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Describing and Interpreting the Space of Classroom Learning in Problem-Solving-Based Mathematics Instruction: Variation as an Analytical Lens

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Abstract: The field of mathematics education research has been promoting problem-solving-based mathematics instruction (PS-based MI) to afford opportunities to develop students' conceptual understanding and problem-solving abilities in mathematics. Given its usefulness, there is still little knowledge in the field about how it can afford such opportunities in real classrooms. In this study, an attempt was made to make in-depth observations of such classrooms from the perspective of variation. We examined the differences in the space of learning provided by two lessons of the same teacher in two Ethiopian primary school classrooms. Based on the literature, we identified three key aspects for analysis: mathematical tasks, lesson structure and classroom interaction patterns. Our analysis showed that, even though both lessons focused on the same topic of solving linear inequalities, they were enacted differently. The lesson that employed a PS-based MI approach constituted a wider space of learning than the lesson employing a conventional approach. This study demonstrates the usefulness of our analytical approach for describing and documenting PS-based MI practice, and for qualitatively interpreting the differences in what is mathematically made available to learn. We suggest that it can provide guidelines for mathematics teachers to reflect upon and to enhance learning spaces in their own classrooms.

Keywords: problem-solving-based mathematics instruction; sequences of mathematical tasks; lesson structure; pattern of classroom interactions; dimension of variation; space of learning; theory of variation

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1. Introduction

The main goal of mathematics education is to improve students' learning in mathematics, and studies on teaching and learning mathematics (e.g., [1–3]) have indicated that students' learning is ultimately determined by the opportunities they have to learn in classrooms. Hiebert and Grouws [2] specifically emphasized classroom instructional strategies and the associated curricular features of instruction as the most determining factors in affording or constraining classroom learning opportunities in mathematics. In this regard, problem-solving-based mathematics instruction (PS-based MI), in which problem solving is treated as a major means of the development of an understanding of any given mathematical concept or process, is promoted as a valuable approach for affording students' learning opportunities in mathematics (e.g., [4–8,10]).

However, according to Lester [6], the field of mathematics education research still has little information regarding to what extent PS-based MI can contribute to affording classroom learning opportunities. In this regard, there is too little concern for what is happening inside real classrooms where PS-based MI is implemented, and the present studies do not allow us to make systematic and in-depth observations of such classrooms [5,6]. In particular, there has been a lack of fine-grained descriptions of what is possible for students to learn, how their learning is handled and presented to students and how they experience it in PS-based MI. In this study, an effort was made to describe and interpret how PS-based MI can promote classroom learning opportunities (called the space of learning) as seen from the perspective of variation, which focuses on identifying the necessary conditions for learning [10]. Specifically, we aimed to address the following research question: *How do PS-based MI practices promote the space of learning for developing students' conceptual understanding and problem-solving abilities in mathematics in the Ethiopian primary school classroom context?* To answer this research question, we described and compared the space of learning afforded by two lessons enacted by the same primary school mathematics teacher on the same mathematical topic of solving linear inequalities from the perspective of variation.

In the following, theories of variation and the space of learning that guide this study are discussed first. Then, we describe PS-based MI and its role in providing the necessary conditions for learning. After describing the methods for collecting and analyzing classroom data, we provide a detailed description and comparison of the space of learning constituted by the two lessons. We conclude this article by discussing the main results of the study and its implications for research and teaching.

2. Variation and Space of Learning

The variation theory of learning (VToL) was developed by Ference Marton and colleagues to understand how a learner might come to experience and discern a variation of some aspects of a given phenomenon against a background of invariance (e.g., [10–12]). Many recent studies have used the VToL as an analytical lens through which to examine how the patterns of variation and invariance enacted through sequences of mathematical tasks, lesson structure and classroom interactions afford learning opportunities in mathematics classrooms for students' discernment of the object of learning (e.g., [13–18]). It provides opportunities to systematically study links between the teacher's goal regarding the content or capability that students should develop (intended object of learning), what is possible to learn (enacted object of learning) and how students see and understand the object of learning (lived object of learning) [10,11,19].

In parallel with the VToL, a pedagogic theory in teaching mathematics, called Teaching with Variation (TwV) (*bianshi jiaoxue* in Chinese), was developed by Gu Ling-yuan and colleagues [20,21]. Pang et al. [22] juxtaposed and compared the *VToL* and *TwV* by analyzing the same mathematics lesson with the two theoretical lenses. They concluded that both theories give emphasis for the systematic use of patterns of variation (difference) and invariance (sameness) for classroom learning. The strength of both theories, which is different from other theoretical frameworks, is their main focus on the object of learning [17,22]. Although there is an overlap of interests, TwV focuses more on the principles and intentions of lesson design, whereas the VToL provides a lens for the details of what is enacted and what it might enable learners to discern [22]. We therefore believe that, by juxtaposing the two variation frameworks for describing and interpreting the same lesson, we can gain a better understanding of the patterns of variation and invariance enacted in mathematics classrooms. Their interrelationship can help us give fine-grained descriptions of teaching and learning by focusing on what mathematical content or capability is provided for learning and on how it is handled by the teacher and experienced by the students.

According to the VToL, learning is described as the development of a capability of seeing an object of learning in new ways, which requires the discernment of aspects of the object of learning that were not previously discerned [10,12]. As this discernment occurs only when experiencing variation in that aspect, instructions can be analyzed by examining what aspects vary and what is invariant in any learning situation [23]. In Marton's [11] terms, the variation that is presented in the learning situation is called a *dimension of variation* (DoV). It is a dimension across which a range of varying aspects can be experienced [12]. For example, when learning about solving linear inequalities, it is necessary to consider several DoVs, such as the meaning of inequality, properties of inequality, the interrelationship between solving linear equations and inequalities, the nature of the solutions of linear inequalities, solution strategies, etc. The collection of such DoVs in a specific learning situation constitutes a *space of learning*. The notion of a space of learning is used to describe what is possible to learn in relation to an object of learning. It does not describe what learners discern but captures only what is possible to discern in relation to the intended object of learning. It is therefore a description of the enacted object of learning that is directly observable by the researcher [10,11]. An aspect is considered to be made possible to discern if the corresponding DoV is opened up. The alternative is that the aspect is taken for granted and kept invariant. In this regard, a space of learning is formed when necessary DoVs are opened up in the process of learning [11,12,23].

The space of learning is mainly constituted by carefully structured sequences of mathematical tasks and examples enacted by the teacher [18] and by classroom interactions between the teacher and students or among the students themselves [10,11,24]. During the interactions, even though most DoVs are opened up by the teacher, there is also a possibility for students to open up some DoVs against a background of their prior knowledge and experiences when they try to see and discern the object of learning [12,23]. Hence, the patterns of variation and invariance that are provided by tasks and examples, classroom interactions in relation to a specific OoL and the way these are sequenced in mathematics instruction are of decisive importance for what is possible to discern or for the space of learning in light of that object of learning.

3. Problem-Solving-Based Mathematics Instruction

There is a research agreement in mathematics education that teaching mathematical concepts and teaching problem solving should be integrated in mathematics instruction (e.g., [4,5,7,25]). Students can learn mathematical concepts through solving mathematically rich problems, and problem-solving abilities and other procedural skills are developed through the process of learning mathematical concepts and procedures. To this point, many researchers have suggested that, in order to promote mathematical concept understanding and problem-solving abilities, students must be provided with opportunities to engage in cognitively demanding tasks (e.g., [25–27]). It is also suggested that such tasks that embody key mathematical concepts and skills should be provided to students at the beginning of a lesson (e.g., [4,5,28–31]).

However, some cognitive load theorists have criticized that launching a lesson with cognitively demanding tasks, which is not explicitly linked to prior instruction, is problematic and therefore undermines student learning (e.g., [32–35]). This argument is based on the idea that human working memory has a limited capacity to process new information that has not been stored in long-term memory, and it is therefore easily overloaded when required to solve unfamiliar and challenging problems with various interacting elements [35]. This approach is also pedagogically demanding for teachers, as they might be uncertain about how to structure classroom learning sessions (e.g., [31,36,37]).

The task design principles that we formulated and reported elsewhere (submitted manuscript) were therefore an attempt to contribute to the existing knowledge of problem-solving instruction with the consideration of the issue of cognitive load. The principles are formulated based on the interrelationship between the VToL and TwV. They are used to design and implement sequences of PS-based MI tasks that are at an appropriate level of challenge for learners' expertise. In our study, we used the term PS-based MI to specifically describe an instructional approach in which problem solving is used both as a tool for learning mathematical concepts and procedures and as a goal by itself for developing problem-solving abilities. We describe here the four interrelated task design principles in brief.

1. Mathematical tasks should focus on the intended object of learning.

Bringing the identified object of learning and its critical aspects into the forefront of students' attention is a critical step in a learning situation [10,19]. It is through mathematical tasks, more than in any other instructional way, that opportunities to learn the object of learning are made available to students [38]. Therefore, sequences of PS-based MI tasks should be designed purposefully to direct students' focus towards the object of learning and its critical aspects. What is varying and what is not should be enacted through each part of the sequence to guarantee that the tasks provide the opportunity to discern the object of learning.

2. Mathematical tasks should build on students' anchoring knowledge point.

It is argued that students' learning of new mathematical knowledge or solving new problems is influenced by what they already know and experience (called anchoring knowledge point), and there is a need to make connections between them for meaningful learning to happen [10,20,21]. Therefore, mathematical tasks should be designed to support students to move from what they already know and experience to the desired new knowledge. This can help learners to engage actively in their learning, as the unnecessary cognitive load due to the long knowledge distance is reduced [21], while also providing sufficient mathematical challenge during the implementation of the tasks [7].

3. Mathematical tasks should encourage multiple ways of experiencing the object of learning.

Mathematical tasks should be designed to encourage the use of multiple solution strategies and explanations of how these strategies are connected to each other and to the underlying mathematics. Such types of tasks often have multiple solution pathways that allow students to solve tasks in different ways. As students struggle to solve and as they later discuss different strategies, they develop a conceptual understanding about the mathematics embedded in the problems, and their problem-solving abilities are enhanced [4,20]. In this regard, each student has the opportunity to obtain his or her own unique but equally valued correct answer [27]. Such types of tasks can reduce the unnecessary challenge in problem solving, as learners must only recall one of the several solution strategies to proceed.

4. Sequences of mathematical tasks should be unfolded progressively in instruction.

To help students experience the potential of sequences of tasks, they should be unfolded progressively in a mathematics instruction to support them to discern new knowledge and develop problem-solving abilities in a step-by-step and hierarchical way. At the beginning of instruction, students are provided less challenging or familiar problems to activate and connect their anchoring knowledge point to the intended new knowledge. At the next stage in the hierarchy, students are required to work on subtasks, which are aimed at helping them to discern each of the critical aspects of the object of learning separately. Tasks demonstrating the ways in which different critical aspects can interact with each other are presented at later stages of the sequence. This can reduce cognitive load, as working memory only deals with individual critical aspects in the initial phase of instruction, and at a later stage, all aspects are integrated into a whole. At the end, to consolidate their learning, students are required to work on other tasks that are cognitively similar to the previous learning tasks. This way of enactment for the object of learning is aimed at directing students' attention toward each part of the sequence and to make progress from one level to the next in the hierarchy in a way that the process of reasoning on one task can provide a meaningful basis for reasoning on subsequent tasks [10,20,39].

4. Methods

Our problem-solving-based mathematics instruction (PS-based MI) task design study project was conducted based on the principles of classroom design [40,41], in which instructional designs are tested, analyzed and improved by a study group that includes mathematics teachers who implement the instruction in classroom settings. It focused on investigating the processes of students' development of conceptual understanding and problem-solving abilities in mathematics during the iterative phases of design, implementation and analysis [41] through the lens of variation [10,20] (see Figure 1).

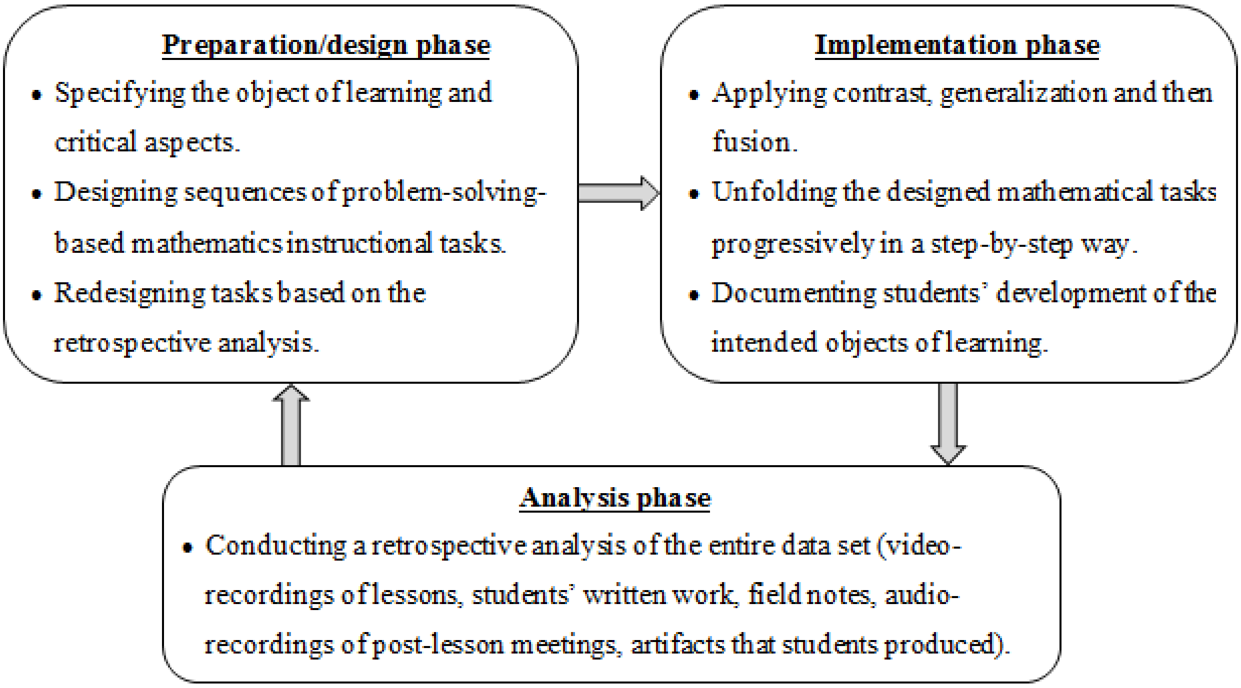

Figure 1. Incorporating theoretical principles of variation into the three design phases of the study.

To analyze the data generated during the three phases of our task design study, we employed a qualitative data analysis approach, which helped us to organize data into themes through an inductive process and to identify patterns among the themes [42]. The results of the analysis of the design and implementation phases of the larger study are reported elsewhere (submitted manuscript). For the purpose of this article, we focus on reporting the analytical process that is specific to describing and interpreting the space of learning constituted by PS-based MI. Here, we describe and compare the space of learning constituted by two lessons of a primary school mathematics teacher—one observed at the beginning of the study (2019) and the other after participating in a PS-based MI task design study group (2020). The lessons were conducted in sixth-grade classes of a regional public primary school in Ethiopia, serving a community with middle to low-class families. In this school, classes were arranged to be homogeneous in terms of the number of students (50 to 53) and the level of their knowledge. What is compared in this study is not what students learn in the two lessons; rather, this study compares what has been made possible to learn from the point of view of the intended objects of learning—developing conceptual understanding and problem-solving abilities in mathematics. This was conducted by analyzing which aspects of the objects of learning were opened up as dimensions of variation (DoVs) and which aspects were kept invariant with respect to making the objects of learning available for students to experience and discern [11].

In order to make the space of learning comparable, the two lessons were designed on the same mathematical topic of solving linear inequalities, and it was taught in grade 6 (12-year-old students) by the same mathematics teacher. The first lesson was individually planned and enacted by the teacher in classroom 1 using the conventional 'explain–practice–memorize' pedagogical approach, whereas the second lesson was designed by the study group and was enacted by the teacher in classroom 2 using the proposed PS-based MI approach. The PS-based MI task design study group consisted of two researchers (one is the first author), the teacher and three other sixth-grade mathematics teachers from governmental primary schools. These schools were chosen for their convenience to be accessed by the researchers. They were located in a large city in the north-west of Ethiopia. The teachers took part in professional development workshops on PS-based MI task design. In the workshops, which were conducted throughout a year, different mathematics lessons were developed through the iterative processes of teaching, observing, revising and re-teaching, targeting to help students experience and discern different objects of learning. The particular mathematical topics of the lessons were chosen because the participating teachers believed that most of their students had difficulty in understanding the intended objects of learning and because the date for teaching the topics in the schools fit well with our study schedule. For the purpose of this study, only one lesson on solving linear inequalities was chosen for analysis. The teacher had nearly 20 years of mathematics teaching experience and a bachelor's degree in mathematics. At the time of data generation, she had been teaching sixth grade mathematics for 5 years in the same school. In this school, she was recognized as a good teacher. The lessons lasted two consecutive regular periods (80 min), and each contained 52 students who were considered to be average when compared to other governmental classrooms in the region.

The qualitative data presented here, which are part of a larger task design study, were collected through classroom observations which were video-taped, and field notes were also taken. Video-taping of each lesson was conducted with two cameras in a real classroom setting: the teacher camera and the whole-class camera. The teacher camera was set at the back of the classroom and was directed to record the teacher's activities throughout the lesson, such as while introducing the lesson, during facilitating small group discussions and while orchestrating whole-class discussions. A small sound recorder was also used to capture the voices of the teacher and students during their interactions. It was used mainly to capture the patterns of teacher questioning and student responses. The whole-class camera was placed in front of the classroom to capture the whole class's activities. Two researchers were present in the classroom during the lesson recordings. One operated the teacher camera, and the other observed and took field notes. The two researchers remained in the back of the classroom and did not interact with the students or the teacher. As designed in the larger study, in addition to the video-taped lessons, lesson plans, textbook pages and classroom worksheets were also collected as supplementary data. Thus, different types of data triangulated our understanding of classroom practices and our interpretation of the results.

To analyze the data, one of the researchers (the first author) transcribed the video-record of both lessons, and the other researcher and the teacher checked both transcripts for accuracy. To support the analysis of the transcribed data, we divided each lesson into manageable units of analysis, called episodes. The episodes were formed on the basis of the pedagogical functions they served. Each lesson episode was therefore classified as belonging to one of the following categories of pedagogical functions: to review students' prior mathematical knowledge (*reviewing*), to introduce new mathematical content (*introducing*), to discuss key mathematical ideas (*discussing*), to practice learned concepts, skills and procedures (*practicing*) and to provide a summary of key concepts (*summarizing*). Within each episode, we focused on the actions of the teacher and students with regard to the tasks, examples, illustrations and exercises provided to students, and we also focused on the type of questions asked and responses given by the teacher and students in the different forms of classroom interactions. Specifically, the focus was on identifying different DoVs opened up in each episode. For example, variations in the tasks and examples, solution strategies and representations were labeled and recorded as DoVs with chronological numbers during their appearance throughout the episodes. The two researchers worked together for the identification of the DoVs. The sequences of DoVs opened up in the two lessons were finally summed up, as shown in Tables 1 and 2. In this way, we produced a description of what is mathematically made available to learn within and across episodes with regard to realizing the intended objects of learning.

In our study, we took precautions to protect the autonomy and anonymity of participating schools, teachers and students in any report of the results of the study. It was approved by the Ethical Committee: Norwegian Centre for Research Data (NSD–Norsk senter for forskningsdata) on 14 January 2020, with the following approval code: Notification Form 811119.

5. Results

In this section, the results of the analysis are presented. We describe how the mathematical topic of solving linear inequalities was handled in the two lessons enacted in classroom 1 and classroom 2. The results are organized in terms of the structure of the lessons, the nature of tasks presented to students and the type of patterns of classroom interaction.

5.1. The Space of Learning in Classroom 1

5.1.1. Lesson Structure

The structure of the lesson enacted in classroom 1 included the following stages (called episodes): reviewing, introducing, discussing, practicing and summarizing. Each stage was used as follows: After checking students' homework, the teacher revised the previous lesson on solving linear equations orally. She then introduced the topic of the current lesson, followed by providing short notes on the definition and procedures of solving linear inequalities. She then led whole class discussions and gave examples from the textbook. Next, she ordered students to work in small groups to solve linear inequality exercises, which were similar to the demonstrated examples. The teacher walked around the classroom monitoring students' work and eventually invited four students to show their work on the black board in parallel and provided feedback on the exercises. Finally, the teacher ended the lesson with a brief summary and homework. This lesson was heavily teacher-directed and hence provided limited opportunities for students to engage in discussions or to provide extended explanations. The teacher-led demonstration of examples took much of the lesson time. Students were involved in group discussions only for a short period of time and for a few exercises near the end of the lesson for the purpose of practice. They were mainly involved in co-solving the examples and exercises with the teacher.

5.1.2. Mathematical Tasks Presented to Students

In classroom 1, the teacher provided different examples and exercises taken from the textbook. In episode 2, for example, two examples were given to support the definition and to show how properties of inequalities are used as a solution strategy. In episode 3, the teacher demonstrated four examples, aiming simply to help them learn the procedure to apply the properties of inequalities. The exercises given in episode 4 were intended to be used as exercises on the concept or procedure demonstrated by the teacher. Even though different mathematical examples and exercises were used across the episodes, the teacher performed most of them herself. All examples and exercises seemed to have the same purpose, namely to practice the solution procedures. In this lesson, few DoVs were opened up by the teacher, mainly during work on the first two examples in episode 2. However, while working through the remaining examples and exercises in episodes 3 and 4, similar DoVs were involved.

5.1.3. Patterns of Classroom Interaction

Throughout the lesson in classroom 1, explanations were provided by the teacher, and students' main activity was listening to the teacher. Students (often together with classmates) provided one-word answers to her questions or short responses to her unfinished sentences. Feedback to students' responses was intended to direct them. When students presented their solutions to the exercises during the ending discussion, she followed up on their work with questions, as indicated in the excerpt from the lesson transcript.

Excerpt: Sample teacher–student interaction in a group of classroom 1

(1) T: Why did you add 3 to both sides of the inequality $x - 3 > 10$?
(2) S: To remove 3.
(3) T: Is it to remove 3 or -3?
(4) S: -3.
(5) T: Why did the inequality sign not change?
(6) S: Because that is the addition rule.

(7) T: You are right. How do you determine the final solution set {14, 15, 16, ... }?
(8) S: Because x > 13.
(9) T: Right, but it is also because the given working region (domain) is the set of natural numbers.

This interaction pattern showed that the questions asked by the teacher (e.g., lines 1, 5, 7) pressed for memorized knowledge or procedural explanations. Students provided responses based on procedures (lines 3, 6) and mathematical facts (line 8). Students were not pressed to elaborate on their explanations or justify main mathematical ideas. In this classroom, the potential of the examples and exercises were limited to engaging students to apply known procedures in solving linear inequalities and to provide short answers to the teacher's leading questions. The instructional activities and DoVs opened up in each of the episodes in the lesson are summarized in Table 1.

Table 1. Instructional activities and corresponding DoVs opened up in classroom 1.

Episode	Instructional Activities	Dimensions of Variation
Episode 1: Reviewing/Introducing (5 min)	After checking students' homework given in the previous day's lesson, the teacher asked: *Who can remember what you have learned last period?* One student gave a response: *We learned about linear equations.* She then asked: *What is a linear equation?* Another student said: *That which involves an equal sign.* The teacher then said: *A linear equation is an equation which can be written as ax + b = 0.* She gives two sample linear equations emphasizing that a and b are constants, x is a variable and the value of x is a solution. The teacher announces the title of the lesson: *Solving linear inequalities.* She then asked: *Do you remember the meaning of the signs* $<, \leq, >, \geq, \neq$? One student gave a response. She then said: *An algebraic expression involving one of these mathematical signs is said to be an inequality.*	DoV1: General form of linear equation • the constants (a, b), variable (x) DoV2: The meaning of inequality signs • $<, \leq, >, \geq, \neq$
Episode 2: Discussing (teacher-led + whole-class) (20 min)	The teacher wrote the following definition: *Definition: An inequality is said to be a linear inequality if it is written in one of the following forms:* $ax + b < 0$, $ax + b \leq 0$, $ax + b > 0$, *or* $ax + b \geq 0$, *for* $a \neq 0$. She then provided examples to support the definition. $x + 3 > 0$, $x + 4 \leq 0$, $2x + 1 < 0$ *and* $3x + 7 \geq 0$ *are linear inequalities.* She worked on another example from the textbook: *Which of the following numbers make the inequality* $x + 3 < 5$ *true?* *(a)* -4 *(b)* -1 *(c) 0 (d) 2 (e) 3* She demonstrated how to find solutions by substituting the given numbers. She said: *The numbers* -4, -1 *and 0 make the inequality true. They are solutions to the inequality. There are also other solutions to it. How can we find them?* She wrote the addition, subtraction, multiplication and division properties of inequalities on the blackboard. She then demonstrated the properties as follows. $\frac{2}{3} < \frac{5}{3}$ *and* $\frac{2}{3} - \frac{1}{3} < \frac{5}{3} - \frac{1}{3}$ $4 > 2$ *and* $4 \times 3 > 2 \times 3$ $10 < 15$ *and* $10 \div 5 < 15 \div 5$ The teacher then provided the following example: *Find the solution set for the inequality* $x + 4 < 7$ *in the given regions.* *(a) Set of whole numbers (b) Set of natural numbers* She then found the solution interacting with students. $x + 4 < 7$ $x + 4 - 4 < 7 - 4$ $x < 3$ *(a) solution set = {0, 1, 2} in the set of whole numbers* *(b) solution set = {1, 2} in the set of natural numbers*	DoV3: General form of linear inequalities • inequality signs, constants, variable DoV4: Solutions to linear inequality DoV5: Properties of inequality • addition, subtraction, multiplication, division DoV6: Applying the subtraction property of inequalities DoV7: Solution of inequalities within a specified set

Table 1. *Cont.*

Episode	Instructional Activities	Dimensions of Variation
Episode 3: Discussing (group work + teacher-led demonstration) (20 min)	The teacher provided the following textbook examples: *Find the solution set of each of the following inequalities.* *(a)* $x - 2 > 5$, *x is a whole number* *(b)* $2x < 10$, *x is a natural number* *(c)* $\frac{1}{4}x > 3$, *x is in {0, 1, 2, . . . , 20}* *(d)* $x + \frac{7}{8} < 1$, *x is in {1, 2, 3, . . . }* The teacher demonstrated all the examples to the whole class. *(a)* $x - 2 > 5$ *(b)* $2x < 10$ $x - 2 + 2 > 5 + 2$ $\frac{2x}{2} < \frac{10}{2}$ $x > 7$ $x < 5$ *S.S = {8, 9, 10, . . . }* *S.S = {1, 2, 3, 4}* *(c)* $\frac{1}{4}x > 3$ *(d)* $x + \frac{7}{8} < 1$ $4(\frac{1}{4}x) > 4(3)$ $x + \frac{7}{8} - \frac{7}{8} < 1 - \frac{7}{8}$ $x > 12$ $x < \frac{1}{8}$ *S.S = {13, 14, 15, . . . , 20}* *S.S = { }*	DoV8: Applying the addition property of inequalities DoV9: Applying the multiplication property of inequalities DoV10: Applying the division property of inequalities DoV7: Solution of inequalities within a specified set
Episode 4: Practicing *(15 min)*	She provided the following exercises from the textbook for the students to work first in small groups and then to show to the whole class. *(a)* $x + 4 < 8$, *x is in the set of whole numbers.* *(b)* $x - 2 < 7$, *x is in the set of integers.* *(c)* $x - 3 > 10$, *x is in the set of natural numbers.* *(d)* $\frac{1}{5}x > 2$, *x is in the set of negative integers.* The teacher walked around each group to check their progress. The teacher then invited four students to show their work on the black board, in parallel, and provided feedback on the four tasks.	DoV6: Applying properties of inequality DoV7: Solution of inequalities within a specified set
Summarizing (5 min)	The teacher provided a short summary. *She gave emphasis on the form of writing inequalities, properties of inequalities and the nature of a solution set.* *At the end, she provided homework exercises from the textbook.*	DoV3: The general form of linear inequalities DoV5: Properties of inequality

5.2. The Space of Learning in Classroom 2

5.2.1. Lesson Structure

The lesson in classroom 2 was directed by following the reviewing, introducing, discussing, practicing and summarizing stages (episodes). The teacher first allowed students to work in pairs on tasks to help them activate their prior knowledge and experiences. After announcing the topic of the lesson, the teacher provided sets of problem-solving tasks and facilitated a whole-class discussion to create a shared understanding of the tasks. The teacher then clarified the expectations for the students, such as how students would work individually and in small groups; which products they would prepare for the whole-class discussion; and how classroom interactions should look. Students then engaged in a problem-solving activity, first individually and then in small groups for an extended lesson time. Students were encouraged to ask each other questions while collaborating. Next, the teacher allowed selected students to present their work and helped them put their solutions on the blackboard in a sequence that was easier to share with other students. The teacher asked questions and facilitated a discussion of the different solution strategies. Following each presentation, students were invited to ask the presenters for elaborations and also to challenge them with alternative approaches. By connecting student-generated solution strategies and the associated key mathematical concepts, the teacher summarized the main ideas of the discussion. Before providing a brief summary of the lesson, the teacher allowed students to work independently on exercises, aiming to consolidate their understanding of solving linear inequalities. The structure of this lesson, therefore, provided opportunities for students to engage in multiple modes of discussions, to provide extended explanations and to justify their reasoning.

5.2.2. Mathematical Tasks Presented to Students

Within and across the episodes of the lesson in classroom 2, sequences of mathematical tasks were presented to students in a step-by-step and hierarchical way. Tasks in episode 1, for example, focused on simplifying algebraic expressions and solving linear equations. These tasks helped students to activate their prior mathematical knowledge in linear equations. In episode 2 and 3, other sets of tasks that were linked to students' real-life contexts were presented, intending to help students understand and solve linear inequalities. The tasks in episode 4 provided opportunities for them to practice the strategies they co-developed during the discussions in episodes 2 and 3. Throughout lesson 2, the tasks encouraged students to use multiple solution strategies and representations, which was also an explicit instruction in the tasks. The whole-class discussion provided evidence that students used a variety of strategies such as repeated addition/subtraction and multiplication and representations such as tables, variables and rules to solve the tasks. In this lesson, even though the full set of tasks was provided at once, students were allowed to work only on subtasks from the sequence. Before moving on to the next subtasks, the solutions to the first subtasks were discussed and used as a scaffolding tool for the next subtasks. The gradual introduction of aspects in the tasks while increasing the complexity of expressions, such as increasing the number of terms, the number of operations and the simplification steps involved, attracted students' attention towards what was varied and thus what remained invariant to open up DoVs.

In this classroom, problem solving was used as a context for students to develop the knowledge of mathematical concepts and procedures and, at the same time, to practice and consolidate this knowledge through solving the problems. The saving/withdrawing problem context that involved a situation of linear equality and inequality, for example, was used to develop the meaning of inequalities and the formulation of their algebraic representations. The problem provided a background to develop algebraic representations of linear equations and inequalities containing variables, constants and equality/inequality symbols. The solutions to these linear equations and inequalities, which had first been found by means of arithmetic representations, were regenerated by substituting values into the algebraic expressions. In this way, the interrelationship between the solution to linear equations ($400 + 5w = 565$, and $582 - 8w = 422$) and those of its related inequalities ($400 + 5w > 565$ and $582 - 8w < 422$) was experienced by the students during their transformational work.

5.2.3. Patterns of Classroom Interaction

The lesson in classroom 2 was structured to facilitate discussions through employing multiple modes of interaction. As students worked on the tasks, individually, in pairs or in small groups, the teacher circulated, validating their work, and took note of the various student inputs to introduce variation in aspects of the intended objects of learning. Once students worked on the tasks, the teacher allowed them to present their solutions to the whole class. She used her observation notes to decide which students to present and in what order, so that the whole class could see and connect a range of varied solution approaches. At many points in their interaction, the teacher asked multiple open-ended questions that prompted students to explain and justify their conceptual thinking in discussions, as indicated in the following excerpts.

Excerpt 1: Sample teacher–student interaction in one group of classroom 2

(1) T: What is Mahlet's amount of money at the end of the 5th week?
(2) S: 425 birr.
(3) T: How do you get it?
(4) S: By adding 5 to 400 five times.
(5) T: Why do you add 5 five times?
(6) S: Because Mahlet saves 5 birr per week.
(7) T: Why do you add to 400?
(8) S: Initially she has 400 birr in her account.
(9) T: What is her amount of money at the end of the 10th week?

(10) S: 450 birr.
(11) T: How about at the end of 20th week? Does the use of variables help to get the answer?

Excerpt 2: Sample teacher–student interaction in another group of classroom 2

(12) T: When does Mahlet's amount of money equal 565?
(13) S: At week 33.
(14) T: How do you get it?
(15) S: By dividing 156 by 5.
(16) T: What is 156?
(17) S: 565 − 400 = 156.
(18) T: So, when does it become greater than 565?
(19) S: At week 34.
(20) T: Why?
(21) S: She saves an additional 5 birr.
(22) T: Is there any other answer to it?
(23) S: Yes, 35.
(24) T: How about at week 37?
(25) S: It is greater.
(26) T: How many answers does it have?
(27) S: Many.
(28) T: Is there any other approach to find the answer?

As indicated in the excerpts above, her questions encouraged students to try more than one solution approach (e.g., lines 11, 28). Such pressing questions also helped to particularly highlight how linear inequalities have more than one solution (lines 22–27) and how solving linear equations and linear inequalities are deeply interwoven (e.g., lines 18–21). The teacher's questions also pressed students to make explicit connections between the different solution approaches that were shared by their classmates. For example, after one student presented and explained her group's tabular representation of the solution, the teacher asked the whole class to compare it with the previously presented strategies—repeated addition and multiplication (Is there a similarity or a difference b/n 1st and 2nd work? 1st, 2nd and 3rd work?). As a result of these connections, students were able to see how these three approaches were related to one another and to the mathematics being worked on.

During the classroom interaction, multiple DoVs were opened up by the teacher. For example, both the form of linear inequalities (algebraic representations) and the parts of the inequalities (variables, coefficients and constants) were focused on by the teacher as DoVs. First, only one aspect was varied: the number of weeks (variable). Thus, the initial amounts (constants) and weekly savings/withdrawal amounts (coefficients) were kept invariant so that students could generalize the form of the linear inequalities from the patterns. In this case, the aspect of 'variable' was varied and hence separated, whereas other aspects, such as 'coefficients' and 'constants', were kept invariant. The forms of inequalities were then allowed to vary by involving other coefficients and constants. After varying these aspects one at a time, both were allowed to vary simultaneously. In addition, the teacher focused on other relevant aspects of solving linear inequalities as DoVs, such as the relationship between the parts of the algebraic inequality (operations); the transformation between the parts (properties of inequalities); and the relationship between arithmetic and algebraic representations of the same inequality. In order for the students to identify the relationship between the arithmetic and algebraic representations, they had to make generalizations about the invariant aspect from the patterns of the arithmetic representations. Thus, different patterns of DoVs were opened up by contrasting, separating, generalizing and fusing aspects of the objects of learning. The instructional activities and DoVs that were opened up in each of the lesson episodes in classroom 2 are summarized in Table 2.

Table 2. Instructional activities and corresponding DoVs opened up in lesson 2.

<table>
<tr><th>Episode</th><th>Instructional Activities</th><th>Dimensions of Variation</th></tr>
<tr>
<td>Episode 1: Reviewing/Introducing (10 min)</td>
<td>The teacher started the lesson by providing the following exercises to students to work in pairs.
1. Simplify each of the following algebraic expressions into their lowest terms.
(a) $12y + 12 - 6y$ (b) $2x + 7 + 5x - 15 + 6 - x$
2. Solve each of the following equations.
(a) $z + 5 = 9$ (b) $7n = 14$ (c) $2m - 4 = 6$
After checking students' work, she summarized students' answers to the whole class. She gave emphasis on the nature of algebraic terms, the change in variables and the use of properties of equality to solve one-step and two-step equations.
The teacher announced the title of the lesson: solving linear inequalities. She then asked: What is an inequality? What is the difference b/n an equation and an inequality? Two students gave responses. In her feedback, she emphasized the difference in the meaning of the signs.
She asked: What do you know about bank accounts? One student gave a response. She emphasized the key terms saving and withdrawing.
She then provided the following tasks written on a piece of paper:
Mahlet had 400 birr in her bank account. She saves an additional 5 birr each week in her account. Her brother Yonas had 582 birr in his account. He withdraws 8 birr each week from his savings.
(a) How much money does Mahlet have in her account in the fifth week? How about in the tenth week?
(b) In which week does Mahlet have 565 birr in her account? In which week is it greater than 565 birr?
(c) In which week does Yonas have 422 birr in his account? In which week is it less than 422 birr?
(d) At what week do Mahlet and Yonas have the same amount of money in their accounts? In which week is Mahlet's money greater than Yonas's money?
She then announced: You can start working only on the first two.</td>
<td>DoV1: Adding/subtracting like terms
DoV2: Change in variables
• y, x
DoV3: Properties of equality
• addition, multiplication, subtraction properties
DoV4: The number of steps involved in solving linear equations
• single-step, two-step equations
DoV2: change in variables
• z, n, m
DoV5: Meaning of mathematical signs
• equal to, less than, greater than
DoV6: Context
• mathematical context (adding, subtracting)
• real-life context (saving, withdrawing)</td>
</tr>
<tr>
<td>Episode 2: Discussing (individual + group work + whole-class) (25 min)</td>
<td>The teacher asked questions while checking students work as they worked individually and in small groups, encouraging them to try more than one method. She took note of some students' work as she checked.
After some time had passed, the teacher invited students to present their solutions to the whole class.
She organized and orchestrated the presentation in the following way:
(a)
Strategy 1:
5th week: $400 + 5 + 5 + 5 + 5 + 5 = 425$ birr.
10th week: $400 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 450$ birr.
Strategy 2:
5th week: $5 \times 5 = 25$ birr, so $400 + 25 = 425$ birr
10th week: $5 \times 10 = 50$ birr, so $400 + 50 = 450$ birr
Strategy 3:
<table>
<tr><th>Week</th><th>Money</th></tr>
<tr><td>1st</td><td>405</td></tr>
<tr><td>2nd</td><td>410</td></tr>
<tr><td>3rd</td><td>415</td></tr>
<tr><td>4th</td><td>120</td></tr>
<tr><td>5th</td><td>425</td></tr>
</table>
<table>
<tr><th>Week</th><th>Money</th></tr>
<tr><td>6th</td><td>430</td></tr>
<tr><td>7th</td><td>435</td></tr>
<tr><td>8th</td><td>440</td></tr>
<tr><td>9th</td><td>445</td></tr>
<tr><td>10th</td><td>450</td></tr>
</table>
(b):
Strategy 1:
$400 + 5 + 5 + 5 + \ldots + 5 = 565$
1w 2w 3w 33w
Strategy 2:
$400 + 50 + 50 + 50 + 15 = 565$
10w 20w 30w 33w
Strategy 3:
$565 - 400 = 165$. So, $165 \div 5 = 33$.
The teacher asked questions to the presenting students.
The teacher then connected their solution strategies and produced the following answer:
<table>
<tr><th>Week</th><th colspan="2">Amount in her account</th></tr>
<tr><td>1st</td><td>$400 + 5$</td><td>$400 + 1 \times 5$</td></tr>
<tr><td>2nd</td><td>$400 + 10$</td><td>$400 + 2 \times 5$</td></tr>
<tr><td>3rd</td><td>$400 + 15$</td><td>$400 + 3 \times 5$</td></tr>
<tr><td>4th</td><td>$400 + 20$</td><td>$400 + 4 \times 5$</td></tr>
<tr><td>5th</td><td>$400 + 25$</td><td>$400 + 5 \times 5$</td></tr>
<tr><td>6th</td><td></td><td></td></tr>
<tr><td></td><td colspan="2">$400 + w \times 5$
$400 + 5w$</td></tr>
</table></td>
<td>DoV7: Multiple solution strategies
• repeated addition, multiplication
DoV8: Multiple representations
• making tables, using variables
DoV9: Interconnection between solving linear equations and inequalities
DoV10: Number of solutions
• one solution, many solutions
DoV11: connection between different strategies
DoV12: Representations of inequalities
• arithmetic, algebraic
DoV13: Number pattern to develop the algebraic form of inequalities
• variable (the number of weeks)
• constant (initial amount of money)
• coefficient (weekly saving rate)
DoV3: Properties of equality
• subtraction, addition multiplication, division
DoV14: Properties of inequality
subtraction, addition multiplication, division
DoV6: Context
• mathematical context (adding)
• real-life context (saving)</td>
</tr>
</table>

Table 2. *Cont.*

<table>
<tr><th>Episode</th><th>Instructional Activities</th><th>Dimensions of Variation</th></tr>
<tr><td></td><td>$400 + 5w = 565$
$400 + 5w - 400 = 565 - 400$
$5w + 400 - 400 = 565 - 400$
$5w = 165$
$5w/5 = 165/5$
$w = 33$ weeks.
So, the money in her account is equal to 565 at the 33rd week and is greater than it at weeks 34, 35, 36, and so on.
That is, $400 + 5w > 565$
$400 + 5w - 400 > 565 - 400$
$5w + 400 - 400 > 565 - 400$
$5w > 165$
$5w/5 > 165/5$
So, $w > 33$ week.</td><td></td></tr>
<tr><td>Episode 3:
Discussing (group work + whole class)
(25 min)</td><td>The teacher then asked: Now do the remaining two questions (c and d) in your groups.
She invited three students to present their work to the whole class.
She organized and orchestrated the presentation in the following way:
(c):
Strategy 1:
$58 - 8 - 8 - 8 - \ldots - 8 = 422$
1w 2w 3w 20w
So, his amount of money is equal to 422 at the 20^{th} week and less than it at week 21, 22, 23, and so on.
Strategy 2:
$582 - 422 = 160$. So, $160 \div 8 = 20$.
Strategy 3:
$582 - 8w = 422$, so $w = 20$ weeks.
$582 - 8w < 422$, so $w < 20$ weeks.
(d):
Strategy 1:
<table>
<tr><th>Week</th><th>Mahlet</th><th>Yonas</th></tr>
<tr><td></td><td>400</td><td>582</td></tr>
<tr><td>1st</td><td>405</td><td>574</td></tr>
<tr><td>2nd</td><td>410</td><td>566</td></tr>
<tr><td>3rd</td><td>415</td><td>558</td></tr>
<tr><td>4th</td><td>420</td><td>550</td></tr>
<tr><td>5th</td><td>425</td><td>542</td></tr>
<tr><td>6th</td><td>430</td><td>534</td></tr>
<tr><td>7th</td><td>435</td><td>526</td></tr>
<tr><td>8th</td><td>440</td><td>518</td></tr>
<tr><td>9th</td><td>445</td><td>510</td></tr>
<tr><td>10th</td><td>450</td><td>502</td></tr>
</table>
<table>
<tr><th>Week</th><th>Mahlet</th><th>Yonas</th></tr>
<tr><td>11th</td><td>455</td><td>494</td></tr>
<tr><td>12th</td><td>460</td><td>486</td></tr>
<tr><td>13th</td><td>465</td><td>478</td></tr>
<tr><td>14th</td><td>470</td><td>470</td></tr>
<tr><td>15th</td><td>475</td><td>462</td></tr>
</table>
Strategy 2:
$400 + 5w = 582 - 8w$
$5w + 8w = 582 - 400$
$13w = 182$
$w = 182/13 = 14$ week
Through discussion, the teacher then provided full solution steps to the whole class.
(c) $582 - 8w < 422$
$582 - 8w + 8w < 422 + 8w$
$582 < 422 + 8w$
$582 - 422 < 422 - 422 + 8w$
$160 < 8w$
$160/8 < 8w/8$
$20 < w$
That is, $w > 20$ weeks.
Hence, his money in his account is less than 422 after the 20th week. At the 72nd week, the money in his account will be:
$582 - 8w = 582 - 8(72) = 6$ birr.
That is, at $w > 72$ weeks, he cannot withdraw 8 birr per week for his account.
(d) $400 + 5w > 582 - 8w$
$5w + 8w > 582 - 400$
$13w > 182$
$w > 182/13$
$w > 14$ week
Mahlet's money is equal to Yonas's money at week 14 and is greater at week 15 and later.</td><td>DoV15: Type of linear inequality
• less than, greater than
DoV16: Multiple solution strategies
• repeated subtraction, multiplication
DoV8: Multiple representations
• making tables, using variables
DoV9: Interconnection between solving linear equations and inequalities
DoV10: Number of solutions
• one solution, many solutions
DoV11: connection between different strategies
DoV12: Representations of inequalities
• arithmetic, algebraic
DoV17: Number patterns to develop the algebraic form of inequalities
• variable (number of weeks)
• constant (initial amounts of money)
• coefficient (weekly withdrawal rate)
DoV3: Properties of equality
subtraction, addition multiplication, division
DoV14: Properties of inequality
• subtraction, addition multiplication, division
DoV6: Context
• mathematical context (adding)
• real-life context (saving)</td></tr>
<tr><td>Episode 4:
Practicing (15 min)</td><td>The teacher provided the following exercises and ordered students to work individually.
Solve each of the following inequalities.
(a) $x + 5 < 8$ (b) $y - 3 > 2$ (c) $2z < 10$
(d) $2x - 3 > 7$ (e) $4y - 1 < 3y + 8$
The teacher walked around the classroom to check every student's work.</td><td>DoV15: Type of linear inequality
• less than, greater than
DoV14: Properties of inequality
subtraction, addition, multiplication, division</td></tr>
<tr><td>Episode 5:
Summarizing
(5 min)</td><td>The teacher orally provided a summary through questioning in 5 min.
She gave emphasis on comparing solving equations and inequalities based on the number of solutions and solution strategies.</td><td>DoV18: Difference b/n an equation and an inequality
• equality/inequality of two expressions
• single solution/many solutions</td></tr>
</table>

6. Discussion and Conclusions

The aim of this study was to describe and interpret the differences in the space of learning constituted by two lessons, in which one was taught in classroom 1 at the beginning of the study and the other was taught in classroom 2 after participating in a PS-based MI task design study group. The analysis on what was possible to learn and how it was handled in the two classrooms was conducted by means of the opened DoVs regarding the intended objects of learning, thus developing conceptual understanding and problem-solving abilities in mathematics. By comparing what was happening in the two classrooms, we showed that, even though the same teacher taught the same mathematical topic in the two classrooms, there were differences in which DoVs were opened. This implies that what was possible for students to learn—the space of learning—was different in the two classrooms. Here, we describe and interpret the differences in the space of learning from three aspects.

The first aspect is the focus and the way in which lessons are structured. The lesson in classroom 1 was structured in a traditional way, via teacher explanations of mathematical concepts and procedures followed by student practice and feedback. That is, the teacher first taught the concept of linear inequality and its solution procedures, and students then practiced the application of the procedures through exercises, followed by the teacher's correction of student answers (see Table 1). On the other hand, the lesson in classroom 2 employed a structure with launching–exploring–discussing/summarizing–consolidating stages, as suggested by Sullivan and colleagues [43,44] (see Table 2). That is, the teacher launched the lesson by activating students' prior knowledge and experiences followed by providing sequences of problem-solving tasks. Then, students were allowed to explore the problems first individually and then in small groups. Based on some selected students' presentation of their work, the teacher orchestrated whole-class discussions on student-generated solutions and strategies and provided a summary of the problems. Before the teacher concluded the lesson, students worked on exercises independently for the purpose of practice and consolidation. Unlike classroom 1, which was dominated by teacher explanations, the multiple means of student engagement employed in classroom 2 allowed learning to be accessible to all students [31]. In general, the main focus of the lesson in classroom 1 was on clarifying the general form of linear inequality and the application of procedures to find solutions to it, which limited students' understanding of the mathematical concepts attached to each step of the application of the procedures. In contrast, the focus in classroom 2 was on developing students' understanding of the process and ability of discovering mathematics patterns, developing algebraic representations of mathematical relationships and building connections between mathematics and daily life situations. The focus in the two classrooms, which is described in terms of the critical aspects of the enacted objects of learning, is summarized in Table 3.

Table 3. Critical aspects of the enacted objects of learning in the two classrooms.

Classroom 1	CA1: Understanding the structure of linear inequalities; CA2: Understanding properties of inequalities; CA3: Applying properties of inequalities to find solution sets of linear inequalities within a specified set.
Classroom 2	CA1: Solving real-life problems involving linear inequalities and interpreting the results; CA2: Developing algebraic representations of linear inequalities from the patterns of arithmetic representations of linear equations; CA3: Applying properties of inequalities to solve linear inequalities.

This analysis, therefore, shows that the structure and focus of the lesson in classroom 2 provided a better opportunity (a wider space of learning) than that in classroom 1.

As a second aspect, a comparison of the patterns of variation and invariance created within and between the tasks presented to students made it possible to describe the differences in DoVs that were opened or not in the two lessons. The tasks presented to students in the two classrooms could have the potential to open up many DoVs with regard to solving linear inequalities. However, the way in which some aspects were focused gave students different opportunities to discern the objects of learning. In classroom 1, textbook examples and exercises were used to help students acquire the knowledge of solving linear inequalities with a focus on the use of properties of inequalities. Most of the examples and exercises were demonstrated by the teacher with little involvement of students. They were so similar that almost identical procedures were applied by the teacher, with little variation, hence opening up few DoVs. This was in sharp contrast to the tasks used in classroom 2, in which patterns of variation and invariance in many relevant aspects were created by the use of well-prepared sequences of problem-solving tasks, aiming to help students understand the real meaning of linear inequalities through transferring real-life contexts to mathematics. In this process, they developed algebraic representations of the linear inequalities involved in these contexts and applied their prior mathematical knowledge and problem-solving experiences to find solutions to the problems. Several aspects of the tasks were deliberately exploited by the teacher to generate important DoVs and to make the corresponding aspects of the objects of learning possible to discern. All tasks (given during the reviewing, introducing and practicing stages) were first carried out by students with little guidance from the teacher. In addition, unlike classroom 1, tasks in classroom 2 provided opportunities for students to explore different solution strategies (variations) in solving a single problem (invariant), and to solve different problems (variation) with the same strategy they identified (invariant). By opening up such DoVs (the use of multiple strategies), the space of learning in classroom 2 was much wider than that in classroom 1, which focused on a particular strategy (the use of properties of inequalities). These types of mathematical tasks also helped students to provide explanations and make connections among different strategies. Hence, our analysis showed that the types of mathematical tasks and in what order they are presented to students created different patterns of variation and invariance in the two classrooms, highlighting different aspects of the objects of learning (see Tables 1 and 2).

Similar to our analysis results on the aspects of lesson structure and mathematical tasks, there was also a considerable difference in the two classrooms with respect to patterns of interaction, which is an important factor influencing the space of learning. Even though both teacher-to-student and student-to-student interactions contribute to opening up DoVs, in this study, our analysis focused on the patterns of teacher-to-student interactions. As part of teacher-to-student interactions, teacher questioning patterns can play a significant role in providing students with the opportunity to develop their mathematical thinking and to deepen their conceptual understanding in mathematics [9,45,46]. We examined here two patterns of teacher questioning: funneling and focusing patterns (see [47]). In classroom 1, the teacher guided students to complete the examples and exercises by asking a series of simple and closed-ended questions without giving enough space for them to find the solution and provide relevant explanations by themselves. The teacher provided immediate feedback to students' quick responses (e.g., see the excerpt in Section 5.1.3). Based on this analysis, this interaction pattern is characterized as 'funneling' [47], in which the teacher asked a series of leading questions to funnel students thinking into narrow and specific response patterns that lead them to reach a predetermined end. However, in classroom 2, through a series of open-ended questions, the teacher encouraged students to focus on each of the important aspects of the tasks. The teacher emphasized student responses and guided them through their own thinking to recognize and correct their misunderstandings and also to develop new understandings (e.g., see the excerpts in Section 5.2.3). Thus, the interaction pattern in classroom 2 is characterized as 'focusing' [47]. The questions asked by the teacher opened up DoVs that provided opportunities for students to explore and justify their thinking to the teacher and other students. These opportunities were exploited by the students, and their responses to these questions were in turn used by the teacher to open up further DoVs for exploration and justification. The space of learning in classroom 2 was therefore a collaborative space constituted by both the teacher and the students [23].

Based on our analysis, we identified that the number of DoVs opened in classroom 1 was small (15 openings, in which 9 were distinct), and all were opened by the teacher (see Table 1). However, in classroom 2, several DoVs were opened (31 openings, in which 18 were distinct), which were jointly opened by both the teacher and students (see Table 2). For example, the input from students' small group discussions opened up some DoVs regarding the choice of solution strategies and representations, a dimension that might not be opened otherwise. The teacher used student-generated DoVs as a tool to introduce variations in that aspect to other students in the class. We therefore conclude that the teacher in classroom 2 created a space of variation in relation to solving linear inequalities, which included the following: understanding linear inequalities based on their knowledge of linear equations; solving linear inequalities simply by substitution based on the solutions they obtained to the related linear equation; and, finally, modeling inequalities using variables, thus solving by using inequality rules. The same teacher in classroom 1, however, focused only on using inequality rules. Thus, in classroom 2, the teacher created a wider space of learning for students to experience than she did in classroom 1.

To conclude this section, even though the teacher taught the same topic in both classrooms, our analysis results suggest that the two lessons offered two quite different possibilities for learning on solving linear inequalities, as different DoVs were opened up and thus were made possible for students to discern. The nature of mathematical tasks, the structure of lessons used in connection with the enactment of these tasks and the patterns of classroom interaction during the implementation of the tasks might be the most determining factors that can impact the space of learning constituted by PS-based MI. Students' conceptual understanding and problem-solving abilities in mathematics can in turn be impacted by such learning spaces. This study offers both theoretical and practical implications. Theoretically, this study demonstrates how the theory of variation is applied to analyze mathematics lessons in such a fine-grained way and to detect and describe differences in how a teacher handles mathematical content, thus providing a deep understanding of mathematics lessons. We suggest that an analysis informed by variation theory in the ways we present in this study—particularly one that draws on the DoV construct—can help to explain what learning takes place in real classrooms. Practically, a detailed description of PS-based MI and the associated systematic patterns of variation and invariance enacted through constructing DoVs provide guidelines for mathematics teachers to reflect upon and to enhance learning spaces in their own classrooms. By exposing what is happening in real classrooms with respect to the structure of lessons, the nature of tasks and patterns of classroom interaction, in the way that we have illustrated, there are opportunities for school teachers and researchers to think further about what is possible for students to learn, how they experience learning and how learning is handled and presented to them in PS-based MI.

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Institutional Review Board Statement: The study was conducted in accordance with the Declaration of Helsinki, and approved by the Institutional Review Board (or Ethics Committee) of Norwegian Centre for Research Data (NSD–Norsk senter for forskningsdata) on 14 January 2020, with the following approval code: Notification Form 811119.

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