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The Roles of Mathematical Metaphors and Gestures in the Understanding of Abstract Mathematical Concepts

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Abstract

When a new mathematical idea is presented to students in terms of abstract mathematical symbols, they may have difficulty to grasp it. This difficulty arises because abstract mathematical symbols do not directly refer to concretely perceivable objects. But, when the same content is presented in the form of a graph or a gesture that depicts that graph, it is often much easier to grasp. The process of solving a complex mathematical problem can also be facilitated with the use of a graphical representation. Transforming a mathematical problem or concept into a graphical representation is a common problem solving strategy, and we may view it as a kind of mathematical metaphor, in the sense that a certain representation of a mathematical problem is described in terms of a visual representation of that problem. Furthermore, since a graphical representation is visual, it can be depicted by gestures. Therefore, visual and motor systems can be actively employed to process a given problem and find a solution for it. In this way, mathematical metaphor offers us a way to employ a wider range of cognitive resources to understand mathematics.

1. Introduction

In order to describe mathematical concepts, we employ a variety of tools and resources. We use language, mathematical symbols, graphs, tables, coordinate systems, physical objects, and many other things to talk about concepts and to solve mathematical problems. These tools function as mediatory channels that help us describe concepts and represent them in the most effective way.

A given mathematical concept may be described in a variety of ways. For example, a mathematical function is a transformation of one or more input quantities into an output. We may use words or mathematical symbols in the form of $y = f(x)$ to describe a given function. In addition to these, we may use a graph in the Cartesian plane, a diagram consisting of geometric shapes and arrows, a table, or ordered pairs to talk about the same function. Each one of these channels may be suitable to study a certain dimension or a certain behavior of this concept. When we understand the algebraic representation of a mathematical function (shown as $y = f(x)$) in terms of the graphical representation of this function in Cartesian coordinate system, we in fact use a mathematical metaphor, as we structure and understand the algebraic representation of the function in terms of its graphical representation [34].

Among the mediatory channels that are used to understand mathematical concepts, gesture has been perhaps the most-discussed one in recent years (e.g., [14, 30, 35]). Gesture is very useful because it is visible and embodied. Furthermore, it is easily at hand. When a new mathematical definition or idea is presented to students in terms of abstract mathematical symbols, it may be difficult for them to grasp it because abstract symbols do not directly refer to concretely perceivable objects in the environment. But, when the same idea is presented in the form of a graph or via a gestural depiction of the graph, it is often much easier to understand. It has been argued that supporting the process of mathematics learning by body movements offers students the opportunity to understand mathematics in entirely new ways [50]. For example, when concavity of a mathematical function is explained by mathematical symbols, it may be very difficult for students to understand. But, this concept can be easily described by a graph or a hand gesture that depicts the graph. The graphical or gestural representation of concavity is much more comprehensible.

Two representations of a mathematical concept[1] are essentially similar or equivalent, as they represent the same thing (though it is often the case that they emphasize different features of the concept). In a mathematical metaphor, one representation of a concept is described and understood in terms of another representation. This is usually done to describe a difficult-to-understand representation in terms of an easy-to-understand representation. In this way, the use of metaphor can facilitate our understanding of mathematical concepts and structures.

Mathematical metaphors are not inherently different from linguistic metaphors, as the basic definition of metaphor (understanding one thing in terms of another thing) can be applied to both. One reason that two rather different expressions are used to refer to them (linguistic metaphors and mathematical metaphors) is that linguistic metaphors are used in daily communication and literature, but mathematical metaphors are used in mathematics discussions. Another possible reason could be the existence of a mathematical logic behind mathematical metaphors. In the following section, we discuss the question why mathematical metaphors are important.

2. Mathematical metaphors

Some mathematical metaphors such as *numbers are points on a line* and *functions are curves in the Cartesian plane* have been widely discussed in the cognitive science literature; see for example [33]. In all of these cases, one thing is described and understood in terms of another thing. In fact, a single concept may have a very large number of concrete realizations. Some realizations are easier to embody and to understand. For example, when a function is represented in terms of a curve in the Cartesian plane, many aspects of its behavior and characteristics are much easier to understand. The reason is that visual representations are usually the easiest channel for understanding the structure of concepts. This is also the case with the description of numbers in terms of points on a line.

Numbers constitute one specific category of abstract concepts [16]. Abstract concepts are those concepts that do not have concretely perceivable referents in the real world. They cannot be seen, heard, smelled, touched, or tasted.

[1] Throughout this paper, we use the term *concept* to refer to an underlying mathematical phenomenon that we understand or communicate about through representations.

On the other hand, concrete concepts can be directly perceived through our senses. When numbers are described in terms of points on a line, they are visualized. This is an example of a visual metaphor, in which an abstract concept is represented by an image. This representation makes the process of understanding numbers and their operations much easier.

Mathematical metaphor is, then, a process of transforming one representation of a concept, a problem, or an idea (usually an abstract representation) into another representation (usually a concrete representation) with the intent of understanding the former representation in terms of the latter one. While the former representation is abstract, unfamiliar, and typically difficult to understand, the latter one is concrete, familiar, and typically easier to understand. Often, the abstract representation is expressed in terms of abstract mathematical symbols and is often detached from sensory experiences. On the other hand, the concrete representation is usually expressed in terms of perceptible objects. Among perceptible objects that are used to metaphorically represent mathematical ideas, visually perceivable objects are the most common ones.

Visual metaphors are extensively used in mathematics and other branches of science. Visual metaphor is a tool through which an abstract concept can be described in terms of a highly concrete representation. High imageability of concrete concepts [41] could be an important factor that makes them effective base (source) domains for metaphorical descriptions. The imageable representation of an abstract concept or a mathematical problem is inherently isomorphic to its abstract representation. That is, they are essentially the same at a deep abstract level, although they are different at a surface level. In other words, an abstract mathematical problem can be solved through solving its imageable representation.

Mathematical metaphors are a special type of metaphor in that there is an isomorphic relationship between target and source domains. For example, an algebraic problem can be solved by solving its geometrical representation. To do this, first the algebraic problem should be transformed into its geometrical representation. Then, the geometrical representation is solved. In many cases, the geometrical representation offers an easier channel for a deeper understanding of that problem and its solution.

3. Gesture as a mediatory channel to describe concepts

Among the mediatory channels or mediatory resources used to describe mathematical concepts, gestures have attracted the attention of many researchers [47]. Teachers use gestures when they describe mathematical concepts and problems [17, 24, 49]. Students employ gestures to talk about mathematical concepts [8] and to express newly learned material even before they express it through speech [12, 43]. Some works have suggested that gestures play an important and maybe a causal role in the process of learning new material and the development of knowledge [3, 23, 48, 52].

In mathematics discussions, many concepts may be metaphorically described in terms of gestures. For example, when an algebraic function is represented in terms of its graphical representation, we may use gestures to depict the graphical representation. This is a common example of the metaphoric use of gestures. At elementary levels of mathematics education, we may use gestures to describe simple arithmetic operations such as addition and subtraction. These are also metaphoric gestures.

The employment of gestures during knowledge acquisition suggests that certain aspects of knowledge may be embodied [20, 26, 38, 39].[2] It has been argued that mathematical knowledge is embodied in two senses. Firstly, it is grounded through the learner / knower's sensory-motor system; secondly, the process of grounding is rooted in the physical environment [4].[3]

It has been suggested that gestures contribute to the process of thinking by focusing attention on perceptual information [3], by organizing ideas [32, 33], and by activating mental images [55]. Here we ask: how do gestures contribute to the process of learning and the grounding of mathematical knowledge? We specifically focus on two types of gestures that are employed in the process of mathematics learning. In the following section, we look at the ways that these two types of gesture contribute to the process of mathematics learning. Then, we discuss the role of perspective in the process of mental simulation through gestures.

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[2] When we say that knowledge is embodied, we mean that the process of acquiring knowledge is mediated by sensory, perception, and motor systems [21].

[3] Concept grounding is a process through which the mental representation of a concept is linked to referents in the real world.

4. Representational gestures and embodiment

According to McNeill's [37] typology, gestures are categorized into four types: pointing gestures, iconic gestures, metaphoric gestures, and beat gestures. Pointing gestures are used to refer to objects or locations with fingers or hands. Iconic gestures depict the meaning aspects of concepts directly through the shape of hands or the trajectory of moving hands. For example, tracing the shape of a circle refers to an object with the shape of a circle. Or, a cupped hand can be used to refer to an object in the shape of a cup. Metaphoric gestures depict the meaning aspects of concepts indirectly through metaphors. For example, a grasping gesture can be used to refer to the understanding of an idea (the metaphor *grasp an idea*). In fact, a metaphoric gesture literally depicts the semantic content of the base domain of a metaphor. Beat gestures do not reflect semantic dimensions of concepts but accompany speech as a rhythmic tool. Beat gestures have neither a literal nor a metaphorical relationship with the semantic content of words.

Iconic and metaphoric gestures are often put in one category and called representational gestures [4]. Representational gestures can be defined as gestures that depict their semantic content literally or metaphorically [2, 32]. Representational gestures are powerful tools in mathematics because they can offer easily perceivable descriptions of highly abstract mathematical concepts. Representational gestures are tools for making abstract mathematical concepts graspable. Through the rest of this paper, we focus on representational gestures as a tool for grounding mathematical concepts.

As mentioned in Section 1, gestures are used by teachers and learners, and both types have been found to enhance the process of learning. However, results of a study suggested that gestures performed by learners produce better learning results compared to just observing gestures that are produced by the teacher and simply observed by the learners [46]. Although the study reported in [46] involved word memorization, it is sensible to expect that the impact of performing gestures on enhancing memory can be true in other fields such as mathematics. These findings suggest that there could be a significant difference between learning that takes place with the support of performing gestures and the learning that takes place with the support of seeing gestures performed.

According to some studies on embodiment, observing an action involves the activation of the same sensory-motor areas that are involved in the actual doing of that same action [18]; for a review, see [28]. For example, seeing a grasping movement seems to involve the activation of those sensory-motor areas that are involved in the actual doing of grasping. This claim has been supported by several neuroimaging studies that have demonstrated the existence of mirror neurons in humans [5, 11, 19]; also see [6, 15, 22, 44]. Then why does performing gestures produce better results than just observing gestures performed even though the same sensory-motor areas should theoretically be activated in both performing and observing a gesture?

At the very least, we can say that a learner learns best when observing a teacher gesture, then mimicking the gesture. One possible answer for the question posed above is that mirror neurons may fire more strongly when performing a gesture than when observing a gesture. Therefore, when a learner observes a teacher's gesture and then mimics that gesture, it would make sense to expect that the process of embodiment is strengthened.

Another answer could be based on the role of perspective. Some researchers have argued that an action concept can be embodied by adopting the perspective of an agent that carries out that action or the perspective of an observer that sees that action [7]. As mentioned, a representational gesture creates a visible simulation of a concept. In this way, it may contribute to the process of grounding the concept when aligned with its verbal description. This takes place in both the literal and the metaphorical description of a concept. In the literal description of a concept, gestures can reflect a direct description of the semantic content of the concept. In the metaphorical description of a concept, gestures can reflect an indirect or metaphorical description of the semantic content. Metaphoric gestures present a visible description of the base domain of the metaphor.

It seems that in both the literal and the metaphorical description of a concept through gestures, the perspective adopted by the comprehender plays a significant role. If the comprehender adopts the perspective of the agent performing the gesture, the process of understanding and grounding will be more successful. In other words, if the comprehender adopts the perspective of a gesture's agent, the grounding process will be more effective. In fact, the process of embodiment can take place from a variety of perspectives. These perspectives do not produce the same level of understanding and learning.

A certain embodied perspective may produce the highest level of understanding and learning compared to other perspectives. In the following section, we look at some representations of mathematical concepts that can be simulated from a variety of action perspectives.

5. Graphic representations of mathematical concepts and embodiment

Creating graphical representations of mathematical concepts is a very common technique for acquiring a better understanding of these concepts and solving mathematical problems. For example, when a mathematics expert looks at the graphical representation of the function $y = f(x)$ in a Cartesian plane, s/he can acquire a better understanding of the behavior of this function. By looking at a graphical representation of this function, s/he can easily realize how x and y are related to each other, what the extreme points are, where the function is ascending and where it is descending, and many other features of this function. However, it must be noted that this happens for someone who is an expert in mathematics or someone who has some knowledge of mathematics; it is not necessarily the same for novices or learners. Grasping the underlying isomorphism across representations requires some level of expertise and prior understanding — it does not immediately become apparent to everyone. In other words, isomorphism is only perceived by some and not by others. Therefore, in some sense, the isomorphism exists only in the mind's eye of the perceiver.

The process of solving a complex abstract mathematical problem can sometimes be significantly facilitated when it is transformed into a graphical representation [27, 29]. The processing of a graphical representation involves the activation of the perceptual (visual) and motor systems. More specifically, it involves the activation of the visual system as graphical representations are highly imageable. It also involves the activation of the motor system as a graph can be processed through scanning the movement of a point on that graph. The trace of this movement can be simulated by representational gestures. Therefore, when the abstract representation of a mathematical concept is transformed into a graphical representation, that abstract concept is grounded through visual and motor systems. This is particularly the case when the graphical representation of a function is a curve in a Cartesian plane. We can see this as a mathematical metaphor in which one representation of a concept is understood in terms of another representation.

In this mathematical metaphor, the graphical representation is the base domain. The target domain is an equation in the form of $y = f(x)$, which is understood in terms of the graphical representation. Therefore, the function $y = f(x)$ is understood and grounded through the sensory-motor system.

A similar point has been made about fictive motion, a cognitive mechanism through which static concepts are conceptualized in terms of dynamic situations [53]. The sentence *The road runs through the desert* is an example in which the static concept of 'road' is conceptualized as a moving entity. Different researchers have attempted to describe the mechanisms behind understanding such metaphorical descriptions in different ways. Matsumoto has argued that three scenarios could be involved in the processing of such sentences: (1) the movement of the focus of attention, (2) the movement of an imaginary entity, (3) the movement of a specific person (e.g., a speaker or a hearer) [36]. Blomberg and Zlatev have suggested that at least three motivations could be behind fictive motion sentences: enactive perception, visual scanning, and imagination [9]. They introduced three examples to explain these three situations: *the road passes through the plains*, *a line of rocks passes through the plains*, and *the path snakes through the plains*. The concept of 'road' affords motion. Therefore, enactive perception is involved in the understanding of the first sentence. In the second sentence, 'a line of rocks' does not afford motion. People normally cannot walk or run on a line of rocks. Therefore, scanning is involved in the processing of this sentence. In the third sentence, a motion verb has been used creatively. This sentence is a truly metaphorical statement that is processed through imagination. Based on particular interpretations of embodiment, understanding such sentences may involve the activation of the motor system. This is particularly the case when the processing of these sentences is accompanied with gestures that describe their fictive motions.

Lakoff and Núñez argue that certain conceptual metaphors based on fictive motion underlie mathematical concepts and ideas [34]. For example, the mathematical metaphor $f(x)$ *never goes beyond* 1 is based on a fictive motion [4]. When these mathematical metaphors are accompanied by gestures, the comprehender simulates this fictive motion with her/his gestures. This is particularly the case with simulation from the perspective of the agent that performs the gesture. In this way, even highly abstract mathematical concepts can be grounded through mathematical metaphors and the sensory-motor system.

Several of these mathematical metaphors based on fictive motion that can be described by gestures have been discussed in the literature; see for example [39, 40]. In one example, the oscillation of a sequence between two values (a fictive motion) is depicted with a hand moving back and forth. However, we must reemphasize that the perspective of simulation by gestures can play a key role in the process of grounding abstract mathematical concepts. If simulation is done through the perspective of the learner, the learner can achieve better results, as the degree of embodiment could be higher when compared to the perspective of an observer.

Another point that we must note here is that in addition to facilitating the process of grounding abstract concepts, gestures can help us to externalize mathematical knowledge [4]. They can help reduce the load on our working memory and help us manage our mathematical thinking [1, 25, 54] by externalizing information and grounding concepts in the physical environment [4]. In fact, gestures can transfer some parts of the load of cognitive processing into the environment and lighten the load of processing systems [31].

6. Grounding mathematical concepts through representations

Recall that mathematical ideas and objects are essentially abstract. Therefore, we often operate on them through their representations [51]. The grounding of one representation in the physical environment may be easier compared to other representations. However, recall also that all representations are isomorphic to each other. In other words, they are essentially similar at a deep abstract level, although they are different at a superficial or concrete level. When we change one representation of a concept or a problem into another representation, we do not change the essence of that concept or problem. We change one concrete or superficial representation into another to understand it in a better way.

Sometimes a specific representation of a mathematical concept can be better grounded in the physical environment through mediatory channels such as gestures. We can see this in changing the abstract algebraic representation of a concept into a graphical representation. The graphical representation can be depicted by gestures. In this way, the sensory-motor system is employed to ground the initial abstract algebraic representation into concrete environment. Therefore, the algebraic representation is grounded through graphical representation and gestures that depict the graphical representation.

In such cases, the algebraic and graphical representations are essentially the same, although they are different in terms of superficial features. Since these two representations are essentially isomorphic to one another, many things that are true about the graphical representation are also true about algebraic representation. In this way, the algebraic representation, which may be highly abstract, can be grounded in the physical environment through the mediation of a graphical representation and accompanying gestures.

7. The impact of gesture on metaphor processing

Wilson and Gibbs

conducted two experiments to examine the impact of real and imagined gestures on the comprehension of metaphorical phrases, and they found that performing a gesture or even imagining performing a gesture could facilitate one's comprehension of metaphorical phrases related to those gestures [56]. For example, participants of the study were faster in processing the metaphorical phrase *push the argument* when they had just made a pushing gesture. Although gestures were performed just before starting to process the metaphorical phrases, we can hypothesize that gesture could make a significant contribution to the understanding of these phrases even if they were used at the same time that verbal words are expressed. The important point about this study is that in both real and imagined priming situations, the gestures were performed from the perspective of the comprehender. In other words, even in the case of an imagined gesture, the comprehenders imagined that they themselves were performing the gesture related to the metaphorical phrase. This might have had a significant role in priming effect.

Although Wilson and Gibbs worked with daily linguistic metaphors, it would not be a stretch to conjecture that their results could be extended to mathematical and scientific metaphors as well. Whether metaphoric gesture is performed as a prime (performed before processing a metaphor) or at the same time that metaphorical statement is expressed may not create essentially different situations. The metaphoric gesture related to the metaphor could facilitate the process of understanding in both cases. Therefore, we can conclude that if simulation by gesture is made through the perspective of the comprehender, successful learning results can be achieved. This is the case with both real simulation (real gesture) and imagined simulation by gestures (imagined gesture through the perspective of the comprehender).

We should note that some embodiment theories assume that comprehenders usually adopt the perspective of the agent who performs the action [57]. The strong version of embodiment theories assumes that agent's perspective is automatically activated during the processing of action sentences, regardless of the reference of the sentence and other contextual factors [44, 45]. However, others have suggested that the comprehender may adopt a perspective other than the agent's if a self-referential pronoun takes a thematic role other than that of the agent [22]. Moreover, some evidence suggests that when a third party is the agent of an action, and no self-referential pronoun is used, the comprehender may adopt the embodied observer's perspective [10, 42].

8. Summary

In this article, we explored various reasons why performing gestures produces better results in learning than just observing gestures even though the same sensory-motor areas are activated in both performing and observing a gesture. In order to answer our question, we discussed the role of perspective in the simulation of concepts. An action concept can be simulated or embodied by adopting a variety of perspectives, including the perspective of an agent that carries out that action and the perspective of an observer. If the comprehender adopts the perspective of the agent who performs the gesture, the process of understanding and grounding will be more effective. The process of simulation and embodiment can take place by adopting a variety of perspectives. These embodied perspectives may not produce the same depth of learning.

We also discussed the roles of gestures and the comprehender's perspective in mathematical metaphors. We pointed out that the process of solving a complex mathematical problem can often be significantly facilitated when it is transformed into a graphical representation. The processing of a graphical representation involves the activation of the perceptual (visual) and motor systems. This is particularly the case with the metaphorical simulation of mathematical concepts through gestures. The graphical representation of a mathematical concept (such as a curve) can be simulated by a gesture (such as tracing its shape in the air). Therefore, when the abstract representation of a mathematical concept is transformed into a graphical representation, that abstract concept is grounded through visual and motor systems. This can be seen as a mathematical metaphor in which one representation of a concept is understood in terms of another representation.

The graphical representation of the function $y = f(x)$ is an example of mathematical metaphors in which the graphical representation is the base domain. The target domain is an equation in the form of $y = f(x)$. In this mathematical metaphor, the function $y = f(x)$ can be understood and grounded through the comprehender's sensory-motor system. Using gestures to show the trace of a moving object on the graph of this function can contribute to the process of grounding through the sensory-motor system.

Finally, we discussed fictive motions that underlie many mathematical concepts. When these mathematical metaphors are accompanied by gestures, the comprehender simulates this fictive motion with her/his gestures. This is particularly the case with simulation from the perspective of the agent that performs the gesture. In this way, even highly abstract mathematical concepts can be grounded through mathematical metaphors and the sensory-motor system. The job of a creative, mathematically oriented mind is to transform abstract mathematical concepts into appropriate representations that can be effectively grounded through the sensory-motor system. Therefore, mathematical metaphors and in particular representational gestures are useful tools for both the keen mathematician and the keen mathematician-in-training.

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