of a parameterised Lipschitz global process formed via the solution mapping of this problem. We note that the difference here to how the approach was presented in the paper is in the inclusion of the boundary data as one of the components this process acts on. Said idea was proposed because during the analysis we noted that the evolution of the flow was dependent on the boundary data.

The existence of the desired process is due to the following proposition.

Proposition 3.1.2. *Let* $R > 0$ *and* $\mathcal{U} = (L^1 \cap BV)(\mathbb{R}_+; \mathbb{R}) \times BV(\hat{I}; \mathbb{R})$. *Assume the following*

(**BP1**) *There exist positive constants* $\check{v}, \hat{v}, V_L, V_\infty$ *such that* $v \in C^{0,1}(\hat{I} \times \mathbb{R}_+; [\check{v}, \hat{v}])$ *and, for all* $(t,x) \in \hat{I} \times \mathbb{R}_+$,

$$TV(v(\cdot, x); \hat{I}) + TV(v(t, \cdot); \mathbb{R}_+) \leq V_\infty, \\ TV(\partial_x v(t, \cdot); \mathbb{R}_+) + \|\partial_x v(t, \cdot)\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} \leq V_L.$$

(**BP2**) *For all* $w \in \mathcal{W}$, $m(\cdot, \cdot, w) \in C^0(\hat{I} \times \mathbb{R}_+; \mathbb{R})$, *and there exist positive constants* M_∞, M_L *such that, for all* $t \in \hat{I}$ *and* $w, w_1, w_2 \in \mathcal{W}$,

$$TV(m(t, \cdot, w); \mathbb{R}_+) + \|m(t, \cdot, w)\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} \leq M_\infty, \\ \|m(t, \cdot, w_1) - m(t, \cdot, w_2)\|_{L^1(\mathbb{R}_+;\mathbb{R})} \leq M_L d_{\mathcal{W}}(w_1, w_2).$$

(BP3) *For all* $w \in \mathcal{W}$, $q(\cdot,\cdot,w) \in C^0(\hat{I}; L^1(\mathbb{R}_+;\mathbb{R}))$ *and there exist positive constants* Q_1, Q_∞ *and* Q_L *such that, for all* $t \in \hat{I}$ *and* $w, w_1, w_2 \in \mathcal{W}$,

$$\begin{aligned} \|q(t,\cdot,w)\|_{L^1(\mathbb{R}_+;\mathbb{R})} &\le Q_1, \\ TV(q(t,\cdot,w);\mathbb{R}_+) + \|q(t,\cdot,w)\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} &\le Q_\infty, \\ \|q(t,\cdot,w_1) - q(t,\cdot,w_2)\|_{L^1(\mathbb{R}_+;\mathbb{R})} &\le Q_L d_{\mathcal{W}}(w_1,w_2). \end{aligned}$$

Define the metric $d_{\mathcal{U}} : \mathcal{U}^2 \to \mathbb{R}_+$ *by*

$$d_{\mathcal{U}}\left((r,b),(r',b')\right) = \|r - r'\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \hat{v}\|b - b'\|_{L^1(\hat{I};\mathbb{R})}.$$

Setting $\hat{T} = \max \hat{I}$, *define the mappings* $\mathcal{N}^1, \mathcal{N}^\infty, \mathcal{N}^{TV} : \hat{I} \times \mathcal{U} \to \mathbb{R}$ *by*

$$\begin{aligned} \mathcal{N}^1_{t_0}(r,b) &= \|r\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \hat{v}\|b\|_{L^1([t_0,\hat{T}];\mathbb{R})}, \\ \mathcal{N}^\infty_{t_0}(r,b) &= \max\left\{\|r\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}, \|b\|_{L^\infty([t_0,\hat{T}];\mathbb{R})}\right\}, \\ \mathcal{N}^{TV}_{t_0}(r,b) &= TV(r) + |b(t_0+) - r(0+)| + TV(b;(t_0,\hat{T}]). \end{aligned}$$

Then, there exists a local Lipschitz process on $\mathcal{U}$, *defined on the subinterval* $[0,T] \subset \hat{I}$, *parameterised by* $\mathcal{W}$, *whose orbits have as first component the solutions to* (3.1.1) *in the sense of Definition 3.1.1.*

In particular,

$$\mathcal{D} = \left\{(r,b) \in \mathcal{U} \colon \mathcal{N}^1_0(r,b) \le R, \mathcal{N}^\infty_0(r,b) \le R, \mathcal{N}^{TV}_0(r,b) \le R\right\}, \quad (3.1.2)$$

for a suitably large $R > 0$,

$$\mathcal{D}_t = \left\{ (r,b) \in \mathcal{D} : \begin{array}{c} \mathcal{N}^1_t(r,b) \le \alpha_1(t), \\ \mathcal{N}^\infty_t(r,b) \le \alpha_\infty(t), \\ \mathcal{N}^{TV}_t(r,b) \le \alpha_{TV}(t), \\ b(\tau) = 0 \text{ for } \tau \in [0,t) \end{array} \right\}, \quad (3.1.3)$$

with

$$\begin{aligned} \alpha_1(t) &= Re^{-M_\infty(T-t)} - Q_1(T-t)e^{M_\infty t}, \\ \alpha_\infty(t) &= Re^{-(M_\infty+V_L)(T-t)} - Q_\infty(T-t)e^{(M_\infty+V_L)t}, \\ \alpha_{TV}(t) &= R(1 - 2(M_\infty+V_L)(T-t))e^{-(M_\infty+V_L)(T-t)} \\ &\quad - 2Q_\infty(1 + (M_\infty+V_L)t)(T-t)e^{(M_\infty+V_L)t}, \end{aligned}$$

where $T \in \hat{I}$ *is chosen such that* $\alpha_1(0) \ge 0$, $\alpha_\infty(0) \ge 0$ *and* $\alpha_{TV}(0) \ge 0$, *and the Lipschitz constants of Definition 3.0.3 are given by*

$$C_u = M_\infty,$$

$$C_t = \mathcal{L},$$
$$C_w = (2RM_L + 2Q_L + Q_\infty M_L T)\, e^{(M_\infty + V_L)T}.$$

with $\mathcal{L}$ some constant depending on R, T, and the constants in (**BP1**)*,* (**BP2**)*, and* (**BP3**)*.*

Proof. The unique solution of (3.1.1) can be constructed via characteristics. In particular, define $\mathcal{X}(\cdot; t_0, x_0)$ and $\mathcal{T}(\cdot; t_0, x_0)$ as the respective solutions of

$$\dot{x} = v(t,x), \qquad \text{with } x(t_0) = x_0,$$

and

$$t' = \frac{1}{v(t,x)}, \qquad \text{with } t(x_0) = t_0.$$

Then the solution can be written as, see [13, (40)]

$$r(t,x) = \begin{cases} r_0(\mathcal{X}(t_0;t,x))\mathcal{E}_w(t_0,t,x) \\ + \int_{t_0}^t q(\tau, \mathcal{X}(\tau;t,x), w)\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,, & x \geq \mathcal{X}(t;t_0,0), \\ b(\mathcal{T}(0;t,x))\mathcal{E}_w(\mathcal{T}(0;t,x),t,x) \\ + \int_{\mathcal{T}(0;t,x)}^t q(\tau, \mathcal{X}(\tau;t,x), w)\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,, & x < \mathcal{X}(t;t_0,0), \end{cases} \tag{3.1.4}$$

with

$$\mathcal{E}_w(\tau,t,x) = \exp \int_\tau^t \left(m(s, \mathcal{X}(s;t,x), w) - \partial_x v(s, \mathcal{X}(s;t,x))\right) \mathrm{d}s\,.$$

Note that, due to the left continuity of b, $r(t,\cdot)$ is right continuous on the interval $[0, \sigma(t))$. We then define the mapping $P^w : \mathcal{A} \to \mathcal{U}$, parameterised by $w \in \mathcal{W}$, which we endeavour to prove is a process, by

$$P^w(t,t_0)(r_0,b) = (r(t,\cdot), b_t), \qquad \text{with} \qquad b_t(\tau) = b(\tau)\mathbb{1}_{[t,\hat{T}]}(\tau), \tag{3.1.5}$$

with $\mathcal{A}$ generated by the sets $\mathcal{D}_t$.

Note, to ensure that this is a well defined process we disregard, i.e. set to 0, b before the time t. This is okay, as if we consider the boundary value problem with initial data at t, we only care that the solution satisfies the boundary condition for time after t.

The semigroup condition and initial data condition of a global process, see (3.0.2a) and (3.0.2c), are an immediate consequence of the definition. It remains to prove $P^w(t_1,t_0)(r_0,b) \in \mathcal{D}_{t_1}$ for $(r_0,b) \in \mathcal{D}_{t_0}$. For ease of notation, we set $(r(t), b_t) = P^w(t,t_0)(r_0,b)$, with $t \in [t_o, T]$. Furthermore, denote by $\sigma(t)$ the separatrix, i.e.

$$\sigma(t) = \mathcal{X}(t;t_0,0),$$

and we drop notating the w as it has no impact on the calculations.

1. We show that $\mathcal{N}_t^1(r(t), b) \leq \alpha_1(t)$, using the inequality $\mathcal{N}_{t_0}^1(r_0, b) \leq \alpha_1(t_0)$. Recall first the following

$$\begin{aligned}
\partial_{x_0}\mathcal{X}(t;t_0,x_0) &= \exp\int_{t_0}^{t} \partial_x v\left(\tau, \mathcal{X}(\tau;t_0,x_0)\right) \mathrm{d}\tau\,,\\
\partial_{t_0}\mathcal{T}(x;t_0,x_0) &= \frac{v\left(t_0,x_0\right)}{v(\mathcal{T}(x;t_0,x_0),x)}\\
&\qquad \times \exp\int_{t_0}^{\mathcal{T}(x;t_0,x_0)} \partial_x v(\tau, \mathcal{X}(\tau;t_0,x_0))\,\mathrm{d}\tau\,,\\
\partial_{t_0}\mathcal{X}(t;t_0,x_0) &= -v(t_0,x_0)\exp\int_{t_0}^{t}\partial_x v(\tau,\mathcal{X}(\tau;t_0,x_0))\,\mathrm{d}\tau\,,\\
\partial_{x_0}\mathcal{T}(x;t_0,x_0) &= -\frac{1}{v(\mathcal{T}(x;t_0,x_0),x)}\\
&\qquad \times \exp\int_{t_0}^{\mathcal{T}(x;t_0,x_0)} \partial_x v(s,\mathcal{X}(s;t_0,x_0))\,\mathrm{d}s\,.
\end{aligned}$$

Directly using the definition of r, see also [13, (SP.2) in Lemma 3], we obtain

$$\begin{aligned}
&\|r(t)\|_{L^1(\mathbb{R}_+;\mathbb{R})}\\
&\leq \int_0^{\sigma(t)} |b(\mathcal{T}(0;t,x))|\mathcal{E}(\mathcal{T}(0;t,x),t,x)\,\mathrm{d}x \qquad [\text{set } \tau = \mathcal{T}(0;t,x)]\\
&\quad + \int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t} |q(\tau,\mathcal{X}(\tau;t,x))|\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \;[\text{set } \xi = \mathcal{X}(\tau;t,x)]\\
&\quad + \int_{\sigma(t)}^{\infty} |r_0(\mathcal{X}(t_0;t,x))|\mathcal{E}(t_0,t,x)\,\mathrm{d}x \qquad [\text{set } \xi = \mathcal{X}(t_0;t,x)]\\
&\quad + \int_{\sigma(t)}^{\infty}\int_{t_0}^{t} |q(\tau,\mathcal{X}(\tau;t,x))|\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \qquad [\text{set } \xi = \mathcal{X}(\tau;t,x)]\\
&= \int_{t_0}^{t} v(\tau,0)|b(\tau)|e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,0))\mathrm{d}s}\,\mathrm{d}\tau\\
&\quad + \int_{t_0}^{t}\int_0^{\sigma(\tau)} |q(\tau,\xi)|e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,\xi))\mathrm{d}s}\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\quad + \int_0^{\infty} |r_0(\xi)|e^{\int_{t_0}^t m(s,\mathcal{X}(s;t_0,\xi))\mathrm{d}s}\,\mathrm{d}\xi\\
&\quad + \int_{t_0}^{t}\int_{\sigma(\tau)}^{\infty} |q(\tau,\xi)|e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,\xi))\mathrm{d}s}\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leq \left(\hat{v}\,\|b\|_{L^1([t_0,t];\mathbb{R})} + \|r_0\|_{L^1(\mathbb{R}_+;\mathbb{R})}\right)e^{M_\infty(t-t_0)}\\
&\quad + Q_1(t-t_0)e^{M_\infty(t-t_0)},
\end{aligned} \tag{3.1.6}$$

where we have used the substitution on the right when listed. Recall that $\mathcal{N}^1_{t_0}(r_0, b) \leq \alpha_1(t_0)$. Then,

$$\begin{aligned}
\mathcal{N}^1_t(r(t), b_t) &= \|r(t)\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \hat{v}\|b_t\|_{L^1([t,\hat{T}];\mathbb{R})} \\
&\leq \left(\hat{v}\|b\|_{L^1([t_0,t];\mathbb{R})} + \|r_0\|_{L^1(\mathbb{R}_+;\mathbb{R})} + Q_1(t-t_0)\right) e^{M_\infty(t-t_0)} \\
&\quad + \hat{v}\|b_t\|_{L^1([t,\hat{T}];\mathbb{R})} \\
&\leq \mathcal{N}^1_{t_0}(r_0, b)e^{M_\infty(t-t_0)} + Q_1(t-t_0)e^{M_\infty(t-t_0)} \\
&\leq \alpha_1(t_0)e^{M_\infty(t-t_0)} + Q_1(t-t_0)e^{M_\infty(t-t_0)} \\
&= \left(Re^{-M_\infty(T-t_0)} - Q_1(T-t_0)e^{M_\infty t_0}\right) e^{M_\infty(t-t_0)} \\
&\quad + Q_1(t-t_0)e^{M_\infty(t-t_0)} \\
&\leq \alpha_1(t).
\end{aligned}$$

2. We show that $\mathcal{N}^\infty_t(r(t), b_t) \leq \alpha_\infty(t)$, using the inequality $\mathcal{N}^\infty_{t_0}(r_0, b) \leq \alpha_\infty(t_0)$. Using the definition of r, we have

$$\begin{aligned}
&\mathcal{N}^\infty_t(r(t), b_t) \\
&= \max\left\{\|r(t)\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}, \|b_t\|_{L^\infty([t,\hat{T}];\mathbb{R})}\right\} \\
&\leq \max\Big\{ \left(\max\{\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}, \|b\|_{L^\infty([t_0,t];\mathbb{R})}\} + Q_\infty(t-t_0)\right) \\
&\qquad\qquad \times e^{(M_\infty+V_L)(t-t_0)}, \|b_t\|_{L^\infty([t,\hat{T}];\mathbb{R})}\Big\} \\
&\leq \mathcal{N}^\infty_{t_0}(r_0, b)e^{(M_\infty+V_L)(t-t_0)} + Q_\infty(t-t_0)e^{(M_\infty+V_L)(t-t_0)} \\
&\leq \alpha_\infty(t_0)e^{(M_\infty+V_L)(t-t_0)} + Q_\infty(t-t_0)e^{(M_\infty+V_L)(t-t_0)} \\
&\leq \left(Re^{-(M_\infty+V_L)(T-t_0)} - Q_\infty(T-t_0)e^{(M_\infty+V_L)t_0}\right) e^{(M_\infty+V_L)(t-t_0)} \\
&\quad + Q_\infty(t-t_0)e^{(M_\infty+V_L)(t-t_0)} \\
&\leq Re^{-(M_\infty+V_L)(T-t)} - Q_\infty(T-t)e^{(M_\infty+V_L)t} \\
&= \alpha_\infty(t).
\end{aligned} \tag{3.1.7}$$

3. We show that $\mathcal{N}^{TV}_t(r(t), b_t) \leq \alpha_{TV}(t)$. We begin by improving upon the estimate [13, **SP.3** in Lemma 3]. Throughout we make extensive use of the total variation properties found in [14, Lemma A.1]. We have

$$\mathcal{N}^{TV}_t(r(t), b_t) = TV(r(t)) + |b_t(t+) - r(t, 0+)| + TV(b_t; (t, \hat{T}]).$$

Considering the first summand above, we have

$$\begin{aligned}
TV(r(t)) = TV(r(t); [0, \sigma(t))) + |r(t, \sigma(t)+) - r(t, \sigma(t)-)| \\
+ TV(r(t); (\sigma(t), \infty)).
\end{aligned}$$

We compute these three terms separately. First,

$$\begin{aligned}
&TV(r(t);(\sigma(t),\infty))\\
&\le TV(r_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad+\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}e^{(M_\infty+V_L)(t-t_0)}(M_\infty+V_L)(t-t_0)\\
&\quad+(t-t_0)\Big(Q_\infty e^{(M_\infty+V_L)(t-t_0)}\\
&\qquad\qquad+Q_\infty e^{(M_\infty+V_L)(t-t_0)}(M_\infty+V_L)(t-t_0)\Big)\\
&=\big(TV(r_0)+\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}(M_\infty+V_L)(t-t_0)\big)\,e^{(M_\infty+V_L)(t-t_0)}\\
&\quad+Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}. \qquad (3.1.8)
\end{aligned}$$

Using that $0=\mathcal{X}(t_0;t,\sigma(t))$, $t_0=\mathcal{T}(0;t,\sigma(t))$, and the continuity of q, $\mathcal{E}$ and $\mathcal{X}$, we find

$$\begin{aligned}
&|r(t,\sigma(t)+)-r(t,\sigma(t)-)|\\
&\le |r_0(\mathcal{X}(t_0;t,\sigma(t)+))\mathcal{E}(t_0,t,\sigma(t)+)\\
&\quad-b(\mathcal{T}(0;t,\sigma(t)-))\mathcal{E}(\mathcal{T}(0;t,\sigma(t)-),t,\sigma(t)-)|\\
&\quad+\bigg|\int_{t_0}^{t}q(\tau,\mathcal{X}(\tau;t,\sigma(t)+))\mathcal{E}(\tau,t,\sigma(t)+)\,\mathrm{d}\tau\\
&\qquad-\int_{\mathcal{T}(0;t,\sigma(t)-)}^{t}q(\tau,\mathcal{X}(\tau;t,\sigma(t)-))\mathcal{E}(\tau,t,\sigma(t)-)\,\mathrm{d}\tau\bigg|\\
&=|r_0(0+)-b(t_0+)|\,\mathcal{E}\left(t_0,t,\sigma(t)-\right)\\
&\le |r_0(0+)-b(t_0+)|\,e^{(M_\infty+V_L)(t-t_0)}\,. \qquad (3.1.9)
\end{aligned}$$

And finally,

$$\begin{aligned}
&TV(r(t);[0,\sigma(t)))\\
&\le TV(b;(t_0,t])e^{(M_\infty+V_L)(t-t_0)}\\
&\quad+\|b\|_{L^\infty([t_0,t];\mathbb{R})}e^{(M_\infty+V_L)(t-t_0)}(M_\infty+V_L)(t-t_0)\\
&\quad+(t-t_0)\left(Q_\infty e^{(M_\infty+V_L)(t-t_0)}+Q_\infty e^{(M_\infty+V_L)(t-t_0)}(M_\infty+V_L)(t-t_0)\right)\\
&=\big(TV(b;(t_0,t])+\|b\|_{L^\infty([t_0,t];\mathbb{R})}(M_\infty+V_L)(t-t_0)\big)\,e^{(M_\infty+V_L)(t-t_0)}\\
&\quad+Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}. \qquad (3.1.10)
\end{aligned}$$

Adding (3.1.8), (3.1.9) and (3.1.10) we have

$$\begin{aligned}
TV(r(t))\le{}&\left(TV(r_0)+|r_0(0+)-b(t_0+)|+TV(b;(t_0,t])\right)e^{(M_\infty+V_L)(t-t_0)}\\
&+(M_\infty+V_L)\left(\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}+\|b\|_{L^\infty([t_0,t];\mathbb{R})}\right)
\end{aligned}$$

$$\times (t-t_0)e^{(M_\infty+V_L)(t-t_0)}$$
$$+ 2Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}.$$

Hence, as $r_0(0+) = b(t) = b(t-)$, we obtain

$$\begin{aligned}
&\mathcal{N}_t^{TV}(r(t), b_t)\\
&= TV(r(t)) + |b_t(t+) - r(t,0+)| + TV(b_t;(t,\hat{T}])\\
&\leq \left(TV(r_0) + |r_0(0+) - b(t_0+)| + TV(b;(t_0,t])\right)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + (M_\infty+V_L)\left(\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} + \|b\|_{L^\infty([t_0,t];\mathbb{R})}\right)(t-t_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + \Big(|b(t+) - b(t-)| + TV(b;(t,\hat{T}])\Big)e^{(M_\infty+V_L)(t-t_0)}\\
&\leq \Big(TV(r_0) + |r_0(0+) - b(t_0+)| + TV(b;(t_0,\hat{T}])\Big)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + (M_\infty+V_L)\left(\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} + \|b\|_{L^\infty([t_0,t];\mathbb{R})}\right)(t-t_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}\\
&\leq \mathcal{N}_{t_0}^{TV}(r_0, b)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2(M_\infty+V_L)\Big(Re^{-(M_\infty+V_L)(T-t_0)} - Q_\infty(T-t_0)e^{(M_\infty+V_L)t_0}\Big)\\
&\qquad \times (t-t_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}\\
&\leq Re^{-(M_\infty+V_L)(T-t_0)}(1-2(M_\infty+V_L)(T-t_0))e^{(M_\infty+V_L)(t-t_0)}\\
&\quad - 2Q_\infty e^{(M_\infty+V_L)t_0}(1+(M_\infty+V_L)t_0)(T-t_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2(M_\infty+V_L)\Big(Re^{-(M_\infty+V_L)(T-t_0)} - Q_\infty(T-t_0)e^{(M_\infty+V_L)t_0}\Big)\\
&\qquad \times (t-t_0)e^{(M_\infty+V_L)(t-t_0)}\\
&\quad + 2Q_\infty(t-t_0)\left(1+(M_\infty+V_L)(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}\\
&\leq \alpha_{TV}(t).
\end{aligned}$$

It remains to prove the Lipschitz stability estimates required of a Lipschitz parameterised process, see (3.0.5).

1. Lipschitz Continuity in Time The L^1, L^∞ and TV bounds above allow us to apply [13, **(SP.6)** in Lemma 3]. Hence, we know that P^w is Lipschitz continuous in time with respect to the L^1 norm and the Lipschitz constant C_t depends on R, $\hat{T}$ and on the constants in (**BP1**), (**BP2**) and (**BP3**).

2. Lipschitz Continuity w.r.t. Initial and Boundary Data. Denote $(\bar{r}(t), \bar{b}_t) = P(t,t_0)(\bar{r}_0, \bar{b})$, with $(\bar{r}_0, \bar{b}) \in \mathcal{D}_t$. Again using (3.1.4), and via the same calculations used to obtain estimate (3.1.6),

$$\begin{aligned}
&\|r(t,\cdot) - \bar{r}(t,\cdot)\|_{L^1((0,\sigma(t));\mathbb{R})} \\
&= \int_0^{\sigma(t)} |b(\mathcal{T}(0;t,x)) - \bar{b}(\mathcal{T}(0;t,x))| \mathcal{E}(\mathcal{T}(0;t,x),t,x)\,\mathrm{d}x \\
&\le \hat{v} e^{M_\infty (t-t_0)} \|b - \bar{b}\|_{L^1([t_0,t];\mathbb{R})},
\end{aligned}$$

and

$$\begin{aligned}
&\|r(t,\cdot) - \bar{r}(t,\cdot)\|_{L^1((\sigma(t),\infty);\mathbb{R})} \\
&= \int_{\sigma(t)}^{\infty} |r_0(\mathcal{X}(t_0;t,x)) - \bar{r}_0(\mathcal{X}(t_0;t,x))| \mathcal{E}(t_0;t,x)\,\mathrm{d}x \\
&\le e^{M_\infty (t-t_0)} \|r_0 - \bar{r}_0\|_{L^1(\mathbb{R}_+;\mathbb{R})}.
\end{aligned}$$

Combining these estimates, we find

$$\begin{aligned}
&d(P(t,t_0)(r_0,b), P(t,t_0)(\bar{r}_0,\bar{b})) \\
&= \|r(t,\cdot) - \bar{r}(t,\cdot)\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \hat{v}\|b_t - \bar{b}_t\|_{L^1([t,\hat{T}];\mathbb{R})} \\
&\le e^{M_\infty (t-t_0)} (\|r_0 - \bar{r}_0\|_{L^1(\mathbb{R}_+;\mathbb{R})} + \hat{v}\|b - \bar{b}\|_{L^1([t_0,\hat{T}];\mathbb{R})}) \\
&\le e^{M_\infty (t-t_0)} d((r_0,b),(\bar{r}_0,\bar{b})),
\end{aligned}$$

as required.

3. Lipschitz Continuity w.r.t. Parameters. Let $w, \bar{w} \in \mathcal{W}$, and $(t_0, t, (r_0, b)) \in \mathcal{A}$. We now use the notation $r(t)$ and $\bar{r}(t)$ for the first component of $P^w(t,t_0)(r_0,b)$ and $P^{\bar{w}}(t,t_0)(r_0,b)$ respectively. To begin, we have

$$\begin{aligned}
&d_{\mathcal{U}}\left(P^w(t,t_0)(r_0,b), P^{\bar{w}}(t,t_0)(r_0,b)\right) \\
&= \|r(t) - \bar{r}(t)\|_{L^1(\mathbb{R}_+;\mathbb{R})} \\
&\le \int_0^{\sigma(t)} |b\left(\mathcal{T}(0;t,x)\right)| \\
&\qquad \times |\mathcal{E}_w\left(\mathcal{T}(0;t,x),t,x\right) - \mathcal{E}_{\bar{w}}\left(\mathcal{T}(0;t,x),t,x\right)|\,\mathrm{d}x \qquad (3.1.11) \\
&\quad + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^{t} |q\left(\tau, \mathcal{X}(\tau;t,x), w\right) \mathcal{E}_w(\tau,t,x) \\
&\qquad\qquad - q\left(\tau, \mathcal{X}(\tau;t,x), \bar{w}\right) \mathcal{E}_{\bar{w}}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x \qquad (3.1.12) \\
&\quad + \int_{\sigma(t)}^{\infty} |r_0\left(\mathcal{X}(t_0;t,x)\right)| |\mathcal{E}_w(t_0,t,x) - \mathcal{E}_{\bar{w}}(t_0,t,x)|\,\mathrm{d}x \qquad (3.1.13)
\end{aligned}$$

$$+\int_{\sigma(t)}^{\infty}\int_{t_0}^{t}|q\left(\tau,\mathcal{X}(\tau;t,x),w\right)\mathcal{E}_w(\tau,t,x)\\ -q\left(\tau,\mathcal{X}(\tau;t,x),\bar{w}\right)\mathcal{E}_{\bar{w}}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x\,. \qquad (3.1.14)$$

Focusing on term (3.1.11), we use the substitution $\eta(x)=\mathcal{T}(0;t,x)$, implying

$$x=\mathcal{X}(\mathcal{T}(x;\eta(x),0);\eta(x),0)=\mathcal{X}(t;\eta(x),0),$$

and

$$t_0=\mathcal{T}(0;t,\mathcal{X}(t;t_0,0))=\mathcal{T}(0;t,\sigma(t)),$$

thus

$$\int_0^{\sigma(t)}|b\left(\mathcal{T}(0;t,x)\right)|\left|\mathcal{E}_w\left(\mathcal{T}(0;t,x),t,x\right)-\mathcal{E}_{\bar{w}}\left(\mathcal{T}(0;t,x),t,x\right)\right|\mathrm{d}x$$
$$=\int_{t_0}^{t}|v(\eta,0)||b(\eta)|\\ \times\left|\exp\int_\eta^t m(s,\mathcal{X}(s;\eta,0),w)\,\mathrm{d}s-\exp\int_\eta^t m(s,\mathcal{X}(s;\eta,0),\bar{w})\,\mathrm{d}s\right|\mathrm{d}\eta$$
$$\leq e^{M_\infty(t-t_0)}R\\ \times\int_{t_0}^{t}|v(\eta,0)|\int_\eta^t|m(s,\mathcal{X}(s;\eta,0),w)-m(s,\mathcal{X}(s;\eta,0),\bar{w})|\,\mathrm{d}s\,\mathrm{d}\eta$$
$$=e^{M_\infty(t-t_0)}R\int_{t_0}^{t}\int_0^{\mathcal{X}(s;t_0,0)}|m(s,\xi,w)-m(s,\xi,\bar{w})|\\ \times\exp\int_s^{\mathcal{T}(0;s,\xi)}\partial_x v(\tau,\mathcal{X}(\tau;s,\xi))\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}s$$
$$\leq M_L R e^{(M_\infty+V_L)(t-t_0)}(t-t_0)d_{\mathcal{W}}(w,\bar{w}),$$

where we have used (**BP1**), (**BP2**), and that, as the initial data is in $\mathcal{D}_{t_0}$, $\|b\|_{L^\infty(\hat{I};\mathbb{R})}\leq R$.

For term (3.1.12)

$$\int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t}|q\left(\tau,\mathcal{X}(\tau;t,x),w\right)\mathcal{E}_w(\tau,t,x)\\ -q\left(\tau,\mathcal{X}(\tau;t,x),\bar{w}\right)\mathcal{E}_{\bar{w}}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x$$
$$=\int_{t_0}^{t}\int_0^{\sigma(\tau)}|q\left(\tau,y,w\right)\exp\int_\tau^t m(s,\mathcal{X}(s;\tau,y),w)\,\mathrm{d}s\\ -q\left(\tau,y,\bar{w}\right)\exp\int_\tau^t m(s,\mathcal{X}(s;\tau,y),\bar{w})\,\mathrm{d}s\,|\,\mathrm{d}y\,\mathrm{d}\tau$$

$$
\begin{aligned}
&\le e^{M_\infty(t-t_0)}\int_{t_0}^{t}\int_0^{\sigma(\tau)} |q(\tau,y,w)-q(\tau,y,\bar{w})|\,\mathrm{d}y\,\mathrm{d}\tau\\
&\quad+\int_{t_0}^{t}\int_0^{\sigma(\tau)}|q(\tau,y,\bar{w})|\\
&\quad\times\left|\exp\int_\tau^t m(s,\mathcal{X}(s;\tau,y),w)\,\mathrm{d}s-\exp\int_\tau^t m(s,\mathcal{X}(s;\tau,y),\bar{w})\,\mathrm{d}s\right|\mathrm{d}y\,\mathrm{d}\tau\\
&\le Q_L e^{M_\infty(t-t_0)}(t-t_0)d_{\mathcal{W}}(w,\bar{w})\\
&\quad+Q_\infty e^{M_\infty(t-t_0)}\\
&\quad\times\int_{t_0}^{t}\int_0^{\sigma(\tau)}\int_\tau^t |m(s,\mathcal{X}(s;\tau,y),w)-m(s,\mathcal{X}(s;\tau,y),\bar{w})|\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\tau\\
&\le Q_L e^{M_\infty(t-t_0)}(t-t_0)d_{\mathcal{W}}(w,\bar{w})\\
&\quad+Q_\infty e^{(M_\infty+V_L)(t-t_0)}\int_{t_0}^{t}\int_\tau^t\int_{\mathbb{R}_+}|m(s,\xi,w)-m(s,\xi,\bar{w})|\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}\tau\\
&\le\left(Q_L+\frac{1}{2}Q_\infty M_L(t-t_0)\right)e^{(M_\infty+V_L)(t-t_0)}(t-t_0)d_{\mathcal{W}}(w,\bar{w}),
\end{aligned}
$$

where we have used (**BP1**), (**BP2**), and (**BP3**).

For term (3.1.13), we use a similar scheme and obtain

$$
\begin{aligned}
&\int_{\sigma(t)}^{\infty}|r_0(\mathcal{X}(t_0;t,x))|\,|\mathcal{E}_w(t_0,t,x)-\mathcal{E}_{\bar{w}}(t_0,t,x)|\,\mathrm{d}x\\
&=\int_0^\infty |r_0(y)|\Bigg|\exp\int_{t_0}^t m(s,\mathcal{X}(s;t_0,y),w)\,\mathrm{d}s\\
&\qquad\qquad-\exp\int_{t_0}^t m(s,\mathcal{X}(s;t_0,y),\bar{w})\,\mathrm{d}s\Bigg|\,\mathrm{d}y\\
&\le Re^{M_\infty(t-t_0)}\int_0^\infty\int_{t_0}^t|m(s,\mathcal{X}(s;t_0,y),w)-m(s,\mathcal{X}(s;t_0,y),\bar{w})|\,\mathrm{d}s\,\mathrm{d}y\\
&\le RM_L e^{(M_\infty+V_L)(t-t_0)}(t-t_0)d_{\mathcal{W}}(w,\bar{w}),
\end{aligned}
$$

where we have used (**BP1**), (**BP2**), and that, as the initial data lies in $\mathcal{D}$, $\|r_0\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}\le R$.

Finally, for term (3.1.14),

$$
\begin{aligned}
&\int_{\sigma(t)}^{\infty}\int_{t_0}^{t}|q(\tau,\mathcal{X}(\tau;t,x),w)\mathcal{E}_w(\tau,t,x)\\
&\qquad\qquad-q(\tau,\mathcal{X}(\tau;t,x),\bar{w})\mathcal{E}_{\bar{w}}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x\\
&\le\int_{t_0}^{t}\int_{\sigma(\tau)}^{\infty}|q(\tau,y,w)\exp\int_\tau^t m(s,\mathcal{X}(s;\tau,y),w)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - q\,(\tau, y, \bar{w}) \exp \int_{\tau}^{t} m(s, \mathcal{X}(s;\tau,y), \bar{w})\,\mathrm{d}s \,|\, \mathrm{d}y\,\mathrm{d}\tau \\
&\le e^{M_\infty (t-t_0)} \int_{t_0}^{t} \int_{\sigma(\tau)}^{\infty} |q(\tau,y,w) - q(\tau,y,\bar{w})|\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad + Q_\infty e^{M_\infty (t-t_0)} \\
&\qquad \times \int_{t_0}^{t} \int_{\sigma(\tau)}^{\infty} \int_{\tau}^{t} |m(\tau, \mathcal{X}(s;\tau,y), w) - m(\tau, \mathcal{X}(s;\tau,y), \bar{w})|\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\tau \\
&\le Q_L e^{M_\infty (t-t_0)} (t-t_0) d_{\mathcal{W}}(w,\bar{w}) \\
&\quad + Q_\infty M_L \frac{1}{2} e^{(M_\infty + V_L)(t-t_0)} (t-t_0)^2 d_{\mathcal{W}}(w,\bar{w}) \\
&= \left(Q_L + Q_\infty M_L \frac{1}{2}(t-t_0) \right) e^{(M_\infty + V_L)(t-t_0)} (t-t_0) d_{\mathcal{W}}(w,\bar{w}),
\end{aligned}
$$

where we have used (**BP1**), (**BP2**), and (**BP3**).

Combining these four estimates together, we have

$$
d_{\mathcal{U}}\left(P^{w}(t,t_0)(r_0,b), P^{\bar{w}}(t,t_0)(r_0,b)\right) \le C_w (t-t_0) d_{\mathcal{W}}(w,\bar{w}),
$$

with

$$
C_w = (2RM_L + 2Q_L + Q_\infty M_L T)\, e^{(M_\infty + V_L)T},
$$

as required. □

The next step is to ensure that this process, when coupled with another, satisfies the PDE in some sense. This is given in the next result.

Proposition 3.1.3. *Let $\mathcal{U} = (L^1 \cap BV)(\mathbb{R}_+;\mathbb{R}) \times BV(\hat{I};\mathbb{R})$, and assume* (**BP1**), (**BP2**) *and* (**BP3**). *Let P^u be some Lipschitz process on $\mathcal{W}$, parameterised by $u = (r,b) \in \mathcal{U}$. Set $P : \mathcal{A} \to \mathcal{U} \times \mathcal{W}$, with $P = ((P_{1,r}, P_{1,b}), P_2)$, to be the process generated by Theorem 3.0.4 by the coupling of the process P^w from Proposition 3.1.2 with P^u.*

Then, for $(t, t_0, (u_0, w_0)) \in \mathcal{A}$,

$$
P_{1,b}(t,t_0)((r_0,b_0),w_0)(\cdot) = b_0(\cdot)\mathbb{1}_{[t,T]}(\cdot),
$$

and the function $r : [t_0,T] \to (L^1 \cap BV)(\mathbb{R}_+;\mathbb{R})$, given by

$$
r(t,\cdot) = P_{1,r}(t,t_0)((r_0,b_0),w_0),
$$

is a solution, in the sense of Definition 3.1.1, to

$$
\partial_t r + \partial_x (v(t,x) r) = \bar{m}(t,x) r + \bar{q}(t,x), \tag{3.1.15}
$$

for $(t,x) \in [t_0,T] \times \mathbb{R}_+$, *under the boundary and initial value conditions*

$$r(t,0) = b_0(t), \qquad \textit{for } t \in [t_0,T],$$
$$r(t_0,x) = r_0(x), \qquad \textit{for } x \in \mathbb{R}_+,$$

where

$$\bar{m}(t,x) = m(t,x,P_2(t,t_0)((r_0,b_0),w_0)),$$
$$\bar{q}(t,x) = q(t,x,P_2(t,t_0)((r_0,b_0),w_0)).$$

Proof. We begin by showing

$$P_{1,b}(t,t_0)((r_0,b_0),w_0)(\cdot) = b_0(\cdot)\mathbb{1}_{[t,T]}(\cdot).$$

First, we recall that, via Theorem 3.0.4, the global process P inherits tangency conditions with respect to its constituent parts via the generating local flow. In other words, $h \mapsto P_1(s+h,s)$ and $h \mapsto P^w(s+h,s)$ are first order tangent at any admissible initial data, i.e.

$$\frac{1}{\tau} d_{\mathcal{U}}(P_1(t_0+\tau,t_0)(u,w), P^w(t_0+\tau,t_0)u) \le \mathcal{O}(\tau), \tag{3.1.16}$$

and hence converges to zero as $\tau \to 0+$.

For simplicity, we write $P^{w_0}(t,t_0)(r_0,b_0) = (r_t,b_t)$, and notice that by definition for any admissible r_0 and w_0,

$$b_t(s) = b_0(s)\mathbb{1}_{[t,T]}(s) \qquad \text{for all } s \in [0,T].$$

Note, in this notation $b_{t_0} = b_0$. Further, as $b_0 \in L^\infty([0,T];\mathbb{R})$, the map $t \mapsto b_t$ is Lipschitz continuous in $L^1([0,T];\mathbb{R})$.

We can thus define a process Π_b on the admissible b by

$$\Pi_b(t,t_0)b_0 = b_t.$$

Π_b is then $L^1([0,T];\mathbb{R})$ Lipschitz continuous in time and with respect to b_0 with Lipschitz constant $L = \max\{\|b_0\|_{L^\infty([0,T];\mathbb{R})}, 1\}$.

Setting $((r(s),b(s)),w(s)) = P(s,t_0)((r_0,b_0),w_0)$, our goal is thus to show $b(t) = \Pi_b(t,t_0)b_0$. We can apply [4, Theorem 2.9] to obtain

$$\begin{aligned}
&\|b(t) - \Pi_b(t,t_0)b_0\|_{L^1([0,T];\mathbb{R})} \\
&\le L \int_{t_0}^{t} \liminf_{h\to 0+} \frac{1}{h} \|b(s+h) - \Pi_b(s+h,s)b(s)\|_{L^1([0,T];\mathbb{R})} \,\mathrm{d}s \\
&\le \frac{L}{\hat{v}} \int_{t_0}^{t} \liminf_{h\to 0+} \frac{1}{h} d_{\mathcal{U}}\big(P_1(s+h,s)((r(s),b(s)),w(s)),
\end{aligned}$$

$$P^{w(s)}(s+h,s)(r(s),b(s))\big)\,\mathrm{d}s$$
$$= 0,$$

where in the last step we have used (3.1.16).

It remains to show that r is a solution of (3.1.15). Fix some arbitrary $t_0 \in [0,T]$, $(\rho_0, b_0) \in \mathcal{U}$, and $w_0 \in \mathcal{W}$. As $\bar{m}$ and $\bar{q}$ inherit analogous assumptions to those of (**BP2**) and (**BP3**), the existence of a process

$$\Pi : \{(s,s_0) \in [t_0,T]^2 \mid s \geq s_0\} \times (L^1 \cap BV)(\mathbb{R}_+;\mathbb{R}) \to (L^1 \cap BV)(\mathbb{R}_+;\mathbb{R}),$$

satisfying

$$\begin{cases} \partial_t \rho + \partial_x (v(t,x)\rho) = \bar{m}(t,x)\rho + \bar{q}(t,x), & (t,x) \in [s_0,T] \times \mathbb{R}_+, \\ \rho(t,0) = b_0(t), & t \in [s_0,T], \\ \rho(s_0,x) = \rho_0(x), & x \in \mathbb{R}_+, \end{cases} \tag{3.1.17}$$

is given by [13]. Recalling the proof of Proposition 3.1.2, Π is Lipschitz continuous in time and with respect to initial data in the set $\mathcal{D}$. Note that choosing $\rho_0 = r_0$ and $s_0 = t_0$, Π gives the solution of (3.1.15). We thus wish to show $\Pi(t,t_0)r_0 = r(t)$ for any admissible $t > 0$, with $r(t) = P_{1,r}(t,t_0)r_0$.

Now, via [4, Theorem 2.9],

$$\begin{aligned} &\|r(t) - \Pi(t,t_0)r_0\|_{L^1(\mathbb{R}_+;\mathbb{R})} \\ &\leq L_\Pi \int_{t_0}^{t} \liminf_{h\to 0+} \frac{1}{h}\|r(\tau+h) - \Pi(\tau+h,\tau)r(\tau)\|_{L^1(\mathbb{R}_+;\mathbb{R})}\,\mathrm{d}\tau\,, \end{aligned}$$

with L_Π the Lipschitz constant of Π. Note that, setting $u_0 = (r_0, b_0)$,

$$r(\tau+h) = P_{1,r}(\tau+h,t_0)(u_0,w_0) = P_{1,r}(\tau+h,\tau)P(\tau,t_0)(u_0,w_0),$$

making use of the semigroup condition for global processes. Then, from the tangency condition (3.0.6), for $\tau \in [t_0,t]$ and $0 < h \leq |t-t_0|$,

$$\begin{aligned} &\frac{1}{h}\|P_{1,r}(\tau+h,\tau)P(\tau,t_0)(u_0,w_0) \\ &\qquad - P^{P_2(\tau,t_0)(u_0,w_0)}(\tau+h,\tau)(r(\tau),b_\tau)\|_{L^1(\mathbb{R}_+;\mathbb{R})} \\ &\leq \frac{1}{h} d\big(P(\tau+h,\tau)P(\tau,t_0)(u_0,w_0), \\ &\qquad (P^{P_2(\tau,t_0)(u_0,w_0)}(\tau+h,\tau)P_1(\tau,t_0)(u_0,w_0), \\ &\qquad P^{P_1(\tau,t_0)(u_0,w_0)}(\tau+h,\tau)P_2(\tau,t_0)(u_0,w_0))\big) \end{aligned}$$

$$\leq \mathcal{O}(h).$$

So, via the triangle inequality

$$\begin{aligned}
&\liminf_{h\to 0+} \frac{1}{h} \|r(\tau+h) - \Pi(\tau+h,\tau) r(\tau)\|_{L^1(\mathbb{R}_+;\mathbb{R})} \\
&\leq \liminf_{h\to 0+} \frac{1}{h} \|P^{P_2(\tau,t_0)(u_0,w_0)}(\tau+h,\tau)(r(\tau),b_\tau) \\
&\qquad\qquad - \Pi(\tau+h,\tau) r(\tau)\|_{L^1(\mathbb{R}_+;\mathbb{R})}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|r(t) - \Pi(t,t_0) r_0\|_{L^1(\mathbb{R}_+;\mathbb{R})} \\
&\leq L_\Pi \int_{t_0}^t \liminf_{h\to 0+} \frac{1}{h} \|P^{P_2(\tau,t_0)(u_0,w_0)}(\tau+h,\tau)(r(\tau),b_\tau) \\
&\qquad\qquad - \Pi(\tau+h,\tau) r(\tau)\|_{L^1(\mathbb{R}_+;\mathbb{R})} \,\mathrm{d}\tau\,.
\end{aligned} \tag{3.1.18}$$

Our goal is now to obtain estimates for the integrand to demonstrate that the limit is zero.

Again, using [13, (40)], we have a solution formula for Π. In particular,

$$\Pi(t,s_0)(\rho_0,b_0) = \begin{cases} \rho_0(\mathcal{X}(s_0;t,x))\bar{\mathcal{E}}(s_0,t,x) \\ + \int_{s_0}^t \bar{q}(\tau,\mathcal{X}(\tau;t,x))\bar{\mathcal{E}}(\tau,t,x)\,\mathrm{d}\tau\,, & x \geq \mathcal{X}(t;s_0,0), \\ b_0(\mathcal{T}(0;t,x))\bar{\mathcal{E}}(\mathcal{T}(0;t,x),t,x) \\ + \int_{\mathcal{T}(0;t,x)}^t \bar{q}(\tau,\mathcal{X}(\tau;t,x))\bar{\mathcal{E}}(\tau,t,x)\,\mathrm{d}\tau\,, & x < \mathcal{X}(t;s_0,0). \end{cases}$$

with

$$\bar{\mathcal{E}}(s_0,t,x) = \exp \int_{s_0}^t (\bar{m}(s,\mathcal{X}(s;t,x)) - \partial_x v(s,\mathcal{X}(s;t,x)))\,\mathrm{d}s\,.$$

Thus, combining this with the solution formula (3.1.4), we have, denoting $\sigma_1(\tau) = \mathcal{X}(\tau+h;\tau,0)$ and $w_\tau = P_2(\tau,t_0)(u_0,w_0)$,

$$\begin{aligned}
&\|P^{w_\tau}(\tau+h,\tau)(r(\tau),b_\tau) - \Pi(\tau+h,\tau) r(\tau)\|_{L^1([\sigma_1(\tau),+\infty);\mathbb{R})} \\
&= \int_{\sigma_1(t)}^\infty |r(\tau,\mathcal{X}(\tau;\tau+h,x))| |\bar{\mathcal{E}}(\tau,\tau+h,x) - \mathcal{E}_{w_\tau}(\tau,\tau+h,x)|\,\mathrm{d}x \\
&\quad + \int_{\sigma_1(\tau)}^\infty \int_\tau^{\tau+h} \big|q(s,\mathcal{X}(s;\tau+h,x),w_\tau)\mathcal{E}_{w_\tau}(s,\tau+h,x) \\
&\qquad\qquad - \bar{q}(s,\mathcal{X}(s;\tau+h,x))\bar{\mathcal{E}}(s,\tau+h,x)\big|\,\mathrm{d}s\,\mathrm{d}x\,. \\
&\leq R \|\mathcal{E}_{w_\tau}(\tau,\tau+h,\cdot) - \bar{\mathcal{E}}(\tau,\tau+h,\cdot)\|_{L^1([\sigma_1(\tau),+\infty);\mathbb{R})}
\end{aligned} \tag{3.1.19}$$

$$
\begin{aligned}
&+ \int_{\sigma_1(\tau)}^{\infty} \int_{\tau}^{\tau+h} \big| q(s, \mathcal{X}(s;\tau+h,x), w_\tau) \mathcal{E}_{w_\tau}(s,\tau+h,x) \\
&\qquad\qquad - \bar{q}(s, \mathcal{X}(s;\tau+h,x)) \bar{\mathcal{E}}(s,\tau+h,x) \big| \, \mathrm{d}s \, \mathrm{d}x \,, \quad (3.1.20)
\end{aligned}
$$

using that $\|r(\tau)\|_\infty \leq R$.

Considering (3.1.19), we have

$$
\begin{aligned}
&\|\mathcal{E}_{w_\tau}(\tau,\tau+h,\cdot) - \bar{\mathcal{E}}(\tau,\tau+h,\cdot)\|_{L^1([\sigma_1(\tau),+\infty);\mathbb{R})} \\
&= \int_{\sigma_1(\tau)}^{\infty} \left| \exp \int_{\tau}^{\tau+h} (m(s,\mathcal{X}(s;\tau+h,x), w_\tau) - \partial_x v(s, \mathcal{X}(s;\tau+h,x))) \, \mathrm{d}s \right. \\
&\left. - \exp \int_{\tau}^{\tau+h} (m(s,\mathcal{X}(s;\tau+h,x), w_s) - \partial_x v(s, \mathcal{X}(s;\tau+h,x))) \, \mathrm{d}s \right| \mathrm{d}x \\
&\leq e^{(M_\infty + V_L)T} \int_{\sigma_1(\tau)}^{\infty} \int_{\tau}^{\tau+h} |m(s,\mathcal{X}(s;\tau+h,x), w_\tau) \\
&\qquad\qquad - m(s,\mathcal{X}(s;\tau+h,x), w_s)| \, \mathrm{d}s \, \mathrm{d}x \\
&\leq e^{(M_\infty + 2V_L)T} \int_{\tau}^{\tau+h} M_L d_{\mathcal{W}}(P_2(s,t_0)(u_0,w_0), P_2(\tau,t_0)(u_0,w_0)) \, \mathrm{d}s \\
&\leq e^{(M_\infty + 2V_L)T} M_L \mathrm{Lip}(P) \int_{\tau}^{\tau+h} (s-\tau) \, \mathrm{d}s \\
&\leq e^{(M_\infty + 2V_L)T} M_L \mathrm{Lip}(P) \frac{h^2}{2}.
\end{aligned}
$$

For (3.1.20), we find

$$
\begin{aligned}
&\int_{\sigma_1(\tau)}^{\infty} \int_{\tau}^{\tau+h} \big| q(s, \mathcal{X}(s;\tau+h,x), w_\tau) \mathcal{E}_{w_\tau}(s,\tau+h,x) \\
&\qquad\qquad - \bar{q}(s, \mathcal{X}(s;\tau+h,x)) \bar{\mathcal{E}}(s,\tau+h,x) \big| \, \mathrm{d}s \, \mathrm{d}x \\
&\leq \int_{\tau}^{\tau+h} \int_{\sigma_1(\tau)}^{\infty} \big| q(s, \mathcal{X}(s;\tau+h,x), w_\tau) - \bar{q}(s, \mathcal{X}(s;\tau+h,x)) \big| \\
&\qquad\qquad \times \mathcal{E}_{w_\tau}(s,\tau+h,x) \, \mathrm{d}x \, \mathrm{d}s \\
&+ \int_{\tau}^{\tau+h} \int_{\sigma_1(\tau)}^{\infty} |\bar{q}(s, \mathcal{X}(s;\tau+h,x)| \\
&\qquad\qquad \times \big| \mathcal{E}_{w_\tau}(s,\tau+h,x) - \bar{\mathcal{E}}(s,\tau+h,x) \big| \, \mathrm{d}x \, \mathrm{d}s \\
&\leq e^{(M_\infty + 2V_L)T} \int_{\tau}^{\tau+h} \int_{0}^{\infty} \big| q(s,x,w_\tau) - \bar{q}(s,x) \big| \, \mathrm{d}x \, \mathrm{d}s \\
&\quad + e^{(M_\infty + 2V_L)T} M_L Q_\infty \mathrm{Lip}(P) \frac{h^3}{2} \\
&\leq e^{(M_\infty + 2V_L)T} Q_L \mathrm{Lip}(P) \frac{h^2}{2} + e^{(M_\infty + 2V_L)T} M_L Q_\infty \mathrm{Lip}(P) \frac{h^3}{2}.
\end{aligned}
$$

One can similarly show that

$$\|P^{w_\tau}(\tau+h,\tau)(r(\tau),b_\tau) - \Pi(\tau+h,\tau)r(\tau)\|_{L^1([0,\sigma_1(\tau));\mathbb{R})} \leq C\frac{h^2}{2},$$

for some constant $C > 0$. Substitution of these estimates into (3.1.18) gives

$$\|r(t) - \Pi(t,t_0)r_0\|_{L^1(\mathbb{R}_+;\mathbb{R})} = 0,$$

as required. □

Note. While this result contained the coupling of the process given by Proposition 3.1.2 with only one other arbitrary parameterised Lipschitz process, the ideas can be extended to the coupling of an arbitrary number of processes. Hence systems of arbitrary size can be considered.

Further, the decision to allow the second process in the coupling to be arbitrary is exactly what allows one to consider systems of many different types of equations, assuming that a similar result is proved for each of the component processes.

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Part II

Research Papers

Paper 1

Lipschitz stability for the Hunter–Saxton equation

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Lipschitz stability for the Hunter–Saxton equation

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Abstract

We study Lipschitz stability in time for α-dissipative solutions to the Hunter–Saxton equation, where $\alpha \in [0, 1]$ is a constant. We define metrics in both Lagrangian and Eulerian coordinates, and establish Lipschitz stability for those metrics.

1 Introduction

In this paper, we investigate the Lipschitz stability of α-dissipative solutions of the initial value problem for the Hunter–Saxton equation,

$$u_t(x,t) + uu_x(x,t) = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(y,t)\ dy - \int_{x}^{+\infty} u_x^2(y,t)\ dy\right), \tag{HS}$$

with initial data $u(x, 0) = u_0(x)$.

This equation was introduced by Hunter and Saxton as a model for the nonlinear instability in the director field of a nematic liquid crystal [13]. Further, it is connected to the high frequency limit of the Camassa–Holm equation [6].

Solutions to (HS) may develop singularities, known as wave breaking, in finite time. That is, $u_x \to -\infty$ spatially pointwise, while u remains continuous and bounded.

One defines the energy density of the solution to be u_x^2. Then, at wave breaking, one sees that some of the energy will concentrate on a set of measure zero. Hence, the energy density in general is not absolutely continuous. Instead, the energy is described by a positive Radon measure. The question then becomes, how does one define the solution past wave breaking? This is determined by how one manipulates the energy past wave breaking. In general, one has the freedom to take as much energy away as one pleases [11]. Two important cases are well studied. Conservative solutions, whom lose no energy past wave breaking, and dissipative solutions, whom remove the energy that

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has concentrated on sets of measure zero at wave breaking. For both the conservative [2, 14], and dissipative case [1], existence of solutions has been shown. Uniqueness for the dissipative case was shown in [5]. Further, the dissipative case is the solution with maximal energy loss for a given initial data, as shown in [4]. The method used in this paper has been applied to the Camassa–Holm equation to prove similar results [12, 8], and existence in the case in which only part of the energy may be removed [10]. A different approach was used to show existence and uniqueness to the differentiated Hunter Saxton equation, $v_t + uv_x = -\frac{1}{2}v^2, v = u_x$ under the assumption that $u(0,t) = 0$ for all t, on the positive real line, with compactly supported initial data[15]. Note that solutions of this equation, extended antisymmetric to the whole real line, must not necessarily be solutions to (HS), due to the requirement that $u(0,t) = 0$ for all time, which we do not have.

We are more concerned with the stability of solutions. This builds upon the work of [11], for which Lipschitz stability was shown for a given time-dependant distance. We intend to overcome a few assumptions of this paper. Namely, we wish to include the possibility of breaking at time zero, to build a metric that relies on the current energy of the system, rather than the past energy, and to rid the requirement of a purely absolutely continuous initial energy measure in the dissipative case. Lipschitz stability was found for the conservative case using different metrics in [14, 3].

Solutions to the problem are found using a generalization of the method of characteristics. For explanatory purposes, formally suppose for now that u is smooth, and its energy density is given by u_x^2. Following the work of [14], we shift from the Eulerian variable u to Lagrangian variables (y, U, V), whom satisfy

$$\begin{aligned} y_t(\xi,t) &= u(y(\xi,t),t), \\ U(\xi,t) &= u(y(\xi,t),t), \\ V(\xi,t) &= \int_{-\infty}^{y(\xi,t)} u_x^2(z,t)\, dz, \end{aligned}$$

which we can define as long as the energy for the solution u does not concentrate on sets of measure zero, i.e. until wave breaking happens. This then gives

$$y_t(\xi,t) = U(\xi,t), \tag{1a}$$

$$U_t(\xi,t) = \frac{1}{2}V(\xi,t) - \frac{1}{4}\lim_{\xi\to\infty} V(\xi,t), \tag{1b}$$

$$V_t(\xi,t) = 0. \tag{1c}$$

This is a system of ordinary differential equations (ODEs) with initial data

$$y(\xi,0) = y_0(\xi), \tag{2a}$$

$$U(\xi,0) = U_0(\xi) = u_0(y_0(\xi)), \tag{2b}$$

$$V(\xi,0) = V_0(\xi) = \int_{-\infty}^{y_0(\xi)} u_x^2(z,0)\, dz. \tag{2c}$$

Assuming energy does not initially concentrate on sets of measure zero, one can take $y_0(\xi) = \xi$.

Wave breaking then occurs when at least two characteristics meet. The time at which wave breaking occurs is given by

$$\tau(\xi) = \begin{cases} -2\frac{y_\xi(\xi,0)}{U_\xi(\xi,0)}, & U_\xi(\xi,0) < 0, \\ 0, & U_\xi(\xi,0) = 0 = y_\xi(\xi,0), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

Up until wave breaking, the solution in Lagrangian coordinates is obtained by solving (1). After wave breaking, how one continues is determined by how one manipulates the energy. For conservative solutions, one continues the solution using (1), retaining the energy in the system. For dissipative solutions, characteristics that interact lose their energy and stick together, given by setting $V_\xi(\xi,t) = 0$ for $t \geq \tau(\xi)$. We consider the case of α-dissipative solutions, for whom $V_\xi(\xi,t) = (1-\alpha)V_\xi(\xi,0)$ for $t \geq \tau(\xi) > 0$. In particular, the system (1) is replaced by

$$y_t(\xi,t) = U(\xi,t), \tag{4a}$$

$$U_t(\xi,t) = \frac{1}{2}V(\xi,t) - \frac{1}{4}\lim_{\xi\to\infty} V(\xi,t), \tag{4b}$$

where

$$V(\xi,t) = \int_{-\infty}^{\xi} V_\xi(\eta,0)(1-\alpha\mathbb{1}_{\{r\in\mathbb{R} | t\geq\tau(r)>0\}}(\eta))\, d\eta.$$

The more general α-dissipative solution [11] considers the situation in which $\alpha : \mathbb{R} \to [0,1)$, i.e. that the drop in energy depends on the position of the particle.

There is no unique way of defining the initial characteristic $y_0(\xi)$. One cannot assume $y_0(\xi) = \xi$, as this doesn't account for energy initially concentrating on sets of measure zero. Due to this, one defines a transformation from Eulerian to Lagrangian coordinates, as seen in [11]. In Section 2, we introduce the spaces we will be working in, and the mappings used to transform from Eulerian to Lagrangian coordinates and back. In addition, we state some known results we will make use of later in the paper. As the solution at time t depends on how the energy was initially distributed, one must introduce an additional energy variable, ν, which will provide a barrier we must overcome in our construction for the Eulerian metric. Additionally, transforming from Eulerian to Lagrangian variables introduces an extra coordinate, hence multiple Lagrangian coordinates represent the same Eulerian coordinates, thus we introduce equivalence classes, whose elements are related by a relabelling.

Section 3 focuses on the construction of a metric which is Lipschitz in time for the Lagrangian coordinate system. For conservative solutions, the metric can be defined using the normal $L^\infty(\mathbb{R})$, $L^1(\mathbb{R})$, and $L^2(\mathbb{R})$ norms, as no energy in the system has been lost, leading to a smooth metric [14]. For dissipative solutions, energy may have suddenly dropped in the past, and the challenge is constructing a metric which doesn't jump upwards over these drops in energy, doesn't split apart the multiple Lagrangian solutions representing the same Eulerian solution, and which renders the flow Lipschitz continuous in time, giving the solutions are continuous with respect to the initial data in our metric.

Finally, Section 4 contains our main result. Using the construction in Lagrangian coordinates we can define a metric in Eulerian coordinates. This then inherits the Lipschitz continuity in time from our previous metric. However, the metric must account for all possible drops in energy that could have occurred in the past, that is, all possible past energy densities ν.

2 The Lagrangian and Eulerian variables

Before continuing, we define the sets in which the Eulerian and Lagrangian coordinates lie. We follow the construction in [2]. We begin by defining the Banach space and associated norm

$$E \coloneqq \{f \in L^\infty(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}, \qquad \|f\|_{E_2} = \|f\|_\infty + \|f'\|_2,$$

and define

$$H_i \coloneqq H^1(\mathbb{R}) \times \mathbb{R}^i, \qquad i = 1, 2,$$

with the norms

$$\|(f,a)\|_{H_1} = \sqrt{\|f\|_{H^1}^2 + |a|^2}, \qquad \|(f,a,b)\|_{H_2} = \sqrt{\|f\|_{H^1}^2 + |a|^2 + |b|^2},$$

where $H^1(\mathbb{R})$ is the usual Sobolev space. We then split $\mathbb{R}$ into $(-\infty, 1)$,and $(-1, \infty)$, and choose $\chi^-, \chi^+ \in C^\infty(\mathbb{R})$ satisfying the following three properties

- $\chi^- + \chi^+ = 1$,
- $0 \le \chi^+ \le 1$,
- $\operatorname{supp}(\chi^-) \subset (-\infty, 1)$ and $\operatorname{supp}(\chi^+) \subset (-1, \infty)$.

We now introduce the mappings

$$R_1 : H_1 \to E \qquad (f, a) \mapsto f + a \cdot \chi^+, \tag{5a}$$

$$R_2 : H_2 \to E \qquad (f, a, b) \mapsto f + a \cdot \chi^+ + b \cdot \chi^-. \tag{5b}$$

These mappings are linear and continuous, due to functions in $H^1(\mathbb{R})$ being continuous. They are also injective. We show this for R_2, and R_1 follows with $b = 0$. If we have two equal elements F and G in the codomain, then there exists $f, g \in H^1(\mathbb{R})$, and $a_f, b_f, a_g, b_g \in \mathbb{R}$ such that

$$f(\xi) + a_f \cdot \chi^+(\xi) + b_f \cdot \chi^-(\xi) = F(\xi) = G(\xi) = g(\xi) + a_g \cdot \chi^+(\xi) + b_g \cdot \chi^-(\xi).$$

for all $\xi \in \mathbb{R}$. Taking the limits at $\pm\infty$, we find $a_f = a_g$ and $b_f = b_g$. It then immediately follows that $f = g$ as required.

From these we define the following Banach spaces and associated norms,

$$E_1 \coloneqq R_1(H_1), \qquad \|f\|_{E_1} = \|R_1^{-1}(f)\|_{H_1},$$

$$E_2 \coloneqq R_2(H_2), \qquad \|f\|_{E_2} = \|R_2^{-1}(f)\|_{H_2}.$$

Remark 2.1 (The choice of χ does not change E_1)**.** *Consider χ^+ and $\hat{\chi}^+$ satisfying the above conditions. Define R_1 and $\hat{R}_1$ as one would expect, reflecting* (5a)*. We show $R_1(H_1) = \hat{R}_1(H_1)$. Consider $f \in R_1(H_1)$. Then there exists $g \in H^1(\mathbb{R})$ and $a \in \mathbb{R}$ such that*

$$f = g + a \cdot \chi^+.$$

Noting that $\chi^+ - \hat{\chi}^+$ is in $C_c^\infty(\mathbb{R})$, we have

$$f - a \cdot \hat{\chi}^+ = g + a \cdot (\chi^+ - \hat{\chi}^+) \in H^1(\mathbb{R}),$$

therefore $f = f - a \cdot \hat{\chi}^+ + a \cdot \hat{\chi}^+$ is in $\hat{R}_1(H_1)$, thus demonstrating $R_1(H_1) \subset \hat{R}_1(H_1)$. The same approach can be used to show $\hat{R}_1(H_1) \subset R_1(H_1)$.

It can also be shown that E_2 does not rely on the choice of χ^- and χ^+.

Using these, we define the Banach space B, and associate with it the expected norm

$$B := E_2 \times E_2 \times E_1 \times E_1, \quad \|(f_1, f_2, f_3, f_4)\|_B = \|f_1\|_{E_2} + \|f_2\|_{E_2} + \|f_3\|_{E_1} + \|f_4\|_{E_1}.$$

Wave breaking may occur at time zero, or may have even occurred in the past. The measure μ corresponds to the energy of the system at time zero. To model previous wave breaking and the corresponding energy loss, an additional energy measure ν must be supplied. This variable carries the initial energy forward in time (i.e. $\nu_t(\mathbb{R}, t) = 0$, as we will see when mapping from Lagrangian to Eulerian coordinates). Corresponding to ν when transforming to Lagrangian coordinates, a variable H is introduced. This will also preserve the energy forward in time. The variable V corresponds to the current energy μ. This variable is necessary for the construction of a semigroup of solutions in Lagrangian coordinates.

We begin with the set of Eulerian coordinates:

Definition 2.2 (Set of Eulerian coordinates - $\mathcal{D}$)**.** *The set $\mathcal{D}$ contains all Eulerian variables $Y = (u, \mu, \nu)$ satisfying the following*

- $u \in E_2$,
- $\mu \leq \nu \in \mathcal{M}^+(\mathbb{R})$,
- $\mu\big((-\infty, x)\big) - \chi_+(x)\mu(\mathbb{R}) \in L^2(\mathbb{R})$,
- $\mu_{ac} = u_x^2 \, dx$,
- *If* $\alpha = 1, \nu_{ac} = \mu = u_x^2 \, dx$,
- *If* $0 \leq \alpha < 1$, $\frac{d\mu}{d\nu}(x) \in \{1, 1 - \alpha\}$, *and* $\frac{d\mu}{d\nu} = 1$ *if* $u_x(x) < 0$,

where $\mathcal{M}^+(\mathbb{R})$ is the set of all finite, positive Radon measures on $\mathbb{R}$.

Followed by the Lagrangian coordinates:

Definition 2.3 (Set of Lagrangian coordinates - $\mathcal{F}$)**.** *Let the set $\mathcal{F}$ be the set of all $X = (y, U, H, V)$, where $(y - id, U, H, V) \in B$, satisfying the following properties*

- $y - id, U, H, V \in W^{1,\infty}(\mathbb{R})$,
- $y_\xi, H_\xi \geq 0$, *and there exists a constant* c *such that* $0 < c < y_\xi + H_\xi$ *a.e.,*
- $y_\xi V_\xi = U_\xi^2$,
- $0 \leq V_\xi \leq H_\xi$ *a.e.,*
- *If* $\alpha = 1, y_\xi(\xi) = 0$ *implies* $V_\xi(\xi) = 0$, $y_\xi(\xi) > 0$ *implies* $V_\xi(\xi) = H_\xi(\xi)$ *a.e.,*
- *If* $0 \leq \alpha < 1$, *there exists* $\kappa : \mathbb{R} \to \{(1-\alpha), 1\}$ *such that* $V_\xi(\xi) = \kappa(\xi) H_\xi(\xi)$ *a.e., with* $\kappa(\xi) = 1$ *for* $U_\xi(\xi) < 0$.

Define the set $\mathcal{F}_0$ *as*

$$\mathcal{F}_0 \coloneqq \big\{X \in \mathcal{F} \mid y + H = id\big\}.$$

The α-dissipative solution $X(t)$ for the equation (HS) in Lagrangian variables is then given by the following ODE system, with initial data $X(0) \in \mathcal{F}$,

$$y_t(\xi, t) = U(\xi, t), \tag{6a}$$

$$U_t(\xi, t) = \frac{1}{2} V(\xi, t) - \frac{1}{4} \lim_{\xi \to \infty} V(\xi, t), \tag{6b}$$

$$H_t(\xi, t) = 0, \tag{6c}$$

$$V(\xi, t) = \int_{-\infty}^{\xi} V_\xi(\eta, 0)(1 - \alpha \mathbb{1}_{\{r \in \mathbb{R} \mid t \geq \tau(r) > 0\}})(\eta)\, d\eta, \tag{6d}$$

for whom existence and uniqueness was shown in [11], in addition to the fact that the wave breaking time is given by

$$\tau(\xi) = \begin{cases} -2\frac{y_\xi(\xi,0)}{U_\xi(\xi,0)}, & U_\xi(\xi, 0) < 0, \\ 0, & U_\xi(\xi, 0) = 0 = y_\xi(\xi, 0), \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

Transforming from Eulerian to Lagrangian coordinates and back is achieved by the following mappings, which are inverses, with respect to equivalence classes, of each other [14, 11] and which developed from the transformations defined for the Camassa–Holm equation in [12].

Definition 2.4 (Mapping $L : \mathcal{D} \to \mathcal{F}_0$)**.** *The following defines the mapping* $L : \mathcal{D} \to \mathcal{F}_0$, *from Eulerian to Lagrangian coordinates,*

$$y(\xi) = \sup\{x \in \mathbb{R} \mid x + \nu\big((-\infty, x)\big) < \xi\}, \tag{8a}$$

$$U(\xi) = u(y(\xi)), \tag{8b}$$

$$H(\xi) = \xi - y(\xi), \tag{8c}$$

$$V(\xi) = \int_{-\infty}^{\xi} H_\xi(\eta) \frac{d\mu}{d\nu} \circ (y(\eta))\, d\eta. \tag{8d}$$

Definition 2.5 (Mapping $M : \mathcal{F} \to \mathcal{D}$)**.** *The following defines the mapping $M : \mathcal{F} \to \mathcal{D}$, from Lagrangian to Eulerian coordinates,*

$$u(x) = U(\xi), \qquad \text{for all } \xi \in \mathbb{R} \text{ such that } x = y(\xi), \tag{9a}$$

$$\mu = y_{\#}(V_\xi \; d\xi), \tag{9b}$$

$$\nu = y_{\#}(H_\xi \; d\xi). \tag{9c}$$

Here, we have used the push forward measure for a measurable function f and μ-measurable set $f^{-1}(A)$, i.e.,

$$f_{\#}(\mu)(A) := \mu(f^{-1}(A)).$$

The mapping L introduces an additional coordinate when mapping from Eulerian to Lagrangian coordinates, hence the mapping is not one-to-one. On the other hand, one can introduce an equivalence relation on $\mathcal{F}$, equating Lagrangian coordinates representing the same Eulerian coordinates.

Definition 2.6 (Equivalence relation on $\mathcal{F}$)**.** *Let G be the group of homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f - id \in W^{1,\infty}(\mathbb{R}), \qquad f^{-1} - id \in W^{1,\infty}(\mathbb{R}), \qquad f_\xi - 1 \in L^2(\mathbb{R}). \tag{10}$$

We define the group action $\bullet : \mathcal{F} \times G \to \mathcal{F}$, called the relabelling of X by f, as

$$(X, f) \mapsto X \bullet f = (y \circ f, U \circ f, H \circ f, V \circ f).$$

Hence, one defines the equivalence relation $\sim$ on $\mathcal{F}$ by

$$X_A \sim X_B \text{ if there exists } f \in G \text{ such that } X_A = X_B \bullet f.$$

Finally, define the mapping $\Pi : \mathcal{F} \to \mathcal{F}_0$, which gives one representative in $\mathcal{F}_0$ for each equivalence class,

$$\Pi(X) = X \bullet (y + H)^{-1}.$$

Note. *We have used in our definition for Π that $(y + H)^{-1} \in G$. We will simply write ΠX, though this is not a linear transformation.*

Lemma 2.7. *[11, Proposition 3.5] Let $X, \tilde{X} \in \mathcal{F}$, and assume $X \sim \tilde{X}$, then*

$$M(X) = M(\tilde{X}).$$

Proof. Let $f \in G$ be such that $\tilde{X} = X \bullet f$. As f is a bijection,

$$\begin{aligned} \tilde{u}(x) &= \tilde{U}(\xi), && \text{for all } \xi \in \mathbb{R} \text{ such that } x = \tilde{y}(\xi), \\ &= (U \circ f)(\xi), && \text{for all } \xi \in \mathbb{R} \text{ such that } x = (y \circ f)(\xi), \\ &= U(\eta), && \text{for all } \eta = f(\xi) \in \mathbb{R} \text{ such that } x = y(\eta), \\ &= u(x). \end{aligned}$$

For any Borel set $A \subset \mathbb{R}$, we have, using the substitution $\eta = f(\xi)$,

$$\begin{aligned}\tilde{\mu}(A) &= \int_{(y\circ f)^{-1}(A)} (V\circ f)_\xi(\xi)\, d\xi \\ &= \int_{y^{-1}(A)} V_\xi(\eta)\, d\eta = \mu(A).\end{aligned}$$

The proof for ν follows from the same calculations as μ. □

Relabelling can be done either initially, or after a given time, and one obtains the same solution, as the following proposition states.

Proposition 2.8. *[11, Proposition 3.7] Define the solution operator $S_t : \mathcal{F} \to \mathcal{F}$, $X \mapsto S_t(X)$ as giving the solution at time t to the ODE system* (6) *with initial data $X \in \mathcal{F}$. Then*

$$S_t(X \bullet f) = S_t(X) \bullet f,$$

for any $f \in G$.

For completeness, we include the definition of a weak α-dissipative solution to (HS). Existence of solutions, using the generalized method of characteristics, was found in [11].

Definition 2.9. *(u,μ,ν) is a weak α-dissipative solution to* (HS) *with initial data $(u_0,\mu_0,\nu_0) \in \mathcal{D}$, if $(u,\mu,\nu) \in \mathcal{D}$ satisfies the initial data, and*

$$u \in C^{0,\frac{1}{2}}(\mathbb{R}\times[0,T],\mathbb{R}), \qquad \text{for all } T \in [0,+\infty), \tag{11a}$$

$$\nu \in C_{weak*}([0,+\infty],\mathcal{M}^+(\mathbb{R})), \tag{11b}$$

$$\nu(t)(\mathbb{R}) = \nu_0(\mathbb{R}), \qquad \text{for all } t \in [0,+\infty), \tag{11c}$$

$$d\mu(t) = d\mu_{ac}(t)^- + (1-\alpha)d\mu_s(t)^-, \tag{11d}$$

$$\mu(s) \overset{*}{\rightharpoonup} \mu(t), \qquad \text{for all } t \in [0,+\infty) \text{ from above}, \tag{11e}$$

$$\mu(s) \overset{*}{\rightharpoonup} \mu(t)^-, \qquad \text{for all } t \in [0,+\infty) \text{ from below}, \tag{11f}$$

and, for all test functions $\varphi \in C_0^\infty(\mathbb{R}\times[0,+\infty))$, (HS) *is satisfied in the distributional sense, that is*

$$\int_0^\infty \int_{\mathbb{R}} \Big(u\varphi_t + \frac{1}{2}u^2\varphi_x - \frac{1}{4}\Big(\int_{-\infty}^x u_x^2\, dy - \int_x^\infty u_x^2\, dy\Big)\varphi\Big)\, dxdt = -\int_{\mathbb{R}} u_0\varphi_0\, dx, \tag{12}$$

where $\varphi_0(x) = \varphi(x,0)$. Further, for each non-negative test function $\phi \in C_0^\infty(\mathbb{R}\times[0,+\infty))$, one must have

$$\int_0^{+\infty}\int_{\mathbb{R}} (\phi_t + u\phi_x)\, d\mu(t)dt \geq -\int_{\mathbb{R}} \phi_0\, d\mu_0. \tag{13}$$

For a complete work through of an α-dissipative problem, see Example A.1.

3 Lipschitz stability in Lagrangian coordinates

We now have the necessary prerequisites to start constructing a metric in Lagrangian coordinates such that the solution to the ODE system (6) is Lipschitz continuous.

Before constructing our metric, we ease the notation. Given X_i, $X_j \in \mathcal{F}$, we define the following sets

$$A_i(t) := \big\{\xi \in \mathbb{R} \mid U_{i,\xi}(\xi,t) \geq 0\big\}, \tag{14a}$$
$$A_{i,j}(t) := A_i(t) \cap A_j(t), \tag{14b}$$
$$B_{i,j}(t) := \big\{\xi \in \mathbb{R} \mid t < \tau_i(\xi) = \tau_j(\xi) < +\infty\big\}, \tag{14c}$$
$$\Omega_{i,j}(t) := A_{i,j}(t) \cup B_{i,j}(t). \tag{14d}$$

We use these to split the real line into two halves. Define, for X_1, $X_2 \in \mathcal{F}$,

$$G_{12}(\xi,t) := |V_{1,\xi} - V_{2,\xi}|(\xi,t)\mathbb{1}_{\Omega_{12}(t)}(\xi) + \big(V_{1,\xi} \vee V_{2,\xi}\big)(\xi,t)\mathbb{1}_{\Omega_{12}^c(t)}(\xi), \tag{15}$$

where we have used the notation $a \vee b = \max\{a,b\}$.

We can now define our metric $d : \mathcal{F}^2 \to \mathbb{R}$ as

$$\begin{aligned} d(X_1,X_2) := {}& \|y_1 - y_2\|_\infty + \|U_1 - U_2\|_\infty + \|y_{1,\xi} - y_{2,\xi}\|_2 \\ &+ \|U_{1,\xi} - U_{2,\xi}\|_2 + \|H_1 - H_2\|_\infty + \|G_{12}\|_1 + \|G_{12}\|_2. \end{aligned} \tag{16}$$

A naive approach would be to use the $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ norms of $V_{1,\xi} - V_{2,\xi}$. However upon wave breaking, these norms could suddenly jump upwards. Consider, for instance, the fully dissipative case, i.e. $\alpha = 1$, with X_1 and X_2 in $\mathcal{F}$ such that $V_{1,\xi} = V_{2,\xi}$ initially. Suppose the first does not break, while the second does. The norm $\|V_{1,\xi} - V_{2,\xi}\|_1$ would initially be zero and would jump upwards and hence become strictly positive after wave breaking. We avoid this by using the norms of G_{12} instead. These are designed to drop after wave breaking in every situation, and thus they are shrinking as time moves forward.

To ensure that d is indeed a metric, we must confirm that the triangle inequality is satisfied for the G_{12} terms.

Proposition 3.1. *The function $d : \mathcal{F}^2 \to \mathbb{R}$ given by* (16) *satisfies the triangle inequality.*

Proof. The triangle inequality is immediate for all the norms in d with the exception of the $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ norms of G_{12}. To ensure these satisfy the triangle inequality, we show that, for all X_1, X_2, $X_3 \in \mathcal{F}$, we have

$$G_{13}(\xi,t) \leq G_{12}(\xi,t) + G_{23}(\xi,t).$$

We introduce the following notation

$$g_{12}(\xi,t) = |V_{1,\xi} - V_{2,\xi}|(\xi,t)\mathbb{1}_{\Omega_{12}(t)}(\xi),$$
$$\tilde{g}_{12}(\xi,t) = \big(V_{1,\xi} \vee V_{2,\xi}\big)(\xi,t)\mathbb{1}_{\Omega_{12}^c(t)}(\xi),$$

which yields

$$G_{12}(\xi,t) = g_{12}(\xi,t) + \tilde{g}_{12}(\xi,t).$$

We begin by noting the following:

- If $\xi \in \Omega_{13}(t)$, then $\xi \in \Omega_{12}(t) \cap \Omega_{23}(t)$ or $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}^c(t)$, but not both.
- If $\xi \in \Omega_{13}^c(t)$, then $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}^c(t)$, unless one of the following two cases occurs:
  - If $\xi \in A_{12}(t)$ and $\xi \notin A_3(t)$, or $\xi \in B_{12}(t)$, then $\xi \in \Omega_{12}(t) \cap \Omega_{23}^c(t)$.
  - If $\xi \in A_{23}(t)$ and $\xi \notin A_1(t)$, or $\xi \in B_{23}(t)$, then $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}(t)$.

Note the sets ξ ends up in are all disjoint.

Further, for $a, b, c \geq 0$, we have the following inequalities,

$$|a - b| \leq a \vee b, \tag{17a}$$

$$a \vee b \leq a \vee c + |b - c|. \tag{17b}$$

We hence strategically use the required inequality for each of the cases above:

- If $\xi \in \Omega_{13}(t)$, then either $\xi \in \Omega_{12}(t) \cap \Omega_{23}(t)$, and
$$|V_{1,\xi} - V_{3,\xi}|(\xi, t) \leq |V_{1,\xi} - V_{2,\xi}|(\xi, t) + |V_{2,\xi} - V_{3,\xi}|(\xi, t)$$
or $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}^c(t)$ and
$$|V_{1,\xi} - V_{3,\xi}|(\xi, t) \leq (V_{1,\xi} \vee V_{2,\xi})(\xi, t) + (V_{2,\xi} \vee V_{3,\xi})(\xi, t),$$
giving
$$\begin{aligned} g_{13}(\xi, t) \leq \Big(& g_{12}(\xi, t)\mathbb{1}_{\Omega_{23}(t)}(\xi, t) + g_{23}(\xi, t)\mathbb{1}_{\Omega_{12}(t)}(\xi, t) \\ & + \tilde{g}_{12}(\xi, t)\mathbb{1}_{\Omega_{23}^c(t)}(\xi, t) + \tilde{g}_{23}(\xi, t)\mathbb{1}_{\Omega_{12}^c(t)}(\xi, t)\Big)\mathbb{1}_{\Omega_{13}(t)}(\xi, t). \end{aligned}$$
- If $\xi \in \Omega_{13}^c(t)$, we either have $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}^c(t)$ and
$$(V_{1,\xi} \vee V_{3,\xi})(\xi, t) \leq (V_{1,\xi} \vee V_{2,\xi})(\xi, t) + (V_{2,\xi} \vee V_{3,\xi})(\xi, t),$$
$\xi \in \Omega_{12}(t) \cap \Omega_{23}^c(t)$ and
$$(V_{1,\xi} \vee V_{3,\xi})(\xi, t) \leq |V_{1,\xi} - V_{2,\xi}|(\xi, t) + (V_{2,\xi} \vee V_{3,\xi})(\xi, t),$$
or $\xi \in \Omega_{12}^c(t) \cap \Omega_{23}(t)$ and
$$(V_{1,\xi} \vee V_{3,\xi})(\xi, t) \leq (V_{1,\xi} \vee V_{2,\xi})(\xi, t) + |V_{2,\xi} - V_{3,\xi}|(\xi, t),$$
giving
$$\begin{aligned} \tilde{g}_{13}(\xi, t) \leq \Big(& \tilde{g}_{12}(\xi, t)(\mathbb{1}_{\Omega_{23}(t)}(\xi, t) + \mathbb{1}_{\Omega_{23}^c(t)}(\xi, t)) + g_{12}(\xi, t)\mathbb{1}_{\Omega_{23}^c(t)}(\xi, t) \\ & + \tilde{g}_{23}(\xi, t)(\mathbb{1}_{\Omega_{12}(t)}(\xi, t) + \mathbb{1}_{\Omega_{12}^c(t)}(\xi, t)) + g_{23}(\xi, t)\mathbb{1}_{\Omega_{12}^c(t)}(\xi, t)\Big) \\ & \times \mathbb{1}_{\Omega_{13}^c(t)}(\xi, t). \end{aligned}$$

As each part of these sums lie on disjoint sets, we indeed have

$$G_{13}(\xi,t) \leq G_{12}(\xi,t) + G_{23}(\xi,t), \quad \text{for all } (\xi,t) \in \mathbb{R} \times [0,+\infty).$$

As all the involved functions are positive, one can apply both the $L^1(\mathbb{R})$ and the $L^2(\mathbb{R})$ norm on either side of the above inequality, and use the triangle inequality, to obtain the required result. □

We are now ready to establish stability.

Theorem 3.2. *Let $X_1(t)$ and $X_2(t)$ be the solutions of the system* (6) *with initial data $X_1(0)$ and $X_2(0)$ in $\mathcal{F}$, respectively. Then*

$$d(X_1(t), X_2(t)) \leq e^t d(X_1(0), X_2(0)).$$

Proof. We derive inequalities for each of the terms in our metric. To do this, we focus first on the metric $D : \mathcal{F}^2 \to \mathbb{R}$, given by

$$\begin{aligned} D(X_1, X_2) &:= d(X_1, X_2) - \|G_{12}\|_1 - \|G_{12}\|_2 \\ &= \|y_1 - y_2\|_\infty + \|U_1 - U_2\|_\infty + \|y_{1,\xi} - y_{2,\xi}\|_2 \\ &\quad + \|U_{1,\xi} - U_{2,\xi}\|_2 + \|H_1 - H_2\|_\infty. \end{aligned} \tag{18}$$

We do not need an estimate for the norm involving H, as it is constant in time. Beginning with the y terms, we have from (6)

$$|(y_1 - y_2)(\xi,t)| \leq |(y_1 - y_2)(\xi,0)| + \int_0^t |(U_1 - U_2)(\xi,s)|\, ds,$$

and hence

$$\|(y_1 - y_2)(\cdot,t)\|_\infty \leq \|(y_1 - y_2)(\cdot,0)\|_\infty + \int_0^t \|(U_1 - U_2)(\cdot,s)\|_\infty\, ds. \tag{19}$$

We also have,

$$\|(y_{1,\xi} - y_{2,\xi})(\cdot,t)\|_2 \leq \|(y_{1,\xi} - y_{2,\xi})(\cdot,0)\|_2 + \int_0^t \|(U_{1,\xi} - U_{2,\xi})(\cdot,s)\|_2\, ds, \tag{20}$$

which follows immediately from the Lagrangian ODE system (6), and using Minkowski's integral inequality.

Set $V_\infty(t) := \lim_{\xi\to+\infty} V(\xi,t)$. Then we have for the U terms,

$$(U_1 - U_2)(\xi,t) = (U_1 - U_2)(\xi,0) + \int_0^t (U_{1,t} - U_{2,t})(\xi,s)\, ds, \tag{21}$$

and for the integral on the RHS,

$$\begin{aligned} \int_0^t (U_{1,t} - U_{2,t})(\xi,s)\, ds &= \int_0^t \frac{1}{2}(V_1 - V_2)(\xi,s) - \frac{1}{4}(V_{1,\infty} - V_{2,\infty})(s)\, ds \\ &= \int_0^t \frac{1}{4}(V_1 - V_2)(\xi,s) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}(V_1-V_2)(\xi,s)-\frac{1}{4}(V_{1,\infty}-V_{2,\infty})(s)\ ds\\
=\frac{1}{4}\int_0^t\Big[&\int_{-\infty}^{\xi}(V_{1,\xi}-V_{2,\xi})(\eta,s)\ d\eta\\
&-\int_{\xi}^{+\infty}(V_{1,\xi}-V_{2,\xi})(\eta,s)\ d\eta\Big]\ ds.
\end{aligned}
$$

Substituting into (21) and taking the absolute value, we have

$$
|U_1-U_2|(\xi,t)\leq|U_1-U_2|(\xi,0)+\frac{1}{4}\int_0^t\int_{\mathbb{R}}|V_{1,\xi}-V_{2,\xi}|(\eta,s)\ d\eta\ ds.
$$

Concentrating on the integral on the RHS, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}}|V_{1,\xi}-V_{2,\xi}|(\eta,s)\ d\eta&\leq\int_{\Omega_{12}(s)}|V_{1,\xi}-V_{2,\xi}|(\eta,s)\ d\eta\\
&\quad+\int_{\Omega_{12}^c(s)}(V_{1,\xi}\vee V_{2,\xi})(\eta,s)\ d\eta\\
&=\int_{\mathbb{R}}G_{12}(\eta,s)\ ds.
\end{aligned}
$$

Thus, after taking the $L^{\infty}(\mathbb{R})$ norm, we end up with

$$
\|(U_1-U_2)(\cdot,t)\|_{\infty}\leq\|(U_1-U_2)(\cdot,0)\|_{\infty}+\frac{1}{4}\int_0^t\|G_{12}(\cdot,s)\|_1\ ds. \tag{22}
$$

For the $L^2(\mathbb{R})$ norm involving the U_{ξ}'s, we use Minkowski's integral inequality, giving

$$
\|(U_{1,\xi}-U_{2,\xi})(\cdot,t)\|_2\leq\|(U_{1,\xi}-U_{2,\xi})(\cdot,0)\|_2+\frac{1}{2}\int_0^t\|(V_{1,\xi}-V_{2,\xi})(\cdot,s)\|_2\ ds.
$$

Using that we integrate on two disjoint sets and (17a), we have

$$
\begin{aligned}
\Big(\int_{\mathbb{R}}|V_{1,\xi}-V_{2,\xi}|^2(\xi,s)\ d\xi\Big)^{\frac{1}{2}}&\leq\Big(\int_{\Omega_{12}(s)}|V_{1,\xi}-V_{2,\xi}|^2(\xi,s)\ d\xi\\
&\quad+\int_{\Omega_{12}^c(s)}\big(V_{1,\xi}\vee V_{2,\xi}\big)^2(\xi,s)\ d\xi\Big)^{\frac{1}{2}}\\
&=\Big(\int_{\mathbb{R}}|G_{12}(\xi,s)|^2\ d\xi\Big)^{\frac{1}{2}},
\end{aligned}
$$

and hence

$$
\|U_{1,\xi}(\cdot,t)-U_{2,\xi}(\cdot,t)\|_2\leq\|U_{1,\xi}(\cdot,0)-U_{2,\xi}(\cdot,0)\|_2+\frac{1}{2}\int_0^t\|G_{12}(\cdot,s)\|_2\ ds. \tag{23}
$$

Combining (19), (20), (22), and (23) together, yields

$$
D(X_1(t),X_2(t))\leq D(X_1(0),X_2(0)) \tag{24}
$$

$$+\int_0^t \left(D(X_1(s), X_2(s)) + \frac{1}{4}\|G_{12}(\cdot,s)\|_1 + \frac{1}{2}\|G_{12}(\cdot,s)\|_2 \right) ds.$$

Thus, it remains to show that $G_{12}(\xi,t)$ is a decreasing function with respect to time.

As, for all $\xi \in \mathbb{R}$, the $V_\xi(\xi,t)$ are decreasing functions in time, $(V_{1,\xi} \vee V_{2,\xi})(\xi,t)$ is a decreasing function in time. Should no wave breaking occur, then the difference $|V_{1,\xi} - V_{2,\xi}|(\xi,t)$ will remain unchanged. Should both break at the same time, then the difference will decrease, as after wave breaking

$$|V_{1,\xi} - V_{2,\xi}|(\xi,t) = (1-\alpha)|V_{1,\xi} - V_{2,\xi}|(\xi,0).$$

Finally, one has to deal with the case of being in $\Omega_{12}^c(0)$ initially, then ending in $\Omega_{12}(t)$, as can happen if one has broken (or will never break) and the other one will break in the future. Define $a \wedge b := \min\{a,b\}$. After breaking, one can write the difference as

$$|V_{1,\xi} - V_{2,\xi}|(\xi,t) = \big(V_{1,\xi} \vee V_{2,\xi}\big)(\xi,t) - \big(V_{1,\xi} \wedge V_{2,\xi}\big)(\xi,t) \leq \big(V_{1,\xi} \vee V_{2,\xi}\big)(\xi,0)$$

due to the fact that, as mentioned previously, the maximum is a decreasing function of time, and the V_ξ's are both positive. Thus one can conclude

$$\|G_{12}(\cdot,t)\|_1 \leq \|G_{12}(\cdot,0)\|_1 \quad \text{and} \quad \|G_{12}(\cdot,t)\|_2 \leq \|G_{12}(\cdot,0)\|_2. \tag{25}$$

Combining this with inequality (24) and recalling (18), one has

$$d(X_1(t), X_2(t)) \leq d(X_1(0), X_2(0)) + \int_0^t d(X_1(s), X_2(s))\, ds$$

and Grönwall's inequality gives the required result. □

This metric faces a major problem: Although two different members of an equivalence class in Lagrangian coordinates represent the same element in Eulerian coordinates, they may have a distance greater than zero. This is demonstrated in the following example.

Example 3.3. *Consider the HS equation with initial data,*

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 1-x, & 0 < x \leq 1, \\ 0, & 1 < x, \end{cases} \qquad \nu_0 = \mu_0 = u_{0,x}^2(x)\, dx.$$

As our initial characteristic we can use $y_0(\xi) = \xi$, since neither energy concentrates on sets of measure zero nor $u_{0,x}(x)$ is unbounded. Furthermore, $U_0(\xi) = u_0(y_0(\xi)) = u_0(\xi)$ by (2). We then find, using (7), that wave breaking will only occur for $\xi \in (0,1)$ and, in particular, $\tau(\xi) = 2$ for all $\xi \in (0,1)$. For $t < 2$, i.e. before wave breaking occurs, the solution is given by (6) and reads

$$V(\xi,t) = \begin{cases} 0, & \xi \leq 0, \\ \xi, & 0 < \xi \leq 1, \\ 1, & 1 < \xi, \end{cases} \qquad U(\xi,t) = \begin{cases} 1 - \frac{1}{4}t, & \xi \leq 0, \\ 1 - \frac{1}{4}t + \frac{(t-2)}{2}\xi, & 0 < \xi \leq 1, \\ \frac{1}{4}t, & 1 < \xi, \end{cases}$$

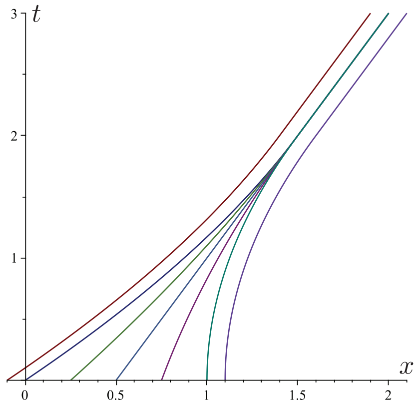


Figure 1: Characteristics $y(\xi,t)$ for Example 3.3, for $t \in [0,3]$, in the dissipative case, i.e. $\alpha = 1$. Note how the characteristics for $\xi \in (0,1)$, meet in one point at $t = 2$, and remain stuck together as all the concentrated energy is lost.

and

$$y(\xi,t) = \begin{cases} t - \frac{1}{8}t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{8}t^2 + \frac{(t-2)^2}{4}\xi, & 0 < \xi \leq 1, \\ \frac{1}{8}t^2 + \xi & 1 < \xi. \end{cases}$$

Wave breaking does not occur at $t = 0$, *and thus* $H(\xi,t) = V(\xi,t)$ *for* $t < 2$. *See Figure* 1 *for a plot of* $y(\xi,t)$

On the other hand, we can define the initial data in Lagrangian coordinates using Definition 2.4. *This yields, using* (6), *for* $t < 2$

$$\hat{V}(\xi,t) = \begin{cases} 0, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 2, \\ 1, & 2 < \xi, \end{cases} \qquad \hat{U}(\xi,t) = \begin{cases} 1 - \frac{1}{4}t, & \xi \leq 0, \\ 1 - \frac{1}{4}t + \frac{(t-2)}{4}\xi, & 0 < \xi \leq 2, \\ \frac{1}{4}t, & 2 < \xi, \end{cases}$$

and

$$\hat{y}(\xi,t) = \begin{cases} t - \frac{1}{8}t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{8}t^2 + \frac{(t-2)^2}{8}\xi, & 0 < \xi \leq 2, \\ -1 + \frac{1}{8}t^2 + \xi, & 2 < \xi. \end{cases}$$

This time wave breaking occurs for all $\xi \in (0,2)$, *and again* $\tau(\xi) = 2$ *for all* $\xi \in (0,2)$. *Once again,* $H(\xi,t) = V(\xi,t)$ *for* $t < 2$.

We now wish to identify the relabelling function connecting our two solutions, which will then imply that these two solutions belong to the same equivalence class. Importantly, the distance between these two solutions is positive. Using Definition 2.6 *and Proposition* 2.8, *we see that we need to identify a homeomorphism* f *satisfying* (10) *such that*

$$(y,U,H,V)(\xi,t) = (\hat{y},\hat{U},\hat{H},\hat{V})(f(\xi),t).$$

Since $\hat{y}(\xi,0)+\hat{H}(\xi,0)=\xi$, we see that $f\in G$ is given by

$$f(\xi)=\begin{cases}\xi, & \xi\le 0,\\ 2\xi, & 0<\xi\le 1,\\ 1+\xi, & 1<\xi.\end{cases}$$

For completions sake, we compute the solution using Definition 2.5 *and obtain in both cases that the solution for $t<2$ is given by*

$$u(x,t)=\begin{cases}1-\frac{1}{4}t, & x\le t-\frac{1}{8}t^2,\\ \frac{-4-t+4x}{2(t-2)}, & t-\frac{1}{8}t^2<x\le 1+\frac{1}{8}t^2,\\ \frac{1}{4}t, & 1+\frac{1}{8}t^2<x.\end{cases}$$

To resolve this issue, we introduce the function $J:\mathcal{F}^2\to\mathbb{R}$, given by

$$J(X_1,X_2)=\inf_{f,g\in G}\big(d(X_1,X_2\bullet f)+d(X_1\bullet g,X_2)\big). \tag{26}$$

This function satisfies the requirement that two elements of the same equivalence class have a distance of zero. Sadly, one cannot conclude that J satisfies the triangle inequality. To resolve this issue, one constructs a metric by taking the infimum over finite sequences.

Definition 3.4 (A metric over equivalence classes in $\mathcal{F}$)**.** *Define the metric $d_{\mathcal{F}}:\mathcal{F}^2\to\mathbb{R}$ as follows*

$$d_{\mathcal{F}}(X_A,X_B):=\inf_{\hat{\mathcal{F}}(X_A,X_B)}\left\{\sum_{n=1}^{N}J\big(X_n,X_{n-1}\big)\right\},$$

where the infimum is taken over the set $\hat{\mathcal{F}}(X_A,X_B)$ of finite sequences of arbitrary length $\{X_i\}_{i=0}^N$ in $\mathcal{F}_0$, such that $X_0=\Pi X_A$ and $X_N=\Pi X_B$.

The following lemma ensures that $d_{\mathcal{F}}$ is indeed a metric.

Lemma 3.5. *Let $X_A,X_B\in\mathcal{F}$ and set $(\hat{X}_A,\hat{X}_B):=(\Pi X_A,\Pi X_B)$. We then have*

$$\|\hat{X}_A-\hat{X}_B\|\le\frac{5}{2}d_{\mathcal{F}}(X_A,X_B)\le 5d(\hat{X}_A,\hat{X}_B), \tag{27}$$

where

$$\|X_A-X_B\|:=\|y_A-y_B\|_\infty+\|U_A-U_B\|_\infty+\|H_A-H_B\|_\infty+\|V_A-V_B\|_\infty. \tag{28}$$

Proof. The ideas of this proof follow the ones of [9, Lemma 3.2]. As

$$d_{\mathcal{F}}(X_A,X_B)=d_{\mathcal{F}}(\Pi X_A,\Pi X_B),$$

we assume for our calculations that $X_A,\ X_B\in\mathcal{F}_0$.

For the upper bound, consider the sequence containing just X_A and X_B. Then

$$d_{\mathcal{F}}(X_A,X_B)\le J(X_A,X_B)=\inf_{f,g\in G}\big(d(X_A,X_B\bullet f)+d(X_A\bullet g,X_B)\big)$$

$$\leq 2d(X_A, X_B),$$

where in the last inequality, we have chosen $f = g = \mathrm{id}$.

For the lower bound, we begin by showing that, for any $X_A, X_B \in \mathcal{F}_0$,

$$\|X_A - X_B\| \leq \frac{5}{2} J(X_A, X_B).$$

First, for any $X \in \mathcal{F}_0$, one has $X \in C^{0,1}(\mathbb{R})^4$, as $Z = (y - \mathrm{id}, U, V, H) \in W^{1,\infty}(\mathbb{R})^4$. Furthermore, $\|y_\xi\|_\infty$, $\|U_\xi\|_\infty$, $\|V_\xi\|_\infty$, and $\|H_\xi\|_\infty$ are all bounded from above by 1, as $0 \leq y_\xi, H_\xi \leq 1$, $0 \leq V_\xi \leq H_\xi$, and $U_\xi^2 = y_\xi V_\xi \leq 1$ almost everywhere. Hence, we have

$$|y(\xi_1) - y(\xi_2)| + |U(\xi_1) - U(\xi_2)| + |V(\xi_1) - V(\xi_2)| + |H(\xi_1) - H(\xi_2)| \leq 4|\xi_1 - \xi_2|,$$

which implies, that for any $f \in G$,

$$\begin{aligned} \|X_A - X_B\| &\leq \|X_A - X_A \bullet f\| + \|X_A \bullet f - X_B\| \\ &\leq 4\|\mathrm{id} - f\|_\infty + \|X_A \bullet f - X_B\|. \end{aligned} \tag{29}$$

Then, using that $X_A \in \mathcal{F}_0$, which implies $y_A + H_A = \mathrm{id}$, and similarly for X_B, we get

$$\|\mathrm{id} - f\|_\infty = \|y_B + H_B - (y_A + H_A) \circ f\|_\infty \leq \|X_A \bullet f - X_B\|.$$

Substituting into (29), we thus end up with

$$\|X_A - X_B\| \leq 5\|X_A \bullet f - X_B\|. \tag{30}$$

Note that we have, for any X_1, $X_2 \in \mathcal{F}$, that

$$\begin{aligned} |V_1(\xi) - V_2(\xi)| &= \left| \int_{-\infty}^{\xi} (V_{1,\xi}(\eta) - V_{2,\xi}(\eta)) \, d\eta \right| \\ &\leq \int_{\mathbb{R}} |V_{1,\xi}(\xi) - V_{2,\xi}(\xi)| \, d\xi \\ &\leq \|G_{12}\|_1 \end{aligned}$$

or equivalently

$$\|V_1 - V_2\|_\infty \leq \|G_{12}\|_1. \tag{31}$$

Recalling (28), setting $V_1 = V_A \circ f$ and $V_2 = V_B$ in (31), and substituting into the RHS of (30), we get

$$\|X_A - X_B\| \leq 5d(X_A \bullet f, X_B). \tag{32}$$

A similar process reveals, for any $g \in G$, that

$$\|X_A - X_B\| \leq 5\|X_A - X_B \bullet g\| \leq 5d(X_A, X_B \bullet g). \tag{33}$$

Combining (32) and (33) together, and taking the infimum over all f, $g \in G$, we end up with

$$2\|X_A - X_B\| \leq 5J(X_A, X_B), \tag{34}$$

as required.

Consider X_A, $X_B \in \mathcal{F}_0$. Given $\epsilon > 0$, there exists a finite sequence $\{X_n\}_{n=0}^N$ in $\mathcal{F}_0$ with $X_0 = X_A$ and $X_N = X_B$, such that

$$\sum_{n=1}^{N} J(X_n, X_{n-1}) < d_{\mathcal{F}}(X_A, X_B) + \epsilon.$$

Using (34), we have

$$2\|X_A - X_B\| \leq 2\sum_{n=1}^{N} \|X_n - X_{n-1}\| \leq 5\sum_{n=1}^{N} J(X_n, X_{n-1}) < 5d_{\mathcal{F}}(X_A, X_B) + 5\epsilon.$$

Since the above inequality holds for any $\epsilon > 0$, the claim follows. □

The following lemma contains two estimates for J, which play en essential role when establishing the Lipschitz stablity in time for $d_{\mathcal{F}}$.

Lemma 3.6. *For X_A, $X_B \in \mathcal{F}$, and $f \in G$ with $\|f_\xi\|_\infty^{\frac{1}{2}} \leq C$ for some $C > 1$, it holds that*

$$J(X_A \bullet f, X_B) \leq CJ(X_A, X_B).$$

As a consequence, for solutions $X_A(t), X_B(t) \in \mathcal{F}$ of (6) *with initial data $X_A(0), X_B(0) \in \mathcal{F}_0$, it holds that*

$$J(\Pi X_A(t), \Pi X_B(t)) \leq e^{\frac{1}{2}t} J(X_A(t), X_B(t)).$$

Proof. The proof follows the ideas of the one for [14, Lemma 4.8]. First, note for f, $h \in G$, and g_A, $g_B \in L^\infty(\mathbb{R})$,

$$\|g_A \circ f - g_B \circ h\|_\infty = \|g_A - g_B \circ h \circ f^{-1}\|_\infty. \tag{35}$$

Importantly, due to the group properties, $w := h \circ f^{-1}$ is in G. We use this relation for the $L^\infty(\mathbb{R})$ terms involving y, U, and H in d. Hence we focus on the $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$ terms.

Beginning with $L^2(\mathbb{R})$ terms, for f, $h \in G$, we have

$$\begin{aligned}\|(y_A \circ f)_\xi - (y_B \circ h)_\xi\|_2^2 &= \int_{\mathbb{R}} |(y_A \circ f)_\xi - (y_B \circ h)_\xi|^2(\xi)\, d\xi \\ &= \int_{\mathbb{R}} |y_{A,\xi} \circ f f_\xi - y_{B,\xi} \circ h h_\xi|^2(\xi)\, d\xi.\end{aligned}$$

Using the substitution $\eta = f(\xi)$, for which $d\xi = \frac{1}{f_\xi \circ f^{-1}(\eta)} d\eta$, we have

$$\begin{aligned}&\|(y_A \circ f)_\xi - (y_B \circ h)_\xi\|_2^2 \\ &= \int_{\mathbb{R}} |y_{A,\xi}(f_\xi \circ f^{-1}) - (y_{B,\xi} \circ h \circ f^{-1})(h_\xi \circ f^{-1})|^2(\eta) \frac{1}{f_\xi \circ f^{-1}(\eta)} d\eta.\end{aligned}$$

Using that $w = h \circ f^{-1} \in G$ has the derivative $w_\eta(\eta) = \frac{h_\xi \circ f^{-1}(\eta)}{f_\xi \circ f^{-1}(\eta)}$, we get

$$\begin{aligned} \|(y_A \circ f)_\xi - (y_B \circ h)_\xi\|_2^2 &= \int_{\mathbb{R}} |(y_A)_\eta - (y_B \circ w)_\eta|^2(\eta) f_\xi \circ f^{-1}(\eta)\, d\eta \\ &\leq \|f_\xi\|_\infty \|(y_A)_\eta - (y_B \circ w)_\eta\|_2^2 \end{aligned} \tag{36}$$

Similarly, one has

$$\|(U_A \circ f)_\xi - (U_B \circ h)_\xi\|_2^2 \leq \|f_\xi\|_\infty \|(U_A)_\eta - (U_B \circ w)_\eta\|_2^2. \tag{37}$$

For the final two norms, we need to introduce some new notation to keep everything clear. Let X_1 be an element of $\mathcal{F}$, and using a relabelling $f \in G$ define $X_2 = X_1 \circ f$. Then we have

$$\begin{aligned} A_2 &= \{\xi \in \mathbb{R} \mid U_{2,\xi}(\xi) \geq 0\} \\ &= \{\xi \in \mathbb{R} \mid U_{1,\xi}(f(\xi)) f_\xi(\xi) \geq 0\} \\ &= \{\xi \in \mathbb{R} \mid U_{1,\xi}(f(\xi)) \geq 0\} \\ &= \{f^{-1}(\xi) \in \mathbb{R} \mid U_{1,\xi}(\xi) \geq 0\} = f^{-1}(A_1). \end{aligned}$$

Using this, we define Ω for a relabelled solution. Given $X_i, X_j \in \mathcal{F}$ for some labels i, j, and their respective relabellings $f, h \in G$, we define

$$\Omega_{i,j}^{f,h} = (f^{-1}(A_i) \cap h^{-1}(A_j)) \cup \{\xi \in \mathbb{R} \mid 0 < \tau_i(f(\xi)) = \tau_j(h(\xi)) < +\infty\}.$$

From the same substitution as before, and using the definition of G_{12},

$$\begin{aligned} &\|(V_A \circ f - V_B \circ h)_\xi \mathbb{1}_{\Omega_{AB}^{f,h}} + ((V_A \circ f)_\xi \vee (V_B \circ h)_\xi) \mathbb{1}_{\Omega_{AB}^{f,h,c}}\|_1 \\ &= \int_{\mathbb{R}} \big|(V_{A\xi} f_\xi \circ f^{-1} - (V_{B,\xi} \circ w) h_\xi \circ f^{-1}) \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w}} \\ &\qquad + (V_{A\xi} f_\xi \circ f^{-1}) \vee \big((V_{B,\xi} \circ w) h_\xi \circ f^{-1}\big) \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w,c}}\big| \frac{1}{|f_\xi \circ f^{-1}|} d\eta \\ &= \int_{\mathbb{R}} \big|(V_{A\xi} - (V_B \circ w)_\xi) \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w}} + V_{A\xi} \vee (V_B \circ w)_\xi \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w,c}}\big|\, d\eta, \end{aligned} \tag{38}$$

and similarly to before,

$$\begin{aligned} &\|(V_A \circ f - V_B \circ h)_\xi \mathbb{1}_{\Omega_{AB}^{f,h}} + (V_{A,\xi} \vee (V_B \circ h)_\xi) \mathbb{1}_{\Omega_{AB}^{f,h,c}}\|_2^2 \\ &\leq \|f_\xi\|_\infty \|(V_A - V_B \circ w)_\xi \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w}} + (V_{A,\xi} \vee (V_B \circ w)_\xi) \mathbb{1}_{\Omega_{AB}^{\mathrm{id},w,c}}\|_2^2. \end{aligned} \tag{39}$$

Combining (35), (36), (37), (38), and (39) together, we have for $f, h \in G$ and $w = h \circ f^{-1}$,

$$d(X_A \bullet f, X_B \bullet h) \leq \|f_\xi\|_\infty^{\frac{1}{2}} d(X_A, X_B \bullet w).$$

For all these estimates, f is involved in the w, so to ensure we can take the infimum, we assume that $\|f_\xi\|_\infty^{\frac{1}{2}} \leq C$ for some $C > 1$.

$$J(X_A \bullet f, X_B) = \inf_{f_1, f_2} \left(d(X_A \bullet f, X_B \bullet f_1) + d(X_A \bullet (f \circ f_2), X_B)\right)$$

$$\leq \inf_{w_1,w_2} (Cd(X_A, X_B \bullet w_1) + Cd(X_A \bullet w_2, X_B))$$
$$= C \inf_{w_1,w_2} (d(X_A, X_B \bullet w_1) + d(X_A \bullet w_2, X_B)) = CJ(X_A, X_B),$$

where we have used the fact that w_1 and w_2 above are still in the group G, and that given $f \in G$ for each $g \in G$, there are $h, l \in G$ such that $g = f \circ h = l \circ f$.

Given t and slightly abusing the notation, denote by $(y+H)^{-1}(\xi,t)$ the inverse of $(y+H)(\cdot,t)$. Recalling (10), we have $(y+H)^{-1}(\cdot,t) \in G$. Furthermore, $(y_\xi + H_\xi)^{-1}(\xi,0) = 1$ as $X(0) \in \mathcal{F}_0$. Choose $\xi \in \mathbb{R}$ and drop it in the notation in the following calculation. We see that

$$\frac{d}{dt}\left[\frac{1}{y_\xi(t) + H_\xi(t)}\right] = -\frac{U_\xi(t)}{(y_\xi(t) + H_\xi(t))^2} \leq \frac{1}{y_\xi(t) + H_\xi(t)} \frac{\sqrt{y_\xi(t)V_\xi(t)}}{y_\xi(t) + H_\xi(t)}$$
$$\leq \frac{1}{y_\xi(t) + H_\xi(t)} \frac{\frac{1}{2}(y_\xi(t) + H_\xi(t))}{y_\xi(t) + H_\xi(t)}$$

so

$$\frac{d}{dt}\left[\frac{1}{y_\xi(t) + H_\xi(t)}\right] \leq \frac{1}{2}\frac{1}{y_\xi(t) + H_\xi(t)},$$

and hence

$$\frac{1}{y_\xi(t) + H_\xi(t)} \leq e^{\frac{1}{2}t}.$$

Then, one has

$$\left[(y+H)^{-1}(\xi,t)\right]_\xi = \frac{1}{(y_\xi + H_\xi)(t, (y+H)^{-1}(\xi,t))} \leq e^{\frac{1}{2}t},$$

and the result follows by using the relabeling function $f(\xi,t) = (y+H)^{-1}(\xi,t)$,

$$J\big(\Pi X_A(t), \Pi X_B(t)\big) = J\big((X_A \bullet (y_A + H_A)^{-1})(t), (X_B \bullet (y_B + H_B)^{-1})(t)\big)$$
$$\leq e^{\frac{1}{4}t} J(X_A(t), (X_B \bullet (y_B + H_B)^{-1})(t))$$
$$\leq e^{\frac{1}{2}t} J\big(X_A(t), X_B(t)\big).$$

□

We can now obtain stability in Lagrangian coordinates.

Theorem 3.7. *Let $X_A(t)$, $X_B(t) \in \mathcal{F}$ be the solutions of the system* (6) *with initial data $X_A(0)$, $X_B(0) \in \mathcal{F}_0$, respectively. Then*

$$d_{\mathcal{F}}(X_A(t), X_B(t)) \leq e^{\frac{3}{2}t} d_{\mathcal{F}}(X_A(0), X_B(0)).$$

Proof. Let $\epsilon > 0$. There exists a finite sequence $\{X_n(t)\}_{n=0}^N$ in $\mathcal{F}$ of solutions to (6), whose initial data lies in $\mathcal{F}_0$, and a sequence of relabelling functions $\{f_n\}_{n=0}^{N-1}, \{g_n\}_{n=1}^N$ in G such that

$$\sum_{n=1}^N \big(d(X_n(0), X_{n-1}(0) \bullet f_{n-1}) + d(X_n(0) \bullet g_n, X_{n-1}(0))\big)$$

$$< d_{\mathcal{F}}(X_A(0), X_B(0)) + \epsilon. \tag{40}$$

From Definition 3.4 and Lemma 3.6, it thus follows that

$$\begin{aligned} d_{\mathcal{F}}(X_A(t), X_B(t)) &\leq \sum_{n=1}^{N} J(\Pi X_n(t), \Pi X_{n-1}(t)) \\ &\leq e^{\frac{1}{2}t} \sum_{n=1}^{N} J(X_n(t), X_{n-1}(t)). \end{aligned}$$

Hence, from (26), Proposition 2.8, and Theorem 3.2, we have

$$\begin{aligned} &d_{\mathcal{F}}(X_A(t), X_B(t)) \\ &\leq e^{\frac{1}{2}t} \sum_{n=1}^{N} \big(d(X_n(t), X_{n-1}(t) \bullet f_{n-1}) + d(X_n(t) \bullet g_n, X_{n-1}(t)) \big) \\ &\leq e^{\frac{3}{2}t} \sum_{n=1}^{N} \big(d(X_n(0), X_{n-1}(0) \bullet f_{n-1}) + d(X_n(0) \bullet g_n, X_{n-1}(0)) \big) \\ &< e^{\frac{3}{2}t} (d_{\mathcal{F}}(X_A(0), X_B(0)) + \epsilon), \end{aligned}$$

where for the final inequality we have used (40). As such a result can be constructed for ϵ arbitrarily small, we have

$$d_{\mathcal{F}}(X_A(t), X_B(t)) \leq e^{\frac{3}{2}t} d_{\mathcal{F}}(X_A(0), X_B(0)),$$

as required. □

4 Equivalence relation in Eulerian variables and Lipschitz stability

We define the metric $d_{\mathcal{D}} : \mathcal{D}^2 \to \mathbb{R}$ on Eulerian coordinates as follows,

$$d_{\mathcal{D}}(Y_1, Y_2) = d_{\mathcal{F}}(L(Y_1), L(Y_2)), \tag{41}$$

for $Y_i = (u_i, \mu_i, \nu_i) \in \mathcal{D}$. An immediate consequence of Theorem 3.7 is the following.

Corollary 4.1. *Let $Y_1(t), Y_2(t) \in \mathcal{D}$ be the α-dissipative solutions at time t of the partial differential equation* (HS)*, with initial data $Y_1(0), Y_2(0) \in \mathcal{D}$, then*

$$d_{\mathcal{D}}(Y_1(t), Y_2(t)) \leq e^{\frac{3}{2}t} d_{\mathcal{D}}(Y_1(0), Y_2(0)).$$

As mentioned earlier, the variable ν was necessarily added to represent the past energy in the system. However, we do not supply the initial energy distribution ν. The following example demonstrates that if we have two different past energy measures, our distance will be greater than zero, yet we have the same solution (u, μ) in Eulerian coordinates.

Example 4.2. *Consider the same u_0 as in Example 3.3, but with different initial energy measures, namely*

$$\nu_0 = u_{0,x}^2(x)dx + \delta_2,$$

and

$$\mu_0 = u_{0,x}^2(x)dx + (1-\alpha)\delta_2.$$

For $\alpha \neq 0$, this models the case where wave breaking takes place at $t = 0$. That is, energy is initially concentrated at the point $x = 2$, and an α-part of it dissipates immediately giving rise to the difference between ν_0 and μ_0.

Then, we have

$$\nu_0((-\infty, x)) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x \leq 2, \\ 2, & 2 < x, \end{cases}$$

and energy initially concentrates at $x = 2$. Thus we must define our initial conditions using the mapping L given by Definition 2.4. We then obtain

$$y_0(\xi) = \begin{cases} \xi, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 2, \\ -1+\xi, & 2 < \xi \leq 3, \\ 2, & 3 < \xi \leq 4, \\ -2+\xi, & 4 < \xi, \end{cases} \quad U_0(\xi) = \begin{cases} 1, & \xi \leq 0, \\ 1 - \frac{1}{2}\xi, & 0 < \xi \leq 2, \\ 0, & 2 < \xi. \end{cases}$$

and using $H_0(\xi) = \xi - y_0(\xi)$ and (6c), gives

$$H_0(\xi) = H(\xi, t) = \begin{cases} 0, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 2, \\ 1, & 2 < \xi \leq 3, \\ -2+\xi, & 3 < \xi \leq 4, \\ 2, & 4 < \xi. \end{cases}$$

Using formula (7), we find that wave breaking occurs twice. For $\xi \in (3,4)$, wave breaking occurs initially, i.e. $\tau(\xi) = 0$ and for $\xi \in (0,2)$ we have $\tau(\xi) = 2$. Using (8d) and (6d), we get, for $t < 2$,

$$V(\xi, t) = \begin{cases} 0, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 2, \\ 1, & 2 < \xi \leq 3, \\ -2+3\alpha+(1-\alpha)\xi, & 3 < \xi \leq 4, \\ 2-\alpha, & 4 < \xi. \end{cases}$$

We then solve the Lagrangian ODE problem (6) for $t \in [0,2)$, and find

$$U(\xi,t) = \begin{cases} 1 - \frac{1}{4}(2-\alpha)t, & \xi \leq 0, \\ 1 - \frac{1}{4}(2-\alpha)t + \frac{1}{4}(t-2)\xi, & 0 < \xi \leq 2, \\ \frac{1}{4}\alpha t, & 2 < \xi \leq 3, \\ -\frac{1}{4}\left(6-7\alpha\right)t + \frac{1}{2}(1-\alpha)t\xi, & 3 < \xi \leq 4, \\ \frac{1}{4}\left(2-\alpha\right)t, & 4 < \xi, \end{cases}$$

and

$$y(\xi,t) = \begin{cases} t - \frac{1}{8}(2-\alpha)t^2 + \xi, & \xi \leq 0, \\ t - \frac{1}{8}(2-\alpha)t^2 + \frac{1}{8}(t-2)^2\xi, & 0 < \xi \leq 2, \\ -1 + \frac{1}{8}\alpha t^2 + \xi, & 2 < \xi \leq 3, \\ 2 - \frac{1}{8}(6-7\alpha)t^2 + \frac{1}{4}(1-\alpha)t^2\xi, & 3 \leq \xi < 4, \\ -2 + \frac{1}{8}(2-\alpha)t^2 + \xi, & 4 < \xi, \end{cases}$$

see Figure 2. Note that, for any $t \in (0,2)$ and $\alpha \neq 1$ the function $y(\cdot,t)$ is strictly increasing and hence invertible. In particular, one has, slightly abusing the notation,

$$y^{-1}(x,t) = \begin{cases} -t + \frac{1}{8}(2-\alpha)t^2 + x, & x \leq t - \frac{1}{8}(2-\alpha)t^2, \\ \frac{-8t+(2-\alpha)t^2+8x}{(t-2)^2}, & t - \frac{1}{8}(2-\alpha)t^2 < x \leq 1 + \frac{1}{8}\alpha t^2, \\ 1 - \frac{1}{8}\alpha t^2 + x, & 1 + \frac{1}{8}\alpha t^2 < x \leq 2 + \frac{1}{8}\alpha t^2, \\ \frac{-16+(6-7\alpha)t^2+8x}{2(1-\alpha)t^2}, & 2 + \frac{1}{8}\alpha t^2 < x \leq 2 + \frac{1}{8}(2-\alpha)t^2, \\ 2 - \frac{1}{8}(2-\alpha)t^2 + x & 2 + \frac{1}{8}(2-\alpha)t^2 < x, \end{cases}$$

and inserting this into $U(\xi,t)$ we obtain the solution for $t \in (0,2)$,

$$u(x,t) = \begin{cases} 1 - \frac{1}{4}(2-\alpha)t, & x \leq t - \frac{1}{8}(2-\alpha)t^2, \\ \frac{-4-\alpha t+4x}{2(t-2)}, & t - \frac{1}{8}(2-\alpha)t^2 < x \leq 1 + \frac{1}{8}\alpha t^2, \\ \frac{1}{4}\alpha t, & 1 + \frac{1}{8}\alpha t^2 < x \leq 2 + \frac{1}{8}\alpha t^2, \\ \frac{2x-4}{t}, & 2 + \frac{1}{8}\alpha t^2 < x \leq 2 + \frac{1}{8}(2-\alpha)t^2, \\ \frac{1}{4}(2-\alpha)t & 2 + \frac{1}{8}(2-\alpha)t^2 < x. \end{cases}$$

The following calculations are for $\alpha \neq 1$. Using the mapping M, given by Definition 2.5, we can calculate μ and ν for $t \in (0,2)$. For any Borel set A of $\mathbb{R}$, we get

$$\begin{aligned} \mu(A,t) &= \int_{y^{-1}(A,t)} V_\xi(\xi,t)\, d\xi \\ &= \int_{y^{-1}(A,t)} \frac{1}{2}\mathbb{1}_{(0,2]}(\xi)\, d\xi + \int_{y^{-1}(A,t)} (1-\alpha)\mathbb{1}_{(3,4]}(\xi)\, d\xi \\ &= \int_{y^{-1}(A\cap(t-\frac{1}{8}(2-\alpha)t^2, 1+\frac{1}{8}\alpha t^2],t)} \frac{1}{2}\, d\xi \\ &\quad + \int_{y^{-1}(A\cap(2+\frac{1}{8}\alpha t^2, 2+\frac{1}{8}(2-\alpha)t^2],t)} (1-\alpha) d\xi \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_A \mathbb{1}_{(t-\frac{1}{8}(2-\alpha)t^2, 1+\frac{1}{8}\alpha t^2]}(y^{-1}(x,t))_x \, dx \\
&\qquad + (1-\alpha)\int_A \mathbb{1}_{(2+\frac{1}{8}\alpha t^2, 2+\frac{1}{8}(2-\alpha)t^2]}(y^{-1}(x,t))_x \, dx \\
&= \frac{4}{(t-2)^2}\int_A \mathbb{1}_{(t-\frac{1}{8}(2-\alpha)t^2, 1+\frac{1}{8}\alpha t^2]}(x) \, dx \\
&\qquad + \frac{4}{t^2}\int_A \mathbb{1}_{(2+\frac{1}{8}\alpha t^2, 2+\frac{1}{8}(2-\alpha)t^2]}(x) \, dx \\
&= \int_A u_x^2(x,t) \, dx,
\end{aligned}
$$

and for ν, we find

$$
\begin{aligned}
\nu(A,t) &= \int_{y^{-1}(A,t)} H_\xi(\xi,t) \, d\xi \\
&= \int_{y^{-1}(A,t)} \left(\frac{1}{2}\mathbb{1}_{(0,2]}(\xi) + \mathbb{1}_{(3,4]}(\xi)\right) d\xi \\
&= \int_A u_x^2(x,t) \, dx + \alpha \int_{y^{-1}(A\cap(2+\frac{1}{8}\alpha t^2, 2+\frac{1}{8}(2-\alpha)t^2], t)} d\xi \\
&= \mu(A,t) + 4\frac{\alpha}{(1-\alpha)t^2}\int_A \mathbb{1}_{(2+\frac{1}{8}\alpha t^2, 2+\frac{1}{8}(2-\alpha)t^2]}(x) \, dx.
\end{aligned}
$$

Similar calculations yield for $\alpha = 1$ and any Borel set A of $\mathbb{R}$,

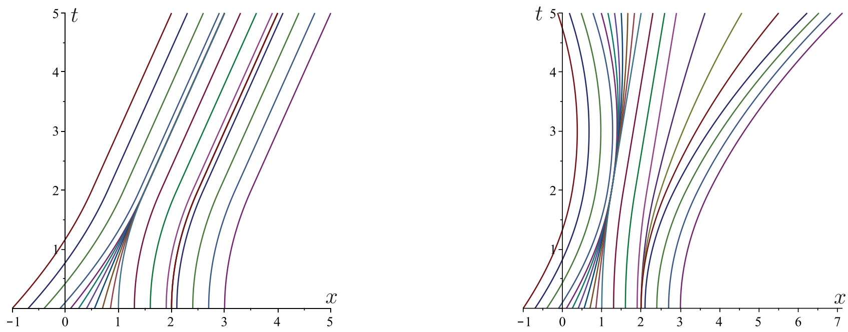
$$
\begin{aligned}
\mu(A,t) &= \int_A u_x^2(x,t) \, dx, \\
\nu(A,t) &= \mu(A,t) + \delta_{\{2+\frac{t^2}{8}\}}(A).
\end{aligned}
$$

We can now compare this example with $\alpha = 1$ to Example 3.3. Both choices of ν_0 lead to the same solution (u,μ) in Eulerian coordinates. So, for the given initial data (u_0,μ_0), there is an equivalence class consisting of triplets (u_0,μ_0,ν_0) leading to the same solution (u,μ). However, different choices of ν lead to quadruples in Lagrangian coordinates that cannot be identified using relabeling and hence their distance with respect to $d_{\mathcal{D}}$, cf. (41), will be greater than zero.

We do not know ν, hence when going backwards in time our metric in Eulerian coordinates can only be defined using u and μ. We define the metric in a similar way to how we defined our J in the previous section. We first define the set $\mathcal{D}_{0,M}$, which is our original set $\mathcal{D}$ without the ν, with an additional assumption that our energy measure is bounded. This will be necessary to ensure that our construction satisfies the definition of a metric. Let

$$
\mathcal{D}_{0,M} := \left\{ (u,\mu) \in E_2 \times \mathcal{M}^+(\mathbb{R}) \;\middle|\; \begin{array}{c} \mu_{ac} = u_x^2 \, dx, \mu(\mathbb{R}) \leq M, \\ \text{and } \mu = u_x^2 \, dx \text{ if } \alpha = 1 \end{array} \right\}. \tag{42}
$$

Then, for $\hat{Y} = (u,\mu) \in \mathcal{D}_{0,M}$, define the set $\mathcal{V}(\hat{Y})$ to be the set of all $\nu \in \mathcal{M}^+(\mathbb{R})$ satisfying



(a) $\alpha = 1$ (b) $\alpha = 0.5$

Figure 2: Plots of the characteristics for the initial data in Example 4.2. Note the initial density causes characteristics to grow from a single point in the $\alpha = 0.5$ case, while in the $\alpha = 1$ case the loss of energy causes them to stick together.

- $\mu \leq \nu \in \mathcal{M}^+(\mathbb{R})$,
- $\mu\big((-\infty, x)\big) - \chi_+(x)\mu(\mathbb{R}) \in L^2(\mathbb{R})$
- If $\alpha = 1$, $\nu_{ac} = \mu = u_x^2\ dx$,
- If $0 \leq \alpha < 1$, $\frac{d\mu}{d\nu}(x) \in \{1, 1-\alpha\}$, and $\frac{d\mu}{d\nu} = 1$ if $u_x(x) < 0$.

Consider $(u, \mu) \in \mathcal{D}_{0,M}$. We note the following inequality,

$$\int_{\mathbb{R}} u_x^2(x)\ dx \leq \mu(\mathbb{R}) \leq M. \tag{43}$$

Define the mapping $J_{\mathcal{D}} : \mathcal{D}_{0,M}^2 \to \mathbb{R}$ as

$$J_{\mathcal{D}}(\hat{Y}_1, \hat{Y}_2) = \inf_{(\nu_1,\nu_2)\in\mathcal{V}(\hat{Y}_1)\times\mathcal{V}(\hat{Y}_2)} d_{\mathcal{D}}((u_1, \mu_1, \nu_1), (u_2, \mu_2, \nu_2)). \tag{44}$$

We encounter a similar problem as to our metric on the previous set of equivalence classes in $\mathcal{F}$. We cannot conclude that the triangle inequality is satisfied for this distance.

Following a similar construction as before, we define the metric $d_M : \mathcal{D}_{0,M}^2 \to \mathbb{R}$ by

$$d_M(\hat{Y}_A, \hat{Y}_B) := \inf_{\hat{\mathcal{D}}(Y_A, Y_B)} \sum_{n=1}^{N} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}), \tag{45}$$

where the infimum is taken over $\hat{\mathcal{D}}(\hat{Y}_1, \hat{Y}_2)$, the set of all finite sequences $\{\hat{Y}_i\}_{i=1}^N$ in $\mathcal{D}_{0,M}$ satisfying $\hat{Y}_0 = \hat{Y}_A$ and $\hat{Y}_N = \hat{Y}_B$. The following result ensures this is a metric.

Lemma 4.3. *The function $d_M : \mathcal{D}_{0,M}^2 \to \mathbb{R}$ given by* (45) *defines a metric on $\mathcal{D}_{0,M}$.*

Proof. Symmetry is immediate, as the distance d_M, if you dig deep enough, is constructed of metrics.
The triangle inequality is more challenging. Let $\hat{Y}_A$, $\hat{Y}_B$, $\hat{Y}_C \in \mathcal{D}_{0,M}$. Choose $\epsilon > 0$. Select two sequences

- $\{\hat{Y}_i\}_{i=0}^{N_1}$ in $\hat{\mathcal{D}}(\hat{Y}_A, \hat{Y}_B)$, and
- $\{\hat{Y}_i\}_{i=N_1}^{N_2}$ in $\hat{\mathcal{D}}(\hat{Y}_B, \hat{Y}_C)$,

where $N_1, N_2 \in \mathbb{N}$ and $N_1 < N_2$, such that

- $\sum_{n=1}^{N_1} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}) \leq d_M(\hat{Y}_A, \hat{Y}_B) + \epsilon$, and
- $\sum_{n=N_1+1}^{N_2} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}) \leq d_M(\hat{Y}_B, \hat{Y}_C) + \epsilon$.

Then

$$\begin{aligned} d_M(\hat{Y}_A, \hat{Y}_C) \leq \sum_{n=1}^{N_2} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}) &= \sum_{n=1}^{N_1} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}) + \sum_{n=N_1+1}^{N_2} J_{\mathcal{D}}(\hat{Y}_n, \hat{Y}_{n-1}) \\ &\leq d_M(\hat{Y}_A, \hat{Y}_B) + d_M(\hat{Y}_B, \hat{Y}_C) + 2\epsilon. \end{aligned}$$

As one can make a similar construction for any $\epsilon > 0$, the inequality involving the RHS and LHS is satisfied for any $\epsilon > 0$, and hence

$$d_M(\hat{Y}_A, \hat{Y}_C) \leq d_M(\hat{Y}_A, \hat{Y}_B) + d_M(\hat{Y}_B, \hat{Y}_C).$$

It remains to show the zero condition, that is

$$d_M(\hat{Y}_A, \hat{Y}_B) = 0 \quad \text{if and only if} \quad \hat{Y}_A = \hat{Y}_B.$$

First, set $\hat{Y} = \hat{Y}_A = \hat{Y}_B$, and let $\nu \in \mathcal{V}(\hat{Y})$, we have

$$0 \leq d_M(\hat{Y}, \hat{Y}) \leq d_{\mathcal{D}}((\hat{u}, \hat{\mu}, \nu), (\hat{u}, \hat{\mu}, \nu)) = 0.$$

Thus we obtain the backward implication for this statement. The forward implication is more challenging.

Suppose $d_M(\hat{Y}_A, \hat{Y}_B) = 0$. Let $\epsilon > 0$, and select a sequence $\{Y_n\}_{n=0}^N$ in $\mathcal{D}$ with $\mu_n(\mathbb{R}) \leq M$ for all n, $(u_0, \mu_0) = (u_A, \mu_A)$, and $(u_N, \mu_N) = (u_B, \mu_B)$, such that

$$\sum_{n=1}^{N} d_{\mathcal{D}}(Y_n, Y_{n-1}) < d_M(\hat{Y}_A, \hat{Y}_B) + \frac{2}{5}\epsilon = \frac{2}{5}\epsilon.$$

Such a sequence exists because of the definition of the infimum.

Setting $X_n = L(Y_n)$, and using Lemma 3.5 together with (41), we have

$$\sum_{n=1}^{N} \|X_n - X_{n-1}\| \leq \frac{5}{2} \sum_{n=1}^{N} d_{\mathcal{D}}(Y_n, Y_{n-1}) < \epsilon. \tag{46}$$

Immediately from the definition of the norm $\|\cdot\|$, given by (28), we have that

$$\sum_{n=1}^{N} \|y_n - y_{n-1}\|_\infty < \epsilon \quad \text{and} \quad \sum_{n=1}^{N} \|U_n - U_{n-1}\|_\infty < \epsilon. \tag{47}$$

Let $X_A = X_0 = L(Y_0)$ and $X_B = X_N = L(Y_N)$. Note that y_A and y_B are continuous and increasing, by Definition (2.4). Thus for any $x \in \mathbb{R}$, there are ξ_A and ξ_B such that $y_A(\xi_A) = x = y_B(\xi_B)$. Substituting this into the difference of the u's, we get

$$\begin{aligned}
|u_A(x) - u_B(x)| &= |u_A(y_A(\xi_A)) - u_B(y_A(\xi_A))| \\
&\leq |u_A(y_A(\xi_A)) - u_B(y_B(\xi_A))| + |u_B(y_B(\xi_A)) - u_B(y_A(\xi_A))| \\
&= |U_A(\xi_A) - U_B(\xi_A)| + \left| \int_{y_A(\xi_A)}^{y_B(\xi_A)} u_{B,x}(x)\, dx \right| \\
&\leq \|U_A - U_B\|_\infty + \sqrt{|y_A(\xi_A) - y_B(\xi_A)|} \sqrt{\left| \int_{y_A(\xi_A)}^{y_B(\xi_A)} u_{B,x}^2(x)\, dx \right|} \\
&\leq \sum_{n=1}^{N} \|U_n - U_{n-1}\|_\infty + \sqrt{\sum_{n=1}^{N} \|y_n - y_{n-1}\|_\infty} \sqrt{M} \\
&< \epsilon + \sqrt{\epsilon M},
\end{aligned}$$

where we have used the Cauchy Schwartz inequality to split our integral, and (43). As this is satisfied for any $\epsilon > 0$, one has $u_A = u_B$.

We now show $\mu_A = \mu_B$. From [7, Section 7.3], we need only to show that

$$\int_{\mathbb{R}} f(x)\, d\mu_A(x) = \int_{\mathbb{R}} f(x)\, d\mu_B(x), \qquad \text{for all } f \in C_0(\mathbb{R}), \tag{48}$$

where $C_0(\mathbb{R})$ denotes the set of all continuous functions whom vanish at $\pm\infty$. Using that $C_c^\infty(\mathbb{R})$ is a dense subset of $C_0(\mathbb{R})$, it suffices to show (48) for any $f \in C_c^\infty(\mathbb{R})$.

Let $f \in C_c^\infty(\mathbb{R})$, then

$$\begin{aligned}
\int_{\mathbb{R}} f(x)(d\mu_A - d\mu_B)(x) &= \int_{\mathbb{R}} [(f \circ y_A)(\xi) V_{A,\xi}(\xi) - (f \circ y_B)(\xi) V_{B,\xi}(\xi)]\, d\xi \\
&= \int_{\mathbb{R}} (f \circ y_A)(\xi)(V_{A,\xi}(\xi) - V_{B,\xi}(\xi))\, d\xi \\
&\quad + \int_{\mathbb{R}} [(f \circ y_A)(\xi) - (f \circ y_B)(\xi)] V_{B,\xi}(\xi)\, d\xi
\end{aligned}$$

We show these two integrals equal zero.

For the first of these two integrals use integration by parts,

$$\int_{\mathbb{R}} (f \circ y_A)(\xi)(V_{A,\xi}(\xi) - V_{B,\xi}(\xi))\, d\xi = -\int_{\mathbb{R}} y_{A,\xi}(\xi)(f' \circ y_A)(\xi)(V_A(\xi) - V_B(\xi))\, d\xi.$$

Using that $0 \leq y_{A,\xi} \leq 1$, we have that

$$\int_{\mathbb{R}} |y_{A,\xi}(\xi)(f' \circ y_A)(\xi)(V_A(\xi) - V_B(\xi))| \, d\xi \leq \|f'\|_1 \|V_A - V_B\|_\infty \leq \|f'\|_1 \epsilon,$$

where we have used that (46) implies

$$\|V_A - V_B\|_\infty \leq \sum_{n=1}^{N} \|X_n - X_{n-1}\| < \epsilon.$$

For the second integral, we use

$$\int_{\mathbb{R}} |(f \circ y_A)(\xi) - (f \circ y_B)(\xi)| V_{B,\xi}(\xi) \, d\xi \\ \leq \|(f \circ y_A)(\xi) - (f \circ y_B)(\xi)\|_\infty \|V_{B,\xi}\|_1.$$

We have that $\|V_{B,\xi}\|_1 \leq M$. Also,

$$|(f \circ y_A)(\xi) - (f \circ y_B)(\xi)| \leq \left| \int_{y_A(\xi)}^{y_B(\xi)} f'(\eta) \, d\eta \right| \leq \|y_B - y_A\|_\infty \|f'\|_\infty < \epsilon \|f'\|_\infty,$$

and thus

$$\|(f \circ y_A)(\xi) - (f \circ y_B)(\xi)\|_\infty \|V_{B,\xi}\|_1 < \epsilon \|f'\|_\infty M.$$

Once again, this is true for any $\epsilon > 0$, and hence the integrals are zero, concluding the proof. □

From this, we can conclude with our final Lipschitz stability result.

Theorem 4.4. *Let $\hat{Y}_A(t) = (u_A, \mu_A)(t)$ and $\hat{Y}_B(t) = (u_B, \mu_B)(t)$ be α-dissipative solutions at time t to the problem*

$$u_t(x,t) + u u_x(x,t) = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(y,t) \, dy - \int_{x}^{+\infty} u_x^2(y,t) \, dy \right), \tag{49}$$

with initial data $\hat{Y}_A(0), \hat{Y}_B(0) \in \mathcal{D}_{0,M}$ respectively. Then

$$d_M(\hat{Y}_A(t), \hat{Y}_B(t)) \leq e^{\frac{3}{2}t} d_M(\hat{Y}_A(0), \hat{Y}_B(0)).$$

Proof. Let $\epsilon > 0$, and choose a finite sequence $\{Y_i(t)\}_{i=0}^N$ of α-dissipative solutions to the partial differential equation (49) in $\mathcal{D}$, with initial data$\{Y_i(0)\}_{i=0}^N$ in $\mathcal{D}$ satisfying $(u_0, \mu_0)(0) = (u_A, \mu_A)(0), (u_N, \mu_N)(0) = (u_B, \mu_B)(0)$, $\mu_i(\mathbb{R}) \leq M$ for all $i = 1, \ldots, N$, and such that

$$\sum_{n=1}^{N} d_{\mathcal{D}}(Y_n(0), Y_{n-1}(0)) < d_M(\hat{Y}_A(0), \hat{Y}_B(0)) + \epsilon.$$

Then, we have using Corollary 4.1

$$d_M(\hat{Y}_A(t), \hat{Y}_B(t)) \leq \sum_{n=1}^{N} d_{\mathcal{D}}\Big(Y_n(t), Y_{n-1}(t)\Big)$$

$$
\begin{aligned}
&\leq e^{\frac{3}{2}t} \sum_{n=1}^{N} d_{\mathcal{D}}\Big(Y_n(0), Y_{n-1}(0)\Big) \\
&< e^{\frac{3}{2}t}(d_M(\hat{Y}_A(0), \hat{Y}_B(0)) + \epsilon).
\end{aligned}
$$

As one can construct such a relation for any $\epsilon > 0$, we obtain the required result. □

A Examples

Example A.1. *We compute an α-dissipative example with $\alpha = \frac{1}{3}$. Given*

$$
u_0(x) = \begin{cases} 1, & x \leq -2, \\ -1 - x, & -2 < x \leq -1, \\ 0, & -1 < x \leq 1, \\ 1 - x, & 1 < x \leq 2, \\ -1, & 2 < x, \end{cases} \qquad \mu_0 = \nu_0 = u_{0,x}^2(x)\ dx,
$$

so that

$$
\mu_0((-\infty, x)) = \nu_0((-\infty, x)) = \begin{cases} 0, & x \leq -2, \\ x + 2, & -2 < x \leq -1, \\ 1, & -1 < x \leq 1, \\ x, & 1 < x \leq 2, \\ 2, & 2 < x, \end{cases}
$$

then the transformation L, given by Definition 2.4, yields

$$
y_0(\xi) := \begin{cases} \xi, & \xi \leq -2, \\ -1 + \frac{1}{2}\xi, & -2 < \xi \leq 0, \\ -1 + \xi, & 0 < \xi \leq 2, \\ \frac{1}{2}\xi, & 2 < \xi \leq 4, \\ -2 + \xi, & 4 < \xi, \end{cases} \qquad U_0(\xi) = \begin{cases} 1, & \xi \leq -2, \\ -\frac{1}{2}\xi, & -2 < \xi \leq 0, \\ 0, & 0 < \xi \leq 2, \\ 1 - \frac{1}{2}\xi, & 2 < \xi \leq 4, \\ -1, & 4 < \xi, \end{cases}
$$

and

$$
V_0(\xi) = H_0(\xi) = \begin{cases} 0, & \xi \leq -2, \\ 1 + \frac{1}{2}\xi, & -2 < \xi \leq 0, \\ 1, & 0 < \xi \leq 2, \\ \frac{1}{2}\xi, & 2 < \xi \leq 4, \\ 2, & 4 < \xi. \end{cases}
$$

Next, we determine for which points $\xi \in \mathbb{R}$ wave breaking will occur and when. Using (7), we have

$$
\tau(\xi) = \begin{cases} 2, & \xi \in (-2, 0) \cup (2, 4), \\ \infty, & \textit{otherwise}. \end{cases}
$$

Computing the solution using (6)*, one obtains*

$$y(\xi,t)=\begin{cases}\begin{cases}t-\frac{1}{4}t^2+\xi, & \xi\leq -2,\\ -1+\frac{(t-2)^2}{8}\xi, & -2<\xi\leq 0,\\ -1+\xi, & 0<\xi\leq 2,\\ t-\frac{1}{4}t^2+\frac{(t-2)^2}{8}\xi, & 2<\xi\leq 4,\\ -2-t+\frac{1}{4}t^2+\xi, & 4<\xi,\end{cases} & 0\leq t<2,\\ \begin{cases}\frac{1}{3}+\frac{2}{3}t-\frac{1}{6}t^2+\xi, & \xi\leq -2,\\ -1+\frac{(t-2)^2}{12}\xi, & -2<\xi\leq 0,\\ -1+\xi & 0<\xi\leq 2,\\ \frac{1}{3}+\frac{2}{3}t-\frac{1}{6}t^2+\frac{(t-2)^2}{12}\xi, & 2<\xi\leq 4,\\ -\frac{7}{3}-\frac{2}{3}t+\frac{1}{6}t^2+\xi, & 4<\xi,\end{cases} & 2\leq t,\end{cases}$$

$$U(\xi,t)=\begin{cases}\begin{cases}1-\frac{1}{2}t, & \xi\leq -2,\\ \frac{(t-2)}{4}\xi, & -2<\xi\leq 0,\\ 0, & 0<\xi\leq 2,\\ 1-\frac{1}{2}t+\frac{(t-2)}{4}\xi, & 2<\xi\leq 4,\\ -1+\frac{1}{2}t, & 4<\xi,\end{cases} & 0\leq t<2,\\ \begin{cases}\frac{2}{3}-\frac{1}{3}t, & \xi\leq -2,\\ \frac{(t-2)}{6}\xi, & -2<\xi\leq 0,\\ 0, & 0<\xi\leq 2,\\ \frac{2}{3}-\frac{1}{3}t+\frac{(t-2)}{6}\xi, & 2<\xi\leq 4,\\ -\frac{2}{3}+\frac{1}{3}t, & 4<\xi,\end{cases} & 2\leq t,\end{cases}$$

$$H(\xi,t)=H_0(\xi),\quad 0\leq t,$$

and

$$V(\xi,t)=\begin{cases}H(\xi), & 0\leq t<2,\\ \begin{cases}0, & \xi\leq -2,\\ \frac{2}{3}+\frac{1}{3}\xi, & -2<\xi\leq 0,\\ \frac{2}{3}, & 0<\xi\leq 2,\\ \frac{1}{3}\xi, & 2<\xi\leq 4,\\ \frac{4}{3}, & 4<\xi,\end{cases} & 2\leq t.\end{cases}$$

Using Definition 2.5*, we can finally compute the solution* (u,μ,ν)*, which is* *given by*

$$u(x,t) = \begin{cases} \begin{cases} 1-\frac{1}{2}t, & x \le -2+t-\frac{1}{4}t^2, \\ \frac{2+2x}{t-2}, & -2+t-\frac{1}{4}t^2 < x \le -1, \\ 0, & -1 < x \le 1, \\ \frac{-2+2x}{t-2}, & 1 < x \le 2-t+\frac{1}{4}t^2, \\ -1+\frac{1}{2}t, & 2-t+\frac{1}{4}t^2 < x, \end{cases} & t < 2, \\ 0, & t = 2 \\ \begin{cases} \frac{2}{3}-\frac{1}{3}t, & x \le -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2, \\ \frac{2+2x}{t-2}, & -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2 < x \le -1, \\ 0, & -1 < x \le 1, \\ \frac{-2+2x}{t-2}, & 1 < x \le \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2, \\ -\frac{2}{3}+\frac{1}{3}t, & \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2 < x, \end{cases} & 2 < t, \end{cases}$$

$$\mu(t,(-\infty,x)) = \begin{cases} \begin{cases} 0, & x \le -2+t-\frac{1}{4}t^2, \\ 1+\frac{4+4x}{(t-2)^2}, & -2+t-\frac{1}{4}t^2 < x \le -1, \\ 1, & -1 < x \le 1, \\ 1+\frac{-4+4x}{(t-2)^2}, & 1 < x \le 2-t+\frac{1}{4}t^2, \\ 2, & 2-t+\frac{1}{4}t^2 < x, \end{cases} & t < 2, \\ \begin{cases} 0, & x \le -1, \\ \frac{2}{3}, & -1 < x \le 1, \\ \frac{4}{3}, & 1 < x, \end{cases} & t = 2 \\ \begin{cases} 0, & x \le -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2, \\ \frac{2}{3}+\frac{4+4x}{(t-2)^2}, & -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2 < x \le -1, \\ \frac{2}{3}, & -1 < x \le 1, \\ \frac{2}{3}+\frac{-4+4x}{(t-2)^2}, & 1 < x \le \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2, \\ \frac{4}{3}, & \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2 < x, \end{cases} & 2 < t, \end{cases}$$

and

$$\nu(t,(-\infty,x)) = \begin{cases} \begin{cases} 0, & x \leq -2+t-\frac{1}{4}t^2, \\ 1+\frac{4+4x}{(t-2)^2}, & -2+t-\frac{1}{4}t^2 < x \leq -1, \\ 1, & -1 < x \leq 1, \\ 1+\frac{-4+4x}{(t-2)^2}, & 1 < x \leq 2-t+\frac{1}{4}t^2, \\ 2, & 2-t+\frac{1}{4}t^2 < x, \end{cases} & t < 2, \\ \begin{cases} 0, & x \leq -1, \\ 1, & -1 < x \leq 1, \\ 2, & 1 < x, \end{cases} & t = 2 \\ \begin{cases} 0, & x \leq -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2, \\ 1+\frac{6+6x}{(t-2)^2}, & -\frac{5}{3}+\frac{2}{3}t-\frac{1}{6}t^2 < x \leq -1, \\ 1, & -1 < x \leq 1, \\ 1+\frac{-6+6x}{(t-2)^2}, & 1 < x \leq \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2, \\ 2, & \frac{5}{3}-\frac{2}{3}t+\frac{1}{6}t^2 < x, \end{cases} & 2 < t. \end{cases}$$

Notice that ν carries the initial energy forward in time, while μ is the actual energy in the system at the current time. Thus the difference in the two is the lost energy.

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Paper 2

A Lipschitz Metric For α-Dissipative Solutions To The Hunter–Saxton Equation

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A Lipschitz Metric for α-Dissipative Solutions to the Hunter–Saxton Equation

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Abstract

We explore the Lipschitz stability of solutions to the Hunter–Saxton equation with respect to the initial data. In particular, we study the stability of α-dissipative solutions constructed using a generalised method of characteristics approach, where α is a function determining the energy loss at each position in space.

1 Introduction

In this work we study particular solutions to the Hunter–Saxton equation, which is given by

$$u_t(x,t) + uu_x(x,t) = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(y,t)\,dy - \int_{x}^{+\infty} u_x^2(y,t)\,dy\right). \tag{HS}$$

To be precise, our goal is to define a metric for which α-dissipative solutions, constructed using a generalised method of characteristics, are Lipschitz continuous with respect to the initial data.

Equation (HS) was first introduced by Hunter and Saxton to model nonlinear instability in the director field for nematic liquid crystals [11]. The physical applications of this equation are not the focus of this paper, however.

Solutions to this equation may experience singularities in finite time. Specifically, a solution u will remain bounded and continuous, while its spatial derivative will diverge to $-\infty$ at certain points. Parts of the energy, calculated using the energy density function u_x^2, initially spread over sets of positive measure, will concentrate onto sets of Lebesgue measure zero. These singularities are referred to as "wave-breaking", and how one treats the concentrated energy after these points in time determines the solution.

Discarding the concentrated energy, one obtains dissipative solutions, for which existence and uniqueness have been shown [1, 4]. On the other hand, one could retain the energy, obtaining so called conservative solutions, in which

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case existence has been shown in [2, 12], and uniqueness in [5]. Finally, one could choose to remove an α proportion of the energy, with $\alpha \in [0, 1]$. These are known as α-dissipative solutions, for whom existence has been established in [8].

This paper focuses on the importance of the energy in the notion of a solution to our problem. To be precise, an α-dissipative solution to the Cauchy problem of (HS) is a pair (u, μ) satisfying

$$u_t + uu_x = \frac{1}{4}\left(\int_{-\infty}^{x} d\mu - \int_{x}^{+\infty} d\mu\right) \tag{1a}$$

$$\mu_t + (u\mu)_x \leq 0, \tag{1b}$$

in the distributional sense. The second measure valued PDE inequality tracks the energy, and correspondingly the variable μ is a positive finite Radon measure representing the current energy.

To motivate where equation (1b) comes from, formally consider $u \in C^2(\mathbb{R} \times [0, +\infty))$, such that $\mu = u_x^2(\cdot, t) \in L^2(\mathbb{R})$ for all times $t \geq 0$. Then

$$(u_x^2)_t = 2u_x u_{xt} = 2u_x(-(uu_x)_x + \frac{1}{2}u_x^2) = -u_x^3 - 2uu_x u_{xx} = -(uu_x^2)_x. \tag{2}$$

In other words, equation (1b) is satisfied with equality, and $\mu(t, \mathbb{R}) = \mu(0, \mathbb{R})$ for all times $t \geq 0$. This is thus a fully conservative solution. In reality, global solutions experience weaker regularity than we have assumed here, due to wave-breaking. Furthermore, discarding part of the concentrated energy at wave breaking yields a loss of energy resulting in (1b).

The prequel to this piece of work [9] takes α to be constant, and a Lipschitz metric in time was constructed. However, we had to assume that the two solutions one is measuring the distance between share the same α. This paper continues this work, constructing a new Lipschitz stable metric for the case where α is now possibly different for both solutions, and is a function of space. In this scenario, the amount of energy lost is determined by the point where the energy concentrates. In particular, we are looking for a metric d that satisfies the estimate

$$d\big((u_A(t), \mu_A(t)), (u_B(t), \mu_B(t))\big) \leq e^{At} d\big((u_A(0), \mu_A(0)), (u_B(0), \mu_B(0))\big), \tag{3}$$

for all $t \geq 0$. Here $A \in \mathbb{R}$ is some positive constant. The method we use is developed from [12], where a Lipschitz metric for conservative solutions has been found using ideas from [2]. An alternative construction making use of pseudo-inverses was employed in [3]. In [1], a metric satisfying property (3) has also been found for dissipative solutions, in addition to Lipschitz continuity with respect to the variable t. This metric uses an optimal transportation approach, constructing a Wasserstein / Kantorovich-Rubenstein inspired cost function over transportation plans, and minimising over said plans.

A generalised method of characteristics is used to construct α-dissipative solutions to (HS) and to define a metric. Up until wave breaking occurs, we can define our Lagrangian coordinates (y, U, V) by

$$y_t(\xi,t) = u(y(\xi,t),t), \tag{4a}$$
$$U(\xi,t) = u(y(\xi,t),t), \tag{4b}$$
$$V(\xi,t) = \int_{-\infty}^{y(\xi,t)} u_x^2(z,t)\,dz, \tag{4c}$$

with ξ a parameter, the so called "label" of the characteristic $y(\xi,t)$. From the classical sense of Lagrangian coordinates, we may sometimes refer to ξ as a "particle".

This leads to an ODE system, given by

$$y_t(\xi,t) = U(\xi,t), \tag{5a}$$
$$U_t(\xi,t) = \frac{1}{2}V(\xi,t) - \frac{1}{4}\lim_{\xi\to\infty} V(\xi,t), \tag{5b}$$
$$V_t(\xi,t) = 0. \tag{5c}$$

Assuming that no wave breaking occurs at time zero, one can take as initial data $y(\xi,0) = \xi$. The first two variables y and U represent respectively the position and velocity of particles $\xi \in \mathbb{R}$ as usual, while the third variable V corresponds to the μ in Eulerian variables, and represents the cumulative energy up to particle ξ.

To demonstrate where the third ODE comes from, once again formally consider a sufficiently smooth solution u such that (2) is satisfied. Then, differentiating (4c) with respect to time,

$$\begin{aligned} V_t(\xi,t) &= U(\xi,t)u_x^2(y(\xi,t),t) + \int_{-\infty}^{y(\xi,t)} (u_x^2(x,t))_t\,dx \\ &= U(\xi,t)u_x^2(y(\xi,t),t) - U(\xi,t)u_x^2(y(\xi,t),t) = 0. \end{aligned}$$

Wave breaking in Lagrangian coordinates corresponds to a collection of characteristics colliding. Specifically, wave breaking occurs at time $\tau(\xi)$ for $\xi \in \mathbb{R}$ when $y_\xi(\xi,\tau(\xi)) = 0$. In the case of piecewise affine and continuous solutions in Lagrangian coordinates, this corresponds to intervals where the function $y(\cdot,\tau)$ is constant. The desire to characterise this behaviour at time zero is what prevents us from simply taking $y(\xi,0) = \xi$, as such initial data does not contain concentrated particles initially. This problem is overcome by applying the mappings between Eulerian and Lagrangian coordinates, introduced in [10] in the context of the Camassa–Holm equation.

For a given $\xi \in \mathbb{R}$, the wave breaking time $\tau(\xi)$ can be calculated using the initial data for the ODE system (5). In particular,

$$\tau(\xi) = \begin{cases} -2\frac{y_\xi(\xi,0)}{U_\xi(\xi,0)}, & U_\xi(\xi,0) < 0, \\ 0, & U_\xi(\xi,0) = 0 = y_\xi(\xi,0), \\ +\infty, & \text{otherwise}. \end{cases}$$

For a fully conservative solution the system (5) determines the solution for all time. On the opposite end of the spectrum, a fully dissipative solution corresponds to the system formed by equations (5a) and (5b), but setting $V_\xi(\xi, t) = 0$ for $t \geq \tau(\xi)$.

In the general case, $\alpha : \mathbb{R} \to [0, 1]$, the energy loss at wave breaking is dependent on the particles position at time $\tau(\xi)$, and is given by $\alpha(y(\xi, \tau(\xi)))$. The α-dissipative solution is thus given by (5a) and (5b), and setting

$$V(\xi, t) = \int_{-\infty}^{\xi} V_\xi(\eta, 0)\big(1 - \alpha(y(\eta, \tau(\eta)))\mathbb{1}_{\{r\in\mathbb{R} \mid t\geq\tau(r)>0\}}(\eta)\big)\, d\eta.$$

Using this, one obtains the conservative solution in the case $\alpha \equiv 0$, and the fully dissipative solution in the case $\alpha \equiv 1$.

The construction of our metric will take advantage of the approachable properties of these Lagrangian coordinates. The general idea is as follows. First, we establish how one transforms between Eulerian and Lagrangian coordinates. Second, we define a suitable metric in Lagrangian coordinates, satisfying Lipschitz stability with respect to the initial Lagrangian data, similar to inequality (3). Finally, we define a suitable metric in Eulerian coordinates by measuring the distance of the corresponding Lagrangian coordinates, inheriting the Lipschitz stability we require.

The paper is organised as follows. Section 2 begins with the setup of the relevant spaces for our problem, in both Lagrangian and Eulerian coordinates.

To solve our problem we will need to introduce a secondary dummy measure ν. This will also be a positive finite Radon measure, which is bounded from below by μ and which plays a key role when defining the transformations between Eulerian and Lagrangian coordinates. In Lagrangian variables this will correspond to a function H. Importantly, ν is a necessity in the construction of our Lipschitz metric, but does not form part of the solution to (HS). Therefore we will need to consider equivalence classes with respect to ν when constructing our metric in Eulerian coordinates.

Energy concentrating on sets of measure zero must be accounted for in the definition of the initial characteristics. Thus the next step in Section 2 is to introduce a mapping from Eulerian to Lagrangian coordinates, and vice versa, that account for this initial energy concentration. For three Eulerian coordinates, there will be a corresponding four Lagrangian coordinates. Hence there will be some redundancy, in that multiple Lagrangian coordinates will correspond to the same set of Eulerian coordinates. These will form a set of equivalence classes, related by a group of homeomorphisms called relabelling functions.

Throughout the second section we will introduce relevant established results that we make use of.

In Section 3, we construct a semi-metric in Lagrangian coordinates that satisfies Lipschitz continuity with respect to the initial data. This will form a central part of the construction of our metric.

We will see that the semi-metric we construct in Section 3 is far from optimal, since the distance between two elements of the same equivalence class, i.e. two elements representing the same Eulerian coordinates, can be positive.

In Section 4, we overcome this issue and detail how we construct the metric in Lagrangian coordinates. Additionally, we establish the Lipschitz continuity with respect to the initial data in the Lagrangian setting.

In the final section, Section 5, we return to Eulerian coordinates, using our metric in Lagrangian coordinates to define a Lipschitz metric in time. In this section we have to take equivalence classes with respect to the dummy variable ν into account.

2 Lagrangian and Eulerian coordinates

Before we can begin our construction of the metric, we must set up our Eulerian and Lagrangian coordinate spaces. In addition, we must examine the Lagrangian ODE problem, what it means to be a solution in Eulerian coordinates, and introduce relevant results from past literature. This follows the construction outlined in [2] and [8].

We begin by introducing an important set. Let E be the Banach space of $L^\infty(\mathbb{R})$ functions with $L^2(\mathbb{R})$ weak derivatives, with an associated norm $\|\cdot\|_E$,

$$E \coloneqq \{f \in L^\infty(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}, \quad \|f\|_E = \|f\|_\infty + \|f'\|_2.$$

Furthermore, define $H_i \coloneqq H^1(\mathbb{R}) \times \mathbb{R}^i$ for $i = 1, 2$, and $H_0 = L^2(\mathbb{R}) \times \mathbb{R}$, with the norms

$$\|(f, x)\|_{H_i} = \sqrt{\|f\|_{H^1}^2 + |x|^2}, \quad \|(f, x)\|_{H_0} = \sqrt{\|f\|_2^2 + |x|^2}.$$

We split the real line into two overlapping sets $(-\infty, 1)$ and $(-1, \infty)$, and pick two functions χ^+ and χ^- in $C^\infty(\mathbb{R})$ satisfying the following three properties,

- $\chi^- + \chi^+ = 1$,
- $0 \leq \chi^+ \leq 1$,
- $\operatorname{supp}(\chi^-) \subset (-\infty, 1)$, and $\operatorname{supp}(\chi^+) \subset (-1, \infty)$,

called a partition of unity.

Using these two functions, we define the following two linear, continuous, and injective mappings,

$$R_1 : H_1 \to E \quad (f, a) \overset{R}{\mapsto} \hat{f} = f + a \cdot \chi^+, \tag{6}$$

$$R_2 : H_2 \to E \quad (f, a, b) \mapsto \hat{f} = f + a \cdot \chi^+ + b \cdot \chi^-. \tag{7}$$

They define the following two Banach spaces, which are subsets of E,

$$E_1 \coloneqq R_1(H_1), \quad \|\hat{f}\|_{E_1} = \|R_1^{-1}(\hat{f})\|_{H_1},$$

$$E_2 \coloneqq R_2(H_2), \quad \|\hat{f}\|_{E_2} = \|R_2^{-1}(\hat{f})\|_{H_2}.$$

Note, from (6), operation R is well defined for elements of H_0. We define the set E_0, and the corresponding norm $\|\cdot\|_{E_0}$, by

$$E_0 \coloneqq R(H_0), \qquad \|f\|_{E_0} = \|R^{-1}(f)\|_{H_0}.$$

Finally, our α must lie in the following set,

$$\Lambda \coloneqq W^{1,\infty}(\mathbb{R};[0,1)) \cup \{1\}. \tag{8}$$

Avoiding functions which attain values on $[0,1]$, with 1 inclusive, is necessary to ensure that the mappings between Eulerian and Lagrangian coordinates are invertible with respect to equivalence classes. See Example B.2.

With this setup done, we can define the space of Eulerian coordinates.

Definition 2.1 (Set of Eulerian coordinates - $\mathcal{D}$)**.** *Let $\alpha \in \Lambda$. The set $\mathcal{D}^\alpha$ contains all Y, with $Y = (u,\mu,\nu)$, satisfying the following*

- $u \in E_2$,
- $\mu \leq \nu \in \mathcal{M}^+(\mathbb{R})$,
- $\mu_{\mathrm{ac}} \leq \nu_{\mathrm{ac}} \in \mathcal{M}^+(\mathbb{R})$,
- $\mu_{\mathrm{ac}} = u_x^2\, dx$,
- $\mu\left((-\infty,x)\right) \in E_0$,
- $\nu\left((-\infty,x)\right) \in E_0$,
- *If* $\alpha \equiv 1, \nu_{ac} = \mu = u_x^2\, dx$,
- *If* $\alpha \in W^{1,\infty}(\mathbb{R};[0,1))$, *then* $\frac{d\mu}{d\nu}(x) > 0$, *and* $\frac{d\mu_{\mathrm{ac}}}{d\nu_{\mathrm{ac}}}(x) = 1$ *if* $u_x(x) < 0$, *for any* $x \in \mathbb{R}$,

where $\mathcal{M}^+(\mathbb{R})$ is the set of all finite, positive Radon measures on $\mathbb{R}$.

The set $\mathcal{D}$ is defined as

$$\mathcal{D} \coloneqq \{Y^\alpha \coloneqq (Y,\alpha) \mid \alpha \in \Lambda \text{ and } Y \in \mathcal{D}^\alpha\} = \bigcup_{\alpha\in\Lambda} \left(\mathcal{D}^\alpha \times \{\alpha\}\right).$$

Finally, for M, $L \geq 0$, the subset $\mathcal{D}_M^L$ is given by

$$\mathcal{D}_M^L \coloneqq \{Y^\alpha \in \mathcal{D} \mid \mu(\mathbb{R}) \leq M \text{ and } \|\alpha'\|_\infty \leq L\}. \tag{9}$$

Before defining the Lagrangian coordinates, we introduce a new Banach space B,

$$B \coloneqq E_2 \times E_2 \times E_1 \times E_1,\ \|(f_1,f_2,f_3,f_4)\|_B = \|f_1\|_{E_2} + \|f_2\|_{E_2} + \|f_3\|_{E_1} + \|f_4\|_{E_1}.$$

Definition 2.2 (Set of Lagrangian coordinates - $\mathcal{F}$)**.** *Let $\alpha \in \Lambda$. The set $\mathcal{F}^\alpha$ contains all $X = (y,U,H,V)$ such that $(y - id, U, H, V) \in B$, satisfying*

- $y - id, U, H, V \in W^{1,\infty}(\mathbb{R})$,

- $y_\xi, H_\xi \geq 0$, *and there exists a constant* c *such that* $0 < c < y_\xi + H_\xi$ *a.e.*,
- $y_\xi V_\xi = U_\xi^2$,
- $0 \leq V_\xi \leq H_\xi$ *a.e.*,
- *If* $\alpha \equiv 1, y_\xi(\xi) = 0$ *implies* $V_\xi(\xi) = 0$, $y_\xi(\xi) > 0$ *implies* $V_\xi(\xi) = H_\xi(\xi)$ *a.e.*,
- *If* $0 \leq \alpha < 1$, *there exists* $\kappa : \mathbb{R} \to (0,1]$ *such that* $V_\xi(\xi) = \kappa(y(\xi))H_\xi(\xi)$ *a.e., with* $\kappa(y(\xi)) = 1$ *for* $U_\xi(\xi) < 0$.

The set $\mathcal{F}$ *is defined as*

$$\mathcal{F} \coloneqq \{X^\alpha \coloneqq (X, \alpha) \mid \alpha \in \Lambda \text{ and } X \in \mathcal{F}^\alpha\} = \bigcup_{\alpha \in \Lambda} (\mathcal{F}^\alpha \times \{\alpha\}).$$

Finally, for M, $L \geq 0$, *the subset* $\mathcal{F}_M^L$ *is given by*

$$\mathcal{F}_M^L \coloneqq \{X \in \mathcal{F} \mid \|V\|_\infty \leq M \text{ and } \|\alpha'\|_\infty \leq L\}. \tag{10}$$

For $\alpha \in \Lambda$, define the set $\mathcal{F}_0^\alpha$ and $\mathcal{F}_0$ as

$$\mathcal{F}_0^\alpha \coloneqq \big\{X \in \mathcal{F}^\alpha \mid y + H = \mathrm{id}\big\},$$

and

$$\mathcal{F}_0 \coloneqq \big\{X^\alpha \coloneqq (X, \alpha) \in \mathcal{F} \mid y + H = \mathrm{id}\big\} = \bigcup_{\alpha \in \Lambda} (\mathcal{F}_0^\alpha \times \{\alpha\}).$$

Similar, we set $\mathcal{F}_{0,M}^L = \mathcal{F}_0 \cap \mathcal{F}_M^L$.

In the general case, where wave breaking can occur, the α-dissipative solutions to the Hunter–Saxton equation in Lagrangian coordinates are defined as follows.

Definition 2.3 (α-Dissipative Solution)**.** *Let* $X_0^\alpha = (X_0, \alpha) \in \mathcal{F}$. *We say that* $X^\alpha = (X, \alpha)$ *is an* α*-dissipative solution with the given initial data* X_0^α *if* $X(t) \in \mathcal{F}^\alpha$ *for all* $t \geq 0$ *and satisfies*

$$y_t(\xi, t) = U(\xi, t), \tag{11a}$$

$$U_t(\xi, t) = \frac{1}{2}V(\xi, t) - \frac{1}{4}V_\infty(t), \tag{11b}$$

$$H_t(\xi, t) = 0, \tag{11c}$$

$$V(\xi, t) = \int_{-\infty}^{\xi} V_\xi(\eta, 0)(1 - \alpha(y(\eta, \tau(\eta))) \mathbb{1}_{\{r \in \mathbb{R} \mid t \geq \tau(r) > 0\}}(\eta))\, d\eta, \tag{11d}$$

with initial data $X(0) = X_0$, *where* $V_\infty(t) = \lim_{\xi \to +\infty} V(\xi, t)$.

Observe that α is independent of time in the above definition, but is essential since the ODE system (11) depends heavily on the choice of α. Furthermore, note that the derivative V_ξ is in general a discontinuous function in time for particles $\xi \in \mathbb{R}$ experiencing wave breaking.

Existence and uniqueness for the system (11) has been shown in [8], with the additional fact that the wave breaking time for a particle $\xi \in \mathbb{R}$ is given by

$$\tau(\xi) = \begin{cases} -2\frac{y_\xi(\xi,0)}{U_\xi(\xi,0)}, & U_\xi(\xi,0) < 0, \\ 0, & U_\xi(\xi,0) = 0 = y_\xi(\xi,0), \\ +\infty, & \text{otherwise.} \end{cases} \tag{12}$$

We will now introduce some simple estimates that we will make use of later on.

Lemma 2.4. *Consider two α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ with initial data $X_{0,A}^{\alpha_A}$ and $X_{0,B}^{\alpha_B}$ in $\mathcal{F}$. Then for each fixed $\xi \in \mathbb{R}$ the following estimates hold*

$$|y_A(\xi,t) - y_B(\xi,t)| \leq |y_A(\xi,0) - y_B(\xi,0)| + \int_0^t |U_A(\xi,s) - U_B(\xi,s)|\,ds, \tag{13a}$$

$$\begin{aligned} |U_A(\xi,t) - U_B(\xi,t)| \leq{}& |U_A(\xi,0) - U_B(\xi,0)| \\ &+ \frac{1}{4}\int_0^t \|V_{A,\xi}(\cdot,s) - V_{B,\xi}(\cdot,s)\|_1\,ds. \end{aligned} \tag{13b}$$

Proof. The first estimate is immediate from the ODE system (11). We focus on the second. For a fixed $\xi \in \mathbb{R}$,

$$\begin{aligned} U_A(\xi,t) - U_B(\xi,t) &= U_A(\xi,0) - U_B(\xi,0) \\ &\quad + \int_0^t \bigg(\frac{1}{2}(V_A(\xi,s) - V_B(\xi,s)) \\ &\qquad\qquad - \frac{1}{4}(V_{A,\infty}(s) - V_{B,\infty}(s))\bigg)\,ds \\ &= U_A(\xi,0) - U_B(\xi,0) \\ &\quad + \frac{1}{4}\int_0^t \bigg(\int_{-\infty}^{\xi} (V_{A,\xi}(\eta,s) - V_{B,\xi}(\eta,s))\,d\eta \\ &\qquad\qquad - \int_{\xi}^{\infty} (V_{A,\xi}(\eta,s) - V_{B,\xi}(\eta,s))\,d\eta\bigg)\,ds. \end{aligned}$$

Hence

$$|U_A(\xi,t) - U_B(\xi,t)| \leq |U_A(\xi,0) - U_B(\xi,0)| + \frac{1}{4}\int_0^t \|V_{A,\xi}(\cdot,s) - V_{B,\xi}(\cdot,s)\|_1\,ds,$$

as required. □

As a consequence, we have the following corollary.

Corollary 2.5. *For two α-dissipative solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ with initial data $X_{0,A}^{\alpha_A}$ and $X_{0,B}^{\alpha_B}$ in $\mathcal{F}$, we have*

$$\|y_A(t) - y_B(t)\|_\infty \leq \|y_A(0) - y_B(0)\|_\infty + \int_0^t \|U_A(s) - U_B(s)\|_\infty \, ds, \tag{14a}$$

$$\|U_A(t) - U_B(t)\|_\infty \leq \|U_A(0) - U_B(0)\|_\infty + \frac{1}{4}\int_0^t \|V_{A,\xi}(s) - V_{B,\xi}(s)\|_1 \, ds, \tag{14b}$$

$$\|y_{A,\xi}(t) - y_{B,\xi}(t)\|_2 \leq \|y_{A,\xi}(0) - y_{B,\xi}(0)\|_2 + \int_0^t \|U_{A,\xi}(s) - U_{B,\xi}(s)\|_2 \, ds, \tag{14c}$$

$$\|U_{A,\xi}(t) - U_{B,\xi}(t)\|_2 \leq \|U_{A,\xi}(0) - U_{B,\xi}(0)\|_2 + \frac{1}{2}\int_0^t \|V_{A,\xi}(s) - V_{B,\xi}(s)\|_2 \, ds. \tag{14d}$$

2.1 Mappings between Eulerian and Lagrangian coordinates

The goal now is to introduce a way of mapping from Eulerian to Lagrangian coordinates and back. These mappings were developed from similar ones for the more complicated Camassa–Holm equation [10], and will be central in using a metric in Lagrangian coordinates to define a metric in Eulerian coordinates.

Definition 2.6 (Mapping $\hat{L} : \mathcal{D} \to \mathcal{F}_0$)**.** *The mapping $\hat{L} : \mathcal{D} \to \mathcal{F}_0$, from Eulerian to Lagrangian coordinates, is defined by*

$$\hat{L}(Y^\alpha) = \hat{L}((Y, \alpha)) = (X, \alpha) = X^\alpha$$

with $X = (y, U, H, V)$ given by

$$y(\xi) = \sup\{x \in \mathbb{R} \mid x + \nu\big((-\infty, x)\big) < \xi\}, \tag{15a}$$

$$U(\xi) = u(y(\xi)), \tag{15b}$$

$$H(\xi) = \xi - y(\xi), \tag{15c}$$

$$V(\xi) = \int_{-\infty}^{\xi} H_\xi(\eta) \frac{d\mu}{d\nu} \circ y(\eta) \, d\eta. \tag{15d}$$

Definition 2.7 (Mapping $\hat{M} : \mathcal{F} \to \mathcal{D}$)**.** *The mapping $\hat{M} : \mathcal{F} \to \mathcal{D}$, from Lagrangian to Eulerian coordinates, is defined by*

$$\hat{M}(X^\alpha) = \hat{M}((X, \alpha)) = (Y, \alpha) = Y^\alpha$$

with $Y = (u, \mu, \nu)$ given by

$$u(x) = U(\xi), \qquad \text{for all } \xi \in \mathbb{R} \text{ such that } x = y(\xi), \tag{16a}$$

$$\mu = y_\#(V_\xi \, d\xi), \tag{16b}$$

$$\nu = y_\#(H_\xi \, d\xi). \tag{16c}$$

Here, we have used the push forward measure for a measurable function f and a μ-measurable set $f^{-1}(A)$, i.e.,

$$f_{\#}(\mu)(A) := \mu(f^{-1}(A)).$$

The mapping $\hat{L}$ maps four Eulerian coordinates $((u,\mu,\nu),\alpha)$ to five Lagrangian coordinates $((y,U,H,V),\alpha)$. Hence there is some redundancy here. That is to say, a set of Lagrangian coordinates can represent the same Eulerian coordinates. This set is an equivalence class, whose elements are related by what is referred to as a "relabelling".

Definition 2.8 (Relabelling)**.** *Let $\mathcal{G}$ be the group of homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f - \mathrm{id} \in W^{1,\infty}(\mathbb{R}), \qquad f^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R}), \qquad f_\xi - 1 \in L^2(\mathbb{R}). \tag{17}$$

We define the group action $\circ : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$, called the relabelling of $X^\alpha \in \mathcal{F}$ by f, as

$$(X^\alpha, f) \mapsto X^\alpha \circ f = ((y \circ f, U \circ f, H \circ f, V \circ f), \alpha).$$

Hence, one defines the equivalence relation $\sim$ on $\mathcal{F}$ by

$$X_A^{\alpha_A} \sim X_B^{\alpha_B} \text{ if there exists } f \in \mathcal{G} \text{ such that } X_A^{\alpha_A} = X_B^{\alpha_B} \circ f.$$

Finally, define the mapping $\Pi : \mathcal{F} \to \mathcal{F}_0$, which gives one representative in $\mathcal{F}_0$ for each equivalence class,

$$\Pi(X^\alpha) = X^\alpha \circ (y + H)^{-1}.$$

Under these equivalence classes, the mappings $\hat{L}$ and $\hat{M}$ are inverses of one another [8, 12].

Lemma 2.9. *Let $Y^\alpha \in \mathcal{D}$, and $\hat{L}(Y^\alpha) = X^\alpha$. Then, for any $f \in \mathcal{G}$,*

$$\hat{M}(X^\alpha) = Y^\alpha = \hat{M}(X^\alpha \circ f).$$

Further, the relabelling is carried forward in time by the solution, see [8, Proposition 3.7].

Lemma 2.10. *Denote by $S_t : \mathcal{F} \to \mathcal{F}, X_0^\alpha \mapsto S_t(X_0^\alpha)$ for $t \in [0, +\infty)$ the solution operator defined in Definition* 2.3 *through the ODE system* (11)*. Then, for any initial data $X_0^\alpha \in \mathcal{F}$, and any relabelling function $f \in \mathcal{G}$,*

$$S_t(X_0^\alpha \circ f) = S_t(X_0^\alpha) \circ f.$$

At this point, we should explore what a solution to the Hunter–Saxton equation can look like.

Example 2.11. *Consider as initial data*

$$u_0(x) = \begin{cases} 1 + x, & -1 < x \leq 0, \\ 1 - x, & 0 < x \leq 1, \\ 0, & \textit{otherwise}, \end{cases} \qquad \nu_0 = \mu_0 = u_{0,x}^2 \, dx,$$

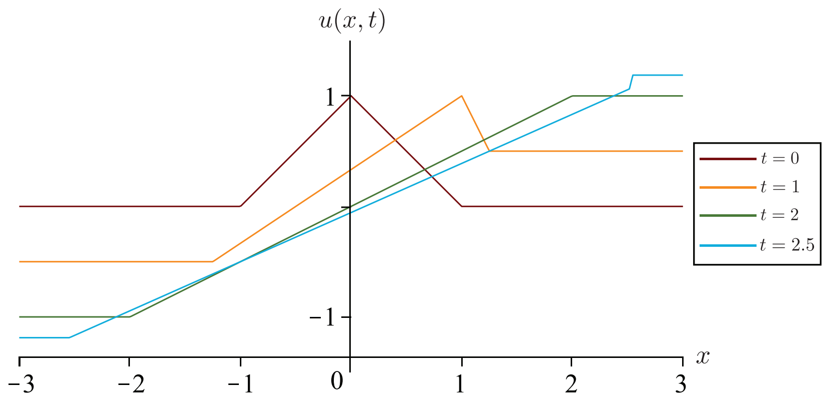

Figure 1: Plots of u, as given by (18), at different times.

and $\alpha \in \Lambda$ *such that* $\alpha(2) = \frac{1}{2}$.

The corresponding α*-dissipative solution is given by*

$$u(x,t) = \begin{cases} \begin{cases} -\frac{1}{2}t, & x \leq -\frac{1}{4}t^2 - 1, \\ \frac{2-t+2x}{t+2}, & -\frac{1}{4}t^2 - 1 < x \leq t, \\ \frac{-2-t+2x}{t-2}, & t < x \leq \frac{1}{4}t^2 + 1, \\ \frac{1}{2}t, & \frac{1}{4}t^2 + 1 < x, \end{cases} & 0 \leq t < 2, \\ \begin{cases} -\frac{1}{4} - \frac{3}{8}t, & x \leq -\frac{3}{16}t^2 - \frac{1}{4}t - \frac{3}{4}, \\ \frac{2-t+4x}{2(t+2)}, & -\frac{3}{16}t^2 - \frac{1}{4}t - \frac{3}{4} < x \leq \frac{1}{16}t^2 + \frac{3}{4}t + \frac{1}{4}, \\ \frac{-2-t+2x}{t-2}, & \frac{1}{16}t^2 + \frac{3}{4}t + \frac{1}{4} < x \leq \frac{3}{16}t^2 + \frac{1}{4}t + \frac{3}{4}, \\ \frac{1}{4} + \frac{3}{8}t, & \frac{3}{16}t^2 + \frac{1}{4}t + \frac{3}{4} < x, \end{cases} & 2 < t. \end{cases} \tag{18}$$

with

$$\mu(t) = u_x^2(t)\,dx + \frac{1}{2}\delta_2 \mathbb{1}_{\{t=2\}}(t).$$

See Figure 1 for plots of u *at different times.*

Note that the third interval shrinks into the single point $x = 2$ *as* $t \to 2$*, and the derivative* $u_x \to -\infty$ *as* $t \to 2$*. Of course we retain that* u *is a distributional solution regardless of the value of* u *at this point. However,* $u(\cdot,t) \in H^1(\mathbb{R})$ *and therefore* $u(2,2) = 1$.

Furthermore, note that all $\alpha \in \Lambda$*, which satisfy* $\alpha(2) = \frac{1}{2}$*, yield the same* α*-dissipative solution. This is due to wave breaking occurring once at* $(t,x) = (2,2)$ *for all of these* α*-dissipative solutions. As a consequence, it is vital to consider* Y^α *instead of* Y*, when constructing our metric.*

With our notation in place, we introduce the definition of an α-dissipative solution for (HS).

Definition 2.12 (α-Dissipative Solution)**.** *Let $Y_0^\alpha = (Y_0, \alpha) = ((u_0, \mu_0, \nu_0), \alpha)$ be in $\mathcal{D}$. We say $Y^\alpha = (Y, \alpha) = ((u, \mu, \nu), \alpha)$ is a weak solution with the given initial data Y_0^α if the following conditions are satisfied,*

$$u \in C^{0,\frac{1}{2}}(\mathbb{R} \times [0, T]), \qquad \text{for any } T \geq 0, \tag{19a}$$

$$\nu \in C_{weak*}([0, +\infty); \mathcal{M}^+(\mathbb{R})), \tag{19b}$$

$$Y(t) \in \mathcal{D}^\alpha, \qquad \text{for any } t \in [0, +\infty), \tag{19c}$$

$$Y(0) = Y_0, \tag{19d}$$

$$\nu(t)(\mathbb{R}) = \nu_0(\mathbb{R}), \qquad \text{for any } t \in [0, +\infty). \tag{19e}$$

Further, u must satisfy (HS) *in the distributional sense, that is, for any test function $\varphi \in C_c^\infty(\mathbb{R} \times [0, +\infty))$ with $\varphi(x, 0) = \varphi_0(x)$,*

$$\int_0^{+\infty} \int_{\mathbb{R}} \left[u\varphi_t + \frac{1}{2}u^2\varphi_x + \frac{1}{4}\left(\int_{-\infty}^x d\mu - \int_x^{+\infty} d\mu \right)\varphi \right] dx\, dt = -\int_{\mathbb{R}} u_0\varphi_0\, dx, \tag{20}$$

and μ must satisfy

$$\int_0^{+\infty} \int_{\mathbb{R}} [\phi_t + u\phi_x]\, d\mu(t)\, dt \geq -\int_{\mathbb{R}} \phi_0\, d\mu_0, \tag{21}$$

for every non-negative test function

$$\phi \in C_c^\infty(\mathbb{R} \times [0, +\infty); [0, +\infty)), \quad \text{with} \quad \phi(x, 0) = \phi_0(x).$$

Finally, we say that Y^α is an α-dissipative solution if Y^α is a weak solution and if for each $t \in [0, +\infty)$,

$$d\mu(t) = d\mu_{ac}^-(t) + (1 - \alpha(x))d\mu_s^-(t), \tag{22a}$$

$$\mu(s) \overset{*}{\rightharpoonup} \mu(t), \qquad \text{as } s \downarrow t, \tag{22b}$$

$$\mu(s) \overset{*}{\rightharpoonup} \mu^-(t), \qquad \text{as } s \uparrow t. \tag{22c}$$

Note. *If $Y^\alpha(t)$ is a conservative solution, then* (21) *will be an equality.*

Bringing everything together, define $T_t : \mathcal{D} \to \mathcal{D}$ for $t \in [0, +\infty)$ as

$$T_t Y_0^\alpha = (\hat{M} \circ S_t \circ \hat{L})Y_0^\alpha.$$

Then T_t associates to each initial data $Y_0^\alpha = (Y_0, \alpha) \in \mathcal{D}$ an α-dissipative solution in the sense of Definition 2.12. The proof can be found in [8, Theorem 3.14]. Henceforth when referring to α-dissipative solutions in Eulerian coordinates, we refer to the solutions given by T_t.

Finally, it is important to observe that u and μ are independent of ν and therefore it is possible to introduce equivalence classes in Eulerian coordinates.

Lemma 2.13. *Let $Y_A^{\alpha_A}$ and $Y_B^{\alpha_B}$ be two α-dissipative solutions with initial data $Y_{A,0}^{\alpha_A}$ and $Y_{0,B}^{\alpha_B}$ in $\mathcal{D}$. If*

$$u_{0,A} = u_{0,B}, \quad \mu_{0,A} = \mu_{0,B} \quad \text{and} \quad \alpha_A = \alpha_B, \tag{23}$$

then

$$u_A(\cdot, t) = u_B(\cdot, t) \quad \text{and} \quad \mu_A(t) = \mu_B(t) \quad \text{for all } t \geq 0.$$

Proof. Without loss of generality assume that $\mu_{0,A} = \nu_{0,A}$.

Introduce $X_{0,i}^{\alpha_i} = ((y_{0,i}, U_{0,i}, V_{0,i}, H_{0,i}), \alpha_i) = \hat{L}(Y_{0,i}^{\alpha_i})$ for $i = A, B$. We claim there exists an increasing and Lipschitz continuous function g such that

$$(y_{0,A} \circ g, U_{0,A} \circ g, V_{0,A} \circ g) = (y_{0,B}, U_{0,B}, V_{0,B}). \tag{24}$$

By assumption $V_{0,A}(\xi) = H_{0,A}(\xi)$ for all $\xi \in \mathbb{R}$ and hence

$$y_{0,A}(\xi) + V_{0,A}(\xi) = \xi \quad \text{for all } \xi \in \mathbb{R}.$$

For $V_{0,B}(\xi)$, on the other hand, we have that there exists a function $\kappa : \mathbb{R} \to [0, 1]$ such that

$$V_{0,B,\xi}(\xi) = \kappa(y_{0,B}(\xi))H_{0,B,\xi}(\xi) \quad \text{for all } \xi \in \mathbb{R},$$

which implies that

$$\begin{aligned} y_{0,B}(\xi) + V_{0,B}(\xi) &= y_{0,B}(\xi) + H_{0,B}(\xi) + V_{0,B}(\xi) - H_{0,B}(\xi) \\ &= \xi - \int_{-\infty}^{\xi} (1 - \kappa(y_{0,B}(\eta)))H_{0,B,\xi}(\eta)d\eta, \end{aligned}$$

where the function on the right hand side is increasing and Lipschitz continuous with Lipschitz constant at most one. Introduce

$$g(\xi) = \xi - \int_{-\infty}^{\xi} (1 - \kappa(y_{0,B}(\eta)))H_{0,B,\xi}(\eta)d\eta,$$

then

$$y_{0,B}(\xi) + V_{0,B}(\xi) = g(\xi) = y_{0,A}(g(\xi)) + V_{0,A}(g(\xi)) \quad \text{for all } \xi \in \mathbb{R}. \tag{25}$$

Next, we establish that $y_{0,A}(g(\xi)) = y_{0,B}(\xi)$ for all $\xi \in \mathbb{R}$. Assume the opposite, i.e., there exists $\bar{\xi} \in \mathbb{R}$ such that $y_{0,A}(g(\bar{\xi})) \neq y_{0,B}(\bar{\xi})$ and without loss of generality we assume that

$$y_{0,A}(g(\bar{\xi})) < y_{0,B}(\bar{\xi}). \tag{26}$$

Since (15) implies for $i = A, B$,

$$\mu_{0,i}((-\infty, y_{0,i}(\xi))) \leq V_{0,i}(\xi) \leq \mu_{0,i}((-\infty, y_{0,i}(\xi)]) \quad \text{for all } \xi \in \mathbb{R},$$

we have, recalling (23) and using (25),

$$\begin{aligned} \mu_{0,A}((-\infty, y_{0,B}(\bar{\xi}))) = \mu_{0,B}((-\infty, y_{0,B}(\bar{\xi}))) &\leq V_{0,B}(\bar{\xi}) \\ &< V_{0,A}(g(\bar{\xi})) \leq \mu_{0,A}((-\infty, y_{0,A}(g(\bar{\xi}))]) \end{aligned}$$

Since this is only possible if $y_{0,B}(\bar{\xi})) \leq y_{0,A}(g(\bar{\xi}))$, we end up with a contradiction to (26). Thus $y_{0,A} \circ g = y_{0,B}$ and, by Definition 2.6, $V_{0,A} \circ g = V_{0,B}$ and

$$U_{0,A} \circ g = u \circ y_{0,A} \circ g = u \circ y_{0,B} = U_{0,B}, \tag{27}$$

which finishes the proof of (24).

Next, we show

$$(y_A, U_A, V_A)(g(\xi), t) = (y_B, U_B, V_B)(\xi, t) \quad \text{for all } \xi \in \mathbb{R} \text{ and } t \geq 0. \tag{28}$$

Therefore, observe that the system of ordinary differential equations given by (11a)–(11c) is a closed system for (y, U, V) and hence H does not influence the time evolution of (y, U, V). Furthermore, recalling that $\alpha_A = \alpha_B$ and repeating the argument of [8, Proposition 3.7], one finds (28).

Finally, we can apply the mapping M to go back to Eulerian coordinates as follows. Let $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, then there exists $\xi \in \mathbb{R}$ such that

$$y_A(g(\xi), t) = x = y_B(\xi, t)$$

and hence

$$u_A(x, t) = U_A(g(\xi), t) = U_B(\xi, t) = u_B(x, t).$$

Furthermore, let $\bar{\xi} = \sup\{\eta \mid y_B(\eta, t) < x\}$, then $g(\bar{\xi}) = \sup\{\eta \mid y_A(\eta, t) < x\}$ and therefore

$$\mu_A((-\infty, x), t) = \int_{-\infty}^{g(\bar{\xi})} V_{A,\xi}(\eta, t) d\eta = V_A(g(\bar{\xi}), t) = V_B(\bar{\xi}, t) = \mu_B((-\infty, x), t).$$

□

We can now define a new set that will contain triplets $Z^\alpha = (Z, \alpha) = ((u, \mu), \alpha)$ that form the solution to (HS).

Definition 2.14 (Equivalence classes in $\mathcal{D}$)**.** *The set $\mathcal{D}_0$ contains all $Z^\alpha = (Z, \alpha) = ((u, \mu), \alpha) \in E_2 \times \mathcal{M}^+(\mathbb{R}) \times \Lambda$ satisfying*

- $\mu_{ac} = u_x^2\, dx$,
- $\mu = u_x^2\, dx$ *if* $\alpha = 1$,
- $\mu((-\infty, x)) \in E_0$.

Then, for each $Z^\alpha = (Z, \alpha) = ((u, \mu), \alpha) \in \mathcal{D}_0$ we define the set

$$\mathcal{V}(Z^\alpha) := \{\nu \in \mathcal{M}^+(\mathbb{R}) \mid ((Z, \nu), \alpha) \in \mathcal{D}\},$$

i.e. the equivalence class of all ν related by having the same $Z^\alpha = ((u, \mu), \alpha)$.

Finally, for M, $L \geq 0$, define $\mathcal{D}^L_{0,M}$ by

$$\mathcal{D}^L_{0,M} := \{Z^\alpha \in \mathcal{D}_0 \mid \mu(\mathbb{R}) \leq M \text{ and } \|\alpha'\|_\infty \leq L\}.$$

Note. *$\mathcal{D}$ can be written as*

$$\mathcal{D} = \{((u, \mu, \nu), \alpha) \mid ((u, \mu), \alpha) \in \mathcal{D}_0 \text{ and } \nu \in \mathcal{V}((u, \mu), \alpha)\}.$$

Note. *Under the present setting, uniqueness of fully dissipative solutions has been established in [4]. For the conservative case, uniqueness was shown in [5].*

3 A metric in Lagrangian coordinates

Our first goal is to introduce a metric in the space of Lagrangian coordinates that is Lipschitz stable with respect to initial Lagrangian coordinates in the sense of equivalence classes.

We begin our approach by introducing a semi-metric, i.e. dropping the triangle inequality requirement, on the set of Lagrangian coordinates. The most important condition of this mapping is that it is Lipschitz continuous with respect to the initial data in $\mathcal{F}$. We will then, in the next section, use this semi-metric to define a metric on the space of equivalence classes in Lagrangian coordinates, ensuring that Lagrangian coordinates representing the same Eulerian coordinates have a distance of zero.

We introduce important sets that our construction will take advantage of.

3.1 Some important sets

For two α-dissipative solutions $X_i^{\alpha_i}$, $X_j^{\alpha_j}$, with labels i and j, define the sets

$$\mathcal{A}_i(t) = \mathcal{A}(X_i^{\alpha_i};t) = \{\xi \in \mathbb{R} \mid U_{i,\xi}(\xi,t) \geq 0\}, \tag{29a}$$

$$\mathcal{A}_{i,j}(t) = \mathcal{A}_i(t) \cap \mathcal{A}_j(t), \tag{29b}$$

$$\mathcal{B}_{i,j}(t) = \mathcal{B}(X_i^{\alpha_i}, X_j^{\alpha_j};t) = \{\xi \in \mathbb{R} \mid t < \tau_i(\xi) = \tau_j(\xi) < \infty\}, \tag{29c}$$

$$\Omega_{i,j}(t) = \Omega(X_i^{\alpha_i}, X_j^{\alpha_j};t) = \mathcal{A}_{i,j}(t) \cup \mathcal{B}_{i,j}(t). \tag{29d}$$

Should X_i, X_j be just elements of $\mathcal{F}$ (with no time dependence), take $t = 0$ in the definitions, and naturally these will no longer be dependent on time.

We can describe the contents of these sets as follows

- $\mathcal{A}_{i,j}(t)$ contains the particles ξ for which no wave breaking will occur for both solutions at any point in the future.
- $\mathcal{B}_{i,j}(t)$ contains the ξ for which wave breaking will occur in both solutions at the same time in the future.
- $\Omega_{i,j}^c(t)$ contains everything else, i.e. particles for which wave breaking occurs at different times in the future, or for which only one of the two will break.

Importantly, these three sets form a disjoint union of the entire real line and are independent of the choice of α.

Furthermore, elements ξ of the sets $\mathcal{B}_{i,j}(t)$ and $\Omega_{i,j}(t)$ remain in their respective set until both have broken and ξ enters $\mathcal{A}_{i,j}(t)$.

A natural question is "how do these sets change after a relabelling of the Lagrangian coordinates?". To begin answering this question, we introduce the following notation:

For $X_i^{\alpha_i}$ and $X_j^{\alpha_j}$ in $\mathcal{F}$, and $f, h \in \mathcal{G}$ define

$$\mathcal{A}_i^f(t) = \mathcal{A}(X_i^{\alpha_i} \circ f; t) \tag{30a}$$
$$\mathcal{A}_{i,j}^{f,h}(t) = \mathcal{A}_i^f(t) \cap \mathcal{A}_j^h(t), \tag{30b}$$
$$\mathcal{B}_{i,j}^{f,h}(t) = \mathcal{B}(X_i^{\alpha_i} \circ f, X_j^{\alpha_j} \circ h; t) \tag{30c}$$
$$\Omega_{i,j}^{f,h}(t) = \mathcal{A}_{i,j}^{f,h}(t) \cup \mathcal{B}_{i,j}^{f,h}(t). \tag{30d}$$

If f and h are the identity functions, then this notation collapses back to that in (29).

Consider two functions f and h in $\mathcal{G}$, the set of relabelling functions, as given by Definition 2.8. Such functions are continuous and strictly monotonically increasing, i.e. $f_\xi(\xi) > 0$, almost everywhere, cf. [10, Lemma 3.2].

Let $X^\alpha \in \mathcal{F}$. Then

$$\begin{aligned} \mathcal{A}^f &= \{\xi \in \mathbb{R} \mid (U \circ f)_\xi(\xi) \geq 0\} \\ &= \{\xi \in \mathbb{R} \mid (U_\xi \circ f)(\xi) f_\xi(\xi) \geq 0\} \\ &= \{\xi \in \mathbb{R} \mid (U_\xi \circ f)(\xi) \geq 0\} \\ &= \{\xi \in \mathbb{R} \mid f(\xi) \in \mathcal{A}\} = f^{-1}(\mathcal{A}), \end{aligned} \tag{31}$$

or equivalently, $\mathcal{A} = f(\mathcal{A}^f)$.

Inspired by the previous calculation, we look at the other sets. We have, as f is bijective, for $X_A^{\alpha_A}$ and $X_B^{\alpha_B} \in \mathcal{F}$,

$$f(\mathcal{A}_{A,B}^{f,h}) = f(\mathcal{A}_A^f) \cap f(\mathcal{A}_B^h) = \mathcal{A}_A \cap \mathcal{A}_B^{h\circ f^{-1}} = \mathcal{A}_{A,B}^{\mathrm{id}, h\circ f^{-1}}. \tag{32}$$

We also have a relation for the breaking times after relabelling. Once again take $X_A^{\alpha_A}$ and $X_B^{\alpha_B} \in \mathcal{F}$, and suppose that $f(\eta) \in \mathcal{A}_A^c$. Defining temporarily $X_C^{\alpha_C} = X_A^{\alpha_A} \circ f$, the wave breaking time after relabelling is given by

$$\tau_C(\eta) = -2\frac{(y_A \circ f)_\eta(\eta)}{(U_A \circ f)_\eta(\eta)} = -2\frac{y_{A,\xi}(f(\eta)) f_\eta(\eta)}{U_{A,\xi}(f(\eta)) f_\eta(\eta)} = \tau_A(f(\eta)) \text{ a.e.},$$

which gives us

$$\begin{aligned} f(\mathcal{B}_{A,B}^{f,h}) &= \{f(\xi) \mid \xi \in \mathbb{R} \text{ and } 0 < \tau_A(f(\xi)) = \tau_B(h(\xi)) < +\infty\} \\ &= \{\xi \in \mathbb{R} \mid 0 < \tau_A(\xi) = \tau_B((h \circ f^{-1})(\xi)) < +\infty\} = \mathcal{B}_{A,B}^{\mathrm{id}, h\circ f^{-1}}. \end{aligned} \tag{33}$$

This has the immediate consequence

$$f(\Omega_{A,B}^{f,h,c}(t)) = \Omega_{A,B}^{\mathrm{id}, h\circ f^{-1}, c}(t). \tag{34}$$

3.2 Construction of a semi-metric for Lagrangian coordinates

We now begin the first step of the construction of our metric, measuring the distance between two α-dissipative solutions, where $\alpha \in \Lambda$.

We cannot simply use a metric based on the norms of the Banach space E. This is a consequence of the discontinuities in time of the derivatives V_ξ. For two solutions $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$, the difference $\|V_{A,\xi}(t) - V_{B,\xi}(t)\|_1$ can increase in time and in particular, it can have a jump of positive height.

To resolve this issue, we introduce a new function $G_{A,B}(\xi,t)$ that will decrease in time, and only drops can occur.

Let $X_A^{\alpha_A}$, $X_B^{\alpha_B}$ be two α-dissipative solutions. The following functions will all contribute to the function $G_{A,B}(\xi,t)$.

$$g_{A,B}(\xi,t) = g(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi,t) = |V_{A,\xi}(\xi,t) - V_{B,\xi}(\xi,t)|, \tag{35}$$

$$\begin{aligned}\hat{g}_{A,B}(\xi,t) &= \hat{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi,t)\\ &= |V_{A,\xi}(\xi,t) - V_{B,\xi}(\xi,t)| + \|\alpha_A - \alpha_B\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi,t)\\ &\quad + \|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi,t)\Big(|y_A(\xi,t) - y_B(\xi,t)|\\ &\qquad + |U_A(\xi,t) - U_B(\xi,t)|\Big),\end{aligned} \tag{36}$$

$$\begin{aligned}\bar{g}_{A,B}(\xi,t) &= \bar{g}(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi,t)\\ &= |V_{A,\xi}(\xi,t) - V_{B,\xi}(\xi,t)|\\ &\quad + \big(V_{A,\xi} \wedge V_{B,\xi}\big)(\xi,t)(\alpha_A(\xi)\mathbb{1}_{\mathcal{A}_A^c(t)}(\xi) + \alpha_B(\xi)\mathbb{1}_{\mathcal{A}_B^c(t)}(\xi))\\ &\quad + \|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi,t)\\ &\qquad \times \Big(|y_A(\xi,t) - \mathrm{id}(\xi)|\mathbb{1}_{\mathcal{A}_A^c(t)}(\xi)\\ &\qquad\quad + |y_B(\xi,t) - \mathrm{id}(\xi)|\mathbb{1}_{\mathcal{A}_B^c(t)}(\xi)\\ &\qquad\quad + |U_A(\xi,t)|(\mathbb{1}_{\mathcal{A}_A^c(t)}(\xi) + \mathbb{1}_{\mathcal{A}_B^c(t)}(\xi))\\ &\qquad\quad + |U_B(\xi,t)|(\mathbb{1}_{\mathcal{A}_A^c(t)}(\xi) + \mathbb{1}_{\mathcal{A}_B^c(t)}(\xi))\Big),\end{aligned} \tag{37}$$

where

$$\alpha'_{A,B} = \alpha'_A \vee \alpha'_B.$$

Here we use a shorthand notation for the minimum and the maximum. For $a, b \in \mathbb{R}$,

$$a \wedge b = \min\{a,b\} \quad \text{and} \quad a \vee b = \max\{a,b\}.$$

Proposition 3.1. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be two α-dissipative solutions with initial* *data* $X_{0,A}^{\alpha_A} \in \mathcal{F}_0$ *and* $X_{0,B}^{\alpha_B} \in \mathcal{F}$. *Define*

$$\begin{aligned} G_{A,B}(\xi,t) &= G(X_A^{\alpha_A}, X_B^{\alpha_B})(\xi,t) \\ &= g_{A,B}(\xi,t)\mathbb{1}_{\mathcal{A}_{A,B}(t)}(\xi) + \hat{g}_{A,B}(\xi,t)\mathbb{1}_{\mathcal{B}_{A,B}(t)}(\xi) \\ &\quad + \bar{g}_{A,B}(\xi,t)\mathbb{1}_{\Omega^c_{A,B}(t)}(\xi) \\ &\quad + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (V_{A,\xi} \wedge V_{B,\xi})(\xi,t) \\ &\qquad \times (\|V_{A,\xi}(\cdot,t)\|_1 + \|V_{B,\xi}(\cdot,t)\|_1 + 1) \\ &\qquad \times (\mathbb{1}_{\mathcal{A}^c_A(t)}(\xi) + \mathbb{1}_{\mathcal{A}^c_B(t)}(\xi))\mathbb{1}_{\mathcal{B}^c_{A,B}(t)}(\xi) \end{aligned} \tag{38}$$

and let

$$\begin{aligned} M_{A,B} &= \max(\|V_A(\cdot,0)\|_\infty, \|V_B(\cdot,0)\|_\infty) \\ &= \max(\sup_{t\geq 0}\|V_A(\cdot,t)\|_\infty, \sup_{t\geq 0}\|V_B(\cdot,t)\|_\infty). \end{aligned} \tag{39}$$

Then

$$\|(V_{A,\xi} - V_{B,\xi})(\cdot,t)\|_i \leq \|G_{A,B}(\cdot,t)\|_i \qquad \text{for } i = 1,2, \tag{40}$$

and $G_{A,B}$ *is a decreasing function over breaking times, i.e.*

$$G_{A,B}(\xi,\tau(\xi)) \leq \lim_{t\uparrow\tau(\xi)} G_{A,B}(\xi,t).$$

Furthermore,

$$\begin{aligned} \|G_{A,B}(\cdot,t)\|_1 &\leq \|G_{A,B}(\cdot,0)\|_1 \\ &\quad + \int_0^t (\|G_{A,B}(\cdot,s)\|_1 + \frac{1}{4}M_{A,B}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(\cdot,s)\|_1)\, ds, \end{aligned} \tag{41}$$

and

$$\begin{aligned} \|G_{A,B}(\cdot,t)\|_2 &\leq \|G_{A,B}(\cdot,0)\|_2 \\ &\quad + \int_0^t (\|G_{A,B}(\cdot,s)\|_2 + \frac{1}{4}\sqrt{M_{A,B}}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(\cdot,s)\|_1)\, ds. \end{aligned} \tag{42}$$

Proof. Relationship (40) is an immediate consequence of the definition of $G_{A,B}$.

We have tactically constructed $G_{A,B}$ such that it can be split into four parts. The first three are defined on disjoint sets whose union is the entire real line, and the final term is necessary in order to obtain (41) and (42).

For a function $h : \mathbb{R} \to \mathbb{R}$ we use the notation, $h(t-) := \lim_{s\uparrow t} h(s)$. Furthermore, we drop the ξ for ease of readability, in the following computations.

We begin by demonstrating that $G_{A,B}$ decreases over breaking times $\tau(\xi)$.

Consider $\xi \in \mathcal{A}_{A,B}(t)$ for all time. These particles do not experience wave breaking, thus the energy at these points is retained, and hence

$$g_{A,B}(t) = |V_{A,\xi}(t) - V_{B,\xi}(t)|$$

is constant. For other values of ξ things are not so simple.

For $\xi \in \mathcal{B}_{A,B}(0)$, at time $\tau(\xi)$, we have

$$\begin{aligned}
g_{A,B}(\tau) &= |V_{A,\xi}(\tau) - V_{B,\xi}(\tau)| \\
&= |V_{A,\xi}(\tau-)(1-\alpha_A(y_A(\tau-))) - V_{B,\xi}(\tau-)(1-\alpha_B(y_B(\tau-)))| \\
&\leq |V_{A,\xi}(\tau-) - V_{B,\xi}(\tau-)|(1-\alpha_A(y_A(\tau-))) \\
&\quad + |\alpha_A(y_A(\tau-)) - \alpha_B(y_B(\tau-))| V_{B,\xi}(\tau-) \\
&\text{or} \\
&\leq |V_{A,\xi}(\tau-) - V_{B,\xi}(\tau-)|(1-\alpha_B(y_B(\tau-))) \\
&\quad + |\alpha_A(y_A(\tau-)) - \alpha_B(y_B(\tau-))| V_{A,\xi}(\tau-).
\end{aligned}$$

Using that, for any $t \in [0, +\infty)$,

$$\begin{aligned}
|\alpha_A(y_A(t)) - \alpha_B(y_B(t))| &\leq |\alpha_A(y_A(t)) - \alpha_B(y_A(t))| + |\alpha_B(y_A(t)) - \alpha_B(y_B(t))| \\
&\leq \|\alpha_A - \alpha_B\|_\infty + \|\alpha_B'\|_\infty |y_A(t) - y_B(t)|
\end{aligned}$$

and similarly

$$|\alpha_A(y_A(t)) - \alpha_B(y_B(t))| \leq \|\alpha_A - \alpha_B\|_\infty + \|\alpha_A'\|_\infty |y_A(t) - y_B(t)|$$

we find that

$$\begin{aligned}
g_{A,B}(\tau) &\leq |V_{A,\xi}(\tau-) - V_{B,\xi}(\tau-)| \\
&\quad + |\alpha_A(y_A(\tau-)) - \alpha_B(y_B(\tau-))| \big(V_{A,\xi}(\tau-) \wedge V_{B,\xi}(\tau-)\big) \\
&\leq |V_{A,\xi}(\tau-) - V_{B,\xi}(\tau-)| + \|\alpha_A - \alpha_B\|_\infty \big(V_{A,\xi}(\tau-) \wedge V_{B,\xi}(\tau-)\big) \\
&\quad + \|\alpha_{A,B}'\|_\infty |y_A(\tau-) - y_B(\tau-)| \big(V_{A,\xi}(\tau-) \wedge V_{B,\xi}(\tau-)\big) \\
&\leq \hat{g}_{A,B}(\tau-).
\end{aligned}$$

For $\xi \in \Omega_{A,B}^c(0)$, we consider two possibilities. First, we can have one solution breaking at time $\tau(\xi)$, and the other never breaking. Suppose $X_A^{\alpha_A}$ breaks at $\tau_A(\xi)$, then

$$\begin{aligned}
g_{A,B}(\tau_A) &= |V_{A,\xi}(\tau_A) - V_{B,\xi}(\tau_A)| \\
&= |V_{A,\xi}(\tau_A-)(1-\alpha_A(y_A(\tau_A-))) - V_{B,\xi}(\tau_A-)| \\
&\leq |V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| + \alpha_A(y_A(\tau_A-))(V_{A,\xi}(\tau_A-) \wedge V_{B,\xi}(\tau_A-)) \\
&\leq |V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| \\
&\quad + (\alpha_A(y_A(\tau_A-)) - \alpha_A(\mathrm{id}) + \alpha_A(\mathrm{id}))(V_{A,\xi}(\tau_A-) \wedge V_{B,\xi}(\tau_A-)) \\
&\leq |V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| \\
&\quad + (\|\alpha_A'\|_\infty |y_A(\tau_A-) - \mathrm{id}| + \alpha_A(\mathrm{id}))(V_{A,\xi}(\tau_A-) \wedge V_{B,\xi}(\tau_A-)) \\
&\leq \bar{g}_{A,B}(\tau_A-).
\end{aligned}$$

The last case is where both break at different times. Suppose $X_A^{\alpha_A}$ breaks first, and $X_B^{\alpha_B}$ second. At time τ_B, we can use the previous result, hence

$$g_{A,B}(\tau_B) \leq \bar{g}_{A,B}(\tau_B-).$$

At time τ_A, we have

$$\begin{aligned}
\bar{g}_{A,B}(\tau_A) &= |(1-\alpha_A(y_A(\tau_A-)))V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| \\
&\quad + \alpha_B(\mathrm{id})((1-\alpha_A(y_A(\tau_A-)))V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\quad + \|\alpha'_{A,B}\|_\infty\Big(|y_B(\tau_A-)-\mathrm{id}| + |U_A(\tau_A-)| + |U_B(\tau_A-)|\Big) \\
&\qquad\qquad \times ((1-\alpha_A(y_A(\tau_A-)))V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\le |V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| + \alpha_B(\mathrm{id})(V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\quad + (\alpha_A(y_A(\tau_A-)) - \alpha_A(\mathrm{id}) + \alpha_A(\mathrm{id}))(V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\quad + \|\alpha'_{A,B}\|_\infty\Big(|y_B(\tau_A-)-\mathrm{id}| + |U_A(\tau_A-)| + |U_B(\tau_A-)|\Big) \\
&\quad \times (V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\le |V_{A,\xi}(\tau_A-) - V_{B,\xi}(\tau_A-)| \\
&\quad + (\alpha_A(\mathrm{id}) + \alpha_B(\mathrm{id}))(V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\quad + \|\alpha'_{A,B}\|_\infty\Big(|y_A(\tau_A-)-\mathrm{id}| + |y_B(\tau_A-)-\mathrm{id}| \\
&\qquad\qquad + |U_A(\tau_A-)| + |U_B(\tau_A-)|\Big) \\
&\qquad\qquad \times (V_{A,\xi}(\tau_A-)\wedge V_{B,\xi}(\tau_A-)) \\
&\le \bar{g}_{A,B}(\tau_A-).
\end{aligned}$$

The final term in $G_{A,B}$ is decreasing in time, because $\|V_{i,\xi}(\cdot,t)\|_1$ with $i=1,2$ is decreasing and the sets $(\mathcal{A}_i^c \cap \mathcal{B}_{A,B}^c)(t) = (\mathcal{A}_i \cup \mathcal{B}_{A,B})^c$, with $i=A,B$, are shrinking in time, and thus the respective indicator functions are decreasing in time.

Hence we have that $G_{A,B}(\tau) \le G_{A,B}(\tau-)$ for all breaking times τ.

We now wish to obtain our estimate backwards in time. We consider an arbitrary time t, and construct different estimates depending on what set ξ is in at time t. As we know that $G_{A,B}(\xi,t)$ decreases over breaking times, we can employ a strategy of constructing an estimate backwards to the most recent breaking time $\tau(\xi)$, or zero if no breaking occurs in the past. Assuming we hit another breaking time, ξ may enter a different set, and we can then employ our estimate for that set.

To make our strategy clearer we consider the first case, that is $\xi \in \mathcal{A}_{A,B}(t)$. In this case particle ξ experienced wave breaking in the past for at least one, or neither, of the solutions. Set $\hat{\tau}(\xi)$ to be the largest of the two breaking times, or zero if neither broke. Then

$$g_{A,B}(t) = |V_{A,\xi}(t) - V_{B,\xi}(t)| = |V_{A,\xi}(\hat{\tau}) - V_{B,\xi}(\hat{\tau})| = g_{A,B}(\hat{\tau}).$$

If $\hat{\tau}(\xi) > 0$, depending on which set ξ sat in before $\tau(\xi)$, we can employ one of our previous estimates. For example, if ξ was in $\mathcal{B}_{A,B}(t)$ for $t < \tau(\xi)$, we can use that

$$g_{A,B}(\tau) = |V_{A,\xi}(\tau) - V_{B,\xi}(\tau)| \le \hat{g}_{A,B}(\tau-).$$

We can then employ the next estimate we calculate.

Consider $\xi \in \mathcal{B}_{A,B}(t)$. Then, using the estimates we have obtained in Lemma 2.4, we have

$$\begin{aligned}
\hat{g}_{A,B}(t) \leq \hat{g}_{A,B}(0)& \\
&+ \|\alpha'_{A,B}\|_\infty \int_0^t (V_{A,\xi}(s) \wedge V_{B,\xi}(s)) \\
&\qquad \times \left(|U_A(s) - U_B(s)| + \frac{1}{4}\|V_{A,\xi}(s) - V_{B,\xi}(s)\|_1 \right) ds \\
\leq \hat{g}_{A,B}(0)& \\
&+ \|\alpha'_{A,B}\|_\infty \int_0^t (V_{A,\xi}(s) \wedge V_{B,\xi}(s)) \\
&\qquad \times \left(|U_A(s) - U_B(s)| + \frac{1}{4}\|G_{A,B}(s)\|_1 \right) ds \\
\leq \hat{g}_{A,B}(0)& \\
&+ \int_0^t \left(\hat{g}_{A,B}(s) + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (V_{A,\xi}(s) \wedge V_{B,\xi}(s))\|G_{A,B}(s)\|_1 \right) ds.
\end{aligned}$$

Finally, we consider $\xi \in \Omega^c_{A,B}(t)$. We have

$$\begin{aligned}
|y_A(t) - \mathrm{id}|(V_{A,\xi}(t) \wedge V_{B,\xi}(t)) &\leq |y_A(0) - \mathrm{id}|(V_{A,\xi}(0) \wedge V_{B,\xi}(0)) \\
&\quad + \int_0^t |U_A(s)|(V_{A,\xi}(s) \wedge V_{B,\xi}(s))\, ds
\end{aligned}$$

and

$$\begin{aligned}
|U_A(t)|(V_{A,\xi}(t) \wedge V_{B,\xi}(t)) &\leq |U_A(0)|(V_{A,\xi}(0) \wedge V_{B,\xi}(0)) \\
&\quad + \frac{1}{4}\int_0^t \|V_{A,\xi}(s)\|_1 (V_{A,\xi}(s) \wedge V_{B,\xi}(s))\, ds.
\end{aligned}$$

Assume without loss of generality that $\tau_A(\xi) < \tau_B(\xi)$. First, we consider $\tau_A(\xi) < t < \tau_B(\xi)$. Then

$$\begin{aligned}
\bar{g}_{A,B}(t) &= |V_{A,\xi}(t) - V_{B,\xi}(t)| + \alpha_B(\mathrm{id})\big(V_{A,\xi}(t) \wedge V_{B,\xi}(t)\big) \\
&\quad + \|\alpha'_{A,B}\|_\infty \big(|y_B(t) - \mathrm{id}| + |U_A(t)| + |U_B(t)|\big)\big(V_{A,\xi}(t) \wedge V_{B,\xi}(t)\big) \\
&\leq \bar{g}_{A,B}(\tau_A) + \|\alpha'_{A,B}\|_\infty \int_{\tau_A}^t \big(V_{A,\xi}(s) \wedge V_{B,\xi}(s)\big) \\
&\qquad \times \big(|U_B(s)| + \frac{1}{4}\|V_{A,\xi}(s)\|_1 + \frac{1}{4}\|V_{B,\xi}(s)\|_1\big)\, ds.
\end{aligned}$$

For the case where $t < \tau_A(\xi) < \tau_B(\xi)$, we find

$$\begin{aligned}\bar{g}_{A,B}(t) &= |V_{A,\xi}(t) - V_{B,\xi}(t)| + (\alpha_A(\mathrm{id}) + \alpha_B(\mathrm{id}))\big(V_{A,\xi}(t) \wedge V_{B,\xi}(t)\big) \\ &\quad + \|\alpha'_{A,B}\|_\infty\big(V_{A,\xi}(t) \wedge V_{B,\xi}(t)\big)\big(|y_A(t) - \mathrm{id}| + |y_B(t) - \mathrm{id}| \\ &\qquad + 2|U_A(t)| + 2|U_B(t)|\big) \\ &\le \bar{g}_{A,B}(0) + \|\alpha'_{A,B}\|_\infty \int_0^t \big(V_{A,\xi}(s) \wedge V_{B,\xi}(s)\big) \\ &\qquad \times \big(|U_A(s)| + |U_B(s)| \\ &\qquad + \frac{1}{2}\|V_{A,\xi}(s)\|_1 + \frac{1}{2}\|V_{B,\xi}(s)\|_1\big)\, ds.\end{aligned}$$

The case where one breaks and the other does not can be analysed in a similar manner. In the end, we see that for any t such that the final wave breaking time has not occurred, we have

$$\begin{aligned}\bar{g}_{A,B}(t) &\le \bar{g}_{A,B}(0) \\ &\quad + \|\alpha'_{A,B}\|_\infty \int_0^t (V_{A,\xi}(s) \wedge V_{B,\xi}(s)) \\ &\qquad \times \left(|U_A(s)| + |U_B(s)| + \frac{1}{4}\|V_{A,\xi}(s)\|_1 + \frac{1}{4}\|V_{B,\xi}(s)\|_1\right) \\ &\qquad \times (\mathbb{1}_{\mathcal{A}_A^c(s)} + \mathbb{1}_{\mathcal{A}_B^c(s)})\, ds \\ &\le \bar{g}_{A,B}(0) + \int_0^t \bar{g}_{A,B}(s) ds \\ &\quad + \frac{1}{4}\|\alpha'_{A,B}\|_\infty \int_0^t (V_{A,\xi}(s) \wedge V_{B,\xi}(s))\left(\|V_{A,\xi}(s)\|_1 + \|V_{B,\xi}(s)\|_1\right) \\ &\qquad \times (\mathbb{1}_{\mathcal{A}_A^c(s)} + \mathbb{1}_{\mathcal{A}_B^c(s)})\mathbb{1}_{\mathcal{B}_{A,B}^c(s)}\, ds\end{aligned}$$

As pointed out earlier, the final term in (38) is decreasing with respect to time.

Combining all these estimates together, we have

$$\begin{aligned}G_{A,B}(\xi, t) &\le G_{A,B}(\xi, 0) \\ &\quad + \int_0^t \Bigg(G_{A,B}(\xi, s) + \frac{1}{4}\|\alpha'_{A,B}\|_\infty(V_{A,\xi}(\xi, s) \wedge V_{B,\xi}(\xi, s)) \\ &\qquad \times \|G_{A,B}(s)\|_1 \mathbb{1}_{\mathcal{B}_{A,B}(s)}(\xi)\Bigg)\, ds.\end{aligned} \tag{43}$$

Taking the L^1 norm with respect to ξ of (43), we have

$$\begin{aligned}\|G_{A,B}(t)\|_1 &\le \|G_{A,B}(0)\|_1 \\ &\quad + \int_0^t \left(\|G_{A,B}(s)\|_1 + \frac{1}{4}M_{A,B}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(s)\|_1\right) ds.\end{aligned}$$

Taking the L^2 norm with respect to ξ of (43), we have

$$\|G_{A,B}(t)\|_2 \le \|G_{A,B}(0)\|_2 + \int_0^t \left(\|G_{A,B}(s)\|_2 + \frac{1}{4}\sqrt{M_{A,B}}\|\alpha'_{A,B}\|_\infty \|G_{A,B}(s)\|_1 \right) ds,$$

where we have used Minkowski's inequality, and that, as $|V_{A,\xi}(\xi,t)| \le 1$ by assumption,

$$\int_{\mathbb{R}} (V_{A,\xi} \wedge V_{B,\xi})^2(\xi,t)\, d\xi \le \int_{\mathbb{R}} (V_{A,\xi} \wedge V_{B,\xi})(\xi,t)\, d\xi \le M_{A,B}.$$

□

We then define our norm $D : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$\begin{aligned} D(X_A^{\alpha_A}, X_B^{\alpha_B}) &= \|y_A - y_B\|_\infty + \|U_A - U_B\|_\infty \\ &\quad + \|y_{A,\xi} - y_{B,\xi}\|_2 + \|U_{A,\xi} - U_{B,\xi}\|_2 \\ &\quad + \|H_A - H_B\|_\infty + \frac{1}{4}\|G_{A,B}\|_1 + \frac{1}{2}\|G_{A,B}\|_2 \\ &\quad + \|\alpha_A - \alpha_B\|_\infty. \end{aligned} \tag{44}$$

Note. *Note that $G_{A,B}$, and hence D, does not satisfy the triangle inequality. D, however, satisfies the other properties in the definition of a metric on the space of Lagrangian coordinates. Thus, it is a semi-metric.*

As we will see in Section 4, the triangle inequality is not necessary for our final metric construction. This is due to Lemma A.1.

Lemma 3.2. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}$ be α-dissipative solutions with initial data $X_{0,A}^{\alpha_A} \in \mathcal{F}_0$ and $X_{0,B}^{\alpha_B} \in \mathcal{F}$, respectively. Then*

$$D(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \le e^{C_{A,B}t} D(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}),$$

with

$$C_{A,B} = 2 + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (M_{A,B} + 2\sqrt{M_{A,B}}) \tag{45}$$

and $M_{A,B}$ given by (39).

Proof. We have, combining (40) with Corollary 2.5,

$$\|U_A(t) - U_B(t)\|_\infty \le \|U_A(0) - U_B(0)\|_\infty + \frac{1}{4}\int_0^t \|G_{A,B}(s)\|_1\, ds,$$

$$\|U_{A,\xi}(t) - U_{B,\xi}(t)\|_2 \le \|U_{A,\xi}(0) - U_{B,\xi}(0)\|_2 + \frac{1}{2}\int_0^t \|G_{A,B}(s)\|_2\, ds.$$

Combining these inequalities with our estimates from Corollary 2.5 and Proposition 3.1, we have

$$\begin{aligned} D(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) &\le D(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}) \\ &\quad + \big(2 + \frac{1}{4}\|\alpha'_{A,B}\|_\infty (M_{A,B} + 2\sqrt{M_{A,B}})\big) \int_0^t D(X_A^{\alpha_A}(s), X_B^{\alpha_B}(s))\, ds. \end{aligned} \tag{47}$$

The result then follows from Grönwall's inequality. □

One final result we will make use of in the next section is as follows.

Lemma 3.3. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be two α-dissipative solutions with initial data $X_{0,A}^{\alpha_A}$ and $X_{0,B}^{\alpha_B}$ in $\mathcal{F}_0$. Given $t \geq 0$, let $f \in \mathcal{G}$ such that $\Pi(X_A^{\alpha_A}(t)) = X_A^{\alpha_A}(t) \circ f$ and $h \in \mathcal{G}$. Then,*

$$\begin{aligned} D(\Pi(X_A^{\alpha_A}(t)), X_B^{\alpha_B}(t) \circ h) &= D(X_A^{\alpha_A}(t) \circ f, X_B^{\alpha_B}(t) \circ h) \\ &\leq e^{(2\bar{M}_{A,B}+\frac{1}{4})t} D(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t) \circ w), \end{aligned} \tag{48}$$

where $w = h \circ f^{-1} \in \mathcal{G}$ and $\bar{M}_{A,B} = M_{A,B} \vee 1$.

Proof. To begin with note that while h can be any function in $\mathcal{G}$, the function f is unique and depends on the chosen time t. In particular one has, see e.g. [9], that

$$0 \leq f_\xi(\xi) \leq e^{\frac{1}{2}t} \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{49}$$

Furthermore, the group property implies, that $f^{-1}(\xi) = (y_A + H_A)(\xi, t)$, and hence (11), (13b), and $X_{0,A}^{\alpha_A}$ in $\mathcal{F}_0$ yield

$$\begin{aligned} |f^{-1}(\xi) - \xi| &= |(y_A + H_A)(\xi, t) - (y_A + H_A)(\xi, 0)| \\ &\leq \int_0^t |U_A(\xi, s)| ds \\ &\leq |U_A(\xi, t)|t + \frac{1}{4}\|V_A(0)\|_\infty t^2 \\ &\leq |U_A(\xi, t)|t + \frac{1}{4} M_{A,B} t^2 \end{aligned} \tag{50}$$

for all $\xi \in \mathbb{R}$.

Keeping these estimates in mind, we drop the t in $X_A^{\alpha_A}(t)$ and $X_B^{\alpha_B}(t)$ for ease in readability.

It is immediate that

$$\|y_A \circ f - y_B \circ h\|_\infty = \|y_A - y_B \circ w\|_\infty, \tag{51a}$$
$$\|U_A \circ f - U_B \circ h\|_\infty = \|U_A - U_B \circ w\|_\infty, \tag{51b}$$
$$\|H_A \circ f - H_B \circ h\|_\infty = \|H_A - H_B \circ w\|_\infty. \tag{51c}$$

Note that, for any function $F : \mathbb{R} \to \mathbb{R}$ differentiable at $\xi \in \mathbb{R}$,

$$(F \circ h)_\xi \circ f^{-1}(\xi) = (F \circ w)_\xi(\xi) f_\xi \circ f^{-1}(\xi).$$

Thus, after using the substitution $\eta = f(\xi)$ and (49),

$$\begin{aligned} \|(y_A \circ f)_\xi - (y_B \circ h)_\xi\|_2^2 &= \int_\mathbb{R} |(y_A \circ f)_\xi - (y_B \circ h)_\xi|^2 \circ f^{-1} (f^{-1})_\xi(\eta) d\eta \\ &= \int_\mathbb{R} |y_{A,\xi}(\eta) - (y_B \circ w)_\xi(\eta)|^2 f_\xi \circ f^{-1}(\eta)\, d\eta \\ &\leq \|f_\xi\|_\infty \|y_{A,\xi} - (y_B \circ w)_\xi\|_2^2 \\ &\leq e^{\frac{1}{2}t} \|y_{A,\xi} - (y_B \circ w)_\xi\|_2^2. \end{aligned} \tag{52}$$

And similarly, one finds

$$\|(U_A \circ f)_\xi - (U_B \circ h)_\xi\|_2^2 \le e^{\frac{1}{2}t} \|U_{A,\xi} - (U_B \circ w)_\xi\|_2^2. \tag{53}$$

We wish to show that

$$G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1}(\xi) \le A(t) G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w)(\xi) f_\xi \circ f^{-1}(\xi), \tag{54}$$

for some positive function $A(t)$.

For the characteristic functions, we have

$$\mathbb{1}_{\mathcal{A}_{A,B}^{f,h}} \circ f^{-1} = \mathbb{1}_{\mathcal{A}_{A,B}^{\mathrm{id},w}}, \quad \mathbb{1}_{\mathcal{B}_{A,B}^{f,h}} \circ f^{-1} = \mathbb{1}_{\mathcal{B}_{A,B}^{\mathrm{id},w}}, \tag{55}$$

and

$$\mathbb{1}_{\Omega_{A,B}^{f,h,c}} \circ f^{-1} = \mathbb{1}_{\Omega_{A,B}^{\mathrm{id},w,c}}, \tag{56}$$

which follow from (32), (33), and (34).

For g and $\hat{g}$, given by (35), (36),

$$\begin{aligned} g(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} &= |V_{A,\xi} - (V_B \circ w)_\xi| f_\xi \circ f^{-1} \\ &= g(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1}, \end{aligned} \tag{57}$$

$$\begin{aligned} \hat{g}(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h) \circ f^{-1} = \Big[& |V_{A,\xi} - (V_B \circ w)_\xi| \\ & + \|\alpha_A - \alpha_B\|_\infty \big(V_{A,\xi} \wedge (V_B \circ w)_\xi\big) \\ & + \|\alpha'_{A,B}\|_\infty \big(V_{A,\xi} \wedge (V_B \circ w)_\xi\big) \\ & \times \Big(|y_A - y_B \circ w| + |U_A - U_B \circ w|\Big)\Big] f_\xi \circ f^{-1} \\ = {} & \hat{g}(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) f_\xi \circ f^{-1}. \end{aligned} \tag{58}$$

For $\bar{g}$ given by (37),

$$
\begin{aligned}
&\bar{g}(X_A^{\alpha_A}\circ f, X_B^{\alpha_B}\circ h)\circ f^{-1}\\
&= \Big[|V_{A,\xi}-(V_B\circ w)_\xi|\\
&\quad + \big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\big((\alpha_A\circ f^{-1})\mathbb{1}_{\mathcal{A}_A^c} + (\alpha_B\circ f^{-1})\mathbb{1}_{\mathcal{A}_B^{w,c}}\big)\\
&\quad + \|\alpha'_{A,B}\|_\infty\big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\\
&\qquad \times \Big(|y_A-f^{-1}|\mathbb{1}_{\mathcal{A}_A^c} + |y_B\circ w - f^{-1}|\mathbb{1}_{\mathcal{A}_B^{w,c}}\\
&\qquad\quad + (|U_A|+|U_B\circ w|)(\mathbb{1}_{\mathcal{A}_A^c}+\mathbb{1}_{\mathcal{A}_B^{w,c}})\Big)\Big] f_\xi\circ f^{-1}\\
&\leq \Big[|V_{A,\xi}-(V_B\circ w)_\xi|\\
&\quad + \big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\\
&\qquad \times \Big((\|\alpha'_{A,B}\|_\infty(|U_A|t+\frac{1}{4}M_{A,B}t^2)+\alpha_A)\mathbb{1}_{\mathcal{A}_A^c}\\
&\qquad\quad + (\|\alpha'_{A,B}\|_\infty(|U_A|t+\frac{1}{4}M_{A,B}t^2)+\alpha_B)\mathbb{1}_{\mathcal{A}_B^{w,c}}\Big)\\
&\quad + \|\alpha'_{A,B}\|_\infty\big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\\
&\qquad \times \Big((|y_A-\mathrm{id}|+|U_A|t+\frac{1}{4}M_{A,B}t^2)\mathbb{1}_{\mathcal{A}_A^c}\\
&\qquad\quad + (|y_B\circ w-\mathrm{id}|+|U_A|t+\frac{1}{4}M_{A,B}t^2)\mathbb{1}_{\mathcal{A}_B^{w,c}}\\
&\qquad\quad + (|U_A|+|U_B\circ w|)(\mathbb{1}_{\mathcal{A}^c}+\mathbb{1}_{\mathcal{A}_B^{w,c}})\Big)\Big] f_\xi\circ f^{-1},\\
&= \bar{g}(X_A^{\alpha_A}, X_B^{\alpha_B}\circ w) f_\xi\circ f^{-1}\\
&\qquad + 2\|\alpha'_{A,B}\|_\infty\big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\\
&\qquad\qquad \times (|U_A|t+\frac{1}{4}M_{A,B}t^2)(\mathbb{1}_{\mathcal{A}_A^c}+\mathbb{1}_{\mathcal{A}_B^{w,c}}) f_\xi\circ f^{-1},
\end{aligned}
\tag{59}
$$

where we used (50). Finally, for the last term in G, we apply the same strategy. We get

$$
\begin{aligned}
&\Big[\big((V_A\circ f)_\xi\wedge (V_B\circ h)_\xi\big)\\
&\quad \times (\|(V_A\circ f)_\xi\|_1+\|(V_B\circ h)_\xi\|_1+1)(\mathbb{1}_{\mathcal{A}_A^{f,c}}+\mathbb{1}_{\mathcal{A}_B^{h,c}})\mathbb{1}_{\mathcal{B}_{A,B}^{f,h}}\Big]\circ f^{-1}\\
&= \big(V_{A,\xi}\wedge (V_B\circ w)_\xi\big)\\
&\quad \times (\|V_{A,\xi}\|_1+\|(V_B\circ w)_\xi\|_1+1)(\mathbb{1}_{\mathcal{A}_A^c}+\mathbb{1}_{\mathcal{A}_B^{w,c}})\mathbb{1}_{\mathcal{B}_{A,B}^{\mathrm{id},w}} f_\xi\circ f^{-1},
\end{aligned}
\tag{60}
$$

where we have used substitution to deal with the $L^1(\mathbb{R})$ terms present inside this term.

Thus, combining (55), (56), (57), (58), (59), and (60), we find

$$G(X_A^{\alpha_A}\circ f, X_B^{\alpha_B}\circ h)\circ f^{-1} \leq (1+2t+2M_{A,B}t^2)G(X_A^{\alpha_A}, X_B^{\alpha_B}\circ w,) f_\xi\circ f^{-1},$$

exactly as desired in (54). Taking the $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ norms, and with the substitution $\eta = f(\xi)$, we have

$$\|G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h)\|_1 \leq (1 + 2t + 2M_{A,B}t^2)\|G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w)\|_1 \tag{61}$$

and

$$\|G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B} \circ h)\|_2 \leq (1 + 2t + 2M_{A,B}t^2)e^{\frac{1}{4}t}\|G(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w)\|_2. \tag{62}$$

Combining (51), (52), (53). (61), and (62), we have

$$\begin{aligned} D(X_A^{\alpha_A} \circ f, &X_B^{\alpha_B} \circ h) \\ &\leq (1 + 2t + 2M_{A,B}t^2)e^{\frac{1}{4}t} D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w) \\ &\leq e^{2\bar{M}_{A,B}t} e^{\frac{1}{4}t} D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ w). \end{aligned}$$

□

4 Towards a metric

We have two issues we strive to resolve in this section. First, the mapping constructed in the previous section is not a metric, but it is a semi-metric. Second, Lagrangian coordinates that represent the same Eulerian coordinates, i.e. lie in the same equivalence class, do not in general have a distance of zero. In other words, this is a semi-metric over the whole space of Lagrangian coordinates, but not over the space of equivalence classes. In resolving the second issue, we resolve the first.

We begin with a helpful observation from the proof of Lemma 2.4 and (40).

Proposition 4.1. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be in $\mathcal{F}$. Then*

$$\|V_A - V_B\|_\infty \leq \|V_{A,\xi} - V_{B,\xi}\|_1 \leq \|G_{A,B}\|_1.$$

Define $J : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$J(X_A^{\alpha_A}, X_B^{\alpha_B}) = \inf_{f,g \in \mathcal{G}} \left(D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ f) + D(X_A^{\alpha_A} \circ g, X_B^{\alpha_B}) \right). \tag{63}$$

We begin by noting that J is zero when measuring the distance between members of the same equivalence class. Indeed, suppose that $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ in $\mathcal{F}$ share the same equivalence class. That is, there exist f_A and f_B in $\mathcal{G}$ such that

$$X_A^{\alpha_A} \circ f_A = X_B^{\alpha_B} \quad \text{and} \quad X_B^{\alpha_B} \circ f_B = X_A^{\alpha_A}.$$

Then

$$D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ f_B) + D(X_A^{\alpha_A} \circ f_A, X_B^{\alpha_B}) = D(X_A^{\alpha_A}, X_A^{\alpha_A}) + D(X_B^{\alpha_B}, X_B^{\alpha_B}) = 0,$$

and hence the infimum will be zero.

We will make use of a slight modification of a result that has already been established in [7, Lemma 3.2].

Lemma 4.2. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be in $\mathcal{F}_0$. Then, for any relabelling function $f \in \mathcal{G}$,*

$$\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq 5\|X_A^{\alpha_A} \circ f - X_B^{\alpha_B}\|,$$

where the norm $\|\cdot\|$ is given by,

$$\|X^\alpha\| = \|y - \mathrm{id}\|_\infty + \|U\|_\infty + \|H\|_\infty + \frac{1}{4}\|V\|_\infty + \|\alpha\|_\infty \quad \textit{for any } X^\alpha \in \mathcal{F}.$$

Hence, we have

$$2\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq 5\|X_A^{\alpha_A} \circ f - X_B^{\alpha_B}\| + 5\|X_A^{\alpha_A} - X_B^{\alpha_B} \circ g\|. \tag{64}$$

Using Proposition 4.1, we have

$$\begin{aligned}
\|X_A^{\alpha_A} \circ f - X_B^{\alpha_B}\| &= \|y_A \circ f - y_B\|_\infty + \|U_A \circ f - U_B\|_\infty \\
&\quad + \|H_A \circ f - H_B\|_\infty + \frac{1}{4}\|V_A \circ f - V_B\|_\infty \\
&\quad + \|\alpha_A - \alpha_B\|_\infty \\
&\leq \|y_A \circ f - y_B\|_\infty + \|U_A \circ f - U_B\|_\infty \qquad (65)\\
&\quad + \|H_A \circ f - H_B\|_\infty + \frac{1}{4}\|G(X_A^{\alpha_A} \circ f, X_B^{\alpha_B})\|_1 \\
&\quad + \|\alpha_A - \alpha_B\|_\infty \\
&\leq D(X_A^{\alpha_A} \circ f, X_B^{\alpha_B}).
\end{aligned}$$

Thus, substituting this inequality into (64), we see

$$\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq \frac{5}{2}\big(D(X_A^{\alpha_A} \circ f, X_B^{\alpha_B}) + D(X_A^{\alpha_A}, X_B^{\alpha_B} \circ g)\big) \tag{66}$$

and after taking the infimum over all $f, g \in \mathcal{G}$, we have the following.

Corollary 4.3. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be in $\mathcal{F}_0$. Then*

$$\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq \frac{5}{2} J(X_A^{\alpha_A}, X_B^{\alpha_B}).$$

Thus the restriction of J to $\mathcal{F}_0 \times \mathcal{F}_0$ is a semi-metric.

Using this semi-metric we are able to construct a metric on the more restricted set $\mathcal{F}_M^L$, given by (10). For M, $L > 0$, introduce $\hat{d} : \mathcal{F}_M^L \times \mathcal{F}_M^L \to \mathbb{R}$, defined by

$$\hat{d}(X_A^{\alpha_A}, X_B^{\alpha_B}) \coloneqq \inf_{\hat{\mathcal{F}}(X_A^{\alpha_A}, X_B^{\alpha_B})} \sum_{n=1}^{N} J(X_n^{\alpha_n}, X_{n-1}^{\alpha_{n-1}}), \tag{67}$$

where $\hat{\mathcal{F}}(X_A^{\alpha_A}, X_B^{\alpha_B})$ is the set of finite sequences $\{X_n^{\alpha_n}\}_{n=0}^N$ of arbitrary length in $\mathcal{F}_{0,M}^L$, satisfying $X_0^{\alpha_0} = \Pi X_A^{\alpha_A}$ and $X_N^{\alpha_N} = \Pi X_B^{\alpha_B}$.

Note. *Let $X_A^{\alpha_A}$, $X_B^{\alpha_B} \in \mathcal{F}_M^L$. Then, directly from the definition we have*

$$\hat{d}(X_A^{\alpha_A}, X_B^{\alpha_B}) = \hat{d}(\Pi X_A^{\alpha_A}, \Pi X_B^{\alpha_B}).$$

Note. *$\hat{d}$ inherits from J that if both $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ are in the same equivalence class, then $\hat{d}(X_A^{\alpha_A}, X_B^{\alpha_B})$ is zero. Indeed, consider the finite sequence $X_0^{\alpha_0} = \Pi X_A^{\alpha_A}$ and $X_1^{\alpha_1} = \Pi X_B^{\alpha_B} = \Pi X_A^{\alpha_A}$.*

It remains to ensure that $\hat{d}$ satisfies the identity of indiscernibles. That is we need to prove the implication

$$\hat{d}(X_A^{\alpha_A}, X_B^{\alpha_B}) = 0 \implies X_A^{\alpha_A} \sim X_B^{\alpha_B}, \tag{68}$$

meaning if the distance between the two elements is zero, then both Lagrangian coordinates lie in the same equivalence class.

Using Corollary 4.3 and Lemma A.1, with $F = J$,we get the following result, which confirms (68).

Corollary 4.4. *The function $\hat{d} : \mathcal{F}_M^L \times \mathcal{F}_M^L \to \mathbb{R}$ defined by (67) is a metric. Furthermore, for any $X_A^{\alpha_A}, X_B^{\alpha_B} \in \mathcal{F}_{0,M}^L$ it satisfies*

$$\frac{2}{5}\|X_A^{\alpha_A} - X_B^{\alpha_B}\| \leq \hat{d}(X_A^{\alpha_A}, X_B^{\alpha_B}) \leq J(X_A^{\alpha_A}, X_B^{\alpha_B}).$$

The following lemma will form the bridge that allows us to use the Lipschitz stability estimate we have obtained for D to prove Lipschitz stability with respect to $\hat{d}$.

Lemma 4.5. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be two α-dissipative solutions with initial data $X_{0,A}^{\alpha_A}$ and $X_{0,B}^{\alpha_B}$ in $\mathcal{F}_0$, respectively. Then*

$$J(\Pi X_A^{\alpha_A}(t), \Pi X_B^{\alpha_B}(t)) \leq e^{(4\bar{M}_{A,B} + \frac{1}{2})t} J(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)),$$

where $\bar{M}_{A,B} = M_{A,B} \vee 1$.

Proof. To ease digestion, we drop α and t in the notation for this proof. Furthermore, we set $C \coloneqq 2\bar{M}_{A,B} + \frac{1}{4}$ and for $i = A$, B, let $f_i \in \mathcal{G}$ such that $\Pi X_i^{\alpha_i} = X_i^{\alpha_i} \circ f_i$.

From Lemma 3.3, we have,

$$\begin{aligned} J(\Pi X_A, \Pi X_B) &= \inf_{f_1, f_2} \left(D(X_A \circ f_A \circ f_1, \Pi X_B) + D(X_A \circ f_A, (\Pi X_B) \circ f_2) \right) \\ &\leq \inf_{f_1, f_2} \left(D(X_A \circ f_1, \Pi X_B) + e^{Ct} D(X_A, (\Pi X_B) \circ f_2 \circ f_A^{-1}) \right) \\ &\leq e^{Ct} \inf_{f_1, f_2} \left(D(X_A \circ f_1, \Pi X_B) + D(X_A, (\Pi X_B) \circ f_2) \right) \\ &= e^{Ct} J(X_A, \Pi X_B), \end{aligned} \tag{69}$$

where we are using that $f_A \circ f_1$ lies in $\mathcal{G}$ for any $f_1 \in \mathcal{G}$ and that any element $f \in \mathcal{G}$ can be written as $f = f_A \circ g$ for some $g \in \mathcal{G}$, which implies that $g = f_A^{-1} \circ f$.

We can then do the same again, but now we swap the roles of X_B and X_A,

$$\begin{aligned} J(X_A, \Pi X_B) &= \inf_{f_1,f_2} \big(D(X_A \circ f_1, X_B \circ f_B) + D(X_A, X_B \circ f_B \circ f_2)\big) \\ &\leq \inf_{f_1,f_2} \big(e^{Ct} D(X_A \circ f_1 \circ f_B^{-1}, X_B) + D(X_A, X_B \circ f_2)\big) \\ &\leq e^{Ct} \inf_{f_1,f_2} \big(D(X_A \circ f_1, X_B) + D(X_A, X_B \circ f_2)\big) \\ &= e^{Ct} J(X_A, X_B). \end{aligned} \tag{70}$$

Substituting (70) into (69), we obtain the required result. □

Thus we can now show our Lipschitz stability result.

Theorem 4.6. *Let $X_A^{\alpha_A}$ and $X_B^{\alpha_B}$ be two α-dissipative solutions with initial data $X_{0,A}^{\alpha_A}$ and $X_{0,B}^{\alpha_B}$ in $\mathcal{F}_{0,M}^L$, respectively. Then*

$$\hat{d}(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \leq e^{R_M^L t} \hat{d}(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}), \tag{71}$$

where

$$R_M^L := 4\bar{M} + \frac{5}{2} + \frac{1}{4} L(M + 2\sqrt{M}), \tag{72}$$

with $\bar{M} = M \vee 1$.

Proof. Let $\epsilon > 0$. Consider a finite sequence $\{X_{0,n}^{\alpha_n}\}_{n=0}^N \in \hat{\mathcal{F}}(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B})$ and a sequence of relabelling functions $\{f_n\}_{n=0}^{N-1}, \{g_n\}_{n=1}^N$ in $\mathcal{G}$ such that

$$\sum_{n=1}^N \big(D(X_{0,n}^{\alpha_n}, X_{0,n-1}^{\alpha_{n-1}} \circ f_{n-1}) + D(X_{0,n}^{\alpha_n} \circ g_n, X_{0,n-1}^{\alpha_{n-1}})\big) < \hat{d}(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}) + \epsilon.$$

Set $X_n^{\alpha_n}(t) = S_t X_{0,n}^{\alpha_n}$. Then, by Lemma 2.10, $X_A^{\alpha_A}(t) = S_t X_{0,0}^{\alpha_0}$, and $X_B^{\alpha_B}(t) = S_t X_{0,N}^{\alpha_N}$. Furthermore, $X_n^{\alpha_n}(t) \in \mathcal{F}_M^L$ for all $t \geq 0$ and all n. Thus, using Lemmas 2.10, 3.2 and 4.5,

$$\begin{aligned} &\hat{d}(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \\ &\leq \sum_{n=1}^N J(\Pi X_n^{\alpha_n}(t), \Pi X_{n-1}^{\alpha_{n-1}}(t)) \\ &\leq e^{(4\bar{M}+\frac{1}{2})t} \sum_{n=1}^N J(X_n^{\alpha_n}(t), X_{n-1}^{\alpha_{n-1}}(t)) \\ &\leq e^{(4\bar{M}+\frac{1}{2})t} \sum_{n=1}^N \big(D(X_n^{\alpha_n}(t), X_{n-1}^{\alpha_{n-1}}(t) \circ f_{n-1}) + D(X_n^{\alpha_n}(t) \circ g_n, X_{n-1}^{\alpha_{n-1}}(t))\big) \\ &\leq e^{(4\bar{M}+\frac{1}{2})t} \sum_{n=1}^N e^{(2+\frac{1}{4}L(M+2\sqrt{M}))t} \big(D(X_n^{\alpha_n}(0), X_{n-1}^{\alpha_{n-1}}(0) \circ f_{n-1}) \\ &\qquad\qquad + D(X_n^{\alpha_n}(0) \circ g_n, X_{n-1}^{\alpha_{n-1}}(0))\big) \\ &\leq e^{(4\bar{M}+\frac{5}{2}+\frac{1}{4}L(M+2\sqrt{M}))t} (\hat{d}(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}) + \epsilon.). \end{aligned}$$

The final result follows, as this inequality is true for any $\epsilon > 0$. □

4.1 A simplification in the case α is a constant

In the case where $\alpha \in [0,1] \subset \Lambda$, i.e. α is a constant, the construction can be simplified.

First, define the subset of $\mathcal{F}$ containing elements for which α is constant,

$$\mathcal{F}_c \coloneqq \{X^\alpha \in \mathcal{F} \mid \alpha \in [0,1]\}.$$

For $X^\alpha \in \mathcal{F}_c$, we introduce the two functions

$$V_\xi^d(\xi,t) \coloneqq \alpha V_\xi(\xi,t)\mathbb{1}_{\mathcal{A}^c(t)}(\xi), \qquad V_\xi^c(\xi,t) \coloneqq (1-\alpha\mathbb{1}_{\mathcal{A}^c(t)}(\xi))V_\xi(\xi,t).$$

The second function V_ξ^c is in fact constant, so the time dependence can be dropped. Note also $V_\xi(\xi,t) = V_\xi^c(\xi) + V_\xi^d(\xi,t)$.

Using this, we can introduce a simpler function $G : \mathcal{F}_c \times \mathcal{F}_c \to [0,+\infty)$, given by

$$\begin{aligned} G_{A,B}(\xi) &= G\left(X_A^{\alpha_A}, X_B^{\alpha_B}\right)(\xi) \\ &= |V_{A,\xi}(\xi) - V_{B,\xi}(\xi)|\mathbb{1}_{\mathcal{A}_{A,B}}(\xi) \\ &\quad + \left(|V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + |V_{A,\xi}^d(\xi) - V_{B,\xi}^d(\xi)|\right)\mathbb{1}_{\mathcal{B}_{A,B}}(\xi) \\ &\quad + \left(|V_{A,\xi}^c(\xi) - V_{B,\xi}^c(\xi)| + V_{A,\xi}^d(\xi) \vee V_{B,\xi}^d(\xi)\right)\mathbb{1}_{\Omega_{A,B}^c}(\xi), \end{aligned}$$

for any $X_A^{\alpha_A}$, $X_B^{\alpha_B} \in \mathcal{F}^c$. It satisfies

$$|V_{A,\xi}(\xi) - V_{B,\xi}(\xi)| \leq |G_{A,B}(\xi)|.$$

We can then define a metric $D : \mathcal{F}_c \times \mathcal{F}_c \to \mathbb{R}$ by

$$\begin{aligned} D(X_A^{\alpha_A}, X_B^{\alpha_B}) \coloneqq{} & \|y_A - y_B\|_\infty + \|U_A - U_B\|_\infty + \|H_A - H_B\|_\infty \\ & + \|y_{A,\xi} - y_{B,\xi}\|_2 + \|U_{A,\xi} - U_{B,\xi}\|_2 \\ & + \frac{1}{4}\|G_{A,B}\|_1 + \frac{1}{2}\|G_{A,B}\|_2 + |\alpha_A - \alpha_B|. \end{aligned} \tag{73}$$

The construction throughout Section 3 and Section 4 can be repeated, yielding the following result. For any two α-dissipative solutions $X_A^{\alpha_A}$, $X_B^{\alpha_B}$ with initial data $X_{0,A}^{\alpha_A}$, $X_{0,B}^{\alpha_B} \in \mathcal{F}_c \cap \mathcal{F}_0$,

$$\hat{d}(X_A^{\alpha_A}(t), X_B^{\alpha_B}(t)) \leq e^{\frac{3}{2}t}\hat{d}(X_{0,A}^{\alpha_A}, X_{0,B}^{\alpha_B}).$$

Note that here $L = 0$ and that the exponent is independent of M. This is also why we can consider any initial data in $\mathcal{F}_c \cap \mathcal{F}_0$ and not only in $\mathcal{F}_c \cap \mathcal{F}_{0,M}^L$.

5 A return to Eulerian coordinates

Using our metric in Lagrangian coordinates, we shall now define our metric in Eulerian coordinates. The problem we have to overcome is the fact that a solution to the α-dissipative Hunter–Saxton problem consists of a pair (u,μ), and the additional dummy measure ν is only necessary for the construction of said solution.

Before we tackle this issue, we note an immediate corollary of our previous theorem. Define the metric $d_{\mathcal{D}} : \mathcal{D}_M^L \times \mathcal{D}_M^L \to \mathbb{R}$ by

$$d_{\mathcal{D}}(Y_A^{\alpha_A}, Y_B^{\alpha_B}) \coloneqq \hat{d}(\hat{L}(Y_A^{\alpha_A}), \hat{L}(Y_B^{\alpha_B})). \tag{74}$$

We then have the following result which is an immediate consequence of Theorem 4.6.

Corollary 5.1. *Let $Y_A^{\alpha_A}$, $Y_B^{\alpha_B}$ be two α-dissipative solutions with initial data $Y_{0,A}^{\alpha_A}$ and $Y_{0,B}^{\alpha_B}$ in $\mathcal{D}_M^L$, respectively. Then*

$$d_{\mathcal{D}}(Y_A^{\alpha_A}(t), Y_B^{\alpha_B}(t)) \leq e^{R_M^L t} d_{\mathcal{D}}(Y_{0,A}^{\alpha_A}, Y_{0,B}^{\alpha_B}),$$

with R_M^L given by (72).

Recalling Definition 2.14, our construction now follows a very similar path to that of the Lagrangian metric. We begin by defining a function $\hat{J} : \mathcal{D}_{0,M}^L \times \mathcal{D}_{0,M}^L \to \mathbb{R}$, given by

$$\hat{J}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) \coloneqq \inf_{(\nu_A,\nu_B)\in\mathcal{V}(Z_A^{\alpha_A})\times\mathcal{V}(Z_B^{\alpha_B})} d_{\mathcal{D}}(((Z_A,\nu_A),\alpha_A),((Z_B,\nu_B),\alpha_B)), \tag{75}$$

which no longer depends on the choice of ν. In a similar vain to J, this function is zero when measuring the distance between two elements of the same equivalence class in $\mathcal{D}$.

We cannot conclude that $\hat{J}$ satisfies the triangle inequality. Using the same strategy as before, we define the function $\bar{d} : \mathcal{D}_{0,M}^L \times \mathcal{D}_{0,M}^L \to \mathbb{R}$ by

$$\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) = \inf_{\hat{\mathcal{D}}(Z_A^{\alpha_A}, Z_B^{\alpha_B})} \sum_{i=1}^{N} \hat{J}(Z_i^{\alpha_i}, Z_{i-1}^{\alpha_{i-1}}), \tag{76}$$

where $\hat{\mathcal{D}}(Z_A^{\alpha_A}, Z_B^{\alpha_B})$ denotes the set of all finite sequences of arbitrary length $\{Z_i^{\alpha_i}\}_{i=0}^N$ in $\mathcal{D}_{0,M}^L$ satisfying $Z_0^{\alpha_0} = Z_A^{\alpha_A}$ and $Z_N^{\alpha_N} = Z_B^{\alpha_B}$.

From Lemma A.1, we can only conclude that $\bar{d}$ is a pseudo-metric, as inequality (85) is not satisfied. It therefore remains to prove the implication

$$\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) = 0 \implies Z_A^{\alpha_A} = Z_B^{\alpha_B}.$$

We introduce now the bounded Lipschitz norm on the set of finite Radon measures $\mathcal{M}(\mathbb{R})$,

$$\|\mu\|_{\mathcal{M}} = \sup_{\phi\in\mathcal{L}} \left| \int_{\mathbb{R}} \phi(x)\, d\mu \right|, \tag{77}$$

where

$$\mathcal{L} = \{\phi \in W^{1,\infty}(\mathbb{R}) \mid \|\phi\|_{1,\infty} \leq 1\}.$$

Lemma 5.2. *For $Z_A^{\alpha_A} = ((u_A, \mu_A), \alpha_A)$ and $Z_B^{\alpha_B} = ((u_B, \mu_B), \alpha_B)$ in $\mathcal{D}_0$, define the norm*

$$\|Z_A^{\alpha_A} - Z_B^{\alpha_B}\|_{\mathcal{D}_0} := \|u_A - u_B\|_\infty + \|\mu_A - \mu_B\|_{\mathcal{M}} + \|\alpha_A - \alpha_B\|_\infty.$$

Then, for any $Z_A^{\alpha_A}$, $Z_B^{\alpha_B} \in \mathcal{D}_{0,M}^L$,

$$\|Z_A^{\alpha_A} - Z_B^{\alpha_B}\|_{\mathcal{D}_0} \leq \left(5 + 2\bar{M}\right) \bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \frac{\sqrt{5M}}{\sqrt{2}} \sqrt{\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B})}, \tag{78}$$

where $\bar{M} = 1 \vee M$.

Proof. Let $\epsilon > 0$. Consider a sequence

$$\{Y_k^{\alpha_k}\}_{k=0}^N = \{((Z_k, \nu_k), \alpha_k)\}_{k=0}^N = \{((u_k, \mu_k, \nu_k), \alpha_k)\}_{k=0}^N \quad \text{in } \mathcal{D}_M^L$$

satisfying $Z_0^{\alpha_0} = Z_A^{\alpha_A}$ and $Z_N^{\alpha_N} = Z_B^{\alpha_B}$ such that

$$\sum_{k=1}^N d_{\mathcal{D}}(Y_k^{\alpha_k}, Y_{k-1}^{\alpha_{k-1}}) \leq \bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \epsilon.$$

Set $X_k^{\alpha_k} = \hat{L}(Y_k^{\alpha_k})$ for $k = 0, \ldots, N$. Notice that from the definition of $\hat{L}$, $X_k^{\alpha_k} \in \mathcal{F}_0$. Then, from Corollary 4.4

$$\begin{aligned}
\|\alpha_A - \alpha_B\|_\infty \leq \|X_0^{\alpha_0} - X_N^{\alpha_N}\| &\leq \frac{5}{2}\hat{d}(X_0^{\alpha_0}, X_N^{\alpha_N}) \\
&= \frac{5}{2} d_{\mathcal{D}}(Y_0^{\alpha_0}, Y_N^{\alpha_N}) \\
&\leq \frac{5}{2} \sum_{k=1}^N d_{\mathcal{D}}(Y_k^{\alpha_k}, Y_{k-1}^{\alpha_{k-1}}) \\
&\leq \frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \frac{5}{2}\epsilon.
\end{aligned} \tag{79}$$

This holds for any $\varepsilon > 0$, and thus

$$\|\alpha_A - \alpha_B\|_\infty \leq \frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}). \tag{80}$$

From the continuity and increasing nature of y_0, for any $x \in \mathbb{R}$ there exists a $\xi \in \mathbb{R}$ such that $y_0(\xi) = x$. It then follows that

$$\begin{aligned}
|u_A(x) - u_B(x)| &= |u_A(y_0(\xi)) - u_B(y_0(\xi))| \\
&\leq |u_A(y_0(\xi)) - u_B(y_N(\xi))| + |u_B(y_N(\xi)) - u_B(y_0(\xi))| \\
&= |U_0(\xi) - U_N(\xi)| + \left| \int_{y_0(\xi)}^{y_N(\xi)} u_{B,x}(\eta)\, d\eta \right| \\
&\leq \|U_0 - U_N\|_\infty + \sqrt{|y_0(\xi) - y_N(\xi)|} \left(\int_{\mathbb{R}} u_{B,x}^2(\eta)\, d\eta \right)^{\frac{1}{2}} \\
&\leq \|X_0^{\alpha_0} - X_N^{\alpha_N}\| + \sqrt{\|X_0^{\alpha_0} - X_N^{\alpha_N}\|}\sqrt{M} \\
&\leq \frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \frac{5}{2}\epsilon + \sqrt{M}\sqrt{\frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \frac{5}{2}\epsilon},
\end{aligned}$$

where we used (79). This holds for any $\epsilon > 0$, and thus

$$\|u_A - u_B\|_\infty \leq \frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \sqrt{M}\sqrt{\frac{5}{2}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B})}. \tag{81}$$

Consider any $\phi \in \mathcal{L}$ and $k = 1, \dots, N$. Then,

$$\left|\int_{\mathbb{R}} \phi(x)\, d(\mu_k - \mu_{k-1})\right| = \left|\int_{\mathbb{R}} (\phi \circ y_k)(\xi) V_{k,\xi}(\xi) - (\phi \circ y_{k-1})(\xi) V_{k-1,\xi}(\xi)\, d\xi\right|.$$

After using $\xi = f(\eta)$, where $f \in \mathcal{G}$ is some relabelling function, we find

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \phi(x)\, d(\mu_k - \mu_{k-1})\right| \\
&= \left|\int_{\mathbb{R}} (\phi \circ y_k \circ f)(\xi)(V_k \circ f)_\xi(\xi) - (\phi \circ y_{k-1})(\xi) V_{k-1,\xi}(\xi)\, d\xi\right| \\
&\leq \left|\int_{\mathbb{R}} (\phi \circ y_k \circ f)(\xi)((V_k \circ f)_\xi(\xi) - V_{k-1,\xi}(\xi))\, d\xi\right| \\
&\quad + \left|\int_{\mathbb{R}} ((\phi \circ y_k \circ f)(\xi) - (\phi \circ y_{k-1})(\xi)) V_{k-1,\xi}(\xi)\, d\xi\right|.
\end{aligned}$$

Focusing on the first integral, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} (\phi \circ y_k \circ f)(\xi)((V_k \circ f)_\xi(\xi) - V_{k-1,\xi}(\xi))\, d\xi\right| &\leq \|\phi\|_\infty \|(V_k \circ f)_\xi - V_{k-1,\xi}\|_1 \\
&\leq \|(V_k \circ f)_\xi - V_{k-1,\xi}\|_1 \\
&\leq \|G(X_k^{\alpha_k} \circ f, X_{k-1}^{\alpha_{k-1}})\|_1,
\end{aligned}$$

where the final inequality follows from (40) in Proposition 3.1.

For the second integral

$$\begin{aligned}
&\left|\int_{\mathbb{R}} ((\phi \circ y_k \circ f)(\xi) - (\phi \circ y_{k-1})(\xi)) V_{k-1,\xi}(\xi)\, d\xi\right| \\
&\leq \int_{\mathbb{R}} |(y_k \circ f)(\xi) - y_{k-1}(\xi)| V_{k-1,\xi}(\xi)\, d\xi \\
&\leq M\|y_k \circ f - y_{k-1}\|_\infty
\end{aligned}$$

where we have used $\|\phi\|_{1,\infty} \leq 1$,and $\|V_{k-1,\xi}\|_1 \leq M$. Thus after taking the sum of these two inequalities, from a similar argument to that used for (65), we find

$$\left|\int_{\mathbb{R}} \phi(x)\, d(\mu_k - \mu_{k-1})\right| \leq 4\bar{M} D\left(X_k^{\alpha_k} \circ f, X_{k-1}^{\alpha_{k-1}}\right), \tag{82}$$

Swapping the k and $k-1$ terms, and replacing f by another relabelling function $g \in \mathcal{G}$, we get

$$\left|\int_{\mathbb{R}} \phi(x)\, d(\mu_k - \mu_{k-1})\right| \leq 4\bar{M} D\left(X_k^{\alpha_k}, X_{k-1}^{\alpha_{k-1}} \circ g\right). \tag{83}$$

Thus, summing (82) and (83), and taking the infimum over all $f, g \in \mathcal{G}$, we find

$$\left|\int_{\mathbb{R}} \phi(x)\, d(\mu_k - \mu_{k-1})\right| \leq 2\bar{M} J(X_k^{\alpha_k}, X_{k-1}^{\alpha_{k-1}}),$$

and hence we can apply the same argument as in the proof of Lemma A.1 for the left inequality of (86), obtaining

$$\begin{aligned}\left|\int_{\mathbb{R}} \phi(x) d(\mu_k - \mu_{k-1})\right| &\leq 2\bar{M}\hat{d}(X_k^{\alpha_k}, X_{k-1}^{\alpha_{k-1}})\\ &= 2\bar{M} d_{\mathcal{D}}(Y_k^{\alpha_k}, Y_{k-1}^{\alpha_{k-1}}),\end{aligned}$$

Taking the infimum over all $\phi \in \mathcal{L}$, and from the definition of $\|\cdot\|_{\mathcal{M}}$, see (77), we have that

$$\begin{aligned}\|\mu_A - \mu_B\|_{\mathcal{M}} &\leq \sum_{k=1}^{N} \|\mu_k - \mu_{k-1}\|_{\mathcal{M}}\\ &\leq 2\bar{M} \sum_{k=1}^{N} d_{\mathcal{D}}(Y_k^{\alpha_k}, Y_{k-1}^{\alpha_k - 1})\\ &\leq 2\bar{M}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}) + \epsilon.\end{aligned}$$

Once again as this construction can be done for any $\epsilon > 0$, we can conclude

$$\|\mu_A - \mu_B\|_{\mathcal{M}} \leq 2\bar{M}\bar{d}(Z_A^{\alpha_A}, Z_B^{\alpha_B}). \tag{84}$$

Summing up (80), (81), and (84), we get (78). □

With everything set up, we can finish with our main theorem.

Theorem 5.3. *Let $Z_A^{\alpha_A} = ((u_A, \mu_A), \alpha_A)$ and $Z_B^{\alpha_B} = ((u_B, \mu_B), \alpha_B)$ be two α-dissipative solutions to* (HS)*, constructed via the generalised method of characteristics, with initial data $Z_{0,A}^{\alpha_A}$ and $Z_{0,B}^{\alpha_B}$ in $\mathcal{D}_{0,M}^L$, respectively. Then*

$$\bar{d}(Z_A^{\alpha_A}(t), Z_B^{\alpha_B}(t)) \leq e^{R_M^L t}\bar{d}(Z_{0,A}^{\alpha_A}, Z_{0,B}^{\alpha_B}).$$

with R_M^L given by (72).

Proof. Let $\epsilon > 0$. Given $Z_{0,A}^{\alpha_A}$ and $Z_{0,B}^{\alpha_B}$ in $\mathcal{D}_{0,M}^L$, there exists a sequence

$$\{Y_{0,k}^{\alpha_k}\}_{k=0}^N = \{(Z_{0,k}, \nu_{0,k}), \alpha_k)\}_{k=0}^N = \{((u_{0,k}, \mu_{0,k}, \nu_{0,k}), \alpha_k)\}_{k=0}^N \quad \text{in } \mathcal{D}_M^L$$

such that $Z_{0,0}^{\alpha_0} = Z_{0,A}^{\alpha_A}$, $Z_{0,N}^{\alpha_N} = Z_{0,B}^{\alpha_B}$, and

$$\sum_{i=k}^{N} d_{\mathcal{D}}(Y_{0,k}^{\alpha_k}, Y_{0,k-1}^{\alpha_{k-1}}) \leq \bar{d}(Z_{0,A}^{\alpha_A}, Z_B^{\alpha_B}) + \epsilon.$$

Denote by $Y_k^{\alpha_k}$ for $k = 0, \dots, N$ the α dissipative solution with initial data $Y_{0,k}^{\alpha_k}$. Then, from Corollary 5.1,

$$\begin{aligned}\bar{d}(Z_A^{\alpha_A}(t), Z_B^{\alpha_B}(t)) &\leq \sum_{k=1}^{N} d_{\mathcal{D}}(Y_k^{\alpha_k}(t), Y_{k-1}^{\alpha_{k-1}}(t)) \\ &\leq e^{R_M^L t} \sum_{k=1}^{N} d_{\mathcal{D}}(Y_{0,k}^{\alpha_k}, Y_{0,k-1}^{\alpha_{k-1}}) \\ &\leq e^{R_M^L t}(\bar{d}(Z_{A,0}^{\alpha_A}, Z_B^{\alpha_B}) + \epsilon),\end{aligned}$$

and as this construction can be done for any $\epsilon > 0$, the result holds. □

5.1 A simplification in the case α is constant.

Using Section 4.1 as basis, one can repeat the construction from this section, yielding the following result. For any two α-dissipative solutions $Z_A^{\alpha_A}$, $Z_B^{\alpha_B}$ with initial data $Z_{0,A}^{\alpha_A}$, $Z_{0,B}^{\alpha_B}$ in $\mathcal{D}_0$,

$$\bar{d}(Z_A^{\alpha_A}(t), Z_B^{\alpha_B}(t)) \leq e^{\frac{3}{2}t}\bar{d}(Z_{0,A}^{\alpha_A}, Z_{0,B}^{\alpha_B}).$$

Note that here $L = 0$ and that the exponent is independent of M. This is also why we can consider any initial data in $\mathcal{D}_0$ and not only in $\mathcal{D}_{0,M}^L$.

Appendices

Appendix A Important results

The following result is a well established construction of a pseudo-metric on the quotient of a metric space. For instance, the idea was used in [6] for the periodic Camassa–Holm equation.

Lemma A.1. *Let $X \subseteq Y$, with Y a normed space, and suppose*

$$\|x_A - x_B\| \leq CF(x_A, x_B), \qquad \textit{for all } x_A, x_B \in X, \tag{85}$$

for some function $F : X \times X \to \mathbb{R}^+$ and some constant $C > 0$. If F satisfies for all $x_A, x_B \in X$

- $x_A = x_B \implies F(x_A, x_B) = 0$,
- $F(x_A, x_B) = F(x_B, x_A)$,

then the function $d : X \times X \to \mathbb{R}^+$ given by

$$d(x_A, x_B) := \inf \left\{ \sum_{k=1}^{N} F(x_k, x_{k-1}) \;\middle|\; x_k \in X, x_0 = x_A, x_N = x_B, N \in \mathbb{N} \right\}$$

is a metric, and

$$\frac{1}{C}\|x_A - x_B\| \leq d(x_A, x_B) \leq F(x_A, x_B) \tag{86}$$

for all $x_A, x_B \in X$.

Should (85) *not be satisfied, but the rest of the conditions are, then we can only conclude that* d *is a pseudo-metric. That is, we cannot say* $d(x_A, x_B) = 0$ *implies* $x_A = x_B$, *but every other condition of a metric is satisfied.*

Proof. Symmetry is immediate from the assumptions, as well as the fact that if $x_A = x_B$, then $d(x_A, x_B) = 0$. We begin by showing if $d(x_A, x_B) = 0$, then $x_A = x_B$. Let $\epsilon > 0$. Choose a sequence $\{x_k\}_{k=0}^N$ such that

$$\sum_{k=1}^{N} F(x_k, x_{k-1}) \leq d(x_A, x_B) + \epsilon.$$

Then, by our assumption

$$\|x_A - x_B\| \leq \sum_{k=1}^{N} \|x_k - x_{k-1}\| \leq \sum_{k=1}^{N} CF(x_k, x_{k-1}) \leq Cd(x_A, x_B) + C\epsilon.$$

This inequality is satisfied for any $\epsilon > 0$, hence

$$\|x_A - x_B\| \leq Cd(x_A, x_B),$$

and so, if $d(x_A, x_B) = 0$, $\|x_A - x_B\| = 0$. Thus $x_A = x_B$ as required.

The right hand estimate of (86) is obtained immediately by considering the sequence $x_0 = x_A$ and $x_1 = x_B$ in the definition of d.

Next, we have the triangle inequality. Consider $x_A, x_B, x_C \in X$, and let $\epsilon > 0$. Take two sequences, $\{x_k\}_{k=0}^N$ and $\{x_k\}_{k=N}^M$, with $M > N$, $x_0 = x_A$, $x_N = x_B$ and $x_M = x_C$, such that

$$\sum_{k=1}^{N} F(x_k, x_{k-1}) \leq d(x_A, x_B) + \epsilon.$$

and

$$\sum_{k=N+1}^{M} F(x_k, x_{k-1}) \leq d(x_B, x_C) + \epsilon.$$

Then

$$\begin{aligned} d(x_A, x_C) &\leq \sum_{k=1}^{M} F(x_k, x_{k-1}) \\ &\leq \sum_{k=1}^{N} F(x_k, x_{k-1}) + \sum_{k=N+1}^{M} F(x_k, x_{k-1}) \\ &\leq d(x_A, x_B) + d(x_B, x_C) + 2\epsilon. \end{aligned}$$

Hence, as this construction can be done for any $\epsilon > 0$, we have

$$d(x_A, x_B) \leq d(x_A, x_B) + d(x_B, x_C),$$

as required. □

Appendix B Examples

We now explore some examples to demonstrate notable details about the constructed metric.

To begin, we note a limitation or advantage of our metric, dependent on ones perspective. Specifically, the role the α plays in the solution is dependent on whether wave breaking actually occurs. Due to the difference of the α measured in our metric, this means one could have a positive distance even if the u's and μ's are the same for all time.

Example B.1. *Consider the initial data*

$$u_0(x) = \begin{cases} 1, & x \leq -2, \\ -1-x, & -2 < x \leq -1, \\ 0, & -1 < x \leq 1, \\ 1-x, & 1 < x \leq 2, \\ -1, & 2 < x, \end{cases} \qquad \mu_0 = \nu_0 = u_{0,x}^2(x)\,dx,$$

and from this we can calculate the cumulative energy,

$$\mu_0((-\infty, x)) = \nu_0((-\infty, x)) = \begin{cases} 0, & x \leq -2, \\ 2+x, & -2 < x \leq -1, \\ 1, & -1 < x \leq 1, \\ x, & 1 < x \leq 2, \\ 2, & 2 < x. \end{cases}$$

Let $\alpha_A \equiv \frac{1}{3}$, as in Example A.1 in [9], and $\alpha_B : \mathbb{R} \to [0,1)$ such that $\alpha_B(1) = \alpha_B(-1) = \frac{1}{3}$, but $\alpha_B \neq \alpha_A$.

Transforming, using the mapping $\hat{L}$ from Definition (2.6)*, we obtain the initial data in Lagrangian coordinates,*

$$y_0(\xi) \coloneqq \begin{cases} \xi, & \xi \leq -2, \\ -1+\frac{1}{2}\xi, & -2 < \xi \leq 0, \\ -1+\xi, & 0 < \xi \leq 2, \\ \frac{1}{2}\xi, & 2 < \xi \leq 4, \\ -2+\xi, & 4 < \xi, \end{cases} \qquad U_0(\xi) = \begin{cases} 1, & \xi \leq -2, \\ -\frac{1}{2}\xi, & -2 < \xi \leq 0, \\ 0, & 0 < \xi \leq 2, \\ 1-\frac{1}{2}\xi, & 2 < \xi \leq 4, \\ -1, & 4 < \xi, \end{cases}$$

and

$$V_0(\xi) = H_0(\xi) = \begin{cases} 0, & \xi \leq -2, \\ 1+\frac{1}{2}\xi, & -2 < \xi \leq 0, \\ 1, & 0 < \xi \leq 2, \\ \frac{1}{2}\xi, & 2 < \xi \leq 4, \\ 2, & 4 < \xi. \end{cases}$$

Determining the wave breaking times using (12)*, we get*

$$\tau(\xi) = \begin{cases} 2, & \xi \in (-2,0) \cup (2,4), \\ +\infty, & \text{otherwise.} \end{cases}$$

We can then calculate the solution using the ODE system (11)*, and one obtains for either* α_A *or* α_B *that*

$$y(\xi,t) = \begin{cases} \begin{cases} t - \frac{1}{4}t^2 + \xi, & \xi \leq -2, \\ -1 + \frac{1}{8}(t-2)^2\xi, & -2 < \xi \leq 0, \\ -1 + \xi, & 0 < \xi \leq 2, \\ t - \frac{1}{4}t^2 + \frac{1}{8}(t-2)^2\xi, & 2 < \xi \leq 4, \\ -2 - t + \frac{1}{4}t^2 + \xi, & 4 < \xi, \end{cases} & 0 \leq t < 2, \\ \begin{cases} \frac{1}{3} + \frac{2}{3}t - \frac{1}{6}t^2 + \xi, & \xi \leq -2, \\ -1 + \frac{1}{12}(t-2)^2\xi, & -2 < \xi \leq 0, \\ -1 + \xi & 0 < \xi \leq 2, \\ \frac{1}{3} + \frac{2}{3}t - \frac{1}{6}t^2 + \frac{1}{12}(t-2)^2\xi, & 2 < \xi \leq 4, \\ -\frac{7}{3} - \frac{2}{3}t + \frac{1}{6}t^2 + \xi, & 4 < \xi, \end{cases} & 2 \leq t, \end{cases}$$

$$U(\xi,t) = \begin{cases} \begin{cases} 1 - \frac{1}{2}t, & \xi \leq -2, \\ \frac{1}{4}(t-2)\xi, & -2 < \xi \leq 0, \\ 0, & 0 < \xi \leq 2, \\ 1 - \frac{1}{2}t + \frac{1}{4}(t-2)\xi, & 2 < \xi \leq 4, \\ -1 + \frac{1}{2}t, & 4 < \xi, \end{cases} & 0 \leq t < 2, \\ \begin{cases} \frac{2}{3} - \frac{1}{3}t, & \xi \leq -2, \\ \frac{1}{6}(t-2)\xi, & -2 < \xi \leq 0, \\ 0, & 0 < \xi \leq 2, \\ \frac{2}{3} - \frac{1}{3}t + \frac{1}{6}(t-2)\xi, & 2 < \xi \leq 4, \\ -\frac{2}{3} + \frac{1}{3}t, & 4 < \xi, \end{cases} & 2 \leq t, \end{cases}$$

$$H(\xi,t) = H_0(\xi), \quad 0 \leq t,$$

and

$$V(\xi,t) = \begin{cases} H(\xi), & 0 \leq t < 2, \\ \begin{cases} 0, & \xi \leq -2, \\ \frac{2}{3} + \frac{1}{3}\xi, & -2 < \xi \leq 0, \\ \frac{2}{3}, & 0 < \xi \leq 2, \\ \frac{1}{3}\xi, & 2 < \xi \leq 4, \\ \frac{4}{3}, & 4 < \xi, \end{cases} & 2 \leq t. \end{cases}$$

See Figure 2 *for a plot of the characteristics* y*.*

This example demonstrates that the choice of the metric plays an important role when comparing two solutions. These two solutions remain the same for all time. However the distance given in our metric, constructed using (44), will be positive, as $\alpha_A \neq \alpha_B$.

This phenomenon occurs if, at points $x \in \mathbb{R}$ where wave breaking occurs, $\alpha_A(x) = \alpha_B(x)$. Or in other words, replacing α_A by α_B or vice versa has no

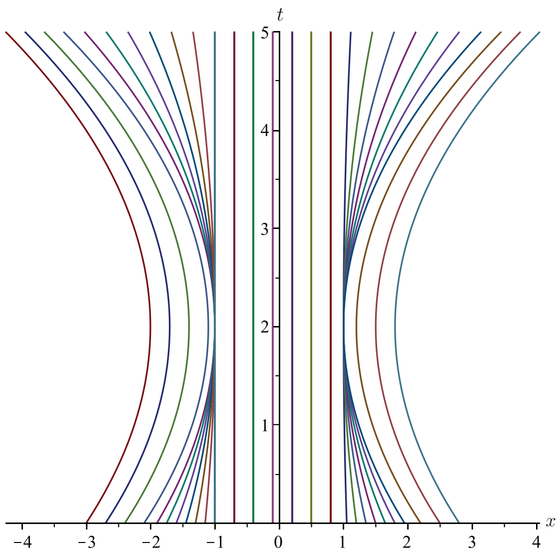

Figure 2: Plot of the characteristics $y(\xi,t)$ for different values of ξ. Note the concentration of characteristics at the wave breaking time $t=2$, and the subsequent spreading due to only partial energy loss.

impact on the solutions in that case. Therefore, one could argue that following our construction with D, given by (44), replaced by

$$\hat{D}(X_A, X_B) = D(X_A^{\alpha_A}, X_B^{\alpha_B}) - \|\alpha_A - \alpha_B\|_\infty,$$

might be more appropriate for certain purposes.

In the next example, we demonstrate why we restrict ourselves from choosing $\alpha : \mathbb{R} \to [0,1]$, i.e. such that points of wave breaking can be fully dissipative and other points can be partially dissipative or conservative.

Example B.2. *We consider as initial data,*

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 1-x, & 0 < x \leq \frac{1}{2}, \\ \frac{3}{2} - 2x, & \frac{1}{2} < x \leq 1, \\ -\frac{1}{2}, & 1 < x, \end{cases} \qquad \mu_0 = \nu_0 = u_{0,x}^2\, dx, \tag{87}$$

and assume the following values of $\alpha : [0,1) \to \mathbb{R}$:

$$\alpha\left(\frac{13}{16}\right) = 1 \quad \text{and} \quad \alpha(1) = \frac{1}{2}.$$

The points here are chosen tactically to be where wave breaking occurs in the future.

We begin by calculating the cumulative energy function. We have

$$u_{0,x}^2(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x \leq \frac{1}{2}, \\ 4, & \frac{1}{2} < x \leq 1, \\ 0, & 1 < x, \end{cases}$$

and

$$\mu_0\big((-\infty, x)\big) = \nu_0\big((-\infty, x)\big) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq \frac{1}{2}, \\ -\frac{3}{2} + 4x, & \frac{1}{2} < x \leq 1, \\ \frac{5}{2}, & 1 < x. \end{cases}$$

Thus, using the transformation $\hat{L}$ *from Definition* 2.6,

$$y_0(\xi) = \begin{cases} \xi, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 1, \\ \frac{3}{10} + \frac{1}{5}\xi, & 1 < \xi \leq \frac{7}{2}, \\ \xi - \frac{5}{2}, & \frac{7}{2} < \xi, \end{cases} \qquad U_0(\xi) = \begin{cases} 1, & \xi \leq 0, \\ 1 - \frac{1}{2}\xi, & 0 < \xi \leq 1, \\ \frac{9}{10} - \frac{2}{5}\xi, & 1 < \xi \leq \frac{7}{2}, \\ -\frac{1}{2}, & \frac{7}{2} < \xi, \end{cases}$$

and

$$H_0(\xi) = V_0(\xi) = \begin{cases} 0, & \xi \leq 0, \\ \frac{1}{2}\xi, & 0 < \xi \leq 1, \\ -\frac{3}{10} + \frac{4}{5}\xi, & 1 < \xi \leq \frac{7}{2}, \\ \frac{5}{2}, & \frac{7}{2} < \xi. \end{cases} \tag{88}$$

Thus, we can calculate the times at which wave breaking occurs. Using (12),

$$\tau(\xi) = \begin{cases} 2, & \xi \in (0,1), \\ 1, & \xi \in (1, \frac{7}{2}), \\ +\infty, & \textit{otherwise}. \end{cases}$$

With everything in place, we can solve the ODE system (11), *giving*

$$y(\xi,t) = \begin{cases} \begin{cases} -\frac{5}{16}t^2 + t + \xi, & \xi \leq 0, \\ -\frac{5}{16}t^2 + t + \frac{1}{8}(t-2)^2\xi, & 0 < \xi \leq 1, \\ \frac{3}{10} + \frac{9}{10}t - \frac{31}{80}t^2 + \frac{1}{5}(t-1)^2\xi & 1 < \xi \leq \frac{7}{2}, \\ -\frac{5}{2} - \frac{1}{2}t + \frac{5}{16}t^2 + \xi, & \frac{7}{2} < \xi, \end{cases} & 0 \leq t < 1, \\ \begin{cases} \frac{1}{4} + \frac{1}{2}t - \frac{1}{16}t^2 + \xi, & \xi \leq 0, \\ \frac{1}{4} + \frac{1}{2}t - \frac{1}{16}t^2 + \frac{1}{8}(t-2)^2\xi, & 0 < \xi \leq 1, \\ \frac{3}{4} + \frac{1}{16}t^2, & 1 < \xi \leq \frac{7}{2}, \\ -\frac{11}{4} + \frac{1}{16}t^2 + \xi, & \frac{7}{2} < \xi, \end{cases} & 1 \leq t < 2, \\ \begin{cases} \frac{3}{8} + \frac{3}{8}t - \frac{1}{32}t^2 + \xi, & \xi \leq 0, \\ \frac{3}{8} + \frac{3}{8}t - \frac{1}{32}t^2 + \frac{1}{16}(t-2)^2\xi, & 0 < \xi \leq 1, \\ \frac{5}{8} + \frac{1}{8}t + \frac{1}{32}t^2, & 1 < \xi \leq \frac{7}{2}, \\ -\frac{23}{8} + \frac{1}{8}t + \frac{1}{32}t^2 + \xi, & \frac{7}{2} < \xi, \end{cases} & 2 \leq t, \end{cases} \tag{89}$$

$$U(\xi,t)=\begin{cases}\begin{cases}1-\frac{5}{8}t, & \xi\le 0,\\ 1-\frac{5}{8}t+\frac{1}{4}(t-2)\xi, & 0<\xi\le 1,\\ \frac{2}{5}(t-1)\xi+\frac{9}{10}-\frac{31}{40}t, & 1<\xi\le\frac{7}{2},\\ -\frac{1}{2}+\frac{5}{8}t, & \frac{7}{2}<\xi,\end{cases} & 0\le t<1,\\ \begin{cases}\frac{1}{2}-\frac{1}{8}t, & \xi\le 0,\\ \frac{1}{2}-\frac{1}{8}t+\frac{1}{4}(t-2)\xi, & 0<\xi\le 1,\\ \frac{1}{8}t, & 1<\xi,\end{cases} & 1\le t<2,\\ \begin{cases}\frac{3}{8}-\frac{1}{16}t, & \xi\le 0,\\ \frac{3}{8}-\frac{1}{16}t+\frac{1}{8}(t-2)\xi, & 0<\xi\le 1,\\ \frac{1}{8}+\frac{1}{16}t, & 1<\xi,\end{cases} & 2\le t,\end{cases} \tag{90}$$

$$H(\xi,t)=H_0(\xi),\quad 0\le t,$$

and

$$V(\xi,t)=\begin{cases}H(\xi) & 0\le t<1,\\ \begin{cases}0, & \xi\le 0,\\ \frac{1}{2}\xi, & 0<\xi\le 1,\\ \frac{1}{2}, & 1<\xi,\end{cases} & 1\le t<2,\\ \begin{cases}0, & \xi\le 0,\\ \frac{1}{4}\xi, & 0<\xi\le 1,\\ \frac{1}{4}, & 1<\xi,\end{cases} & 2\le t.\end{cases} \tag{91}$$

We can transform back into Eulerian coordinates using the mapping M from Definition 2.7, giving at $t=2$,

$$\nu\big((-\infty,x),2\big)=\begin{cases}0, & x\le 1,\\ \frac{5}{2}, & 1<x,\end{cases} \tag{92}$$

$$\mu\big((-\infty,x),2\big)=\begin{cases}0, & x\le 1,\\ \frac{1}{4}, & 1<x,\end{cases} \tag{93}$$

and

$$u(x,2)=\frac{1}{4}. \tag{94}$$

Transforming back to Lagrangian coordinates, setting

$$\bar{X}:=\hat{L}\left(u(\cdot,2),\mu(\cdot,2),\nu(\cdot,2)\right),$$

we obtain

$$\bar{y}(\xi)=\begin{cases}\xi, & \xi\le 1,\\ 1, & 1<\xi\le\frac{7}{2},\\ -\frac{5}{2}+\xi, & \frac{7}{2}<\xi,\end{cases}\qquad \bar{U}(\xi)=\frac{1}{4}, \tag{95}$$

$$\bar{H}(\xi)=\begin{cases}0, & \xi\le 1,\\ -1+\xi, & 1<\xi\le\frac{7}{2},\\ \frac{5}{2}, & \frac{7}{2}<\xi,\end{cases} \tag{96}$$

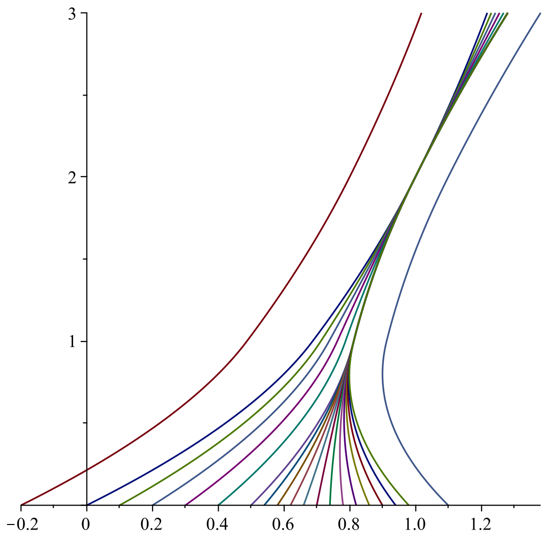

Figure 3: Plot of the characteristics $y(t,\xi)$ for different values of ξ. In comparison to Figure 2, there are now two wave-breaking times, with the first corresponding to full energy dissipation, thus no fan is released, and the second to half the energy being lost.

and

$$\bar{V}(\xi) = \begin{cases} 0, & \xi \leq 1, \\ -\frac{1}{10} + \frac{1}{10}\xi, & 1 < \xi \leq \frac{7}{2}, \\ \frac{1}{4}, & \frac{7}{2} < \xi. \end{cases} \tag{97}$$

And finally we can observe the issue. After transforming to Eulerian coordinates and back, the Lagrangian coordinates are no longer connected by a relabelling function.

Indeed, one sees that in constructing an $f \in \mathcal{G}$ such that $\bar{y} \circ f = y(\cdot, 2)$ and $\bar{V} \circ f = V(\cdot, 2)$, that one must have

$$f(\xi) = \begin{cases} 1+\xi, & \xi \leq 0, \\ 1 + \frac{5}{2}\xi, & 0 < \xi \leq 1, \\ \frac{7}{2}, & 1 < \xi \leq \frac{7}{2}, \\ \xi, & \frac{7}{2} < \xi, \end{cases}$$

however, $\bar{H} \circ f \neq H$ and $f \notin \mathcal{G}$.

In this final example we demonstrate that the choice of ν has no affect on the final solution.

Example B.3. *Consider as initial data*

$$u_0(x) = \begin{cases} 1, & x \leq -1, \\ -x, & -1 < x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x, \end{cases}$$

with

$$\mu_0 = u_{0,x}^2 \, dx + \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}}, \qquad \text{and} \qquad \nu_0 = \mu_0 + 3\mathbb{1}_{(0,1]} u_{0,x}^2 \, dx + \delta_{\frac{1}{2}}.$$

In this example we consider $\alpha = \frac{1}{2}$ and drop it from the notation of coordinates for simplicity. Set $X_{A,0} = \hat{L}(u_0, \mu_0, \mu_0)$ and $X_{B,0} = \hat{L}(u_0, \mu_0, \nu_0)$. Then

$$y_{A,0}(\xi) = \begin{cases} \xi, & \xi \leq -1, \\ -\frac{1}{2} + \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ -\frac{1}{2}, & 0 < \xi \leq 1, \\ -1 + \frac{1}{2}\xi, & 1 < \xi \leq 3, \\ \frac{1}{2}, & 3 < \xi \leq 4, \\ -\frac{3}{2} + \frac{1}{2}\xi, & 4 < \xi \leq 5, \\ -4 + \xi, & 5 < \xi, \end{cases} \qquad y_{B,0}(\xi) = \begin{cases} \xi, & \xi \leq -1, \\ -\frac{1}{2} + \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ -\frac{1}{2}, & 0 < \xi \leq 1, \\ -1 + \frac{1}{2}\xi, & 1 < \xi \leq 2, \\ -\frac{2}{5} + \frac{1}{5}\xi, & 2 < \xi \leq \frac{9}{2}, \\ \frac{1}{2}, & \frac{9}{2} < \xi \leq \frac{13}{2}, \\ -\frac{4}{5} + \frac{1}{5}\xi, & \frac{13}{2} < \xi \leq 9, \\ -8 + \xi, & 9 < \xi, \end{cases}$$

$$U_{A,0}(\xi) = \begin{cases} 1, & \xi \leq -1, \\ \frac{1}{2} - \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ \frac{1}{2}, & 0 < \xi \leq 1, \\ 1 - \frac{1}{2}\xi, & 1 < \xi \leq 2, \\ -1 + \frac{1}{2}\xi, & 2 < \xi \leq 3, \\ \frac{1}{2}, & 3 < \xi \leq 4, \\ -\frac{3}{2} + \frac{1}{2}\xi, & 4 < \xi \leq 5, \\ 1, & 5 < \xi, \end{cases} \qquad U_{B,0}(\xi) = \begin{cases} 1, & \xi \leq -1, \\ \frac{1}{2} - \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ \frac{1}{2}, & 0 < \xi \leq 1, \\ 1 - \frac{1}{2}\xi, & 1 < \xi \leq 2, \\ -\frac{2}{5} + \frac{1}{5}\xi, & 2 < \xi \leq \frac{9}{2}, \\ \frac{1}{2}, & \frac{9}{2} < \xi \leq \frac{13}{2}, \\ -\frac{4}{5} + \frac{1}{5}\xi, & \frac{13}{2} < \xi \leq 9, \\ 1, & 9 < \xi, \end{cases}$$

$$H_{A,0}(\xi) = \begin{cases} 0, & \xi \leq -1, \\ \frac{1}{2} + \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ \frac{1}{2} + \xi, & 0 < \xi \leq 1, \\ 1 + \frac{1}{2}\xi, & 1 < \xi \leq 3, \\ -\frac{1}{2} + \xi, & 3 < \xi \leq 4, \\ \frac{3}{2} + \frac{1}{2}\xi, & 4 < \xi \leq 5, \\ 4, & 5 < \xi, \end{cases} \qquad H_{B,0}(\xi) = \begin{cases} 0, & \xi \leq -1, \\ \frac{1}{2} + \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ \frac{1}{2} + \xi, & 0 < \xi \leq 1, \\ 1 + \frac{1}{2}\xi, & 1 < \xi \leq 2, \\ \frac{2}{5} + \frac{4}{5}\xi, & 2 < \xi \leq \frac{9}{2}, \\ -\frac{1}{2} + \xi, & \frac{9}{2} < \xi \leq \frac{13}{2}, \\ \frac{4}{5} + \frac{4}{5}\xi, & \frac{13}{2} < \xi \leq 9, \\ 8, & 9 < \xi, \end{cases}$$

and

$$V_{A,0}(\xi) = H_{A,0}(\xi), \qquad V_{B,0}(\xi) = \begin{cases} 0, & \xi \leq -1, \\ \frac{1}{2} + \frac{1}{2}\xi, & -1 < \xi \leq 0, \\ \frac{1}{2} + \xi, & 0 < \xi \leq 1, \\ 1 + \frac{1}{2}\xi, & 1 < \xi \leq 2, \\ \frac{8}{5} + \frac{1}{5}\xi, & 2 < \xi \leq \frac{9}{2}, \\ \frac{1}{4} + \frac{1}{2}\xi, & \frac{9}{2} < \xi \leq \frac{13}{2}, \\ \frac{11}{5} + \frac{1}{5}\xi, & \frac{13}{2} < \xi \leq 9, \\ 4, & 9 < \xi. \end{cases}$$

Calculating $\tau(\xi)$ *via* (12) *for both sets of initial data, one finds*

$$\tau_A(\xi) = \tau_B(\xi) = \begin{cases} 2, & \xi \in (-1,0) \cup (1,2), \\ +\infty, & \textit{otherwise}. \end{cases}$$

Solving the ODE system (11) *with this initial data, one finds*

$$V_A(\xi,t) = \begin{cases} V_{A,0}(\xi), & t < 2 \\ \begin{cases} 0, & \xi \leq -1, \\ \frac{1}{4} + \frac{1}{4}\xi, & -1 < \xi \leq 0, \\ \frac{1}{4} + \xi, & 0 < \xi \leq 1, \\ 1 + \frac{1}{4}\xi, & 1 < \xi \leq 2, \\ \frac{1}{2} + \frac{1}{2}\xi, & 2 < \xi \leq 3, \\ -1 + \xi, & 3 < \xi \leq 4, \\ 1 + \frac{1}{2}\xi, & 4 < \xi \leq 5, \\ \frac{7}{2}, & 5 < \xi, \end{cases} & 2 \leq t. \end{cases}$$

$$U_A(\xi,t) = \begin{cases} \begin{cases} 1 - t, & \xi \leq -1, \\ \frac{1}{2} - \frac{3}{4}t + \frac{1}{4}(t-2)\xi, & -1 < \xi \leq 0, \\ \frac{1}{2} - \frac{3}{4}t + \frac{1}{2}t\xi, & 0 < \xi \leq 1, \\ 1 - \frac{1}{2}t + \frac{1}{4}(t-2)\xi, & 1 < \xi \leq 2, \\ -1 - \frac{1}{2}t + \frac{1}{4}(t+2)\xi, & 2 < \xi \leq 3, \\ \frac{1}{2} - \frac{5}{4}t + \frac{1}{2}t\xi, & 3 < \xi \leq 4, \\ -\frac{3}{2} - \frac{1}{4}t + \frac{1}{4}(t+2)\xi, & 4 < \xi \leq 5, \\ 1 + t, & 5 < \xi, \end{cases} & t < 2, \\ \begin{cases} \frac{3}{4} - \frac{7}{8}t, & \xi \leq -1, \\ \frac{1}{2} - \frac{3}{4}t + \frac{1}{8}(t-2)\xi, & -1 < \xi \leq 0, \\ \frac{1}{2} - \frac{3}{4}t + \frac{1}{2}t\xi, & 0 < \xi \leq 1, \\ \frac{3}{4} - \frac{3}{8}t + \frac{1}{8}(t-2)\xi, & 1 < \xi \leq 2, \\ -\frac{3}{4} - \frac{5}{8}t + \frac{1}{4}(t+2)\xi, & 2 < \xi \leq 3, \\ \frac{3}{4} - \frac{11}{8}t + \frac{1}{2}t\xi, & 3 < \xi \leq 4, \\ -\frac{5}{4} - \frac{3}{8}t + \frac{1}{4}(t+2)\xi, & 4 < \xi \leq 5, \\ \frac{5}{4} + \frac{7}{8}t, & 5 < \xi, \end{cases} & 2 \leq t, \end{cases} \tag{98}$$

$$y_A(\xi,t)=\begin{cases}\begin{cases}t-\frac{1}{2}t^2+\xi, & \xi\le -1,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{8}(t-2)^2\xi, & -1<\xi\le 0,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{4}t^2\xi, & 0<\xi\le 1,\\ -1+t-\frac{1}{4}t^2+\frac{1}{8}(t-2)^2\xi, & 1<\xi\le 2,\\ -1-t-\frac{1}{4}t^2+\frac{1}{8}(t+2)^2\xi, & 2<\xi\le 3,\\ \frac{1}{2}+\frac{1}{2}t-\frac{5}{8}t^2+\frac{1}{4}t^2\xi, & 3<\xi\le 4,\\ -\frac{3}{2}-\frac{3}{2}t-\frac{1}{8}t^2+\frac{1}{8}(t+2)^2\xi, & 4<\xi\le 5,\\ -4+t+\frac{1}{2}t^2+\xi, & 5<\xi,\end{cases} & t<2,\\ \begin{cases}\frac{1}{4}+\frac{3}{4}t-\frac{7}{16}t^2+\xi, & \xi\le -1,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{16}(t-2)^2\xi, & -1<\xi\le 0,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{4}t^2\xi, & 0<\xi\le 1,\\ -\frac{3}{4}+\frac{3}{4}t-\frac{3}{16}t^2+\frac{1}{16}(t-2)^2\xi, & 1<\xi\le 2,\\ -\frac{5}{4}-\frac{3}{4}t-\frac{5}{16}t^2+\frac{1}{8}(t+2)^2\xi, & 2<\xi\le 3,\\ \frac{1}{4}+\frac{3}{4}t-\frac{11}{16}t^2+\frac{1}{4}t^2\xi, & 3<\xi\le 4,\\ -\frac{7}{4}-\frac{5}{4}t-\frac{3}{16}t^2+\frac{1}{8}(t+2)^2\xi, & 4<\xi\le 5,\\ -\frac{17}{4}+\frac{5}{4}t+\frac{7}{16}t^2+\xi, & 5<\xi,\end{cases} & 2\le t.\end{cases} \tag{99}$$

One then finds

$$u(x,t)=\begin{cases}\begin{cases}1-t, & x\le -1+t-\frac{1}{2}t^2,\\ \frac{t+2x}{(t-2)}, & -1+t-\frac{1}{2}t^2<x\le -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2,\\ \frac{2-t+4x}{2t}, & -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2<x\le -\frac{1}{2}+\frac{1}{2}t-\frac{1}{8}t^2,\\ \frac{2x}{t-2}, & -\frac{1}{2}+\frac{1}{2}t-\frac{1}{8}t^2<x\le 0,\\ \frac{2x}{t+2}, & 0<x\le \frac{1}{2}+\frac{1}{2}t+\frac{1}{8}t^2,\\ \frac{-2-t+4x}{2t}, & \frac{1}{2}+\frac{1}{2}t+\frac{1}{8}t^2<x\le \frac{1}{2}+\frac{1}{2}t+\frac{3}{8}t^2,\\ \frac{t+2x}{t+2}, & \frac{1}{2}+\frac{1}{2}t+\frac{3}{8}t^2<x\le 1+t+\frac{1}{2}t^2,\\ 1+t, & 1+t+\frac{1}{2}t^2<x,\end{cases} & t<2,\\ \begin{cases}\frac{3}{4}-\frac{7}{8}t, & x\le -\frac{3}{4}+\frac{3}{4}t-\frac{7}{16}t^2,\\ \frac{t+2x}{t-2}, & -\frac{3}{4}+\frac{3}{4}t-\frac{7}{16}t^2<x\le -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2,\\ \frac{2-t+4x}{2t}, & -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2<x\le -\frac{1}{2}+\frac{1}{2}t-\frac{1}{8}t^2,\\ \frac{2x}{(t-2)}, & -\frac{1}{2}+\frac{1}{2}t-\frac{1}{8}t^2<x\le -\frac{1}{4}+\frac{1}{4}t-\frac{1}{16}t^2,\\ \frac{2-t+4x}{2(t+2)}, & -\frac{1}{4}+\frac{1}{4}t-\frac{1}{16}t^2<x\le \frac{1}{4}+\frac{3}{4}t+\frac{1}{16}t^2,\\ \frac{-2-3t+8x}{4t}, & \frac{1}{4}+\frac{3}{4}t+\frac{1}{16}t^2<x\le \frac{1}{4}+\frac{3}{4}t+\frac{5}{16}t^2,\\ \frac{2+t+4x}{2(t+2)}, & \frac{1}{4}+\frac{3}{4}t+\frac{5}{16}t^2<x\le \frac{3}{4}+\frac{5}{4}t+\frac{7}{16}t^2,\\ \frac{5}{4}+\frac{7}{8}t, & \frac{3}{4}+\frac{5}{4}t+\frac{7}{16}t^2<x,\end{cases} & 2\le t,\end{cases} \tag{100}$$

and, for $t>0$,

$$\mu(t)=u_x^2(\cdot,t)\ dx+\frac{1}{4}(\delta_{-1}+\delta_0)\mathbb{1}_{\{t=2\}}(t). \tag{101}$$

On the other hand,

$$V_B(\xi,t)=\begin{cases} V_{B,0}(\xi), & t<2,\\ \begin{cases} 0, & \xi\leq -1,\\ \frac{1}{4}+\frac{1}{4}\xi, & -1<\xi\leq 0,\\ \frac{1}{4}+\xi, & 0<\xi\leq 1,\\ 1+\frac{1}{4}\xi, & 1<\xi\leq 2,\\ \frac{11}{10}+\frac{1}{5}\xi, & 2<\xi\leq \frac{9}{2},\\ -\frac{1}{4}+\frac{1}{2}\xi, & \frac{9}{2}<\xi\leq\frac{13}{2},\\ \frac{17}{10}+\frac{1}{5}\xi, & \frac{13}{2}<\xi\leq 9,\\ \frac{7}{2}, & 9<\xi, \end{cases} & 2\leq t, \end{cases}$$

$$U_B(\xi,t)=\begin{cases} \begin{cases} 1-t, & \xi\leq -1,\\ \frac{1}{2}-\frac{3}{4}t+\frac{1}{4}(t-2)\xi, & -1<\xi\leq 0,\\ \frac{1}{2}-\frac{3}{4}t+\frac{1}{2}t\xi, & 0<\xi\leq 1,\\ 1-\frac{1}{2}t+\frac{1}{4}(t-2)\xi, & 1<\xi\leq 2,\\ -\frac{2}{5}-\frac{1}{5}t+\frac{1}{10}(t+2)\xi, & 2<\xi\leq\frac{9}{2},\\ \frac{1}{2}-\frac{7}{8}t+\frac{1}{4}t\xi, & \frac{9}{2}<\xi\leq\frac{13}{2},\\ -\frac{4}{5}+\frac{1}{10}t+\frac{1}{10}(t+2)\xi, & \frac{13}{2}<\xi\leq 9,\\ 1+t, & 9<\xi, \end{cases} & t<2,\\ \begin{cases} \frac{3}{4}-\frac{7}{8}t, & \xi\leq -1,\\ \frac{1}{2}-\frac{3}{4}t+\frac{1}{8}(t-2)\xi, & -1<\xi\leq 0,\\ \frac{1}{2}-\frac{3}{4}t+\frac{1}{2}t\xi, & 0<\xi\leq 1,\\ \frac{3}{4}-\frac{3}{8}t+\frac{1}{8}(t-2)\xi, & 1<\xi\leq 2,\\ -\frac{3}{20}-\frac{13}{40}t+\frac{1}{10}(t+2)\xi, & 2<\xi\leq\frac{9}{2},\\ \frac{3}{4}-t+\frac{1}{4}t\xi, & \frac{9}{2}<\xi\leq\frac{13}{2},\\ -\frac{11}{20}-\frac{1}{40}t+\frac{1}{10}(t+2)\xi, & \frac{13}{2}<\xi\leq 9,\\ \frac{5}{4}+\frac{7}{8}t, & 9<\xi, \end{cases} & 2\leq t, \end{cases}$$

$$y_B(\xi,t)=\begin{cases}\begin{cases}t-\frac{1}{2}t^2+\xi, & \xi\leq -1,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{8}(t-2)^2\xi, & -1<\xi\leq 0,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{4}t^2\xi, & 0<\xi\leq 1,\\ -1+t-\frac{1}{4}t^2+\frac{1}{8}(t-2)^2\xi, & 1<\xi\leq 2,\\ -\frac{2}{5}-\frac{2}{5}t-\frac{1}{10}t^2+\frac{1}{20}(t+2)^2\xi, & 2<\xi\leq\frac{9}{2},\\ \frac{1}{2}+\frac{1}{2}t-\frac{7}{16}t^2+\frac{1}{8}t^2\xi, & \frac{9}{2}<\xi\leq\frac{13}{2},\\ -\frac{4}{5}-\frac{4}{5}t+\frac{1}{20}t^2+\frac{1}{20}(t+2)^2\xi, & \frac{13}{2}<\xi\leq 9,\\ -8+t+\frac{1}{2}t^2+\xi, & 9<\xi,\end{cases} & t<2,\\ \begin{cases}\frac{1}{4}+\frac{3}{4}t-\frac{7}{16}t^2+\xi, & \xi\leq -1,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{16}(t-2)^2\xi, & -1<\xi\leq 0,\\ -\frac{1}{2}+\frac{1}{2}t-\frac{3}{8}t^2+\frac{1}{4}t^2\xi, & 0<\xi\leq 1,\\ -\frac{3}{4}+\frac{3}{4}t-\frac{3}{16}t^2+\frac{1}{16}(t-2)^2\xi, & 1<\xi\leq 2,\\ -\frac{13}{20}-\frac{3}{20}t-\frac{13}{80}t^2+\frac{1}{20}(t+2)^2\xi, & 2<\xi\leq\frac{9}{2},\\ \frac{1}{4}+\frac{3}{4}t-\frac{1}{2}t^2+\frac{1}{8}t^2\xi, & \frac{9}{2}<\xi\leq\frac{13}{2},\\ -\frac{21}{20}-\frac{11}{20}t-\frac{1}{80}t^2+\frac{1}{20}(t+2)^2\xi, & \frac{13}{2}<\xi\leq 9,\\ -\frac{33}{4}+\frac{5}{4}t+\frac{7}{16}t^2+\xi, & 9<\xi,\end{cases} & 2\leq t,\end{cases}$$

and one sees that the transformation M yields again (u,μ) given by (100) *and* (101).

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Paper 3

On the coupling of well posed differential models

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On the Coupling of Well Posed Differential Models

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Abstract

Consider the coupling of 2 evolution equations, each generating a global process. We prove that the resulting system generates a new global process. This statement can be applied to differential equations of various kinds. In particular, it also yields the well posedness of a predator–prey model, where the coupling is in the differential terms, and of an epidemiological model, which does not fit previous well posedness results.

Keywords: Processes in Metric Spaces; Well Posedness of Evolution Equations; Coupled Problems.

MSC 2020: 34G20; 35M30; 35L65; 35F30.

1 Introduction

A variety of models describing the evolution in time of real situations is obtained coupling simpler models devoted to specific subsystems. In this paper we provide a framework where the well posedness of the *"big"* model follows from that of its parts.

Predictive models consisting of couplings of evolution equations, possibly of different types, are very common in the applications of mathematics. Here we only note that their use ranges, for instance, from epidemiology [8, 9, 11], to traffic modeling [14, 20], to several specific engineering applications [13, 27].

In this manuscript, the core result is set in a metric space, so that linearity plays no role whatsoever. This also allows the range of applicability of the general theorem to encompass, for instance, ordinary, partial and measure differential equations. In each of these cases, we obtain stability estimates tuned to the metric structure typical of the specific evolution equation considered, which can be, for example, the Euclidean norm in $\mathbb{R}^n$, the $\mathbf{L^1}$ norm in spaces of $\mathbf{BV}$ functions or some Wasserstein type distance between measures.

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At the abstract level, the starting point is provided by the framework of evolution equations in metric spaces, see [2, 3, 4, 10, 22, 23]. In this setting, an evolution equation is well posed as soon as it generates a *Global Process*, i.e., a Lipschitz continuous solution operator, see Definition 2.2. In other words, global processes substitute, in the time dependent case, semigroups that, in the autonomous case, have as trajectories the solutions to evolution equations.

Assume that two evolution equations are given, each depending on a parameter and each generating a global process, also depending on that parameter. We now let the parameter in an equation vary in time according to the other equation: a coupling between the two models is thus obtained. Theorem 2.6 ensures the well posedness of this coupled model, in the sense that it generates a new global process.

The assumptions required in this abstract construction are then verified in 5 sample situations: ordinary differential equations, initial and boundary value problems for renewal equations, measure valued balance laws and scalar conservation laws. Thus, we prove that any coupling of these equations results in a well posed model. Indeed, in each of these cases, we provide a full set of detailed stability estimates compatible with the abstract results. Note that assumptions ensuring global in time existence results are also provided.

Finally, we consider specific cases. First, we briefly show that Theorem 2.6 comprises the case of the traffic model introduced in [20], where a scalar conservation law is coupled to an ordinary differential equation.

Then, we detail the case of a predator–prey model inspired by [7], namely

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \Big(\rho\, V\big(t, x, p(t)\big)\Big) = -\eta\left(\big\|p(t) - x\big\|\right)\, \rho(t,x) \\ \dot p = U\left(t, p, \rho(t)\right) . \end{cases} \tag{1.1}$$

While we refer to § 4.1 for a detailed explanation of the terms in (1.1), here we remark that in (1.1) the coupling is not only in the source term of the partial differential equations, but also in the convective term, where no nonlocal term is involved (V is a function defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $P(t) \in \mathbb{R}^n$).

Then, we apply the general construction to a recent epidemiological model presented in [11] whose well posedness, to our knowledge, was not proved at the time of this writing. In this case, the coupling involves a boundary value problem for a renewal equation, see § 4.2.

For all basic results on evolution equations in metric spaces, we refer to the extended treatises [2, 3, 22], whose wide bibliographies also give a detailed view on the whole field. Below, we follow the approach outlined in [4, 10, 23]. The different frameworks differ in their approaches but offer similar results. Related to Theorem 2.6 is, for instance, [22, Theorem 26]. However, here we follow a more quantitative approach to the various stability estimates.

We expect that also other equations fit in the framework introduced in Section 2. Natural candidates are, for instance, measure differential equations [24, 25] and their coupling with ordinary differential equations as considered in [16]. A further class of couplings is that in [13], consisting of ordinary and partial differential equations similar to those comprised in § 3.3. Very likely to comply with the present structure is also the general class of traffic models presented in [18].

This work is organized as follows. Section 2, once the basic notation is introduced, presents the general result. Each of the paragraphs in Section 3 is devoted to a particular evolution equation: its well posedness is proved obtaining those estimates that allow the application of Theorem 2.6. Specific models are then dealt with in Section 4. Finally, proofs are in the final Section 5.

2 Definitions and Abstract Results

Below we rely on the framework established in [4, 10, 23], see [2, 3, 22] for an alternative, essentially equivalent, setting. Let (X, d) be a metric space and I be a real interval. First, a *local flow* on X provides a sort of tangent vector field to X.

Definition 2.1 ([10, Definition 2.1]). *Given $\delta > 0$ and a closed set $\mathcal{D} \subseteq X$, a* local flow *is a continuous map $F\colon [0, \delta] \times I \times \mathcal{D} \mapsto X$, such that $F(0, t_o)\, u = u$ for any $(t_o, u) \in I \times \mathcal{D}$ and which is Lipschitz in its first and third arguments uniformly in the second, i.e. there exists a $\mathbf{Lip}(F) > 0$ such that for all $\tau, \tau' \in [0, \delta]$ and $u, u' \in \mathcal{D}$*

$$d\left(F(\tau, t_o)u, F(\tau', t_o)u'\right) \leq \mathbf{Lip}(F) \cdot \Big(d(u, u') + \left|\tau - \tau'\right|\Big) . \tag{2.1}$$

Given an evolution equation, a *global process* is a candidate for the solution operator, i.e., for the mapping assigning to initial datum u at time t_o and to time t the solution evaluated at time t.

Definition 2.2 ([10, Definition 2.5]). *Fix a family of sets $\mathcal{D}_{t_o} \subseteq \mathcal{D}$ for all $t_o \in I$, and a set*

$$\mathcal{A} = \left\{(t, t_o, u) \colon t \geq t_o,\ t_o, t \in I \text{ and } u \in \mathcal{D}_{t_o}\right\} . \tag{2.2}$$

A global process *on X is a map $P\colon \mathcal{A} \mapsto X$ such that, for all $u \in \mathcal{D}_{t_o}$ and $t_o, t_1, t_2 \in I$ with $t_2 \geq t_1 \geq t_o$,*

$$P(t_o, t_o)u = u \tag{2.3}$$

$$P(t_1, t_o)u \in \mathcal{D}_{t_1} \tag{2.4}$$

$$P(t_2, t_1) \circ P(t_1, t_o)u = P(t_2, t_o)u. \tag{2.5}$$

In Theorem 2.4 below, a global process is constructed from a local flow by means of a suitable extension of *Euler Polygonals* to metric spaces.

Definition 2.3 ([10, Definition 2.3]). *Let F be a local flow. Fix $u \in \mathcal{D}$, $t_o \in I$, $\tau \in [0, \delta]$ with $t_o + \tau \in I$. For every $\varepsilon > 0$, let $k = \lfloor \tau/\varepsilon \rfloor$, where the symbol $\lfloor \cdot \rfloor$ denotes the integer part. An Euler ε-polygonal is*

$$F^\varepsilon(\tau, t_o)\, u = F(\tau - k\varepsilon, t_o + k\varepsilon) \circ \bigcirc_{h=0}^{k-1} F(\varepsilon, t_o + h\varepsilon)\, u \tag{2.6}$$

whenever it is defined.

Above, we used the notation $\bigcirc_{h=0}^{k} f_h = f_k \circ f_{k-1} \circ \ldots \circ f_1 \circ f_0$.

For a local flow F, its corresponding Euler ε-polygonal F^ε, and any $t_o \in I$, introduce the notation:

$$\mathcal{D}_{t_o}^3 = \left\{ u \in \mathcal{D} \colon \begin{array}{l} F^{\varepsilon_3}(\tau_3, t_o + \tau_1 + \tau_2) \circ F^{\varepsilon_2}(\tau_2, t_o + \tau_1) \circ F^{\varepsilon_1}(\tau_1, t_o) u \\ \text{is in } \mathcal{D} \text{ for all } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \left]0, \delta\right] \text{ and all} \\ \tau_1, \tau_2, \tau_3 \geq 0 \text{ such that } t_o + \tau_1 + \tau_2 + \tau_3 \in I \end{array} \right\}. \tag{2.7}$$

The next result provides the basis for our construction of solutions to coupled problems.

Theorem 2.4 ([10, Theorem 2.6])**.** *Let (X, d) be a complete metric space and $\mathcal{D}$ be a closed subset of X. Assume that for the local flow $F \colon [0, \delta] \times I \times \mathcal{D} \mapsto X$ there exist*

1. *a non decreasing map $\omega \colon [0, \delta] \to \mathbb{R}_+$ with $\int_0^\delta \frac{\omega(\tau)}{\tau}\, d\tau < +\infty$ such that*

$$d\Big(F(k\tau, t_o + \tau) \circ F(\tau, t_o) u, F\left((k+1)\tau, t_o\right) u\Big) \leq k\, \tau\, \omega(\tau) \tag{2.8}$$

whenever $\tau \in [0, \delta]$, $k \in \mathbb{N}$ and the left hand side above is well defined;

2. *a positive constant L such that*

$$d\left(F^\varepsilon(\tau, t_o) u_1, F^\varepsilon(\tau, t_o) u_2\right) \leq L\, d(u_1, u_2) \tag{2.9}$$

whenever $\varepsilon \in \left]0, \delta\right]$, $u_1, u_2 \in \mathcal{D}$, $\tau \geq 0$, $t_o, t_o + \tau \in I$ and the left hand side above is well defined.

Then, there exists a family of sets $\mathcal{D}_{t_o}$, for $t_o \in I$, and a unique global process (as in Definition 2.2) $P \colon \mathcal{A} \to X$ with the following properties:

1. *$\mathcal{D}_{t_o}^3 \subseteq \mathcal{D}_{t_o}$ for any $t_o \in I$, with $\mathcal{D}_{t_o}^3$ as defined in (2.7);*
2. *P is Lipschitz continuous with respect to $(t, t_o, u) \in \mathcal{A}$;*
3. *P is tangent to F in the sense that for all $(t_o + \tau, t_o, u) \in \mathcal{A}$, with $\tau \in \left]0, \delta\right]$:*

$$\frac{1}{\tau}\, d\left(P(t_o + \tau, t_o) u, F(\tau, t_o) u\right) \leq \frac{2L}{\ln(2)} \int_0^\tau \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi\ . \tag{2.10}$$

A general condition to ensure that $\mathcal{A}$ is non empty is [10, Condition **(D)**]. Below, in the examples we consider, it explicitly stems out that $\mathcal{A} \neq \emptyset$.

We now head towards considering processes depending on parameters.

Definition 2.5. *Let $(\mathcal{U}, d_\mathcal{U})$ and $(\mathcal{W}, d_\mathcal{W})$ be metric spaces. A* Lipschitz Process on $\mathcal{U}$ parametrized by $w \in \mathcal{W}$ *is a family of maps $P^w \colon \mathcal{A}_\mathcal{U} \to \mathcal{U}$, with*

$$\begin{aligned} \mathcal{I} &= \Big\{(t, t_o) \in I \times I \colon t \geq t_o\Big\}, \\ \mathcal{A}_\mathcal{U} &= \Big\{(t, t_o, u) \colon (t, t_o) \in \mathcal{I},\ u \in \mathcal{D}_{t_o}^\mathcal{U}\Big\}, \\ \mathcal{D}_t^\mathcal{U} &\subseteq \mathcal{U}, \end{aligned}$$

such that for all $w \in \mathcal{W}$, P^w is a Global Process in the sense of Definition 2.2 *and there exist positive constants C_u, C_t, C_w such that*

$$d_{\mathcal{U}}\left(P^w(t,t_o)u_1, P^w(t,t_o)u_2\right) \le e^{C_u(t-t_o)}\, d_{\mathcal{U}}(u_1,u_2)\,, \tag{2.11}$$

$$d_{\mathcal{U}}\left(P^w(t_1,t_o)u, P^w(t_2,t_o)u\right) \le C_t\, |t_2-t_1|\,, \tag{2.12}$$

$$d_{\mathcal{U}}\left(P^{w_1}(t,t_o)u_o, P^{w_2}(t,t_o)u_o\right) \le C_w\,(t-t_o)\, d_{\mathcal{W}}(w_1,w_2)\,. \tag{2.13}$$

We equip the product space $\mathcal{U}\times\mathcal{W}$ with the distance

$$d\left((u',w'),(u'',w'')\right) = d_{\mathcal{U}}(u',u'') + d_{\mathcal{W}}(w',w'').$$

Theorem 2.6. *Let $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$ be complete. Let $P^w\colon \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$ be a Lipschitz Process on $\mathcal{U}$ parametrized by $w \in \mathcal{W}$, and let $P^u\colon \mathcal{A}_{\mathcal{W}} \to \mathcal{W}$ be a Lipschitz Process on $\mathcal{W}$ parametrized by $\mathcal{U}$. Let C_u, C_w, and C_t be constants that satisfy* (2.11)–(2.12)–(2.13) *for both processes. Then,*

1. *Introducing*

$$\mathcal{A}_F = \left\{\left(\tau, t_o, (u,w)\right) : \tau \ge 0,\ t_o, t_o+\tau \in I,\ (u,w) \in \mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o}\right\},$$

the map

$$\begin{array}{cccc} F & : & \mathcal{A}_F & \to & \mathcal{U}\times\mathcal{W} \\ & & \left(\tau, t_o, (u,w)\right) & \mapsto & \left(P^w(t_o+\tau,t_o)u, P^u(t_o+\tau,t_o)w\right) \end{array} \tag{2.14}$$

is a local flow on $\mathcal{U}\times\mathcal{W}$.

2. *F satisfies the assumptions of Theorem* 2.4 *with*

$$L = e^{(C_u+C_w)T} \quad \text{and} \quad \omega(\tau) = C_t\, C_u\, \tau \tag{2.15}$$

hence F generates a unique global process $P\colon \mathcal{A} \to \mathcal{U}\times\mathcal{W}$, for a suitable $\mathcal{A} \subseteq I\times I\times\mathcal{U}\times\mathcal{W}$, satisfying properties 1., 2. and 3. in Theorem 2.4.

3. *For all $t_o \in I$ and $\tau \ge 0$ with $t_o+\tau \ge t_o$, we have*

$$F(\tau,t_o)(\mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o}) \subseteq (\mathcal{D}^{\mathcal{U}}_{t_o+\tau} \times \mathcal{D}^{\mathcal{W}}_{t_o+\tau}) \tag{2.16}$$

hence the process P is defined on $\mathcal{A}$ with

$$\mathcal{A} \supseteq \left\{\left(\tau,t_o,(u,w)\right) : \begin{array}{c} \tau\ge 0,\ t_o, t_o+\tau \in I, \\ (u,w) \in \mathcal{D}^{\mathcal{U}}_{t_o}\times\mathcal{D}^{\mathcal{W}}_{t_o} \end{array}\right\}. \tag{2.17}$$

The proof is deferred to § 5.1.

An analogous result can be proved defining the local flow F by means of local flows $F^{\mathcal{U}}_w$ and $F^{\mathcal{W}}_u$, provided these local flows satisfy the assumptions of Theorem 2.4 and have a Lipschitz continuous dependence on the parameter.

Theorem 2.7. *Consider two complete metric spaces* $(\mathcal{U}, d_{\mathcal{U}})$ *and* $(\mathcal{W}, d_{\mathcal{W}})$*. Let*

$$F^w : [0,\delta] \times I \times \mathcal{D}^{\mathcal{U}} \to \mathcal{U}, \qquad \textit{and} \qquad F^u : [0,\delta] \times I \times \mathcal{D}^{\mathcal{W}} \to \mathcal{W},$$

be local flows parametrized by $w \in \mathcal{W}$ *and* $u \in \mathcal{U}$*, respectively, so that there exists* $\mathcal{L}$ *such that for all* $\tau \in [0,\delta]$ *and* $t \in I$*,*

$$\begin{array}{rclll} d_{\mathcal{U}}\left(F^{w_1}(\tau,t)u, F^{w_2}(\tau,t)u\right) & \leq & \mathcal{L}\, d_{\mathcal{W}}(w_1,w_2) & u \in \mathcal{D}^{\mathcal{U}} & w_1, w_2 \in \mathcal{W} \\ d_{\mathcal{W}}\left(F^{u_1}(\tau,t)w, F^{u_2}(\tau,t)w\right) & \leq & \mathcal{L}\, d_{\mathcal{U}}(u_1,u_2) & u \in \mathcal{D}^{\mathcal{W}} & u_1, u_2 \in \mathcal{U} \end{array}$$

Then, setting $\mathcal{D} = \mathcal{D}^{\mathcal{U}} \times \mathcal{D}^{\mathcal{W}}$*, the coupling*

$$\begin{array}{rcccc} \hat{F} & : & [0,\delta] \times I \times \mathcal{D} & \to & \mathcal{U} \times \mathcal{W} \\ & & \big(\tau, t, (u,w)\big) & \mapsto & \big(F^w(t,t_o)u, F^u(t,t_o)w\big) \end{array}$$

is a local flow in the sense of Definition 2.1*. If moreover* F^w *and* F^u *satisfy assumptions* 1 *and* 2 *in Theorem* 2.4*, then* $\hat{F}$ *is tangent to the local flow* F *defined in* (2.14) *by means of the processes* P^w *and* P^u *defined through Theorem* 2.4*.*

As a direct consequence of Theorem 2.7, by means of [5, Theorem 2.9], we have that whenever Theorem 2.6 applies, if $\hat{F}$ generates a global process $\hat{P}$, then $\hat{P}$ coincides with the process P constructed in Theorem 2.6.

3 General Cauchy Problems

In the paragraphs below we consider differential equations depending on parameters that generate parametrized Lipschitz processes in the sense of Definition 2.5. Thus, any coupling of the processes below meets the requirements of Theorem 2.6 and generates a new Lipschitz process. Moreover, we verify that this new process eventually yields solutions to the coupled problem.

Throughout, $\hat{I}$ is a real interval containing 0. If $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm, while $\|x\|_V$ is the norm of x in the Banach space V. The open, respectively closed, ball centered at x with radius r is $B(x,r)$, respectively $\overline{B(x,r)}$.

3.1 Ordinary Differential Equations

This brief paragraph mainly serves as a paradigm for the subsequent ones. Indeed, we begin by considering the classical Cauchy problem for an ordinary differential equation

$$\left\{ \begin{array}{ll} \dot{u} = f(t,u,w) & t \in \hat{I} \\ u(t_o) = u_o \end{array} \right. \qquad \text{with} \qquad f : \hat{I} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n, \tag{3.1}$$

where $t_o \in \hat{I}$, $u_o \in \mathbb{R}^n$ and the parameter w is fixed in $\mathcal{W}$.

Definition 3.1. *A map* $u : I \to \mathbb{R}^n$ *is a solution to* (3.1) *if* $t_o \in I \subseteq \hat{I}$*,* $u(t_o) = u_o$*, for a.e.* $t \in I$*,* u *is differentiable at* t *and* $\dot{u}(t) = f\left(t, u(t), w\right)$*.*

The well posedness of (3.1) is an elementary result which we state below to allow subsequent couplings of (3.1) with other equations within the framework of Theorem 2.6.

Proposition 3.2. *Let $R > 0$. Define $\mathcal{D} = \overline{B(0,R)}$ in $\mathbb{R}^n$ and consider the Cauchy problem* (3.1) *under the assumptions*

(ODE1) *For all $u \in \mathcal{D}$ and all $w \in \mathcal{W}$, the map $t \mapsto f(t,u,w)$ is measurable.*

(ODE2) *There exist positive F_L, F_∞ such that for all $t \in \hat{I}$, $u_1, u_2 \in \mathcal{D}$ and $w_1, w_2 \in \mathcal{W}$*

$$\left\| f(t,u_1,w_1) - f(t,u_2,w_2) \right\| \le F_L \left(\|u_1 - u_2\| + d_{\mathcal{W}}(w_1,w_2) \right), \tag{3.2}$$
$$\sup_{w\in\mathcal{W}} \left\| f(\cdot,\cdot,w) \right\|_{\mathbf{L}^\infty(\hat{I}\times\hat{\mathcal{D}};\mathbb{R}^n)} \le F_\infty \,. \tag{3.3}$$

Then, there exists $T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathbb{R}^n$ pametrized by $\mathcal{W}$ in the sense of Definition 2.5*, whose orbits solve* (3.1) *according to Definition* 3.1*, with*

$$\begin{aligned} &T \le R/(2F_\infty) \ , \quad C_u = F_L \,, \quad C_t = F_\infty \,, \quad C_w = F_L\, e^{F_L T} \,, \\ &\mathcal{D}_t = \overline{B\left(0, R - (T-t) \sup_{w\in\mathcal{W}} \left\| f(\cdot,\cdot,w) \right\|_{\mathbf{L}^\infty(\hat{I}\times\hat{\mathcal{D}};\mathbb{R}^n)}\right)} \,. \end{aligned} \tag{3.4}$$

Long time existence is also available.

Corollary 3.3. *Assume $\sup \hat{I} = +\infty$ and that, for every $R > 0$,* **(ODE1)** *and* **(ODE2)** *hold with $F_\infty = F_\infty(R)$ satisfying*

$$\limsup_{R\to+\infty} \frac{F_\infty(R)}{R\ln(R)} < +\infty.$$

Then, for all $t_o \in \hat{I}$, the solution to (3.1) *exists for every $t \ge t_o$.*

The proof is deferred to § 5.2. We now verify that Theorem 2.6 applies to the coupling of (3.1) with other Lipschitz Processes.

Proposition 3.4. *Set $\mathcal{U} = \mathbb{R}^n$. Assume that* **(ODE1)**–**(ODE2)** *hold. Let P^u be a Lipschitz Process on $\mathcal{W}$ parametrized by $u \in \mathcal{U}$. Call $P\colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem* 2.6 *coupling P^w, generated by* (3.1)*, and P^u. If $([t_o,T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then*

$$\begin{array}{rcl} u: \ [t_o,T] & \to & \mathbb{R}^n \\ t & \mapsto & P_1(t,t_o)(u_o,w_o) \end{array}$$

solves

$$\left\{ \begin{array}{l} \dot{u} = \bar{f}(t,u) \\ u(t_o) = u_o \end{array} \right. \quad \text{where } \bar{f}(t,u) = f\left(t, u, P_2(t,t_o)(u_o,w_o)\right)$$

in the sense of Definition 3.1.

The proof is deferred to § 5.2.

A particular case of Proposition 3.2 of interest is the following.

Corollary 3.5. *Let $R > 0$. Define $\hat{\mathcal{D}} = \overline{B(0,R)}$ in $\mathcal{U} = \mathbb{R}^n$. Choose $\mathcal{W} = \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^M)$ and fix $\eta \in \mathbf{L}^\infty(\hat{I} \times \mathbb{R}^N;\mathbb{R})$. Consider the Cauchy problem* (3.1) *with*

$$f(t,u,w) = g\left(t, u, \int_{\mathbb{R}^N} \eta(t,x)\, w(x)\,\mathrm{d}x\right) \tag{3.5}$$

under the assumptions:

(NL1) *For all $u \in \hat{\mathcal{D}}$ and $W \in \mathbb{R}^M$, the map $t \mapsto g(t,u,W)$ is measurable.*

(NL2) *There exist positive L_g and G_∞ such that for all $t \in \hat{I}$, $u_1, u_2 \in \hat{\mathcal{D}}$ and $W_1, W_2 \in \mathbb{R}^M$*

$$\begin{aligned} \left\| g(t,u_1,W_1) - g(t,u_2,W_2) \right\| &\le G_L \left(\|u_1 - u_2\| + \|W_1 - W_2\| \right), \\ \sup_{\hat{I}\times\hat{\mathcal{D}}\times\mathbb{R}^M} \left\| g(t,u,W) \right\| &\le G_\infty . \end{aligned}$$

Then, given the interval $I = [0,T]$ with $T = \frac{R}{2G_\infty}$ and, for every $t \in I$, the domain

$$\mathcal{D}_t = \overline{B\left(0, R - (T-t)\|g\|_{\mathbf{L}^\infty(\hat{I}\times\hat{\mathcal{D}}\times\mathbb{R}^M;\mathbb{R}^n)}\right)}, \tag{3.6}$$

problem (3.1)–(3.5) *generates a Lipschitz Process on $\mathbb{R}^n$ pametrized by $w \in \mathcal{W}$, with constants in* (2.11)–(2.12)–(2.13) *given by*

$$\begin{aligned} C_u &= G_L \left(1 + \|\eta\|_{\mathbf{L}^\infty(\hat{I}\times\mathbb{R}^N;\mathbb{R})}\right), \quad C_t = G_\infty \\ C_w &= G_L \left(1 + \|\eta\|_{\mathbf{L}^\infty(\hat{I}\times\mathbb{R}^N;\mathbb{R})}\right) \exp\left(G_L(1 + \|\eta\|_{\mathbf{L}^\infty(\hat{I}\times\mathbb{R}^N;\mathbb{R})})\,\hat{T}\right). \end{aligned} \tag{3.7}$$

The proof is a direct consequence of Proposition 3.2 and is hence omitted. Note that also Proposition 3.4 is immediately extended to the case of (3.5). The analog of Corollary 3.3 in this setting is given by the following result, whose proof is omitted, since it is identical to that of Corollary 3.3.

Corollary 3.6. *Assume $[0,+\infty) \subseteq \hat{I}$ and that, for every $R > 0$,* **(NL1)** *and* **(NL2)** *hold with $G_\infty = G_\infty(R)$ satisfying*

$$\limsup_{R\to+\infty} \frac{G_\infty(R)}{R\ln(R)} < +\infty.$$

Then the solution to (3.1)*, with vector field* (3.5)*, exists for every $t \ge t_o$.*

3.2 The Initial Value Problem for a Renewal Equation

We examine the following initial value problem for a first order partial differential equation

$$\begin{cases} \partial_t u + \mathrm{div}_x \left(v(t,x,w)\, u\right) = m(t,x,w)u + q(t,x,w) & (t,x) \in \hat{I} \times \mathbb{R}^n, \\ u(t_o,x) = u_o(x), & x \in \mathbb{R}^n \end{cases} \tag{3.8}$$

for $u_o \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$ and $t_o \in \hat{I}$. Proofs are deferred until § 5.3.

Definition 3.7. *For a fixed $w \in \mathcal{W}$, a function $u \in \mathbf{C^0}\left([t_o, T]; \mathbf{L^1}(\mathbb{R}^n;\mathbb{R})\right)$, where $[t_o, T] \subseteq \hat{I}$, is a solution to* (3.8) *if:*

1. *for any test function $\varphi \in \mathbf{C_c^\infty}(]t_o, T[\times \mathbb{R}^n; \mathbb{R})$,*

$$\int_{t_o}^{T} \int_{\mathbb{R}^n} \Big(u(t,x)\, \partial_t \varphi(t,x) + u(t,x)\, v(t,x,w) \cdot \nabla_x \varphi(t,x) \\ + \big(m(t,x,w)\, u(t,x) + q(t,x,w)\big)\, \varphi(t,x) \Big)\, \mathrm{d}x\, \mathrm{d}t = 0;$$

2. *$u(t_o, x) = u_o(x)$ for a.e. $x \in \mathbb{R}^n$.*

Proposition 3.8. *Let $R > 0$ and set $\mathcal{U} = \mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$. Define*

$$\mathcal{D} = \left\{ u \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}) \colon \max\left\{ \|u\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}, \|u\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})}, \mathrm{TV}(u) \right\} \le R \right\}.$$

Consider the Cauchy problem (3.8) *under the assumptions*

(IP1) *For all $w \in \mathcal{W}$, $v(\cdot,\cdot,w) \in \mathbf{C^0}(\hat{I} \times \mathbb{R}^n; \mathbb{R}^n)$, $v(t,\cdot,w) \in \mathbf{C^2}(\mathbb{R}^n;\mathbb{R}^n)$ for all $t \in \hat{I}$ and there exist positive constants V_1, V_L, V_∞ such that for all $t \in \hat{I}$*

$$\left\| v(t,\cdot,w) \right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^n)} \le V_\infty\,; \qquad \left\| \nabla v(t,\cdot,w) \right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \le V_L\,;$$
$$\left\| \nabla \nabla \cdot v(t,\cdot,w) \right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)} \le V_1\,.$$

and, for all $w_1, w_2 \in \mathcal{W}$ and $t \in \hat{I}$,

$$\left\| v(t,\cdot,w_1) - v(t,\cdot,w_2) \right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^n)} \le V_L\, d_{\mathcal{W}}(w_1,w_2),$$
$$\left\| \nabla \cdot \big(v(t,\cdot,w_1) - v(t,\cdot,w_2) \big) \right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \le V_L\, d_{\mathcal{W}}(w_1,w_2).$$

(IP2) *For all $w \in \mathcal{W}$, $m(\cdot,\cdot,w) \in \mathbf{C^0}(\hat{I} \times \mathbb{R}^n; \mathbb{R})$ and there exist positive constants M_∞, M_L such that for all $t \in \hat{I}$ and for all $w, w_1, w_2 \in \mathcal{W}$*

$$\begin{array}{rcl} \left\| m(t,\cdot,w) \right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \mathrm{TV}\left(m(t,\cdot,w)\right) & \le & M_\infty\,; \\ \left\| m(t,\cdot,w_1) - m(t,\cdot,w_2) \right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \le & M_L\, d_{\mathcal{W}}(w_1,w_2)\,. \end{array}$$

(IP3) *For all $w \in \mathcal{W}$, $q(\cdot,\cdot,w) \in \mathbf{L^1}\left(\hat{I}; \mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})\right)$ and there exist positive constants Q_∞, Q_1, Q_L such that for all $t \in \hat{I}$ and for all $w, w_1, w_2 \in \mathcal{W}$,*

$$\begin{array}{rcl} \left\| q(t,\cdot,w) \right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \mathrm{TV}\left(q(t,\cdot,w)\right) & \le & Q_\infty\,; \\ \left\| q(t,\cdot,w) \right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \le & Q_1, \\ \left\| q(t,\cdot,w) - q(t,\cdot,w_2) \right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \le & Q_L\, d(w_1,w_2)\,. \end{array}$$

Then, there exists $T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathcal{U}$ pametrized by $\mathcal{W}$ in the sense of Definition 2.5, whose orbits solve (3.8) in the sense of Definition 3.7, with

$$C_u = M_\infty, \tag{3.9}$$

$$C_t = V_\infty R e^{(M_\infty + 2V_L)}T + Q_1 e^{M_\infty}T + (M_\infty + V_L) R e^{(M_\infty + V_L)}T, \tag{3.10}$$

$$\begin{aligned} C_w = {} & [V_L(2R + Q_\infty)(1 + (V_1 + M_\infty)T) \\ & + (Q_L + (M_L + V_L)(R + Q_\infty T))] e^{(M_\infty + V_L)T}, \end{aligned} \tag{3.11}$$

and

$$\mathcal{D}_t = \left\{ u \in \mathcal{D} \colon \begin{array}{rl} \|u\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \leq \alpha_1(t) \\ \|u\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} & \leq \alpha_\infty(t) \\ \mathrm{TV}(u) & \leq \alpha_{\mathrm{TV}}(t) \end{array} \right\}, \tag{3.12}$$

where

$$\begin{array}{rcl} \alpha_1(t) & = & R e^{-M_\infty (T-t)} - Q_1 (T-t) e^{M_\infty t}, \\ \alpha_\infty(t) & = & R e^{-(M_\infty + V_L)(T-t)} - Q_\infty e^{(M_\infty + V_L)t} (T-t), \\ \alpha_{\mathrm{TV}}(t) & = & R e^{-(M_\infty + V_L)(T-t)} \big(1 - (M_\infty + V_1)(T-t)\big) \\ & & - Q_\infty e^{(M_\infty + V_L)t} \big(1 + (M_\infty + V_1)t\big)(T-t). \end{array} \tag{3.13}$$

Corollary 3.9. *Assume $[0,+\infty) \subseteq \hat{I}$ and that* **(IP1)**, **(IP2)**, *and* **(IP3)** *hold. Then the solution to (3.8) exists for every $t \geq t_o$.*

Continuing now to the act of coupling this Lipschitz process with another.

Proposition 3.10. *Set $\mathcal{U} = \mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$. Assume that* **(IP1)**–**(IP2)**–**(IP3)** *hold. Let P^u be a Lipschitz process on $\mathcal{W}$, parametrised by $u \in \mathcal{U}$. Call $P \colon \mathcal{A} \to \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}) \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the process generated in Theorem 2.6 by the coupling of process P^w, found in Proposition 3.8, with P^u. If $([t_o,T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then the map*

$$\begin{array}{rcl} u \colon [t_o,T] & \to & (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}) \\ t & \mapsto & P_1(t,t_o)(u_o,w_o) \end{array}$$

solves

$$\left\{ \begin{array}{ll} \partial_t u + \mathrm{div}_x \big(\bar{v}(t,x)\, u\big) = \bar{m}(t,x) u + \bar{q}(t,x) & (t,x) \in [t_o,T] \times \mathbb{R}^n, \\ u(t_o,x) = u_o(x), & x \in \mathbb{R}^n \end{array} \right.$$

in the sense of Definition (3.8), where

$$\bar{m}(t,x) = m\big(t,x,P_2(t,t_o)(u_o,w_o)\big), \qquad \bar{q}(t,x) = q\big(t,x,P_2(t,t_o)(u_o,w_o)\big), \\ \bar{v}(t,x) = v\big(t,x,P_2(t,t_o)(u_o,w_o)\big).$$

3.3 The Boundary Value Problem for a Linear Balance Law

Consider the model

$$\begin{cases} \partial_t u + \partial_x \big(v(t,x)\, u\big) = m(t,x,w)\, u + q(t,x,w) & (t,x) \in \hat{I} \times \mathbb{R}_+ \\ u(t,0) = b(t) & t \in \hat{I} \\ u(t_o,x) = u_o(x) & x \in \mathbb{R}_+ . \end{cases} \tag{3.14}$$

where $u_o \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$, $t_o \in \hat I$ and $w \in \mathcal{W}$. Throughout, we choose left continuous representatives of **BV** functions. Proofs are deferred to § 5.4.

Definition 3.11. *For a fixed $w \in \mathcal{W}$, a function $u \in \mathbf{C^0}\left([t_o,T];\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})\right)$, with $[t_o,T] \subseteq \hat I$, such that $u(t) \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R})$ for a.e. $t \in [t_o,T]$ is a solution to* (3.14) *if:*

1. *For all $\varphi \in \mathbf{C_c^\infty}(]t_o,T[\, \times \mathring{\mathbb{R}}_+;\mathbb{R})$*

$$\int_{t_o}^{T}\int_{\mathbb{R}_+}\Big(u(t,x)\,\partial_t\varphi(t,x) + v(t,x)\,u(t,x)\,\partial_x\varphi(t,x) + \big(m(t,x,w)\,u(t,x) + q(t,x,w)\big)\;\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t \;=\; 0\,.$$

2. *For a.e. $x \in \mathbb{R}_+$, $u(t_o,x) = u_o(x)$.*
3. *For a.e. $t \in [t_o,T]$, $\lim_{x\to 0+} u(t,x) = b(t)$.*

Proposition 3.12. *Let $\mathcal{U} = \mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$ and fix $b \in \mathbf{BV}(\hat I;\mathbb{R})$. For $R > 0$, define*

$$\mathcal{D} = \left\{u \in \mathcal{U}\colon \max\left\{\|u\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})},\ \|u\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})},\ \mathrm{TV}(u) + \left|b(\sup\hat I) - u(0)\right|\right\} \le R\right\}. \tag{3.15}$$

Assume

(BP1) *There exist positive constants $\check v, \hat v, V_1, V_\infty$ such that for all $v \in \mathbf{C^{0,1}}(\hat I \times \mathbb{R}_+;[\check v,\hat v])$ and for all $(t,x) \in \hat I \times \mathbb{R}_+$*

$$\begin{aligned}\mathrm{TV}\left(v(\cdot,x);\hat I\right) + \mathrm{TV}\left(v(t,\cdot)\right) &\le V_\infty\,,\\ \mathrm{TV}\left(\partial_x v(t,\cdot)\right) + \left\|\partial_x v(t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\le V_L\,.\end{aligned}$$

(BP2) *For all $w \in \mathcal{W}$, $m(\cdot,\cdot,w) \in \mathbf{C^0}(\hat I \times \mathbb{R}_+;\mathbb{R})$ and there exist M_∞, M_L such that for all $t \in \hat I$, $w, w_1, w_2 \in \mathcal{W}$,*

$$\begin{aligned}\mathrm{TV}\left(m(t,\cdot,w)\right) + \left\|m(t,\cdot,w)\right\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\le M_\infty\,,\\ \left\|m(t,\cdot,w_1) - m(t,\cdot,w_2)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\le M_L\, d_{\mathcal{W}}(w_1,w_2)\,.\end{aligned}$$

(BP3) *For all $w \in \mathcal{W}$, $q(\cdot,\cdot,w) \in \mathbf{C^0}\left(\hat I;\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})\right)$ and there exist Q_1, Q_∞ such that for all $t \in \hat I$ and $w, w_1, w_2 \in \mathcal{W}$, and*

$$\begin{aligned}\left\|q(t,\cdot,w)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\le Q_1\,,\\ \mathrm{TV}\left(q(t,\cdot,w)\right) + \left\|q(t,\cdot,w)\right\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\le Q_\infty\,,\\ \left\|q(t,\cdot,w_1) - q(t,\cdot,w_2)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\le Q_L\, d_{\mathcal{W}}(w_1,w_2)\,.\end{aligned}$$

(BP4) *$b \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\hat{I};\mathbb{R})$, is left continuous, and there exist positive constants B_1 and B_∞ such that*

$$\begin{aligned} \|b\|_{\mathbf{L^1}(\hat{I};\mathbb{R})} &\leq B_1\,, \\ \mathrm{TV}(b) + \|b\|_{\mathbf{L^\infty}(\hat{I};\mathbb{R})} &\leq B_\infty\,. \end{aligned}$$

Then, there exists $R, T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathcal{U}$, parametrized by $\mathcal{W}$ in the sense of Definition 2.5, whose orbits solve (3.14) in the sense of Definition 3.11, with

$$\begin{aligned} C_u &= M_\infty \\ C_t &= [\hat{v}(B_1 + 2R + R(M_\infty + V_L)T) + M_\infty R + Q_1] e^{M_\infty T}\,, \\ C_w &= [B_\infty M_L + \hat{v}\, Q_L + \frac{1}{2}\hat{v}\, Q_\infty\, M_L\, T \\ &\qquad + M_L\, R + Q_L + \frac{1}{2}\, M_L\, Q_\infty\, T] e^{M_\infty T}\,, \\ \mathcal{D}_t &= \left\{ r \in \mathcal{U} \colon \begin{array}{l} \|u\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \alpha_1(t)\,,\ \|u\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} \leq \alpha_\infty(t)\,, \\ \mathrm{TV}(u) + \left|b(t) - u(0)\right| \leq \alpha_{TV}(t) \end{array} \right\} \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \alpha_1(t) &= Re^{-M_\infty(T-t)} - (\hat{v}B_\infty + Q_1)(T-t)e^{M_\infty t} \\ \alpha_\infty(t) &= Re^{-M_\infty(T-t)} - Q_\infty(T-t) \\ \alpha_{\mathrm{TV}}(t) &= R\left(1 - (M_\infty + V_L)(T-t)\right) e^{(M_\infty + V_L)(T-t)} \\ &\quad -2Q_\infty(1 + (M_\infty + V_L)t)(T-t)e^{(M_\infty+V_L)t} \\ &\quad -B_\infty(M_\infty + V_L)(T-t)e^{(M_\infty+V_L)t} - \mathrm{TV}(b;[t,T])e^{(M_\infty+V_L)t}\,. \end{aligned}$$

A result entirely analogous to Corollary 3.9 can be proved also in the case of (3.14).

Proposition 3.13. *Set $\mathcal{U} = \mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$. Assume* **(BP1)**–**(BP2)**–**(BP3)**–**(BP4)**. *Let P^u be a Lipschitz process on $\mathcal{W}$, parametrised by $u \in \mathcal{U}$. Set $P \colon \mathcal{A} \to \mathcal{U} \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, to be the process generated in Theorem 2.6 by the coupling of the process P^w, constructed in Proposition 3.12, with P^u. If $\left(t, t_o, (u_o, w_o)\right) \in \mathcal{A}$, then*

$$\begin{aligned} u \colon [t_o, T] &\to \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}) \\ t &\mapsto P_1(t, t_o)\,(u_o, w_o) \end{aligned} \tag{3.17}$$

is a solution to

$$\begin{cases} \partial_t u + \partial_x \left(v(t,x)\, u\right) = \bar{m}(t,x)\, u + \bar{q}(t,x) & (t,x) \in [t_o, T] \times \mathbb{R}_+ \\ u(t,0) = b(t) & t \in [t_o, T] \\ u(t_o, x) = u_o(x) & x \in \mathbb{R}_+ \end{cases} \tag{3.18}$$

in the sense of Definition 3.11, where

$$\begin{aligned} \bar{m}(t,x) &= m\left(t, x, P_2(t,t_o)\,(u_o, w_o)\right)\,, \\ \bar{q}(t,x) &= q\left(t, x, P_2(t,t_o)\,(u_o, w_o)\right)\,. \end{aligned} \tag{3.19}$$

3.4 Measure Valued Balance Laws

Following [6], consider the following measure valued balance law

$$\partial_t \mu + \partial_x \big(b(t,\mu,w)\mu\big) + c(t,\mu,w)\mu = \int_{\mathbb{R}_+} \big(\eta(t,\mu,w)\big)(y)\, \mathrm{d}\mu(y) \tag{3.20}$$

for $t \in \hat I$, with $\mu(t_o) = \mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, the set of bounded, positive Radon measures on $\mathbb{R}_+$ equipped with the following distance, induced by the dual norm of $\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})$, see [6, § 2]:

$$d_{\mathcal{M}}(\mu_1,\mu_2) = \sup_{\substack{\varphi \in \mathbf{C^1}(\mathbb{R}_+;\mathbb{R}) \\ \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1}} \int_{\mathbb{R}_+} \varphi \, \mathrm{d}(\,\mu_1 - \mu_2). \tag{3.21}$$

We refer to [15] for basic measure theoretic results. Below, if X is a Banach space, then $\mathbf{BC}(\hat I; X)$ is the space of bounded continuous functions with the supremum norm. $\mathbf{BC}^{\alpha,\mathbf{1}}(\hat I \times \mathcal{M}^+(\mathbb{R}_+); X)$ is the space of X valued functions which are bounded with respect to the $\|\cdot\|_X$ norm, Hölder continuous with exponent α with respect to time and Lipschitz continuous in the measure variable with respect to $d_{\mathcal{M}}$ in (3.21). These spaces are equipped with the norms

$$\begin{aligned}
\|f\|_{\mathbf{BC}(\hat I;X)} &= \sup_{t\in\hat I} \left\| f(t) \right\|_X , \\
\|f\|_{\mathbf{BC}^{\alpha,\mathbf{1}}(\hat I \times \mathcal{M}^+(\mathbb{R}_+);X)} &= \sup_{t\in \hat I, \mu \in \mathcal{M}^+(\mathbb{R}_+)} \Big(\left\| f(t,\mu) \right\|_X + \mathbf{Lip}\left(f(t,\cdot)\right) \\
&\qquad\qquad + \mathbf{H}\left(f(\cdot,\mu)\right) \Big), \\
\|f\|_{(\mathbf{BC}\cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))} &= \sup_{x\in\mathbb{R}_+} \left\| f(x) \right\|_{\mathcal{M}(\mathbb{R}_+)} + \mathbf{Lip}(f) ,
\end{aligned}$$

where, with a slight abuse of notation,

$$\begin{aligned}
\mathbf{Lip}\left(f(t,\cdot)\right) &= \sup_{\substack{\mu_1,\mu_2 \in \mathcal{M}^+(\mathbb{R}_+) \\ \mu_1 \neq \mu}} \Big(\left\| f(t,\mu_1) - f(t,\mu_2) \right\|_X \Big/ d_{\mathcal{M}}(\mu_1,\mu_2) \Big) , \\
\mathbf{H}\left(f(\cdot,\mu)\right) &= \sup_{s_1,s_2 \in \hat I} \Big(\left\| f(s_1,\mu) - f(s_2,\mu) \right\|_X \Big/ |s_1 - s_2|^\alpha \Big) , \\
\mathbf{Lip}(f) &= \sup_{\substack{x_1,x_2\in\mathbb{R}_+ \\ x_1\neq x_2}} \Big(d_{\mathcal{M}}\left(f(x_1), f(x_2)\right) \Big/ \|x_2 - x_1\| \Big) .
\end{aligned}$$

Definition 3.14. *Given $T \in \hat I$ with $T > t_o$ and $w \in \mathcal{W}$, a function $\mu\colon [t_o,T] \to \mathcal{M}^+(\mathbb{R}_+)$ is a* weak solution *to* (3.20) *on the time interval $[t_o,T]$ if μ is narrowly continuous with respect to time (i.e., for every bounded function $\psi \in \mathbf{C^0}(\mathbb{R}_+;\mathbb{R})$, the map $t \mapsto \int_{\mathbb{R}_+} \psi(x)\,\mathrm{d}\mu(t,x)$ is continuous), and for all $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})\big([t_o,T] \times \mathbb{R}_+;\mathbb{R}\big)$, the following equality holds:*

$$\int_{t_o}^{T} \int_{\mathbb{R}_+} \Big(\partial_t \varphi(t,x) + \big(b(t,\mu,w)\big)(x)\, \partial_x \varphi(t,x)$$

$$
\begin{aligned}
& - \Big(c(t,\mu,w)\big)\,(x)\;\varphi(t,x)\Big)\,\mathrm{d}\mu\,(t,x)\,\mathrm{d}t \\
& + \int_{t_o}^{T}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,x)\,\mathrm{d}\big[\eta(t,\mu,w)(y)\big]\,(x)\right)\mathrm{d}\mu\,(t,y)\,\mathrm{d}t \\
= & \int_{\mathbb{R}_+}\varphi(T,x)\;\mathrm{d}\mu\,(T,x) - \int_{\mathbb{R}_+}\varphi(t_o,x)\;\mathrm{d}\mu_o\,(x)\,.
\end{aligned}
$$

Proposition 3.15. *Let $R>0$. Set $\mathcal{U}=\mathcal{M}^+(\mathbb{R})$ and let*

$$\mathcal{D}=\left\{\mu\in\mathcal{M}^+(\mathbb{R}_+)\colon\,\mu(\mathbb{R}_+)\le R\right\}.$$

Consider the Cauchy problem (3.20) *under the assumptions, for some positive constant $\hat{L}$,*

(MVBL1) *For every $w\in\mathcal{W}$, $b(\cdot,\cdot,w)\in\mathbf{BC}^{\alpha,1}(\hat{I}\times\mathcal{D};\mathbf{W}^{\mathbf{1},\infty}\left(\mathbb{R}_+;\mathbb{R}\right))$. Further, for every $w,w_1,w_2\in\mathcal{W}$, $t\in\hat{I}$, and $\mu\in\mathcal{D}$, $b(t,\mu,w)(0)\ge 0$, and, for some $B>0$,*

$$
\begin{aligned}
\left\|b(t,\mu,w)\right\|_{\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R})} &\le B\,,\\
\left\|b(\cdot,\mu,w_1)-b(\cdot,\mu,w_2)\right\|_{\mathbf{BC}(\hat{I};\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R}))} &\le \hat{L}\,d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

(MVBL2) *For every $w\in\mathcal{W}$,*

$$c(\cdot,\cdot,w)\in\mathbf{BC}^{\alpha,1}(\hat{I}\times\mathcal{D};\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R}))$$

Further, there exists a positive constant $C\ge 0$ such that, for all $w,w_1,w_2\in\mathcal{W}$, $\mu\in\mathcal{D}$ and $t\in\hat{I}$,

$$
\begin{aligned}
\left\|c(t,\mu,w)\right\|_{\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R})} &\le C\,,\\
\left\|c(\cdot,\mu,w_1)-c(\cdot,\mu,w_2)\right\|_{\mathbf{BC}(\hat{I};\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R}))} &\le \hat{L}\,d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

(MVBL3) *For all $w\in\mathcal{W}$, and setting $\mathcal{B}=(\mathbf{BC}\cap\mathbf{W}^{\mathbf{1},\infty})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))$,*

$$\eta(\cdot,\cdot,w)\in\mathbf{BC}^{\alpha,1}\left(\hat{I}\times\mathcal{D};\;\mathcal{B}\right).$$

Further, there exists an $E>0$ such that, for all $w,w_1,w_2\in\mathcal{W}$, $t\in\hat{I}$, and $\mu\in\mathcal{D}$,

$$
\begin{aligned}
\left\|\eta(t,\mu,w)\right\|_{\mathcal{B}} &\le E\,,\\
\left\|\eta(\cdot,\mu,w_1)-h(\cdot,\mu,w_2)\right\|_{\mathbf{BC}(\hat{I};\mathcal{B})} &\le \hat{L}\,d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

Then, there exist $T>0$, such that $[0,T]\subseteq\hat{I}$, and a Lipschitz Process on $\mathcal{M}^+(\mathbb{R}^n)$, pametrized by $\mathcal{W}$ in the sense of Definition 2.5 *whose orbits solve* (3.20) *in the sense of Definition* 3.14, *with*

$$
\begin{aligned}
C_u = 3(B+C+E)\,,\quad C_t = (B+C+E)\,e^{2(B+C+E)T}R\,,\\
C_w = C^*(T,B,C,E)\;R\;L\;e^{5(B+C+E)T}\,,\\
\mathcal{D}_t = \left\{\mu\in\mathcal{D}\colon\,\mu(\mathbb{R}_+)\le Re^{-3(B+C+E)(T-t)}\right\}.
\end{aligned}
\tag{3.22}
$$

The proof is a direct consequence of [6, Theorem 2.10] and, hence, it is omitted. In particular, C^* in (3.22) is the constant defined in [6, Item (iv), Theorem 2.10].

Proposition 3.16. *Set $\mathcal{U} = \mathcal{M}^+(\mathbb{R}^n)$. Fix $T > 0$ and assume that* **(MVBL1)**–**(MVBL2)**–**(MVBL3)** *hold. Let P^u be a Lipschitz process on $\mathcal{W}$, parametrised by $u \in \mathcal{U}$. Call $P\colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem* 2.6 *coupling P^w, found in Proposition* 3.15*, and P^u. If $([t_o, T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then the map*

$$\begin{array}{rccc} \mu & : & [t_o,T] & \to & \mathcal{M}^+(\mathbb{R}^n) \\ & & t & \mapsto & P_1(t,t_o)(\mu,w) \end{array} \tag{3.23}$$

solves the measure valued balance law

$$\begin{cases} \partial_t \mu + \partial_x \left(\bar{b}(t,\mu)\,\mu\right) + \bar{c}(t,\mu)\,\mu = \int_{\mathbb{R}_+} \left(\bar{\eta}(t,\mu)\right)(y)\,\mathrm{d}\mu(y) & t \in \hat{I} \\ \mu(t_o) = \mu_o \end{cases}$$

in the sense of Definition 3.14*, where*

$$\bar{b}(t,\mu) = b\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right), \qquad \bar{c}(t,\mu) = c\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right),$$
$$\bar{\eta}(t,\mu) = \eta\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right).$$

The proof is deferred to § 5.5.

3.5 Scalar NonLinear Conservation Laws

We now consider the following scalar nonlinear conservation law in one space dimension:

$$\begin{cases} \partial_t u + \partial_x f(t,u,w) = 0 & (t,x) \in \hat{I} \times \mathbb{R}, \\ u(t_o,x) = u_o(x) & x \in \mathbb{R} \end{cases} \tag{3.24}$$

for $t_o \in \hat{I}$, $u_o \in \mathbf{L^1}(\mathbb{R};\mathbb{R})$, $w \in \mathcal{W}$, with $f\colon \hat{I} \times \mathbb{R} \times \mathcal{W} \to \mathbb{R}$ a given function.

Definition 3.17. *Fix $w \in \mathcal{W}$ and $[t_o,T] \subseteq \hat{I}$. We say that a map $u \in \mathbf{C^0}\left([t_o,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ is a solution to problem* (3.24) *if it is a Kružkov-Entropy solution, i.e.*

$$\begin{aligned} &\int_{t_o}^{T} \int_{\mathbb{R}} \Big[|u-k|\,\partial_t\varphi + \operatorname{sign}(u-k)\,\left(f(t,u,w) - f(t,k,w)\right)\,\partial_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \geq \int_{\mathbb{R}} \left|u(T,x)-k\right|\varphi(T,x)\,\mathrm{d}x - \int_{\mathbb{R}} \left|u_o(x)-k\right|\varphi(t_o,x)\,\mathrm{d}x\,, \end{aligned} \tag{3.25}$$

for all non-negative test functions $\varphi \in \mathbf{C_c^\infty}(\hat{I} \times \mathbb{R};\mathbb{R}_+)$, and for all $k \in \mathbb{R}$.

Proposition 3.18. *Let $R > 0$ and t_o, T be such that $[t_o,T] \subseteq \hat{I}$. Choose $\mathcal{U} = \mathbf{L^1}(\mathbb{R};\mathbb{R})$ and define $\mathcal{D} = \left\{u \in \mathcal{U}\colon \operatorname{TV}(u) \leq R\right\}$. Consider the Cauchy problem*

$$\begin{cases} \partial_t u + \partial_x f(u,w) = 0 & (t,x) \in [t_o,T] \times \mathbb{R}, \\ u(t_o,x) = u_o(x) & x \in \mathbb{R} \end{cases} \tag{3.26}$$

under the assumptions

(CL1) *For all $w \in \mathcal{W}$, the map $u \mapsto f(u,w)$ is piecewise twice continuously differentiable.*

(CL2) *There exists a positive F_L such that for all $u_1, u_2 \in \mathbb{R}$ and all $w, w_1, w_2 \in \mathcal{W}$*

$$\begin{aligned} \left|f(u_1,w) - f(u_2,w)\right| &\leq F_L\, |u_1 - u_2| \\ \mathbf{Lip}\left(f(\cdot,w_1) - f(\cdot,w_2)\right) &\leq F_L\, d_{\mathcal{W}}(w_1,w_2) \end{aligned}$$

Then, there exists a Lipschitz Process on $\mathbf{L^1}(\mathbb{R};\mathbb{R})$, pametrized by $\mathcal{W}$, whose orbits are solutions to (3.24) *in the sense of Definition* 3.17*, with constants in* (2.11)–(2.12)–(2.13)

$$C_u = 0\,, \quad C_t = F_L\, R\,, \quad C_w = F_L\, R\,, \quad \mathcal{D}_t = \mathcal{D}\,.$$

The proof is classical and follows, for instance, from [17, Theorem 2.14 and Theorem 2.15].

Remark 3.19. The present treatment is limited to *homogeneous*, i.e., with a flux independent of x, conservation laws. Note that general 2×2 systems of conservation laws can *not* be approached by means of Theorem 2.6 while, for instance, we do comprehend a non local coupling of the form

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f\left(u, \int_{\mathbb{R}} w\, \mathrm{d}x\right) = 0 \\ u(0,x) = u_o(x) \end{array} \right. \qquad \left\{ \begin{array}{l} \partial_t w + \partial_x g\left(w, \int_{\mathbb{R}} u\, \mathrm{d}x\right) = 0 \\ w(0,x) = w_o(x)\,. \end{array} \right.$$

Proposition 3.20. *Set $\mathcal{U} = \mathbf{L^1}(\mathbb{R};\mathbb{R})$. Assume that* **(CL1)**–**(CL2)** *hold. Let P^u be a Lipschitz process on $\mathcal{W}$, parametrised by $u \in \mathcal{U}$. Call $P \colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem* 2.6 *coupling P^w, generated by* (3.26)*, to P^u. If $([t_o,T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then*

$$\begin{array}{rrl} u: & [t_o,T] \to & \mathbf{L^1}(\mathbb{R};\mathbb{R}) \\ & t \mapsto & P_1(t,t_o)(u_o,w_o) \end{array} \qquad \textit{solves} \qquad \left\{ \begin{array}{l} \partial_t u + \partial_x \bar f(t,u) = 0 \\ u(t_o) = u_o, \end{array} \right.$$

in the sense of Definition 3.17*, where $\bar f(t,u) = f\left(u, P_2(t,t_o)(u_o,w_o)\right)$.*

The proof is left until § 5.6.

4 Specific Coupled Problems

The abstract framework developed in Section 2, thanks to the proofs in the subsequent paragraphs, allows to prove the Lipschitz well posedness of several models.

As a first example, consider the model introduced in [20], where a large and slow vehicle positioned at $y = y(t)$ affects the overall traffic density $\rho = \rho(t,x)$.

The resulting model [20, Formula (2.1)] consists in the coupling of the Lighthill-Whitam [21] and Richards [26] macroscopic model decribing the evolution of ρ coupled with an ordinary differential equation for y, that is

$$\begin{cases} \partial_t \rho + \partial_x f\left(x, y(t), \rho\right) = 0 \\ \dot{y} = w\left(\rho(t, y)\right) \end{cases} \tag{4.1}$$

Clearly, this coupled problem fits in Theorem 2.6 thanks to Proposition 3.20 and Proposition 3.4, once the functions f and w meet reasonable requirements.

In the next paragraphs, we consider in particular the case of a predator–prey system (§ 4.1) and that of an epidemiological model (§ 4.2). To our knowledge, this latter well posedness is first proved here.

4.1 Predators and Prey

On the basis of the games introduced in [7] we consider the following predator–prey model:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, V\left(t, x, p(t)\right)\right) = -\eta\left(\left\|p(t) - x\right\|\right)\, \rho(t, x) \\ \rho(0, x) = \bar{\rho}(x) \end{cases} \tag{4.2a}$$

where

$$\begin{cases} \dot{p} = U\left(t, p, \rho(t)\right) \\ p(0) = \bar{p}. \end{cases} \tag{4.2b}$$

We consider a specific example, letting $\rho = \rho(t, x)$ be the density of some prey species moving in $\mathbb{R}^N$ and $p = p(t)$ be the position in $\mathbb{R}^N$ of a predator hunting it. To escape the predator, prey adopt a strategy defined by the speed

$$V(t, x, p) = -\frac{p - x}{\alpha + \|p - x\|^2}\, \psi\left(\|p - x\|^2\right) \tag{4.3}$$

where the term $\dfrac{p - x}{\alpha + \|p - x\|^2}$ stands for the escape direction of the prey. The positive term α in the denominator smooths the normalization. The function ψ describes the relevance of the predator p to the prey at x as a function of the distance $\|p - x\|$. The function $\eta = \eta\left(\|p - x\|\right)$ describes the effect of the feeding of the predator at p on the prey at x. On the other hand, the predator hunts moving towards the region of highest (mean) prey density, i.e., with speed

$$U(t, p, \rho) = \left(\nabla\varphi * \rho\right)(p)\,, \tag{4.4}$$

where φ is an averaging kernel.

Here, we show that (4.2) fits in the general framework presented in Section 2. Indeed, with reference to § 3.2, set

$$\begin{array}{rclrclrcl} \mathcal{U} & = & \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\,, & u & = & \rho\,, & v(t, x, w) & = & V(t, x, w)\,, \\ \mathcal{W} & = & \mathbb{R}^N\,, & w & = & p\,, & m(t, x, w) & = & -\eta\left(\|w - x\|\right)\,, \\ & & & & & & q(t, x, w) & = & 0\,, \end{array} \tag{4.5}$$

while with reference to § 3.1, set

$$\begin{array}{rclrcl} \mathcal{U} & = & \mathbb{R}^N , & u & = & p , \\ \mathcal{W} & = & \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) , & w & = & \rho , \end{array} \qquad f(t,u,w) = U(t,u,w) . \tag{4.6}$$

Proposition 4.1. *Fix positive $\alpha, r_\rho, r_p, r_\eta$ and mollifiers*

(V) *Let V be as in* (4.3) *with $\psi \in \mathbf{C_c^\infty}(\mathbb{R}^N;\mathbb{R}_+)$, with $\operatorname{spt}\psi \subseteq B(0,r_\rho)$ and $\int_{B(0,r_\rho)} \psi \, \mathrm{d}\xi = 1$.*

(U) *Let U be defined in* (4.4) *with $\varphi \in \mathbf{C_c^\infty}(\mathbb{R};\mathbb{R})$, positive, with $\operatorname{spt}\varphi \subseteq [-r_p, r_p]$ in* (4.4)*.*

(η) *$\eta \in \mathbf{C_c^\infty}(\mathbb{R}^N;\mathbb{R})$, positive, with $\operatorname{spt}\eta \subseteq B(0,r_\eta)$.*

Then, conditions **(IP1)**–**(IP2)**–**(IP3)** *and* **(ODE1)**–**(ODE2)** *are all satisfied. Therefore, model* (4.2) *defines a unique global process in the sense of Definition* 2.2*.*

Proof. Consider first **(IP1)**. By (4.3), V is a smooth function and the exponential factor ensures all the required boundedness conditions. We also have that $\left\|\nabla_p V\right\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^{N\times N})}$ is bounded, proving the first Lipschitz requirement in **(IP1)**. Prove now the latter inequality:

$$\begin{aligned} & \int_{\mathbb{R}^N} \left| \nabla_x \cdot \big(V(t,x,p_1) - V(t,x,p_2)\big) \right| \mathrm{d}x \\ = & \int_{\mathbb{R}^N} \left| \nabla_x \cdot V(t,x,p_1) - \nabla_x \cdot V(t,x,p_2) \right| \mathrm{d}x \\ = & \int_{B(p_1,r_p)\cup B(p_2,r_p)} \left| \nabla_x \cdot V(t,x,p_1) - \nabla_x \cdot V(t,x,p_2) \right| \mathrm{d}x \\ \le & \int_{B(p_1,r_p)\cup B(p_2,r_p)} \sup_{p\in\mathbb{R}^N} \left\| \nabla_p \nabla_x \cdot V(t,x,p) \right\| \mathrm{d}x \ \left\|p_2 - p_1\right\| \end{aligned}$$

proving also the latter requirement in **(IP1)**.

To prove **(IP2)**, we compute

$$\left\|m(t,\cdot,w)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \mathrm{TV}\left(m(t,\cdot,w)\right) = \max_{B(0,r_\eta)} |\eta| + \left\|\eta'\right\|_{\mathbf{L^1}(B(0,r_\eta);\mathbb{R})},$$

and

$$\begin{aligned} \left\|m(t,\cdot,w_1) - m(t,\cdot,w_2)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \le \int_{B(w_1,r_\eta)\cup B(w_2,r_\eta)} \sup_{B(0,r_\eta)} \left|\eta'\right| \left\|w_2 - w_1\right\| \mathrm{d}x \\ & \le \mathcal{O}(1) \left\|\eta'\right\|_{\mathbf{L}^\infty(B(0,r_\eta);\mathbb{R})} \left\|w_2 - w_1\right\|. \end{aligned}$$

Clearly, due to (4.5), **(IP3)** is immediate.

The regularity required in **(ODE1)** is immediate. Pass to the Lipschitz estimate:

$$\left\|U(t,p_1,\rho_1) - U(t,p_2,\rho_2)\right\|$$

$$
\begin{aligned}
&\leq \left\|U(t,p_1,\rho_1) - U(t,p_1,\rho_2)\right\| + \left\|U(t,p_1,\rho_2) - U(t,p_2,\rho_2)\right\| \\
&= \left\|\left(\nabla\varphi * (\rho_1 - \rho_2)\right)(p_1)\right\| + \left\|(\nabla\varphi * \rho_2)(p_1) - (\nabla\varphi * \rho_2)(p_2)\right\| \\
&\leq \left\|\nabla\varphi * (\rho_1 - \rho_2)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} + \left\|\nabla^2\varphi * \rho_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^{N\times N})} \|p_1 - p_2\| \\
&\leq \|\nabla\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|\rho_1 - \rho_2\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} + \left\|\nabla^2\varphi * \rho_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^{N\times N})} \|p_1 - p_2\| .
\end{aligned}
$$

Finally, the latter boundedness in **(ODE2)** is proved as follows:

$$
\sup_{\rho\in\mathcal{D}_\rho} \left\|U(\cdot,\cdot,\rho)\right\| \leq \sup_{\rho\in\mathcal{D}_\rho} \|\nabla\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}
$$

completing the proof by the definition of $\mathcal{D}_\rho$.

By Proposition 3.8, the balance law in (4.2) defines a global process P_1. Similarly, Proposition 3.2 ensures that the ordinary differential equation in (4.2) generates a global process P_2. Now, Proposition 3.10 and Proposition 3.4 ensure that the global process P obtained from P_1 and P_2 through Theorem 2.6 yields a solution to the coupled problem (4.2). □

4.2 Modeling Vaccination Strategies

Consider the model presented in [11, § 2]:

$$
\begin{cases}
\dot S = -\rho_S\, I\, S - p(t) \\
\partial_t V + \partial_\tau V = -\rho_V\, I\, V \\
\dot I = (\rho_S\, S + \int_0^{T_*} \rho_V\, V) I - \vartheta\, I - \mu\, I \\
\dot R = \vartheta\, I + V(t,T_*) \\
V(t,0) = p(t)\,.
\end{cases}
\tag{4.7}
$$

It describes a population consisting of susceptibles, $S = S(t)$, of infected that are also infective, $I = I(t)$, and recovered individuals, $R = R(t)$. The vaccination rate is $p = p(t)$ and vaccinated individuals need a time T_* to get immunized. More precisely, $V = V(t,\tau)$ is the number of individuals at time t vaccinated at time $t-\tau$, for $\tau \in [0,T_*]$. Thus, at time T_*, vaccinated individual enter the R population.

The positive constants ρ_S, ϑ and μ quantify the infectivity rate, the recovery rate and the mortality rate, respectively. The function $\rho_V = \rho_V(\tau)$ describes the infectivity rate of individuals vaccinated after time τ from being dosed.

Note that model (4.7) is triangular, in the sense that the evolution of the R population results from that of the other ones, without affecting them.

Model (4.7), once the R population is omitted, fits in the abstract framework presented in Section 2. Indeed, with reference to the notation used in § 3.1, we pose

$$
\begin{aligned}
&\mathcal{U} = \mathbb{R}^2\,, \qquad \mathcal{W} = \mathbf{L}^1([0,T_*];\mathbb{R})\,, \qquad u = \begin{bmatrix} S \\ I \end{bmatrix}, \qquad w = V\,, \\
&f(t,u,w) = \begin{bmatrix} -\rho_S\, u_1\, u_2 - p(t) \\ \left(\rho_S\, u_1 + \int_0^{T_*} \rho_V(\tau)\, w(\tau)\, \mathrm{d}\tau - \vartheta - \mu\right) u_2 \end{bmatrix},
\end{aligned}
\tag{4.8}
$$

while with reference to § 3.3, we set

$$\begin{array}{l}\mathcal{U} = \mathbf{L^1}([0,T_*];\mathbb{R})\\ \mathcal{W} = \mathbb{R}^2,\end{array} \qquad x = \tau\,, \quad u = V\,, \quad w = \left[\begin{array}{c} S \\ I \end{array}\right], \tag{4.9a}$$

and

$$\begin{aligned} v(t,x) &= 1\\ m(t,x,w) &= -\rho_V(x)\, w_2\\ q(t,x,w) &= 0\\ b(t) &= p(t). \end{aligned} \tag{4.9b}$$

The well posedness of (4.7) now follows once we verify that Proposition 3.4 and Proposition 3.13 can be applied.

Proposition 4.2. *Fix positive r, T_*, ρ_S and choose $p \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R})$, $\rho_V \in \mathbf{BV}([0,T_*];\mathbb{R})$. Then, problem (4.7) defines a unique global process P, in the sense of Definition 2.2, defined on all initial data*

$$S_o, I_o, R_o \in [0,r] \quad \textit{and} \quad V_o \in \mathbf{L^1}([0,T_*];\mathbb{R}_+)$$

with

$$\mathrm{TV}(V_o) + \|V_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq r\,.$$

P is Lipschitz continuous as a function of time and of the initial data, with respect to the Euclidean norm in (S_o, I_o, R_o) and to the $\mathbf{L^1}$ norm in V.

Proof. Verifying **(ODE1)** is immediate. The Lipschitz continuity required in **(ODE2)** follows from the boundedness $u \in \mathcal{D}_\mathcal{U}$, which is a closed ball in $\mathcal{U} = \mathbb{R}^2$ and from the choice of ρ_V, see § 3.1. Hence, Proposition 3.2 applies.

Conditions **(BP1)** and **(BP3)** are immediate. The first requirement in **(BP2)** follows from the choice of ρ_V and the boundedness of $\mathcal{D}_\mathcal{U}$. The second is ensured by the linearity of m and the boundedness of ρ_V. Since p has bounded variation, **(BP4)** is satisfied on any bounded time interval. Hence, also Proposition 3.12 can be applied.

Then, Proposition 3.4 and Proposition 3.13, through Theorem 2.6, ensure the well posedness of the coupled system (4.8)–(4.9).

We now verify the well posedness of the R component. From (4.7), using (5.30), we have

$$V(t,\tau) = \left\{\begin{array}{ll} V_o(\tau + t_o - t)\, \exp\left(-\int_{t_o}^t \rho_V(s)\, I(s)\, \mathrm{d}s\right) & \text{if } t \leq \tau + t_o\,,\\ p(t-\tau)\, \exp\left(-\int_{t-\tau}^t \rho_V(s)\, I(s)\, \mathrm{d}s\right) & \text{if } t > \tau + t_o\,. \end{array}\right.$$

This shows that the map $t \mapsto V(t,T_*)$ is sufficiently regular for the equation for R, namely $\dot R = \vartheta\, I(t) + V(t,T_*)$, to be explicitly solved: $R(t) = R_o + \int_0^t \left(I(s) + V(s,T_*)\right) \mathrm{d}s$. Thus, the full model (4.7) is well posed. □

5 Technical Details

5.1 Proofs for Section 2

Proof of Theorem 2.6. We begin by showing F is a local flow in the sense of Definition 2.1. F is continuous as it is a pairing of two continuous functions. Further

$$F(0,t_o)(u,w) = \big(P^w(t_o,t_o)u, P^u(t_o,t_o)w\big) = (u,w)\,.$$

We prove the Lipschitz continuity in time and with respect to initial conditions of F:

$$\begin{aligned}
& d\left(F(\tau_1,t_o)(u_1,w_1), F(\tau_2,t_o)(u_2,w_2)\right) \\
& \le d_{\mathcal{U}}\left(P^{w_1}(t_o+\tau_1,t_o)u_1, P^{w_1}(t_o+\tau_1,t_o)u_2\right) \\
& \quad + d_{\mathcal{U}}\left(P^{w_1}(t_o+\tau_1,t_o)u_2, P^{w_2}(t_o+\tau_1,t_o)u_2\right) \\
& \quad + d_{\mathcal{U}}\left(P^{w_2}(t_o+\tau_1,t_o)u_2, P^{w_2}(t_o+\tau_2,t_o)u_2\right) \\
& \quad + d_{\mathcal{W}}\left(P^{u_1}(t_o+\tau_1,t_o)w_1, P^{u_1}(t_o+\tau_1,t_o)w_2\right) \\
& \quad + d_{\mathcal{W}}\left(P^{u_1}(t_o+\tau_1,t_o)w_2, P^{u_2}(t_o+\tau_1,t_o)w_2\right) \\
& \quad + d_{\mathcal{W}}\left(P^{u_2}(t_o+\tau_1,t_o)w_2, P^{u_2}(t_o+\tau_2,t_o)w_2\right) \\
& \le e^{C_u\tau_1}\, d_{\mathcal{U}}(u_1,u_2) + C_w\,\tau_1\, d_{\mathcal{W}}(w_1,w_2) + C_t\,|\tau_1-\tau_2| \\
& \quad + e^{C_u\tau_1}\, d_{\mathcal{W}}(w_1,w_2) + C_w\,\tau_1\, d_{\mathcal{U}}(u_1,u_2) + C_t\,|\tau_1-\tau_2| \\
& \le (e^{C_u\delta} + C_w\,\delta)\; d\left((u_1,w_1),(u_2,w_2)\right) + 2\,C_t\,|\tau_1-\tau_2|\,.
\end{aligned}$$

Thus F is indeed a local flow in the sense of Definition 2.1, with $\mathbf{Lip}(F) = e^{C_u\delta} + C_w\,\delta + 2C_t$.

We now show that F satisfies the assumptions of Theorem 2.4. Consider (2.8):

$$\begin{aligned}
& d\left(F(k\tau,t_o+\tau)\circ F(\tau,t_o)(u,w), F((k+1)\tau,t_o)(u,w)\right) \\
& = d_{\mathcal{U}}\Big(P^{P^u(t_o+\tau,t_o)w}(t_o+(k+1)\tau,t_o+\tau)P^w(\tau,t_o)u, \\
& \qquad P^w\left(t_o+(k+1)\tau,t_o\right)u\Big) \qquad (5.1) \\
& \quad + d_{\mathcal{W}}\Big(P^{P^w(t_o+\tau,t_o)u}(t_o+(k{+}1)\tau,t_o+\tau)P^u(t_o+\tau,t_o)w, \\
& \qquad P^u\left(t_o+(k{+}1)\tau,t_o\right)w\Big). \qquad (5.2)
\end{aligned}$$

We consider only the term (5.1), since the latter is entirely similar. By (2.5), we have

$$P^w\left(t_o+(k+1)\tau,t_o\right)u = P^w(t_o+(k+1)\tau,t_o+\tau)\;P^w(t_o+\tau,t_o)u\,,$$

hence, via (2.12) and (2.13),

$$d_{\mathcal{U}}\left(P^{P^u(t_o+\tau,t_o)w}(t_o+(k+1)t,t_o+\tau)P^w(t_o+\tau,t_o)u, P^w\left(t_o+(k+1)\tau,t_o\right)u\right)$$

$$
\begin{aligned}
&= d_{\mathcal{U}}\Big(P^{P^u(t_o+\tau,t_o)w}(t_o+(k+1)\tau,t_o+\tau)P^w(t_o+\tau,t_o)u,\\
&\qquad P^w(t_o+(k+1)\tau,t_o+\tau)P^w(t_o+\tau,t_o)u\Big)\\
&\le C_w\, k\,\tau\, d_{\mathcal{W}}\left(P^u(t_o+\tau,t_o)w,w\right)\\
&\le k\,\tau\, C_t\, C_w\tau\,.
\end{aligned}
\tag{5.3}
$$

Combining (5.3) with the analogous estimate bounding (5.2), we end up with

$$
d\Big(F(k\tau,t_o+\tau)\circ F(\tau,t_o)(u,w),F\left((k+1)\tau,t_o\right)(u,w)\Big)\le k\,\tau\,\omega(\tau)
$$

where ω is as in (2.15). Thus (2.8) is satisfied.

We consider the second condition in Theorem 2.4, namely (2.9). Note that Euler polygonals for the local flow F, see Definition 2.3, can be written recursively, as

$$
F^\varepsilon(\tau,t_o)(u,w)=F(\tau-k\varepsilon,t_o+k\varepsilon)\circ F^\varepsilon(k\varepsilon,t_o)(u,w)\,.
$$

For any $\tau\in[0,\delta]$ and for any (u,w), $(\bar u,\bar w)$ in $\mathcal{U}\times\mathcal{W}$, we have

$$
\begin{aligned}
d\left(F(\tau,t_o)(u,w),F(\tau,t_o)(\bar u,\bar w)\right) &= d_{\mathcal{U}}(P^w(t_o+\tau,t_o)u,P^{\bar w}(t_o+\tau,t_o)\bar u)\\
&\quad + d_{\mathcal{W}}(P^u(t_o+\tau,t_o)w,P^{\bar u}(t_o+\tau,t_o)\bar w)\,.
\end{aligned}
$$

For the first of these summands, by the triangle inequality, we have

$$
\begin{aligned}
&d_{\mathcal{U}}\left(P^w(t_o+\tau,t_o)u,P^{\bar w}(t_o+\tau,t_o)\bar u\right)\\
&\le d_{\mathcal{U}}\left(P^w(t_o+\tau,t_o)u,P^w(t_o+\tau,t_o)\bar u\right)+d_{\mathcal{U}}\left(P^w(t_o+\tau,t_o)\bar u,P^{\bar w}(t_o+\tau,t_o)\bar u\right)\\
&\le e^{C_u\tau}\, d_{\mathcal{U}}(u,\bar u)+C_w\,\tau\, d_{\mathcal{W}}(w,\bar w)\,.
\end{aligned}
$$

The second term is estimated analogously, leading to

$$
d\left(F(\tau,t_o)(u,w),F(\tau,t_o)(\bar u,\bar w)\right)\le\Big(e^{C_u\tau}+C_w\,\tau\Big)\ d\left((u,w),(\bar u,\bar w)\right)\,. \tag{5.4}
$$

Estimate (5.4) is of use in the following:

$$
\begin{aligned}
&d\left(F^\varepsilon(\tau,t_o)(u,w),F^\varepsilon(\tau,t_o)(\bar u,\bar w)\right)\\
&= d\left(F(\tau-k\varepsilon,t_o+k\varepsilon)F^\varepsilon(k\varepsilon,t_o)(u,w),F(\tau-k\varepsilon,t_o+k\varepsilon)F^\varepsilon(k\varepsilon,t_o)(\bar u,\bar w)\right)\\
&\le\Big(e^{C_u(\tau-k\varepsilon)}+C_w\,(\tau-k\varepsilon)\Big)\ d\left(F^\varepsilon(k\varepsilon,t_o)(u,w),F^\varepsilon(k\varepsilon,t_o)(\bar u,\bar w)\right)\,.
\end{aligned}
$$

It remains to estimate the distance in the latter right hand side. We have for any $k\in\mathbb{N}\setminus\{0\}$,

$$
F^\varepsilon(k\varepsilon,t_o)(u,w)=F(\varepsilon,t_o)\ F^\varepsilon\left((k-1)\varepsilon,t_o\right)(u,w),
$$

and thus using iteratively (5.4),

$$
d\left(F^\varepsilon(k\varepsilon,t_o)(u,w),F^\varepsilon(k\varepsilon,t_o)(\bar u,\bar w)\right)
$$

$$\begin{aligned}
&\leq \left(e^{C_u\varepsilon} + C_w\,\varepsilon\right) d\left(F^\varepsilon((k-1)\varepsilon, t_o)(u,w), F^\varepsilon((k-1)\varepsilon, t_o)(\bar u, \bar w)\right)\\
&\leq \left(e^{C_u\varepsilon} + C_w\,\varepsilon\right)^k d\left((u,w),(\bar u,\bar w)\right)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d\left(F^\varepsilon(\tau,t_o)(u,w), F^\varepsilon(\tau,t_o)(\bar u,\bar w)\right)\\
&\leq \left(e^{C_u(\tau-k\varepsilon)} + C_w\,(\tau-k\varepsilon)\right)\left(e^{C_u\varepsilon} + C_w\,\varepsilon\right)^k\; d\left((u,w),(\bar u,\bar w)\right).
\end{aligned}$$

Hence, (2.9) is satisfied provided there exists a positive L such that for all $\varepsilon > 0$ and $t \in [0,T]$

$$\left(e^{C_u(\tau-k\varepsilon)} + C_w\,(\tau-k\varepsilon)\right)\left(e^{C_u\varepsilon} + C_w\,\varepsilon\right)^k \leq L\,,$$

where $k = \lfloor \frac{\tau}{\varepsilon} \rfloor$. Indeed, since $e^a + b \leq e^{a+b}$ for all $a,b \in \mathbb{R}_+$, we have

$$\begin{aligned}
\left(e^{C_u(\tau-k\varepsilon)} + C_w(\tau-k\varepsilon)\right)\left(e^{C_u\varepsilon} + C_w\,\varepsilon\right)^k &\leq e^{(C_u+C_w)(\tau-k\varepsilon)}\left(e^{(C_u+C_w)\varepsilon}\right)^k\\
&= e^{(C_u+C_w)\tau},
\end{aligned}$$

so that $L = e^{(C_u+C_w)\delta}$.

Finally, note that (2.16) directly follows from the definition (2.14) of F, together with the properties $P^w(t_o+\tau,t_o)\mathcal{D}^{\mathcal{U}}_{t_o} \subseteq \mathcal{D}^{\mathcal{U}}_{t_o+\tau}$, which holds for all $w \in \mathcal{W}$, and $P^u(t_o+\tau,t_o)\mathcal{D}^{\mathcal{W}}_{t_o} \subseteq \mathcal{D}^{\mathcal{W}}_{t_o+\tau}$, which holds for all $u \in \mathcal{U}$. Therefore, with reference to (2.7), we have $\mathcal{D}^3_{t_o} \supseteq (\mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o})$ and Condition 1. in Theorem 2.4 completes the proof of (2.17). □

Proof of Theorem 2.7. The continuity of $\hat F$ is immediate. The Lipschitz continuity follows from the triangle inequality and a Lipschitz constant is $\mathbf{Lip}(\hat F) = \mathcal{L} + \max\{\mathbf{Lip}(F^w)\,,\ \mathbf{Lip}(F^u)\}$. Hence, $\hat F$ is a local flow according to Definition 2.1.

Concerning the tangency condition, compute

$$\begin{aligned}
\frac{1}{\tau}\, d\left(\hat F(\tau,t_o)(u,w), F(\tau,t_o)(u,w)\right) &= \frac{1}{\tau}\, d_{\mathcal{U}}\left(F^w(\tau,t_o)u, P^w(t_o+\tau,t_o)u\right)\\
&\quad + \frac{1}{\tau}\, d_{\mathcal{W}}\left(F^u(\tau,t_o)w, P^u(t_o+\tau,t_o)w\right)
\end{aligned}$$

and the first order tangency condition (2.10) allows to complete the proof. □

5.2 Proofs for Section 3.1

Proof of Corollary 3.3. For $k \in \mathbb{N}$, define $R_k = 2^k$ and $\hat{\mathcal{D}}^k = \overline{B\,(0,R_k)}$. Fix $u_o \in \mathbb{R}^n$. There exists $\bar k \in \mathbb{N} \setminus \{0\}$ such that $\|u_o\| \leq R_{\bar k-1}$. We proceed recursively.

For $k = \bar k$**:** consider the process P^w_k, given by Proposition 3.2, according to the choice $R_k = 2^k$. By Proposition 3.2 we know that $P^w_k(t,0)u_o$ is defined for every $t \in [t_o, T_k]$, where $T_k = \frac{R_k}{2F_\infty(R_k)}$. Define $u_k = P^w_k(T_k,t_o)u_o \in \hat{\mathcal{D}}^k$.

For $k > \bar{k}$: assume $u_{k-1} \in \hat{\mathcal{D}}^{k-1}$ and consider the process P_k^w, given by Proposition 3.2, according to the choice $R_k = 2^k$. By Proposition 3.2 we know that $P_k^w(t, t_o)u_{k-1}$ is defined for every $t \in [t_o, T_k]$, where $T_k = \frac{R_k}{2F_\infty(R_k)}$. Define $u_k = P_k^w(T_k, t_o)u_{k-1} \in \hat{\mathcal{D}}^k$.

Define the function

$$u(t) = \begin{cases} P_{\bar{k}}^w(t, t_o)u_o & \text{if } t \in [t_o, T_{\bar{k}}] \\ P_k^w(t - \sum_{h=\bar{k}}^{k-1} T_h, 0)u_{k-1} & \text{if } \sum_{h=\bar{k}}^{k-1} T_h < t \leq \sum_{h=\bar{k}}^{k} T_h, \end{cases}$$

which clearly is a solution to (3.1). Computing

$$\sum_{k=\bar{k}}^{+\infty} T_k = \sum_{k=\bar{k}}^{+\infty} \frac{2^{k-1}}{F_\infty(2^k)} \geq O(1) \sum_{k=\bar{k}}^{+\infty} \frac{2^{k-1}}{2^k \log(2^k)} = O(1) \sum_{k=\bar{k}}^{+\infty} \frac{1}{k} = +\infty,$$

shows that the solution u is defined for every $t \geq t_o$. □

Proof of Proposition 3.4. Let F_1 be the first component of the local flow F defined in (2.14).

Let $t \in [0, T]$ be a Lebesgue point of the map $t \mapsto f\big(t, P(t, t_o)(u_o, w_o)\big)$. Choose h small so that $t + h \in [0, T]$ and set $(u, w) = P(t, t_o)(u_o, w_o)$. Then,

$$\begin{aligned}
&\left\| \frac{P_1(t+h, t_o)(u_o, w_o) - P_1(t, t_o)(u_o, w_o)}{h} - f(t, P(t, t_o)(u_o, w_o)) \right\| \\
&= \left\| \frac{P_1(t+h, t)(u, w) - u}{h} - f(t, u, w) \right\| \\
&\leq \left\| \frac{P_1(t+h, t)(u, w) - F_1(h, t)(u, w)}{h} \right\| + \left\| \frac{F_1(h, t)(u, w) - u}{h} - f(t, u, w) \right\| \\
&= R_1(h) + R_2(h)\,.
\end{aligned}$$

Considering first the term R_1, we use estimate (2.10), giving

$$\begin{aligned}
R_1(h) &= \left\| \frac{P_1(t+h, t)P(t, t_o)(u_o, w_o) - F_1(h, t)P(t, t_o)(u_o, w_o)}{h} \right\| \\
&\leq \frac{2\,L}{\ln 2} \int_0^h \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi \longrightarrow 0, \text{ as } h \to 0
\end{aligned}$$

with L and ω as in (2.15). For R_2, we have

$$\begin{aligned}
R_2(h) &= \left\| \frac{1}{h} \int_0^h f(t+\tau, F_1(\tau, t)(u, w), w)\, \mathrm{d}\tau - f(t, u, w) \right\| \\
&= \left\| \frac{1}{h} \int_0^h \big[f(t+\tau, F_1(\tau, t)(u, w), w) - f(t, u, w) \big]\, \mathrm{d}\tau \right\| \\
&\leq \left\| \frac{1}{h} \int_0^h \big[f(t+\tau, F_1(\tau, t)(u, w), w) - f(t+\tau, P_1(t+\tau, t)(u, w), w) \big] d\tau \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\frac{1}{h}\int_0^h \big[f(t+\tau,P_1(t+\tau,t)(u,w),w)\right.\\
&\qquad\left. -f(t+\tau,P_1(t+\tau,t)(u,w),P_2(t+\tau,t)(u,w))\big]\,\mathrm{d}\tau\right\|\\
&+\left\|\frac{1}{h}\int_0^h \big[f(t+\tau,P_1(t+\tau,t)(u,w),P_2(t+\tau,t)(u,w))-f(t,u,w)\big]\mathrm{d}\tau\right\|\\
&= R_{2,1}(h)+R_{2,2}(h)+R_{2,3}(h).
\end{aligned}
$$

We have, as f is Lipschitz continuous, and using (2.10)–(2.15), that

$$
\begin{aligned}
R_{2,1}(h) &\le \frac{L_f}{h}\int_0^h \left\|F_1(\tau,t)(u,w)-P_1(t+\tau,t)(u,w)\right\|\mathrm{d}\tau\\
&\le \frac{2\,L}{\ln 2}\,\frac{L_f}{h}\int_0^h \tau\int_0^\tau \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;\;d\tau\\
&\to 0 \text{ as } h\to 0+\,;\\
R_{2,2}(h) &\le \frac{L_f}{h}\int_0^h \left\|P_2(t+\tau,t)(u,w)-P_2(t,t)(u,w)\right\|\mathrm{d}\tau\\
&\le \frac{L_f\cdot L_P}{h}\int_0^h \tau\;\mathrm{d}\tau\\
&\to 0 \text{ as } h\to 0+\,;\\
R_{2,3}(h) &\le \int_0^h \frac{1}{h}\left\|f\left(t+\tau,P(t+\tau,t_o)(u_o,w_o)\right)-f(t,P(t,t_o)(u_o,w_o))\right\|\mathrm{d}\tau\\
&\to 0 \text{ as } h\to 0+\,,
\end{aligned}
$$

the latter convergence following from the choice of t as a Lebesgue point. □

5.3 Proofs for § 3.2

With reference to (3.8) and (3.14), introduce for $\bar t, t\in \hat I$ and $\bar x, x\in\mathbb{R}_+$ the characteristics

$$
t\mapsto \mathcal{X}(t;\bar t,\bar x) \text{ solves } \begin{cases}\dot x = v(t,x,w)\\ x(\bar t)=\bar x,\end{cases} \tag{5.5a}
$$

and

$$
t\mapsto \mathcal{T}(x;\bar x,\bar t) \text{ solves } \begin{cases}t' = 1/v(t,x,w)\\ t(\bar x)=\bar t\,,\end{cases} \tag{5.5b}
$$

and in the sequel we omit the dependence on w. As is well known, see for instance [12, Lemma 5] and the references therein, the unique solution to (3.8) is

$$
u(t,x) = u_o\left(\mathcal{X}(t_o;t,x)\right)\,\mathcal{E}_w(t_o,t,x)+\int_{t_o}^t q\left(s,\mathcal{X}(s;t,x),w\right)\,\mathcal{E}_w(s,t,x)\,\mathrm{d}s \tag{5.6}
$$

where the characteristics $\mathcal{X}$ are defined by (5.5) and

$$\mathcal{E}_w(\tau, t, x) = \exp \int_\tau^t \Big(m\left(s, \mathcal{X}(s; t, x), w\right) - \operatorname{div} v\left(s, \mathcal{X}(s; t, x)\right)\Big) \,\mathrm{d}s\,.$$

Below, we often use the substitution $y \leftrightarrow x$, where

$$y = \mathcal{X}(t; t_o, x) \quad \text{with Jacobian} \quad J(t, y), \tag{5.7}$$

given by

$$J(t, y) = \exp\left(\int_t^{t_o} \nabla \cdot v\left(s, \mathcal{X}(s; \tau, y)\right) \mathrm{d}s\right),$$

for more details see for instance [12, Proof of Proposition 3].

Lemma 5.1. *Assume* **(IP1)** *holds and use the notation* (5.5). *Let* $u \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$. *Then, for all* $t_o, t \in \hat{I}$

$$\int_{\mathbb{R}^n} \Big| u\left(\mathcal{X}(t; t_o, x)\right) - u(x)\Big| \,\mathrm{d}x \le \frac{V_\infty}{V_L} \left(e^{V_L |t - t_o|} - 1\right) \operatorname{TV}(u)\,. \tag{5.8}$$

This Lemma is an extension of [5, Lemma 2.3] to $\mathbb{R}^n$.

Proof of Lemma 5.1. Along the same lines of [1, Lemma 3.24], thanks to [1, Theorem 3.9], we assume that $u \in (\mathbf{C^1} \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$. Then, using the change of coordinates (5.7),

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big| u\left(\mathcal{X}(t; t_o, x)\right) - u(x)\Big| \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \int_{t_o}^t \nabla u\left(\mathcal{X}(\tau; t_o, x)\right)\, v\left(\tau, \mathcal{X}(\tau; t_o, x)\right) \mathrm{d}\tau \right| \mathrm{d}x \\
&\le \left| \int_{t_o}^t \int_{\mathbb{R}^n} \Big\| \nabla u\left(\mathcal{X}(\tau; t_o, x)\right)\Big\| \, \Big\| v\left(\tau, \mathcal{X}(\tau; t_o, x)\right)\Big\| \,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\le \left| \int_{t_o}^t \int_{\mathbb{R}^n} \|\nabla u(y)\| \, \|v(\tau, y)\| \, \exp\left| \int_\tau^t \nabla \cdot v\left(\tau, \mathcal{X}(s; \tau, y)\right) \mathrm{d}s \right| \mathrm{d}y\,\mathrm{d}\tau \right| \\
&\le V_\infty \,\|\nabla u\|_{\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^n)} \left| \int_{t_o}^t e^{V_L (t-\tau)} \,\mathrm{d}\tau \right| \\
&= \frac{V_\infty}{V_L} \left(e^{V_L |t - t_o|} - 1\right) \|\nabla u\|_{\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^n)}\,,
\end{aligned}$$

which yields (5.8). □

Define the parameterized mapping P^w by

$$P^w : \begin{array}{ccl} \mathcal{A} & \to & \mathcal{U} \\ (t, t_o, u_o) & \mapsto & u(t) \end{array} \quad \text{where} \quad u(t) \text{ is given by (5.6);} \tag{5.9}$$

Below, by **(IP1)** and **(IP2)**, for all $t, \tau \in \hat{I}$, $x \in \mathbb{R}^n$ and $w \in \mathcal{W}$, we use the uniform estimate

$$0 \le \mathcal{E}_w(\tau, t, x) \le e^{(M_\infty + V_L)|t - \tau|}\,. \tag{5.10}$$

Lemma 5.2. *For all $w \in \mathcal{W}$, P^w in* (5.9) *is a global process according to Definition* 2.2.

Proof of Lemma 5.2. That P^w satisfies (2.3) is an immediate consequence of its definition (5.6). The uniqueness of the solution ensures that (2.5) is satisfied.

Fix $t_o, t \in I$, with $t_o \leq t$, and $r_o \in \mathcal{D}_{t_o}$. It remains to show (2.4), that is, $u(t) = P^w(t, t_o)u_o \in \mathcal{D}_t$ for each $w \in \mathcal{W}$.

1. We begin by showing that, if $\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_1(t_o)$, then $\left\|u(t)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_1(t)$. Making use of **(IP2)**–**(IP3)**–(3.13)–(5.6)–(5.7), see also [12, Proposition 3, **(H3)**],

$$\begin{aligned}
&\left\|u(t)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \\
&\leq \left(\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} + \left\|q(\cdot,\cdot,w)\right\|_{\mathbf{L^1}([t_o,t]\times\mathbb{R}^n;\mathbb{R})}\right) \\
&\quad \times \exp\left(\int_{t_o}^{t} \left\|m(\tau,\cdot,w)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau\right) \\
&\leq \left(\alpha_1(t_o) + Q_1(t-t_o)\right) e^{M_\infty(t-t_o)} \\
&\leq \left(Re^{-M_\infty(T-t_o)} - Q_1(T-t_o)e^{M_\infty t_o} + Q_1(t-t_o)\right) e^{M_\infty(t-t_o)} \\
&\leq Re^{-M_\infty(T-t)} - Q_1(T-t)e^{M_\infty t} \\
&= \alpha_1(t)\,,
\end{aligned} \tag{5.11}$$

as required.

2. Assuming now that $\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_\infty(t_o)$, we show that $\left\|u(t)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_\infty(t)$, We use (3.13)–(5.6), see also [12, Proposition 3, **(H4)**], together with **(IP1)**, **(IP2)**, **(IP3)** and (5.10). Then,

$$\begin{aligned}
\left\|u(t)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} &\leq \left(\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^{t} \left\|q(s,\cdot,w)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s\right) \\
&\quad \times \exp\left(\int_{t_o}^{t} \left(\left\|m(s,\cdot,w)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \left\|\nabla\cdot v(s)\right\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})}\right) \mathrm{d}s\right) \\
&\leq \left(\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + Q_\infty(t-t_o)\right) e^{(M_\infty+V_L)(t-t_o)} \\
&\leq \left(\alpha_\infty(t_o) + Q_\infty(t-t_o)\right) e^{(M_\infty+V_L)(t-t_o)} \\
&\leq \left(Re^{-(M_\infty+V_L)(T-t_o)} - Q(T-t_o)e^{(M_\infty+V_L)t_o}\right) e^{(M_\infty+V_L)(t-t_o)} \\
&+ Q(t-t_o)e^{(M_\infty+V_L)(t-t_o)} \\
&\leq Re^{-(M_\infty+V_L)(T-t)} - Q(T-t)e^{(M_\infty+V_L)t} \\
&= \alpha_\infty(t)\,,
\end{aligned}$$

as required.

3. Finally, we show that, if $u_o \in \mathcal{D}_{t_o}$, then $\mathrm{TV}\left(u(t)\right) \leq \alpha_{TV}(t)$. We use **(IP1)**–**(IP2)**–**(IP3)**–(3.13)–(5.6)–(5.7)–(5.10), see also [12, Formula (31)]:

$$\begin{aligned}\mathrm{TV}\left(u(t)\right) \leq \Bigg[& \mathrm{TV}(u_o) + \int_{t_o}^{t} \mathrm{TV}\left(q(s,\cdot,w)\right) \mathrm{d}s \\ & + \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^{t} \left\|q(s,\cdot,w)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}s \right) \\ & \times \int_{t_o}^{t} \left(\mathrm{TV}\left(m(s,\cdot,w)\right) + \left\|\nabla\nabla \cdot v(s)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \right) \mathrm{d}s \Bigg] e^{(M_\infty + V_L)|t-\tau|}\end{aligned} \tag{5.12}$$

Since $u_o \in \mathcal{D}_{t_o}$, by (3.12), $\mathrm{TV}(u_o) \leq \alpha_{\mathrm{TV}}(t_o)$ and we have that (5.12) becomes

$$\begin{aligned}& \mathrm{TV}\left(u(t)\right) \\ \leq & \Bigg[\alpha_{\mathrm{TV}}(t_o) + Q_\infty (t-t_o) \\ & + \left(Re^{-(M_\infty+V_L)(T-t_o)} - Q_\infty e^{(M_\infty+V_L)t_o}(T-t_o) + Q_\infty (t-t_o) \right) \\ & \times (M_\infty + V_1)(t-t_o) \Bigg] e^{(M_\infty+V_L)(t-t_o)} \\ \leq & R\left(1 - (M_\infty + V_1)(T-t)\right) e^{-(M_\infty+V_L)(T-t)} \\ & - Q_\infty \left(1 + (M_\infty+V_1)t\right)(T-t_o)\, e^{(M_\infty+V_L)t} \\ & + Q_\infty (t-t_o)\left(1 + (M_\infty+V_1)(t-t_o)\right) e^{(M_\infty+V_L)(t-t_o)} \\ \leq & R\left(1 - (M_\infty + V_1)(T-t)\right) e^{-(M_\infty+V_L)(T-t)} \\ & - Q_\infty \left(1 + (M_\infty+V_1)t\right)(T-t)\, e^{(M_\infty+V_L)t} \\ = & \alpha_{\mathrm{TV}}(t)\,,\end{aligned}$$

completing the proof of (2.4). □

Proof of Proposition 3.8. We define the mapping P^w by (5.9). That this defines a process is a consequence of Lemma 5.2.

It remains to show the three Lipschitz continuity estimates (2.11), (2.12), and (2.13).

1. Lipschitz continuity w.r.t initial data By the linear structure of (3.8), from (5.11) we immediately have

$$\left\|P^w(t,t_o)(u_o - \bar{u}_o)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \leq e^{M_\infty(t-t_o)} \left\|u_o - \bar{u}_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}$$

which is compatible with the choice of C_u in (3.9).

2. Lipschitz continuity in time By direct computations based on (5.6), for $t \geq t_o$:

$$\left\|P^w(t,t_o)u_o - u_o\right\|_{\mathbf{L}^1(\mathbb{R}_+;\mathbb{R})}$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^n} \left| u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x) \right| \mathcal{E}_w(t_o,t,x)\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^n} \int_{t_o}^{t} \left| q\left(\tau, \mathcal{X}(\tau;t,x), w\right) \right| \mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} \left| u_o(x) \right| \left| \mathcal{E}_w(t_o,t,x) - 1 \right| \mathrm{d}x
\end{aligned}$$

and we consider the latter three terms separately. First, use (5.10) and Lemma 5.1, for $t \ge t_o$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x) \right| \mathcal{E}_w(t_o,t,x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \left| u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x) \right| \mathrm{d}x \; e^{(M_\infty + V_L)(t-t_o)} \\
&\le \frac{V_\infty}{V_L} \left(e^{V_L |t-t_o|} - 1 \right) \mathrm{TV}(u_o)\, e^{(M_\infty + V_L)(t-t_o)} \\
&\le V_\infty \; \mathrm{TV}(u_o)\, e^{(M_\infty + 2V_L)(t-t_o)}\, (t-t_o)\,.
\end{aligned}$$

To deal with the second term, use the change of coordinates (5.7) and **(IP2)**–**(IP3)**:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{t_o}^{t} \left| q\left(\tau, \mathcal{X}(\tau;t,x), w\right) \right| \mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{t_o}^{t} \left| q(\tau,y,w) \right| \exp\left(\int_{\tau}^{t} m\left(s, \mathcal{X}(s;\tau,y), w\right) \mathrm{d}s \right) \mathrm{d}\tau\,\mathrm{d}y \\
&\le Q_1\, e^{M_\infty (t-t_o)}\, (t-t_o)\,.
\end{aligned}$$

Finally, the third term is treated as follows, by (5.10):

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| u_o(x) \right| \left| \mathcal{E}_w(t_o,t,x) - 1 \right| \mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \left| u_o(x) \right| e^{(M_\infty + V_L)(t-t_o)} (M_\infty + V_L)(t-t_o)\,\mathrm{d}x \\
&\le (M_\infty + V_L)\, \|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\, e^{(M_\infty + V_L)(t-t_o)} (t-t_o)\,.
\end{aligned}$$

Adding up, we have

$$\begin{aligned}
\left\| P^w(t,t_o)u_o - u_o \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \le{} & V_\infty \; \mathrm{TV}(u_o)\, e^{(M_\infty + 2V_L)(t-t_o)}\, (t-t_o) \\
& + Q_1\, e^{M_\infty (t-t_o)}\, (t-t_o) \\
& + (M_\infty + V_L)\, \|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\, e^{(M_\infty + V_L)(t-t_o)} (t-t_o),
\end{aligned}$$

which agrees with the choice of C_t in (3.10).

3. Lipschitz continuity w.r.t parameters From [12, **(H5)**], using **(IP1)**, **(IP2)**, and **(IP3)**,

$$\left\| P_w(t,t_o)u_o - P_{\bar{w}}(t,t_o)u_o \right\|_{\mathbf{L^1}(\mathbb{R}^n:\mathbb{R})}$$

$$
\begin{aligned}
&\le \int_{t_o}^{t} \left\| v(\tau,\cdot,w_1) - v(\tau,\cdot,w_2) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \mathrm{d}\tau \\
&\quad \times \Bigg[\left\| u_o \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} + \mathrm{TV}(u_o) \\
&\qquad + \int_{t_o}^{t} \left(\max_{\omega=w_1,w_2} \left\| q(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \max_{\omega=w_1,w_2} \mathrm{TV}(q(\tau,\cdot,\omega)) \right) \mathrm{d}\tau \Bigg] \\
&\quad \times \exp\Bigg(\int_{t_o}^{t} \Big(\max_{\omega=w_1,w_2} \left\| m(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \\
&\qquad\qquad + \max_{\omega=w_1,w_2} \left\| \nabla v(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \Big) \mathrm{d}\tau \Bigg) \\
&\quad \times \left[1 + \int_{t_o}^{t} \max_{\omega=w_1,w_2} \left(\left\| \nabla\nabla\cdot v(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} + \mathrm{TV}(m(\tau,\cdot,\omega)) \right) \mathrm{d}\tau \right] \\
&\quad + \Bigg[\int_{t_o}^{t} \left\| q(\tau,\cdot,w) - q(\tau,\cdot,\bar w) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \\
&\qquad + \int_{t_o}^{t} \Big(\left\| m(\tau,\cdot,w) - m(\tau,\cdot,\bar w) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\
&\qquad\qquad + \left\| \nabla\cdot \left(v(\tau,\cdot,w) - v(\tau,\cdot,\bar w) \right) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \Big) \mathrm{d}\tau \\
&\qquad \times \left(\left\| u_o \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^{t} \max_{\omega=w,\bar w} \left\| q(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \right) \Bigg] \\
&\quad \times \exp\left(\int_{t_o}^{t} \max_{\omega=w,\bar w} \left\| m(\tau,\cdot,\omega) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \mathrm{d}\tau \right) \\
&\le \Big[V_L(2R+Q_\infty)\left(1+(V_1+M_\infty)(t-t_o)\right) \\
&\quad + (Q_L + (M_L+V_L)\left(R+Q_\infty(t-t_o)\right)) \Big] e^{(M_\infty+V_L)(t-t_o)} (t-t_o) d_{\mathcal{W}}(w_1,w_2) \\
&\le \Big[V_L(2R+Q_\infty)(1+(V_1+M_\infty)\hat T) \\
&\quad + (Q_L + (M_L+V_L)(R+Q_\infty \hat T)) \Big] e^{(M_\infty+V_L)\hat T} (t-t_o) d_{\mathcal{W}}(w_1,w_2),
\end{aligned}
$$

in agreement with the choice of C_w in (3.11).

Choice of T. The time T has to be chosen so that $\alpha_1(0) > 0$, $\alpha_\infty(0) > 0$ and $\alpha_{\mathrm{TV}}(0) > 0$. Clearly, by (3.13), for T sufficiently small, these requirements are all met. □

Proof of Corollary 3.9. Note that the constants defined in **(IP1)**, **(IP2)**, and **(IP3)** do not depend on R. Moreover T has to be chosen such that

$\alpha_1(0) > 0$, $\alpha_\infty(0) > 0$ and $\alpha_{\rm TV}(0) > 0$, which are equivalent to

$$\left\{\begin{array}{l} Re^{-M_\infty T} - Q_1 T > 0 \\ Re^{-(M_\infty + V_L)T} - Q_\infty T > 0 \\ Re^{-(M_\infty + V_L)T}\left(1 - (M_\infty + V_1)\, T\right) - Q_\infty T > 0. \end{array}\right.$$

The proof ends setting $T = \min\left\{\frac{1}{2(M_\infty + V_1)}, \frac{\ln(2)}{M_\infty + V_L}\right\}$, provided R is sufficiently big. □

Proof of Proposition 3.10. The Lipschitz continuity of P ensured by Theorem 2.6 shows that P_1 is $\mathbf{L^1}$–Lipschitz continuous, and hence it lies in $\mathbf{C^0}([t_o, T]; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}))$ as required.

We focus our attention now on the first item in Definition 3.7, the second being immediate. To ease reading, for any test function $\varphi \in \mathbf{C_c^\infty}(]t_o, T[\times\mathbb{R}^n; \mathbb{R})$ we introduce the notation

$$\mathcal{I}_\varphi(u, w) = u\,\partial_t\varphi + u\,v\cdot\nabla_x\varphi + \left(m(\cdot,\cdot,w)\,u + q(\cdot,\cdot,w)\right)\varphi. \tag{5.13}$$

We want to prove that, for any $\varphi \in \mathbf{C_c^\infty}(]t_o, T[\,\times\mathbb{R}^n; \mathbb{R})$,

$$\int_{\mathbb{R}^n}\int_{t_o}^{T} \mathcal{I}_\varphi\left(P(t, t_o)(u_o, w_o)\right) \mathrm{d}t\ \mathrm{d}x = 0.$$

We begin by discretising the time domain. For a given $k \in \mathbb{N}\setminus\{0\}$ and $i = 0, \ldots, k$, introduce $t_i = t_o + i(T - t_o)/k$ and $(\tilde u_i, \tilde w_i) = P(t_{i-1}, t_o)(u_o, w_o)$. Splitting the integral then gives

$$\begin{aligned} &\int_{t_o}^{T}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(P(t, t_o)(u_o, w_o)\right) \mathrm{d}x\ \mathrm{d}t \\ &= \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big(\mathcal{I}_\varphi\left(P(t, t_{i-1})(\tilde u_i, \tilde w_i)\right) - \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right)\Big) \mathrm{d}x\ \mathrm{d}t \\ &\quad + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right) \mathrm{d}x\ \mathrm{d}t\ . \end{aligned} \tag{5.14}$$

We compute the terms on the last two lines separately, our goal is to show that they both converge to zero as $k \to \infty$.

For the first,

$$\begin{aligned} &\mathcal{I}_\varphi\left(P(t, t_{i-1})(\tilde u_i, \tilde w_i)\right) - \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right) \\ &= \partial_t\varphi\left(P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right) && (5.15) \\ &\quad + \Big(P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) v(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i)) \\ &\qquad - F_1(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i) v(t, x, F_2(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i))\Big)\cdot\nabla_x\varphi && (5.16) \\ &\quad + \Big(m\left(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i)\right) P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) && (5.17) \end{aligned}$$

$$- m\left(t, x, F_2(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right) F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\Big)\varphi \tag{5.18}$$

$$+ \Big(q\left(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i)\right) - q\left(t, x, F_2(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right)\Big)\varphi. \tag{5.19}$$

Recall that the tangency condition (2.10) ensures

$$\frac{1}{t-t_{i-1}}\left\|P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \le I_1(t)$$
$$\frac{1}{t-t_{i-1}} d_{\mathcal{W}}\left(P_2(t, t_{i-1})(\tilde u_i, \tilde w_i), F_2(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right) \le I_1(t)$$

with

$$I_1(t) = \frac{L}{\ln(2)}\int_{t_{i-1}}^{t} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,,$$

with L and ω defined as in (2.15), so that, considering (5.15),

$$\left|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} (\partial_t\varphi\ )\left(P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right)\mathrm{d}x\ \mathrm{d}t\right|$$
$$\le \|\partial_t\varphi\ \|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R})}$$
$$\times\int_{t_{i-1}}^{t_i}\left\|P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\mathrm{d}t \tag{5.20}$$
$$\le \frac{L}{\ln(2)}\|\partial_t\varphi\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R})}\,(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,. \tag{5.21}$$

Considering the next term (5.16),

$$\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big[P_1(t, t_{i-1})(\tilde u_i, \tilde w_i)v(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i))$$
$$- F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)v(t, x, F_2(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i))\Big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big[P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\Big] \tag{5.22}$$
$$\times v(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i))\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x$$
$$+ \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i) \tag{5.23}$$
$$\times\Big[v(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i)) - v(t, x, F_2(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i))\Big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\,.$$

For (5.22), using **(IP1)** and the same approach as for (5.21), we get

$$\Bigg|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big[P_1(t, t_{i-1})(\tilde u_i, \tilde w_i) - F_1(t-t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)\Big]$$
$$\times v(t, x, P_2(t, t_{i-1})(\tilde u_i, \tilde w_i))\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\Bigg|$$

$$\leq \frac{L}{\ln(2)} V_\infty \left\|\nabla_x\varphi\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R}^n)} (t_i - t_{i-1})^2 \int_0^{t_i - t_{i-1}} \frac{\omega(\xi)}{\xi} \,\mathrm{d}\xi\,. \tag{5.24}$$

For the second term (5.23), using **(IP1)** again, we have,

$$\begin{aligned}
&\Big|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} F_1(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\\
&\times\Big[v(t,x,P_2(t,t_{i-1})(\tilde u_i,\tilde w_i)) - v(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i))\Big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\Big|\\
&\leq \int_{t_{i-1}}^{t_i} \left\|F_1(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\|\nabla_x\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\\
&\qquad\times V_L d_{\mathcal{W}}\left(P_2(t,t_{i-1})(\tilde u_i,\tilde w_i),F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)\mathrm{d}t\,\mathrm{d}x\\
&\leq \frac{L}{\ln(2)} R\|\nabla_x\varphi\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R}^n)} V_L (t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.
\end{aligned}\tag{5.25}$$

Pass to (5.17)–(5.18) and using again (5.15):

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big|\Big(m\left(t,x,P_2(t,t_{i-1})(\tilde u_i,\tilde w_i)\right)P_1(t,t_{i-1})(\tilde u_i,\tilde w_i)\\
&\qquad -m\left(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)F_1(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\Big)\varphi\Big|\,\mathrm{d}x\,\mathrm{d}t\\
\leq\;& \int_{t_{i-1}}^{t_i}\left\|m\left(t,\cdot,P_2(t,t_{i-1})(\tilde u_i,\tilde w_i)\right)-m\left(t,\cdot,F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&\qquad\times\left\|P_1(t,t_{i-1})(\tilde u_i,\tilde w_i)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}t\\
&+\int_{t_{i-1}}^{t_i}\left\|m\left(t,\cdot,F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\\
&\qquad\times\left\|P_1(t,t_{i-1})(\tilde u_i,\tilde w_i)-F_1(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}t\\
\leq\;& M_L\,R\,\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\int_{t_{i-1}}^{t_i} d_{\mathcal{W}}\left(P_2(t,t_{i-1})(\tilde u_i,\tilde w_i),F_2(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)\mathrm{d}t\\
&+M_\infty\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\int_{t_{i-1}}^{t_i}\left\|P_1(t,t_{i-1})(\tilde u_i,\tilde w_i)-F_1(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\mathrm{d}t\\
\leq\;& \frac{L}{\ln(2)}\left(M_L\,R+M_\infty\right)\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;.
\end{aligned}\tag{5.26}$$

Concerning (5.19), the tangency condition (2.10) implies

$$\begin{aligned}
&\left|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}[q(t,x,P_2(t,t_{i-1})(\tilde u,\tilde w))-q(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde u,\tilde w))]\varphi(t)\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\leq Q_L\|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n)}\int_{t_{i-1}}^{t_i} d_{\mathcal{W}}\left(P_2(t,t_{i-1})(\tilde u,\tilde w_i),F_2(t-t_{i-1},t_{i-1})(\tilde u,\tilde w_i)\right)\mathrm{d}t\\
&\leq\frac{L}{\ln(2)}\,Q_L\,\|\varphi\|_{L^\infty([t_o,T]\times\mathbb{R}^n)}(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;.
\end{aligned}\tag{5.27}$$

Computing the sum over all time intervals, we get:

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \Big(\mathcal{I}_\varphi\left(P(t,t_{i-1})(\tilde u_i, \tilde w_i)\right) - \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right)\Big)\, \mathrm{d}x\, \mathrm{d}t \\
&\le \sum_{i=1}^{k} [(5.21)] + [(5.24)] + [(5.25)] + [(5.26)] + [(5.27)] \\
&\le \frac{L}{\ln(2)}\, \mathcal{C} \int_0^{(T-t_o)/k} \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi \, \sum_{i=1}^{k} (t_i - t_{i-1})^2 \\
&= \frac{L}{\ln(2)}\, \mathcal{C} \int_0^{(T-t_o)/k} \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi \, \frac{(T-t_o)^2}{k} \\
&\underset{k\to+\infty}{\longrightarrow} 0\,,
\end{aligned}
$$

where $\mathcal{C}$ depends on the test function φ and the constants from **(IP1)**-**(IP2)**-**(IP3)**.

Pass now to estimate (5.14). Temporarily, for $i = 0, \ldots, k$, we define $(u_i(t), w_i(t)) = F(t - t_{i-1}, t_{i-1})(\tilde u_i, \tilde w_i)$. Then $u_i(t) = P^{\tilde w_i}(t, t_{i-1})\tilde u_i$, and thus it satisfies

$$
\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\psi(u_i(t), \tilde w_i)\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall\, \psi \in \mathbf{C}^\infty_{\mathbf{c}}(]t_{i-1}, t_i[\, \times \mathbb{R}^n; \mathbb{R})\,. \tag{5.28}
$$

Then, each summand in (5.14) can be estimated as follows:

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde u_i,\tilde w_i)\right) \mathrm{d}x\, \mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(u_i(t), \tilde w_i\right) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \Big[\left(m(t,x,\tilde w_i) - m(t,x,w_i(t))\right) u_i(t) \\
&\qquad\qquad + \left(q(t,x,\tilde w_i) - q(t,x,w_i(t))\right) \Big] \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} u_i(t) \left(v(t,x,w_i(t)) - v(t,x,\tilde w_i)\right) \cdot \nabla_x \varphi\, \mathrm{d}x\, \mathrm{d}t \\
&\le \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(u_i(t), \tilde w_i\right) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)} \int_{t_{i-1}}^{t_i} (M_L\, R + Q_L)\, d_{\mathcal{W}}\left(\tilde w_i, w_i(t)\right) \mathrm{d}t \\
&\quad + \|\nabla_x \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)} \int_{t_{i-1}}^{t_i} V_L R\, d_{\mathcal{W}}\left(\tilde w_i, w_i(t)\right) \mathrm{d}t \\
&\le \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(u_i(t), \tilde w_i\right) \mathrm{d}x\, \mathrm{d}t
\end{aligned}
$$

$$+ \|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)} \frac{1}{2}(M_L\, R + Q_L)\,\mathcal{C}\,(t_i - t_{i-1})^2$$
$$+ \|\nabla_x\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)} \frac{1}{2}\, V_L\, R\,\mathcal{C}\,(t_i - t_{i-1})^2\,, \tag{5.29}$$

where $\mathcal{C}$ is the Lipschitz constant of $t \mapsto w(t)$ and we used the equality $w(t_{i-1}) = \tilde{w}_i$. The latter two summands in (5.29) are treated as the terms above.

Concerning the first summand, consider $\chi_\varepsilon \in \mathbf{C}^\infty_\mathbf{c}(]t_{i-1}, t_i[; [0,1])$ satisfying $\chi_\varepsilon(t) = 1$, for $t \in\,]t_{i-1}+\varepsilon, t_i - \varepsilon[$, and define $\varphi_\varepsilon = \varphi \cdot \chi_\varepsilon$. Then,

$$\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\big(u_i(t),\tilde{w}_i\big)\,\mathrm{d}x\,\mathrm{d}t = \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_{\varphi-\varphi_\varepsilon}\big(u_i(t),\tilde{w}_i\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_{\varphi_\varepsilon}\big(u_i(t),\tilde{w}_i\big)\,\mathrm{d}x\,\mathrm{d}t\ .$$

The second term here vanishes, by (5.28). We then have

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_{\varphi-\varphi_\varepsilon}\big(u_i(t),\tilde{w}_i\big)\,\mathrm{d}x\,\mathrm{d}t\\ =\ &\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Bigg[u_i\,\partial_t(\varphi-\varphi_\varepsilon) + u_i\, v(t,x,\tilde{w}_i)\cdot\nabla_x(\varphi(t,x)-\varphi_\varepsilon(t,x))\\ &\qquad + \big(m(t,x,\tilde{w}_i)\,u_i + q(t,x,\tilde{w}_i)\big)\,(\varphi(t,x)-\varphi_\varepsilon(t,x))\Bigg]\,\mathrm{d}x\,\mathrm{d}t\ .\end{aligned}$$

Via a use of the Dominated Convergence Theorem, the last two terms here tend to zero as $\varepsilon \to 0$, since $\chi_\varepsilon \to 1$ a.e. on $[t_{i-1}, t_i]$. For the first term, by the construction of χ_ε and the $\mathbf{L^1}$ continuity in time of u_i,

$$\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} u_i\,\partial_t(\varphi-\varphi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^n}\big(u_i(t_i,x)\,\varphi(t_i,x) - u_i(t_{i-1},x)\,\varphi(t_{i-1},x)\big)\,\mathrm{d}x\,\mathrm{d}t\ .$$

Passing to the sum (5.14), and remembering that

$$u_i(t_{i-1},x) = \tilde{u}_i = P_1(t_{i-1},t_o)(u_o,w_o),$$

we get

$$\begin{aligned}&\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\big(u_i(t),\tilde{w}_i\big)\,\mathrm{d}x\,\mathrm{d}t\\ &= \sum_{i=1}^{k-1}\int_{\mathbb{R}}\Big[F_1(t_i - t_{i-1}, t_{i-1})P(t_{i-1},t_o)(u_o,w_o) - P_1(t_i,t_o)(u_o,w_o)\Big]\\ &\qquad\times\varphi(t_i,x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{k-1} (t_i - t_{i-1}) \frac{2L}{\ln(2)} \int_0^{t_i - t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \, \left\| \varphi(t_i) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \\
&\leq \frac{2L}{\ln(2)} \, \|\varphi\|_{\mathbf{L}^\infty([t_o,T];\mathbb{R}^n;\mathbb{R})} \, (T - t_o) \, \int_0^{(T-t_o)/k} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \\
&\underset{k\to+\infty}{\longrightarrow} 0,
\end{aligned}
$$

as required. □

5.4 Proofs for § 3.3

Similar to the previous sections, for each $w \in \mathcal{W}$ the unique solution to (3.14) in the sense of Definition 3.11 is

$$
u(t,x) = \begin{cases}
u_o\left(\mathcal{X}(t_o;t,x)\right) \, \mathcal{E}_w(t_o,t,x) & \\
\quad + \displaystyle\int_{t_o}^{t} q\left(\tau, \mathcal{X}(\tau;t,x), w\right) \, \mathcal{E}_w(\tau,t,x) \, \mathrm{d}\tau & x \geq \mathcal{X}(t;t_o,0) \\
b\left(\mathcal{T}(0;t,x)\right) \, \mathcal{E}_w\left(\mathcal{T}(0;t,x),t,x\right) & \\
\quad + \displaystyle\int_{\mathcal{T}(0;t,x)}^{t} q\left(\tau, \mathcal{X}(\tau;t,x), w\right) \, \mathcal{E}_w(\tau,t,x) \, \mathrm{d}\tau & x < \mathcal{X}(t;t_o,0)
\end{cases}
\tag{5.30}
$$

where now

$$
\mathcal{E}_w(\tau,t,x) = \exp \int_\tau^t \Big(m\left(s, \mathcal{X}(s;t,x), w\right) - \partial_x v\left(s, \mathcal{X}(s;t,x)\right) \Big) \, \mathrm{d}s \, . \tag{5.31}
$$

Working under the assumptions of Proposition 3.12, we define the parametrised mapping P^w, which we propose is a process, by

$$
\begin{array}{rccl}
P^w : & \mathcal{A} & \to & \mathcal{U} \\
& (t,t_o,u_o) & \mapsto & u(t)
\end{array}
\qquad \text{where} \qquad u(t) \text{ is given by (5.30);} \tag{5.32}
$$

where $\mathcal{A}$ is generated by the sets $\mathcal{D}_t$ as given by (3.16).

Lemma 5.3. *The mapping P^w as defined in* (5.32) *is a process in the sense of Definition* 2.2.

Proof of Lemma 5.3. Fix $w \in \mathcal{W}$. Conditions (2.3) and (2.5) are an immediate consequence of (5.32). It remains to show (2.4). As the choice of $w \in \mathcal{W}$ has no impact on this result, we omit references to w.

Define $\sigma(t) = X(t;t_o,0)$, and for a fixed $t \in I$, $J_1 = [0,\sigma(t)[$, and $J_2 = [\sigma(t),+\infty[$.

1. We first show that, if $\|u_o\|_{\mathbf{L}^1(\mathbb{R}_+;\mathbb{R})} \leq \alpha_1(t_o)$, then $\left\|u(t)\right\|_{\mathbf{L}^1(\mathbb{R}_+;\mathbb{R})} \leq \alpha_1(t)$.

To begin, we have

$$
\begin{aligned}
\left\|u(t)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq{} & \int_0^{\sigma(t)} \left|b(\mathcal{T}(0;t,x))\, \mathcal{E}(\mathcal{T}(0;t,x),t,x)\right| \mathrm{d}x \\
& + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t \left|q(\tau,\mathcal{X}(\tau;t,x))\, \mathcal{E}(\tau,t,x)\right| \mathrm{d}\tau\, \mathrm{d}x \\
& + \int_{\sigma(t)}^{+\infty} \left|u_o(\mathcal{X}(t_o;t,x))\mathcal{E}(t_o,t,x)\right| \mathrm{d}x \\
& + \int_{\sigma(t)}^{+\infty} \int_{t_o}^t \left|q(\tau,\mathcal{X}(\tau;t,x))\mathcal{E}(\tau;t,x)\right| \mathrm{d}\tau\, \mathrm{d}x \\
={} & \int_{t_o}^t \left|v(\eta,0)\right| \left|b(\eta)\right| \exp \int_\eta^t m(s,\mathcal{X}(s;0,\eta))\ \ \mathrm{d}s\, \mathrm{d}\eta \\
& + \int_{t_o}^t \int_0^{\sigma(\tau)} \left|q(\tau,\xi)\right| \exp \int_\tau^t m(s,\mathcal{X}(s;t,0))\, \mathrm{d}s\, \mathrm{d}\xi\, \mathrm{d}\tau \\
& + \int_0^{+\infty} \left|u_o(\xi)\right| \exp \int_{t_o}^t m(s,\mathcal{X}(s;t_o,\xi))\, \mathrm{d}s\, \mathrm{d}\xi \\
& + \int_{t_o}^t \int_{\sigma(t)}^{+\infty} \left|q(\tau,\xi)\right| \exp \int_\tau^t m(s,\mathcal{X}(s;\tau,\xi))\, \mathrm{d}s\, \mathrm{d}\xi\, \mathrm{d}\tau \\
\leq{} & \Big(\left\|u_o\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} + (\hat{v}B_\infty + Q_1)(t-t_o)\Big)\, e^{M_\infty(t-t_o)}.
\end{aligned}
\tag{5.33}
$$

Inserting the fact that $\left\|u_o\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \alpha_1(t_o)$ into (5.33), we have

$$
\begin{aligned}
\left\|u(t)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\leq \Big(\left\|u_o\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} + (\hat{v}B_\infty + Q_1)(t-t_o)\Big)\, e^{M_\infty(t-t_o)} \\
&\leq \Big(Re^{-M_\infty(T-t_o)} - (\hat{v}B_\infty + Q_1)(T-t_o)e^{M_\infty t_o} \\
&\qquad + (\hat{v}B_\infty + Q_1)(t-t_o)\Big)e^{M_\infty(t-t_o)} \\
&\leq Re^{-M_\infty(T-t)} - (\hat{v}B_\infty + Q_1)(T-t)e^{M_\infty t} \\
&= \alpha_1(t)
\end{aligned}
$$

2. We now show that if $\left\|u_o\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \leq \alpha_\infty(t_o)$ and $B_\infty \leq \alpha_\infty(t_o)$, then $\left\|u(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \leq \alpha_\infty(t)$.

We have, directly from (5.30),

$$
\begin{aligned}
\left\|u(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} &\leq \Big(\max\Big\{\left\|u_o\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}, B_\infty\Big\} + Q_\infty(t-t_o)\Big)\, e^{M_\infty(t-t_o)} \\
&\leq \left(\alpha_\infty(t_o) + Q_\infty(t-t_o)\right) e^{M_\infty(t-t_o)} \\
&\leq \Big(Re^{-M_\infty(T-t_o)} - Q_\infty(T-t_o) + Q_\infty(t-t_o)\Big)\, e^{M_\infty(t-t_o)} \\
&\leq Re^{-M_\infty(T-t)} - Q_\infty(T-t) \\
&= \alpha_\infty(t).
\end{aligned}
$$

3. Finally, we demonstrate that if $\mathrm{TV}(u_o) + \left|u_o(0) - b(t_o)\right| \leq \alpha_{TV}(t_o)$, then $\mathrm{TV}(u) + \left|u(t,0) - b(t)\right| \leq \alpha_{TV}(t)$.

The left continuity of b implies the right continuity of $u(t,\cdot)$ at 0, and hence

$$\begin{aligned}
\mathrm{TV}\left(u(t)\right) &= \mathrm{TV}\left(u(t);]0,+\infty[\right) \\
&\leq \mathrm{TV}\left(u(t);]0,\sigma(t)[\right) && (5.34)\\
&\quad + \left|u(t,\sigma(t)-) - u(t,\sigma(t)+)\right| && (5.35)\\
&\quad + \mathrm{TV}\left(u(t);]\sigma(t),+\infty[\right) . && (5.36)
\end{aligned}$$

We calculate the three terms (5.34), (5.35) and (5.36) separately.

Beginning with (5.34), we have

$$\begin{aligned}
\mathrm{TV}\left(u(t);]0,\sigma(t)[\right) \leq\; & \mathrm{TV}\left(b(\mathcal{T}(0;t,x))\mathcal{E}(\mathcal{T}(0;t,x),t,x);]0,\sigma(t)[\right) \\
& + \mathrm{TV}\left(\int_{\mathcal{T}(0;t,x)}^{t} q(\tau,\mathcal{X}(\tau;t,x))\mathcal{E}(\tau,t,x)\,\mathrm{d}\tau\,;]0,\sigma(t)[\right) \\
\leq\; & \left(\mathrm{TV}(b;]t_o,t[) + \|b\|_{\mathbf{L}^\infty([t_o,t];\mathbb{R})}(M_\infty + V_L)(t-t_o)\right) \\
& \times e^{(M_\infty+V_L)(t-t_o)} \\
& + Q_\infty(t-t_o)(1+(M_\infty+V_L)(t-t_o))e^{(M_\infty+V_L)(t-t_o)}
\end{aligned}$$

For the second term (5.35),

$$\begin{aligned}
&\left|u\left(t,\sigma(t)+\right) - u\left(t,\sigma(t)-\right)\right| \\
\leq &\left|u_o\left(\mathcal{X}\left(t_o;t,\sigma(t)+\right)\right)\mathcal{E}\left(t_o,t,\sigma(t)+\right)\right. \\
&\qquad \left. - b\left(\mathcal{T}\left(0;t,\sigma(t)-\right)\right)\mathcal{E}\left(\mathcal{T}\left(0;t,\sigma(t)-\right),t,\sigma(t)-\right)\right| \\
&+\left|\int_{t_o}^{t} q\left(\tau,\mathcal{X}\left(\tau;t,\sigma(t)+\right),w\right)\mathcal{E}\left(\tau,t,\sigma(t)+\right)\mathrm{d}\tau\right. \\
&\qquad \left. - \int_{\mathcal{T}(0;t,\sigma(t)-)}^{t} q\left(\tau,\mathcal{X}\left(\tau;t,\sigma(t)-\right),w\right)\mathcal{E}\left(\tau,t,\sigma(t)-\right)\mathrm{d}\tau\right| \\
= &\left|u(t_o,0+) - b(t_o+)\right|\mathcal{E}\left(t_o,t,\sigma(t)-\right) \\
\leq &\left(\left|u(t_o,0) - b(t_o)\right| + \left|b(t_o-) - b(t_o+)\right|\right)e^{(M_\infty+V_L)(t-t_o)} .
\end{aligned}$$

Note that $\mathrm{TV}\left(b;]t_o,t[\right) + \left|b(t_o-) - b(t_o+)\right| = \mathrm{TV}(b;[t_o,t[)$ from the left continuity of b.

For the final term (5.36), we find

$$\begin{aligned}
\mathrm{TV}\left(u(t);]\sigma(t),+\infty[\right) \leq \Big(&\mathrm{TV}(u_o;]0,+\infty[) + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}(M_\infty + V_L)(t-t_o) \\
&+ Q_\infty(1+(M_\infty+V_L)(t-t_o))(t-t_o)\Big)e^{(M_\infty+V_L)(t-t_o)}.
\end{aligned}$$

Finally, notice that, as $u(t,0) = b(t)$, we have

$$\mathrm{TV}\left(u(t,\cdot);]0,+\infty[\right) + \left|u(t,0) - b(t)\right| = \mathrm{TV}\left(u(t,\cdot)\right),$$

and thus we need only to show $\mathrm{TV}\left(u(t,\cdot)\right) \le \alpha_{TV}(t)$. Using these three estimates, we obtain

$$\begin{aligned}
\mathrm{TV}\left(u(t)\right) \le & \Big(\mathrm{TV}\left(u_o;]0,+\infty[\right) + \left|u(t,0) - b(t_o)\right| \\
& + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}(M_\infty + V_L)(t - t_o) \\
& + \mathrm{TV}(b; [t_o, t[) + B_\infty(M_\infty + V_L)(t - t_o) \\
& + 2Q_\infty(1 + (M_\infty + V_L)(t - t_o))(t - t_o)\Big) e^{(M_\infty+V_L)(t-t_o)} \\
\le & \Big(\alpha_{TV}(t_o) + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}(M_\infty + V_L)(t - t_o) \\
& + \mathrm{TV}(b; [t_o, t[) + B_\infty(M_\infty + V_L)(t - t_o) \\
& + 2Q_\infty(1 + (M_\infty + V_L)(t - t_o))(t - t_o)\Big) e^{(M_\infty+V_L)(t-t_o)} \\
\le & \Big(R(1 - 2(M_\infty + V_L)(T - t_o))e^{(M_\infty+V_L)(T-t_o)} \\
& - 2Q_\infty(1 + (M_\infty + V_L)t_o)(T - t_o)e^{(M_\infty+V_L)t_o} \\
& - B_\infty(M_\infty + V_L)(T - t_o)e^{(M_\infty+V_L)t_o} \\
& - \mathrm{TV}(b; [t_o, T])e^{(M_\infty+V_L)t_o} \\
& + \Big(Re^{-(M_\infty+V_L)(T-t_o)} - Q_\infty(T - t)e^{(M_\infty+V_L)t}\Big) \\
& \quad \times (M_\infty + V_L)(t - t_o) \\
& + \mathrm{TV}(b; [t_o, t[) + B_\infty(M_\infty + V_L)(t - t_o) \\
& + 2Q_\infty(1 + (M_\infty + V_L)(t - t_o))(t - t_o)\Big) e^{(M_\infty+V_L)(t-t_o)} \\
\le & \ R(1 - (M_\infty + V_L)(T - t))e^{(M_\infty+V_L)(T-t)} \\
& - 2Q_\infty(1 + (M_\infty + V_L)t)(T - t)e^{(M_\infty+V_L)t} \\
& - B_\infty(M_\infty + V_L)(T - t)e^{(M_\infty+V_L)t} \\
& - \mathrm{TV}(b; [t, T])e^{(M_\infty+V_L)t} \\
= & \ \alpha_{TV}(t),
\end{aligned}$$

as required. □

Proof of Proposition 3.12. The mapping P^w, as given by (5.32), is a process for any $w \in \mathcal{W}$ by Lemma 5.3. It remains to show that P^w is a Lipschitz process on $\mathcal{U}$ parametrised by $w \in \mathcal{W}$, i.e., it satisfies (2.11), (2.12), and (2.13), with C_u, C_t and C_w given by (3.16).

1. Lipschitz Continuity w.r.t. Initial Data. Consider two initial data $u_1, u_2 \in \mathcal{D}$, $t_o, t \in I$ with $t_o < t$, and $w \in \mathcal{W}$.

To begin, assume that $x \in [0, \sigma(t)[$. Then, it is easy to see from (5.30) that

$$\left|P^w(t,t_o)u_1 - P^w(t,t_o)u_2\right|(x) = 0\,,$$

as b, q and m are independent of the choice of initial data u_o. Similarly, for $x \in [\sigma(t), +\infty[$,

$$\left|P^w(t,t_o)u_1 - P^w(t,t_o)u_2\right|(x) = \left|u_1(\mathcal{X}(t_o;t,x)) - u_2(\mathcal{X}(t_o;t,x))\right| \mathcal{E}_w(t_o,t,x)\,.$$

Thus, using the substitution $y = \mathcal{X}(t_o;t,x)$,

$$\begin{aligned}
& d_{\mathcal{U}}\left(P^w(t,t_o)u_1, P^w(t,t_o)u_2\right) \\
&= \int_{\sigma(t)}^{+\infty} \left|u_1(\mathcal{X}(t_o;t,x)) - u_2(\mathcal{X}(t_o;t,x))\right|\mathcal{E}_w(t_o,t,x)\,\mathrm{d}x \\
&= \int_0^{+\infty} \left|u_1(y) - u_2(y)\right| e^{\int_{t_o}^t m(s,\mathcal{X}(s;t_o,y),w)\mathrm{d}s}\,\mathrm{d}y \\
&\le e^{M_\infty(t-t_o)} \left\|u_1(0) - u_2(0)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})}\,.
\end{aligned}$$

2. Lipschitz Continuity w.r.t. Time. Consider $u_o \in \mathcal{D}$, $t_o, t \in I$, and $w \in \mathcal{W}$.

We have

$$\begin{aligned}
d_{\mathcal{U}}(P^w(t,t_o)u_o, u_o) \quad \le \quad & \left\|P^w(t,t_o)u_o - u_o\right\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\
& + \left\|P^w(t,t_o)u_o - u_o\right\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)}\,.
\end{aligned} \tag{5.37}$$

Focusing on the first term of (5.37), using (5.30), **(BP1)**, **(BP2)**, **(BP3)**, **(BP4)**, and that $u_o \in \mathcal{D}$,

$$\begin{aligned}
& \left\|P^w(t,t_o)u_o - u_o\right\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\
&\le \int_0^{\sigma(t)} \left|b(\mathcal{T}(0;t,x))\mathcal{E}_w(\mathcal{T}(0;t,x),t,x) - u_o(x)\right| \mathrm{d}x \\
&\quad + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t \left|q(\tau, \mathcal{X}(\tau;t,x), w)\mathcal{E}_w(\tau,t,x)\right| \mathrm{d}\tau\,\mathrm{d}x \\
&= \int_{t_o}^t v(y,0)|b(y)e^{\int_y^t m(s,\mathcal{X}(s;y,0),w)\mathrm{d}s} - u_o(\mathcal{X}(t;0,y))e^{\int_y^t \partial_x v(s,\mathcal{X}(s;y,0))\mathrm{d}s}|\,\mathrm{d}y \\
&\quad + Q_1 e^{M_\infty(t-t_o)}(t-t_o) \\
&\le \hat{v}(B_1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} + Q_1)e^{M_\infty(t-t_o)}(t-t_o) \\
&\quad + \int_{t_o}^t v(y,0)|u_o(\mathcal{X}(t;0,y))||e^{\int_y^t m(s,\mathcal{X}(s;y,0),w)\mathrm{d}s} - e^{\int_y^t \partial_x v(s,\mathcal{X}(s;y,0))\mathrm{d}s}|\,\mathrm{d}y \\
&\le \hat{v}(B_1 + R + Q_1)e^{M_\infty T}(t-t_o) \\
&\quad + \hat{v}\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \int_{t_o}^t (M_\infty + V_L)(t-y)e^{(M_\infty+V_L)(t-y)}\,\mathrm{d}x \\
&\le \hat{v}(B_1 + R + Q_1)e^{M_\infty T}(t-t_o) + \hat{v}R(M_\infty + V_L)(t-t_o)^2 e^{(M_\infty+V_L)(t-t_o)}\,.
\end{aligned}$$

For the second term of (5.37), once again from (5.30),

$$\begin{aligned}
&\left\|P^w(t,t_o)u_o - u_o\right\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)}\\
&\le \int_{\sigma(t)}^{+\infty} \left|u_o(\mathcal{X}(t_o;t,x))\mathcal{E}_w(t_o,t,x) - u_o(x)\right| \mathrm{d}x\\
&\quad + \int_{\sigma(t)}^{+\infty}\int_{t_o}^{t} |q(\tau,\mathcal{X}(\tau;t,x),w)|\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x\\
&\le \int_{\sigma(t)}^{+\infty} \left|u_o(\mathcal{X}(t_o;t,x)) - u_o(x)\right|\mathcal{E}_w(t_o,t,x)\,\mathrm{d}x\\
&\quad + \int_{\sigma(t)}^{+\infty} |u_o(x)|\left|\mathcal{E}_w(t_o,t,x)-1\right|\mathrm{d}x\\
&\quad + \int_{t_o}^{t}\int_{\sigma(t)}^{+\infty} |q(\tau,\xi,w)|e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,\xi),w)\mathrm{d}s}\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\le \left[\hat v\,\mathrm{TV}(u_o;\mathbb{R}_+) + M_\infty\|u_o\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} + Q_1\right] e^{M_\infty(t-t_o)}(t-t_o)\\
&\le \left[\hat v R + M_\infty R + Q_1\right] e^{M_\infty(t-t_o)}(t-t_o)\,,
\end{aligned}$$

where we have made use of (A.4).

Concluding, we thus have

$$\begin{aligned}
&d_{\mathcal{U}}(P^w(t,t_o)u_o,u_o)\\
&\qquad\le \left[\hat v(B_1+2R+R(M_\infty+V_L)T) + M_\infty R + Q_1\right] e^{M_\infty T}(t-t_o).
\end{aligned}$$

3. Lipschitz Continuity w.r.t. Parameters. Consider $u_o \in \mathcal{D}$, $t_o, t \in I$ and $w_1, w_2 \in \mathcal{W}$.

We have

$$\begin{aligned}
&d_{\mathcal{U}}(P^{w_1}(t,t_o)u_o, P^{w_2}(t,t_o)u_o)\\
&\le \left\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)}\\
&\quad + \left\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)}
\end{aligned} \tag{5.38}$$

For the first term of (5.38),

$$\left\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)}$$

$$\le \int_0^{\sigma(t)} \left|b(\mathcal{T}(0;t,x))\right| \left|\mathcal{E}_{w_1}(\mathcal{T}(0;t,x),t,x) - \mathcal{E}_{w_2}(\mathcal{T}(0;t,x),t,x)\right| \mathrm{d}x \tag{5.39}$$

$$+ \int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t} |q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)|\,\mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}x \tag{5.40}$$

$$+ \int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t} \left|q(\tau,\mathcal{X}(\tau;t,x),w_2)\right| \left|\mathcal{E}_{w_2}(\tau,t,x) - \mathcal{E}_{w_1}(\tau,t,x)\right| \mathrm{d}x\,. \tag{5.41}$$

Focussing first on (5.39), we use **(BP2)**, and get

$$\int_0^{\sigma(t)} \left|b\left(\mathcal{T}(0;t,x)\right)\right| \left|\mathcal{E}_{w_1}(\mathcal{T}(0;t,x),t,x) - \mathcal{E}_{w_2}(\mathcal{T}(0;t,x),t,x)\right| \mathrm{d}x$$

$$
\begin{aligned}
&= \int_{t_o}^{t} v(y,0)\left|b(y)\right|\left|\mathcal{E}_{w_1}(y,t,\mathcal{X}(t;0,y)) - \mathcal{E}_{w_2}(y,t,\mathcal{X}(t;0,y))\right| \mathrm{d}y \\
&\le B_\infty e^{M_\infty(t-t_o)} \int_{t_o}^{t}\int_{y}^{t} v(y,0)\left|m(s,\mathcal{X}(s;y,0),w_1) - m(s,\mathcal{X}(s;y,0),w_2)\right| \mathrm{d}s\,\mathrm{d}y \\
&= B_\infty e^{M_\infty(t-t_o)} \int_{t_o}^{t}\int_{0}^{\sigma(s)} \left|m(s,\xi,w_1) - m(s,\xi,w_2)\right| \mathrm{d}\xi\,\mathrm{d}s \\
&\le B_\infty M_L e^{M_\infty(t-t_o)}(t-t_o) d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

For (5.40), using **(BP3)**,

$$
\begin{aligned}
&\int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t} \left|q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)\right| \mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \int_{t_o}^{t}\int_{0}^{\sigma(\tau)} \left|v(y,\tau)\right| \left|q(\tau,y,w_1) - q(\tau,y,w_2)\right| e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,y),w_1)\mathrm{d}s}\,\mathrm{d}y\,\mathrm{d}\tau \\
&\le Q_L\,\hat{v}\,e^{M_\infty(t-t_o)}\,d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

Finally, for (5.41), we have

$$
\begin{aligned}
&\int_0^{\sigma(t)}\int_{\mathcal{T}(0;t,x)}^{t} \left|q(\tau,\mathcal{X}(\tau;t,x),w_2)\right| \left|\mathcal{E}_{w_2}(\tau,t,x) - \mathcal{E}_{w_1}(\tau,t,x)\right| \mathrm{d}\tau\,\mathrm{d}x \\
&= \int_{t_o}^{t}\int_{0}^{\sigma(\tau)} \left|q(\tau,\xi,w_2)\right| \left|e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,\xi),w_2)\mathrm{d}s} - e^{\int_\tau^t m(s,\mathcal{X}(s;\tau,\xi),w_1)\mathrm{d}s}\right| \mathrm{d}\xi\,\mathrm{d}\tau \\
&\le Q_\infty e^{M_\infty(t-t_o)} \int_{t_o}^{t}\int_{0}^{\sigma(\tau)}\int_{\tau}^{t} \left|m(s,\mathcal{X}(s;\tau,\xi),w_1) - m(s,\mathcal{X}(s;\tau,\xi),w_2)\right| \mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\le Q_\infty e^{M_\infty(t-t_o)} \int_{t_o}^{t}\int_{\tau}^{t}\int_{\mathcal{X}(s;t_o,0)}^{\mathcal{X}(s;\tau,0)} \left|m(s,y,w_1) - m(s,y,w_2)\right| \mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\tau \\
&\le Q_\infty M_L e^{M_\infty(t-t_o)} \frac{1}{2}(t-t_o)^2 d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L^1}(J_1;\mathbb{R}_+)} \\
&\le \left[B_\infty M_L + \hat{v} Q_L + \frac{1}{2} Q_\infty M_L (t-t_o)\right] e^{M_\infty(t-t_o)}(t-t_o) d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
\tag{5.42}
$$

Focusing now on the second term of (5.38), we have

$$
\begin{aligned}
&\left\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R})} \\
&\le \int_{\sigma(t)}^{+\infty} \left|u_o\left(\mathcal{X}(t_o;t,x)\right)\right| \left|\mathcal{E}_{w_1}(t_o,t,x) - \mathcal{E}_{w_2}(t_o,t,x)\right| \mathrm{d}x \qquad (5.43) \\
&\quad + \int_{\sigma(t)}^{+\infty}\int_{t_o}^{t} \left|q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)\right| \mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \qquad (5.44)
\end{aligned}
$$

$$+\int_{\sigma(t)}^{+\infty}\int_{t_o}^{t}|q(\tau,\mathcal{X}(\tau;t,x),w_2)||\mathcal{E}_{w_1}(\tau,t,x)-\mathcal{E}_{w_2}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x\,. \tag{5.45}$$

Looking at term (5.43),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty}\left|u_o(\mathcal{X}(t_o;t,x))\right|\left|\mathcal{E}_{w_1}(t_o,t,x)-\mathcal{E}_{w_2}(t_o,t,x)\right|\mathrm{d}x\\
&\le \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}e^{M_\infty(t-t_o)}\\
&\quad\times\int_0^{+\infty}\int_{t_o}^{t}\left|m(s,\mathcal{X}(s;t_o,y),w_1)-m(s,\mathcal{X}(s;t_o,y),w_2)\right|\mathrm{d}s\,\mathrm{d}x\\
&=\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}e^{M_\infty(t-t_o)}\int_{t_o}^{t}\int_{\sigma(s)}^{+\infty}\left|m(s,y,w_1)-m(s,y,w_2)\right|\mathrm{d}y\,\mathrm{d}s\\
&\le M_L R e^{M_\infty(t-t_o)}(t-t_o)d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

Next, for the term (5.44),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty}\int_{t_o}^{t}\left|q(\tau,\mathcal{X}(\tau;t,x),w_1)-q(\tau,\mathcal{X}(\tau;t,x),w_2)\right|\mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x\\
&\le e^{M_\infty(t-t_o)}\int_{t_o}^{t}\int_{\sigma(\tau)}^{+\infty}\left|q(\tau,y,w_1)-q(\tau,y,w_2)\right|\mathrm{d}y\,\mathrm{d}\tau\\
&\le Q_L e^{M_\infty(t-t_o)}(t-t_o)d_{\mathcal{W}}(w_1,w_2).
\end{aligned}$$

Finally, for term (5.45),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty}\int_{t_o}^{t}|q(\tau,\mathcal{X}(\tau;t,x),w_2)||\mathcal{E}_{w_1}(\tau,t,x)-\mathcal{E}_{w_2}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x\\
&\le Q_\infty e^{M_\infty(t-t_o)}\\
&\quad\times\int_{t_o}^{t}\int_{\sigma(t)}^{+\infty}\int_{\tau}^{t}\left|m(s,\mathcal{X}(s;\tau,\xi),w_1)-m(s,\mathcal{X}(s;\tau,\xi),w_2)\right|\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&=Q_\infty e^{M_\infty(t-t_o)}\int_{t_o}^{t}\int_{\tau}^{t}\int_{\sigma(s)}^{+\infty}\left|m(s,y,w_1)-m(s,y,w_2)\right|\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\tau\\
&\le\frac{1}{2}M_L Q_\infty e^{M_\infty(t-t_o)}(t-t_o)^2 d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

Thus, combining these estimates together we have

$$\begin{aligned}
&\left\|P^{w_1}(t,t_o)u_o-P^{w_2}(t,t_o)u_o\right\|_{\mathbf{L}^1(J_1;\mathbb{R}_+)}\\
&\qquad\le\left[M_L R+Q_L+\frac{1}{2}M_L Q_\infty(t-t_o)\right]e^{M_\infty(t-t_o)}d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}\tag{5.46}$$

Due to the assumption $u_o\in\mathcal{D}$, we have $\|u_o\|_{\mathbf{L}^1(\mathbb{R}_+;\mathbb{R})}\le R$. Hence, substituting (5.42) and (5.46) into (5.38), and as $(t-t_o)<T$, we get

$$d_{\mathcal{U}}(P^{w_1}(t,t_o)u_o,P^{w_2}(t,t_o)u_o)\le C_w(t-t_o)d_{\mathcal{W}}(w_1,w_2) \tag{5.47}$$

where C_w is as in (3.16), as required. □

Proof of Proposition 3.13. For fixed $t_o \in I$, $u_o \in \mathcal{U}$, and $w \in \mathcal{W}$, define by $\Pi_{(t_o,u_o,w_o)} : \{(s,s_o) \in [t_o,T]^2 : s \geq s_o\} \times \mathcal{U} \to \mathcal{U}$ to be the process with $s \mapsto \Pi_{(t_o,u_o,w_o)}(s,s_o)\rho_o$ being the solution of

$$\begin{cases} \partial_t \rho + \partial_x \big(v(t,x)\,\rho\big) = \bar{m}(t,x)\,\rho + \bar{q}(t,x) & (t,x) \in [s_o,T] \times \mathbb{R}_+ \\ \rho(t,0) = b_o(t) & t \in [s_o,T] \\ \rho(s_o,x) = \rho_o(x) & x \in \mathbb{R}_+ \end{cases} \tag{5.48}$$

with $\bar{m}$ and $\bar{q}$ the given by (3.19). For notational simplicity, we will write $\Pi_{(t_o,u_o,w_o)} = \Pi$ when the (t_o,u_o,w_o) when no confusion arises.

The mapping Π is Lipschitz continuous with respect to time and initial data, for some constant $\mathcal{L} > 0$, as $\bar{m}$ and $\bar{q}$ satisfy correspondingly **(BP2)** and **(BP3)**, which do not explicitly depend on w.

By this construction, $t \mapsto \Pi_{(t_o,u_o,w_o)}(t,t_o)u_o$ is the solution of (3.18).

From [5, Theorem 2.9], we have

$$\begin{aligned} &\left\| u(t) - \Pi_{(t_o,u_o,w_o)}(t,t_o)u_o \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \\ &\leq \mathcal{L} \int_{t_o}^{t} \liminf_{h\to 0+} \frac{1}{h} \left\| u(\tau+h) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau) \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \mathrm{d}\tau \\ &= \mathcal{L} \int_{t_o}^{t} \liminf_{h\to 0+} \frac{1}{h} \Big\| P_1(\tau+h,\tau)P(\tau,t_o)(u_o,w_o) \\ &\qquad\qquad - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau) \Big\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \mathrm{d}\tau\,. \end{aligned}$$

Thus it suffices to show, for any $0 \leq t_o \leq \tau \in [0,T]$, that

$$\liminf_{h\to 0+} \frac{1}{h} \left\| P_1(\tau+h,\tau)P(\tau,t_o)(u_o,w_o) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau) \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} = 0.$$

The tangency condition (2.10) ensures that

$$\begin{aligned} &\frac{1}{h} \left\| P_1(\tau+h,\tau)u(\tau) - P^{P_2(\tau,t_o)(u_o,w_o)}(\tau+h)u(\tau) \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \\ &\leq \mathcal{O}(1) \int_0^h \frac{\omega(\xi)}{\xi}\, d\xi \to 0 \end{aligned}$$

as $h \to 0$.

Further, it can be shown, using formula (5.30), that

$$\left\| P^{P_2(\tau,t_o)(u_o,w_o)}(\tau+h,\tau)u(\tau) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau) \right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \mathcal{O}(1)\, h^2\,,$$

with the constant $\mathcal{O}(1)$ depending on the constants laid out in **(BP1)**-**(BP4)**, R and T. Thus this also converges to zero as $h \to 0$, completing our proof. □

5.5 Proofs for § 3.4

Lemma 5.4. *Let $f \in \mathbf{BC}(\mathbb{R}_+;\mathbb{R})$. For any $\eta \in \mathbb{N} \setminus \{0\}$ there exists a function $f_\eta \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$ such that*

- $\left\| f'_\eta \right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \leq \frac{2}{\eta} \left\| f \right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}$,
- $f_\eta \to f$ *pointwise, as* $\eta \to 0$,
- $\left\| f_\eta \right\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} \leq \left(1 + \frac{2}{\eta}\right) \left\| f \right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})}$.

Proof of Lemma 5.4. Consider $f_\eta(x) = \frac{1}{\eta} \int_0^\eta f(x+y)\, dy$. □

Lemma 5.5. *The mapping μ defined by* (3.23) *in Proposition* 3.16 *is narrowly continuous.*

Proof of Lemma 5.5. Choose $f \in \mathbf{BC}(\mathbb{R}_+)$ and fix $t \in \mathbb{R}^+$. Let $\varepsilon > 0$ and for $\eta > 0$ define $f_\eta \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$ as in Lemma 5.4. Then, setting $M_\eta = \left\| f_\eta \right\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}$, so that $\mathbf{Lip}\left(\frac{f_\eta}{M_\eta}\right) \leq 1$, we have

$$
\begin{aligned}
& \left| \int_{\mathbb{R}_+} f(x)\, \mathrm{d}\big(P_1(t,t_o)\mu_o - P_1(s,t_o)\mu_o\big)(x) \right| \\
\leq\; & \left| \int_{\mathbb{R}_+} \big(f(x) - f_\eta(x)\big)\, \mathrm{d}\big(P_1(t,t_o)\mu_o - P_1(s,t_o)\mu_o\big)(x) \right| \\
& + M_\eta \left| \int_{\mathbb{R}_+} \frac{f_\eta(x)}{M_\eta}\, \mathrm{d}(P_1(t,t_o)\mu_o - P_1(s,t_o)\mu_o)(x) \right| \\
\leq\; & \int_{\mathbb{R}_+} \left| f(x) - f_\eta(x) \right| \mathrm{d}\Big(\left| P_1(t,t_o)\mu_o - P_1(s,t_o)\mu_o \right| \Big)(x) \\
& + M_\eta\, d_{\mathcal{M}}(P_1(t,t_o)\mu_o, P_1(s,t_o)\mu_o) \\
\leq\; & \int_{\mathbb{R}_+} \left| f(x) - f_\eta(x) \right| \mathrm{d}\Big(\left| P_1(t,t_o)\mu_o - P_1(s,t_o)\mu_o \right| \Big)(x) \\
& + \mathbf{Lip}(P) \left(1 + \frac{2}{\eta}\right) \left\| f \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} |t-s|
\end{aligned}
$$

By the Dominated Convergence Theorem, the first term can be bounded by $\varepsilon/2$ for η small. Then, choose s so that also the latter summand above is bounded by $\varepsilon/2$. □

Proof of Proposition 3.16.

The Narrow Continuity: This is a consequence of Lemma 5.5.

Distributional Solution: To simplify calculations we define, for a test function $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_o,T] \times \mathbb{R};\mathbb{R}))$,

$$\mathcal{I}_\varphi(\mu,w) = \int_{\mathbb{R}_+} \left(\partial_t\varphi(\cdot,x) + b(\cdot,\mu,w)(x)\partial_x\varphi(\cdot,x) - c(\cdot,\mu,w)(x)\varphi(\cdot,x)\right) d\mu(\cdot,x)$$
$$+ \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(\cdot,x) d[\eta(\cdot,\mu,w)(y)](x)\right) d\mu(\cdot,y).$$

By a density argument, it suffices to check the integral equality in Definition 3.14 for $\varphi \in \mathbf{C^1_c}([t_o,T] \times \mathbb{R}_+;\mathbb{R}))$. We discretise the time domain. For a spacing $k \in \mathbb{N}$, and $i = 0,\ldots,k$, we introduce the grid points $t_i = t_o + \frac{i(T-t_o)}{k}$, and the associated $(\tilde\mu_i,\tilde w_i) = P(t_{i-1},t_o)(u_o,w_o)$. We then split the integral,

$$\int_{t_o}^T \mathcal{I}_\varphi\left(P(t,t_o)(\mu_o,w_o)\right) \mathrm{d}t$$
$$= \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \underbrace{\left[\mathcal{I}_\varphi\left(P(t,t_{i-1})(\tilde\mu_i,\tilde w_i)\right) - \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\right)\right]}_{A_{1,i}(t)} \mathrm{d}t \tag{5.49}$$
$$+ \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \underbrace{\mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\right)}_{A_{2,i}(t)} \mathrm{d}t\ . \tag{5.50}$$

Our first goal is to demonstrate that (5.49) vanishes in the limit $k \to \infty$. Focusing on $A_{1,i}$, we split the integral to get

$$A_{1,i}(t)$$
$$= \int_{\mathbb{R}_+} \partial_t\varphi(t,x)\, \mathrm{d}\left(P_1(t,t_{i-1})(\tilde\mu_i,\tilde w_i) - F_1(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\right)(x) \tag{5.51}$$
$$+ \int_{\mathbb{R}_+} b\left(t,P(t,t_{i-1})(\tilde\mu_i,\tilde w_i)\right)(x)\partial_x\varphi(t,x)\, \mathrm{d}P_1(t,t_{i-1})(\tilde\mu_i,\tilde w_i)\,(x)$$
$$- \int_{\mathbb{R}_+} b(t,F(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i))(x) \tag{5.52}$$
$$\times\, \partial_x\varphi(t,x)\, \mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\,(x)$$
$$+ \int_{\mathbb{R}_+} c(t,F(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i))(x)\varphi(t,x)\, \mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\,(x)$$
$$- \int_{\mathbb{R}_+} c(t,P(t,t_{i-1})(\tilde\mu_i,\tilde w_i))(x)\varphi(t,x)\, \mathrm{d}P_1(t,t_{i-1})(\tilde\mu_i,\tilde w_i)\,(x) \tag{5.53}$$
$$+ \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\quad \mathrm{d}[\eta(t,P(t,t_{i-1})(\tilde\mu_i,\tilde w_i))(y)]\,(x)\right) \mathrm{d}P_1(t,t_{i-1})(\tilde\mu_i,\tilde w_i)\,(y)$$
$$- \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\, \mathrm{d}[\eta(t,F(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i))(y)]\,(x)\right) \tag{5.54}$$
$$\mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde\mu_i,\tilde w_i)\,(y)\,.$$

We now deal with each of these terms separately. To simplify the notation we will set

$$\begin{array}{rcccl} P_i(t) & \equiv & (\mu_{i,P}(t), w_{i,P}(t)) & = & P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i), \\ F_i(t) & \equiv & (\mu_{i,F}(t), w_{i,F}(t)) & = & F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,. \end{array} \tag{5.55}$$

We will make extensive use of the relation (2.10), which gives

$$d(P_i(t),F_i(t)) \le \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{w(\xi)}{\xi}\,\mathrm{d}\xi \tag{5.56}$$

for L as in (2.15). For (5.51),

$$\begin{aligned} &\left| \int_{\mathbb{R}_+} \partial_t\varphi(t,x)\; d\left(\mu_{i,P}(t)-\mu_{i,F}(t)\right)(x) \right| \\ &\qquad\qquad \le \|\partial_t\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} d_{\mathcal{M}}\left(\mu_{i,P}(t),\mu_{i,F}(t)\right) \\ &\qquad\qquad \le \|\partial_t\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;. \end{aligned}$$

Next, for (5.52), calling $L_b = \sup_{t\in[0,T],w\in\mathcal{W}} \mathbf{Lip}(b(t,\cdot,w))$,

$$\begin{aligned} &\left| \int_{\mathbb{R}_+} b(t,P_i(t))(x)\partial_x\varphi(t,x)\; d\mu_{i,P}(t)(x) \right. \\ &\qquad \left. - \int_{\mathbb{R}_+} b(t,F_i(t))(x)\partial_x\varphi(t,x)\; d\mu_{i,F}(t)(x) \right| \\ &= \left| \int_{\mathbb{R}_+} \left[b(t,P_i(t))(x) - b(t,F(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(x)\right] \partial_x\varphi(t,x)\; d\mu_{i,P}(t)(x) \right. \\ &\qquad \left. + \int_{\mathbb{R}_+} b(t,F_i(t))(x)\partial_x\varphi(t,x)\; d\left(\mu_{i,P}(t)-\mu_{i,F}(t)\right)(x) \right| \\ &\le \|\partial_x\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} (RL_b + R\hat{L} + B)\; d(P_i(t),F_i(t)) \\ &\le \|\partial_x\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} (RL_b + R\hat{L} + B) \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;. \end{aligned}$$

Repeat the same calculations for (5.53) and set $L_c = \sup_{t\in[0,T],w\in\mathcal{W}} \mathbf{Lip}\left(c(t,\cdot,w)\right)$,

$$\begin{aligned} &\left| \int_{\mathbb{R}_+} c(t,F_i(t))(x)\varphi(t,x)\; dw_{i,F}(t)(x) - \int_{\mathbb{R}_+} c(t,P_i(t))(x)\varphi(t,x)\; d\mu_{i,P}(t)(x) \right| \\ &\le \|\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} (RL_c + R\hat{L} + C) \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;. \end{aligned}$$

Finally, for the term (5.54), we find

$$\left| \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\; d[\eta(t,P_i(t))(y)](x) \right)\; d\mu_{i,P}(t)(y) \right.$$

$$
\begin{aligned}
&\quad - \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\ d[\eta(t,F_i(t))(y)](x) \right)\ dw_{i,F}(t)(y) \Bigg| \\
&= \Bigg| \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\ d[\eta(t,P_i(t))(y) - \eta(t,F_i(t))(y)](x) \right)\ d\mu_{i,P}(t)(y) \\
&\quad + \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\ d[\eta(t,F_i(t))(y)](x) \right)\ d\left(\mu_{i,P}(t) - w_{i,F}(t)\right)(y) \Bigg| \\
&\le \|\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} R \left(\sup_{\substack{t\in[0,T]\\ w\in\mathcal{W}}} \mathbf{Lip}(\eta(t,\cdot,w)) + \hat{L} + E \right) \\
&\quad \times \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi\,.
\end{aligned}
$$

Combining these four estimates together, we have for a constant $\mathcal{C}$, independent of k,

$$
\left| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} A_{1,i}(t)\,\mathrm{d}t \right| \le \mathcal{C} \sum_{i=1}^{k} \frac{(t_i - t_{i-1})^2}{2} \int_0^{\frac{T-t_o}{k}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi \to 0 \text{ as } k \to +\infty\,.
$$

Now,

$$
\begin{aligned}
&A_{2,i}(t)| \\
&= \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right) \\
&= \mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right) \\
&+ \int_{\mathbb{R}_+} (b(t,\mu_{i,F}(t),w_{i,F}(t))(x) - b(t,\mu_{i,F}(t),\tilde{w}_i)(x))\partial_x\varphi(t,x)\ d\mu_{i,F}(t)(x) \\
&+ \int_{\mathbb{R}_+} (c(t,\mu_{i,F}(t),\tilde{w}_i)(x) - c(t,\mu_{i,F}(t),w_{i,F}(t))(x))\varphi(t,x)\ d\mu_{i,F}(t)(x) \\
&+ \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x)\, d[\eta(t,\mu_{i,F}(t),w_{i,F}(t))(y) - \eta(t,\mu_{i,F}(t),\tilde{w}_i)(y)](x) \right) d\mu_{i,F}(t)(x)
\end{aligned}
$$

and hence

$$
\begin{aligned}
A_{2,i}(t) \le{}& \mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right) \\
&+ \hat{L}R\left(2\|\varphi\|_{\mathbf{W^{1,\infty,}}(\mathbb{R}_+;\mathbb{R})} + \|\partial_x\varphi\|_{\mathbf{W^{1,\infty,}}(\mathbb{R}_+;\mathbb{R})}\right) \\
&\times \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,. \qquad (5.57)
\end{aligned}
$$

The second term will thus converge to zero in the summation. Hence we concentrate on the summation of the first term.

In the next calculation, we will use the fact

$$
\int_{\mathbb{R}_+} \varphi(T,x)\,\mathrm{d}\left(\mu_{k,F}(T) - P_1(T,t_o)(u_o,w_o)\right)(x)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+} \varphi(T,x)\,\mathrm{d}\Big(F_1(T-t_{k-1},t_{k-1})P(t_{k-1},t_o)(u_o,w_o) \\
&\qquad\qquad - P_1(T,t_{k-1})P(t_{k-1},t_o)(u_o,w_o)\Big)(x) \\
&\le \left\|\varphi(T)\right\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})} \frac{2L}{\ln 2}\frac{T-t_o}{k}\int_0^{\frac{T-t_o}{k}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi \\
&\to 0, \text{ as } k\to\infty.
\end{aligned}
$$

Focusing on the summation of the first term in (5.57)

$$
\begin{aligned}
&\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right)\mathrm{d}t \\
&= \sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d\mu_{i,F}(t_i)(x) - \int_{\mathbb{R}_+}\varphi(t_{i-1},x)\ d\tilde{\mu}_i(x)\right) \\
&= \int_{\mathbb{R}_+}\varphi(T,x)\ d\mu_{T,F}(T)(x) - \int_{\mathbb{R}_+}\varphi(t_o,x)\ d\mu_o(x) \\
&\quad + \sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d(\mu_{i,F}(t_i)-\tilde{\mu}_{i+1})(x)\right) \\
&\underset{k\to+\infty}{\longrightarrow} \int_{\mathbb{R}_+}\varphi(T,x)\, d(P_1(T,t_o)(u_o,w_o))(x) - \int_{\mathbb{R}_+}\varphi(t_o,x)\ d\mu_o(x),
\end{aligned}
$$

where we use that

$$
\begin{aligned}
&\sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d(\mu_{i,F}(t_i)-\tilde{\mu}_{i+1})(x)\right) \\
&\qquad\qquad \le \|\varphi\|_{\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})}\frac{2L}{\ln 2}T\int_0^{\frac{T-t_o}{k}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi \underset{k\to+\infty}{\longrightarrow} 0\,, \qquad (5.58)
\end{aligned}
$$

completing the proof. ☐

5.6 Proofs for § 3.5

Proof of Proposition 3.20. We assume for simplicity that both processes P^u and P^w share the same constants C_u, C_w, C_t in (2.11)–(2.12)–(2.13).

The properties of P ensured by Theorem 2.6 show that we have $P_1 \in \mathbf{C^0}([t_o,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}))$ as required by Definition 3.17.

Introduce the following notation. For any $k\in\mathbb{R}$ and $\varphi\in\mathbf{C_c^\infty}(\hat{I}\times\mathbb{R};\mathbb{R}_+)$, denote

$$
\begin{aligned}
\mathcal{I}_{\varphi,k}(u,w) &= \int_{\mathbb{R}}\left[|u-k|\,\partial_t\varphi + q_k(u,w)\,\partial_x\varphi\right]\mathrm{d}x\,, \\
q_k(u,w) &= \operatorname{sign}(u-k)\ \left(f(u,w)-f(k,w)\right)\,.
\end{aligned}
$$

Fix $N \in \mathbb{N} \setminus \{0\}$ and, for every $i \in \{0, \ldots, N\}$, define $t_i = t_o + i\frac{T-t_o}{N}$ and, for $t \in [t_{i-1}, T]$,

$$\begin{array}{rcccl} & & (\tilde u_i, \tilde w_i) & = & P(t_{i-1}, t_o)(u_o, w_o)\,, \\ \bar P_i(t,x) & \equiv & (u_{i,P}(t,x), w_{i,P}(t)) & = & P(t, t_{i-1})(\tilde u_i, \tilde w_i)(x)\,, \\ \bar F_i(t,x) & \equiv & (u_{i,F}(t,x), w_{i,F}(t)) & = & \left(P^{\tilde w_i}\,(t, t_{i-1})\,\tilde u_i(x), P^{\tilde u_i}\,(t, t_{i-1})\,\tilde w_i\right)\,. \end{array} \tag{5.59}$$

We now prove in 2 steps that

$$\begin{aligned} \int_{t_o}^{T} \mathcal{I}_{\varphi,k}(P\,(t,t_o)\,(u_o,w_o))\,\mathrm{d}t \geq & \int_{\mathbb{R}} \left|P_1\,(T,t_o)\,(u_o,w_o) - k\right| \varphi(T,x)\,\mathrm{d}x \\ & - \int_{\mathbb{R}} \left|u_o(x) - k\right| \varphi(0,x)\,\mathrm{d}x\,. \end{aligned} \tag{5.60}$$

Step 1: We prove the inequality

$$\int_{t_o}^{T} \mathcal{I}_{\varphi,k}(P\,(t,t_o)\,(u_o,w_o))\,\mathrm{d}t \geq \limsup_{N\to+\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde w_i)\,\mathrm{d}t\,. \tag{5.61}$$

To this aim, write

$$\int_{t_o}^{T} \mathcal{I}_{\varphi,k}(P\,(t,t_o)\,(u_o,w_o))\,\mathrm{d}t$$

$$= \int_{t_o}^{T} \int_{\mathbb{R}} \left|P_1\,(t,t_o)\,(u_o,w_o)(x) - k\right| \partial_t\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t \tag{5.62}$$

$$+ \int_{t_o}^{T} \int_{\mathbb{R}} q_k\left(P\,(t,t_o)\,(u_o,w_o)(x)\right) \partial_x\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t \tag{5.63}$$

We proceed towards the estimate of (5.62). For every $i \in \{1, \ldots, N\}$ and $k \in \mathbb{R}$, using (2.10) with L and ω given by (2.15), we have

$$\begin{aligned} & \left| \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left[\left|u_{i,P}(t,x) - k\right| \partial_t\varphi(t,x) - \left|u_{i,F}(t,x) - k\right| \partial_t\varphi(t,x) \right] \mathrm{d}x\ \mathrm{d}t \right| \\ \leq & \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,P}(t,x) - u_{i,F}(t,x)\right| \partial_t\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t \\ \leq & \frac{2L}{\ln(2)} \|\partial_t\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\ \mathrm{d}t \\ \leq & \frac{L}{\ln(2)} \frac{(T-t_o)^2}{N^2} \|\partial_t\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,. \end{aligned}$$

Therefore, the term (5.62) is estimated as:

$$\int_{t_o}^{T} \int_{\mathbb{R}} \left|P_1\,(t,t_o)\,(u_o,w_o)(x) - k\right| \partial_t\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,P}(t,x) - k\right| \partial_t \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \\
&\geq \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,F}(t,x) - k\right| \partial_t \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \right] \\
&\qquad\qquad - \frac{L}{\ln(2)} \frac{(T-t_o)^2}{N} \left\|\partial_t \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi
\end{aligned}
$$

and the last term converges to 0 as $N \to +\infty$. Thus, the term (5.62) is estimated as follows:

$$
[(5.62)] \geq \limsup_{N\to+\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,F}(t,x) - k\right| \partial_t \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \, . \tag{5.64}
$$

We pass now to the term (5.63). For every $i \in \{1, \ldots, N\}$ and $k \in \mathbb{R}$, since q_k is Lipschitz continuous [19, Lemma 3] and using (2.10), L_f from **(CL2)**, L and ω from (2.15),

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k\left(\bar{P}_i(t,x)\right) \partial_x \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t - \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i) \, \partial_x \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \\
&= \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left[q_k\left(\bar{P}_i(t,x)\right) - q_k\left(u_{i,F}(t,x), w_{i,P}(t)\right)\right] \partial_x \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \\
&+ \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left[q_k\left(u_{i,F}(t,x), w_{i,P}(t)\right) - q_k(u_{i,F}(t,x), \tilde{w}_i)\right] \partial_x \varphi(t,x) \, \mathrm{d}x \; \mathrm{d}t \\
&\leq L_f \left\|\partial_x \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,P}(t,x) - u_{i,F}(t,x)\right| \mathrm{d}x \; \mathrm{d}t \\
&+ \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \Big| f\left(u_{i,F}(t,x), w_{i,P}(t)\right) - f(u_{i,F}(t,x), \tilde{w}_i) \\
&\qquad\qquad - \Big(f\left(k, w_{i,P}(t)\right) - f(k, \tilde{w}_i) \Big) \Big| \; \left|\partial_x \varphi(t,x)\right| \mathrm{d}x \; \mathrm{d}t \\
&\leq L_f \frac{2L}{\ln(2)} \left\|\partial_x \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \; \mathrm{d}t \\
&+ L_f \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,F}(t,x) - k\right| \cdot d_{\mathcal{W}}(w_{i,P}(t), \tilde{w}_i) \cdot \left|\partial_x \varphi(t,x)\right| \, \mathrm{d}x \; \mathrm{d}t \\
&\leq L_f \frac{2L}{\ln(2)} \left\|\partial_x \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \frac{(t_i - t_{i-1})^2}{2} \int_0^{t_i - t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \\
&+ L_f \, C_t (R+k) \left\|\partial_x \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \, \mathrm{d}t \\
&\leq \frac{L_f}{2} \left\|\partial_x \varphi\right\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \left(L_f \, C_t (R+k) + \frac{2L}{\ln(2)} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \right) \frac{(T-t_o)^2}{N^2} \, .
\end{aligned}
$$

Therefore, (5.63) is estimated as

$$\begin{aligned}
&\int_{t_o}^{T} \int_{\mathbb{R}} q_k \left(P\left(t, t_o\right)\left(u_o, w_o\right)(x)\right) \partial_x \varphi(t, x) \, \mathrm{d}x \, \mathrm{d}t \\
\geq &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t, x), \tilde{w}_i) \, \partial_x \varphi(t, x) \, \mathrm{d}x \, \mathrm{d}t \\
&- \frac{L_f}{2} \|\partial_x \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \left(L_f \, C_t(R+k) + \frac{2L}{\ln(2)} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi\right) \frac{(T-t_o)^2}{N^2}
\end{aligned}$$

and the last term converges to 0 as $N \to +\infty$. Thus,

$$[(5.63)] \geq \limsup_{N \to +\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t, x), \tilde{w}_i) \, \partial_x \varphi(t, x) \, \mathrm{d}x \, \mathrm{d}t \, . \tag{5.65}$$

Combining (5.64) and (5.65), the proof of Step 1, namely (5.61), is completed.

Step 2: Now we prove that

$$\begin{aligned}
&\liminf_{N \to +\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i) \, \mathrm{d}t \\
\geq &\int_{\mathbb{R}} \left|P_1(T, t_o)(u_o, w_o)(x) - k\right| \varphi(T, x) \, \mathrm{d}x - \int_{\mathbb{R}} \left|u_o(x) - k\right| \varphi(t_o, x) \, \mathrm{d}x
\end{aligned} \tag{5.66}$$

Fix $i \in \{1, \ldots, N\}$. For $\varepsilon > 0$ sufficiently small, consider a function $\chi_\varepsilon \in \mathbf{C_c^\infty}\left(]t_{i-1}, t_i[; [0,1]\right)$ such that $\chi_\varepsilon(t) = 1$ for $t \in [t_{i-1} + \varepsilon, t_i - \varepsilon]$ and define $\varphi_\varepsilon = \varphi \cdot \chi_\varepsilon$. Then, by Definition 3.17 and the choice of χ_ε, we have that for every $\varepsilon > 0$ sufficiently small,

$$\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi_\varepsilon,k}(u_{i,F}(t, x), \tilde{w}_i) \, \mathrm{d}t \geq 0 \, .$$

This implies that

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i) \, \mathrm{d}t \\
= &\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi-\varphi_\varepsilon,k}(u_{i.F}(t), \tilde{w}_i) \, \mathrm{d}t + \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi_\varepsilon,k}(u_{i.F}(t), \tilde{w}_i) \, \mathrm{d}t \\
\geq &\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi-\varphi_\varepsilon,k}(u_{i.F}(t), \tilde{w}_i) \, \mathrm{d}t \\
= &\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left|u_{i,F}(t, x) - k\right| \partial_t \left(\varphi - \varphi_\varepsilon\right)(t, x) \, \mathrm{d}x \, \mathrm{d}t & (5.67) \\
&+ \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t, x), \tilde{w}_i) \, \partial_x(\varphi - \varphi_\varepsilon)(t, x) \, \mathrm{d}x \, \mathrm{d}t & (5.68)
\end{aligned}$$

for every $\varepsilon > 0$ sufficiently small. Moreover the continuity in time of $u_{i,F}$ implies that

$$\lim_{\varepsilon\to 0^+}[(5.67)] = \int_{\mathbb{R}} \left|u_{i,F}(t_i,x) - k\right| \varphi(t_i,x)\, \mathrm{d}x - \int_{\mathbb{R}} \left|u_{i,F}(t_{i-1},x) - k\right| \varphi(t_{i-1},x)\, \mathrm{d}x\,,$$

while, by the Dominated Convergence Theorem, we deduce that

$$\lim_{\varepsilon\to 0^+}[(5.68)] = \lim_{\varepsilon\to 0^+} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i)\, \partial_x(\varphi - \varphi_\varepsilon)(t,x)\, \mathrm{d}x\ \mathrm{d}t = 0\,.$$

Therefore, we get

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\ \mathrm{d}t \\ &\geq \int_{\mathbb{R}} \left|u_{i,F}(t_i,x) - k\right| \varphi(t_i,x)\, \mathrm{d}x - \int_{\mathbb{R}} \left|u_{i,F}(t_{i-1},x) - k\right| \varphi(t_{i-1},x)\ \mathrm{d}x\ . \end{aligned}$$

Summing over i, we obtain that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\, \mathrm{d}t \\ &\geq \sum_{i=1}^{N} \int_{\mathbb{R}} \left|u_{i,F}(t_i,x) - k\right| \varphi(t_i,x)\, \mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}} \left|u_{i,F}(t_{i-1},x) - k\right| \varphi(t_{i-1},x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \left|u_{N,F}(T,x) - k\right| \varphi(T,x)\, \mathrm{d}x - \int_{\mathbb{R}} \left|u_o(x) - k\right| \varphi(t_o,x)\, \mathrm{d}x \qquad (5.69) \\ &\quad + \sum_{i=1}^{N-1} \int_{\mathbb{R}} \Big(\left|u_{i,F}(t_i,x) - k\right| - \left|u_{i+1,F}(t_i,x) - k\right|\Big)\ \varphi(t_i,x)\, \mathrm{d}x\ . \qquad (5.70) \end{aligned}$$

We now estimate the first term in (5.69):

$$\begin{aligned} &\int_{\mathbb{R}} \left|u_{N,F}(T,x) - k\right| \varphi(T,x)\, \mathrm{d}x - \int_{\mathbb{R}} \left|P_1(T,t_o)(u_o,w_o)(x) - k\right| \varphi(T,x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(\left|F_1(T - t_{N-1}, t_{N-1})(\tilde{u}_{N-1}, \tilde{w}_{N-1})(x) - k\right| \\ &\qquad - \left|P_1(T,t_o)(u_o,w_o)(x) - k\right|\Big)\varphi(T,x)\, \mathrm{d}x \end{aligned}$$

and, using L and ω as in (2.15), we get

$$\begin{aligned} &\Bigg|\int_{\mathbb{R}} \Big(\left|F_1(T - t_{N-1}, t_{N-1})(\tilde{u}_{N-1}, \tilde{w}_{n-1})(x) - k\right| \\ &\qquad - \left|P_1(T,t_o)(u_o,w_o)(x) - k\right|\Big)\varphi(T,x)\, \mathrm{d}x\Bigg| \end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} \bigl| F_1(T-t_{N-1},t_{N-1}) P(t_{N-1},t_o)(u_o,w_o)(x) \\
&\qquad - P_1(T,t_{N-1}) P(t_{N-1},t_o)(u_o,w_o)(x) \bigr| \, \varphi(T,x) \, \mathrm{d}x \\
&\le \frac{2L}{\ln(2)} \, \frac{T-t_o}{N} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \\
&\to 0 \quad \text{as} \quad N \to +\infty \,.
\end{aligned}
$$

We now estimate (5.70) using (5.59) and (2.10)

$$
\begin{aligned}
&\sum_{i=1}^{N-1} \int_{\mathbb{R}} \Bigl| \bigl| u_{i,F}(t_i,x) - k \bigr| - \bigl| u_{i+1,F}(t_i,x) - k \bigr| \Bigr| \, \varphi(t_i,x) \, \mathrm{d}x \\
&\le \sum_{i=1}^{N-1} \int_{\mathbb{R}} \bigl| u_{i,F}(t_i,x) - u_{i+1,F}(t_i,x) \bigr| \, \varphi(t_i,x) \, \mathrm{d}x \\
&= \sum_{i=1}^{N-1} \int_{\mathbb{R}} \Bigl| P^{\tilde{w}_i}(t_i,t_{i-1}) \tilde{u}_i(x) - P_1(t_i,t_{i-1}) \tilde{u}_i(x) \Bigr| \, \varphi(t_i,x) \, \mathrm{d}x \\
&\le \|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \sum_{i=1}^{N-1} \Bigl\| P^{\tilde{w}_i}(t_i,t_{i-1}) \tilde{u}_i - P_1(t_i,t_{i-1}) \tilde{u}_i \Bigr\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \\
&\le \frac{2\,L}{\ln 2} \|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \sum_{i=1}^{N-1} (t_i - t_{i-1}) \int_0^{t_i - t_{i-1}} \frac{\omega(\tau)}{\tau} \, \mathrm{d}\tau \\
&\le \frac{2\,L}{\ln 2} \|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} (T-t_o) \int_0^{(T-t_o)/N} \frac{\omega(\tau)}{\tau} \, \mathrm{d}\tau \\
&\to 0 \quad \text{as } N \to +\infty \,.
\end{aligned}
$$

The obtained estimates for (5.69) and (5.70), as $N \to +\infty$, proved Step 2, namely (5.66). □

A Appendix: BV Estimates

We gather here a few estimates on **BV** functions used in the proofs.

Lemma A.1. *Recall the following elementary estimates on* **BV** *functions, see also [8, § 4.2] or [1]:*

$$
\left.\begin{array}{r} u \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R}) \\ w \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R}) \end{array}\right\} \Rightarrow \mathrm{TV}(u\,w) \le \mathrm{TV}(u) \|w\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} + \|u\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \mathrm{TV}(w) \tag{A.1}
$$

$$
\left.\begin{array}{r} \varphi \in \mathbf{C^{0,1}}(\mathbb{R}^n;\mathbb{R}) \\ u \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R}^n) \end{array}\right\} \Rightarrow \mathrm{TV}(\varphi \circ u) \le \mathbf{Lip}(\varphi)\ \mathrm{TV}(u) \tag{A.2}
$$

$$
\left.\begin{array}{l} u \in \mathbf{L^1}(\hat{I};\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})) \\ u(t) \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R}) \end{array}\right\} \Rightarrow \mathrm{TV}\left(\int_{t_o}^t u(\tau,\cdot)\ \mathrm{d}\tau \right) \le \int_{t_o}^t \mathrm{TV}\bigl(u(\tau)\bigr)\ \mathrm{d}\tau \tag{A.3}
$$

$$\left.\begin{array}{l} u \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R}) \\ \delta \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}_+) \end{array}\right\} \Rightarrow \int_{\mathbb{R}_+} \left| u\left(x+\delta(x)\right) - u(x) \right| \mathrm{d}x \le \mathrm{TV}(u) \left\|\delta\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \tag{A.4}$$

and in (A.3) *we have* $t_o, t \in \hat{I}$ *with* $t_o \le t$*.*

Proof of Lemma A.1. Inequality (A.1) follows from [1, Formula (3.10)]. The one dimensional proof follows. For any partition $(x_i)_{i=0}^N$ of $\mathbb{R}_+$, we have

$$\begin{aligned} & \sum_{i=1}^N \left| u(x_i)w(x_i) - u(x_{i-1})w(x_{i-1}) \right| \\ \le\ & \sum_{i=1}^N \left| u(x_i) - u(x_{i-1}) \right| \left| w(x_i) \right| + \sum_{i=1}^N \left| w(x_i) - w(x_{i-1}) \right| \left| u(x_{i-1}) \right| \\ \le\ & \left\|w\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \sum_{i=1}^N \left| u(x_i) - u(x_{i-1}) \right| + \left\|u\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \sum_{i=1}^N \left| w(x_i) - w(x_{i-1}) \right| \\ \le\ & \mathrm{TV}(u) \left\|w\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} + \left\|u\right\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} \mathrm{TV}(w)\,, \end{aligned}$$

and taking the supremum over all such sequence, we get our required result.

The definition of total variation directly implies (A.2) and (A.3). For a proof of (A.4) see for instance [5, Lemma 2.3]. □

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Conflict of Interest

The author declares no conflicts of interest in this paper.

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Appendix A

Useful well known results

A.1 Results from a variety of literature

We first outline a well known Sobolev inequality

Lemma A.1.1. *Consider f in $H^1(\mathbb{R})$. Then*

$$\|f\|_\infty \leq \|f\|_{H^1(\mathbb{R})}.$$

Proof. The proof is done for $f \in C_c^\infty(\mathbb{R})$, and can be extended via the density of said set in $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$.

We have

$$f^2(x) = \int_{-\infty}^{x} 2f(x)f'(x)\,\mathrm{d}x \leq 2\|f\|_2\|f'\|_2 \leq (\|f\|_2^2 + \|f'\|_2^2) = \|f\|_{H^1(\mathbb{R})}^2,$$

for any $x \in \mathbb{R}$, and the result follows by taking the square root. □

We make use of the follow properties of BV functions. This is a repetition of the Appendix of the Paper 3.

Lemma A.1.2. *Let $u, w \in BV(\mathbb{R}_+;\mathbb{R})$, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Then*

$$TV(uw) \leq TV(u)\|w\|_{L^\infty(\mathbb{R}_+;\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}TV(w),$$

and

$$TV(\varphi \circ u) \leq LTV(u),$$

with L satisfying

$$|\varphi(u_1) - \varphi(u_2)| \leq L|u_1 - u_2|, \qquad \text{for all } u_1, u_2 \in u(\mathbb{R}).$$

Proof. The first statement is shown in the Appendix of Paper 3.

For the second, for any partition of the $\{x_i\}_{i=0}^N$ of $\mathbb{R}_+$,

$$\sum_{i=1}^{N} |\varphi \circ u(x_i) - \varphi \circ u(x_{i-1})| \leq \sum_{i=1}^{N} L|u(x_i) - u(x_{i-1})|,$$

and the result follows from taking in the infimum over all such partitions. □

Lemma A.1.3. *Let $I \subseteq \mathbb{R}_+$. For any $u \in L^1(I; L^1(\mathbb{R}_+;\mathbb{R}))$, with $u(t,\cdot) \in BV(\mathbb{R}_+;\mathbb{R})$,*

$$TV\left(\int_{t_o}^{t} u(\tau,\cdot)\,\mathrm{d}\tau\right) \leq \int_{t_o}^{t} TV(u(\tau))\,\mathrm{d}\tau\,.$$

A.2 Terminology - metrics

The terminology for generalisations of metrics is sometimes inconsistent, so in this section we outline the terms that we use.

For completeness, the classical definition of a metric is also given.

Definition A.2.1. Let X be some set.

A metric is a non-negative mapping $d : X^2 \to \mathbb{R}_+$ satisfying the following three properties, for any $x, y, z \in X$,

$$d(x,y) = 0 \quad \text{if and only if} \quad x = y; \qquad \text{(Identity Axiom)}$$
$$d(x,y) = d(y,x); \qquad \text{(Symmetry)}$$
$$d(x,z) \leq d(x,y) + d(y,z). \qquad \text{(Triangle Inequality)}$$

A semi-metric is a mapping $d : X^2 \to \mathbb{R}_+$ that satisfies the identity axiom and symmetry, but need not satisfy the triangle inequality.

A pseudo-metric is a mapping $d : X^2 \to \mathbb{R}_+$ satisfying symmetry, the triangle inequality, but needs only to satisfy the backwards direction of the identity axiom, i.e. that $d(x,x) = 0$ for any $x \in X$.

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