Precise geoid determination over Sweden using the Stokes–Helmert method and improved topographic corrections

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Abstract. Four different implementations of Stokes' formula are employed for the estimation of geoid heights over Sweden: the Vincent and Marsh (1974) model with the high-degree reference gravity field but no kernel modifications; modified Wong and Gore (1969) and Molodenskii et al. (1962) models, which use a high- degree reference gravity field and modification of Stokes' kernel; and a least-squares (LS) spectral weighting proposed by Sjöberg (1991). Classical topographic correction formulae are improved to consider long- wavelength contributions. The effect of a Bouguer shell is also included in the formulae, which is neglected in classical formulae due to planar approximation. The gravimetric geoid is compared with global positioning system (GPS)-levelling-derived geoid heights at 23 Swedish Permanent GPS Network SWEPOS stations distributed over Sweden. The LS method is in best agreement, with a 10.1-cm mean and ±5.5-cm standard deviation in the differences between gravimetric and GPS geoid heights. The gravimetric geoid was also fitted to the GPS-levelling-derived geoid using a four-parameter transformation model. The results after fitting also show the best consistency for the LS method, with the standard deviation of differences reduced to ±1.1 cm. For comparison, the NKG96 geoid yields a 17-cm mean and ±8-cm standard deviation of agreement with the same SWEPOS stations. After four-parameter fitting to the GPS stations, the standard deviation reduces to ±6.1 cm for the NKG96 geoid. It is concluded that the new corrections in this study improve the accuracy of the geoid. The final geoid heights range from 17.22 to 43.62 m with a mean value of 29.01 m. The standard errors of the computed geoid heights, through a simple error propagation of standard errors of mean anomalies, are also computed. They range from ±7.02 to ±13.05 cm. The global root-mean-square error of the LS model is the other estimation of the accuracy of the final geoid, and is computed to be ±28.6.

Keywords: Geoid height – Stokes' formula – Modification – Topographic correction – Downward continuation

1 Introduction

The boundary-value problem in physical geodesy can be solved by Stokes' well-known formula for the anomalous gravity potential, with the geoidal height calculated using Bruns' formula. The geoid represents a vertical datum for orthometric heights used in many countries. An accurate geoid is also of interest in many other geophysical applications.

With the advent of the global positioning system (GPS), the geoid has become more important. Gravimetrically determined geoid heights can be applied to orthometric height determination by GPS. This proce- dure replaces costly conventional levelling operations with quicker and cheaper GPS surveys, as long as the geoid height has been computed to a high accuracy.

This paper is concerned with the determination of the geoid height over an area limited by latitudes 54° and 70°N, and longitudes 10° and 25°E, including the whole of Sweden. In order to estimate the geoid heights to a high accuracy (centimeter-level), Stokes' theory is re- fined to avoid some approximations used in the existing techniques (see e.g. Vanicek et al. l996a).

The contents of this paper are as follows. First, geoid height determination by gravity and GPS-levelling data is discussed. Stokes' theory (Stokes 1849) enables us to compute the geoid height, N, from the gravity anomaly data, Δg. However, the major drawback of Stokes' formula is that it requires coverage of gravity over the whole Earth. To diminish this problem, the integration area is limited to a spherical cap with radius ψ_0 around the computation point, and the truncation error committed is reduced by a modification of Stokes' kernel function. In the modified Stokes formula, the long-to-medium-wavelength components of the geoid are typically determined from a global Earth Geopotential Model, EGM96 (Lemoine et al. 1996) in this study, whereas the short-wavelength contributions are obtained from terrestrial gravity and topographic information.

Thereafter, the most important part of this study is presented: the topographic corrections. Stokes' formula requires that (1) the effects of masses exterior to the geoid are primarily removed (or at least reduced onto the geoid) and (2) the gravity be referred to the geoid. The topographic and atmospheric masses violate the first requirement. Our main contribution is the improvement of the classical formulae for the topographic corrections, which suffer from the planar approximation and omission of some long-wavelength contributions. To satisfy the second requirement, the corrected gravity anomaly at the topography must be analytically continued downward to the geoid. The effect of the atmosphere (direct and indirect) is presented as a correction that is applied to the geoid directly. An error estimate of the final geoid model, by propagating the estimated errors of the terrestrial mean gravity anomalies, is also presented.

2 Geoid determination

2.1 Stokes' theory for the original and higher-degree reference field and kernel modifications

The gravimetric geoid height determination usually employs the original Stokes formula. When working with the Stokes kernel, we are supposed to evaluate a surface convolution integral over the whole Earth. This is, of course, an impractical requirement. Therefore, the area of integration is usually limited to a spherical cap around the computation point. This truncation causes an error in the computed geoid height, called the truncation error. This error can be reduced by introducing a modification to the Stokes kernel. The lack of a global coverage of gravity data can be compensated by a combination of terrestrial gravity with a global EGM; in essence the long-wavelength geoid height contributions would be determined from a geopotential model and the short-wavelength information from terrestrial gravity and topographic data.

The following different combinations of the geopotential model with the Stokes integral are experimented with (see also Fan 1989): (1) using a high-degree reference field, but no kernel modification (see e.g. Vincent and Marsh 1974), (2) combination of Stokes' kernel modification with a high-degree reference field (see e.g. Molodenskii et al. 1962; Wong and Gore 1969), and (3) minimizing the global mean square error (MSE) of the truncation error, the potential coefficients, and terrestrial gravity anomalies in a least-squares (LS) sense (see e.g. Sjöberg 1986, 1991).

Assuming a surface spherical cap of integration, σ_0, with geocentric angle, ψ_0, around the computation point, a general geoid height estimator that combines Stokes' kernel modification and the high-degree reference gravity field can be written as (Vanicek and Sjöberg 1991)

$$\widetilde{N} = \frac{c}{2\pi}\iint_{\sigma_0} S_N(\psi)\,(\Delta g - \Delta g_M)d\sigma + c\sum_{n=2}^{M} s_n^*\Delta g_n \tag{1}$$

where

$S_N(\psi)$ = the modified spherical Stokes function = $S(\psi) - \sum_{k=2}^{N}\frac{2k+1}{2} s_k P_k(\cos\psi)$

$s_0, s_1 \dots \dots, s_k$ = modification parameters

$s_2^*, \dots \dots, s_n^*$ = parameters which are obtained for each geoid height estimator

$S(\psi) = \sum_{k=2}^{\infty}\frac{2k+1}{k-1}P_k(\cos\psi)$

$P_k(\cos\psi)$ = kth Legendre polynomial

ψ = spherical distance between computation and running points

Δg = the gravity anomaly at the geoid level derived from the observed magnitude of gravity at the Earth's surface

Δg_n = nth Laplace harmonic of Δg determined from potential coefficients

Δg_M = gravity anomaly computed from a global EGM

$c = \frac{R}{2\gamma}$

R = mean geoid radius

γ = an approximation of the global mean value of normal gravity in Bruns' formula

M = degree of the global EGM

N = degree of kernel modification.

Also

$$S_{N+1}(\psi) = \sum_{k=N+1}^{\infty}\frac{2k+1}{k-1}P_k(\cos\psi) = S(\psi) - \sum_{k=2}^{N}\frac{2k+1}{k-1}P_k(\cos\psi) \tag{2}$$

Different choices of the modification parameters s_k and s_n^* lead to different solutions as follows. Following Molodenskii et al. (1962), who made a modification to the spherical Stokes kernel, Vanicek and Kleusberg (1987) made a modification to the spheroidal Stokes kernel [see Eq. (2)]. Then, the modification parameters, s_k, were determined from the system of linear equations

$$\sum_{n=2}^{N}\frac{2n+1}{2}e_{kn}(\psi_0)s_n(\psi_0) = Q_K^N(\psi_0) \tag{3}$$

Here Paul's function (Paul 1973)

$$e_{kn}(\psi_0) = \int_{\psi=\psi_0}^{\pi} P_n(\cos\psi)\,P_k(\cos\psi)\sin\psi\,d\psi \tag{4}$$

and the Vanicek and Kleusberg (or spheroidal Molodenskii) truncation coefficients are evaluated from

$$Q_K^N(\psi_0) = Q_k(\psi_0) - \sum_{j=2}^{N}\frac{(2j+1)}{(j-1)}e_{kj}(\psi_0) \tag{5}$$

where

$$Q_k(\psi_0) = \int_{\psi=\psi_0}^{\pi} S(\psi)\,P_k(\cos\psi)\sin\psi\,d\psi \tag{6}$$

are the Molodenskii truncation coefficients. Furthermore, $s_n^* = s_k$ in Eq. (1). Also, the Molodenskii-modified spheroidal Stokes function

$$S_N^s(\psi) = S_{N+1}(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k P_k(\cos\psi) \tag{7}$$

is used instead of $S_N(\psi)$ in Eq. (1). This procedure is here called the Molodenskii et al. method.

The modified Wong and Gore (1969) method employs a high-degree residual field and modified Stokes' kernel with $s_k = s_n^* = 2/(n-1)$ in Eq. (1). The term ""modified" means that the high-degree reference gravity field and kernel modification are combined in this model.

The Vincent and Marsh (1974) choices of the arbitrary parameters are $s_k = 0$ and $s_n^* = 2/(n-1)$in Eq. (1). This method corresponds to a high-degree reference gravity field and no kernel modification. These three estimators use a higher-degree reference field in Stokes' integral [see Eq. (1)] by the subtraction of the long-wavelength contribution of gravity anomalies (computed from a global EGM) from the terrestrial gravity anomalies. This subtraction is time consuming and has to be done for each computation point (especially for large values of *M)*. Molodenskii et al. (1962) used the original Pizzetti reference field, and Jekeli (1981) and Sjöberg (1984) also emphasize this point. Therefore, the LS estimator below uses a model reference gravity field of degree and order 2.

Sjöberg (1986, 1991) proposes LS modification of Stokes' formula, which reduces the truncation error, erroneous terrestrial gravity data and potential harmonic errors in an LS sense. One such estimate of geoid height is given by

$$\widetilde{N}_1 = \frac{c}{2\pi} \iint_{\sigma_0} S_N'(\psi)\, \Delta g d\sigma + c \sum_{n=2}^{M} (Q_{Nn} + s_n') \Delta g_n \tag{8}$$

where

$$S_N'(\psi) = S(\psi) - \sum_{n=2}^{N} \frac{2n+1}{2}\, s_n' P_k(\cos\psi) \tag{9}$$

Q_{Nn} in Eq. (8) can be written as

$$Q_{Nn} = Q_n - \sum_{k=2}^{N} \frac{2k+1}{2} s_k' e_{nk} \tag{10}$$

The expected MSE of the LS estimator is minimized, resulting in the parameters s_n' given by the following system of linear symmetric equations:

$$\sum_{r=2}^{N} a_{kr} s_r' = h_k \qquad k = 2,3, \dots, N \tag{11}$$

where

$$a_{kr} = (\sigma_k^2 + dc_k)\delta_{kr} - \frac{2r+1}{2} \sigma_k^2 e_{kr} - \frac{2k+1}{2} \sigma_r^2 e_{rk} + \frac{2k+1}{2} \frac{2r+1}{2} \sum_{n=2}^{\infty} e_{nk} e_{nr} (\sigma_n^2 + c_n) \tag{12}$$

and

$$h_k = \frac{2\sigma_k^2}{k-1} - Q_k \sigma_k^2 + \frac{2k+1}{2} \sum_{n=2}^{\infty} (Q_n e_{nk} (\sigma_n^2 + c_n) - \frac{2}{n-1} e_{nk} \sigma_n^2) \tag{13}$$

where

$$c_n = \frac{1}{4\pi}\iint_\sigma \Delta g_n^2 \, d\sigma \quad (14)$$

and σ_n^2 is the nth gravity anomaly error degree variance, dc_n is the expected MSE of Δg_n and δ_{kr} is Kronecker's delta. The gravity anomaly degree variance c_n can be computed from a global EGM as

$$c_n = \frac{(GM)^2}{a^4}(n-1)^2 \sum_{m=0}^{n}(C_{nm}^2 + S_{nm}^2) \quad (15)$$

where GM is the product of the universal gravitational constant G and the mass of the Earth, M, a is the equatorial radius of the reference ellipsoid, and C_{nm} and S_{nm} are the potential coefficients of degree n and order m. The gravity anomaly error degree variance, due to erroneous potential coefficients, is computed from

$$dc_n = \frac{(GM)^2}{a^4}(n-1)^2 \sum_{m=0}^{n}\left(\delta_{C_{nm}}^2 + \delta_{S_{nm}}^2\right) \quad (16)$$

where δ_{Cnm} and δ_{Snm} are the standard deviations of potential coefficients taken from a global EGM. The error degree variances for the terrestrial gravity anomalies (σ_n^2) can be estimated from the knowledge of an error degree covariance function. One covariance function is, for example, given by Sjöberg (1986)

$$C(\psi) = c_1\left[\frac{1-\Omega}{\sqrt{1-2\Omega\cos\psi+\Omega^2}} - (1-\Omega) - (1-\Omega)\Omega\cos\psi\right] \quad (17)$$

where σ_n^2 can be expressed by

$$\sigma_n^2 = c_1(1-\Omega)\Omega^n \quad (18)$$

The parameters c_1 and Ω can be determined from a knowledge of the error variance $C(0)$ and the correlation length ξ; the value of the argument for which $C(\psi)$ has decreased to half of its value at $\psi=0$ (Moritz, 1980). The value of $C(0)=10$ mGal^2 and a correlation length of 0.1° are used in this study.

2.2 GPS-levelling geoid height

The geoid height N can be directly estimated on land through space techniques with combination of the ellipsoidal height h, computed from GPS, and orthometric height H, computed from precise levelling, by the following well-known formula

$$N = h - H \quad (19)$$

It has to be noted that if the normal height system is used instead of the orthometric heights H, then the height computed by Eq. (19) is the quasi-geoid rather than the geoid. This is the case in this study over Sweden, where the RH70 normal height system is used at the GPS stations. To correct for this separation between the orthometric height H and normal height H^N, the formula (Sjöberg 1995)

$$H_P - H_P^N = -\frac{H_P \Delta g^B}{\gamma} + \frac{H_P^2}{2\gamma}\left(\frac{\partial \Delta g^F}{\partial H}\right)_P \tag{20}$$

is used where Δg^B and Δg^F are the Bouguer and free-air anomalies, respectively, and (Heiskanen and Moritz 1967)

$$\left(\frac{\partial \Delta g^F}{\partial H}\right)_P = \frac{R^2}{2\pi}\int\int_\sigma \frac{\Delta g^F - \Delta g_P^F}{\ell_0^3} d\sigma - \frac{2}{R}\Delta g_P^F \tag{21}$$

where ℓ_0 is the spatial distance between the computation point, P, and the running point, and σ is the unit sphere.

The other correction, which affects the geoid height determination using Eq. (19), is the postglacial rebound of the crust and mantle in the Fennoscandian area. The GPS-levelling stations used in this study are at the benchmarks of the Swedish Permanent GPS Network (SWEPOS). The zero-point epoch of these GPS stations is adjusted to the EUREF-89, which refers to epoch 1989.0, but the Swedish height system RH70 refers to epoch 1970.0. Therefore, the orthometric heights should be reduced to 1989.0. The absolute rate of the land uplift, referred to the ellipsoid, is determined by (Sjöberg and Fan 1986)

$$\dot{h} = \dot{H}_a + \dot{H}_e + \dot{N} = \dot{H}_e + 1.07\dot{H}_a \tag{22}$$

where $\dot{H}_e$ is rate of eustatic rise of sea level (1 mm/yr), $\dot{H}_a$ is the apparent rate of land uplift relative to mean sea level, and $\dot{N}$ is the rate of change of the geoid height in Fennoscandia. $\dot{N}$ was estimated to be about 10% of the land uplift rate (Sjöberg 1983). As this rate seems to be somewhat too high in the central uplift region, a factor of 0.07 between the geoid and land uplift rates is used in this study (Nahavandchi and Sjöberg 1998a). The values of $\dot{H}_a$ are estimated over Sweden by Mäkinen et al. (1986) (see also Sjöberg et al. 1988).

In addition, a fitting process between the gravimetric and GPS-levelling geoid was conducted. The geoid height change ΔN corresponding to a general seven-parameter datum transformation will be independent of the rotations, and in geographical coordinates is of the form (Heiskanen and Moritz 1967)

$$\Delta N = N_{\mathrm{Grav}} - N_{\mathrm{GPS}} = \Delta X \cos\phi \cos\lambda + \Delta Y \cos\phi \sin\lambda + \Delta Z \sin\phi + kR \tag{23}$$

where ϕ and λ are geographical coordinates, ΔX, ΔY, ΔZ are the three translations and k is the scale factor. Equation (23) represents a very useful regression formula, which may be used for transforming a regional gravimetric geoid to a set of GPS-levelling geoid heights. However, it should be noted that some long-wavelength geoid height, vertical datum and GPS errors will be absorbed by the parameters.

3 Corrections to gravimetric geoid determination

The application of Stokes' formula for the computation of the geoid height requires that the disturbing potential is harmonic outside the geoid. This is satisfied by removing the effects of external masses or reducing them inside the geoid (direct effect). The effects of masses are then restored after applying Stokes' integral (indirect effect). The formulae derived for topographic corrections here are based on a constant topographic density. These formulae can also be generalized to a laterally variable density simply by putting it within the surface integrals on the direct and indirect topographic effects. Geoid height determination by Stokes' formula also requires that the gravity anomalies, Δg, must refer to the geoid. In order to satisfy this second condition, the gravity anomalies available on the topography of the Earth are reduced to the geoid. This reduction is called downward continuation.

The corrections mentioned above, combined with the idea of Stokes-Helmert integration, are realized by the formula (Heiskanen and Moritz 1967, p. 324)

$$N = \frac{R}{4\pi\gamma} \iint_\sigma S(\psi) \Delta g^{H^*} d\sigma + \delta N_I \tag{24}$$

where Δg^{H^*} is the gravity anomaly including the topographic correction and reduced to the geoid and δN_I is the indirect effect on the geoid. In this study, we have decided to use the Helmert second condensation method to replace the external masses by a condensed layer placed on the geoid. For more details, see e.g. Wichiencharoen (1982), Vanicek and Martinec (1994) and Nahavandchi and Sjöberg (1998b). The notation Δg^H at the ground level can be expressed via

$$\Delta g^H = \Delta g + \delta \Delta g_{\mathrm{dir}} \tag{25}$$

where Δg is the surface free-air anomaly and $\delta \Delta g_{\mathrm{dir}}$ is the direct topographic effect determined at the topography. The notation Δg^{H^*} is the analytically downward-continued Δg^H from the topography to the geoid. This process can be achieved with a Taylor expansion. It should be mentioned that Eq. (24) can also be rewritten as (Sjöberg 2000)

$$N = \frac{R}{4\pi\gamma} \iint_\sigma S(\psi)(\Delta g + \delta \Delta g^*_{\mathrm{dir}} + \delta \Delta g_{\mathrm{dc}}) d\sigma + \delta N_I \tag{26}$$

where $\delta \Delta g^*_{\mathrm{dir}}$ is the direct topographical effect on gravity anomaly which is referred to a point on the geoid and $\delta \Delta g_{\mathrm{dc}}$ is the correction due to the downward continuation of the free-air anomaly Δg. However, Eq. (24) is preferred to Eq. (26) from the numerical point of view (Nahavandchi and Sjöberg, in press).

The effects of the atmosphere on the geoid height determination, the truncation error, as well as ellipsoidal correction are also studied in this section.

3.1 Topographic corrections

3.1.1 Direct topographic correction in Stokes' formula

The correction due to removing the gravitational effects of the masses above the geoid is here simply called the direct effect. The Helmert second condensation method is used, which preserves the mass but changes the potential of the topography.

The classical integral formula for direct effect determination at point P, on the topography, can be approximated from (see Vanicek et al. 1986; Vanicek and Kleusberg 1987)

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{classic}}(H_P) = \frac{\mu R^2}{2}\iint_\sigma \frac{H^2 - H_P^2}{\ell_0^3}\, d\sigma \tag{27}$$

where $\mu = G\rho_0$, and ρ_0 is the density of topography, assumed to be constant, H and H_P are the orthometric heights of the running and computation points, respec tively, and $\ell_0 = R\sqrt{2(1-\cos\psi)} = 2R\sin\frac{\psi}{2}$.

In a strict sense, Eq. (27) can only be used for the far zone integration area, where $\ell_0 \gg H$, and the effect of the near zone and a Bouguer shell (which cannot be derived from a planar model) are completely missing (Martinec and Vanicek 1994a). It should also be mentioned that the power series of height *H* used in the integration is limited to the second order.

Sjöberg (1994, 1995) developed the direct effect in spherical harmonics to power H^2, and Nahavandchi and Sjöberg (1998b) extended this approach to power H^3. The latter result can be summarized as

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{NS}^*}(H_P) \doteq -\frac{\pi\mu}{2R}\left[5H_P^2 + 3\overline{H_P^2} + 2\sum_{n,m}^{M'} n(H^2)_{nm}Y_{nm}(P)\right]$$
$$+\frac{\pi\mu}{2R^2}\left[\frac{28}{3}H_P^3 + \frac{9}{2}\overline{H_P^2}H_P - \frac{1}{2}\overline{H_P^3} + H_P\sum_{n,m}^{M'} n(2n+9)(H^2)_{nm}Y_{nm}(P) - \frac{1}{3}\sum_{n,m}^{M'} n(2n+7)(H^3)_{nm}Y_{nm}(P)\right] \tag{28}$$

where Y_{nm} are fully normalized spherical harmonics obeying the following rule:

$$\frac{1}{4\pi}\int\int_\sigma Y_{nm}Y_{n'm'}\, d\sigma = \begin{cases} 1 & \text{if } n = n' \text{ and } m = m' \\ 0 & \text{Otherwise} \end{cases} \tag{29}$$

and

$$(H^\upsilon)_{nm} = \frac{1}{4\pi}\iint_\sigma H_P^\upsilon Y_{nm}\, d\sigma, \qquad \upsilon = 1,2,3, \ldots. \tag{30}$$

$$H_P^\upsilon = \sum_{n,m}(H^\upsilon)_{nm}\, Y_{nm}(P) \tag{31}$$

$$\overline{H_P^\upsilon} = \sum_{n,m}\frac{1}{2n+1}(H^\upsilon)_{nm}\, Y_{nm}(P) \tag{32}$$

In Eq. (28), M' is the maximum degree of height coefficients in a spherical harmonic representation. In order to compare Eq. (27) with Eq. (28) we only use the second power of elevation *H* in Eq. (28), i.e.

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{NS}^*}(H_P) \doteq -\frac{2\pi\mu}{R}\sum_{n,m}^{M'}\frac{(n+2)(n+1)}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{33}$$

Equation (33) is related to a point at the geoid. Rewriting this formula for a point *P* at the topography, we obtain

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{NS}}(H_P) \doteq -\frac{2\pi\mu}{R}\sum_{n,m}^{M'}\left(\frac{R}{r}\right)^{n+1}\frac{(n+2)(n+1)}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{34}$$

This harmonic presentation of the direct topographic effect [Eqs. (33) and (34)] is simple for the computations. It is also free from the problems encountered in integral formulae, such as the singularity at the computation point. However, the harmonic expansion series of H^2 (and H^3) will include the long wavelengths. The incorporation of all significant contributions from both short and long wavelengths requires an expansion in spherical harmonics of H^2 (and H^3) to very high degrees, which is practically difficult and ruins the simplicity of this method. Nahavandchi and Sjöberg (1998b) show that the dominant part of the power series in Eq. (28) is the second power of elevation H. For example, the contribution from the harmonic expansion series H^3 on the geoid was within 9 cm in the Himalayas for the EGM96 model.

Equation (34) can also be written as a surface integral (Nahavandchi 2000a; Sjöberg 2000)

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{new}}(H_P) = -\frac{4\pi\mu}{R}H_P^2 - \frac{3\mu}{8}\iint_\sigma \frac{H^2-H_P^2}{\ell_0}d\sigma + \frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2-H^2}{\ell^3}\left(1-\frac{3H_P^2}{\ell^2}\right)d\sigma \tag{35}$$

where $\ell = \sqrt{r_P^2 + r^2 - 2r_P r\cos\psi)}$, and $r_P = R + H_P$. The above formula can also be written as

$$\Delta g_{\mathrm{dir}}^{\mathrm{new}}(H_P) = -\frac{5\pi\mu}{2R}H_P^2 - \frac{3\pi\mu}{2R}\overline{H_P^2} + \frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2-H^2}{\ell^3}\left(1-\frac{3H_P^2}{\ell^2}\right)d\sigma \tag{36}$$

3.1.2 Direct topographic correction for potential coefficients

In determining the geoidal undulations from a global EGM, we must expect a bias of the external harmonic series when applied at the geoid within the topographic masses. This bias can be estimated by removing the effects of topographical masses, which implies a direct effect on the geopotential model. Helmert's second condensation method is used for reducing the masses (Vanicek et al. 1995). The direct effect on geopotential to the third power of elevation is estimated directly on the geoid to be (Nahavandchi and Sjöberg 1998b)

$$\delta N_{\mathrm{dir}}^{\mathrm{M}}(H_P) \doteq -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{M'}\sum_{m=-n}^{n}\frac{n+2}{2n+1}(H^2)_{nm}Y_{nm}(P) - \frac{2\pi\mu}{R\gamma}\sum_{n=0}^{M'}\sum_{m=-n}^{n}\frac{(n+2)(n+1)}{3(2n+1)}(H^3)_{nm}Y_{nm}(P) \tag{37}$$

with the same notations as previously defined. Sjöberg (1994, 1996) directly derived the total (direct and indirect) effect on the geoid to power H^2.

3.2 Primary indirect topographic effect

The effect of restoration of the reduced masses on the geoid is the indirect effect. The classical formula for determining the indirect topographic effect on the geoid for Helmert's second condensation method is (Wichiencharoen 1982)

$$\delta N_{\mathrm{I}}^{\mathrm{classic}}(P) = -\frac{\pi\mu H^2}{\gamma} - \frac{\mu R^2}{6\gamma}\iint_\sigma \frac{H^3-H_P^3}{\ell_0^3}d\sigma \tag{38}$$

with the same notations as before.

Sjöberg (1994, 1995) developed the indirect topographic effect in spherical harmonics to power H^2, and Nahavandchi and Sjöberg (1998b) extended this approach to power H^3. The spherical harmonic presentation of indirect topographic effect is derived as (Nahavandchi and Sjöberg 1998b)

$$\delta N_{\mathrm{I}}^{\mathbf{NS}}(P) = -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\frac{n-1}{2n+1}H_n^2(P) + \frac{2\pi\mu}{3R\gamma}\sum_{n=0}^{\infty}\frac{n(n-1)}{(2n+1)}H_n^3(P) \tag{39}$$

where

$$H_n^{\nu}(P) = \frac{2n+1}{4\pi}\iint_{\sigma} H^{\nu} P_n(\cos\psi)\, d\sigma, \qquad \nu = 2,3 \tag{40}$$

The classical formula [Eq. (38)] is not practical for computation, as it requires an integration over the whole Earth to include long-wavelength contributions. It also suffers from planar approximation (Martinec and Vanicek 1994b, Sjöberg and Nahavandchi 1999). On the other hand, the spherical harmonic presentation of indirect effect [Eq. (39)] needs a very high maximum degree of expansion, to consider all short- and long- wavelength information. Therefore, a compromise between these two methods is derived as (Sjöberg and Nahavandchi 1999)

$$\Delta\delta N_{\mathrm{I}}(P) = \delta N_{\mathrm{I}}^{\text{classic}} - \delta N_{\mathrm{I}}^{\text{New}} = -\frac{3\pi\mu}{\gamma}H_P^2 - \frac{3R\mu}{4\gamma}\iint_{\sigma}\frac{H^2-H_P^2}{\ell_0}d\sigma - \frac{\mu}{8\gamma}\iint_{\sigma}\frac{H^3-H_P^3}{\ell_0}d\sigma \tag{41}$$

or

$$\delta N_{\mathrm{I}}^{\mathbf{New}}(P) = \delta N_{\mathrm{I}}^{\mathbf{classic}}(P) - \Delta\delta N_{\mathrm{I}}(P) \tag{42}$$

where $\Delta\delta N_{\mathrm{I}}$ in spectral form is approximated as

$$\Delta\delta N_{\mathrm{I}}(P) = -\frac{3\pi\mu}{\gamma}\overline{H_P^2} + \frac{\pi\mu}{2R\gamma}\left(H_P^3 - \overline{H_P^3}\right) \tag{43}$$

3.3 Secondary indirect topographic effect

The secondary indirect topographic effect is a free-air correction of gravity from geoid to co-geoid, i.e. $2\gamma\delta N_{\mathbf{Dir}}/R \doteq 2\gamma\delta\boldsymbol{\zeta}_{\mathrm{I}}/R$, where $\delta\boldsymbol{\zeta}_{\mathrm{I}}$ is the indirect topographic effect on the height anomaly and $\delta N_{\mathbf{Dir}}$ is the direct topographic effect on the geoid. This yields the following correction, directly on the geoid, to the third power of elevation H (Nahavandchi and Sjöberg 1998b):

$$\delta N_{\mathrm{I2}} \doteq \frac{4\pi\mu}{\gamma}\sum_{n,m}^{M'}\frac{n+2}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) - \frac{\pi\mu}{R\gamma}H_P\sum_{n,m}^{M'}\frac{4n^2+2n+3}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) + \frac{2\pi\mu}{3R\gamma}\sum_{n,m}^{M'}\frac{2n^2-8n-3}{(2n+1)(n-1)}(H^3)_{nm}Y_{nm}(P) \tag{44}$$

According to the authors' experience, the secondary indirect topographic effect is at least two orders of magnitude smaller than the direct topographic effect and, therefore, is expressed with a spherical harmonic presentation [Eq. (44)].

3.4 Atmospheric effect

The effect of the mass of the atmosphere must also be removed prior to the application of Stokes' formula. This corresponds to the direct atmospheric effect. After the application of Stokes' formula, the effect of restoring the atmosphere should be applied. Sjöberg (1993) emphasized that there could be additional significant direct and indirect atmospheric effects stemming from a more detailed treatment of the Earth's topography than is made in the classical International Association of Geodesy (IAG) approach (Moritz 1980, p. 422). A new approach was derived by Sjöberg (1998). Furthermore, Sjöberg and Nahavandchi (2000) derived the total of the direct and indirect effects, called the total atmospheric effect, to the modified Stokes formula, implying the combination with potential coefficients. This total atmospheric effect is derived as
(Sjöberg and Nahavandchi 2000)

$$\delta N^{a}_{\text{total}} = c_1 \sum_{n=0}^{1} (s_n + Q_{Mn}) \frac{n+2}{2n+1} H_n(P) - c_1 \sum_{n=2}^{M} \left(\frac{2}{n-1} - Q_{Mn} - s_n \right) H_n(P) \\ - c_1 \sum_{n=M+1}^{\infty} \left(\frac{2}{n-1} - \frac{n+2}{2n+1} Q_{Mn} \right) H_n(P) \quad (45)$$

where $c_1 = (2\pi R \rho^0 G)/\gamma$ and ρ^0 is the density of the atmosphere at the radius of sea level. This formula can be added directly to the geoid height as a correction due to the effect of the atmosphere.

3.5 Downward continuation of the Helmert gravity anomaly

In order to obtain the boundary values in Stokes' formula, the gravity anomalies Δg^H at the topography have to be reduced onto the geoid. This reduction is the downward continuation. The main problems with down- ward continuation are the masses between the topography and the geoid and the irregularity of the density distribution, which causes the disturbing potential to be non-harmonic outside the geoid. Vanicek et al. (1996b) examined downward continuation of the Helmert gravity anomaly and found out that the determination of this effect is a well-posed problem for $5' \times 5'$ geographic cells.

Prior to the downward continuation, the gravity anomalies are corrected for the direct topographic correction (resulting in the Helmert gravity anomaly at the topography). The application of the direct effect makes the gravity anomalies smoother and thus better suited to downward continuation.

After Bjerhammar (1962), a fictious field of gravity anomalies Δg^{H^*} is assumed on the geoid, which generate the gravity anomalies Δg^H on the topography. These two anomalies can be related by the Poisson formula (including the spherical harmonics of degrees zero and one) (Kellogg 1929, MacMillan 1930)

$$\Delta g^H = \frac{R}{4\pi} \iint_\sigma \Delta g^{H^*} K(r, \psi, R)\, d\sigma \quad (46)$$

where $K(r, \psi, R)$ is the spherical Poisson kernel described by

$$K(r, \psi, R) = \sum_{n=0}^{\infty} (2n+1) \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi) = R \frac{r^2 - R^2}{\ell^3} \quad (47)$$

In Eq. (47), a spherical approximation is used. The Helmert gravity anomalies Δg^{H} at the topography are known, and the Helmert gravity anomalies Δg^{H^*} at the geoid are desired. In this sense, Eq. (46) can be solved in different ways; for example, by a linear approximation. We have used an iterative process by transforming the Poisson integral to a system of 48 510 linear algebraic equations to solve Eq. (46) (see Heiskanen and Moritz 1967; Bjerhammar 1969; Vanicek et al. 1996b).

The Poisson kernel which dominates the behavior of Poisson's integral tapers off rapidly with increasing of distance from the computation point. Therefore, it need only be integrated over a small spherical cap ψ_0 instead of e whole Earth. Rewriting the integral in Eq. (46) gives

$$\Delta g^{H} = \frac{R}{4\pi} \iint_{\sigma_0} \Delta g^{H^*} K(r, \psi, R)\, d\sigma + \frac{R}{4\pi} \iint_{\sigma - \sigma_0} \Delta g^{H^*} K(r, \psi, R)\, d\sigma \tag{48}$$

where σ_0 denotes the spherical cap of radius ψ_0. If the second term on the right-hand side of Eq. (48) is neglected, this yields a truncation error. This truncation error is reduced using Molodenskii's truncation modification technique (Molodenskii et al. 1962). A spherical cap with radius equal to 1° assures us that the contribution from the rest of the world is small (Vanicek et al. 1996b; Nahavandchi 1998a). In order to minimize the effect of the distant gravity data (outside the spherical cap), the modification of Poisson's kernel is introduced in the same way as the modified Stokes' kernel was used in the Stokes integration (Vanicek et al. 1996b). The low-degree harmonics Δg_L^H are subtracted from the gravity anomaly Δg^{H} at the topography. These can be computed from the EGM96 global model as (Vanicek et al. 1996b; Nahavandchi 1998a)

$$\Delta g_L^H = \gamma \sum_{n=2}^{L} (n-1) \left(\frac{R}{r}\right)^{n+2} \sum_{m=-n}^{n} A_{nm}^{*}\, Y_{nm}(P) \tag{49}$$

where A_{nm}^{*} are the potential coefficients taken from a global EGM and corrected for the direct topographic correction [see Eq. (37)].

Then Eq. (48) can be rewritten as

$$\Delta g^{H} = \frac{R}{4\pi} \iint_{\sigma_0} \Delta g^{H^*} K^{M}(r, \psi, R, \psi_0)\, d\sigma + dg \tag{50}$$

where $dg = \delta g_T + \Delta g_L^H$ is the sum of the truncation error and the low-degree harmonics of the gravity anomaly. The value of L = 20 is selected in this study, referring to a relatively low-degree reference field. The truncation error can be computed from a global gravity model using (Vanicek et al. 1996b; Nahavandchi 1998a)

$$\delta g_T = -\frac{R\gamma}{2r} \sum_{n=2}^{L} \sum_{m=-n}^{n} (n-1) \overline{Q}_n(r, R, \psi_0) A_{nm}^{*} Y_{nm}(P) \tag{51}$$

where

$$\bar{Q}_n(r, R, \psi_0) = \int_{\psi=\psi_0}^{\pi} K^{M}(r, R, \psi, \psi_0)\, P_n(\cos\psi) \sin\psi\, d\psi \tag{52}$$

and the modified Poisson kernel in a spectral form is

$$K^{M}(r, R, \psi, \psi_0) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \bar{Q}_n(r, R, \psi_0) P_n(\cos\psi) \tag{53}$$

3.6 Truncation error in Stokes' formula

Because of the limited gravity coverage over the whole Earth's surface and the huge computation time, the integration area in Stokes' integral is split into a reasonably small cap σ_0 (with radius ψ_0) around the computation point and the rest of the world. The short-wavelength part of the geoid can then be written as

$$N_{\Delta g} = \frac{R}{4\pi\gamma}\iint_{\sigma} S_M(\psi)\,\Delta g d\sigma = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S_M(\psi)\,\Delta g d\sigma + \frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_0} S_M(\psi)\,\Delta g d\sigma \tag{54}$$

As the contribution from the rest of the world is sufficiently small (considering the use of modified Stokes' formula), the second term on the right-hand side (the truncation error) of Eq. (54) can be evaluated from a global gravity model as

$$\delta N = \frac{R}{2\pi}\sum_{n=2}^{M}(n-1)\,Q_n^M \Delta g_n \tag{55}$$

Notice that this correction is already included in the LS estimation [Eq. (8)].

3.7 Ellipsoidal correction for terrestrial gravity data

As the quantities of the anomalous gravity field are relatively small, we usually neglect the terms of order e^2 and above in computations of the formulae used in physical geodesy. e^2 is the square of the first numerical eccentricity of the reference ellipsoid. Therefore, these expressions hold only for a spherical approximation. The ellipsoidal correction to the terrestrial gravity anomalies, ϵ_s, is evaluated by the formulae given in Moritz (1980) as

$$\epsilon_s = e^2 \Delta g^1 \tag{56}$$

where

$$\Delta g^1 = \frac{1}{R}\sum_{n=2}^{M}\sum_{m=0}^{n}(G_{nm}\cos m\lambda + H_{nm}\sin m\lambda)P_{nm}(\sin\phi) \tag{57}$$

and G_{nm} and H_{nm} are defined in Moritz (1980). As the ellipsoidal correction is relatively small, it can be estimated by using the truncated spherical harmonic coefficients from a global geopotential model.

4 Numerical investigations

4.1 Data sources

The area of study is limited by latitudes 54º N and 70º N and longitudes 10º E and 25º E, which include the whole of Sweden. The intention is to determine the geoid heights in this area. As a preliminary to this computation, all corrections mentioned in Sect. 3 were evaluated. The numerical results are illustrated in graphical form.

The height coefficients $(H)_{nm}$, $(H^2)_{nm}$ and $(H^3)_{nm}$ were determined using Eqs. (30) and (31). For this, a $30' \times 30'$ Digital Terrain Model (DTM) was generated using the Geophysical Explanation Technology (GETECH) $5' \times 5'$ DTM (GETECH 1995a) and averaged using area weighting. Since the interest is in continental elevation coefficients, the heights below sea level were all set to zero. The spherical harmonic coefficients of the topography were computed to degree and order 360. The parametric definitions are: $\mu = G\rho_0$, where $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and $\rho_0 = 2670\ kg/m^3$, R=6371 km, and $\gamma = 981$ Gal. In all the integral formulas, the $2.5' \times 2.5'$ DTM (GETECH 1995b) is used. The EGM96 global geopotential model to degree and order 360 is used in the computations.

The gravity data over Scandinavia were provided by the National Survey and Cadastre of Denmark (KMS). Bjerhammar's deterministic method (Bjerhammar 1973) was used to compute mean free-air gravity anomalies over Scandinavia. As the free-air gravity anomalies are highly correlated with topography, they are not appropriate for prediction and gridding over land. On the other hand, the Bouguer anomaly field is smoother and less correlated with topography. Therefore, Bouguer anomalies were used over land and free-air anomalies were used over sea for prediction (see Nahavandchi 2000b). DTM height information was used for correcting the height bias (see Nahavandchi 2000b). The mean free-air anomalies, obtained from the above stated prediction, over Scandinavia for $6' \times 10'$ geographic cells range from -125.19 to 193.18 mGal with a mean value of -0.29 mGal and a standard deviation (SD) of ±25.04 mGal (see Nahavandchi 2000b).

4.2 Corrections to the geoid height determination

Table 1 shows the statistics of all corrections mentioned in Sect. 3. Comparison of these results and further elaboration on the subject are given below.

Table 1. The statistics of all corrections mentioned in Sect. 3

	Min	Max	Mean	Standard deviation
Direct topographic effect on gravity (mGal)	−35.38	59.89	10.42	10.26
Direct topographic effect on geopotential (cm)	−13.88	−6.89	−7.92	3.13
Secondary indirect topographic effect (cm)	−1.7	3.1	1.1	1.23
Downward-continuation correction to gravity (mGal)	−42.59	81.97	0.13	19.22
Primary indirect topographic effect (cm)	−1.13	13.46	4.81	3.14
Total atmospheric effect on geoid (cm)	−0.05	1.56	0.95	0.19
Truncation error (cm)	3.66	7.30	5.38	1.09
Ellipsoidal correction (mGal)	0.007	0.055	0.024	0.009

The direct topographic effect on gravity in Stokes' formula, computed from Eq. (36), is plotted in Fig. 1. This effect contains both short- and long-wavelength contributions. The application of the direct topographic effect to the mean free-air anomalies has reduced the original values of (-125.19 to 193.18 mGal) to (-105.26 to 171.53 mGal), i.e. a reduction of 42 mGal of the span. The standard deviation is also reduced from $\pm$30.05 to $\pm$25.04 mGal.

The direct topographic effect on geopotential [Eq. (37)], to degree 360, has been computed directly on the geoid and is illustrated in Fig. 2. This effect is small compared to the effect in Stokes' integral and always negative in this study.

Figure 3 shows the secondary indirect topographic effect. It is computed from Eq. (44) to degree 360. This effect has been computed directly on the geoid using a spherical harmonics

presentation of the heights. It is two orders of magnitude smaller than the direct topographic effect (Fig. 2) in Stokes' formula. This effect is relatively small and contributes very little to the final geoid, but has a systematic effect that must be considered when an accurate geoid is desired.
The ellipsoidal correction is computed from Eqs. (56) and (57) using EGM96. It is even smaller than the secondary indirect effect. Its contribution to the geoid height has a maximum value of 0.07 cm.

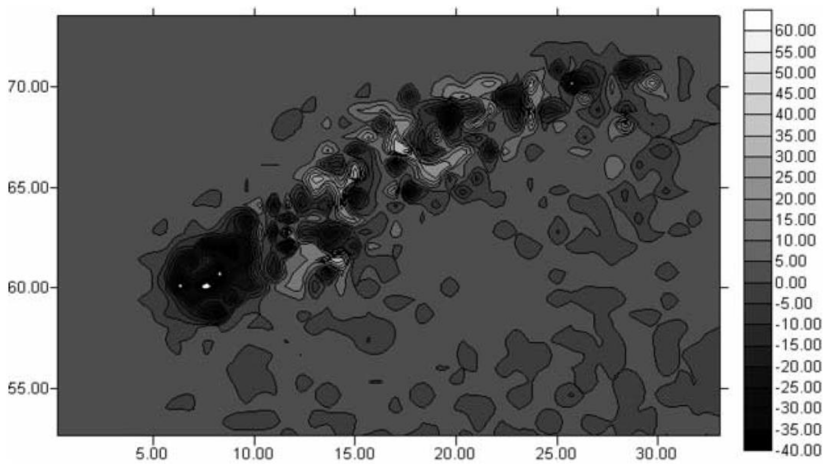

Fig. 1. Direct topographic effect on gravity in Stokes' formula. Contour interval 5 mGal

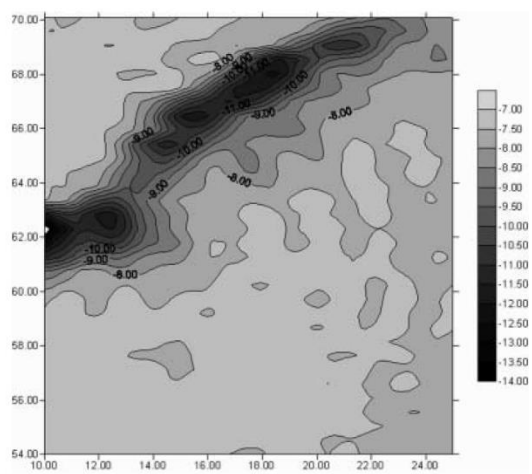

Fig. 2. Direct topographic effect on geopotential. Contour interval 0.5 cm

The application of the direct topographic effect is inevitable for the theoretical justification of the harmonic downward continuation used in this study. The contribution of downward-continuation correction to be gravity anomaly has a very-short-wavelength character and mostly large values. Equation (50) was used for the computation of this effect. It should be noted that due to the very-high-frequency nature of this effect, its plot will not be useful, therefore is not shown here. Downward continuation correction on the geoid always gives positive values in this study.

The primary indirect topographic effect, computed from Eqs. (42) and (43), is illustrated in Fig. 4. The contributions are from both short- and long-wavelength parts of the geoid.

The total atmospheric effect on the geoid, including the direct and indirect atmospheric effects, as computed from Eq. (45), is depicted in Fig. 5. However, the zero- and first-degree harmonic contributions to the total atmospheric geoid effect are not included in the solution (The zero-degree effect is about −5.57 m, whilst the first-degree effect reaches 2−3 cm; see Sjöberg and Nahavandchi 2000.) The total atmospheric effect has been evaluated on the geoid and can be directly added to the final geoid.

The correction due to the truncation error, for an integration cap of 6º, is computed from Eq. (55) using EGM96. It has a quite significant contribution and must be added to the final geoid. This correction has already been considered in the LS model, therefore it is not applied in this model.

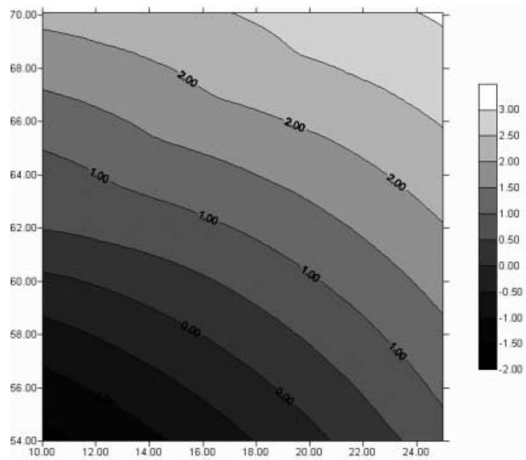

Fig. 3. Secondary indirect topographic effect on the geoid. Contour interval 0.5 cm

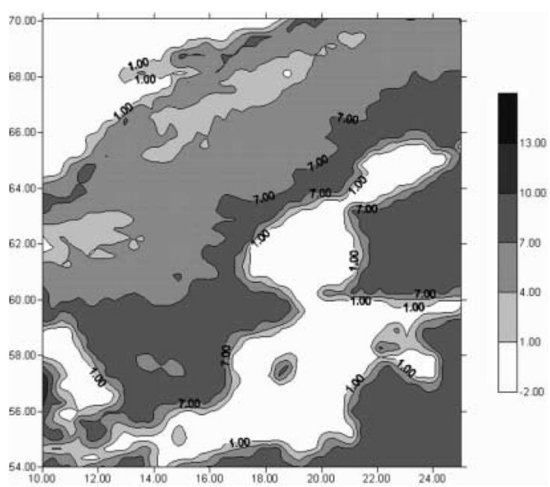

Fig. 4. Primary indirect topographic effect. Contour interval 3 cm

4.3 Geoid determination with the modified kernel

Nahavandchi (1998b) shows the power of the LS estimator over the other modified Stokes' kernels. The mean and standard deviation of the differences between the geoid derived from the LS estimator and the GPS- derived geoid heights at 23 SWEPOS GPS stations are found to be 10.1 and ±5.5 cm, respectively. They are computed to be 13.1 and ±7.1 cm for the Molodenskii et al. model, 18.2 and ±11.2 cm for the modified Wong and Gore estimator, and 23.1 and ±16.1 cm for the Vincent and

Marsh model, respectively (see Nahavandchi 1998b). Therefore, the LS estimator is used for the final gravimetric geoid height determination.

Nahavandchi (1998b) also used EGM96 to compare different geoid estimators by changing the maximum degree of expansion (*M)* of the reference field. This analysis resulted in the best solutions with large values of *M*. In order to obtain further insight into this analysis, the observed geoid height values at 23 SWEPOS GPS stations are compared with the gravimetric geoid heights determined from different values of *M* (*M* = 20, 180, 360). The LS model is used to estimate the gravimetric geoid. All corrections are applied in this investigation. Table 2 shows the statistics of the differences between gravimetric and GPS-levelling geoid heights. The results of Table 2 justify our belief that a reference field of an order as high as possible, in this study, gives the best results at the GPS stations. Standard deviation and mean value for *M* = 360 are computed to be ±5.5 and 10.1 cm, respectively, while they are found to be ±11.9 and 18.1 cm for *M* =20.

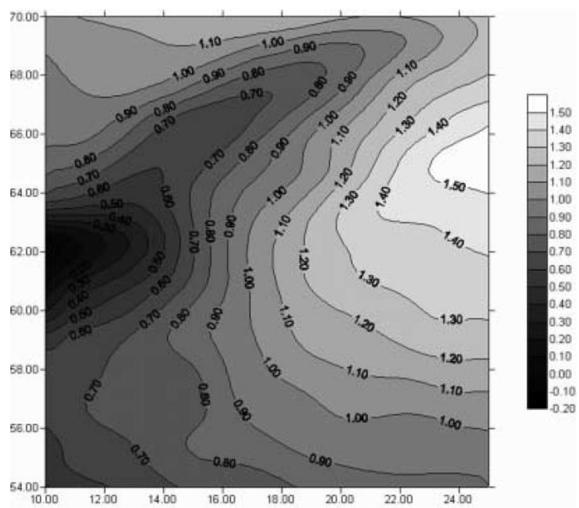

Fig. 5. Total atmospheric effect on the geoid. Contour interval 0.1 cm

Some studies propose using a reference field constructed by applying only satellite-derived harmonics (Vanicek and Kleusberg 1987, Vanicek et al. 1996a). They argue that a reference field with a higher degree and order than 20 by 20 is constructed using the same terrestrial gravity data that is used in geoid height determination referred to this reference surface. Another disadvantage of a combined field is its spatial inhomogeneity. However, other researchers use a combined reference field of an order as high as possible (see e.g. Despotakis 1987; Zhao 1989; Forsberg 1990; Fan 1993; Forsberg et al. 1996, etc.) The use of a satellite- only model versus a combined model (to degree and order e.g. 20 by 20) is not investigated here. Figure 6 shows the plot of the geoid heights determined using the LS model with the values of $M = N = 360$. It has been computed on a $6' \times 10'$ grid. The integration cap is selected equal to 6^{o}. The Geodetic Reference System (GRS80) normal field and its reference ellipsoid were used in this study. Therefore, the final gravimetric geoid heights are referred to the GRS80 ellipsoid. In this plot all corrections have been carried out, resulting in the final geoid which ranges from 17.22 to 43.62 m with a mean value of 29.01 m.

4.4 Statistical error propagation

An internal error propagation might be useful in providing the statistical error properties of the gravimetric geoid heights, by propagating the estimated errors of the mean terrestrial gravity anomaly data, $\sigma^2_{\Delta g}$. Assuming that the variances of the mean gravity anomaly data are known and

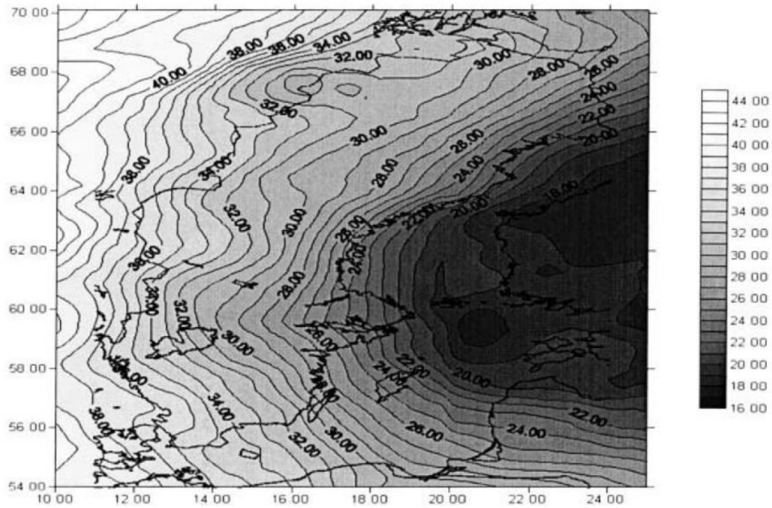

Fig. 6. Final gravimetric geoid heights determined by the LS estimator. Contour interval 1 m

uncorrelated, using the law of error propagation on Eq. (54), the effect of the terrestrial data error on geoid height, $\sigma^2_{N_{\Delta g}}$, can be evaluated from

$$\sigma^2_{N_{\Delta g}} = \left(\frac{R\Delta\phi\Delta\lambda}{4\pi\gamma}\right)^2 \sum_i (S_M(\psi)\cos\phi)^2_i \sigma^2_{\Delta g}(\phi,\lambda)_i \quad (58)$$

Index i accounts for the number of cells in which the contributions from gravity anomalies are considered. The statistics of these internal geoid undulation error estimates are given in Table 3. These standard deviations are highly correlated, especially over short distances, and range from 7.0 to 13.1 cm with an RMS value of ±9.2 cm. This represents the accuracy estimate of the gravimetric geoid height obtained from the internal error propagation. It should be noted that the errors in all the employed corrections are considered much smaller than the error in mean free-air gravity anomalies and are neglected in this computation. Also, the errors due to the low-frequency reference field are not applied here. This method of estimating the geoidal height errors may not be accurate but represents valuable information on the expected relative accuracies. It also indicates locations where the gravity anomalies have poor quality and quantity.

Table 2. Statistics of differences between gravimetric and GPS-levelling geoid height with different values of M. Units in cm

	$M = 360$	$M = 180$	$M = 20$
Min	4.4	8.1	9.5
Max	26.0	28.1	36.1
Mean	10.1	12.2	18.1
Standard deviation	5.5	7.1	11.9

The accuracy of the geoid height estimations can also be investigated using its global MSE. In the case of the LS model, we have (Sjöberg 1991)

$$\overline{\delta N}^2 = \left(\frac{R}{2\gamma}\right)\sum_{n=2}^{N}\left[\left(\frac{2}{n-1} - s_n^* - Q_{Nn}\right)^2 \sigma_n^2 + Q_{Nn}^2 c_n + s'^2_n d_{c_n}\right] +$$
$$+\left(\frac{R}{2\gamma}\right)^2 \sum_{n=N+1}^{n_{\max}}\left[\left(\frac{2}{n-1} - Q_{Nn}\right)^2 \sigma_n^2 + Q_{Nn}^2 c_n\right] \quad (59)$$

For $n_{\max} = 1000$, maximum degree of modification $N = 360$, and truncation radius of $\psi_0 = 6°$, the global RMS error is computed to be ±28.6 cm. This result includes the errors from truncation, erroneous terrestrial gravity data, and potential coefficients.

However, it should be noted that the most reliable way to estimate the accuracy of the gravimetric geoid height is to compare its result with externally derived geoid height data from GPS levelling. This is investigated in Nahavandchi (1998b).

Table 3. Statistics of geoidal height errors through propagating the estimated errors of mean gravity anomalies. Units in cm

Min	Max	Mean	Standard deviation	RMS
7.0	13.1	9.1	0.9	9.2

4.5 Comparisons

The geoidal undulations were also computed using the other three estimators: Molodenskii et al., modified Wong and Gore, and Vincent and Marsh models. The statistics of differences between these three estimators and the LS model are shown in Table 4. The best agreement is between the LS estimator and the Molodenskii et al. model. The mean of differences is computed to be 2.41 cm with a standard deviation of ±1.40 cm. It should also be mentioned that all corrections are already applied to these geoid estimators.

In order to investigate how each of the direct and indirect topographic effects as well as the atmospheric correction improve the agreement of the gravimetric geoid height with the GPS-levelling stations, a specific analysis with each of these corrections was carried out. Table 5 shows the statistics of the differences between the gravimetric and GPS-levelling geoid heights at 23 GPS stations, including and excluding all corrections. The results in Table 5 show an improvement of 18.6 cm in mean value of differences, and ±5.0 cm in standard deviation.

Thereafter, we investigated (comparing with GPS results) the gravimetric geoid height for each of the direct, indirect and atmospheric effects (Table 6). Results indicate that the direct topographic correction of gravity anomalies significantly reduces the differences between the gravimetric and GPS-levelling geoid heights. An improvement of about 17.5 cm is computed for the mean value of differences. The indirect topographic correction and atmospheric effect also improve the gravimetric geoid heights on GPS stations. Improvements of 6.0 and 0.8 cm were found for the mean value of differences, respectively.

The Sjöberg LS, Molodenskii et al., Wong and Gore, Vincent and Marsh, and NKG96 (Forsberg et al. 1996; the best available geoid model in the region) gravimetric geoid models were evaluated on 23 SWEPOS GPS stations. The corrections due to the difference between the orthometric and normal heights [Eqs. (20) and (21)], and the effect of postglacial rebound [Eq. (22)], were also applied. Table 7 shows the statistics of differences be- tween GPS-levelling and gravimetric geoid heights at 23 GPS stations. The results of Table 7 show the advantages of the LS model over the other methods (see Nahavandchi 1998b).

Table 4. Statistics of differences between three geoid height models with the LS estimator. Units in cm

	Molodenskii et al.	Wong and Gore	Vincent and Marsh
Min	−2.2	−4.1	−10.5
Max	6.1	10.4	20.2
Mean	2.4	4.1	15.1
Standard deviation	1.4	2.3	9.6

Table 5. Statistics of differences between gravimetric and GPS-levelling geoid height, with and without gravimetric geoid height corrections in Sect. 3. Units in cm

	Without corrections	With all corrections
Min	20.1	4.4
Max	40.2	26.0
Mean	28.7	10.1
Standard deviation	10.5	5.5

Table 6. Statistics of differences between gravimetric and GPS-levelling geoid height with direct, indirect and atmospheric corrections. Units in cm

	Direct effect + downward-continuation correction	Indirect effect	Atmospheric effect
Min	2.1	17.3	19.5
Max	28.5	35.0	39.0
Mean	11.2	22.7	27.9
Standard deviation	7.1	7.3	9.8

The gravimetric geoid is then fitted to the 23 GPS-levelling geoid heights using a four-parameter transformation model [Eq. (23)]. Table 8 shows the statistics of differences between gravimetric- and GPS-levelling-derived geoid heights, after fitting.

The results of Tables 7 and 8 illustrate an improvement of about ±4.4, ±4.8, ±7.0, ±11.0, and ±2 cm in standard deviations of remaining residuals for LS, Molodenskii et al., Wong and Gore, Vincent and Marsh, and NKG96 models, respectively. The LS model still has the best agreement with the GPS-levelling- derived geoid, compared with the other estimators and the best geoid model in the region (NKG96).

Table 7. Statistics of differences between GPS-levelling- derived and gravimetric geoid height models at 23 GPS stations. Units in cm

	LS	Molodenskii et al.	Wong and Gore	Vincent and Marsh	NKG96
Min	4.4	5.8	8.2	4.2	−0.7
Max	26.0	32.8	33.2	65.2	33.5
Mean	10.1	13.1	18.2	23.1	17.0
Standard deviation	5.5	7.1	11.2	16.1	8.1

Table 8. Statistics of differences between GPS-levelling- derived and gravimetric geoid height models after fitting to 23 GPS stations within the 6^0 integration area. Units in cm

	LS	Molodenskii et al.	Wong and Gore	Vincent and Marsh	NKG96
Min	−3.1	−4.2	−6.1	−9.4	−11.2
Max	5.2	6.3	9.2	16.0	10.9
Mean	0.0	0.0	0.0	0.0	0.0
Standard deviation	1.1	2.3	4.2	5.1	6.1

5 Discussions and conclusions

Four different estimators have been selected to determine the geoid heights over Sweden. The Vincent and Marsh estimator employs a high-degree reference gravity field with low-degree satellite-derived gravity anomalies subtracted from the terrestrial gravity anomalies. No kernel modification is used in this model. The modified Wong and Gore and Molodenskii et al. estimators use the Stokes kernel modification and a high-degree reference field. The LS model combines the terrestrial and satellite-derived gravity anomalies in an optimum way.

Related to the question of the geoid height determination is the question of direct and indirect topographic corrections. These topographic corrections are improved in this study. Classical integral formulas for the topographic corrections suffer from planar approximations, and some long-wavelength contributions are also missing. Spherical harmonic presentation of topographic correction is simple but needs to be expanded to a very high degree in practice to consider short-wavelength information. Therefore, we present combined correction as a compromise, which considers both short- and long- wavelength contributions. The direct topographic effect on geopotential is considered separately, using a spherical harmonics presentation. Topographic corrections are very significant and have to be applied in both the geopotential model and Stokes' formula.

The atmospheric geoid effect is derived as a total correction, adding direct and indirect atmospheric effects. It has been presented by spherical harmonics as a correction to the modified Stokes formula. Its effect is significant for accurate geoid height determination and always has to be considered.

Helmert anomalies at the topography were reduced on the geoid with the downward continuation correction procedure. On computing this effect, the Poisson kernel is modified. The correction due to the truncation error (due to limiting the integration area) is computed using EGM96 and added to the final solution. The low-frequency contributions, computed from EGM96, are subtracted from the gravity data in the Poisson kernel and downward continued, separately. It is then added to the downward-continued short-wavelength part computed with the iterative process. The results show that this effect is very important, augmented by the height difference between the topography and the geoid. The effect of downward continuation on the geoid is positive everywhere in this study.

The results of the four gravimetric geoid height estimators are compared with GPS-levelling geoid height at SWEPOS GPS stations. The results of the LS estimator agree better with the GPS-levelling geoid than the other methods. It has the smallest differences in mean value (10.1 cm) and standard deviation (±5.5 cm). The global RMS error of this estimator is computed to be ±28.6 cm. After fitting the gravimetric geoids to the GPS-levelling stations with a four-parameter transformation model, the results also show that the LS method has the best consistency, where the standard deviation of differences is reduced to ±1.1 cm.

As a comparison, we have used the best geoid model in the region (NKG96) and also fitted it to the same GPS stations. The results yield ±6.1 cm standard error of agreement with GPS stations. Comparing with the other estimators, it is concluded that all new corrections applied in this study, compared with the classical one (which is used in the NKG96 model), improve the accuracy of the geoid heights in Sweden.

It should be noted that if a precise geoid, say at the centimeter level, is desired, the topographical density correction discussed in Martinec (1993) must be considered in the final computations, especially in mountainous areas. A denser DTM, especially in the mountainous area on the border of Norway and Sweden, is needed to evaluate topographic corrections more precisely. According to our experience, the use of the $2.5' \times 2.5'$ DTM in this study may cause error of several centimeters in the direct topographic effects, in mountainous areas and thus in the resulting geoid heights.

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