Simen Knutsen Furset

Existence, smoothness and numerical approximation for two generalizations of the stochastic heat equation

Master's thesis in Mathematical Studies
Supervisor: Espen Robstad Jakobsen
May 2023

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Contents

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Abstract

In this thesis we consider a class of stochastic evolution equations given formally by $dX(t) + AX(t) = dW(t)$, where $X(t)$ belongs to a Hilbert space H, A is an operator on (a subset of) the Hilbert space and $W(t)$ is a Q-Wiener process on the Hilbert space. We consider existence, regularity and covariance for (weak) solutions in the case where $H = \mathcal{L}^2(\mathcal{D})$ for $\mathcal{D} \subset \mathbb{R}^d$, $A = (\iota^2 - \Delta)^\gamma$ and $W(t)$ has increments that are Gaussian fields on $\mathcal{D}$ with Matern-like covariance structure. This analysis is primarily be based on a lecture note by Kovács and Larsson [16] and a book by Da Prato and Zabczyk [8]. We also consider existence, regularity and covariance of a more general type of stochastic evolution equation described by Kirchner and Willems [15], where we also have a fractional time derivative. These are given formally by $(\frac{\mathrm{d}}{\mathrm{d}t} + (\iota^2 - \Delta)^\gamma)^\delta X(t) = \dot{W}_t$. Finally we consider some approximation results for the finite element method approximations for both the fractional and non-fractional equations. We consider error results of both the strong error in both cases and the error in covariance for the simple heat equation.

Introduction and motivation

Gaussian random field models are common in spatial statistics [11]. Especially popular are those specified by a Matern auto-covariance function, i.e.

$$r_M(h) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} \left(\frac{\sqrt{2\nu}}{\rho} h \right)^\nu K_\nu \left(\frac{\sqrt{2\nu}}{\rho} h \right),$$

where K_ν is a modified Bessel function of the second kind. The three parameters σ, ν and ρ separately control the variance, the smoothness and the correlation range of the process, respectively. Since a hallmark paper by Lindgren, Rue and Lindström [17] in 2011 there has been a great interest in efficient numerical simulation of Matern fields using finite element methods. These methods exploit the fact that Matern fields solve stochastic partial differential equations of the form $c(\kappa^2 - \Delta)^\beta W(x) = \mathrm{d}U(x)$ when considered as an equation on $\mathbb{R}^d$ [21]. $\mathrm{d}U(x)$ here denotes a spatial white noise. The parameters c, β and κ are not the same as in those in auto-covariance function above, but have similar respective interpretations; c controls variance, β controls the smoothness and κ controls the correlation range. Recently there has been interest in developing models for spatial statistics that also incorporate time, while still allowing efficient computation and easy interpretation. The obvious approach of simply extending the Matern field from d to $d+1$ dimensions is undesirable, since it prohibits us from controlling the smoothness and correlation range separately in time and space. The second obvious approach is two construct a process $\{W(t)(x)\}_{t\in[0,T]}$ by defining $W_0(x)=0$ and requiring that $W(t_2)(x)-W(t_2)(x)$ is a Matern field for $t_2>t_1$, similarly to how Wiener processes are constructed. However, this model will not have very interesting time-space interactions and will therefore have limited use in the modelling of complex phenomenon. A more sophisticated idea is to instead consider models that are solutions to stochastic evolution equations. For example we could consider a stochastic heat equation formally expressed by

$$\mathrm{d}X(t)(x)-\Delta X(t)(x)\mathrm{d}t=\mathrm{d}W(t)(x)\,, \tag{1}$$

where $W(t,x)$ is as defined earlier; a Wiener process in time for each x and a Matern field in space for each t. The hope is that the solutions to Equation 1 will inherit the "nice" properties of Matern fields. Various generalizations of the stochastic heat equation have also been considered. In this thesis we consider two such generalization.

Section 1 is a preliminary introduction to some necessary theory. It is assumed that the reader is already familiar with Banach and Hilbert space theory and the theory of random variables on $\mathbb{R}$. We consider the definition of random variables on Banach and Hilbert spaces and generalize the Wiener process on $\mathbb{R}$ to a general Hilbert space H. We "apply" some of these concepts when we introduce the concept of Whittle-Matern fields in Section 1.4. We then discuss the notion of predictability and conditional expectation, both of which will be useful to us later. Finally in Section 1.7 we define the space-time covariance function of a random process on a Hilbert space, inspired by the article by Kirchner and Willems [15]. Section 1 is heavily based on a lecture note by Kovács and Larsson [16] and somewhat on a book by Da Prato and Zabzcyk [8]. The discussion of Whittle-Matern noise in Section 1.4 and the discussion of space-time covariance functions in Section 1.7 are however taken from other sources.

In Section 2 we construct a stochastic integral and discuss some of its properties. A small detour is made into the theory of C_0-semigroups and deterministic evolution equations. We then derive existence and uniqueness results for a very general class of stochastic evolution equations on Hilbert spaces. Section 2 is also heavily based on the lecture note by Kovács and Larsson [16] and the book by Da Prato and Zabzcyk [8], however our treatment of semigroups is based primarily on the treatment of the subject by Engel and Nagel [10] and somewhat on that of Pazy [18].

In Section 3 we consider a "space fractional heat equation", which we formally define as

$$\mathrm{d}X(t)+(\iota^2-\Delta)^\gamma X(t)\mathrm{d}t=\mathrm{d}W(t,x)\,. \tag{2}$$

We consider this equation as an equation on a compact domain $\mathcal{D}\subset\mathbb{R}^d$ and $t\in[0,T]$ and only with initial condition $X(0)=0$ and zero boundary conditions, i.e. $X(t,x)=0$ on $\partial\mathcal{D}$. We then derive existence, uniqueness, as well as spatial and temporal smoothness results for this equation using the theory from Section 2. These results give conditions only involving the parameters $0<\gamma\leq 1$ and $0\leq\beta$ and the dimension d and are thus much more explicit and easier to check than the abstract results from Section 2. Finally we derive the space-time covariance function of the solutions and show that asymptotically they have exponential decay in correlation range both in space and in time. Kovács and Larsson [16] consider the case where $\gamma = 1$ and $\iota = 0$ in their lecture note and the analysis in Sections 3.1 and 3.2 is the authors attempt at extending this analysis to a more general case.

In Section 4 we further generalize the heat equation and consider a "space-time fractional heat equation", which we formally define as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + (\iota^2 - \Delta)^{\gamma}\right)^{\gamma} X(t,x)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t} W(t,x)\,. \tag{3}$$

This equation is outside of the framework established in Section 2, so Section 4.1 and 4.2 describes the theory of the existence and uniqueness such equations. In Section 4.3 we apply this theory to Equation 3 to find results for temporal and spatial smoothness in terms of the parameters β, γ and δ. In Section 4.4 we derive the asymptotic covariance operator of the solutions and show that we have exponential decay of correlation both in time and in space. Section 4 is heavily based on a paper by Kirchner and Willems [15] where they consider a more general version of this equation. However the analysis of spatial and temporal smoothness in Section 4.3 is novel in the sense that the general results of Kirchner and Willems have been derived under more concrete assumptions. The discussion of the covariance function in Section 4.4 is taken from Kirchner and Willems, but the analysis of correlation decay in Section 4.4 is somewhat novel.

In Section 5 we consider how to apply an abstract version of the finite element method to numerically solve the two clases of stochastic evolution equations that we have described. We consider a partial discretization where the spatial part of the solution space is approximated by a finite-dimensional subspace, but no discretization in time is made. Section 5.1 describes our assumption on the finite element space and in Section 5.2 we construct a discrete approximation Δ_h to the Laplacian Δ. In Section 5.3 we derive some approximation results for the eigenvectors and eigenvalues of the discrete Laplacian. In Section 5.4 we apply the eigen-approximations from Section 5.3 to estimate the strong error, first for the heat equation, and then for the generalizations thereof considered in Section 3 and 4. Finally we consider an approximation result for the space-time covariance function in Section 5.4. Our treatment of the finite element method is based heavily on that of Strang and Fix [19], but inspiration has also been taken from Thomeè [20] and some lecture notes by Barth and Lang [3].

Notation

In this thesis we adopt a function notation for random processes, i.e. we use X to denote the process $[0,T] \to \mathcal{L}^2(\Omega, H)$ and $X(t)$ to denote the corresponding H-valued random variable. This notation has the advantage that we can easily extend it to also include the dependence on the probability space Ω. In this case we will write $X(\omega)$ to denote the corresponding map $[0,T] \to H$ and $X(t,\omega)$ to denote the corresponding element in H. Frequently in this thesis the Hilbert space H is a function space, most commonly $\mathcal{L}^2(\mathcal{D})$ for a domain $\mathcal{D} \subset \mathbb{R}^d$. In this case we will also use the notation $X(t,x)$ to denote the $\mathcal{R}$-valued random variable corresponding to $t \in [0,T]$ and $x \in \mathcal{D}$.

We will frequently in this thesis employ estimation results of the type $x \leq C_1 y$ and $y \leq C_2 z$, where we do several estimations all imposing different constants on the estimand. In this case we usually write something along the lines of $x \leq Cy \leq Cz$. It is then left implicit that the constant C changes between estimations. In these cases the dependencies of C are left up to context.

1 Random processes on Hilbert spaces

In this section we cover the basic theory of Hilbert-space valued random variables and processes. The treatment is based heavily on a lecture note by Kovács and Larsson [16].

1.1 Banach-space valued random variables

Let E be a separable Banach space over $\mathbb{R}$. We denote the smallest σ-algebra containing the open sets in U by the symbol $\mathcal{B}(E)$. We can then define a probability measure μ on $(E, \mathcal{B}(E))$ by a map $\mu : \mathcal{B}(E) \longrightarrow \mathbb{R}$ satisfying the axioms of a probability measure

- $\mu(A) \in [0, 1]$ for $A \in \mathcal{B}(E)$
- For a countable collection of pairwise disjoint sets $\{A_i\}_i \subset \mathcal{B}(E)$ we have that $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.

We will study these measures primarily by doing projection into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. More precisely for a functional f in the dual space E^* we consider the real random variable $f : (E, \mathcal{B}(E), \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The map f induces the push-forward probability measure $\mu_f = \mu \circ f^{-1}$ on $\mathbb{R}$. In this thesis we will mostly consider the case where E is a seperable Hilbert space. By the Riesz representation theorem seperable Hilbert spaces are self-dual and the functionals $f \in E^*$ can be completely characterized by considering elements $v \in E$ and defining $f_v(x) = \langle v, x \rangle$. Whenever we work in a Hilbert space we will commonly use H or U to denote it. For Banach spaces we commonly use E or F. Whenever we are discussing Hilbert spaces we always assume that they are seperable.

Definition 1.1.1. ***Gaussian random variable on a Banach space.*** *Let Ω be a probability space equipped with a σ-algebra Σ and a probability measure P. Let E be a Banach space. An E-valued random variable is a $B(E)$-measurable map $X : (\Omega, \Sigma, P) \longrightarrow (E, B(E))$. An E-valued random variable is called Gaussian if for every $f \in E^*$ the projections $f \circ X : \Omega \to \mathbb{R}$ are either a real Gaussian variable or is constant with probability* 1.
If E is a Hilbert space then we only need to consider $f_v = \langle v, X \rangle$, so that X is Gaussian if for every $v \in E$, $f_v \circ X = \langle X, v \rangle$ is a either a real Gaussian variable or is constant with probability 1.

We now restrict our view to that of a Hilbert space H. For an H-valued random variable X and for $v \in E$ we can calculate the mean value $m_v := \mathrm{E}[f_v(X)]$ of the real-valued projection $f_v(X) = \langle v, X \rangle$ by

$$m_v = \int_\Omega f_v(X)\, \mathrm{d}P = \int_\Omega \langle v, X \rangle\, \mathrm{d}P\,.$$

If we assume that $\mathbb{E}\left[\|X\|_H\right] = \int_\Omega \|X\|_H\, \mathrm{d}P < \infty$, then by the Cauchy-Bunyakowsky-Schwarz inequality we have that

$$m_v = \int_\Omega \langle v, X \rangle\, \mathrm{d}P \leq \|v\|_H \int_\Omega \|X\|_H\, \mathrm{d}P < \infty\,,$$

implying that the map $v \mapsto m_v$ is a linear functional $H \to \mathbb{R}$. Applying the same inequality we can also see that the functional is continuous. We can thus apply the Riesz representation theorem to see that there exists an element $m \in H$ such that

$$m_v = \int_\Omega \langle v, X \rangle \, \mathrm{d}P = \langle m, v \rangle \, .$$

We call m the mean of X. This element can also be expressed as the Bochner integral of X since

$$m_v = \int_\Omega \langle v, X \rangle \, \mathrm{d}P = \left\langle \int_\Omega X \, \mathrm{d}P, v \right\rangle = \langle m, v \rangle \, ,$$

so that $m_v = \int_\Omega X dP$. We often denote this integral by $\mathbb{E}[X]$. As we have seen the condition $\mathbb{E}[\|X\|_H] < \infty$ is sufficient to determine that $\mathbb{E}[X]$ exists. The space of random variables on H such that $\mathbb{E}[\|X\|_H] < \infty$ exists is denoted by $\mathcal{L}^1(\Omega, H)$. This space is a Banach space under the norm $\| \cdot \|_{\mathcal{L}^1(\Omega,H)} := \mathbb{E}[\| \cdot \|_H]$.

We can further calculate the covariance of two projections by

$$\mathrm{Cov}(f_u(X), f_v(X)) := \int_\Omega \langle u, X \rangle \langle v, X \rangle \, \mathrm{d}P - m_u m_v \, .$$

We now assume that $\mathbb{E}[\|X\|_H^2] < \infty$ and fix $u \in H$. Similarly to before we can then see that the map $v \mapsto \mathrm{Cov}(f_u(X), f_v(X))$ is a well-defined continuous, linear functional on H since

$$\mathrm{Cov}(f_u(X), f_v(X)) \leq \|u\|_H \|v\|_H \mathrm{Var}(\|X\|_H) < \infty \, ,$$

again by the Cauchy-Bunyakovsky-Schwarz inequality. We can then apply the Riesz representation theorem to $v \mapsto \mathrm{Cov}(f_u(X), f_v(X))$ to conclude that there exists an element q_u, depending on u, such that

$$\mathrm{Cov}(f_u(X), f_v(X)) = \langle q_u, v \rangle \, .$$

We can now define $Q : u \mapsto q_u$. The operator Q is called the covariance operator of X. It is easy to see that Q is a linear operator since for all $x \in H$ we have that

$$\begin{aligned} \langle Q(u+v), x \rangle &= \mathrm{Cov}(f_{u+v}(X), f_x(X)) = \mathrm{Cov}(f_u(X), f_x(X)) + \mathrm{Cov}(f_v(X), f_x(X)) \\ &= \langle Qu, x \rangle + \langle Qv, x \rangle = \langle Qu + Qv, x \rangle \, . \end{aligned}$$

Q is also a bounded operator since

$$\|Qu\|_H^2 = \langle Qu, Qu \rangle = \mathrm{Cov}(f_u(X), f_{Qu}(X)) \leq \|u\|_H \|Qu\|_H \mathrm{Var}(\|X\|_H) \, ,$$

so that $\|Q\|_{L(H)} \leq \mathrm{Var}(\|X\|_H)$. Since the covariance is symmetric we see also that Q is self-adjoint. In addition Q is non-negative definite since

$$0 \leq \mathrm{Var}(\langle X, u \rangle) = \langle Qu, u \rangle \, .$$

In the case where X is Gaussian it can be also shown that $\mathrm{Tr}(Q) < \infty$, see Kovács and Larsson [16]. We use the notation $X \sim N(m, Q)$ to denote that X is a Gaussian variable with mean element m and covarianec operator Q. From now on, whenever we presuppose a covariance operator it will be implicit that the operator has the properties we have listed. We summarize this in a definition.

Definition 1.1.2. ***Covariance operator***. *Let H be a Hilbert space. An operator $Q \in L(H)$ is called a covariance operator if it is self-adjoint, non-negative definite, and has finite trace, i.e.*

$$Tr(Q) := \sum_{k=1}^{\infty} \langle Qe_k, e_k \rangle = \sum_{k=1}^{\infty} \lambda_k < \infty \,,$$

where $\{\lambda_k\}_k$ are the eigenvalues of Q, with corresponding eigenvectors $\{e_k\}_k$. If H is self-adjoint and non-negative definite, but has unbounded trace, we call it an improper covariance operator.

It is worth noting that finite-trace operators are compact, so that we can always find an orthonormal eigenvector basis for the Hilbert space H with respect to a covariance operator. In addition, since covariance operators are non-negative definite, their eigenvalues will be non-negative.

We will now look at two examples of Gaussian random variables on Hilbert spaces.

Example 1.1.1. ***Gaussian random variable on*** $\mathcal{L}^2([0,1])$. *Let $H = \mathcal{L}^2([0,1])$. Define the H-valued random variable X by $X = h + Zg$, where $Z \sim N(0,1)$ and $g, h \in H$. Then for $v \in H$ we have*

$$f_v(X) = \int_0^1 h(x)v(x)\,dx + Z\int_0^1 g(x)v(x)\,dx = m_v + \sigma_v Z \,,$$

such that $f_v(X) \sim N(m_v, \sigma_v^2)$. It follows that X is a Gaussian random variable on E. It is clear that $m = h$ since $\langle h, v\rangle = \int_0^1 h(x)v(x)\,dx = m_v$. We can calculate the covariance of two projections f_u and f_v by

$$Cov(f_u(X), f_v(X)) = Cov(m_u + \sigma_u Z, m_v + \sigma_v Z) = \sigma_v\sigma_u = \int_0^1 g(x)v(x)\,dx \int_0^1 g(x)u(x)\,dx$$
$$= \langle\langle g, u\rangle g, v\rangle = \langle Qu, v\rangle \,,$$

so that $Qu = \langle g, u\rangle g = g(x)\int_0^1 g(x)u(x)\,dx$. Also

$$Tr(Q) = \sum_{k=1}^{\infty} \langle\langle g, e_k\rangle\, g, e_k\rangle = \left\langle g, \sum_{k=1}^{\infty} \langle g, e_k\rangle\, e_k \right\rangle = \langle g, g\rangle = ||g(x)||^2 < \infty \,.$$

Example 1.1.2. ***Gaussian random variable on a*** $\ell^2(\mathbb{R})$. *Let $H = \ell^2(\mathbb{R})$, the Hilbert space of square-integrable sequences equipped with the standard inner product $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$. Then $e_k := \{\delta_{i,k}\}_i$ defines an orthonormal basis for H. Let $\{\beta_k\}_k$ be a sequence of independent standard Gaussian variables and define*

$$X_n := m + \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k e_k \,,$$

where $\{\lambda_k\}_k \subset \mathbb{R}$ and $m \in H$, so that

$$X := \lim_{n\to\infty} X_n = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k e_k = m + \left(\sqrt{\lambda_1}\beta_1, \sqrt{\lambda_2}\beta_2, \sqrt{\lambda_3}\beta_3, ...\right) \,. \qquad (4)$$

The limit X_n converges to X in mean-square assuming $\sum_{k=1}^{\infty}|\lambda_k| < \infty$. For a fixed $v = (v_1, v_2, ...) \in U$ we have that $f_v(X_n) = \sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k\langle e_k, v\rangle = \sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k v_k$, which is Gaussian on $\mathbb{R}$, so that X_n and the limit X are both Gaussian on U. X has mean m since $E[\langle X, u\rangle] = \langle m, u\rangle$ and covariance given by

$$Cov(f_u(X), f_v(X)) = \sum_{k=1}^{\infty}\lambda_k\langle u, e_k\rangle\langle v, e_k\rangle = \sum_{k=1}^{\infty}\lambda_k u_k v_k = \langle Qu, v\rangle$$

where the covariance operator Q is defined by $e_k \mapsto \lambda_k e_k$, so that $Q(u_1, u_2, ...) = (\lambda_1 u_1, \lambda_2 u_2, ...)$.

There is nothing unique about the space $\ell^2(\mathbb{R})$ in Example 1.1.2. We could pick any Hilbert space with an orthonormal basis $\{e_k\}_k$ and perform the same construction. Alternatively we can fix the covariance operator Q a priori. Covariance operators satisfy the requirements of the spectral theorem so we can always find a corresponding eigenvector basis $\{e_k\}_k$. The sum given by Equation 4 will then converge since $\sum_{k=1}^{\infty}|\lambda_k| = \sum_{k=1}^{\infty}\langle Qe_k, e_k\rangle = \mathrm{Tr}(Q) < \infty$. It follows that, for any given Hilbert space H, if we pick an arbitrary element $m \in H$ and a covariance operator Q acting on H there will always exist a Gaussian random variable with m as its mean element and Q as its covariance operator.

We now consider a Gaussian random variable X on a Hilbert space H with mean element m and covariance operator Q. It turns out that every such variable can be decomposed in the manner of Equation 4. To see this, let $\{e_k\}$ be an orthonormal eigenvector basis for Q. For all $\omega \in \Omega$, we can decompose $X(\omega)$ with respect to the basis and get

$$X(\omega) = \sum_k \langle X(\omega), e_k\rangle e_k \, .$$

By assumption X is Gaussian, so $\langle X(\omega), e_k\rangle$ has a Gaussian distribution with mean $\langle m, e_k\rangle$ and variance $\langle Qe_k, e_k\rangle = \lambda_k$. It follows that $\frac{1}{\sqrt{\lambda_k}}\langle X - m, e_k\rangle$ has a standard Gaussian distribution, assuming $\lambda_k > 0$. In the case where $\lambda_k = 0$ we have that $\langle X(\omega) - m, e_k\rangle = 0$ almost surely. We write

$$\beta_k(\omega) = \begin{cases} \frac{1}{\sqrt{\lambda_k}}\langle X(\omega) - m, e_k\rangle, & \text{if } \lambda_k > 0. \\ 0, & \text{if } \lambda_k = 0. \end{cases} ,$$

Then

$$X(\omega) = \sum_k \langle X(\omega), e_k\rangle e_k = \sum_k \sqrt{\lambda_k}\beta_k e_k + \sum_k \langle m, e_k\rangle e_k = m + \sum_k \sqrt{\lambda_k}\beta_k e_k \, .$$

It remains only to show that the β_k's are independent. They are pairwise uncorrelated since

$$E[\beta_k\beta_j] = \frac{1}{\sqrt{\lambda_k}\sqrt{\lambda_j}}E[\langle X-m, e_k\rangle\langle X-m, e_j\rangle] = \frac{1}{\sqrt{\lambda_k}\sqrt{\lambda_j}}\langle Qe_k, e_j\rangle = \begin{cases} 1, & \text{if } k = j. \\ 0, & \text{if } k \neq j. \end{cases} .$$

If we take any finite subselection of β_k's, say $\{\beta_{k_n}\}_{n=1}^{N}$, then this subselection has a joint Gaussian distribution since

$$(a_1, a_2, ..., a_N) \cdot (\beta_{k_1}, \beta_{k_2}, ..., \beta_{k_N}) = \sum_{n=1}^{N} a_n \beta_{k_n} = \sum_{n=1}^{N} a_n \frac{1}{\sqrt{\lambda_{k_n}}}\langle X - m, e_{k_n}\rangle$$

$$= C + \langle X, \sum_{n=1}^{N} \frac{a_n}{\sqrt{\lambda_{k_n}}} e_{k_n} \rangle \, ,$$

which has a Gaussian distribution by the assumption that X is Gaussian. It follows that any finite subselection $\{\beta_{k_n}\}_{n=1}^{N}$ of β_k's is independent. We summarize our discussion in Theorem 1.1.1.

Theorem 1.1.1. ***Karhunen-Loève expansion***. *Let X be a Gaussian random variable on a Hilbert space H with mean element m and covariance operator Q. Let $\{e_k\}_k$ be an orthonormal eigenvector basis for H with respect to Q with corresponding eigenvalues $\{\lambda_k\}_k$. We can then write the random variable X in the form*

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k \, ,$$

where $\{\beta_k\}_k$ is a collection of real, independent, standard Gaussian variables. This expansion is known as the Karhunen-Loève expansion. Conversely, for any covariance operator Q the sum $X = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k$ converges in $\mathcal{L}^2(\Omega, H)$ to a Gaussian random variable with mean element 0 and covariance operator Q.

1.2 Covariance operators with non-finite trace

In some cases it might be of interest to also consider improper covariance operators, i.e. covariance operators with non-finite trace. In this case the Karhunen-Loève expansion $\sum_{k=1}^{\infty} \sqrt{\lambda_L} \beta_k e_k$ gives a divergent sum. The solution to this problem is to extend the Hilbert space by equipping it with a laxer norm. As a an example, let us consider the Hilbert space of square-integrable sequences $\ell^2(\mathbb{R})$. and the operator $Q = I$, the identity operator. The identity has orthonormal eigenvector basis given by $e_k = (\delta_{k,n})_n$, so that $\mathrm{Tr}(Q) = \sum_{k=1}^{\infty} \langle e_k, e_k \rangle_{\ell^2(\mathbb{R})} = \sum_{k=1}^{\infty} 1 = \infty$, so Q is an improper covariance operator. The Karhunen-Loève expansion of $X \sim N(0, Q)$ is the given by

$$X = \sum_{k=1}^{\infty} \beta_k e_k \, .$$

where the elements β_k are i.i.d. real Gaussian variables with variance 1. This sum diverges. As already stated we wish to extend the Hilbert space $\ell^2(\mathbb{R})$ so the sum converges. Consider the extension given by

$$\tilde{U} = \left\{ (a_n)_n \; : \; \sum_{k=1}^{\infty} \frac{a_n^2}{k^2} < \infty \right\} \supset \ell^2(\mathbb{R}) \, ,$$

equipped with the inner product $\langle (a_n)_n, (b_n)_n \rangle_{\tilde{U}} := \left\langle (\frac{a_n}{k})_n, (\frac{b_n}{k})_n \right\rangle_{\ell^2(\mathbb{R})}$. Since we have changed the norm, $(e_k)_k$ is no longer an orthonormal basis, we have to normalize it by defining $f_k := k e_k$. Now consider the operator $\tilde{Q} : \tilde{U} \to \tilde{U}$ defined by $\tilde{Q}(f_k) = \frac{1}{k^2} f_k$. Then $\mathrm{Tr}(\tilde{Q}) = \sum_{k=1}^{\infty} \left\langle \tilde{Q} f_k, f_k \right\rangle_{\tilde{U}} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Now let $X \sim N(0, \tilde{Q})$. By Theorem 1.1.1 we can write

$$X = \sum_{k=1}^{\infty} \frac{1}{k} \beta_k f_k = \sum_{k=1}^{\infty} \beta_k e_k \,,$$

where the β_k's are independent real Gaussian variables with mean 0 and variance 1. Since $\mathrm{Tr}(\tilde{Q}) < \infty$, $X \in \tilde{U}$ almost surely.

In general we might wish to make sense of an improper covariance-operator Q on a general, separable, real Hilbert space U. In the above example we solved the problem by extending the Hilbert space $\ell^2(\mathbb{R})$ to $\tilde{U}$ and defining a new covariance operator $\tilde{Q}$ on $\tilde{U}$. How can we generalize this procedure? First let $\{e_k\}_k$ be a countable, orthonormal, eigenvector basis for U with respect to Q and define

$$\mathcal{A} = \left\{ \sum_k a_k e_k \,:\, (a_k)_k \subset \mathbb{R} \right\}$$

i.e. the set of formal series of $\{e_k\}_k$. Now let $A : \mathcal{A} \to \mathcal{A}$ be defined by $A(e_k) = \frac{1}{k} e_k$. Then we can define

$$\tilde{U} = \{ u \in \mathcal{A} \,:\, ||Au||_U < \infty \} \supset U \,.$$

We equip $\tilde{U}$ with the inner product $\langle u, v \rangle_{\tilde{U}} = \langle Au, Av \rangle_U$. $(\tilde{U}, \langle \cdot, \cdot \rangle_{\tilde{U}})$ is then a separable Hilbert space and $f_k := A^{-1} e_k = k e_k$ is an orthonormal basis for $\tilde{U}$. Define $\tilde{Q} : \tilde{U} \to \tilde{U}$ by $\tilde{Q} = A^2 Q$. Then

$$\mathrm{Tr}(\tilde{Q}) = \sum_{k=1}^{\infty} \left\langle A^2 Q f_k, f_k \right\rangle_{\tilde{U}} = \sum_{k=1}^{\infty} \frac{\lambda_k}{k^2} \leq ||Q||_{L(U)} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \,,$$

since Q is assumed to be bounded. Then by Theorem 1.1.1 $X \sim N(0, \tilde{Q})$ is given by

$$X = \sum_{k=1}^{\infty} \frac{\sqrt{\lambda_k}}{k} \beta_k f_k = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k \,.$$

so that the summarize $\sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k$, which didn't converge in H, converges in $\tilde{H}$. We summarize our findings in a theorem.

Theorem 1.2.1. *Let H be a Hilbert space. Let Q be an improper covariance operator and let $\{e_k\}_k$ be an orthonormal eigenvector basis for H with respect to Q with corresponding eigenvalues $\{\lambda_k\}_k$. Let $\{\beta_k\}_k$ be a family of independent, real, standard Gaussian variables. Then there exists a Hilbert space $\tilde{H} \supset H$ such that the sum*

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k$$

converges to a Gaussian variable in $\tilde{H}$ with covariance operator $\tilde{Q}$.

1.3 Wiener-processes on Hilbert spaces

In the Itô-calculus the real-valued Wiener process plays an important role in constructing the Itô-integral. Our goal is to construct a stochastic integral that can be used to express differential equations on a Hilbert space. To this end we generalize the Wiener-process on $\mathbb{R}$ to a general Hilbert space H.

Definition 1.3.1. ***Wiener-process.*** *Let H be a Hilbert space and let Q be a covariance operator. A Q-Wiener process W on U is a random process $\{W(t)\}_{t\in[0,\infty)}$ satisfying*

- $W(0) = 0$ *almost surely.*
- *The map $t \mapsto W(t)$ is almost surely continuous.*
- *For any finite partition $0 = t_0 < t_1 < t_2 < ... < t_n < \infty$ the increments $\{W(t_{k+1}) - W(t_k)\}_{k=1}^{n-1}$ are independent random variables.*
- *For any $0 \leq s \leq t < \infty$ the increments $W(t) - W(s) \sim N(0, (t-s)Q)$.*

Let W be a Wiener process. For each $t \in [0, \infty)$ $W(t) = W(t) - W(0) \sim N(0, tQ)$. Therefore we can write

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k\,, \tag{5}$$

It can be shown that for every k, $\beta_k(t)$ is a standard Wiener process on $\mathbb{R}$, see Kovács and Larsson [16]. The family of Wiener processes $\{\beta_k(t)\}_k$ is independent. Conversely for a covariance operator Q and an independent family of real, standard, Wiener processes Equation 5 gives a Wiener process on U. Equation 5 converges in $\mathcal{L}^2(\Omega, C([0,T], H))$.

1.4 Whittle-Matern fields

A Hilbert space of particular interest to us is the space of square integrable functions $H = \mathcal{L}^2(\mathcal{D})$, where $\mathcal{D} \subset \mathbb{R}^d$. When working on this space, Hilbert-Schmidt operators $Q : H \to H$ are also integral operators, i.e.

$$Qg(x) = \int_{\mathcal{D}} g(y)r(x,y)\,\mathrm{d}y$$

for some function $r : \mathbb{R}^2 \to \mathbb{R}$. Gaussian random variables on $\mathcal{L}^2(\mathcal{D})$ corresponds to $\mathcal{D}$-indexed Gaussian random fields and the function r is the auto-covariance function of the corresponding field. A common choice of Gaussian field in spatial statistics is the Matern field, with corresponding stationary auto-covariance function

$$r_M(h) = r_M(x-y) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}\left(\frac{\sqrt{2\nu}}{\rho}h\right)^{\nu} K_\nu\left(\frac{\sqrt{2\nu}}{\rho}h\right)\,, \tag{6}$$

where K_ν is a Bessel function of the second kind. One of the advantages of using Matern fields in statistical modelling is that the parameters σ, ρ and ν are easily interpretable. σ controls the variance and ν controls the smoothness of the realized fields. ρ controls the "range" of the fields; the rate at which covariance decays in space. The spectral density function corresponding to Matern fields is

$$f_M(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}^d} e^{-i\omega h} r(h)\,\mathrm{d}h = \frac{c}{2\pi}(\kappa^2+\omega^2)^{-\beta}\,,$$

where $\beta = \nu + \frac{d}{2}$, $\kappa^2 = \frac{\nu}{2\pi^2\rho^2}$, and $c = \sigma^2\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(\nu)}2^{3\nu-1}\nu^\nu\pi^{2\nu+\frac{d}{2}-1}$. Now let $\mathcal{D} = \mathbb{R}^d$ and consider the covariance operator $Q = (\kappa^2 - \Delta)^{-\beta}$, defined on sufficiently smooth subset of $H = \mathcal{L}^2(\mathbb{R}^d)$. Then for $g \in \mathcal{L}^2(\mathbb{R}^d)$ we have that

$$\begin{aligned}
[Qg](x) &= \mathcal{F}^{-1}\left(c(\kappa^2+\omega^2)^{-\beta}\hat{g}(\omega)\right) \\
&= \int_{\mathbb{R}^d} e^{i\omega x} c(\kappa^2+\omega^2)^{-\beta}\frac{1}{2\pi}\int_{\mathbb{R}^d} e^{-i\omega y} g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} e^{i\omega(x-y)}\frac{c}{2\pi}(\kappa^2+\omega^2)^{-\beta}\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

A random variable on $\mathcal{L}^2(\mathbb{R}^d)$ will thus have auto-covariance function

$$r(x,y) = \int_{\mathbb{R}^d} e^{i\omega(x-y)}\frac{c}{2\pi}(\kappa^2+\omega^2)^{-\beta}\,.$$

This corresponds to having spectral density $f(\omega) = \frac{c}{2\pi}(\kappa^2+\omega^2)^{-\beta}$. Thus Gaussian random variables on $\mathcal{L}^2(\mathcal{D})$ with $Q = c(\kappa^2-\Delta)^{-\beta}$ are Matern fields on $\mathbb{R}^d$. This property is no longer true if we consider Q as a covariance operator on $\mathcal{L}^2(\mathcal{D})$ for some strict subset $\mathcal{D}$ of $\mathbb{R}^d$. In general Gaussian random variables on $\mathcal{L}^2(\mathcal{D})$ with $Q = c(\kappa^2-\Delta)^{-\beta}$ are considered generalizations of Matern fields. They are often called Whittle-Matern fields in the literature, named after Peter Whittle, who was among the first who showed the relationship between $Q = c(\kappa^2-\Delta)^{-\beta}$ and the Matern field [21]. Due to the prevalence of Matern fields in spatial statistics, Whittle-Matern fields have received a lot attention in the literature the last decade, partially since they more easily lend themselves to numerical simulation than classical Matern fields [17]. Because of these properties, $Q = c(\kappa^2-\Delta)^{-\beta}$ will be our canonical choice of covariance operator when we consider SPDE's driven by a Q-Wiener process.

1.5 Measurability and predictability

To discuss the measurability of random processes we introduce the notion of normal filtrations.

Definition 1.5.1. ***Normal filtration*** *A filtration is a family of σ-algebras $\{\mathcal{F}_t\}_t$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ when $s \leq t$. A filtration is called normal if for $t \geq 0$*

- $P(A) = 0 \Rightarrow A \in \mathcal{F}_0$
- $\bigcup_{s>t}\mathcal{F}_s = \mathcal{F}_t$

We say that a random process X is adapted to $\{\mathcal{F}_t\}_t$ if the random variable $X(t)$ are $\mathcal{F}_t$-adapted. For a Q-Wiener process W it is also natural to require that the increment $W(t)-W(s)$ (for $s \leq t$) is $\mathcal{F}_s$-independent. If a Q-Wiener process satisfies this requirement we say that it is a "Q-Wiener process with respect to the filtration $\{\mathcal{F}_t\}_t$".

There are two ways to think about random processes. One way is to consider X to be a $[0, T]$-indexed collection $\{X(t)\}_t$ of random variables. The other is to consider X as a random H-valued function, i.e. a measurable map $\Omega \times [0, T] \to (H, B(H))$, where Ω is a probability space. To define measurability in this context, we need to attach a σ-algebra to the product space $\Omega \times [0, T]$. The σ-algebra we consider is the smallest σ-algebra containing sets of the form

$$(s, t] \times F \text{ where } 0 \leq s < t \leq T \text{ and } F \in \mathcal{F}_s \,,$$

and

$$\{0\} \times F \text{ where } F \in \mathcal{F}_0 \,,$$

where $\mathcal{F}_s$ is a normal filtration. We denote this σ-algebra by $\mathcal{P}$. We use this σ-algebra to define predictability.

Definition 1.5.2. ***Predictability***. *Let H be a Hilbert space. We call a map $X : (\Omega \times [0, T], \mathcal{P}) \to (H, \mathcal{B}(H))$ H-predictable if it is measurable.*

The next proposition shows that the predictability requirement is not so strict, but allows for a large class of behaviour.

Theorem 1.5.1. *Adapted processes with almost surely left-continuous sample paths are predictable.*

Proof. Let X be an adapted process with left-continuous sample paths. We define

$$X_n(\omega, t) = X(\omega, 0)\mathcal{I}_{\{0\}}(t) + \sum_{k=0}^{\infty} X(\omega, \frac{k}{2^n})\mathcal{I}_{(k/2^n, (k+1)/2^n]}(t) \,.$$

Since X is adapted, $X(k/2^n)$ is $\mathcal{F}_{k/2^n}$ measurable, so that $X_n(t)$ is $\mathcal{F}_t$-measurable. $X_n(t)$ is thus an adapted process as well. By the almost sure continuity of X, $X_n(\omega, t) \to X(\omega, t)$ almost surely. Now take an open set $U \in \mathcal{B}(H)$. Then

$$X_n^{-1}(U) = \{0\} \times X(\cdot, 0)^{-1}(U) + \bigcup_{k=1}^{\infty} \left(\frac{k}{2^n}, \frac{k+1}{2^n} \right] \times X\left(\cdot, \frac{k}{2^n}\right)^{-1} (U) \,.$$

Since X is adapted $X_n(t)^{-1}(U)$ is $\mathcal{F}_t$-measurable, implying that $X_n^{-1}(U) \in \mathcal{F}$, so that X_n is predictable. Since a limit of measurable maps is measurable it follows that X is predictable also. □

1.6 Conditional expectation and martingales

A type of random process that is of particular interest to us are those known as martingales. Intuitively these are processes that at any given point is equally likely to "increase" or "decrease", whatever that means in a Hilbert space context. We will see later that the Q-Wiener processes we have discussed earlier are examples of martingales. In order to rigorously define martingales we are first going to need a definition of conditional expectation for random variables on Hilbert spaces.

Definition 1.6.1. ***Conditional expectation.*** *Let H be a Hilbert space and let $X : (\Omega, \mathcal{F}) \to H$ be a random variable on H. For a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ we formally define the conditional expectation $E[X|\mathcal{G}]$ by the unique $\mathcal{G}$-measurable random variable Z satisfying*

$$\int_A X dP = \int_A Z dP \text{ for all } A \in \mathcal{G} .$$

It is not a trivial fact that the conditional expectation exists and is unique. For a proof of this fact the reader is referred to Kovács and Larsson [16]. I now state, without proof, a lemma with some important properties of the conditional expectation. For proofs of these properties the reader is again referred to Kovács and Larsson [16].

Lemma 1.6.1. ***Properties of the conditional expectation.***
Let H be a Hilbert space and let $X, Y \in \mathcal{L}^1(\Omega, H)$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra.

1. *If X is $\mathcal{G}$-measurable then $\mathbb{E}\left[XY\middle|\mathcal{G}\right] = X\mathbb{E}\left[Y\middle|\mathcal{G}\right]$.*
2. *If X is $\mathcal{G}$-independent, then $\mathbb{E}\left[X\middle|\mathcal{G}\right] = \mathbb{E}\left[X\right]$.*

The following lemma describes another important property of the conditional expectation. This lemma we do prove.

Proposition 1.6.1. *Let H and U be Hilbert spaces and B a bounded linear operator $U \to H$. Let X be a random variable $(\Omega, \mathcal{F}) \to U$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub σ-algebra of $\mathcal{F}$. Assume that $X, BX \in \mathcal{L}^1(\Omega, H)$. Then $\mathbb{E}\left[BX\middle|\mathcal{G}\right] = B\mathbb{E}\left[X\middle|\mathcal{G}\right]$.*

Proof. By the definition of the conditional expectation, for all $A \in \mathcal{G}$

$$\int_A BX \,\mathrm{d}P = \int_A \mathbb{E}\left[BX\middle|\mathcal{G}\right] \,\mathrm{d}P ,$$

but also by definition

$$\int_A BX \,\mathrm{d}P = B\int_A X \,\mathrm{d}P = B\int_A \mathbb{E}\left[X\middle|\mathcal{G}\right] \,\mathrm{d}P = \int_A B\mathbb{E}\left[X\middle|\mathcal{G}\right] \,\mathrm{d}P .$$

The interchange of the operator B and the Bochner integral is possible due to the assumption that $X, BX \in \mathcal{L}^1(\Omega, H)$. The conclusion follows by the uniqueness of the conditional expectation. ☐

We are now ready to define martingales.

Definition 1.6.2. ***Martingales.*** *Let H be a Hilbert space and $\{\mathcal{F}_t\}_t$ a normal filtration. Let M be an $\{\mathcal{F}_t\}_t$-adapted random process on H. M is called a martingale with respect to $\{\mathcal{F}_t\}$, or an $\mathcal{F}_t$-martingale, if*

- *$\mathbb{E}[||M_t||_H] < \infty$ for all $t \geq 0$.*

- $E[M_t|\mathcal{F}_s] = M_s$ *when* $s \leq t$.

The following martingale inequalities are known as the Burkholder-Davis-Gundy inequality and the sub-martingale inequality and they will both prove useful in our later discussion.

Theorem 1.6.1. ***Martingale inequalities.*** *Let* $\{M(t)\}_t$ *be a* H*-valued martingale with respect to the normal filtration* $\{\mathcal{F}_\sqcup\}_t$*. Take* $p \in (1, \infty]$*. If* $\mathbb{E}[||M(t)||_H^p] < \infty$*, then* $||M(t)||_H^p$ *is a sub-martingale, i.e.*

$$||M(s)||_H^p \leq \mathbb{E}[||M(t)||_H^p|\mathcal{F}_s] \quad s \leq t\,.$$

Also, for $p > 1$ *and* $T > 0$

$$\mathbb{E}\left[\sup_{t\in[0,T]} ||M(t)||_H^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[||M(T)||_H^p]$$

Proof. Let $s \leq t$. Then by Definition 1.6.2 $||M(s)||_H^p = ||\mathbb{E}[M(t)|\mathcal{F}_s]||_H^p$. By the triangle-inequality $||\mathbb{E}[M(t)|\mathcal{F}_s]||_H^p \leq \mathbb{E}[||M(t)||_H|\mathcal{F}_s]^p$. Finally, the map $x \mapsto x^p$ is convex, so by Jensen's inequality $\mathbb{E}[||M(t)||_H|\mathcal{F}_s]^p \leq \mathbb{E}[||M(t)||_H^p|\mathcal{F}_s]$. The second inequality follows from the Burkholder-Davis-Gundy inequality for real sub-martingales, see Bassily [4]. □

We have special interest in the space of almost surely continuous, square-integrable $\mathcal{F}_t$-martingales on H. We will see later that this space plays an important role in the construction of our stochastic integral. We denote this space $\mathcal{M}_T^2(H)$. More precisely we denote by $\mathcal{M}_T^2(H)$ the space of $\mathcal{F}_t$-martingales satisfying

- The path $t \mapsto M(t)$ is almost surely continuous.
- $\sup_{t\in[0,T]} \int_\Omega ||M(t)||_H^2 \mathrm{d}P = \sup_{t\in[0,T]} \mathbb{E}\left[||M(t)||_H^2\right] < \infty$.

The second condition motivates a natural norm on $\mathcal{M}_T^2$, namely

$$||M||_{\mathcal{M}_T^2(H)} := \sup_{t\in[0,T]} \sqrt{\int_\Omega ||M(t)||_H^2 dP} = \sup_{t\in[0,T]} \sqrt{\mathbb{E}\left[||M(t)||_H^2\right]}\,.$$

Our goal is to show that $\mathcal{M}_T^2(H)$ is a Banach space when equipped with this norm. Since for $s \leq t$ and $p \geq 1$ we have that

$$||M(s)||_H^p \leq \mathbb{E}[||M(t)||_H^p|\mathcal{F}_s]$$

it follows by the law of the iterated expectation that

$$\mathbb{E}[||M(s)||_H^p] \leq \mathbb{E}[||M(t)||_H^p]\,,$$

so that the norm $\|\cdot\|_{\mathcal{M}_T^2(H)}$ can alternatively be written as

$$||M||_{\mathcal{M}_T^2(H)} := \sup_{t\in[0,T]} \sqrt{\mathbb{E}\left[||M(t)||_H^2\right]} = \sqrt{\mathbb{E}\left[||M(T)||_H^2\right]}\,.$$

We now consider the Banach space $\chi = \mathcal{L}^2(\Omega, C([0,T], H))$, equipped with the norm $||X||_\chi := \sqrt{\mathbb{E}[||X(t)||_\infty^2]} = \sqrt{\mathbb{E}[\sup_{t\in[0,T]} ||X(t)||_H^2]}$. We will show that this norm is equivalent to $\|\cdot\|_{\mathcal{M}_T^2(H)}$. First we see that

$$||M||_{\mathcal{M}_T^2(H)} = \sqrt{\mathbb{E}\left[||M(T)||_H^2\right]} \leq \sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} ||M(t)||_H^2\right]} = ||M||_\chi \,,$$

and by Theorem 1.6.1

$$||M||_\chi = \sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} ||M(t)||_H^2\right]} \leq \left(\frac{2}{2-1}\right)\sqrt{\mathbb{E}\left[||M(T)||_H^2\right]} = 2||M||_{\mathcal{M}_T^2(H)} \,.$$

It follows that $||\cdot||_{\mathcal{M}_T^2(H)}$ and $||\cdot||_\chi$ are equivalent norms on the space $\mathcal{M}_T^2(H)$. It follows also that elements of $\mathcal{M}_T^2$ are mean-square continuous, so that $\mathcal{M}_T^2$ is a subspace of χ. With this in mind we are ready to show that $\mathcal{M}_T^2(H)$ is a Banach space.

Theorem 1.6.2. *$\mathcal{M}_T^2(H)$ is a Banach space.*

Proof. Let $\{M_n\}_n \subset \mathcal{M}_T^2(H)$ be a Cauchy sequence. Then

$$||M_m - M_n||_\chi \leq 2||M_m - M_n||_{\mathcal{M}_T^2(H)}$$

so that $\{M_n\}_n$ is also a Cauchy sequence in χ. We know χ to be complete, so $M_n \to M$ as $n \to \infty$ for some $M \in \chi$. We now show that $M \in \mathcal{M}_T^2(H)$. We see that

$$\begin{aligned}
0 \leq \mathbb{E}\left[\|\mathbb{E}\left[M(t)|\mathcal{F}_s\right] - M(s)\|_H\right] &= \mathbb{E}\left[\|\mathbb{E}\left[M(t) - M_n(t)|\mathcal{F}_s\right] - (M(s) - M_n(s))\|_H\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\|M(t) - M_n(t)\||\mathcal{F}_s\right]\right] + \mathbb{E}\left[\|M(s) - M_n(s)\|_H\right] \\
&\leq 2\mathbb{E}\left[\sup_{t\in[0,T]} ||M(s) - M_n(s)||_H\right] \\
&\leq 2\sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} ||M(s) - M_n(s)||_H^2\right]} \\
&= 2||M - M_n||_\chi \to 0 \text{ as } n \to \infty \,.
\end{aligned}$$

Thus $\mathbb{E}\left[\|\mathbb{E}\left[M(t)|\mathcal{F}_s\right] - M(s)\|_H\right] = 0$ implying that $\|\mathbb{E}\left[M(t)|\mathcal{F}_s\right] - M(s)\|_H = 0$, so that $\mathbb{E}\left[M(t)|\mathcal{F}_s\right] = M(s)$. It follows that M is a martingale. Since $M \in \chi$, M is almost surely continuous. Since M is a martingale we also know that $\sup_{t\in[0,T]} \mathbb{E}\left[||M(t)||_H^2\right] = \mathbb{E}\left[||M(T)||_H^2\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} ||M(t)||_H^2\right] = ||M||_\chi^2 < \infty$. It follows that $M \in \mathcal{M}_T^2(H)$. Also

$$||M - M_n||_{\mathcal{M}_T^2(H)} \leq 2||M - M_n||_\chi \to 0 \text{ as } n \to \infty \,,$$

so $M_n \to M$ in $\mathcal{M}_T^2(H)$. □

We finally show that Q-Wiener processes are martingales. Let W be a Q-Wiener process with respect to a filtration $\mathcal{F}_t$ Recall first that Q-Wiener processes have independent increments. Therefore

$$\mathbb{E}\left[W(t)|\mathcal{F}_s\right] = \mathbb{E}\left[W(s)|\mathcal{F}_s\right] + \mathbb{E}\left[W(t) - W(s)|\mathcal{F}_s\right] .$$

Since W is adapted to the filtration it follows by Lemma 1.6.1 that $\mathbb{E}\left[W(s)|\mathcal{F}_s\right] = W(s)$. Since $W(t) - W(s)$ is $\mathcal{F}_s$-independent we also know that $\mathbb{E}\left[W(t) - W(s)|\mathcal{F}_s\right] = \mathbb{E}\left[W(t) - W(s)\right] = 0$. It follows that $\mathbb{E}\left[W(t)|\mathcal{F}_s\right] = W(s)$. In addition

$$\|W(t)\|^2_{\mathcal{M}^2_T(H)} = \mathbb{E}\left[\|W(t)\|^2_H\right] = \sum_k \lambda_k \mathbb{E}\left[\beta_k(t)^2\right] = t\mathrm{Tr}(Q) < \infty ,$$

so that W is also an element of $\mathcal{M}^2_T(H)$.

1.7 The space-time covariance function

We have seen that for $u, v \in H$ the covariance operator Q of a random variable X satisfies $\langle Qu, v\rangle = \mathbb{E}\left[\langle X, u\rangle_H \langle X, v\rangle_H\right]$. A natural generalization of this to random processes X would be to imagine a function $r : [0, T] \times [0, T] \to L(H)$ satisfying

$$\langle r(s, t)u, v\rangle_H := \mathbb{E}\left[\langle X(t), u\rangle_H \langle X(s), v\rangle_H\right] .$$

We call r the space-time covariance function of X. This definition of a space-time covariance function for H-valued processes is inspired by Kirchner and Willems [15], though they do not refer to it by this name. In order to avoid confusion between this type of space-time covariance function and the more common type of covariance function that defines the covariance between point values on a random field, we refer to the latter as auto-covariance functions. The discussions of covariance, both here and in later sections are to the authors knowledge somewhat novel and therefore less rigorous and more experimental than that the other parts of the thesis.

Example 1.7.1. *For the Q-Wiener processes W we have discussed we can easily calculate r. Assuming $s < t$, we have that*

$$\begin{aligned}\langle r_W(s, t)u, v\rangle_H &= \mathbb{E}\left[\langle W(s), u\rangle_H \langle W(s), v\rangle_H\right] + \mathbb{E}\left[\langle W(t) - W(s), u\rangle_H \langle W(s), v\rangle_H\right] \\ &= \langle sQu, v\rangle ,\end{aligned}$$

so that in general $r_W(s, t) = \min(s, t)Q$.

The following proposition lists some properties of space-time covariance functions.

Proposition 1.7.1. ***Properties of the space-time covariance function.*** *Let X be a H-valued random process on $[0, T]$, with space-time covariance function r. Then the following properties hold*

- *$r(t, t)$ is a covariance operator.*
- *$r(t, s) = r(s, t)$ for any $t, s \in [0, T]$.*
- *For any finite collection $\{t_k\}_{k=1}^n \subset [0, T]$, with any corresponding coefficients $\{a_k\}_{k=1}^n \subset \mathbb{R}$, the linear combination $\sum_{k,j} a_k a_j r(t_k, t_j)$ is a non-negative definite operator.*

Proof. The first two properties are trivial. For the third property define the H-valued random variable A by $A := \sum_k a_k X(t_k)$. Then the covariance operator Q_A of A is given by $\langle Q_A u, v\rangle_H = \sum_{k,j} a_k a_j \langle r(t_k, t_j)u, v\rangle_H$. We know that Q_A must be non-negative definite, i.e. $\langle Q_A e_k, e_k\rangle \geq 0$. It follows that $\sum_{k,j} a_k a_j r(t_k, t_j)$ must be non-negative definite also. □

When we later consider solutions to stochastic heat equations in Section 3 and 4 we will especially interested in the asymptotic covariance structure. Intuitively we are not interested in the effects of any initial condition, but only the covariance properties arising from the equations themselves, i.e. we are interested in the covariance properties for "large" values of t. With this in mind we make the following definition:

Definition 1.7.1. ***Asymptotic space-time covariance function***. *The stationary space-time covariance function $r : [0, \infty) \to L(H)$ is defined by*

$$r(h) = \lim_{t\to\infty} r(t + h, t)\,,$$

where the limit is defined strongly, i.e. we expect that $r(t, t + h)u \to r(h)u$ for every $u \in H$. In addition we also denote $r(0)$ the asymptotic covariance operator.

The asymptotic space-time covariance function $r(h)$ might not always exist. The intuition is that asymptotically the space-time covariance function $r(t, s)$ might depend only on the difference in time t, not the absolute values of t and s. The asymptotic covariance operator $r(0)$ is then simply the covariance operator of the random process $X(t)$ for "large" values of t.

2 Stochastic integration and evolution equations

In this section we construct a stochastic integral that is suitable to the problems we are interested in. This integral will prove important in the analysis of evolution equations with stochastic driving noise in Subsection 3. We consider existence and uniqueness results for these equations. The following treatment is heavily based on the treatment of the topic in the lecture note by Kovács and Larsson [16].

2.1 Constructing the stochastic integral

We are now ready to begin constructing our stochastic integral. Let U and H be Hilbert spaces. We wish to make sense of integrals of the form

$$\int_0^T \phi(t)\,\mathrm{d}W(t)\,,$$

where W is a Q-Wiener process on U and $\phi : [0,T]\times\Omega \to L(U,H)$. The construction of this integral is similar to the standard construction of the Lebesgue integral. The first step is to define the integral for an analogue of simple functions.

Definition 2.1.1. *Let ϕ be an $L(U,H)$-process $[0,T]\times\Omega \in L(U,H)$, and let $\{\mathcal{F}_t\}_t$ be a normal filtration. ϕ is called an elementary process with respect to $\{\mathcal{F}_t\}_t$ if*

$$\phi(t,\omega) = \sum_{n=1}^{N-1} \phi_n(\omega)\mathbb{I}_{(t_n,t_{n+1}]}(t)\,,$$

where $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$ is a partition of $[0,T]$ and $\phi_n(\omega)$ are random operators such that $\phi_n x : (\Omega,\mathcal{F}_{t_n}) \to H$ is measurable for each $x \in U$ and each ϕ_n is of the form

$$\phi_n(\omega) = \sum_{j=1}^{k_n} L_j^n \mathbb{I}_{A_j^n}(\omega)\,,$$

where $L_j^n \in L(U,H)$ and $\{A_j^n\}_{j=1}^{k_n}$ is a partition of Ω for every $n = 0,...,N-1$. Note also that we must have $A_j^n \in \mathcal{F}_{t_n}$ in order for $\phi(t,\omega)$ to be $\{\mathcal{F}_t\}_t$-adapted.

We call the space of elementary processes $\mathcal{E}$. The dependence on the filtration is left implicit in this notation.

Let W be a Q-Wiener process with normal filtration $\{\mathcal{F}_s\}_s$. For $\phi \in \mathcal{E}$ it is easy to define our stochastic integral. We define

$$\int_0^T \phi(t)\,\mathrm{d}W(t) := \sum_{n=0}^{N-1} \phi_n(W(t_{n+1}) - W(t_n))\,.$$

For $t \in [0,T]$ we similarly define

$$\int_0^t \phi(t)\,\mathrm{d}W(t) := \sum_{n=0}^{N-1} \phi_n \Delta W_n(t)\,,$$

where

$$\Delta W_n(s) := \begin{cases} W(t_{n+1}) - W(t_n) & \text{if } t_n \le t_{n+1} \le t \\ W(t) - W(t_n) & \text{if } t_n \le t \le t_{n+1} \\ 0 & \text{if } t \le t_n \le t_{n+1} \end{cases} .$$

We will now show that for elementary processes $\int_0^\cdot \phi(t)dW(t) \in \mathcal{M}_T^2(H)$.

Theorem 2.1.1. *$\int_0^\cdot \phi(t)dW(t) \in \mathcal{M}_T^2(H)$, i.e. $\int_0^\cdot \phi(t)dW(t)$ is mean-square continuous, square-integrable, $\{\mathcal{F}_t\}_t$-adapted, and a martingale.*

Proof. Let ϕ be an elementary process and define

$$M(t) := \int_0^t \phi(t)\,\mathrm{d}W(t) = \sum_{n=1}^{N-1} \phi_n(\omega)\mathbb{I}_{(t_n,t_{n+1}]}(t)\,.$$

We first show that M is continuous. Since W is a.s. continuous by assumption, $\Delta W_n(s)$ is also a.s. continuous. ϕ_n is a bounded operator a.s. and therefore preserves this continuity. Therefore M is a finite sum of a.s. continuous functions and is therefore a.s. continuous. M is square-integrable since for all $t \in [0,T]$

$$\begin{aligned} \mathbb{E}\left[\|M(t)\|_H^2\right] &= \mathbb{E}\left[\left\|\sum_{n=0}^{N-1} \phi_n \Delta W_n(t)\right\|_H^2\right] \\ &\le N\mathbb{E}\left[\sum_{n=0}^{N-1} \|\phi_n\|_{L(U,H)}^2 \|\Delta W_n(t)\|_U^2\right] \\ &\le N \max_{n,j} \|L_j^n\|_{L(U,H)}^2 \sum_{n=0}^{N-1} \mathbb{E}\left[\|\Delta W_n(t)\|_U^2\right] < \infty\,. \end{aligned}$$

Now $\phi_n \Delta W_n(t)$ is $\mathcal{F}_t$ adapted by assumption, so $M(t)$ is also $\mathcal{F}_t$-adapted. Finally we show that M satisfies the martingale property, i.e. we want to show that $\mathbb{E}\left[M(t)\middle|\mathcal{F}_s\right] = M(s)$ for $s < t$. We first decompose $M(t)$ into two parts.

$$M(t) = \sum_{n=0}^{N-1} \phi_n \Delta W_n(t) = \sum_{n=0}^{N-1} \phi_n \Delta W_n(s) + \sum_{n=0}^{N-1} \phi_n(\Delta W_n(t) - \Delta W_n(s))\,,$$

so that by the $\mathcal{F}_t$-adaptability of M

$$\mathbb{E}\left[M(t)\middle|\mathcal{F}_s\right] = M(s) + \mathbb{E}\left[\sum_{n=0}^{N-1} \phi_n(\Delta W_n(t) - \Delta W_n(s))\middle|\mathcal{F}_s\right]\,.$$

It remains to show that the latter part is zero. Now let the index ℓ be such that $s \in (t_\ell, t_{\ell+1}]$. Since $s < t$, we then have that $\Delta W_n(t) - \Delta W_n(s) = 0$ a.s. for $n < \ell$. For $n = \ell$, $\Delta W_n(t) - \Delta W_n(s) = W(t) - W(s)$ if $t \in (t_\ell, t_{\ell+1}]$, and $\Delta W_n(t) - \Delta W_n(s) = W(t_{\ell+1}) - W(s)$ if not. For $n > \ell$, $\Delta W_n(t) - \Delta W_n(s) = \Delta W_n(t)$. Therefore

$$\sum_{n=0}^{N-1} \phi_n(\Delta W_n(t) - \Delta W_n(s)) = \phi_\ell(W(\min(t, t_{\ell+1})) - W(s)) + \sum_{n=\ell+1}^{N-1} \phi_n \Delta W_n(t)\,.$$

Then since $W(\min(t,t_{\ell+1}))-W(s)$ is $\mathcal{F}_s$-independent we see that

$$\mathbb{E}\left[\phi_\ell(W(\min(t,t_{\ell+1}))-W(s))\middle|\mathcal{F}_s\right]=\phi_\ell\mathbb{E}\left[W(\min(t,t_{\ell+1}))-W(s)\right]=0\,.$$

We can see by a similar argument that $\mathbb{E}\left[\phi_n\Delta W_n(t)\middle|\mathcal{F}_s\right]=0$. It follows that $\mathbb{E}\left[M(t)\middle|\mathcal{F}_s\right]=M(s)$ and therefore M is a martingale. □

For elementary processes we also have an analogue of the Itô-isometry. We will use this isometry to extend the stochastic integral to a larger space of processes.

Proposition 2.1.1. ***An Itô-isometry***. *For $\phi\in\mathcal{E}$*

$$\left\|\int_0^\cdot \phi(t)dW(t)\right\|^2_{\mathcal{M}^2_T(H)}=\mathbb{E}\left[\left\|\int_0^T \phi(t)dW(t)\right\|^2_H\right]=\mathbb{E}\left[\int_0^T \|\phi(s)Q^{\frac{1}{2}}\|^2_{L_2(U,H)}\,ds\right]$$

Proof. Firstly

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^T \phi(t)dW(t)\right\|^2_H\right]&=\mathbb{E}\left[\left\langle\sum_{m=0}^{N-1}\phi_m\Delta W_n(T),\sum_{n=0}^{N-1}\phi_n\Delta W_n(T)\right\rangle_H\right]\\&=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\mathbb{E}\left[\langle\phi_m\Delta W_m(T),\phi_n\Delta W_n(T)\rangle_H\right]=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}(*)_{m,n}\,.\end{aligned}$$

Let $\{f_k\}_k$ be an orthonormal basis for H and $\{e_k\}_k$ be an orthonormal basis for U that is an eigenvector basis with respect to Q. Now the summand can be expanded using Parseval's identity $\langle x,y\rangle=\sum_k\langle x,f_k\rangle\langle y,f_k\rangle$,

$$\begin{aligned}(*)_{m,n}&=\mathbb{E}\left[\sum_\ell\langle\phi_m\Delta W_m(T),f_\ell\rangle_H\langle\phi_n\Delta W_n(T),f_\ell\rangle_H\right]\\&=\sum_{\ell,i,j}\mathbb{E}\left[\langle\Delta W_m(T),e_i\rangle_U\langle\phi_m^*f_\ell,e_i\rangle_U\langle\Delta W_n(T),e_j\rangle_U\langle\phi_n^*f_\ell,e_j\rangle_U\right]\,.\end{aligned}$$

We can assume without loss of generality that $m>n$. We know that ϕ_m is $\mathcal{F}_{t_m}$ measurable and $\Delta W_m(T)$ is independent of $\mathcal{F}_{t_m}$. Then by iterated expectation we have that

$$\begin{aligned}(*)_{m,n}&=\sum_{\ell,i,j}\mathbb{E}\big[\mathbb{E}\left[\langle\Delta W_m(T),e_i\rangle_U\langle\phi_m^*f_\ell,e_i\rangle_U\langle\Delta W_n(T),e_j\rangle_U\langle\phi_n^*f_\ell,e_j\rangle_U\,\middle|\mathcal{F}_{t_n}\right]\big]\\&=\mathbb{E}\left[\sum_{\ell,i,j}\langle\phi_m^*f_\ell,e_i\rangle_U\langle\phi_n^*f_\ell,e_j\rangle_U\langle\Delta W_m(T),e_i\rangle_U\,\mathbb{E}\left[\langle\Delta W_n(T),e_j\rangle_U\right]\right]=0\,,\end{aligned}$$

since $\mathbb{E}\left[\langle\Delta W_n(T),e_j\rangle_U\right]=0$. It follows that for $m\neq n$, we have that $(*)_{m,n}=0$. For $m=n$ however, we can use that fac that $\mathrm{Cov}(\Delta W_m(T))=(t_{m+1}-t_m)Q$ to see that

$$
\begin{aligned}
(*)_{m,m} &= \mathbb{E}\left[\sum_{\ell,i,j} \langle \phi_m^* f_\ell, e_i\rangle_U \langle \phi_n^* f_\ell, e_j\rangle_U \, \mathbb{E}\left[\langle \Delta W_n(T), e_i\rangle_U \langle \Delta W_n(T), e_j\rangle_U\right]\right] \\
&= \mathbb{E}\left[\sum_{\ell,i,j} \langle \phi_m^* f_\ell, e_i\rangle_U \langle \phi_m^* f_\ell, e_j\rangle_U \langle (t_{m+1}-t_m)Qe_i, e_j\rangle_U\right] \\
&= \mathbb{E}\left[\sum_{\ell} \left\langle (t_{m+1}-t_m)Q\sum_i \langle \phi_m^* f_\ell, e_i\rangle_U e_i, \sum_j \langle \phi_m^* f_\ell, e_j\rangle_U e_j \right\rangle_U\right] \\
&= \mathbb{E}\left[\sum_{\ell} \langle (t_{m+1}-t_m)Q\phi_m^* f_\ell, \phi_m^* f_\ell\rangle_U\right] \\
&= (t_{m+1}-t_m)\mathbb{E}\left[\sum_{\ell} \left\langle Q^{1/2}\phi_m^* f_\ell, Q^{1/2}\phi_m^* f_\ell\right\rangle_U\right] \\
&= (t_{m+1}-t_m)\mathbb{E}\left[\|Q^{\frac{1}{2}}\phi_m^*\|^2_{L_2(H,U)}\right] \\
&= (t_{m+1}-t_m)\mathbb{E}\left[\|\phi_m Q^{\frac{1}{2}}\|^2_{L_2(U,H)}\right]
\end{aligned}
$$

Putting this together we then have

$$
\begin{aligned}
\mathbb{E}\left[\left\|\int_0^T \phi(t)dW(t)\right\|_H^2\right] &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}(*)_{m,n} = \sum_{m=0}^{N-1}\mathbb{E}\left[\|\phi_m Q^{\frac{1}{2}}\|^2_{L_2(U,H)}\right](t_{m+1}-t_m) \\
&= \mathbb{E}\left[\int_0^T \left\|\phi(s)Q^{\frac{1}{2}}\right\|^2_{L_2(U,H)} \mathrm{d}s\right].
\end{aligned}
$$

The conclusion follows. □

The expression $\mathbb{E}\left[\int_0^T \|\phi(s)Q^{\frac{1}{2}}\|^2_{L_2(U,H)}\,\mathrm{d}s\right]^{\frac{1}{2}}$ defines a semi-norm on $\mathcal{E}$. We denote it by $\|\cdot\|_{\mathcal{E}}$. It is not a norm since $\|\phi\|_{\mathcal{E}} = 0$ only implies that $\phi(t) : U \to H$ is zero on $\mathrm{Im}(Q^{\frac{1}{2}})$. We can "fix" this issue by considering $\mathcal{E}$ instead as a quotient space where $\phi \sim \Psi$ if $\Psi(t) = \phi(t)$ on $\mathrm{Im}(Q^{\frac{1}{2}})$ for all $t \in [0,T]$. This has the consequence of making elements in $\mathcal{E}$ and its closure $L_2(Q^{\frac{1}{2}}(U), H)$-valued processes.

Having established our integral for $\phi \in \mathcal{E}$, we can use the isometry to extend it to all elements in the abstract closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ under $\|\cdot\|_{\mathcal{E}}$. If we have an element $x \in \overline{\mathcal{E}}$ we can by definition find a sequence $\{x_n\}_n \subset \mathcal{E}$ such that $x_n \to x$ in $\|\cdot\|_{\mathcal{E}}$. We can then define

$$
\int_0^{\cdot} x(s)\,\mathrm{d}W(s) := \lim_{n\to\infty}\int_0^{\cdot} x_n(s)\,\mathrm{d}W(s)\,.
$$

$\{\int_0^{\cdot} x_n(s)\,\mathrm{d}W(s)\}_n$ is a Cauchy-sequence in $\mathcal{M}_T^2(H)$ since $\|\int_0^{\cdot} x_m(s)-x_n(s)\,\mathrm{d}W(s)\|_{\mathcal{M}_T^2(H)} = \|x_m(s)-x_n(s)\|_{\mathcal{E}}$ and $\{x_n\}_n$ is a Cauchy-sequence in $\mathcal{E}$. Since $\mathcal{M}_T^2(H)$ is a Hilbert space it thus exists an $M \in \mathcal{M}_T^2(H)$ such that $\int_0^{\cdot} x(s)\,\mathrm{d}W(s) = M$. M is independent of the choice of approximating sequence, since if you have two sequences $\{x_n\}_n$ and $\{y_n\}_n$, both converging to x, then

$$\| \lim_{n\to\infty} \int_0^{\cdot} x_n(s) - y_n(s)\,\mathrm{d}W(s) \|_{\mathcal{M}_T^2(H)} = \|x - x\|_{\mathcal{E}} = 0\,.$$

The following theorem from Kovács and Larsson [16] gives a concrete representation for $\overline{\mathcal{E}}$.

Theorem 2.1.2. *$\mathcal{N}_2(U,H) := \overline{\mathcal{E}}$ is the set of all $L_2(Q^{\frac{1}{2}}(U),H)$-valued processes ϕ such that ϕ is $\mathcal{F}_t$-adapted and right-continuous and $\|\phi\|_{\mathcal{E}} < \infty$.*

We might also be interested in the case where the Q-Wiener process is based on a non-proper covariance Q. As we observed in Theorem 1.2.1 we can then find a (proper) covariance operator $\tilde{Q}$ on an extended Hilbert space $\tilde{U} \supset U$. So let $\tilde{W}$ be a $\tilde{Q}$-Wiener process on $\tilde{U}$ and let $J : Q^{\frac{1}{2}}(U) \to \tilde{U}$ be the embedding $u \mapsto u$. We can then construct a pseudo-inverse J^{-1} of J by projecting $\tilde{U}$ onto $\mathrm{Im}(J)$ before inverting, i.e. for $u \in \tilde{U}$, we define $J^{-1} = \left(J\big|_{\mathrm{Im}(J)} \right)^{-1} P_{\mathrm{Im}(J)}$, where $P_{\mathrm{Im}(J)}$ is the orthogonal projection $\tilde{U} \to \mathrm{Im}(J)$. For $\phi \in \mathcal{N}_2(U,H)$ the integral

$$\int_0^T \phi(s) J^{-1} \mathrm{d}\tilde{W}(s)$$

then exists as long as $\|\phi(\cdot)J^{-1}\|_{\mathcal{E}} < \infty$. This is the case since

$$\|\phi(\cdot)J^{-1}\|_{\mathcal{E}}^2 = \mathbb{E}\left[\int_0^T \|\phi(s)J^{-1}\|_{L_2(U,H)}^2 \mathrm{d}s\right] \le \|J^{-1}\|_{L(U,H)}^2 \|\phi(\cdot)\|_{\mathcal{E}}^2 = \|\phi(\cdot)\|_{\mathcal{E}}^2 < \infty\,.$$

Therefore, if we have an improper covariance operator, we define our stochastic integral by

$$\int_0^T \phi(s)\mathrm{d}W(s) := \int_0^T \phi(s)J^{-1}\mathrm{d}\tilde{W}(s)\,.$$

We now discuss some properties of the stochastic integral. The next result gives us an easy way to calculate the covariance of two stochastic integrals. A proof can be found in Kovács and Larsson [16].

Proposition 2.1.2. *Take $A, B \in \mathcal{N}_2(U,H)$ and take $u, v \in H$. Then*

$$\mathbb{E}\left[\left\langle \int_0^T A(t)\,\mathrm{d}W(t), u \right\rangle_H \left\langle \int_0^T B(t)\,\mathrm{d}W(t), v \right\rangle_H\right] = \mathbb{E}\left[\left\langle \int_0^T A(t)QB(t)^*\,\mathrm{d}t\, u, v \right\rangle_H\right]$$

Another result we will need somewhat frequently is the stochastic Fubini theorem. This result is taken from Da Prato and Zabczyk [8].

Theorem 2.1.3. ***Stochastic Fubini theorem**. Let E be a Banach space equipped with a finite positive measure μ and let ϕ be a mapping $E \to \mathcal{N}_2(U,H)$. If*

$$\int_E \mathbb{E}\left[\int_0^T \|\phi(x,t)Q^{\frac{1}{2}}\|^2_{L_2(U)}\,\mathrm{d}t\right]^{\frac{1}{2}} \mu(\mathrm{d}x) < \infty\,,$$

then

$$\int_E \int_0^T \phi(x,t)\,\mathrm{d}W(t)\,\mu(\mathrm{d}x) = \int_0^T \int_E \phi(x,t)\,\mu(\mathrm{d}x)\,\mathrm{d}W(t)\,.$$

2.2 C_0-semigroups and deterministic evolution equations

We take a small detour from the study of our stochastic integral to study semigroups of bounded operators. These are very useful in the study of initial value problems on Hilbert spaces and are therefore also useful in the study of the stochastic evolution equations we consider in the next subsection. The following introduction is based heavily on the treatment of semigroups in Engel and Nagel [10] with some inspiration taken from Pazy [18]. Proposition 2.2.3 however is taken from Kirchner and Willems [15] and will be needed for the analysis of the space-time fractional heat equation in Section 4. We begin by defining a C_0-semigroup.

Definition 2.2.1. C_0 **- *semigroup***. *Let H be a Hilbert space. A $[0,\infty)$-indexed collection $\{S_t\}_t$ of bounded operators $H \to H$ is called a C_0-semigroup if*

- $S_0 = I$.
- $S_{t+s} = S_t S_s$.
- *For all $x \in H$, the map $t \mapsto S_t x \in H$ is H-continuous.*

The operator defined by $Ax = \lim_{h\to 0} \frac{S(h)x - x}{h}$ is called the generator of the semigroup. The operator A is not (necessarily) defined for all $x \in H$, its domain contains only those x where the limit exists. We denote the domain of A by $D(A)$. We will see that the domain contains important information about the semigroup and the generator of the semigroup is better thought off as a pair $(A, D(A))$, rather than simply the operator A.

The following proposition lists some properties of C_0-semigroups and their generators. Properties 1, 2, and 3 are taken from Engel and Nagel [10] and Property 4 is taken from Pazy [18].

Proposition 2.2.1. *Let $\{S_t\}_t$ be a C_0-semigroup of operators $H \to H$ with generator $(A, D(A))$. Then the following properties hold*

1. *$A : D(A) \to H$ is a linear operator*
2. *For $x \in D(A)$, $S_t x \in D(A)$ and*

$$\frac{d}{dt} S_t x = A S_t x = S_t A x\,.$$

3. For $x \in H$, $\int_0^t S_s x \, ds \in D(A)$ and

$$A\left(\int_0^t S_s x \, ds\right) = S_t x - x\,.$$

If we also have that $x \in D(A)$, then

$$A\left(\int_0^t S_s x \, ds\right) = \int_0^t S_s A x \, ds = S_t x - x\,.$$

4. For $x \in H$

$$\frac{1}{t}\int_0^t S_s x \, ds \to x \text{ as } t \to 0\,.$$

The next proposition tells us that generators of semigroups are closed and linear and have dense domains. The proposition is taken from [10], but we also include the proof here.

Proposition 2.2.2. *Let $\{S_t\}_t$ be a C_0-semigroup of operators $H \to H$. Its generator A is then a closed linear operator and $\overline{D(A)} = H$.*

Proof. To see that that A is a closed operator consider a sequence of $\{x_n\}_n \subset D(A)$ and assume that the sequence $\{Ax_n\}_n$ converges to an element $x \in H$ in $\|\cdot\|_H$. By Proposition 2.2.1 we see that for $t > 0$

$$S_t x_n - x_n = \int_0^t S_s A x_n \, \mathrm{d}s\,.$$

Since $\{Ax_n\}$ is convergent, it must be a bounded sequence, so that $\|Ax_n\| \leq C$. Then $\|S_s A x_n\| \leq C'$ also. Then by dominated convergence, and the strong continuity of S_t, we see that

$$\int_0^t S_s A x_n \, \mathrm{d}s \to \int_0^t S_s y \, \mathrm{d}s\,.$$

In addition, also by the strong continuity of S_t we see that $S_t x_n - x_n \to S_t x - x$. Then $\frac{S_t x - x}{t} = \frac{1}{t}\int_0^t S_s y \, \mathrm{d}s$. Taking the limit as $t \to 0$ we get that

$$\lim_{t\to 0} \frac{S_t x - x}{t} = y\,.$$

It follows that $x \in D(A)$ and $Ax = y$. A is therefore a closed operator. To see that $D(A)$ is dense, note that by Proposition 2.2.1 we know that for $x \in H$, $\frac{1}{t}\int_t S_s x \, \mathrm{d}s \in D(A)$. Hence the sequence $\{\frac{1}{n}\int_0^n S_s x \, \mathrm{d}s\} \subset D(A)$ converges to $x \in H$ by strong continuity. It follows that $\overline{D(A)} = H$. □

It is often straight-forward to verify that a C_0-semigroup has generator A for some domain D. However it is harder to determine whether D is actually the full domain of the generator. The following proposition from Kirchner and Willems [15] will help us determine whether or not a subset $D \subset D(A)$ is "large" or not.

Proposition 2.2.3. *Let $\{S_t\}_t$ be a C_0-semigroup of operators $H \to H$ with generator $(A, D(A))$. If $D \subset D(A)$ is $\|\cdot\|_H$-dense in H and $S_t D \subset D$, then D is $\|\cdot\|_g$-dense in $D(A)$, where $\|\cdot\|_g$ is the graph norm defined by $\|\cdot\|_g := \|\cdot\|_H + \|A(\cdot)\|_H$. We denote the graph closure of D by $\overline{D}^g$.*

Proof. Take $x \in D(A)$. Since D is $\|\cdot\|_H$-dense in H, and $D(A) \subset H$, we can find a sequence $\{x_n\}_n \subset D$ such that $x_n \to x$ in $\|\cdot\|_H$. Since S_s is strongly continuous, both $s \mapsto S_s x_n$ and $s \mapsto S_s(Ax_n)$ are H-continuous maps. It follows that the map $s \mapsto S_s x_n$ is continuous in $\|\cdot\|_g$. Therefore the integral $\int_0^t S_s x_n \, \mathrm{d}s$ is $\|\cdot\|_g$-convergent, implying that $\frac{1}{t}\int_0^t S_s x_n \, \mathrm{d}s \in \overline{D}^g$. We want to show that $\frac{1}{t}\int_0^t S_s x_n \, \mathrm{d}s \to x$ in $\|\cdot\|_g$ as $t \to 0$, implying that $x \in \overline{D}^g$. We consider

$$\left\| \frac{1}{t} \int_0^t S_s x_n \, \mathrm{d}s \right\|_g \leq \left\| \frac{1}{t} \int_0^t S_s x \, \mathrm{d}s - x \right\|_g + \left\| \frac{1}{t} \int_0^t S_s (x_n - x) \, \mathrm{d}s \right\|_g .$$

By the same argument as before the map $s \mapsto S_s x$ is $\|\cdot\|_g$-continuous, so that

$$\left\| \frac{1}{t} \int_0^t S_s x \, \mathrm{d}s \right\|_g \to 0 ,$$

as $t \to 0$. Furthermore

$$\left\| \frac{1}{t} \int_0^t S_s (x_n - x) \, \mathrm{d}s \right\| \leq C \|x - x_n\|_H \to 0 ,$$

as $n \to \infty$. Also, by Proposition 2.2.1

$$\left\| \frac{1}{t} A \int_0^t S_s (x_n - x) \, \mathrm{d}s \right\|_H = \|S_t(x_n - x) + x_n - x\|_H \leq C \|x - x_n\|_H \to 0 ,$$

again as $n \to \infty$. It follows that $\|\frac{1}{t} A \int_0^t S_s(x_n - x) \, \mathrm{d}s\|_g \to 0$ also. It follows that $\frac{1}{t}\int_0^t S_s x_n \, \mathrm{d}s \to x$ implying that $x \in \overline{D}$ as desired. ☐

An important question to us is when an operator A on $D(A)$ generates a C_0-semigroup. The Hille-Yosida theorem, stated below, gives sufficient and necessary conditions for when an operator A generates a C_0-semigroup satisfying a certain bound. The theorem as stated below is from Engel and Nagel [10].

Theorem 2.2.1. ***Hille-Yosida**. A linear operator A on $D(A) \subset H$ generates a C_0-semigroup of operators $\{S_t\}_t$ satisfying $\|S_t\|_{L(H)} \leq M \exp(\omega t)$ for some $M, \omega \in \mathbb{R}$ and for all $t \in [0, \infty)$ if and only if*

1. *A is a closed operator, and $D(A)$ is dense in H under $\|\cdot\|_H$.*
2. *The operator $\lambda I - A$ is invertible for every $\lambda > \omega$ and for every $n \in \mathbb{N}$ the operator $\lambda I - A$ is invertible and satisfies*

$$\|(\lambda I - A)^{-n}\|_{L(H)} \leq \frac{M}{(\lambda - \omega)^n} .$$

We now give example of an operator generating a semigroup. This example, involving the Laplacian, will be of great importance to us later when we analyze various generalizations of the stochastic heat equation.

Example 2.2.1. ***The Laplacian on*** $\mathcal{L}^2(\mathcal{D})$. *Let* $H = \mathcal{L}^2(\mathcal{D})$ *for a bounded domain* $\mathcal{D} \subset \mathbb{R}^d$ *with a smooth boundary* $\partial\mathcal{D}$. *Consider the Laplacian* $\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ *defined on* $D(\Delta) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, *the space of twice (weakly) differentiable functions which are zero on* $\partial\mathcal{D}$. *This space is dense in* $\mathcal{L}^2(\mathcal{D})$. *It is also known that the Laplacian is closed on its domain [9]. The Laplacian thus satisfies the first condition of the Hille-Yosida theorem. By Theorem A.2.1,* $(-\Delta)$ *induces an orthonormal basis of eigenvectors* $\{e_k\}_k$ *on* $D(\Delta)$, *and thus also on* H, *with corresponding eigenvalues* $\{\lambda_k\}_k$. *These eigenvalues are positive and non-decreasing and satisfy the Weyl estimates*

$$C_1 k^{\frac{2}{d}} \leq \lambda_k \leq C_2 k^{\frac{2}{d}} ,$$

implying that for $x \in D(\Delta)$ *and* $n \in \mathbb{N}$ *we have that*

$$\|(\lambda I - \Delta)^{-n} x\|_{\mathcal{L}^2(\mathcal{D})}^2 = \sum_k \frac{\langle x, e_k \rangle_H^2}{(\lambda + \lambda_k)^{2n}} \leq \frac{1}{(\lambda + \inf_k(\lambda_k))^{2n}} \|x\|_{\mathcal{L}^2(\mathcal{D})}^2 ,$$

so that $\|\lambda I + \Delta)^{-n}\|_{L(\mathcal{L}^2(\mathcal{D}))} \leq \frac{1}{(\lambda - \inf_k(\lambda_k))^n}$. *We know that* $\inf_k(\lambda_k) = \lambda_1$ $\{\lambda_k\}_k$ *is an non-decreasing sequence.* Δ *thus satisfies the second condition of the Hille-Yosida theorem with* $M = 1$ *and* $\omega = -\lambda_1$. *It follows that* $(\Delta, D(\Delta))$ *generates a semigroup* $\{S_t\}_t$ *and that* $\|S_t\|_{L(H)} \leq e^{-\lambda_1 t}$.

The above example shows that $(\Delta, D(\Delta))$ generates a semigroup $\{S_t\}_t$. We will now show that this semigroup is given by $S_t := \exp(-\Delta t)$. In fact the following theorem is slightly more general than that, also including the extensions of the Laplacian discussed in Appendix 2.

Theorem 2.2.2. *Let* $g : (-\infty, 0] \to (-\infty, 0]$ *be a measurable function. Let* Δ *and* $D(\Delta)$ *be as assumed in Example 2.2.1. Then the operator* $(g(\Delta), D(g(\Delta)))$ *generates the* C_0*-semigroup given by* $\exp(g(\Delta)t)$, *which is defined on the eigenvector basis* $\{e_k\}_k$ *of* Δ *by* $\exp(g(\Delta)t)e_k := \exp(g(-\lambda_k))e_k$.

Proof. The operators $g(\Delta)$ and the space $D(g(\Delta))$ is well-defined according our discussion in Appendix 2. We first show that $\{\exp(g(\Delta)t)\}_t$ is a C_0-semigroup. We then prove that $(g(\Delta), D(g(\Delta)))$ is its generator.

$\{\exp(g(\Delta)t)\}_t$ **is a** C_0**-semigroup:** Take $x \in H$. Since $\{e_k\}_k$ spans H we know that $x = \sum_k \langle x, e_k \rangle_H e_k$. Then

$$\begin{aligned} \| \exp(g(\Delta)t) e_k\|_H^2 &= \| \sum_k \langle x, e_k \rangle_H \exp(g(-\lambda_k)t) e_k\|_H^2 \\ &= \sum_k \langle x, e_k \rangle_H^2 \exp(-2g(-\lambda_k)t) \\ &\leq \|x\|_H^2 < \infty , \end{aligned}$$

since $\exp(g(-\lambda_k)t) \leq 1$ for all k. So $\exp(g(\Delta)t)$ is a well-defined operator. Clearly $\exp(g(\Delta) * 0) = I$ and $\exp(g(\Delta)(t+s)) = \exp(g(\Delta)t)\exp(g(\Delta)s)$. $\exp(g(\Delta)t)$ is strongly continuous since for $x \in H$

$$\|\exp(g(\Delta)t)x - \exp(g(\Delta)s)x\|_H^2 = \sum_k \langle x, e_k\rangle^2 \left(\exp(g(-\lambda_k)t) - \exp(g(-\lambda_k)s)\right)^2,$$

which goes to zero by the dominated convergence theorem since $\| \langle x, e_k\rangle^2 (\exp(g(-\lambda_k)t) - \exp(g(-\lambda_k)s))^2\|_H \leq 4\|x\|_H^2$. It follows that $\{\exp(\Delta t)\}_t$ is a C_0-semigroup.

$(g(\Delta), D(g(\Delta)))$ **generates** $\exp(g(\Delta)t)$**:** We finally show that $\exp(g(\Delta)t)$ is generated by $(g(\Delta), D(g(\Delta)))$. Let $x \in D(g(\Delta))$. Then

$$\frac{\exp(g(\Delta)h) - I}{h}x = \sum_k \langle x, e_k\rangle \frac{\exp(g(-\lambda_k)h) - 1}{h} e_k = (*)\,.$$

Since $\| \langle x, e_k\rangle \frac{\exp(g(-\lambda_k)h)-1}{h} e_k\|_H \leq \frac{2}{h}\|x\|_H$, we can use the dominated convergence theorem to conclude that

$$(*) \to \sum_k \langle x, e_k\rangle (g(-\lambda_k)e_k) = g(\Delta)x\,,$$

so that $g(\Delta)$ generates the C_0-semigroup $\exp(g(\Delta)t)$. □

The reason C_0-semigroups are important to us is because they provide solutions to evolution equations. Consider the deterministic evolution equation

$$\begin{cases} dy(t) + Ay(t)dt = f(t) \\ y(0) = y_0\,. \end{cases} \tag{7}$$

It is known that if $-A$ generates a C_0-semigroup $\{S_t\}_t$, then the unique (weak) solution to Equation 7 is given by

$$y(t) = S_t y_0 + \int_0^t S_{t-s} f(s)\,\mathrm{d}s\,.$$

A full treatment of this can be found in Engel and Nagel [10].

2.3 Stochastic evolution equations

We are now ready to tackle stochastic evolution equations of the form

$$\begin{cases} \mathrm{d}X(t) + AX(t)\mathrm{d}t = \mathrm{d}W(t) \\ X(0) = x_0\,, \end{cases} \tag{8}$$

where X is an H-valued random process, W is a (possibly improper) H-valued Q-Wiener process, A is an operator defined on a subspace $D(A) \to H$, and x_0 is an $\mathcal{F}_0$ measurable random variable on H. We will also assume $-A$ and $D(A)$ satisfies the conditions of the Hille-Yosida theorem, so that $(-A, D(A))$ generates a C_0-semigroup of operators $\{S_t\}_t$.

It is not immediately obvious what we mean by a solution to Equation 8. In this thesis we consider only so-called weak solutions, defined below.

Definition 2.3.1. ***Weak solution.*** *A random process* $X : [0,T] \to \mathcal{L}^2(\Omega, H)$ *is called a weak solution to Equation 8 on* $[0,T]$ *if* $X(t)$ *is* $\mathcal{F}_t$*-adapted, right-continuous almost surely,* $\mathrm{E}\left[\|X(t)\|^2\right] < \infty$ *for all* $t \in [0,T]$*, and that a.s. for all* $x \in \mathcal{D}(A)$ *and for all* $t \in [0,T]$ *we have that*

$$\langle X(t), x\rangle = \langle x_0, x\rangle + \int_0^t \langle X(s), A^* x\rangle \, \mathrm{d}s + \int_0^t \langle x, \mathrm{d}W(s)\rangle \,, \tag{9}$$

Note that this definition is weak only in the Hilbert space sense and not in the stochastic sense. We require that the Q-Wiener process is defined on an a-priori specified filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. A stochastically weak definition would only require that Equation 9 holds in expectation.

Our goal is now to show that the unique weak solution to Equation 8 is given by

$$X(t) = S_t x_0 + \int_0^t S_{t-s} \, \mathrm{d}W(s) \,, \tag{10}$$

is the unique weak solution to Equation 8. We first show that the integral converges.

Theorem 2.3.1. *Let* H *be a seperable Hilbert space. If*

$$\operatorname{Tr}\left(\int_0^T S_t Q S_t^* \, \mathrm{d}t\right) = \int_0^T \|S_t Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}t < \infty \,,$$

the integral in Equation 10 belongs to $\mathcal{L}^2(\Omega, H)$ *for all* $t \le T$ *and it is mean-square continuous in time.*

Proof. Define $\phi_r(t) := \int_0^t S_{r-s} \, \mathrm{d}W(s)$. Since S_{r-s} is adapted and right continuous (it's deterministic and strongly continuous), and since by the Itô isometry we have that

$$\mathbb{E}\left[\|\phi_r(t)\|_H^2\right] = \int_0^t \|S_{s-r} Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}s = \int_0^t \|S_s Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}s < \infty \,,$$

we see that $\phi_r(t)$ exists for every $0 \le t \le r \le T$ by Theorem 2.1.2. Therefore the random process $M : [0,T] \to \mathcal{L}^2(\Omega, H)$ defined by $M(t) := \phi_t(t) = \int_0^t S_{t-s} \, \mathrm{d}W(s)$ exists. Now assuming $0 \le s < t \le T$ we have that

$$M(t) - M(s) = \int_0^s S_{t-r} - S_{s-r} \, \mathrm{d}W(r) + \int_s^t S_{t-r} \, \mathrm{d}W(r) = (1) + (2) \,.$$

(1) and (2) are independent since the Q-Wiener process has independent increments. Therefore

$$\begin{aligned}\mathbb{E}\left[\|M(t) - M(s)\|^2\right] &= \mathbb{E}\left[\|(1)\|^2\right] + \mathbb{E}\left[\|(2)\|^2\right] \\ &= \int_0^s \|\left(S_{t-s} - I\right) S_{s-r} Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \, \mathrm{d}r + \int_s^t \|S_{t-r} Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \, \mathrm{d}r \,.\end{aligned}$$

The map $f(r) = \|S_{t-r} Q^{\frac{1}{2}}\|^2_{L_2(U,H)}$ is continuous by the strong continuity of S_t. So

$$\int_s^t \|S_{t-r} Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \, \mathrm{d}r \to 0 \text{ as } s \to t \,.$$

Now

$$
\begin{aligned}
\| (S_{t-s} - I)\, S_{s-r} Q^{\frac{1}{2}} \|^2_{L_2(U,H)} &\le \| S_{t-s} - I \|^2_{L(U,H)} \| S_{s-r} Q^{\frac{1}{2}} \|^2_{L_2(U,H)} \\
&\le (\| S_{t-s} \|_{L(U,H)} + \| I \|_{L(U,H)})^2 \| S_{s-r} Q^{\frac{1}{2}} \|^2_{L(U,H)} \\
&\le 2 \max_{s \in [0,T]} \| S(s) \|^2_{L(U,H)} \| S_{s-r} Q^{\frac{1}{2}} \|^2_{L(U,H)} ,
\end{aligned}
$$

so we can apply the dominated convergence theorem to say that

$$
\int_0^s \| (S_{t-s} - I)\, S_{s-r} Q^{\frac{1}{2}} \|^2_{L_2(U,H)} \, \mathrm{d}r \to 0. \text{ as } s \to t \, .
$$

It follows that M is mean-square continuous. □

We now prove that Equation 10 is the unique weak solution to Equation 8.

Theorem 2.3.2. ***Existence and uniqueness of weak solutions.*** *$X : [0,T] \to \mathcal{L}^2(\Omega, H)$ defined by $X(t) = S_t x_0 + \int_0^t S_{t-s} \, dW(s)$ is the unique weak solution to Equation 8 assuming that*

$$
\mathrm{Tr} \left(\int_0^T S_t Q S_t^* \, \mathrm{d}t \right) < \infty \, .
$$

Proof. By Theorem 2.3.1 we know that X is well-defined. We know that $S_t x_0$ is the unique weak solution to Equation 7, i.e. a weak solution to $\mathrm{d}X(t) + AX(t)\mathrm{d}t = 0$, with initial condition $X(0) = x_0$. It follows that $M(t) := \int_0^t S_{t-s} \, \mathrm{d}W(s) = X(t) - S_t x_0$ is the unique weak solution to

$$
\begin{cases}
\mathrm{d}X(t) + AX(t)\mathrm{d}t = \mathrm{d}W(t) \\
X(0) = 0 \, ,
\end{cases}
$$

if and only if X is the unique weak solution to Equation 8. We can therefore assume, without loss of generality, that $x_0 = 0$. Then, by definition, $X(t) = \int_0^t S_{t-s} \, \mathrm{d}W(s)$ is a weak solution if

$$
\langle X(t), x \rangle = \int_0^t \langle X(s), A^* x \rangle \, \mathrm{d}s + \int_0^t \langle x, dW(s) \rangle \, ,
$$

for any arbitrary $x \in D(A)$. We see that

$$
\begin{aligned}
\int_0^t \langle X(s), A^* x \rangle \, \mathrm{d}s &= \int_0^t \left\langle \int_0^s S_{s-r} \, \mathrm{d}W(r), A^* x \right\rangle \mathrm{d}s \\
&= \int_0^t \int_0^s \ell_{A^* x} S_{s-r} \, \mathrm{d}W(r) \, \mathrm{d}s \\
&= \int_0^t \int_0^t \mathbb{I}_{(0,s)}(r) \ell_{A^* x} S_{s-r} \, \mathrm{d}W(r) \, \mathrm{d}s = (*) \, ,
\end{aligned}
$$

where $\ell_\nu := \langle \cdot, \nu \rangle$. We want to apply the stochastic Fubini theorem to $\phi(s,r) := \mathbb{I}_{(0,s)}(r) \ell_{A^* x} S_{s-r}$ to switch the order of integration. Applying the Cauchy-Bunyakovski-Schwarz inequality we have that

$$
\begin{aligned}
\int_0^t \mathbb{E}\left[\int_0^t \|\phi(s,r)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}r\right]^{\frac{1}{2}} \mathrm{d}s &= \int_0^t \left(\int_0^t \|\phi(s,r)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}r\right)^{\frac{1}{2}} \mathrm{d}s \\
&\leq \int_0^t \left(\int_0^s \|S_{s-r}Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}r\right)^{\frac{1}{2}} \|A^*x\|_H\,\mathrm{d}s \\
&\leq t\|A^*x\|_H \left(\int_0^s \|S_rQ^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}r\right)^{\frac{1}{2}} < \infty
\end{aligned}
$$

So we can apply the stochastic Fubini theorem and the fact that by definition $\frac{\mathrm{d}}{\mathrm{d}t}S_tu = AS_tu$ to say that

$$
\begin{aligned}
(*) &= \int_0^t \int_r^t \langle S_{s-r}, A^*x\rangle\,\mathrm{d}s\,\mathrm{d}W(r) = \int_0^t \int_r^t \left\langle I, S^*_{s-r}A^*x\right\rangle \mathrm{d}s\,\mathrm{d}W(r) \\
&= \int_0^t \int_r^t \left\langle I, A^*S^*_{s-r}x\right\rangle \mathrm{d}s\,\mathrm{d}W(r) = \int_0^t \int_r^t \frac{d}{ds}\left\langle I, S^*_{s-r}x\right\rangle \mathrm{d}s\,\mathrm{d}W(r) \\
&= \int_0^t \left\langle I, S^*_{t-r}x\right\rangle - \langle I, x\rangle\,\mathrm{d}W(r) = \left\langle \int_0^t S_{t-r}\,\mathrm{d}W(r), x\right\rangle - \int_0^t \langle x, dW(r)\rangle \\
&= \langle X(t), x\rangle - \int_0^t \langle x, dW(r)\rangle\ ,
\end{aligned}
$$

so that X is a weak solution. Now assume that there are two weak solutions X and Y. Then

$$
\langle X(t) - Y(t), x\rangle = \int_0^t \langle X(s) - Y_s, A^*x\rangle\,\mathrm{d}s
$$

almost surely for all $t \in [0,T]$, so that a version of $X - Y$ is a weak solution to the deterministic evolution equation

$$
\begin{cases} \mathrm{d}f(t) = Af(t)\mathrm{d}t \\ f(0) = 0\,, \end{cases} ,
$$

which is known to have unique weak solution $f = 0$, implying that $X(t) - Y(t) = 0$ for all $t \in [0,T]$ almost surely, i.e. $P(X(t) = Y(t)) = 1$ for all $t \in [0,T]$. It follows that any weak solution is a version of $\int_0^t S_{t-s}\,\mathrm{d}W(s)$. The conclusion follows. □

3 The space fractional heat equation

In this section we consider the special case where $A : D(A) \to \mathcal{L}^2(\mathcal{D}) =: H$ is defined by $A := (\iota^2 - \Delta)^\gamma$ with $0 < \iota$, $0 < \gamma < 1$ and zero initial condition, i.e. the stochastic evolution equation

$$\begin{cases} \mathrm{d}X(t) + (\iota^2 - \Delta)^\gamma X(t)\mathrm{d}t = \mathrm{d}W(t) \\ X(0) = 0\,, \end{cases} \quad . \tag{11}$$

We call this equation the space fractional heat equation, to contrast it with a further generalization of the heat equation we will consider later. We will consider the operator $(\iota^2 - \Delta)^\gamma$ as acting on the space $D(A) = \dot{H}^1$, described in Appendix 2. $\mathcal{D}$ is here a bounded domain in $\mathbb{R}^d$ with smooth boundary and W is a Q-Wiener process on $\mathcal{L}^2(\mathcal{D})$. We will primarily assume Q to be the Matern covariance operator $Q = c(\kappa^2 - \Delta)^{-\beta}$. The analysis in this section is inspired by the analysis of the simple stochastic heat equation by Kovacs and Larsson [16].

From Example 2.2.1 in Appendix 2 we know that $-\Delta$ has an orthonormal, eigenvector basis $\{e_k\}_k$, spanning H, with a corresponding sequence of real, positive, increasing eigenvalues λ_k, diverging to infinity. We also have the Weyl bounds

$$C_1 k^{2/d} \leq \lambda_k \leq C_2 k^{2/d}\,.$$

It follows that the operator $-A$ also has real, decreasing eigenvalues $-(\iota^2 + \lambda_k)^\beta$ and that $-A$ generates a semigroup $S_t = \exp(-At)$ by Theorem 2.2.2. Since all the eigenvalues are real, we know that A is self-adjoint, i.e. $A = A^*$. For ease of notation we define

$$\mu_k := (\iota^2 + \lambda_k)^\gamma\,.$$

We trivially have the estimate $\lambda_k^\gamma \leq (\iota^2 + \lambda_k)^\gamma = \mu_k$. Since $\frac{(\iota^2+x)^\gamma}{x^\gamma}$ is a decreasing function, we also have the estimate $\mu_k = (\iota^2 + \lambda_k)^\gamma \leq \frac{(\iota^2+\lambda_1)^\gamma}{\lambda_1^\gamma}\lambda_k^\gamma$. The eigenvalues μ_k thus satisfy the estimates

$$\lambda_k^\gamma \leq \mu_k \leq C\lambda_k^\gamma\,,$$

after which we can apply the Weyl bounds to λ_k to also obtain bounds for μ_k.

3.1 Existence of weak solutions

We saw in the previous section that if $\int_0^T \|S_t Q^{\frac{1}{2}}\|^2_{L_2(U)}\,\mathrm{d}t < \infty$, then the stochastic convolution

$$C_t = \int_0^T S_{t-s}\,\mathrm{d}W(s)\,,$$

is the unique weak solution to Equation 11. We will now check that this condition holds. First note that for a test element $x \in U$ we have the estimate

$$\begin{aligned} \int_0^T \|A^{\frac{1}{2}} \exp(-tA)x\|_H^2\,\mathrm{d}t &= \int_0^T \sum_k c\mu_k \exp(-2c\mu_k t)\left\langle x, e_k\right\rangle^2\,\mathrm{d}t \\ &= \sum_k \frac{1}{2}(1 - \exp(-2\mu_k T))\left\langle x, e_k\right\rangle^2 \leq \frac{1}{2}\|x\|_U^2\,. \end{aligned}$$

Now let $\{f_k\}_k$ be an orthonormal basis for $H = \mathcal{L}^2(\mathcal{D})$. We consider the integral

$$\int_0^T \|\exp(-At)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}t = \int_0^T \sum_k \|\exp(-At)Q^{\frac{1}{2}}f_k\|^2_U\,\mathrm{d}t = (*)\,.$$

$A^{\frac{1}{2}}$ and $\exp(-At)$ commute since they share eigenvector basis, and $I = A^{\frac{1}{2}}A^{-\frac{1}{2}}$. We can therefore write

$$(*) = \sum_k \int_0^T \|A^{\frac{1}{2}}\exp(-At)A^{-\frac{1}{2}}Q^{\frac{1}{2}}f_k\|^2_H\,\mathrm{d}t\,.$$

We apply our estimate to write $\int_0^T \|A^{\frac{1}{2}}\exp(-At)A^{-\frac{1}{2}}Q^{\frac{1}{2}}f_k\|^2_H\,\mathrm{d}t \le \frac{1}{2}\|A^{-\frac{1}{2}}Q^{\frac{1}{2}}f_k\|^2_H$, so that

$$(*) \le \frac{1}{2}\sum_k \|A^{-\frac{1}{2}}Q^{\frac{1}{2}}f_k\|^2_H = \frac{1}{2}\|A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{L_2(H)}\,.$$

So we have a weak solution if $\|A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|_{L_2(H)} \le \|A^{-\frac{1}{2}}\|_{L_2(H)}\|Q^{\frac{1}{2}}\|_{L(H)} < \infty$, assuming Q is bounded. It is thus sufficient to have $\|A^{-\frac{1}{2}}\|^2_{L_2(H)} = \sum_k(\iota^2+\lambda_k)^{-\gamma} < \infty$.. Since $\lambda_k \le C_2 k^{\frac{2}{d}}$ we can estimate $(\iota^2+\lambda_k)^{-\gamma} \le \lambda_k^{-\gamma} \le Ck^{\frac{2}{d}\gamma}$. Therefore

$$\|A^{-\frac{1}{2}}\|_{L_2(H)} = \sum_k(\iota^2+\lambda_k)^{-\gamma} \le \sum_k Ck^{-\frac{2\gamma}{d}}\,,$$

which is finite if $-\frac{2\gamma}{d} < -1$, or $d < 2\gamma$. This means that we can guarantee solutions to Equation 11 if we have have a bounded Q and the dimension of our solution space is less than 2γ. However, we can do better by considering a Q with higher regularity. For example we could pick the Whittle-Matern covariance operator discussed in Section 1.4, i.e. $Q = c(\kappa^2-\Delta)^{-\beta}$,. This operator has the eigenvalues $c(\kappa^2+\lambda_k)^{-\beta}$ with the same corresponding eigenvectors e_k as $-\Delta$. We can use the estimate as before to calculate

$$\|A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{L_2(H)} = \sigma^2\sum_k(\iota^2+\lambda_k)^{-\gamma}(\kappa^2+\lambda_k)^{-\beta} \le \sum_k Ck^{-\frac{2\gamma}{d}-\frac{2\beta}{d}}\,.$$

It follows that we have existence of weak solutions if $d < 2\beta + 2\gamma$.

3.2 Regularity of solutions

We are also interested in the spatial regularity, or smoothness, of our solutions. To analyze this we will use the $\dot{H}^s$-spaces discussed in Appendix 2. Functions u in $\dot{H}^s$ are in some sense s times differentiable in space, so by seeing if the condition in Theorem 2.3.1 is finite for the norm $\|\cdot\|_s$, we can see if our solutions are s times differentiable in space. We now proceed to do this. Similarly to before

$$\int_0^T \|\exp(-At)Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)}\,\mathrm{d}t \le \frac{1}{2}\|A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)} = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{L_2(H)}.$$

Assuming Q to be bounded we can estimate $\|(-\Delta)^{\frac{s}{2}}A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|_{L_2(H)} \le C\|(-\Delta)^{\frac{s}{2}}A^{-\frac{1}{2}}\|_{L_2(H)}$. Thus weak solutions exists in $\dot{H}^s$ if

$$\|(-\Delta)^{\frac{s}{2}}A^{-\frac{1}{2}}\|^2_{L_2(H)} = \sum_k \lambda_k^s(\iota^2+\lambda_k)^{-\gamma} \leq C\sum_k k^{\frac{2}{d}(s-\gamma)}\,,$$

i.e. if $\frac{2}{d}(s-\gamma) < -1$ or $s < \gamma - \frac{d}{2}$. As discussed before s is a measure of differentiability so this would imply that assuming only $\|Q^{\frac{1}{2}}\|_{L(H)} < \infty$ we can in general get no more than $\gamma - \frac{d}{2}$ derivatives. If we assume instead that $Q = c(\kappa^2+\lambda_k)^{-\beta}$, i.e. Whittle-Matern noise, we get more regularity. We then have

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}A^{-\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{L_2(H)} &= \lambda_k^s\sum_k(\iota^2+\lambda_k)^{-\gamma}(\kappa^2+\lambda_k)^{-\beta}\\ &\leq C\sum_k k^{\frac{2}{d}(s-\gamma-\beta)}\,,\end{aligned}$$

so that we have $s < \gamma+\beta-\frac{d}{2}$, implying that our solutions are $\lceil \gamma+\beta-\frac{d}{2}-1\rceil$ - times mean square differentiable. We can thus get arbitrary spatial regularity by upscaling either β or γ. We summarize our discussion in a proposition.

Proposition 3.2.1. ***Spatial regularity of*** X**.** *Let X be a weak solution to Equation 11, with $Q = c(\kappa^2-\Delta)^{-\beta}$ and $\gamma \in (0,\infty)$. Assume also that $s < \gamma+\beta-\frac{d}{2}$. Then $X \in \mathcal{L}^2(\Omega, \mathcal{L}^2([0,T], \dot{H}^s))$.*

Note that the existence result from the previous subsection follows from this one if we select $s = 0$. A further natural question to ask is whether our weak solutions are (mean-square) differentiable in time or not. The following proposition suggests that they are not, and that solutions are no more than 1/2-Hölder continuous.

Proposition 3.2.2. ***Temporal regularity of*** X. *Let X be a weak solution to Equation 11 in H with $Q = (\kappa^2-\Delta)^{-\beta}$. Assume that $\nu \in (0,\frac{1}{2})$ and $s \in [0,\infty)$ satisfies $2\gamma\nu + s < \gamma+\beta-\frac{d}{2}$. Then for $t_1, t_2 \in [0,T]$ we have that*

$$\|X(t_2)-X(t_1)\|_{\mathcal{L}^2(\Omega,\dot{H}^s)} \leq C|t_2-t_1|^\nu\,.$$

Proof. We know by Theorem 2.3.2 that since a weak solution exists it can be expressed as

$$X(t) = \int_0^t S_{t-s}\,\mathrm{d}W(s)\,.$$

We assume w.l.o.g. that $t_1 < t_2$. Using the triangle inequality and the Itô isometry we can estimate

$$\begin{aligned}\mathbb{E}\left[\|X(t_2)-X(t_1)\|^2_{\dot{H}^s}\right] &= \mathbb{E}\left[\left\|\int_0^{t_2} S_{t_2-s}\,\mathrm{d}W(s) - \int_0^{t_1} S_{t_1-s}\,\mathrm{d}W(s)\right\|^2_{\dot{H}^s}\right]\\ &= \mathbb{E}\left[\left\|\int_0^{t_2} S_{t_2-s} - \mathbb{I}_{(0,t_1)}(s)S_{t_1-s}\,\mathrm{d}W(s)\right\|_{\dot{H}^s}\right]\\ &= \int_0^{t_2}\|(S_{t_2-t_1}-\mathbb{I}_{(0,t_1)}(s))S_{t_1-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)}\,\mathrm{d}s\\ &= \int_0^{t_1}\|(S_{t_2-t_1}-1)S_{t_1-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)}\,\mathrm{d}s + \int_{t_1}^{t_2}\|S_{t_2-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)}\,\mathrm{d}s\\ &= (1)+(2)\end{aligned}$$

We know that the semigroup S_t has the form $\exp(-tA)$ such that for an eigenvector e_k we have that $S_t e_k = \exp(-(\iota^2 + \lambda_k)^\gamma t) e_k$. We can then write

$$\|(S_{t_2-t_1} - I)S_{t_1-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)} = \sum_k \lambda_k^s (1 - e^{-\mu_k(t_2-t_1)})^2 e^{-2c\mu_k(t_1-s)} c(\kappa^2 + \lambda_k)^{-\beta},$$

and

$$\|S_{t_2-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)} = \sum_k \lambda_k^s e^{-2\mu_k(t_2-s)} (\kappa^2 + \lambda_k)^{-\beta}.$$

We first consider (1). We see that

$$\begin{aligned}
(1) &= \int_0^{t_1} \|(S_{t_2-t_1} - I)S_{t_1-s}Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}s \\
&= \sum_k \lambda_k^s \frac{c}{2\mu_k} (1 - e^{-\mu_k(t_2-t_1)})^2 (1 - e^{-2\mu_k t_1})(\kappa^2 + \lambda_k)^{-\beta}.
\end{aligned}$$

We can then use the bounds $1 - e^{-x} \leq 1$ and $(1 - e^{-x})^2 \leq x^a$ for $a \leq 2$ to estimate

$$(1) \leq C(t_2 - t_1)^a \sum_k \lambda_k^{-\gamma-\beta+\gamma a+s}.$$

For (2) we see that

$$(2) = \int_{t_1}^{t_2} \|S_{t_2-s}Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^s)} \, \mathrm{d}s = \sum_k \lambda_k^s \frac{1}{2c\mu_k} (1 - e^{-2c\mu_k(t_2-t_1)})(\kappa^2 + \lambda_k)^{-\beta}.$$

We can then use the bound $1 - e^{-x} \leq x^a$ for $a \leq 1$, to see that

$$(2) \leq C(t_2 - t_1)^a \sum_k \lambda_k^{-\gamma-\beta+\gamma a+s}.$$

Putting this together we get

$$\begin{aligned}
(1) + (2) &= C(t_2 - t_1)^a \sum_k \lambda_k^{-\gamma-\beta+\gamma a+s} \\
&\leq C(t_2 - t_1)^a \sum_k k^{\frac{2}{d}(-\gamma-\beta+\gamma a+s)},
\end{aligned}$$

which converges if $\frac{2}{d}(-\gamma - \beta + \gamma a + s) < -1$ or $\gamma a + s < \gamma + \beta - \frac{d}{2}$. For (1) we must have $a \leq 2$ and for (2) we must have $a \leq 1$. The requirements on the Hölder coefficient $\nu = \frac{a}{2}$ are thus

$$0 \leq 2\gamma\nu + s < \beta + \gamma - \frac{d}{2},$$

and

$$\nu < \frac{1}{2}.$$

The conclusion follows. □

Intuitively we can think of $\beta + \gamma - \frac{d}{2}$ as the "total regularity" of the solution X. Every time we take a spatial derivative it reduces the total regularity by 1. Temporal regularity "costs" 2γ as much; increasing the Hölder coefficient by x reduces the total regularity by $2\gamma x$. The caveat is that we the additional restriction that you can never have a Hölder coefficient greater than $\frac{1}{2}$. The process X can thus never be mean-square differentiable in time either.

3.3 Covariance properties

The results of Section 3.2 tells us how the parameters β and γ influence the spatial and temporal smoothness of solutions to Equation 11. In this section we calculate the asymptotic space-time covariance function we discussed in Section 1.7 and use it to establish qualitative estimates for how the parameters of Equation 11 influence the rate of decay of correlation in its solutions, first in time and then in space. The analysis in this subsection is to the authors knowledge somewhat novel and is therefore more experimental and less rigorous than that found in other parts of this thesis.

Our first proposition gives us a somewhat concrete representation of the asymptotic space-time covariance function for the space fractional heat equation.

Proposition 3.3.1. *Let X be a weak solution to Equation 11 on $\mathcal{L}^2(\mathcal{D})$ with $Q = (\kappa^2 - \Delta)^{-\beta}$. Then the asymptotic space-time covariance function of X is given by*

$$r(h) = \frac{c}{2} S_h (\iota^2 - \Delta)^{-\gamma} (\kappa^2 - \Delta)^{-\beta} \,.$$

Proof. We can calculate the space-time covariance function from Section 1.7 of the solution X using Proposition 2.1.2. In general for solutions to Equation 8 with $x_0 = 0$ we can calculate

$$\begin{aligned} \langle r(t,s)u, v\rangle_H &= \mathbb{E}\left[\langle X(t), u\rangle_H \langle X(s), v\rangle_H\right] \\ &= \mathbb{E}\left[\left\langle \int_0^T \mathbb{I}_{(0,t)} S_{t-\xi}\, \mathrm{d}\xi, u \right\rangle_H \left\langle \int_0^T \mathbb{I}_{(0,s)} S_{s-\xi}\, \mathrm{d}\xi, v \right\rangle_H\right] \\ &= \left\langle \int_0^T \mathbb{I}_{(0,\min(t,s))}(\xi) S_{t-\xi} Q S_{s-\xi}^*\, \mathrm{d}\xi\, u, v \right\rangle_H \\ &= \left\langle \int_0^{\min(t,s)} S_{|t-s|} S_\xi Q S_\xi^*\, \mathrm{d}\xi\, u, v \right\rangle_H \end{aligned}$$

so that

$$r(t,s) = \int_0^{\min(t,s)} S_{|t-s|} S_\xi Q S_\xi^*\, \mathrm{d}\xi \,.$$

In our case we have $Q = c(\kappa^2 + \Delta)^{-\gamma}$ and $S_h e_k = \exp(-\mu_k h) e_k$, so we can test this operator against an eigenvector e_k to get

$$
\begin{aligned}
r(t,s)e_k &= \int_0^{\min(t,s)} \exp(-\mu_k|t-s|)\exp(-2\mu_k\xi)c(\kappa^2+\lambda_k)^{-\beta}\,\mathrm{d}\xi e_k \\
&= \frac{\exp(-(\iota^2+\lambda_k)^\gamma|t-s|)}{2(\iota^2+\lambda_k)^\gamma}(1-\exp(-2(\iota^2+\lambda_k)^\gamma\min(t,s)))c(\kappa^2+\lambda_k)^{-\beta}e_k\,.
\end{aligned}
$$

Specifically as $t \to \infty$ we get the asymptotic space-time covariance function

$$
r(h)e_k = \lim_{t\to\infty} r(t+h,t)e_k = \frac{c}{2}(\iota^2+\lambda_k)^{-\beta}(\kappa^2+\lambda_k)^{-\gamma}\exp(-(\iota^2+\lambda_k)^\gamma h)e_k\,,
$$

so that the asymptotic covariance operator of $X(t)$ is given by

$$
r(h) = \frac{c}{2}S_h(\iota^2-\Delta)^{-\gamma}(\kappa^2-\Delta)^{-\beta}\,.
$$

□

Note that if $\iota = \kappa$, then $r(0) = \frac{c}{2}(\kappa^2-\Delta)^{-\beta-\gamma}$. This is a Whittle-Matern covariance operator and the spatial correlation range depends on κ $(= \iota)$, i.e. large values of κ induce rapid decay of correlation in space. For the case where $\iota \neq \kappa$, we will consider the limiting case of $\mathcal{D} = \mathbb{R}^d$ and find an upper bound for the auto-covariance function corresponding to $r(0)$, i.e. a bound for the spatial decay in the stationary distribution. We do this in the following proposition.

Proposition 3.3.2. *Let X be a solution to Equation 11 on the domain $\mathcal{D} = \mathbb{R}^d$ with $Q = c(\kappa^2-\Delta)^{-\beta}$ and assume $\kappa \neq \iota$. Denote the auto-covariance function corresponding to the asymptotic covariance operator $r(0)$ of X by R. We then have that*

$$
|R(h)| \le O(e^{-2\pi\iota h}) + O(e^{-2\pi\kappa h}) \text{ as } h \to \infty\,.
$$

Proof. Firstly,

$$
\begin{aligned}
[r(0)g](x) &= \mathcal{F}^{-1}\left(\frac{c}{2}(\iota^2+\omega^2)^{-\gamma}(\kappa^2+\omega^2)^{-\beta}\hat{g}(\omega)\right) \\
&= \int_{\mathbb{R}^d} e^{i\omega x}\frac{c}{2}(\iota^2+\omega^2)^{-\gamma}(\kappa^2+\omega^2)^{-\beta}\frac{1}{2\pi}\int_{\mathbb{R}^d} e^{-i\omega y}g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} e^{i\omega(x-y)}\frac{c}{4\pi}(\iota^2+\omega^2)^{-\gamma}(\kappa^2+\omega^2)^{-\beta}\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}
$$

so that the spectral density corresponding to $r(0)$ is given by $f(\omega) = \frac{c}{4\pi}(\iota^2+\omega^2)^{-\gamma}(\kappa^2+\omega^2)^{-\beta}$. This is a product of two Matern spectral densities. The auto-covariance function R is thus proportional to a convolution of two Matern auto-covariance functions. According to Baricz [2] we have the bound $K_\nu(h) < Ce^{-h}$ for modified Bessel functions of the second kind. The Matern auto-covariance function in Equation 6 therefore satisfies the bound

$$
|r_\kappa(h)| < Ch^\nu e^{-\frac{\sqrt{2\nu}}{\rho}|h|} = Ch^\nu e^{-2\pi\kappa|h|}\,.
$$

We can utilize this to calculate

$$
\begin{aligned}
|R(h)| \propto & \, \Big| \int_{\mathbb{R}^d} r_\kappa(h-x) r_\iota(x) \, \mathrm{d}x \Big| \\
& \leq C \int_{\mathbb{R}} |h-x|^\nu |x|^\nu e^{-2\pi\kappa|h-x|} e^{-2\pi\iota|x|} \, \mathrm{d}x \\
& \leq C \int_{-\infty}^{0} |h-x|^\nu |x|^\nu e^{-2\pi\kappa(h-x)} e^{2\pi\iota x} \, \mathrm{d}x \\
& + C \int_{0}^{h} |h-x|^\nu |x|^\nu e^{-2\pi\kappa(h-x)} e^{-2\pi\iota x} \, \mathrm{d}x \\
& + C \int_{h}^{\infty} |h-x|^\nu |x|^\nu e^{-2\pi\kappa(x-h)} e^{-2\pi\iota x} \, \mathrm{d}x = (I) + (II) + (III) \, .
\end{aligned}
$$

For (I) we can do the variable substitution $x = -x$ and calculate

$$
(I) = Ce^{-2\pi\kappa h} \int_0^\infty |h-x|^\nu |x|^\nu e^{-2\pi(\iota+\kappa)x} \, dx = \frac{C}{2\pi(\iota+\kappa)} e^{-2\pi\kappa h} = O(e^{-2\pi\kappa h}) \, .
$$

For (II) we do similarly, assuming $\iota \neq \kappa$,

$$
(II) = Ce^{-2\pi\kappa h} \int_0^h |h-x|^\nu |x|^\nu e^{2\pi(\kappa-\iota)x} \, dx = \frac{C}{2\pi(\kappa-\iota)} e^{-2\pi\iota h} = O(e^{-2\pi\iota h}) \, .
$$

For (III) we can do the variable substitution $x = h - x$ and calculate

$$
(III) = Ce^{2\pi\kappa h} \int_h^\infty |h-x|^\nu |x|^\nu e^{-2\pi(\kappa+\iota)x} \, \mathrm{d}x = \frac{C}{2\pi(\kappa+\iota)} e^{-2\pi\iota h} = O(e^{-2\pi\iota h}) \, .
$$

It follows that the auto-covariance function corresponding to $r(0)$ is bounded by a function that decays like $O(e^{-2\pi\iota|h|}) + O(e^{-2\pi\kappa|h|})$.

□

Proposition 3.3.2 indicates that the spatial range is controlled by $\min(\iota, \kappa)$, i.e. large values of ι and κ will give rapid decline in correlation range, but only if both are large simultaneously.

Proposition 3.3.3. *Let X be a solution to Equation 11 on $\mathcal{L}^2(\mathcal{D})$. Then for $0 \leq \alpha < 1$ and $x, y \in \mathcal{L}^2(\mathcal{D})$ and for large values of t we have that*

$$
|\mathbb{E}\left[X(t+h,x)X(t,y)\right]| \leq O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h}) \quad \textit{as} \;\; h \to \infty \, .
$$

Proof. For elements u and v we have that $\langle r(h)u, v\rangle_H = \mathbb{E}\left[\langle X(t+h), u\rangle_H \langle X(t), u\rangle_H\right]$. We will consider the case where $u = \delta_x$ and $v = \delta_y$; the delta functions centered at x and y respectively. Then for "large" values of t we have that

$$
\langle r(h)\delta_x, \delta_y\rangle_H = \mathbb{E}\left[\langle X(t+h), \delta_x\rangle_H \langle X(t), \delta_y\rangle_H\right] = \mathbb{E}\left[X(t+h,x)X(t,y)\right] \, .
$$

So we can calculate the pointwise covariances by applying our space-time covariance function to delta functions. Using the Weyl estimates and the properties of the delta function discussed in Appendix A2 we can calculate that for $0 \leq \alpha < 1$ we have

$$
\begin{aligned}
|\langle r(h)\delta_x, \delta_y\rangle| &= |\sum_{k,j} e_k(x)e_j(x)\langle r(h)e_k, e_j\rangle| \\
&\leq \sum_k \frac{C}{2} e^{-(\iota^2+\lambda_k)^\gamma h}(\iota^2+\lambda_k)^{-\beta}(\kappa^2+\lambda_k)^{-\gamma} e_k(x)^2 \\
&= \sum_k \frac{C}{2} e^{-\alpha(\iota^2+\lambda_k)^\gamma h} e^{-(1-\alpha)(\iota^2+\lambda_k)^\gamma h}(\iota^2+\lambda_k)^{-\beta}(\kappa^2+\lambda_k)^{-\gamma} e_k(x)^2 \\
&\leq C e^{-\alpha(\iota^2+\lambda_1)^\gamma h} \sum_k e^{-(1-\alpha)(\iota^2+\lambda_k)^\gamma h} \lambda_k^{-\beta-\gamma+\frac{d-1}{2}}
\end{aligned}
\tag{12}
$$

This sum always convergences since the decay in the exponential $e^{-(1-\alpha)(\iota^2+\lambda_k)^\gamma h}$ as $k \to \infty$ will kill any divergence in the eigenvalues λ_k. This garantuees that $|\langle r(h)\delta_x, \delta_y\rangle| < \infty$ while also establishing that $|\mathbb{E}[X(t+h,x)X(t,y)]| \leq O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h})$ as desired.

□

The decay rate of temporal correlation is therefore exponential and controlled by the parameters ι and γ; large values of ι and γ induce rapid decay of correlation in time. The presence of the first eigenvalue λ_1 shows that the geometry of the domain $\mathcal{D}$ can also impact the rate of temporal correlation decay.

4 The space-time fractional heat equation

As we have now seen, a major problem with the space fractional heat equation is that the temporal regularity is capped at a Hölder continuity of order $\frac{1}{2}$. This problem no longer appears if we also let the temporal derivative be fractional, for example we could consider the equation defined formally by

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t} + (\iota^2 - \Delta)^\gamma\right)^\delta X(t) = \dot{W}(t) \\ X(0) = 0\,, \end{cases} . \tag{13}$$

As is, this equation is only a formal expression; we will need to define what $(\frac{\mathrm{d}}{\mathrm{d}t} - (\iota^2 - \Delta)^\gamma)^\delta$ actually means. A further difficulty is that this equation does not fit in the framework of Section 2.3. We will address both of these issues in the following subsection. We first consider an abstract operator $A\colon D(A) \to H$, assuming only that A and $D(A)$ satisfy the conditions of the Hille-Yosida theorem, implying that $(A, D(A))$ generates some semigroup of operators $\{S_t\}_t$ satisfying $\|S_t\|_{L(H)} \leq Me^{\omega t}$ for some $M \in \mathbb{R}$. In subsections 4.3 and 4.4 we go back to the specific case of $A = (\iota^2 - \Delta)^\gamma$. The following treatment borrows heavily from Kirchner and Willems [15].

4.1 The fractional operator $\left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right)^\delta$

The first step is to make sense of the operator $\left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right)^\delta$ is to define the space that the operator acts upon. We have so far considered A as an operator on (a subset of) the Hilbert space H. We will now consider it as an operator on the Bochner space $\tilde{H} = \mathcal{L}^2([0,T], H)$. For $f \in D(\tilde{A}) \subset \mathcal{L}^2([0,T], H)$ we define the corresponding operator $\tilde{A}\colon D(\tilde{A}) \to \mathcal{L}^2([0,T], H)$ by

$$(\tilde{A}f)(s) = A(f(s)) \text{ a.e. } s \in [0,T].$$

If $f \in \mathcal{L}^2([0,T], D(A))$, then $f(s) \in D(A)$ for almost all $s \in [0,T]$, so $(\tilde{A}f)(s)$ is well-defined. For now it is therefore natural to consider the domain of $\tilde{A}$ to be $\mathcal{L}^2([0,T], D(A))$, though we will see below that this domain can be extended by a graph closure.

We can extend the semigroup $\{S_t\}_t$ generated by A in a similar manner. For $f \in \mathcal{L}^2([0,T], H)$ the extended semigroup $\tilde{S}_t\colon \mathcal{L}^2([0,T], H) \to \mathcal{L}^2([0,T], H)$ is defined to act upon f by

$$(\tilde{S}_t f)(s) = S_t(f(s)) \text{ a.e. } s \in [0,T].$$

The following lemma will show that this extended semigroup $\tilde{S}_t$ is generated by $\tilde{A}$ on an extension of its domain. The lemma is taken from Kirchner and Willems [15].

Lemma 4.1.1. *The operator $(-\tilde{A}, D(\tilde{A}))$, where $D(\tilde{A}) := \overline{\mathcal{L}^2([0,T], D(A))}^g$, generates the semigroup $\{\tilde{S}_t\}_t$ of operators $\tilde{H} \to \tilde{H}$. $\overline{\cdot}^g$ here denotes the closure in the graph norm $\|\cdot\|_g$ defined by $\|\cdot\|_g := \|\cdot\|_{\tilde{H}} + \|\tilde{A}(\cdot)\|_{\tilde{H}}$.*

Proof. We first show that $\{\tilde{S}_t\}_t$ inherits the semigroup properties from S_t, then that $(-\tilde{A}, D(\tilde{A}))$ generates the semigroup $\{\tilde{S}_t\}_t$.

$\{\tilde{S}_t\}_t$ **is a semigroup**: Firstly it is easy to see that $\tilde{S}_0 = I$ and that $\tilde{S}_{t+s} = \tilde{S}_t\tilde{S}_s$. We now show that $\tilde{S}_t v \to v$ in $\|\cdot\|_{\mathcal{L}^2([0,T],H)}$ for every $v \in \tilde{H}$, implying that $\tilde{S}_t$ is strongly continuous. Note first that since $\|S_t\|_{L(H)} \leq Me^{\omega T}$ we have that $\|S_t u - u\|_H \leq (Me^{\omega T} + 1)\|u\|_H$ for $u \in H$. By dominated convergence it follows that for $f \in \tilde{H}$ we have that

$$\|\tilde{S}_t f - f\|_{\tilde{H}}^2 = \int_0^T \|S_t(f(\xi)) - f(\xi)\|_H^2 \,\mathrm{d}\xi \to 0 \text{ as } t \to 0\,,$$

by the strong continuity of S_t, implying that $\tilde{S}_t$ is also strongly continuous.

$(-\tilde{A}, D(\tilde{A}))$ **generates** $\{\tilde{S}_t\}_t$: It remains to show that $(-\tilde{A}, D(\tilde{A}))$ generates $\{\tilde{S}_t\}_t$. For $u \in D(A)$, by Proposition 2.2.1

$$\begin{aligned}\left\|\frac{1}{h}(S_h u - u) + Af\right\|_H &= \left\|\frac{1}{h}\int_0^h S_\xi Au \,\mathrm{d}\xi + Au\right\|_H \\ &\leq \frac{1}{h}\int_0^h \|S_\xi Au\|_H \,\mathrm{d}\xi + \|Au\|_H \\ &\leq (Me^{\omega T} + 1)\|Au\|_H\,.\end{aligned}$$

and thus for $f \in \mathcal{L}^2([0,T], D(A))$

$$\begin{aligned}\left\|\frac{1}{h}(\tilde{S}_h f - f) + \tilde{A}f\right\|_{\tilde{H}}^2 &= \int_0^T \left\|\frac{1}{h}(S_h f(t) - f(t)) + Af(t)\right\|_H^2 \mathrm{d}t \\ &\leq (M+1)^2\int_0^T \|Af(t)\|_H^2 \mathrm{d}t = (Me^{\omega T} + 1)^2\|\tilde{A}f\|_{\tilde{H}}^2 < \infty\,.\end{aligned}$$

We can therefore use the dominated convergence theorem to conclude that

$$\left\|\frac{1}{h}(\tilde{S}_h f - f) + \tilde{A}f\right\|_{\tilde{H}}^2 = \int_0^T \left\|\frac{1}{h}(S_h f(t) - f(t)) + Af(t)\right\|_H^2 \mathrm{d}t \to 0 \text{ as } h \to 0\,.$$

It follows that $\frac{1}{h}(\tilde{S}_h f - f) \to -\tilde{A}f$ for all $f \in \mathcal{L}^2([0,T], D(A))$. Since $\overline{D(A)} = H$, it follow almost immediately that $\overline{\mathcal{L}^2([0,T], D(A))} = \mathcal{L}^2([0,T], H)$. By Proposition 2.2.1 we also know that $\mathcal{L}^2([0,T], D(A))$ is invariant under $\tilde{S}_h$. Therefore, by Proposition 2.2.3, $\overline{\mathcal{L}^2([0,T], D(A))}^g = D(\tilde{A})$ is the domain of the generator of the semigroup $\{\tilde{S}_t\}_t$. The generator of the semigroup is hence $(-\tilde{A}, D(\tilde{A}))$, where $-\tilde{A}$ is extended to $\overline{D(\tilde{A})}^g$ naturally by limit. □

We are also interested in the operator $\frac{\mathrm{d}}{\mathrm{d}t} : D(\frac{\mathrm{d}}{\mathrm{d}t}) \subset \mathcal{L}^2([0,T], H) \to \mathcal{L}^2([0,T], H)$, which is here considered to be the weak derivative. In the next lemma we show that $-\frac{\mathrm{d}}{\mathrm{d}t}$ generates the semigroup $\{R_t\}_t$ of right shift operators defined by

$$(R_t f)(\xi) := \begin{cases} f(\xi - t), & \text{if } \xi - t \geq 0 \\ 0\ , \text{else} \end{cases} =: f(\xi - t)\mathbb{I}_{[t,T]}(\xi)\,.$$

This lemma is also taken from Kirchner and Willems [15]. In the lemma we frequently refer to the space $C_c^\infty((0,T], H)$, the space of functions $(0,T] \to H$ with compact support. These functions are technically not defined at 0, but we adapt the convention that for $g \in C_c^\infty((0,T],H)$ we have that $g(0)=0$, so that g is defined $[0,T] \to H$. This extension preserves both the smoothness of the function g and its compact support on $(0,T]$.

Lemma 4.1.2. *The operator* $(-\frac{d}{dt}, D(\frac{d}{dt}))$*, where* $D(\frac{d}{dt}) := \overline{C_c^\infty((0,T],H)}^g$*, generates the semigroup* $\{R_t\}_t$ *of operators* $\tilde{H} \to \tilde{H}$*.* $\overline{}^g$ *here denotes the closure in the graph norm* $\|\cdot\|_g$ *defined by* $\|\cdot\|_g := \|\cdot\|_{\tilde{H}} + \|\frac{d}{dt}(\cdot)\|_{\tilde{H}}$*.*

Proof. We split the proof into two parts, we first show that $\{R_t\}_t$ is a C_0-semigroup, then that $(-\frac{\mathrm{d}}{\mathrm{d}t}, D(\frac{\mathrm{d}}{\mathrm{d}t}))$ generates R_t.

$\{R_t\}_t$ **is a** C_0**-semigroup**: It is clear that R_t is a linear operator and that $R_0 = I$. We see that the semigroup property holds since for $f \in \tilde{H}$ we have that for $t, s \geq 0$

$$\begin{aligned} R_t R_s f(\xi) &= R_t f(\xi - s)\mathbb{I}_{[s,T]}(\xi) = f(\xi - s - t)\mathbb{I}_{[s,T]}(\xi - t)\mathbb{I}_{[t,T]}(\xi) \\ &= f(\xi - (t+s))\mathbb{I}_{[s+t,T]}(\xi) = R_{t+s} f(\xi)\,. \end{aligned}$$

Finally we show that $\{R_t\}_t$ is strongly continuous. We first consider an element $g \in C_c^\infty((0,T],H)$, i.e. a function $g : [0,T] \to H$ which is infinitely many times $\|\cdot\|_H$-differentiable and is zero outside of some compact set $[a,b] \subset (0,T]$. Since g is continuous on a compact interval, it is uniformly continuous, so for any $\epsilon > 0$, we can find $\delta > 0$ such that $\|g(\xi - h) - g(\xi)\|_H < \frac{\epsilon}{\sqrt{b-a}}$ for $0 \leq h < \delta$ and $\xi \in [h,T]$. Therefore

$$\|\mathcal{R}_h g - g\|_{\tilde{H}}^2 = \int_0^h \|g(\xi)\|_H^2 \,\mathrm{d}\xi + \int_h^T \|g(\xi - h) - g(\xi)\|_H^2 \,\mathrm{d}\xi = (1) + (2)\,.$$

If we take $h < a$ then $(1) = 0$ since g is zero outside $[a,b]$ by assumption. If we also let $h < \delta$, then

$$(2) = \int_a^b \|g(\xi - h) - g(\xi)\|_H^2 ds < \int_a^b \frac{\epsilon^2}{b-a} ds = \epsilon^2\,,$$

so that $\|\mathcal{R}_h g - g\|_{\tilde{H}} < \epsilon$. Therefore R_t is strongly continuous on $C_c^\infty((0,T],H)$. Next we let $f \in \tilde{H}$. Since $C_c^\infty((0,T],H)$ is dense in $\tilde{H}$ we can find, for any $\epsilon > 0$, a $g \in C_c^\infty((0,T],H)$ such that $\|f - g\|_{\mathcal{L}^2([0,T],H)} < \frac{\epsilon}{3}$. This can be shown easily by combining the fact that the space of H-valued simple functions are dense in H [13], and the fact that $C((0,T],\mathbb{R})$ is dense in $L^2((0,T],\mathbb{R})$ [1]. We also pick h such that $\|\mathcal{R}_h g - g\|_{\tilde{H}} < \frac{\epsilon}{3}$. Note also that for $f \in \tilde{H}$

$$\|\mathcal{R}_h f\|_{\tilde{H}}^2 = \int_h^T \|f(\xi - h)\|_H^2 \,\mathrm{d}\xi = \int_0^{T-h} \|f(\xi)\|_H^2 \,\mathrm{d}\xi \leq \|f\|_{\tilde{H}}^2\,,$$

implying that $\|R_t f - R_t g\|_{\tilde{H}} \leq \|f - g\|_{\tilde{H}} < \frac{\epsilon}{3}$. Then by the triangle inequality

$$\|\mathcal{R}_h f - f\|_{\tilde{H}} \leq \|\mathcal{R}_h f - \mathcal{R}_h g\|_{\tilde{H}} + \|\mathcal{R}_h g - g\|_{\tilde{H}} + \|g - f\|_{\tilde{H}} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,.$$

So $\{R_t\}_t$ is strongly continuous on $\mathcal{L}^2([0,T],H)$.

$\{R_t\}_t$ **is generated by** $(-\frac{\mathbf{d}}{\mathbf{d}t}, D(\frac{\mathbf{d}}{\mathbf{d}t}))$: Denote the generator of $\{R_t\}_t$ by $(R, D(R))$. Again we will first consider an element $g \in C_c^\infty((0,T],H)$. The function $\phi_\xi : [0,T] \to$

H, defined by $\phi_\xi(t) := (R_t g)(\xi) = g(\xi - t)\mathbb{I}_{[t,T]}(\xi)$ for an arbitrary $\xi \in [0,T]$, is continuously differentiable on $[0,\infty)$ and

$$\phi_\xi^{'}(t) = -g^{'}(\xi - t)\mathbb{I}_{[t,T]}(\xi) = -R_t\left(\frac{\mathrm{d}}{\mathrm{d}t}g\right)(\xi)\,.$$

Applying this, we can use the fundamental theorem of calculus to see that

$$R_t g(\xi) - g(\xi) = g(\xi - t)\mathbb{I}_{[t,T]}(\xi) - g(\xi)\mathbb{I}_{[0,T]}(\xi) = -\int_0^t \phi_\xi^{'}(s)\,\mathrm{d}s = -\int_0^t R_s\frac{\mathrm{d}}{\mathrm{d}t}g(\xi)\,\mathrm{d}s\,.$$

However, by Proposition 2.2.1 we know that

$$R_t g(\xi) - g(\xi) = R\int_0^t R_s g(\xi)\,\mathrm{d}s\,,$$

so that

$$R\frac{1}{t}\int_0^t R_s g(\xi)\,\mathrm{d}s = -\frac{1}{t}\int_0^t R_s\frac{\mathrm{d}}{\mathrm{d}t}g(\xi)\,\mathrm{d}s\,. \tag{14}$$

Again by Proposition 2.2.1 we know that $\frac{1}{t}\int_0^t R_s g(\xi)\,\mathrm{d}s \to g(\xi)$ as $t \to 0$, so that the right hand side of Equation 14 converges to $-\frac{\mathrm{d}}{\mathrm{d}t}g(\xi)$ as $t \to 0$. Thus the left hand side of Equation 14 also converges to $-\frac{\mathrm{d}}{\mathrm{d}t}g(\xi)$. But since R is the generator of a semigroup it is also a closed operator. Therefore $-\frac{\mathrm{d}}{\mathrm{d}t}g = Rg$, so that $R = -\frac{\mathrm{d}}{\mathrm{d}t}$ on $C_c^\infty((0,T],H)$. Since $R_t C_c^\infty((0,T],H) \subset C_c^\infty((0,T],H)$ and $C_c^\infty((0,T],H)$ is dense in $\mathcal{L}^2([0,T],H)$, it follows by Proposition 2.2.3 that $C_c^\infty((0,T],H)$ is graph-norm dense in $D(R)$. Therefore

$$D(R) = \overline{C_c^\infty((0,T],H)}^g = D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\,,$$

so that $(-\frac{\mathrm{d}}{\mathrm{d}t}, D(\frac{\mathrm{d}}{\mathrm{d}t}))$ generates R_t. □

We now consider the product semigroup $U_t := \tilde{S}_t R_t$. The two semigroups commute since for $f \in \mathcal{L}^2([0,T],H)$ and a.e. $\xi \in [0,T]$ we have that

$$[\tilde{S}_t R_t f](\xi) = \tilde{S}_t f(\xi - t) = (S_t f)(\xi - t) = R_t(S_t f)(\xi) = [R_t\tilde{S}_t f](\xi)\,.$$

Moreover we can find the generator of U_t using Proposition 2.2.1. Assume $J : D(J) \to H$ to be the generator of U_t. Then for all $f \in D(\frac{\mathrm{d}}{\mathrm{d}t}) \cap D(A)$.

$$\begin{aligned} Jf = \lim_{h\to 0}\frac{U_h f - f}{h} &= \lim_{h\to 0}\frac{\tilde{S}_h\mathcal{R}_h f - \mathcal{R}_h f}{h} + \lim_{h\to 0}\frac{\mathcal{R}_h f - f}{h} \\ &= -R_0\tilde{A}f - \frac{\mathrm{d}}{\mathrm{d}t}f = -\left(\tilde{A} + \frac{\mathrm{d}}{\mathrm{d}t}\right)f\,, \end{aligned}$$

so that the generator of U_t is equal to the negative of the sum operator $\mathcal{B} := (\frac{\mathrm{d}}{\mathrm{d}t} + \tilde{A})$ on $D(\frac{\mathrm{d}}{\mathrm{d}t}) \cap D(\tilde{A})$. Note that the full generator might be defined on a larger set containing $D(\frac{\mathrm{d}}{\mathrm{d}t}) \cap D(\tilde{A})$. The reader is referred to Kirchner and Willems [15] for further technical details.

We want to exploit the relationship between $\mathcal{B}$ and the semigroup U_t to define $\mathcal{B}^\delta$. Assuming $\mathcal{B}$ is invertible, we have that for $\delta \in \mathbb{N}$

$$\mathcal{B}^{-\delta} = \frac{1}{(\delta-1)!} \int_0^T s^{\delta-1} U_s \, \mathrm{d}s \,. \tag{15}$$

We will show this by induction. First assume $\delta = 1$. Then by Proposition 2.2.1 and the fact that $U_T = \tilde{S}_T R_T = 0$ we have that

$$\int_0^T U_s x \, \mathrm{d}s = -\mathcal{B}^{-1}(U_T - I)x = \mathcal{B}^{-1}x$$

We now assume that Equation 15 holds for $\delta \leq n$. Note that since $\int_0^t U_s x \, \mathrm{d}s = \mathcal{B}^{-1}(U_t x - x)$ by Proposition 2.2.1 we have that $U_t = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}^{-1}U_t x$. For $\delta = n+1$ we can then calculate

$$\begin{aligned}
\frac{1}{n!}\int_0^T s^n U_s x \, \mathrm{d}s &= s^n \mathcal{B}^{-1} U_s x\Big|_0^T - \frac{1}{(n-1)!}\int_0^T s^{n-1}\mathcal{B}^{-1}U_s x \, \mathrm{d}s \\
&= \mathcal{B}^{-1}\frac{1}{(n-1)!}\int_0^T s^{n-1} U_s x \, \mathrm{d}s \\
&= \mathcal{B}^{-1}\mathcal{B}^{-(n-1)} = \mathcal{B}^{-n} \,,
\end{aligned}$$

so that Equation 15 holds for all $\delta \in \mathbb{N}$ by induction. Of course this calculation only holds because we have assumed that $\mathcal{B}$ is invertible. However, $\mathcal{B} = \frac{\mathrm{d}}{\mathrm{d}t} + \tilde{A}$ is not invertible. We resolve this issue by choosing the above integral as a "canonical" fractional inverse for $\mathcal{B}$. This is possible since $\|U_t\|_{L(\tilde{H})} \leq \|\tilde{S}_t\|_{L(\tilde{H})} \leq \max_{t\in[0,T]} \|S_t\|_{L(H)}$ so that the Bochner integral in fact converges for all $\delta \in (0,\infty)$ even when $\mathcal{B}$ is not invertible. For a $\delta > 0$ we can therefore define

$$\mathcal{B}^{-\delta} := \frac{1}{\Gamma(\delta)} \int_0^T s^{\delta-1} U_s \, \mathrm{d}s \,. \tag{16}$$

We can further define $\mathcal{B}^\delta := (\mathcal{B}^{-\delta})^{-1}$ on $\mathrm{Im}(B^{-\delta})$ and of course $\mathcal{B}^0 := I$. The former is well-defined since the operator $\mathcal{B}^{-\delta}$ is bounded and thus invertible.

Now take $g \in \tilde{H}$. Then according to Equation 16 we have that for $s \in [0,T]$

$$\begin{aligned}
\mathcal{B}^{-\delta} g(t) &= \frac{1}{\Gamma(\delta)}\int_0^T s^{\delta-1} U_s g(t) \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)}\int_0^T s^{\delta-1} \tilde{S}_s R_s g(t) \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)}\int_0^t s^{\delta-1} S_s g(t-s) \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} S_{t-s} g(s) \, \mathrm{d}s \,,
\end{aligned}$$

where we have done a variable substitution in the last line. This form of $\mathcal{B}^{-\gamma}$ is slightly easier to work with. For the discussion of Equation 13 in the next subsection we will also need to know how to calculate the adjoint $\mathcal{B}^{-\delta*}$. Suppose we have $f, g \in \tilde{H}$. Then

$$
\begin{aligned}
\left\langle \mathcal{B}^{-\delta} g, f \right\rangle_{\tilde{H}} &= \frac{1}{\Gamma(\delta)} \left\langle \int_0^T \mathbb{I}_{(0,\cdot)}(s)(\cdot - s)^{\delta-1} \tilde{S}_{\cdot - s} g(s)\, \mathrm{d}s, f \right\rangle_{\tilde{H}} \\
&= \frac{1}{\Gamma(\delta)} \int_0^T \int_0^T \left\langle \mathbb{I}_{(0,t)}(s)(t-s)^{\delta-1} \tilde{S}_{t-s} g(s), f(t) \right\rangle_H \mathrm{d}s\, \mathrm{d}t \\
&= \frac{1}{\Gamma(\delta)} \int_0^T \int_0^T \left\langle g(s), \mathbb{I}_{(s,T)}(s)(t-s)^{\delta-1} \tilde{S}^*_{t-s} f(t) \right\rangle_H \mathrm{d}t\, \mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)} \int_0^T \left\langle g(s), \int_s^T (t-s)^{\delta-1} \tilde{S}^*_{t-s} f(t)\, \mathrm{d}t \right\rangle_H \mathrm{d}s \\
&= \left\langle g, \frac{1}{\Gamma(\delta)} \int_\cdot^T (t-\cdot)^{\delta-1} \tilde{S}^*_{t-\cdot} f(t)\, \mathrm{d}t \right\rangle_{\tilde{H}} .
\end{aligned}
$$

It follows that for $g \in \tilde{H}$ we have that

$$
\mathcal{B}^{-\delta *} g(s) = \int_s^T (t-s)^{\delta-1} \tilde{S}^*_{t-s} f(t)\, \mathrm{d}t .
$$

We summarize these two facts in a lemma.

Lemma 4.1.3. *Assume that $g \in \tilde{H}$. Then*

$$
\mathcal{B}^{-\delta} g(s) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S_{t-s} g(s)\, ds ,
$$

and

$$
\mathcal{B}^{-\delta *} g(s) = \frac{1}{\Gamma(\delta)} \int_s^T (t-s)^{\delta-1} \tilde{S}^*_{t-s} f(t)\, dt .
$$

4.2 Existence and uniqueness of weak solutions

The weak solution of Equation 13 is defined similarly to the case of Equation 8.

Definition 4.2.1. ***Weak solution***. *Let W be a $\mathcal{F}_t$-adapted Q-Wiener process on a Hilbert space H. A random process $X \in \mathcal{L}^2(\Omega, \tilde{H})$ is called a weak solution to Equation 13 if $X(t)$ is an $\mathcal{F}_t$-measurable random variable for all $t \in [0,T]$, X is mean-square continuous in time, and for all $x \in D(\mathcal{B}^{\delta *}) \subset \tilde{H}$*

$$
\left\langle X, \mathcal{B}^{\delta *} x \right\rangle_{\tilde{H}} = \int_0^T \langle \mathrm{d}W(t), x(t) \rangle_H \ \mathrm{d}t \ \ a.s.. \tag{17}
$$

Inspired by Lemma 4.1.3 we select as our candidate solution the stochastic convolution X defined by

$$
X(t) := \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S_{t-s}\, \mathrm{d}W(s) , \tag{18}
$$

for $t \in [0,T]$. We will first show a condition for the existence of this stochastic convolution, and then show that under this condition it is also a weak solution to Equation 13.

Theorem 4.2.1. *Let H be a seperable Hilbert space. If*

$$\int_0^T \|t^{\delta-1} S_t Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}t = C < \infty \,,$$

then the integral in Equation 18 belongs to $\mathcal{L}^2(\Omega, \mathcal{L}^2([0,T], H))$ and is mean-square continuous in time.

Proof. By doing the variable substitution $t - s = h$ and applying the Itô isometry we get

$$\begin{aligned}
\|X\|^2_{\mathcal{L}^2(\Omega,\mathcal{L}^2([0,T],H))} &= \mathbb{E}\left[\int_0^T \left\| \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S_{t-s} \, \mathrm{d}W(s) \right\|_H^2 \mathrm{d}t \right] \\
&= \frac{1}{\Gamma(\delta)^2} \int_0^T \int_0^t \|(t-s)^{\delta-1} S_{t-s} Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{\Gamma(\delta)^2} \int_0^T \int_0^t \|h^{\delta-1} S_h Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}h \, \mathrm{d}t \leq \frac{CT}{\Gamma(\delta)^2} < \infty \,,
\end{aligned}$$

so that $X \in \mathcal{L}^2(\Omega, \mathcal{L}^2([0,T], H))$. The mean-square continuity can be proven by the same technique applied in the proof of Proposition 4.3.2.

□

Theorem 4.2.2. ***Existence and uniqueness***. *Assume that*

$$\int_0^T \|t^{\delta-1} S_t Q^{\frac{1}{2}}\|^2_{L_2(H)} \, \mathrm{d}t < \infty \,.$$

Then the random process $X \in \mathcal{L}^2(\Omega, \tilde{H})$ defined by

$$X(t) := \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S_{t-s} \mathrm{d}W(s) \,,$$

is a weak solution to Equation 13 for $x_0 = 0$. Additionally, for any other weak solution Y, we have that for almost all $t \in [0,T]$, $X(t) = Y(t)$ almost surely.

Proof. We first prove existence, then uniqueness.

Existence: First, according to Theorem 4.2.1, $X \in \mathcal{L}^2(\Omega, \tilde{H})$ and X is mean-square continuous. Fix $x \in D(\mathcal{B}^{\delta *}) \subset \tilde{H} = \mathcal{L}^2([0,T], H)$. Then

$$\langle X, \mathcal{B}^{\delta *} x \rangle_{\tilde{H}} = \frac{1}{\Gamma(\delta)} \left\langle \int_0^{\cdot} (\cdot - s)^{\delta-1} S_{\cdot - s} \mathrm{d}W(s), \mathcal{B}^{\delta *} x \right\rangle_{\tilde{H}}$$

$$= \frac{1}{\Gamma(\delta)} \int_0^T \int_0^T \mathbb{I}_{(0,t)}(s) (t-s)^{\delta-1} \left\langle S_{t-s} \mathrm{d}W(s), \mathcal{B}^{\delta *} x(t) \right\rangle_H \mathrm{d}t = (*) \,.$$

Defining $\ell_\nu = \langle \cdot, \nu \rangle_H$ we can set $\phi(t,s) := \mathbb{I}_{(0,t)}(s)(t-s)^{\delta-1} \ell_{\mathcal{B}^{\delta *} x(t)} S_{t-s}$. Then

$$(*) = \frac{1}{\Gamma(\delta)} \int_0^T \int_0^T \phi(t,s) \, \mathrm{d}W(s) \, \mathrm{d}t \,.$$

We now want to apply the stochastic Fubini theorem from Theorem 2.1.3 to this double integral. To this end we must show that $\int_0^T \mathbb{E}\left[\int_0^T \|\phi(t,s)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\right]^{\frac{1}{2}}\,\mathrm{d}t < \infty$. Applying the Cauchy-Bunyakovsky-Schwarz inequality we see that

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\int_0^T \|\phi(t,s)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\right]^{\frac{1}{2}}\,\mathrm{d}t \\
&= \int_0^T \left(\int_0^T \|\phi(t,s)Q^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\right)^{\frac{1}{2}}\,\mathrm{d}t \\
&\le \int_0^T \left(\int_0^t \|(t-s)^{\gamma-1}S_{t-s}Q^{\frac{1}{2}}\|^2_{L_2(H)}\|\ell_{\mathcal{B}^{\delta*}x(t)}\|^2_{L(H)}\,\mathrm{d}s\right)^{\frac{1}{2}}\,\mathrm{d}t \\
&= \int_0^T \left(\int_0^t \|s^{\delta-1}S_sQ^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\right)^{\frac{1}{2}}\|\mathcal{B}^{\delta*}x(t)\|_H\,\mathrm{d}t \\
&\le \left(\int_0^T\int_0^t \|s^{\delta-1}S_sQ^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|\mathcal{B}^{\delta*}x(t)\|^2_H\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le T^{\frac{1}{2}}\|B^{\delta*}x\|_{\tilde{H}}\left(\int_0^t \|s^{\delta-1}S_sQ^{\frac{1}{2}}\|^2_{L_2(H)}\,\mathrm{d}s\right)^{\frac{1}{2}} < \infty\,.
\end{aligned}$$

We can therefore apply the stochastic Fubini theorem and Lemma 4.1.3 to see that almost surely

$$\begin{aligned}
(*) &= \frac{1}{\Gamma(\delta)}\int_0^T\int_0^T \phi(t,s)\,\mathrm{d}t\,\mathrm{d}W(s) \\
&= \frac{1}{\Gamma(\delta)}\int_0^T\int_0^T \mathbb{I}_{(0,t)}(s)(t-s)^{\delta-1}\ell_{\mathcal{B}^{\delta*}x(t)}S_{t-s}\,\mathrm{d}t\,\mathrm{d}W(s) \\
&= \frac{1}{\Gamma(\delta)}\int_0^T\int_0^T \mathbb{I}_{(s,T)}(t)(t-s)^{\delta-1}\ell_{\mathcal{B}^{\delta*}x(t)}S_{t-s}\,\mathrm{d}t\,\mathrm{d}W(s) \\
&= \int_0^T \left\langle \mathrm{d}W(s), \frac{1}{\Gamma(\delta)}\int_s^T (t-s)^{\delta-1}S^*_{t-s}\mathcal{B}^{\delta*}x(t)\,\mathrm{d}t\right\rangle_H \\
&= \int_0^T \left\langle \mathrm{d}W(s), \mathcal{B}^{-\delta*}\mathcal{B}^{\delta*}x\right\rangle_H = \int_0^T \langle \mathrm{d}W(s), x\rangle_H\,.
\end{aligned}$$

In addition $X(t)$ is $\mathcal{F}_t$-measurable, since the stochastic convolution $\int_0^t (t-s)^{\delta-1}S_{t-s}\,\mathrm{d}W(s)$ is $\{\mathcal{F}_t\}_t$-adapted. Hence by Definition 4.2.1 X is a weak solution to Equation 13.

Uniqueness: Now suppose Y is a weak solution to Equation 13. Take an $x \in \mathcal{L}^2([0,T],H)$. Then by Equation 17 almost surely

$$\langle Y, x\rangle_{\tilde{H}} = \int_0^T \left\langle dW(s), \mathcal{B}^{-\delta*}x\right\rangle_H\,.$$

However, by the argument above

$$\langle X, x\rangle_{\tilde{H}} = \left\langle X, \mathcal{B}^{\delta*}\mathcal{B}^{-\delta*}x\right\rangle_{\tilde{H}} = \int_0^T \left\langle dW(s), \mathcal{B}^{-\delta*}x\right\rangle_H\,,$$

so that $\langle Y, x\rangle_{\tilde{H}} = \langle X, x\rangle_{\tilde{H}}$ almost surely for all $x \in \mathcal{L}^2([0,T],H)$. We will now need to show that $\tilde{H} = \mathcal{L}^2([0,T],H)$ is separable. The space of H-valued simple functions on $[0,T]$ is dense in $\mathcal{L}^2([0,T],H)$ [13]. Since H and $\mathcal{L}^2([0,T],\mathbb{R})$ are both seperable it is then easy to show that $\mathcal{L}^2([0,T],H)$ is also seperable. By this seperability we can then find an orthonormal basis $\{f_k\}_k \subset \tilde{H}$ for $\tilde{H}$. By the sub-additivity of the probability measure P we can then calculate

$$\begin{aligned} P(X \neq Y \text{ in } \tilde{H}) &= P\left(\sum_k |\langle X, f_k\rangle_{\tilde{H}} - \langle Y, f_k\rangle_{\tilde{H}}| f_k \neq 0\right) \\ &= P\left(\bigcup_k \{\langle X, f_k\rangle_{\tilde{H}} \neq \langle Y, f_k\rangle_{\tilde{H}}\}\right) \\ &\leq \sum_k P(\langle X, f_k\rangle_{\tilde{H}} \neq \langle Y, f_k\rangle_{\tilde{H}}) = \sum_k 0 = 0\,. \end{aligned}$$

Thus $P(Y = X \text{ in } \tilde{H}) = 1$, implying that $\mathbb{E}\left[\|Y - X\|_{\tilde{H}}^2\right] = 0$ as well. Thus $Y = X$ in $\mathcal{L}^2(\Omega, \tilde{H})$. By Fubini's theorem we also have that $Y = X$ as elements in $\mathcal{L}^2([0,T], \mathcal{L}^2(\Omega, H))$. By the mean-square continuity of Y and X it thus follows that for all $t \in [0,T]$ we have $\mathbb{E}\left[\|X(t) - Y(t)\|_H^2\right] = 0$ and thus $X(t) = Y(t)$ almost surely.

□

4.3 Regularity of solutions

We will now analyze the spatial and temporal regularity of the weak solutions to Equation 13 with $Q = c(\kappa^2 - \Delta)^{-\beta}$. We also return to considering only $A = (\iota^2 - \Delta)^\gamma$ as in Equation 13, instead of a general operator A. Recall that we denoted the eigenvalues of A by $\mu_k = (\iota^2 + \lambda_k)^\gamma$ in Section 3 and that μ_k satisfies the bounds $\lambda_k^\gamma \leq \mu_k \leq C\lambda_k^\gamma$.

We first investigate the spatial regularity. Our approach is similar to that in the previous section, but this time we simultaneously check existence and smoothness by checking if Equation 13 has solutions in $\dot{H}^s$. This is the case when

$$\int_0^T \|t^{\delta-1} S_t Q^{\frac{1}{2}}\|_{L_2(\dot{H}^s)}^2 \,\mathrm{d}t < \infty.$$

We estimate

$$\begin{aligned} \int_0^T \|t^{\delta-1} S_t Q^{\frac{1}{2}}\|_{L_2(\dot{H}^s)}^2 \,\mathrm{d}t &= \int_0^T \sum_k \lambda_k^s t^{2\delta-2} e^{-2\mu_k t} c(\kappa^2 + \lambda_k)^{-\beta} \,\mathrm{d}t \\ &= \int_0^{2\lambda_k T} \sum_k \lambda_k^s \left(\frac{t}{2\mu_k}\right)^{2\delta-2} e^{-t} c(\kappa^2 + \lambda_k)^{-\beta} \frac{\mathrm{d}t}{2\mu_k} \\ &\leq C \sum_k \lambda_k^{s-\beta+(1-2\delta)\gamma} \left(\int_0^\infty t^{2\delta-2} \exp -t \,\mathrm{d}t\right) \\ &= C\Gamma(2\delta - 2) \sum_k \lambda_k^{s-\beta+(1-2\delta)\gamma} \\ &\leq C\Gamma(2\delta - 2) \sum_k k^{\frac{2}{d}(s-\beta+(1-2\delta)\gamma)}\,, \end{aligned}$$

which converges if $\frac{2}{d}(s - \beta + (1-2\delta)\gamma)$ or $s < 2\delta\gamma - \gamma + \beta - \frac{d}{2}$. We must also require that $\delta > \frac{1}{2}$, so that $\Gamma(2\delta - 2) < \infty$. We summarize this discussion in a proposition.

Proposition 4.3.1. ***Spatial regularity of*** X***.*** *Let* X *be a weak solution to Equation 13, with* $Q = c(\kappa^2 - \Delta)^{-\beta}$ *and* $\delta > \frac{1}{2}$*. Assume also that* s *satisfies* $s < 2\delta\gamma - \gamma + \beta - \frac{d}{2}$*. Then* $X \in \mathcal{L}^2(\Omega, \mathcal{L}^2([0,T], \dot{H}^s))$*.*

Proof. The proposition follows from the preceding discussion and Theorem 4.2.1. □

Next we investigate the temporal regularity of the solutions. We first consider the Hölder-continuity of X.

Proposition 4.3.2. ***Hölder-continuity of*** X***.*** *Let* X *be a weak solution to Equation 13, with* $Q = c(\kappa^2 - \Delta)^{-\beta}$ *and* $\delta > \frac{1}{2}$*. Assume that* $\nu \in [0,1)$ *satisfies* $\nu < \delta - \frac{1}{2} + \frac{\beta}{2\gamma} - \frac{d}{4\gamma}$ *and* $\nu \leq \delta - \frac{1}{2}$*. Then*

$$\|X(t) - X(s)\|_{\mathcal{L}^2(\Omega,H)} \leq C|t-s|^{\nu}.$$

Proof. Our proof will be similar to that of temporal smoothness for the fractional heat equation. Assuming $s < t$ we consider

$$\begin{aligned}
\|X(t) - X(s)\|^2_{\mathcal{L}^2(\Omega,H)} &= \mathbb{E}\left[\|X(t) - X(s)\|_H^2\right] \\
&\propto \mathbb{E}\left[\left\|\int_0^t (t-\xi)^{\delta-1} S_{t-\xi} - (s-\xi)^{\delta-1} S_{s-\xi} \mathbb{I}_{(0,s)}(\xi)\, \mathrm{d}W(\xi)\right\|_H^2\right] \\
&= \int_0^t \|\big((t-\xi)^{\delta-1} S_{t-\xi} - (s-\xi)^{\delta-1} S_{s-\xi} \mathbb{I}_{(0,s)}(\xi)\big) Q^{\frac{1}{2}}\|^2_{L_2(H)}\, \mathrm{d}\xi
\end{aligned}$$

We first consider this integral from 0 to s. Doing the variable substitution $\phi = s - \xi$ we then get

$$\begin{aligned}
&\int_0^s \sum_k \big((t-\xi)^{\delta-1} e^{-\mu_k(t-\xi)} - (s-\xi)^{\delta-1} e^{-\mu_k(s-\xi)}\big)^2 c(\kappa^2 + \lambda_k)^{-\beta}\, \mathrm{d}\xi \\
=& \sum_k c(\kappa^2 + \lambda_k)^{-\beta} \int_0^s \big((t-\xi)^{\delta-1} e^{-\mu_k(t-\xi)} - (s-\xi)^{\delta-1} e^{-\mu_k(s-\xi)}\big)^2\, \mathrm{d}\xi = (*)
\end{aligned}$$

We now define the function $f(x) = x^{\delta-1} e^{-\mu_k x}$ and note that by the fundamental theorem of calculus

$$\begin{aligned}
&|f(t-\xi) - f(s-\xi)| \\
=& \left|\int_{s-\xi}^{t-\xi} f'(\phi)\, \mathrm{d}\phi\right| \\
=& \left|\int_0^{t-s} f'(\phi + \xi - s)\, \mathrm{d}\phi\right| \\
=& \left|\int_0^{t-s} (\delta-1)(\phi - \xi - s)^{\delta-2} e^{-\mu_k(\phi-\xi-s)} - \mu_k(\phi - \xi - s)^{\delta-1} e^{-\mu_k(\phi-\xi-s)}\, \mathrm{d}\phi\right| \\
\leq& \int_0^{t-s} |(\delta-1)(\phi-\xi-s)^{\delta-2} e^{-\mu_k(\phi-\xi-s)}|\, \mathrm{d}\phi + \int_0^{t-s} |\mu_k(\phi-\xi-s)^{\delta-1} e^{-\mu_k(\phi-\xi-s)}|\, \mathrm{d}\phi
\end{aligned}$$

We denote $(1) = |(\delta-1)(\phi-\xi-s)^{\delta-2}e^{-\mu_k(\phi-\xi-s)}|$ and $(2) = |\mu_k(\phi-\xi-s)^{\delta-1}e^{-\mu_k(\phi-\xi-s)}|$. Now note that for $\alpha \geq 0$ the function $x^\alpha e^{-\mu_k x}$ attains its maximum at the point $x = \frac{\alpha}{\mu_k}$. Thus

$$e^{-\mu_k x} = x^{-\alpha}x^\alpha e^{-\mu_k x} \leq x^{-\alpha}\left(\frac{\alpha}{\mu_k}\right)^\alpha e^{-\alpha} \leq Cx^{-\alpha}\mu_k^{-\alpha}\,.$$

It follows that $(1) \leq C(\phi-\xi-s)^{\delta-2-\alpha}\mu_k^{-\alpha}$ and $(2) \leq C(\phi-\xi-s)^{\delta-1-\alpha}\mu_k^{1-\alpha}$. Since $\nu \leq \delta - \frac{1}{2}$ we can select $\alpha = \delta - \frac{1}{2} - \nu$ and calculate

$$\begin{aligned}
\left\|\int_0^{t-s}(1)\,\mathrm{d}\phi\right\|_{\mathcal{L}^2((0,s],\mathbb{R})} &\leq \int_0^{t-s}\|(1)\|_{\mathcal{L}^2((0,s],\mathbb{R})}\,\mathrm{d}\phi \\
&\leq \int_0^{t-s}\left(\int_0^s C^2\mu_k^{-2\alpha}(\phi-\xi-s)^{2\delta-2-2\alpha}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&= C\mu_k^{\nu+\frac{1}{2}-\delta}\int_0^{t-s}\left(\int_0^s(\phi-\xi-s)^{2\nu-3}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&= C\mu_k^{\nu+\frac{1}{2}-\delta}\int_0^{t-s}\left(\int_\phi^{\phi+s}u^{2\nu-3}\,\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&\leq C\mu_k^{\nu+\frac{1}{2}-\delta}\int_0^{t-s}\left(\int_\phi^{\infty}u^{2\nu-3}\,\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&= C\mu_k^{\nu+\frac{1}{2}-\delta}\int_0^{t-s}\phi^{\nu-1}\,\mathrm{d}\phi \\
&= C\mu_k^{\nu+\frac{1}{2}-\delta}(t-s)^\nu\,.
\end{aligned}$$

The integration of $u^{2\nu-3}$ from ϕ to ∞ is justified by $\nu < 1$ and the integration of $\phi^{\nu-1}$ from 0 to $t-s$ is justified by $\nu \geq 0$. Similarly we can select $\alpha = \delta + \frac{1}{2} - \nu$ since then $\delta + \frac{1}{2} - \nu \geq \delta - \frac{1}{2} - \nu \geq 0$. We can then calculate

$$\begin{aligned}
\left\|\int_0^{t-s}(2)\,\mathrm{d}\phi\right\|_{\mathcal{L}^2((0,s],\mathbb{R})} &\leq \int_0^{t-s}\|(2)\|_{\mathcal{L}^2((0,s],\mathbb{R})}\,\mathrm{d}\phi \\
&\leq \int_0^{t-s}\left(\int_0^s C^2\mu_k^{2-2\alpha}(\phi-\xi-s)^{2\delta-4-2\alpha}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&= C\mu_k^{\nu+\frac{1}{2}-\delta}\int_0^{t-s}\left(\int_0^s(\phi-\xi-s)^{2\nu-3}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}\phi \\
&\leq C\mu_k^{\nu+\frac{1}{2}-\delta}(t-s)^\nu\,.
\end{aligned}$$

Further

$$
\begin{aligned}
&\int_0^s \left((t-\xi)^{\delta-1}e^{-\mu_k(t-\xi)} - (s-\xi)^{\delta-1}e^{-\mu_k(s-\xi)}\right)^2 \mathrm{d}\xi \\
&= \int_0^s |f(t-\xi) - f(s-\xi)|^2 \,\mathrm{d}\xi \\
&\le \int_0^s \left(\int_0^{t-s} |(1)| + |(2)| \,\mathrm{d}\phi\right)^2 \mathrm{d}\xi \\
&= \left\| \int_0^{t-s} |(1)| + |(2)| \,\mathrm{d}\phi \right\|_{\mathcal{L}^2((0,s],\mathbb{R})} \\
&\le 2\left\| \int_0^{t-s} |(1)| \,\mathrm{d}\phi \right\|^2_{\mathcal{L}^2((0,s],\mathbb{R})} + 2\left\| \int_0^{t-s} |(2)| \,\mathrm{d}\phi \right\|^2_{\mathcal{L}^2((0,s],\mathbb{R})} \le C\mu_k^{2\nu+1-2\delta}(t-s)^{2\nu} .
\end{aligned}
$$

Applying the Weyl estimates we thus get

$$
\begin{aligned}
(*) &\le C(t-s)^{2\nu} \sum_k (\kappa^2 + \lambda_k)^{-\beta} \mu_k^{2\nu+1-2\delta} \\
&\le C(t-s)^{2\nu} \sum_k \lambda_k^{(2\nu+1-2\delta)\gamma-\beta} \\
&\le C(t-s)^{2\nu} \sum_k k^{\frac{2}{d}((2\nu+1-2\delta)\gamma-\beta)} ,
\end{aligned}
$$

which converges if $\frac{2}{d}((2\nu+1-2\delta)\gamma-\beta) < -1$ or $\nu < \delta - \frac{1}{2} + \frac{\beta}{2\gamma} - \frac{d}{4\gamma}$.
We now consider the integral from s to t. This time we do the variable substitution $2\mu_k(t-\xi) = \phi$ and note that $\exp(-\phi) \le 1$ to get

$$
\begin{aligned}
&\int_s^t \sum_k (t-\xi)^{2\delta-2} e^{-2\mu_k(t-\xi)} c(\kappa^2+\lambda_k)^{-\beta} \,\mathrm{d}\xi \\
=&C \int_0^{2\mu_k(t-s)} \sum_k \mu_k^{1-2\delta} \phi^{2\delta-2} e^{-\phi} c(\kappa^2+\lambda_k)^{-\beta} \,\mathrm{d}\phi \\
\le&C \sum_k \lambda_k^{(1-2\delta)\gamma-\beta} \int_0^{2\mu_k(t-s)} \phi^{2\nu-1} \,\mathrm{d}\phi \\
\le&C \sum_k \lambda_k^{(1-2\delta)\gamma-\beta} (\mu_k(t-s))^{2\nu} \le C(t-s)^{2\nu} \sum_k \lambda_k^{(2\nu+1-2\delta)\gamma-\beta} ,
\end{aligned}
$$

which gives the same convergence requirement as $(*)$, namely $\nu < \delta - \frac{1}{2} + \frac{\beta}{2\gamma} - \frac{d}{4\gamma}$. Note that to get $\phi^{2\delta-2}\exp(-\phi) \le \phi^{2\nu-1}$, we must also require $\nu \le \delta - \frac{1}{2}$. It follows that

$$
\|X(t) - X(s)\|_{\mathcal{L}^2(\Omega,H)} \le C|t-s|^{\nu} ,
$$

as desired.

□

Note that the temporal regularity $\delta - \frac{1}{2} + \frac{\beta}{2\gamma} - \frac{d}{4\gamma}$ and the spatial regularity $2\delta\gamma - \gamma + \beta - \frac{d}{2}$ differ only by a factor of 2γ. The significance of this will be even clearer in the next proposition, where we consider the Hölder-regularity of the n-th derivative of X in the fractional order space $\dot{H}^s$.

Proposition 4.3.3. *Let X be a weak solution to Equation 13, with $Q = c(\kappa^2 - \Delta)^{-\beta}$ and $\delta > \frac{1}{2}$. Assume that $\nu \in [0,1)$, $n \in \mathbb{N}$ and $s \in [0,\infty)$ satisfies $2\gamma(n+\nu)+s < (2\delta-1)\gamma+\beta-\frac{d}{2}$ and $n+\nu < \delta-\frac{1}{2}$. Then $\frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) \in \mathcal{L}^2(\Omega, \dot{H}^s)$ and*

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) - \frac{\mathrm{d}^n}{\mathrm{d}s^n}X(s) \right\|_{\mathcal{L}^2(\Omega,\dot{H}^s)} \leq C|t-s|^{\nu}.$$

Proof. The n-th derivative of X at t can be expressed as

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-\xi)^{\delta-1} S_{t-\xi}\,\mathrm{d}W(\xi) \\
&= \int_0^t \sum_{j=0}^{n} C_j (t-\xi)^{\delta-1-j}(\iota^2-\Delta)^{\gamma(n-j)} S_{t-\xi}\,\mathrm{d}W(\xi).
\end{aligned}$$

We now apply the Itô isometry and exchanging the order of summation to calculate

$$\begin{aligned}
\left\| \frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) \right\|^2_{\mathcal{L}^2(\Omega,\dot{H}^s)} &= \int_0^t \left\| \sum_{j=0}^{n} C_j (t-\xi)^{\delta-1-j}(\iota^2-\Delta)^{\gamma(n-j)} S_{t-\xi} Q^{\frac{1}{2}} \right\|^2_{L_2(\dot{H}^s)} \mathrm{d}\xi \\
&= \int_0^t (t-\xi)^{2\delta-2-2j} \sum_k \sum_j C_j^2 \lambda_k^s \mu_k^{2(n-j)} e^{\mu_k(t-\xi)} (\kappa^2+\lambda_k)^{-\beta}\,\mathrm{d}\xi \\
&= \sum_{j=0}^{n} \sum_k C_j^2 \mu_k^{2(n-j)+1-2\delta+2j} \lambda_k^s (\kappa^2+\lambda_k)^{-\beta} \int_0^t u^{2\delta-2-2j} e^{-u}\,\mathrm{d}\xi \\
&\leq C \sum_{j=0}^{n} C_j^2 \sum_k \lambda_k^{\gamma(2(n-j)+1-2\delta+2j)+s-\beta} \int_0^\infty u^{2\delta-2-2j} e^{-u}\,\mathrm{d}\xi \\
&\leq C \sum_{j=0}^{n} C_j^2 \Gamma(2\delta-2-2j) \sum_k k^{\frac{2}{d}(\gamma(2(n-j)+1-2\delta+2j)+s-\beta)},
\end{aligned}$$

which converges when $\frac{2}{d}(\gamma(2(n-j)+1-2\delta+2j)+s-\beta) < -1$ or $2\gamma n+s < \beta+(2\delta-1)\gamma-\frac{d}{2}$. The assumption $n < \delta-\frac{1}{2}$ is also needed to assure that $\Gamma(2\delta-2-2j) < \infty$ for all $j = 1,...,n$. Thus $\frac{\mathrm{d}^n}{\mathrm{d}t^n}X \in \mathcal{L}^2(\Omega, \dot{H}^s)$.

We now prove that the derivative satisfies the Hölder-bounds. The expected difference $\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) - \frac{\mathrm{d}^n}{\mathrm{d}s^n}X(s)\|_{\mathcal{L}^2(\Omega,H)}$ can then be split into $n+1$ parts using the triangle inequality.

$$\begin{aligned}
\left\| \frac{\mathrm{d}^n}{\mathrm{d}t^n}X(t) - \frac{\mathrm{d}^n}{\mathrm{d}s^n}X(s) \right\|_{\mathcal{L}^2(\Omega,\dot{H}^s)} &\leq \sum_{j=0}^{n}(j) \\
&= \sum_{j=0}^{n} \Bigg\| \int_0^t C_j (t-\xi)^{\delta-1-j}(\iota^2-\Delta)^{\gamma(n-j)} S_{t-\xi}\, dW(\xi) \\
&\quad - \int_0^s C_j (s-\xi)^{\delta-1-j}(\iota^2-\Delta)^{\gamma(n-j)} S_{s-\xi}\, dW(\xi) \Bigg\|_{\mathcal{L}^2(\Omega,\dot{H}^s)}.
\end{aligned}$$

For each (j) we can repeat the argument from Proposition 4.3.2 with only minor modifications to find that

$$(j) \leq K|t-s|^{\nu} \sum_{k} \lambda_k^{(2\nu+1+2n-2j-2(\delta-j))\gamma+s-\beta},$$

which for $\nu \in [0,1)$ converges if $2\gamma(n+\nu)+s < \beta+(2\delta-1)\gamma-\frac{d}{2}$ and $n+\nu \leq \delta-\frac{1}{2}$. □

Just like in Section 3, we can intuitively think of $\beta+(2\delta-1)\gamma-\frac{d}{2}$ as the "total regularity" of the solution X. We then have the same relationship between temporal and spatial regularity that we had for solutions to the space fractional heat equation in Section 3. Every time we take a spatial derivative we reduces the total regularity by 1 and every time we take a temporal derivative we reduce the total regularity by 2γ. We also have the restriction that we can never take more than $\delta-\frac{1}{2}$ temporal derivatives. This is analogous to the restriction $\nu < \frac{1}{2}$ for the Hölder coefficient of X that we have for the space fractional heat equation we discussed in Section 3.

4.4 Covariance properties

In this subsection we discuss the properties of the asymptotic space-time covariance function of the space-time fractional heat equation. The covariance properties of Equation 13 is discussed by Kirchner and Willems in [15], but we go a step further and establish asymptotic bounds for the pointwise covariance in space and time, similarly to what we did in Section 3.3. We do this because we are interested in seeing how and if the parameters β, γ, δ, ι and κ influence the decay in correlation. Like Section 3.3, the analysis in this subsection is to the authors knowledge somewhat novel and is therefore more experimental and less rigorous than that found in other parts of this thesis.

Our first proposition gives us a somewhat concrete representation of the asymptotic space-time covariance function for the space-time fractional heat equation.

Proposition 4.4.1. *Let X be a solution to Equation 13 on $\mathcal{L}^2(\mathcal{D})$ with $Q = c(\kappa^2-\Delta)^{-\beta}$. For $h \geq 0$ define the (possibly unbounded) operator Ξ_h on the eigenvectors of Δ by*

$$\Xi_h e_k = \frac{2^{2-2\delta}}{\Gamma(\delta)^2} \int_0^{\infty} (\xi + 2\mu_k h)^{\delta-1} \xi^{\delta-1} e^{-y} \, \mathrm{d}\xi \, e_k .$$

Then the asymptotic space-time covariance function of X is given by

$$r(h) = \frac{c}{2} \Xi_h S_h (\iota^2-\Delta)^{(1-2\delta)\gamma} (\kappa^2-\Delta)^{-\beta} .$$

Proof. As in Section 3.3, we can use Proposition 2.1.2 to calculate that for $u, v \in H$ we have

$$\begin{aligned} \langle r(t,s)u, v\rangle &= \mathbb{E}\left[\langle X(t), u\rangle_H \langle X(s), v\rangle_H\right] \\ &= \left\langle \frac{1}{\Gamma(\delta)^2} \int_0^{\min(t,s)} (t-\xi)^{\delta-1}(s-\xi)^{\delta-1} S_{t-\xi} Q S_{s-\xi} \, \mathrm{d}\xi u, v \right\rangle_H , \end{aligned}$$

so that $r(t,s) = \frac{1}{\Gamma(\delta)^2}\int_0^{\min(t,s)}(t-\xi)^{\delta-1}(s-\xi)^{\delta-1}S_{t-\xi}QS_{s-\xi}\,\mathrm{d}\xi$. To calculate the asymptotic space-time covariance function we can then take the limit as $t\to\infty$.

$$\begin{aligned} r(h) = \lim_{t\to\infty} r(t+h,t) &= \frac{1}{\Gamma(\delta)^2}\int_0^\infty (t+h-\xi)^{\delta-1}(t-\xi)^{\delta-1}S_{t+h-\xi}QS_{t-\xi}\,d\xi \\ &= \frac{1}{\Gamma(\delta)^2}\int_0^\infty (\xi+h)^{\delta-1}(\xi)^{\delta-1}S_{\xi+h}QS_\xi\,d\xi\,, \end{aligned}$$

where we have performed a variable substitution in the last line. For an eigenvector e_k of $\mathcal{D}$ we can then do another variable substitution and calculate

$$\begin{aligned} r(h)e_k &= \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}\int_0^\infty (\xi+h)^{\delta-1}\xi^{\delta-1}e^{-\mu_k(\xi+h)}e^{-\mu_k\xi}\,\mathrm{d}\xi e_k\,, \\ &= \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}e^{-\mu_k h}\int_0^\infty (\xi+h)^{\delta-1}\xi^{\delta-1}e^{-\mu_k\xi}e^{-\mu_k\xi}\,\mathrm{d}\xi e_k\,, \\ &= \frac{c}{2}e^{-\mu_k h}\left(\frac{2^{2-2\delta}}{\Gamma(\delta)^2}\int_0^\infty (\xi+2\mu_k h)^{\delta-1}\xi^{\delta-1}e^{-\xi}\,\mathrm{d}\xi\right)\mu_k^{1-2\delta}(\kappa^2+\lambda_k)^{-\beta}\,e_k \\ &= \frac{c}{2}\Xi_h S_h(\iota^2-\Delta)^{(1-2\delta)\gamma}(\kappa^2-\Delta)^{-\beta}e_k\,. \end{aligned}$$

It follows that

$$r(h) = \frac{c}{2}\Xi_h S_h(\iota^2-\Delta)^{(1-2\delta)\gamma}(\kappa^2-\Delta)^{-\beta}\,.$$

□

The integral operator Ξ_h makes $r(h)$ somewhat difficult to interpret. Things are simpler if we consider only in the asymptotic covariance operator; i.e. the case $h=0$. We do so in the following proposition. This proposition can also be found in Kirchner and Willems [15].

Proposition 4.4.2. *Let X be a solution to Equation 13 on $\mathcal{L}^2(\mathcal{D})$ with $Q = c(\kappa^2-\Delta)^{-\beta}$. Then the asymptotic covariance operator of $X(t)$ is given by*

$$r(0) = \frac{c}{2}\frac{\Gamma\left(\delta-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\delta)}(\iota^2-\Delta)^{(1-2\delta)\gamma}(\kappa^2-\Delta)^{-\beta}\,.$$

Proof. By Proposition 4.4.1 we know that

$$r(0) = \frac{c}{2}\Xi_0 S_0(\iota^2-\Delta)^{(1-2\delta)\gamma}(\kappa^2-\Delta)^{-\beta}\,.$$

We already know that $S_0 = I$. We need to calculate Ξ_0. For an eigenvector e_k we have that

$$\Xi_0 e_k = \frac{2^{2-2\delta}}{\Gamma(\delta)^2}\int_0^\infty \xi^{2\delta-2}e^{-\xi}\,\mathrm{d}\xi\,e_k = \frac{2^{2-2\delta}}{\Gamma(\delta)^2}\Gamma(2\delta-1)e_k\,.$$

We can then use Legendre's duplication formula to see that

$$\Gamma(2\delta-1) = \Gamma\left(2\left(\delta-\frac{1}{2}\right)\right) = \frac{2^{2\delta-2}\Gamma(\delta)\Gamma(\delta-\frac{1}{2})}{\sqrt{\pi}}\,.$$

Thus $\Xi_0 e_k = \frac{\Gamma(\delta-\frac{1}{2})}{\sqrt{\pi}\Gamma(\delta)} e_k$, so that

$$\Xi_0 = \frac{\Gamma(\delta-\frac{1}{2})}{\sqrt{\pi}\Gamma(\delta)} I\,,$$

where I is the identity operator. It follows that

$$r(0) = \frac{c}{2}\frac{\Gamma(\delta-\frac{1}{2})}{\sqrt{\pi}\Gamma(\delta)}(\iota^2+\lambda_k)^{(1-2\delta)\gamma}(\kappa^2+\lambda_k)^{-\beta}\,.$$

□

Proposition 4.4.2 shows us that we have a similar scenario as in Section 3.3; the spectral density of $r(0)$ is a product of two Matern spectral densities, so that the auto-covariance function of $r(0)$ is a convolution of two Matern auto-covariance functions. Following the same argument as in Section 3.3 we will get the same correlation decay that we saw for the space-fractional heat equation i.e. $O(e^{-2\pi\iota h}) + O(e^{-2\pi\kappa h})$. We summarize this in a proposition.

Proposition 4.4.3. *Let X be a solution to Equation 13 on the domain $\mathcal{D} = \mathcal{R}^d$ with $Q = c(\kappa^2 - \Delta)^{-\beta}$ and assume $\kappa \neq \iota$. Denote by R the auto-covariance function corresponding to the asymptotic covariance operator $r(0)$ of X. We then have that*

$$|R(h)| \leq O(e^{-2\pi\iota h}) + O(e^{-2\pi\kappa h}) \;\; as \;\; h \to \infty\,.$$

This indicates that the spatial range is controlled by $\min(\iota, \kappa)$, just like in the case of the space fractional heat equation in Section 3.

In the next proposition we bound the temporal correlation decay.

Proposition 4.4.4. *Let X be a solution to Equation 13 on $\mathcal{L}^2(\mathcal{D})$. Then for $0 \leq \alpha < 1$ and $x, y \in \mathcal{L}^2(\mathcal{D})$ and for large values of t we have that*

$$|\mathbb{E}\left[X(t+h,x)X(t,y)\right]| \leq O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h}) \;\; as \;\; h \to \infty\,.$$

Proof. First note that since

$$r(h)e_k = \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}\int_0^\infty (\xi+h)^{\delta-1}\xi^{\delta-1}e^{-\mu_k(\xi+h)}e^{-\mu_k\xi}\,\mathrm{d}\xi e_k\,,$$

it follows that e_k is also an eigenvector of $r(h)$. We denote the eigenvalue of $r(h)$ corresponding to the eigenvector e_k by $r_k(h)$. Applying Cauchy–Bunyakovsky–Schwarz to these eigenvalues we get

$$\begin{aligned}|r_k(h)| &= \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}\int_0^\infty (\xi+h)^{\delta-1}\xi^{\delta-1}e^{-\mu_k(\xi+h)}e^{-\mu_k\xi}\,\mathrm{d}\xi \\ &\leq \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}\left(\int_0^\infty (\xi+h)^{2\delta-2}e^{-2\mu_k(\xi+h)}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_0^\infty \xi^{2\delta-2}e^{-2\mu_k\xi}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &= \frac{c}{\Gamma(\delta)^2}(\kappa^2+\lambda_k)^{-\beta}\times(I)^{\frac{1}{2}}\times(II)^{\frac{1}{2}} \;=(*)\end{aligned}$$

For (II) we can simply calculate

$$(II) = \int_0^\infty \xi^{2\delta-2} e^{-2\mu_k \xi} \,\mathrm{d}\xi = (2\mu_k)^{1-2\delta}\Gamma(2\delta - 1)\,.$$

For I we fix $0 \leq \alpha < 1$. We can then estimate

$$\begin{aligned}
(I) &= \int_0^\infty (\xi + h)^{2\delta-2} e^{-2\mu_k(\xi+h)} \,\mathrm{d}\xi \\
&= \int_{2\mu_k h}^\infty \xi^{2\delta-2} e^{-2\mu_k \xi} \,\mathrm{d}\xi \\
&= \int_{2\mu_k h}^\infty \xi^{2\delta-2} e^{-\sqrt{\alpha}\xi} e^{-(1-\sqrt{\alpha})\xi} \,\mathrm{d}\xi \\
&\leq e^{-2\sqrt{\alpha}\mu_k h} \int_{2\mu_k h}^\infty \xi^{2\delta-2} e^{-(1-\sqrt{\alpha})\xi} \,\mathrm{d}\xi \leq C e^{-2\sqrt{\alpha}\mu_k h}
\end{aligned}$$

We can apply this estimate to $(*)$ to get the asymptotic behaviour of $r(h)$ as $h \to \infty$

$$\begin{aligned}
(*) &= \frac{c}{\Gamma(\delta)^2}(\kappa^2 + \lambda_k)^{-\beta}(2\mu_k)^{\frac{1}{2}-\delta}\sqrt{\Gamma(2\delta-1)}\sqrt{(I)} \\
&\leq \frac{c}{\Gamma(\delta)^2}(\kappa^2 + \lambda_k)^{-\beta}(2\mu_k)^{\frac{1}{2}-\delta}\sqrt{\Gamma(2\delta-1)} C \mu_k^{-\frac{1}{2}} e^{-\sqrt{\alpha}\mu_k h} \\
&\leq C \lambda_k^{-\delta\gamma-\beta} e^{-\sqrt{\alpha}\mu_k h}
\end{aligned}$$

Using this we can estimate the temporal correlation decay for "big" t by

$$\begin{aligned}
|\mathbb{E}\left[X(t+h,x)X(t,y)\right]| &= |\left\langle r(h)\delta_x, \delta_y \right\rangle| \\
&= \left| \sum_{k,j} e_k(x) e_j(y) \left\langle r(h) e_k, e_j \right\rangle \right| \\
&\leq \sum_k |e_k(x)||e_k(y)||r_k(h)| \\
&\leq C \sum_k e^{-\sqrt{\alpha}\mu_k h} \lambda_k^{-\delta\gamma-\beta+\frac{d-1}{2}} \\
&\leq C \sum_k e^{-\alpha\mu_k h} e^{-\sqrt{\alpha}(1-\sqrt{\alpha})\mu_k h} \lambda_k^{-\delta\gamma-\beta+\frac{d-1}{2}} \\
&\leq C e^{-\alpha(\iota^2+\lambda_1)^\gamma h} \sum_k e^{-\sqrt{\alpha}(1-\sqrt{\alpha})\mu_k h} \lambda_k^{-\delta\gamma-\beta+\frac{d-1}{2}}\,.
\end{aligned}$$

The sum always converges since the decay in the exponential function kills any potential divergence in the λ_k's. We therefore have the desired asymptotic decay $O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h})$. □

The decay rate of temporal correlation is therefore exponential and controlled by the parameters ι and γ and the first eigenvalue λ_1 of Δ, just like in the case of the space fractional heat equation in Section 3.

5 Numerical estimation

In this section we will consider partial numerical schemes to estimate solutions to the SPDE's we have discussed in previous sections. Abstractly we will consider an approximating problem

$$\mathrm{d}\tilde{X}(t) - A_h\tilde{X}(t)\mathrm{d}t = \Pi_h \mathrm{d}W(t) \tag{19}$$

to Equation 8 on a finite-dimensional subspace $V_h \subset \dot{H}^1$. h is here a "refinement" variable, and the idea is that V_h depends on H in such a way that V_h better approximates H as h becomes smaller. We have replaced the operator $A : D(A) \to H$ with a discrete operator $A_h : V_h \to V_h$ and we have applied a map $\Pi_h : \dot{H}^1 \to V_h$ to the noise $W(t)$. We will discuss the choice of A_h and Π_h in later subsections. This is an abstract version of the popular finite element method. Much of the abstract theory in this section is taken from Strang and Fix [19], and the approach to finite element estimation is inspired by the one used by Bolin, Kirchner and Kovács in [5] and [6]. We will however also use Thomée [20] for some results. Note that this approximation problem only discretizes in space, leaving us with a system of (real) linear differential equations in time. A full numerical scheme would also require a numerical method to solve this system of differential equations.

The approach taken here is to first consider eigenvalue and eigenvector results for the discrete operator A_h and then use these to estimate the semigroup error $\|(e^{At} - e^{A_h t}\Pi)e_k\|_H$. Our choice of discretizing operator is discussed in Section 5.2. Our approach differs from that found in Thomée [20] and other textbooks on the finite element method, but is similar to the approach taken by Bolin, Kirchner and Kovács in [5] and [6] and by Strang and Fix in [19]. The rationale behind this choice of approach is that doing calculations in terms of eigenproperties is less technical and more stylistically similar to the approach taken in the regularity analysis in Sections 3 and 4. We adapt the notation that $\tilde{x}$ refers to the discretized version of x. The $\tilde{}$-notation used in this subsection has no relation to the $\tilde{}$-notation used in Section 4, where the $\tilde{}$-notation referred to the extension of an operator to a larger space.

5.1 The approximating space V_h

In principle we could choose any finite-dimensional subspace $V_h \subset \dot{H}^1$ as approximating space. In practice we want to select a space that actually approximates the space $\dot{H}^1$ well. There is a rich literature on how to construct such spaces. An very common example can be found below. For the purposes of this analysis we will not be concerned with explicit constructions but will instead make some fundamental assumptions on how well V_h approximates $\dot{H}^1$. Our assumptions are summarized below.

Assumption 5.1.1. *Let $V_h \subset \dot{H}^1$ be a finite element space. We assume that there exists a $\|\cdot\|_1$-bounded map $\Pi_h : \dot{H}^1 \to V_h$ such that for all $u \in \dot{H}^1$ and all $s \in [0, 2]$ we have the error bound*

$$\|\Pi_h u - u\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s\|u\|_s = Ch^s\|(-\Delta)^{\frac{s}{2}}u\|_{\mathcal{L}^2(\mathcal{D})}\,.$$

We also assume that $C_1h^{-d} \leq \dim(V_h) \leq C_2h^{-d}$.

Note that if Assumption 5.1.1 is satisfied, then for elements $u \in \dot{H}^1$ the error bound also applies to the $\|\cdot\|_{\mathcal{L}^2(\mathcal{D})}$-orthogonal projection $\mathcal{P}_h : \mathcal{L}^2(\mathcal{D}) \to V_h$, since the orthogonal $\|\cdot\|_{\mathcal{L}^2(\mathcal{D})}$-projection is the best possible projection into $\mathcal{L}^2(\mathcal{D})$, in the sense that $\|u - \mathcal{P}_h u\|_{\mathcal{L}^2(\mathcal{D})} = \min_{v \in V_h} \|u - v\|_{\mathcal{L}^2(\mathcal{D})} \leq \|u - \Pi_h u\|_{\mathcal{L}^2(\mathcal{D})}$.

Since $\dot{H}^1$ is a Hilbert space [1], it makes sense to discuss an orthogonal $\|\cdot\|_1$-projection of $\dot{H}^1$ into V_h. We denote this projection by $\mathcal{R}_h$. We call this projection the Ritz projection.

The following example describes a choice of approximating space V_h that is very common in the literature. We provide only a brief overview of this construction here, with many details left out for the sake brevity.

Example 5.1.1. *We consider a set of important special cases of finite dimensional spaces approximating $\dot{H}^1$. For simplicity we will assume that $\mathcal{D} \subset \mathbb{R}^d$ is a polygonal domain. In dimensions 1, 2 and 3 we can then divide the domain $\mathcal{D}$ into a set of simplexes $\mathcal{T}$: line segments in $\mathbb{R}^1$, triangles in $\mathbb{R}^2$ and tetrahedra in $\mathbb{R}^3$. For dimensions 2 and 3 we assume that the internal angle of the simplexes are bounded from below. We then consider the space of functions that are piecewise linear on the simplexes. We denote the space of such piecewise linear functions by S_h, where h denotes the maximal diameter of the simplexes. As mentioned we consider only dimensions $d \in \{1, 2, 3\}$. The major advantage of this choice of approximating space is that it easy to construct a basis for. Note first that the value of $f \in S_h$ on an element $\tau \in \mathcal{T}$ is decided entirely by the values of f on the corners of τ. The value of f on the inside of τ can then be found by linear interpolation. The dimension N_h of S_h is thus equal to the number of nodes (element corners) in $\mathcal{T}$ and $N_h \propto h^{-d}$. Thus if we denote the nodes by $\{a_n\}_{n=1}^{N_h}$ then there exists a unique function $\phi_n \in S_h$ such that $\phi_k(x_j) = \delta_{j,k}$. The set $\{\phi_n\}_{n=1}^{N}$ forms a (non-orthogonal) basis for S_h. Note that if we impose zero boundary conditions on S_h then the dimension of the space is somewhat reduced, but we still have the proportional relationship $N_h \propto h^{-d}$.*

The space S_h satisfies our assumption with $\Pi_h = \mathcal{P}_h$, the orthogonal projection assuming also that the angles of the simplexes are bounded from below. A proof of this fact can be found in Strang and Fix [19], Thomée [20] or any other suitable book on the finite element method.

5.2 The discrete Laplacian and its eigenproperties

In our approximating problem Equation 19 we have so far only stated that we require A_h to be some operator on the approximating space V_h. Of course, the idea is that A_h is in some sense an approximation to the operator A. Since we are mostly concerned with variations of the heat equation in this paper, we will consider how to construct an approximation $\Delta_h : V_h \to V_h$ to the Laplacian $\Delta : \dot{H}^2 \to H$. We call Δ_h the "discrete Laplacian", and we will later use it to construct "plug-in" approximations for more complicated operators, like $(\iota^2 - \Delta)^\gamma$. The theoretical construction is very simple. We denote by Δ_h the operator $V_h \to V_h$ satisfying

$$\langle \Delta_h u, v \rangle_{\mathcal{L}^2(\mathcal{D})} = \left\langle \Delta^{\frac{1}{2}} u, \Delta^{\frac{1}{2}} v \right\rangle_{\mathcal{L}^2(\mathcal{D})} = \langle u, v \rangle_1 \ ,$$

for elements $u, v \in V_h$. Such an operator exists uniquely by the Riesz representation theorem. This method of approximating the Laplacian is called Galerkin's method, or more generally the method of variations.

How well does the operator Δ_h approximate Δ? We answer this question by comparing their eigenproperties, i.e. we wish to compare $\|\lambda_k - \tilde{\lambda}_k\|_H$ and $\|e_k - \tilde{e}_k\|_H$, where $\{\tilde{\lambda}_k\}_{k=1}^{N_h}$ and $\{\tilde{e}_k\}_{k=1}^{N_h}$ are the eigenvalues and orthonormal eigenvectors of Δ_h. The analysis here is heavily inspired by the the treatment of the topic by Strang and Fix [19]. The discrete Laplacian Δ_h will only have N_h pairs of eigenvectors and eigenvalues, where as Δ will have an infinite sequence. We will therefore assume that the eigenvalues are listed in increasing order and compare the first N_h eigenvalues and eigenvector of Δ with the eigenvalues and eigenvectors of Δ_h. An important tool in our analysis will be the so-called min-max principle. It states that

$$\lambda_k = \min_{A \subset \dot{H}^1} \max_{u \in A} \frac{\left\langle \Delta^{\frac{1}{2}} u, \Delta^{\frac{1}{2}} u \right\rangle_{\mathcal{L}^2(\mathcal{D})}}{\langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})}}$$

where A ranges over all the k-dimensional subsets of the domain $D(\Delta^{\frac{1}{2}}) = \dot{H}^1$ of Δ. The min-max principle is readily applied also the approximating space V_h. We assume that $V_h \subset \dot{H}^1$ satisfies Assumption 5.1.1 with $\Pi_h = \mathcal{R}_h$, the Ritz projection. We can then express the eigenvalues $\{\tilde{\lambda}_k\}_k$ of the discrete Laplacian Δ_h using the min-max principle by

$$\tilde{\lambda}_k = \min_{A \subset V_h} \max_{u \in A} \frac{\langle \Delta_h u, u \rangle_{\mathcal{L}^2(\mathcal{D})}}{\langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})}} = \min_{A \subset V_h} \max_{u \in A} \frac{\langle u, u \rangle_1}{\langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})}} .$$

The only difference between the two eigenvalue calculations is that when we are calculating the eigenvalues of Δ_h we are minimizing only over the subset $V_h \subset \dot{H}^1$. Therefore

$$\lambda_k \leq \tilde{\lambda}_k ,$$

so that we are approximating λ_k from above. Our next goal is to also find an upper bound for $\tilde{\lambda}_k$. To this end we will also need to consider the subspace spanned by $\{e_l\}_{l=1}^k$, the first k eigenvectors of Δ. We denote this space by E_k. The following lemma and its proof is taken from Strang and Fix [19].

Lemma 5.2.1. *Denote the set of unit vectors in E_k by $\mathcal{E}_k$. Define*

$$\sigma_k^h = \max_{u \in \mathcal{E}_k} |2 \langle u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} - \langle u - \mathcal{R}_h u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} | .$$

If $\sigma_k^h < 1$ then

$$\tilde{\lambda}_k \leq \frac{1}{1 - \sigma_k^h} \lambda_k .$$

Proof. We first want to show that the projection $\mathcal{R}_h E_k$ of E_k is k-dimensional. If $\dim(\mathcal{R}_h E_k) < k$, then there exists an $u \in \mathcal{E}_k$, such that $\Pi_h u = 0$. Assume this to be the case. Then

$$\sigma_k^h \geq | \langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})} | = 1 .$$

This contradicts the assumption that $\sigma_k^h < 1$. Therefore $\mathcal{R}_h E_k$ is k-dimensional. We now proceed to show the bound on $\tilde{\lambda}_k$. Since $\mathcal{R}_h E_k \subset V_h$, we see by the min-max principle

$$\tilde{\lambda}_k \leq \max_{u \in \mathcal{R}_h E_k} \frac{\langle u, u \rangle_1}{\langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})}} = \max_{u \in \mathcal{E}_k} \frac{\langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_1}{\langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})}} = (*) \,.$$

$\mathcal{R}_h$ is the orthogonal projection $\mathcal{L}^2(\mathcal{D}) \to V_h$ in the energy norm, so $\langle u - \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 = 0$. It follows that

$$\begin{aligned}
0 \leq \|u - \mathcal{R}_h u\|_1^2 &= \langle u - \mathcal{R}_h u, u - \mathcal{R}_h u \rangle_1 = \langle u, u \rangle_1 + \langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 - 2 \langle u, \mathcal{R}_h u \rangle_1 \\
&= \langle u, u \rangle_1 - \langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 - 2 \langle u - \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 \\
&= \langle u, u \rangle_1 - \langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 \,.
\end{aligned}$$

We can therefore estimate $\langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_1 \leq \langle u, u \rangle_1 = \left\langle \Delta^{\frac{1}{2}} u, \Delta^{\frac{1}{2}} u \right\rangle_{\mathcal{L}^2(\mathcal{D})}$ in the numerator of $(*)$. By a similar calculation, for $u \in \mathcal{E}_k$

$$\begin{aligned}
\langle \mathcal{R}_h u, \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} &= \langle u, u \rangle_{\mathcal{L}^2(\mathcal{D})} - \big(2 \langle u - \mathcal{R}_h u, \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} - \langle u - \mathcal{R}_h u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} \big) \\
&\geq 1 - \max_{u \in \mathcal{E}_k} \big| \langle u - \mathcal{R}_h u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} - 2 \langle u - \mathcal{R}_h u, \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} \big| \\
&= 1 - \sigma_k^h \,.
\end{aligned}$$

Thus

$$(*) \leq \frac{1}{1 - \sigma_k^h} \max_{u \in \mathcal{E}_k} \left\langle \Delta^{\frac{1}{2}} u, \Delta^{\frac{1}{2}} u \right\rangle_{\mathcal{L}^2(\mathcal{D})} \leq \frac{1}{1 - \sigma_k^h} \|\Delta^{\frac{1}{2}}\|_{L(E_k)}^2 \|u\|_{\mathcal{L}^2(\mathcal{D})}^2 = \frac{1}{1 - \sigma_k^h} \lambda_k \,.$$

□

This reduces our problem of finding an upper bound for $\tilde{\lambda}_k$ to the problem of estimating σ_k^h. We do this in the next theorem. The theorem and its proof is taken from Strang and Fix [19].

Theorem 5.2.1. *Eigenvalue estimates*. *The eigenvalues $\{\tilde{e}_k\}_{k=1}^{N_h}$ of the discrete Laplacian $\Delta_h : V_h \to V_h$ satisfies the estimates*

$$\lambda_k \leq \tilde{\lambda}_k \leq \lambda_k + C h^s \lambda_k^{\frac{s}{2}+1} \,.$$

Proof. We first fix $u \in \mathcal{E}_k$. We can then use the Cauchy–Bunyakovsky–Schwarz inequality and Assumption 5.1.1 to find that

$$\langle u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} \leq \|u\|_{\mathcal{L}^2(\mathcal{D})} \|u - \mathcal{R}_h u\|_{\mathcal{L}^2(\mathcal{D})} \leq C h^s \|(-\Delta)^{\frac{s}{2}} u\|_{\mathcal{L}^2(\mathcal{D})} \|u\|_{\mathcal{L}^2(\mathcal{D})} \,.$$

Since $\|\Delta\|_{L(E_k)} = \lambda_k$ and $\|u\|_{\mathcal{L}^2(\mathcal{D})} = 1$, we therefore estimate

$$\langle u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} \leq C h^s \lambda_k^{\frac{s}{2}} \,.$$

Similarly we can use Assumption 5.1.1 to see that

$$\langle u - \mathcal{R}_h u, u - \mathcal{R}_h u \rangle_{\mathcal{L}^2(\mathcal{D})} = \|u - \mathcal{R}_h u\|_{\mathcal{L}^2(\mathcal{D})}^2 \leq C h^{2s} \lambda_k^s \,.$$

This is a higher order estimate than that for other term and hence we can find a constant C such that

$$\sigma_k^h \leq 2|\langle u, u - \mathcal{R}_h u\rangle_{\mathcal{L}^2(\mathcal{D})}| + 2|\langle u - \mathcal{R}_h u, u - \mathcal{R}_h u\rangle_{\mathcal{L}^2(\mathcal{D})}| \leq Ch^s \lambda_k^{\frac{s}{2}}.$$

Since $\frac{1}{1-x} \leq 1 + 2x$ for $x \leq \frac{1}{2}$ we can thus estimate that for small h

$$\frac{1}{1 - \sigma_k^h} \leq 1 + 2\sigma_k^h \leq 1 + Ch^s \lambda_k^{\frac{s}{2}}.$$

By Lemma 5.2.1 our final estimates for $\tilde{\lambda_k}$ is thus

$$\lambda_k \leq \tilde{\lambda_k} \leq \lambda_k + Ch^s \lambda_k^{\frac{s}{2}+1}.$$

The conclusion follows.

□

We now proceed to find error bounds for the eigenvectors of the discrete Laplacian. The following theorem and its proof is taken from Strang and Fix [19].

Theorem 5.2.2. ***Eigenvector estimates.*** *The eigenvectors $\{\tilde{e}_k\}_{k=1}^{N_h}$ of the discrete Laplacian $\Delta_h : V_h \to V_h$ satisfies the estimates*

$$\|e_k - \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \lambda_k^{\frac{s}{2}}.$$

Proof. By the triangle inequality

$$\begin{aligned} &\|e_k - \tilde{e_k}\|_{\mathcal{L}^2(\mathcal{D})} \\ \leq &\|e_k - \mathcal{R}_h e_k\|_{\mathcal{L}^2(\mathcal{D})} + \|\mathcal{R}_h e_k - \langle Re_k, \tilde{e}_k\rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} + \|\langle Re_k, \tilde{e}_k\rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_k - \tilde{e_k}\|_{\mathcal{L}^2(\mathcal{D})} \\ = &(I) + (II) + (III). \end{aligned}$$

We already have the bound $(I) \leq Ch^s \|e_k\|_s = Ch^s \lambda_k^{\frac{s}{2}}$ by Assumption 5.1.1. To bound (II) we first note that by the definition of the discrete Laplacian and the definition of the projection $\mathcal{R}_h$ we have that

$$\begin{aligned} \tilde{\lambda_j} \langle \mathcal{R}_h e_k, \tilde{e_j}\rangle_{\mathcal{L}^2(\mathcal{D})} &= \langle \mathcal{R}_h e_k, \Delta_h \tilde{e}_j\rangle_{\mathcal{L}^2(\mathcal{D})} = \langle \mathcal{R}_h e_k, \tilde{e}_j\rangle_1 = \left\langle \Delta^{\frac{1}{2}} \mathcal{R}_h e_k, \Delta^{\frac{1}{2}} \tilde{e}_j \right\rangle_{\mathcal{L}^2(\mathcal{D})} \\ &= \left\langle \Delta^{\frac{1}{2}} e_k, \Delta^{\frac{1}{2}} \tilde{e}_j \right\rangle_{\mathcal{L}^2(\mathcal{D})} = \langle \Delta e_k, \tilde{e}_j\rangle_{\mathcal{L}^2(\mathcal{D})} = \lambda_k \langle e_k, \tilde{e}_j\rangle_{\mathcal{L}^2(\mathcal{D})}. \end{aligned}$$

Substracting $\lambda_k \langle \mathcal{R}_h e_k, \tilde{e}_j\rangle_{\mathcal{L}^2(\mathcal{D})}$ from both sides we get

$$(\tilde{\lambda_j} - \lambda_k) \langle \mathcal{R}_h e_k, \tilde{e_j}\rangle_{\mathcal{L}^2(\mathcal{D})} = \lambda_k \langle e_k - \mathcal{R}_h e_k, \tilde{e}_j\rangle_{\mathcal{L}^2(\mathcal{D})}.$$

Second we note that assuming the eigenvalues of Δ to be distinct, we can for sufficiently small h guarantee that $|\tilde{\lambda_j} \neq \lambda_k|$ for all $k \neq j$. This is a consequence of Theorem 5.2.1. Therefore there exists a constant ρ, universal in $j \neq k$ and h, such that

$$\frac{\lambda_k}{|\tilde{\lambda_j} - \lambda_k|} \leq \rho.$$

We now combine these two facts to estimate (II). Since $\mathcal{R}_h e_k \in V_h$ we can write $\mathcal{R}_h e_k = \sum_{j=1}^{N_h} \langle Re_k, \tilde{e}_j \rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_j$. By Parseval's identity we thus have

$$
\begin{aligned}
(II)^2 = \|\mathcal{R}_h e_k - \langle \mathcal{R}_h e_k, \tilde{e}_k \rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_k\|^2_{\mathcal{L}^2(\mathcal{D})} &= \| \sum_{j \neq k} \langle \mathcal{R}_h e_k, \tilde{e}_j \rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_j \|^2_{\mathcal{L}^2(\mathcal{D})} \\
&= \sum_{j \neq k} \langle \mathcal{R}_h e_k, \tilde{e}_j \rangle^2_{\mathcal{L}^2(\mathcal{D})} \\
&\leq \sum_{j \neq k} \left(\frac{\lambda_k}{\tilde{\lambda}_j - \lambda_k} \right)^2 \langle e_k - \mathcal{R}_h e_k, \tilde{e}_k \rangle^2_{\mathcal{L}^2(\mathcal{D})} \\
&\leq \rho^2 \sum_{j \neq k} \langle e_k - \mathcal{R}_h e_k, \tilde{e}_k \rangle^2_{\mathcal{L}^2(\mathcal{D})} \\
&\leq C \rho^2 \|e_k - \mathcal{R}_h e_k\|^2_{\mathcal{L}^2(\mathcal{D})} \leq C^2 h^{2s} \lambda_k^s ,
\end{aligned}
$$

again by Assumption 5.1.1. Lastly we bound (III). By the reverse triangle inequality

$$
\begin{aligned}
(III) &= \| \langle e_k, \tilde{e}_k \rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_k - \tilde{e}_k \|_{\mathcal{L}^2(\mathcal{D})} \\
&= | \langle e_k, \tilde{e}_k \rangle_{\mathcal{L}^2(\mathcal{D})} - 1 | \|\tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} \\
&= | \langle e_k, \tilde{e}_k \rangle_{\mathcal{L}^2(\mathcal{D})} - 1 | \\
&= | \| \langle e_k, \tilde{e}_k \rangle_c \tilde{e}_k \|_{\mathcal{L}^2(\mathcal{D})} - \|e_k\|_{\mathcal{L}^2(\mathcal{D})} | \\
&\leq \| \langle e_k, \tilde{e}_k \rangle_{\mathcal{L}^2(\mathcal{D})} \tilde{e}_k - e_k \|_{\mathcal{L}^2(\mathcal{D})} \\
&\leq (I) + (II) \leq C h^s \lambda_k^{\frac{s}{2}} .
\end{aligned}
$$

The conclusion follows.

□

5.3 Strong error estimates for the stochastic heat equation

We are now ready to find strong error estimates for the semi-discretization of the heat equation in Equation 19. The analysis that follows in the next two subsection is largely my own take on the error analysis, though inspiration has been taken from the analysis of the finite element method for deterministic equations done by Strang and Fix in [19] and by Thomeè in [20], lecture notes by Barth and Lang [3], and the articles [5] and [6] by Bolin, Kirchner and Kovács where they develop error bounds for a class of non-temporal SPDE's, using similar methods.

Note first that Equation 19 is an SPDE of the same form that we studied in Section 2.3. We therefore know by Theorem 2.3.2 that Equation 19 has a unique weak solution given by the stochastic convolution

$$
\tilde{X}(t) = \int_0^t \tilde{S}_{t-s} \Pi_h \mathrm{d}W(t) ,
$$

assuming $\tilde{X}(0) = X(0) = 0$. $\tilde{S}_t$ denotes the semigroup generated by (Δ_h, V_h): the discrete Laplacian and the finite element space V_h. since we are now in a finite-dimensional space the existence of this semigroup is uncontroversial, and the semigroup is of the form $\tilde{S}_t = e^{-\Delta_h t}$. The only assumption we make on the map Π_h :

$\dot{H}^1 \to V_h$ is that it satisfies Assumption 5.1.1. A natural choice of Π_h, common in the literature, is the orthogonal projection $\mathcal{P}_h$. The reason we here consider a general projection Π_h instead, is that there are several possible approaches to estimating the Q-Wiener process W. One might for example estimate it by truncating a Karhunen-Loève expansion or by using a finite element method. Since our only assumption on Π_h is that is satisfies Assumption 5.1.1, we are free to select any approximation $\tilde{W}$ to W as long as it satisfies $\|\tilde{W} - W\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \|W\|_s$, almost surely.

The following lemma is a vital component in all the error analysis we do here.

Lemma 5.3.1. *Assume that our finite element space V_h satisfies Assumption 5.1.1. Let e_k be an eigenvector of $-\Delta$. Then for $s \in [0,2]$ we have that*

$$\|(S_t - \tilde{S}_t \Pi_h) e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \lambda_k^{\frac{s}{2}} \,,$$

Proof. We begin by defining the map $\rho_h : \mathcal{L}^2(\mathcal{D}) \to V_h$ by $e_k \mapsto \tilde{e}_k$ for $k \leq N_h$ and $e_k \mapsto 0$ for $k > N_h$. For $k \leq N_h$ we thus have $\|\rho_h e_k - e_k\|_H \leq Ch^s \lambda_k^{\frac{s}{2}}$ by Theorem 5.2.2. For $k > N_h$ we can use the second part of Assumption 5.1.1 and the Weyl bounds to write

$$\|\rho_h e_k - e_k\|_{\mathcal{L}^2(\mathcal{D})} = \|e_k\|_{\mathcal{L}^2(\mathcal{D})} = 1 = h^s h^{-s} \leq Ch^s N_h^{\frac{s}{d}} \leq Ch^s k^{\frac{s}{2}} \leq Ch^s \lambda_k^{\frac{s}{2}} \,.$$

Thus $\|\rho_h e_k - e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \lambda_k^{\frac{s}{2}}$ for all k and ρ_h satisfies Assumption 5.1.1. Using the triangle inequality we can now decompose $\|(S_t - \tilde{S}_t)\Pi_h e_k\|_{\mathcal{L}^2(\mathcal{D})}$ into two parts

$$\begin{aligned}
\|(S_t - \tilde{S}_t \Pi_h) e_k\|_{\mathcal{L}^2(\mathcal{D})} &\leq \|S_t e_k - \tilde{S}_t \rho_h e_k\|_{\mathcal{L}^2(\mathcal{D})} + \|\tilde{S}_t \rho_h e_k - \tilde{S}_t \Pi_h e_k\|_{\mathcal{L}^2(\mathcal{D})} \\
&= (I) + (II) \,.
\end{aligned}$$

For (I) we consider first the case where $k > N_h$. We can then write

$$(I) = \|S_t e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq 1 \leq Ch^s \lambda_k^{\frac{s}{2}} \,,$$

by the same argument as before. In the case $k \leq N_h$ we split (II) into two further parts

$$\begin{aligned}
(I) &= \|e^{-\lambda_k t} e_k - e^{-\tilde{\lambda}_k t} \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} \\
&\leq \|e^{-\lambda_k t} e_k - e^{-\lambda_k t} \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} + \|e^{-\lambda_k t} \tilde{e}_k - e^{-\lambda_k t} \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} \\
&= (III) + (IV) \,.
\end{aligned}$$

In (III) we can use Theorem 5.2.2 and compare the eigenvectors e_k and $\tilde{e}_k$.

$$(III) = e^{-\lambda_k t} \|e_k - \tilde{e}_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \lambda_k^{\frac{s}{2}} \,.$$

For (IV) we will use the Taylor bound $1 - e^{-x} \leq x$. We will also use the fact that the function $e^{-\lambda_k t} t$ attains its maximum at $t = \frac{1}{\lambda_k}$, so that $e^{-\lambda_k t} t \leq \lambda_k^{-1}$.

$$\begin{aligned}
(IV) &= e^{-\lambda_k t}(1 - e^{-(\tilde{\lambda}_k - \lambda_k)t}) \\
&\leq e^{-\lambda_k t} t (\tilde{\lambda}_k - \lambda_k) \\
&\leq C \lambda_k^{-1} h^s \lambda_k^{\frac{s}{2}+1} = Ch^s \lambda_k^{\frac{s}{2}} \,,
\end{aligned}$$

where we used Theorem 5.2.1 to compare the eigenvalues $\tilde{\lambda}_k$ and λ_k. Finally we use the fact that $\|\tilde{S}_t\|_{L(\mathcal{L}^2(\mathcal{D}))} \leq 1$ and estimate (II) by

$$(II) \leq \|\rho_h e_k - e_k\|_{\mathcal{L}^2(\mathcal{D})} + \|\Pi_h e_k - e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s \lambda_k^{\frac{s}{2}} .$$

Putting this together we get

$$\|(S_t - \tilde{S}_t \Pi_h) e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq (III) + (IV) + (II) \leq Ch^s \lambda_k^{\frac{s}{2}} .$$

□

This lemma is similar to Theorem 3.5 in Thomeè [20], but Thomeè only considers $\Pi_h = \mathcal{P}_h$. However, Thomeè's theorem allows for a general $u \in \dot{H}^s$, not just an eigenvector e_k. It also allows you to "trade" spatial blowup λ_k^{α} for temporal blowup $t^{-\alpha}$.

We now proceed to apply Lemma 5.3.1 to bound the error in $\|X(t)-\tilde{X}(t)\|_{\mathcal{L}^2(\Omega,\mathcal{L}^2(\mathcal{D}))}$. Letting Q be the Whittle-Matern covariance operator $(\kappa^2-\Delta)^{-\beta}$, we can for $t \in [0,T]$ apply the Itô isometry and Lemma 5.3.1 to get

$$\begin{aligned}
\|X(t) - \tilde{X}(t)\|^2_{\mathcal{L}^2(\Omega,\mathcal{L}^2(\mathcal{D}))} &= \mathbb{E}\left[\| \int_0^t S_{t-s}\Pi_h \mathrm{d}W(s) - \int_0^t \tilde{S}_{t-s}\Pi_h \mathrm{d}W(s)\|^2_{\mathcal{L}^2(\mathcal{D})}\right] \\
&= \mathbb{E}\left[\| \int_0^t (S_{t-s} - \tilde{S}_{t-s})\Pi_h dW(t)\|^2_{\mathcal{L}^2(\mathcal{D})}\right] \\
&\leq \int_0^t \sum_k \|(S_{t-s} - \tilde{S}_{t-s})\Pi_h Q^{\frac{1}{2}} e_k\|^2_{\mathcal{L}^2(\mathcal{D})} \,\mathrm{d}s \\
&= \int_0^t \sum_k (\kappa^2 + \lambda_k)^{-\beta} \|(S_{t-s} - \tilde{S}_{t-s})\Pi_h e_k\|^2_{\mathcal{L}^2(\mathcal{D})} \,\mathrm{d}s \\
&\leq \int_0^t \sum_k Ch^{2s}\lambda_k^s (\iota^2 + \lambda_k)^{-\beta} \,\mathrm{d}s \\
&\leq Ct\|(-\Delta)^{\frac{s}{2}} Q^{\frac{1}{2}}\|^2_{L_2(\mathcal{L}^2(\mathcal{D}))} h^{2s} ,
\end{aligned}$$

where we have also used the assumption that the projection Π_h is $\|\cdot\|_1$-bounded. Thus

$$\|X(t) - \tilde{X}(t)\|_{\mathcal{L}^2(\Omega,\mathcal{L}^2(\mathcal{D}))} \leq C\sqrt{t}\|\Delta^{\frac{s}{2}} Q^{\frac{1}{2}}\|_{L_2(\mathcal{L}^2(\mathcal{D}))} h^s .$$

We summarize this in the following proposition.

Proposition 5.3.1. *Let X be a solution to the stochastic heat equation $dX(t) - \Delta X(t)\mathrm{d}t = \mathrm{d}W(t)$ with $Q = (\kappa^2 - \Delta)^{-\beta}$. Let $\tilde{X}$ be a solution to by Equation 19 with $A_h = \Delta_h$, also with $Q = (\kappa^2 - \Delta)^{-\beta}$. Assume that Assumption 5.1.1 holds for the finite element space V_h for some map $\Pi_h : \dot{H}^1 \to V_h$. Assume that $s \in [0,2]$ satisfies $s < \beta - \frac{d}{2}$. Then*

$$\|X(t) - \tilde{X}(t)\|_{\mathcal{L}^2(\Omega,H)} \leq C\sqrt{t}h^s .$$

Proof. By our earlier discussion we have that

$$\|X(t) - \tilde{X}(t)\|_{\mathcal{L}^2(\Omega,H)} \leq C\sqrt{t}\|(-\Delta)^{\frac{s}{2}} Q^{\frac{1}{2}}\|_{L_2(\mathcal{L}^2(\mathcal{D}))} h^s .$$

So we only have to show that $\|(-\Delta)^{\frac{s}{2}}Q^{\frac{1}{2}}\|_{L_2(\mathcal{L}^2(\mathcal{D}))} < \infty$. We calculate

$$\|\Delta^{\frac{s}{2}}Q^{\frac{1}{2}}\|^2_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq C\sum_k \lambda_k^{s-\beta} \leq C\sum_k k^{\frac{2}{d}(s-\beta)}\,,$$

which converges if $\frac{2}{d}(s-\beta) < -1$ or $s < \beta - \frac{d}{2}$. The conclusion follows.

□

5.4 Strong error estimates for the fractional heat equations

We have considered two generalizations of the stochastic heat equation in this thesis. In Section 3 we considered the space fractional heat equation where Δ was replaced by $A = (\iota^2 - \Delta)^\gamma$ for $\gamma \leq 1$ and $\iota > 0$. There are two ways we could approach this problem numerically. One way is too construct a new discrete operator A_h defined through $\langle A_h u, v\rangle_{\mathcal{L}^2(\mathcal{D})} = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle_{\mathcal{L}^2(\mathcal{D})}$ for $u, v \in V_h \subset \dot{H}^{\frac{\gamma}{2}}$. This would require us to redo our eigenvalue analysis for this new operator. Constructing the operator A_h numerically might be also cumbersome since $A^{\frac{1}{2}}u$ might be difficult to calculate in general.

A much easier method is to instead use a "plug-in" approach, where we use the discrete Laplacian we have already constructed and consider the discretization $A_h = (\iota^2 - \Delta_h)^\gamma$. Looking back at Lemma 5.3.1, we see that since $(\iota^2 - \Delta_h)^\gamma$ shares eigenvector basis with Δ_h we would get essentially the same result for the semi-group of this new operator. The only complication would be in the estimation of (IV), where we would get

$$(IV) \leq e^{-(\iota^2+\lambda_k)^\gamma t} t\big((\iota^2 + \tilde{\lambda}_k)^\gamma - (\iota^2 + \lambda_k)^\gamma\big)\,.$$

For $\gamma < 1$ the function x^γ is concave so we can simply use the Taylor bound $(x+h)^\gamma - x^\gamma \leq \gamma x^{\gamma-1}h$ to estimate $(\iota^2 - \tilde{\lambda}_k)^\gamma - (\iota^2 - \lambda_k)^\gamma \leq C(\iota^2+\lambda_k)^{\gamma-1}(\tilde{\lambda}_k - \lambda_k)$ and then proceed as before. We then get

$$(IV) \leq Ce^{-(\iota^2+\lambda_k)^\gamma t} t(\iota^2+\lambda_k)^{\gamma-1} h^s \lambda_k^{\frac{s}{2}+1} \leq Ch^s\lambda_k^{\frac{s}{2}}\,.$$

In the last step we have used the inequality $te^{-\mu_k t} \leq \mu_k^{-1}$. In the end we thus get the same convergence order that we get for $\gamma = 1$. We summarize this in a proposition.

Proposition 5.4.1. *Let X be a solution to Equation 11 with $Q = (\kappa^2 - \Delta)^{-\beta}$. Let $\tilde{X}$ be a solution to by Equation 19 with $A_h = (\iota^2 - \Delta_h)^\gamma$, also with $Q = (\kappa^2 - \Delta)^{-\beta}$. Assume that Assumption 5.1.1 holds for the finite element space V_h for some map $\Pi_h : \dot{H}^1 \to V_h$. Assume that $s \in [0, 2]$ satisfies $s < \beta - \frac{d}{2}$. Then*

$$\|X(t) - \tilde{X}(t)\|_{\mathcal{L}^2(\Omega,H)} \leq C\sqrt{t}h^s\,.$$

The second generalization of the stochastic heat equation that we have studied is the space-time fractional heat equations in Section 4. The main source on these equations used in this thesis is the article [15] by Kirchner and Willems. However, they discuss only the regularity of these equations (and their asymptotic covariance properties), not their numerical approximation. To the author's knowledge there is as of May 2023 no literature analysing numerical approximation for this class of SPDE's. The following is a limited attempt at doing such an analysis. Using our "plug-in" approach it is natural to approximate Equation 13 by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + (\iota^2 - \Delta_h)^\gamma\right)^\delta \tilde{X}(t)dt = \Pi_h dW(t)\,, \tag{20}$$

The theory of Section 4.2 suggests that the unique weak solution to this problem is given by the stochastic convolution

$$\tilde{X}(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \tilde{S}_{t-s} \Pi_h \mathrm{d}W(t)\,.$$

We can use this representation together with Lemma 5.3.1 (modified for $(\iota^2 - \Delta)^\gamma$ as discussed previously) to calculate

$$\begin{aligned}
\|X(t) - \tilde{X}(t)\|^2_{\mathcal{L}^2(\Omega,H)} &= \frac{1}{\Gamma(\delta)^2} \mathbb{E}\left[\left\| \int_0^t (t-s)^{\delta-1} (S_{t-s} - \tilde{S}_{t-s}) \Pi_h \mathrm{d}W(t) \right\|^2_{\mathcal{L}^2(\mathcal{D})}\right] \\
&\leq \frac{1}{\Gamma(\delta)^2} \int_0^t (t-s)^{2\delta-2} \sum_k \|(S_{t-s} - \tilde{S}_{t-s}) \Pi_h Q^{\frac{1}{2}} e_k\|^2_{\mathcal{L}^2(\mathcal{D})} \,\mathrm{d}s \\
&\leq C \int_0^t (t-s)^{2\delta-2} \sum_k h^{2s} \|Q^{\frac{1}{2}} e_k\|^2_s \,\mathrm{d}s \\
&\leq C t^{2\delta-1} \|(-\Delta)^{\frac{s}{2}} Q^{\frac{1}{2}}\|^2_{L_2(\mathcal{L}^2(\mathcal{D}))} h^{2s}\,,
\end{aligned}$$

So the only difference we get to Proposition 5.3.1 is in the dependence on t. We therefore still have the same convergence order as before. We summarize this in a proposition.

Proposition 5.4.2. *Let X be a solution to Equation 13 and $\tilde{X}$ be a solution to Equation 20, both with $Q = (\kappa^2 - \Delta)^{-\beta}$. Assume that Assumption 5.1.1 holds for the finite element space V_h for some map $\Pi_h : H \to V_h$. Assume that $s \in [0,2]$ satisfies $s < \beta - \frac{d}{2}$. Then*

$$\|X(t) - \tilde{X}(t)\|_{\mathcal{L}^2(\Omega,\mathcal{L}^2(\mathcal{D}))} \leq C t^{\delta - \frac{1}{2}} h^s\,.$$

Note that we have still only done a semi-discretization of Equation 13; we make no discretization in time. In fact the temporal discretization is trickier to perform in this case. While Equation 19 has a direct interpretation as a system of SODE's, the discretization we have used for the space-time fractional heat equation is harder to interpret. However, it should in principle be possible to discretize in time by numerically estimating the integral $\int_0^t (t-s)^{\delta-1} \tilde{S}_t \Pi_h \mathrm{d}W(t)$ directly. A discussion of this is outside the scope of this thesis.

5.5 Error in space-time covariance function

In Section 2.7 discussed the space-time covariance function r. We saw that in the case of Equation 8 we can write the space-time covariance function as

$$r(t_1, t_2) = S_{|t-s|} \int_0^{\min(t_1,t_2)} S_\xi Q S_\xi \,\mathrm{d}\xi\,.$$

For Equation 19 this would imply that we can calculate a space-time covariance function for the discrete approximation by

$$\tilde{r}(t_1, t_2) = \tilde{S}_{|t-s|} \int_0^{\min(t_1,t_2)} \tilde{S}_\xi \Pi_h Q \tilde{S}_\xi \, \mathrm{d}\xi \,.$$

We will now consider the estimation error of the space-time covariance function in the Hilbert-Schmidt norm. For simplicity we assume $\Pi_h = \mathcal{P}_h$ and we only consider the simple heat equation, not the fractional variants from Sections 3 and 4.

Proposition 5.5.1. *Assume that V_h satisfy Assumption 5.1.1. Let X be a solution of the stochastic heat equation $\mathrm{d}X(t) - \Delta X(t)\mathrm{d}t = \mathrm{d}W(t)$ and let $\tilde{X}$ be a solution to Equation 19 with $A_h = \Delta_h$. Let r and $\tilde{r}$ be the space-time covariance functions of X and $\tilde{X}$ respectively. Let $t_1, t_2 \in [0, T]$. Assume that $s \in [0, 2]$ satisfies $s < 2\beta - \frac{d}{2}$. Then*

$$\|r(t_1, t_2) - \tilde{r}(t_1, t_2)\mathcal{P}_h\|_{L_2(\mathcal{L}^2(\mathcal{D}))} \le C \min(t_1, t_2) h^s \,.$$

Proof. We first expand the Hilbert-Schmidt norm using the definition and apply the formulas for the space-time covariance functions.

$$\begin{aligned}
&\|r(t_1, t_2) - \tilde{r}(t_1, t_2)\mathcal{P}_h\|^2_{L_2(\mathcal{L}^2(\mathcal{D}))} \\
&= \sum_k \|r(t_1, t_2)e_k - \tilde{r}(t_1, t_2)\mathcal{P}_h e_k\|^2_{\mathcal{L}^2(\mathcal{D})} \\
&\le \sum_k \left\| \int_0^{\min(t_1,t_2)} \left(S_{|t-s|} S_\xi Q S_\xi^* - \tilde{S}_{|t-s|} \tilde{S}_\xi \Pi_h Q \tilde{S}_\xi \mathcal{P}_h \right) e_k \, \mathrm{d}\xi \right\|^2_{\mathcal{L}^2(\mathcal{D})} \\
&= \sum_k (*)_k^2
\end{aligned}$$

We decompose $(*)_k$ as follows

$$\begin{aligned}
(*)_k &= \left\| \int_0^{\min(t_1,t_2)} \left(S_{|t-s|} S_\xi Q S_\xi^* - \tilde{S}_{|t-s|} \tilde{S}_\xi \Pi_h Q \tilde{S}_\xi \mathcal{P}_h \right) e_k \, \mathrm{d}\xi \right\|_{\mathcal{L}^2(\mathcal{D})} \\
&\le \int_0^{\min(t_1,t_2)} \left\| \left(S_{|t-s|} S_\xi Q S_\xi - \tilde{S}_{|t-s|} \tilde{S}_\xi \Pi_h Q \tilde{S}_\xi \mathcal{P}_h \right) e_k \right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \\
&\le \int_0^{\min(t_1,t_2)} \left\| \left(S_{|t-s|} - \tilde{S}_{|t-s|} \Pi_h \right) S_\xi Q S_\xi e_k \right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \\
&+ \int_0^{\min(t_1,t_2)} \left\| \tilde{S}_{|t-s|} \left(\Pi_h S_\xi - \tilde{S}_\xi \Pi_h \right) Q S_\xi e_k \right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \\
&+ \int_0^{\min(t_1,t_2)} \left\| \tilde{S}_{|t-s|} \tilde{S}_\xi \Pi_h Q \left(S_\xi - \tilde{S}_\xi \mathcal{P}_h \right) e_k \right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \\
&= (I) + (II) + (III)
\end{aligned}$$

Note that $\|(S_t - \tilde{S}_t \Pi_h)e_k\|_{\mathcal{L}^2(\mathcal{D})} \le \|S_t(I - \Pi_h)e_k\|_{\mathcal{L}^2(\mathcal{D})} + \|(S_t - \tilde{S}_t)\Pi_h e_k\|_{\mathcal{L}^2(\mathcal{D})} \le Ch^s \lambda_k^{\frac{s}{2}}$. For (I) we can therefore do

$$(I) = \int_0^{\min(t_1,t_2)} \left\| \left(S_{|t-s|} - \tilde{S}_{|t-s|} \Pi_h \right) S_\xi Q S_\xi e_k \right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \le Ch^s \min(t_1, t_2) \|Q e_k\|_s \,,$$

where we have also used the fact that $\|AS_\xi QS_\xi e_k\|_{\mathcal{L}^2(\mathcal{D})} = e^{-2\lambda_k t}\|AQe_k\|_{\mathcal{L}^2(\mathcal{D})} \leq \|AQe_k\|_{\mathcal{L}^2(\mathcal{D})}$. For (II) we first observe that $\|(\Pi_h S_t - \tilde{S}_t\Pi_h)e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq \|(\Pi_h - I)S_t e_k\|_{\mathcal{L}^2(\mathcal{D})} + \|(S_t - \tilde{S}_t\Pi_h)e_k\|_{\mathcal{L}^2(\mathcal{D})} \leq Ch^s\lambda_k^{\frac{s}{2}}$. We can then do similarly to get

$$(II) = \int_0^{\min(t_1,t_2)} \left\|\tilde{S}_{|t-s|}\left(\mathcal{P}_h S_\xi - \tilde{S}_\xi\mathcal{P}_h\right)QS_\xi e_k\right\|_{\mathcal{L}^2(\mathcal{D})} \mathrm{d}\xi \leq Ch^s\min(t_1,t_2)\|Qe_k\|_s \,.$$

For (III) we need to use an additional trick. Observe first that $\|\tilde{S}_{|t-s|}\tilde{S}_\xi\Pi_h Q(S_\xi - \tilde{S}_\xi\mathcal{P}_h)\|_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq \|Q(S_\xi - \tilde{S}_\xi\mathcal{P}_h)\|_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq \|QS_\xi(I-\mathcal{P}_h)\|_{L_2(\mathcal{L}^2(\mathcal{D}))} + \|Q(S_\xi - \tilde{S}_\xi)\mathcal{P}_h e_k\|_{L_2(\mathcal{L}^2(\mathcal{D}))}$. We can now do

$$\begin{aligned}\|QS_\xi(I-\mathcal{P}_h)\|_{L_2(\mathcal{L}^2(\mathcal{D}))} &= \|(QS_\xi(I-\mathcal{P}_h))^*\|_{L_2(\mathcal{L}^2(\mathcal{D}))}\\ &= \|(I-\mathcal{P}_h)S_\xi Q\|_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq Ch^s\|(-\Delta)^{\frac{s}{2}}Q\|_{L_2(\mathcal{L}^2(\mathcal{D}))}\,,\end{aligned}$$

using the fact that S_ξ is bounded and that orthogonal projections are self-adjoint. We can repeat calculation for $\|Q(S_\xi - \tilde{S}_\xi)\mathcal{P}_h e_k\|_{\mathcal{L}^2(\lceil)}$ and apply Lemma 5.3.1 to get

$$\|Q(S_\xi - S_\xi)\mathcal{P}_h e_k\|_{\mathcal{L}^2(\lceil)} \leq Ch^s\sum_k\|Qe_k\|_s = Ch^s\|(-\Delta)^{\frac{s}{2}}Q\|_{\mathcal{L}^2(\mathcal{D})}\,.$$

Combining this we get

$$\sum_k (III)^2 = \|Q(S_\xi - \tilde{S}_\xi\mathcal{P}_h)\|_{L_2(\mathcal{L}^2(\lceil))} \leq C^2h^{2s}\min(t_1,t_2)^2\|(-\Delta)^{\frac{s}{2}}Q\|_{\mathcal{L}^2(\mathcal{D})}\,.$$

Combining this estimate for (III) with the previous estimates for (I) and (II) we can now calculate $\sum_k(*)_k^2$.

$$\sum_k(*)_k^2 \leq C\sum_k\left((I)^2 + (II)^2 + (III)^2\right) \leq C^2h^{2s}\min(t_1,t_2)^2\|(-\Delta)^{\frac{s}{2}}Q\|^2_{\mathcal{L}^2(\mathcal{D})}\,.$$

Thus

$$\|r(t_1,t_2) - \tilde{r}(t_1,t_2)\mathcal{P}_h\|_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq= \sqrt{\sum_k(*)_k^2} \leq Ch^s\min(t_1,t_2)\|(-\Delta)^{\frac{s}{2}}Q\|_{\mathcal{L}^2(\mathcal{D})}\,.$$

We therefore need $\|(-\Delta)^{\frac{s}{2}}Q\|_{\mathcal{L}^2(\mathcal{D})} < \infty$ in order to conclude our proof. We calculate

$$\|(-\Delta)^{\frac{s}{2}}Q\|^2_{L_2(\mathcal{L}^2(\mathcal{D}))} \leq C\sum_k\lambda_k^{s-2\beta} \leq C\sum_k k^{\frac{2(s-2\beta)}{d}} < \infty\,,$$

which is satisfied when $\frac{2(s-2\beta)}{d} < -1$ or $s < 2\beta - \frac{d}{2}$. The conclusion follows.

□

Proposition 5.5.1 tells us that we can expect the convergence rate for the space-time covariance function to be β larger than the convergence rate for the strong error, in fact it is slightly more than twice as large. This convergence order can be directly applied to errors of the form $\mathrm{e}(t,s,u,v) := \mathbb{E}\big[\,\langle X(t),u\rangle\,\langle X(s),v\rangle\,\big] - \mathbb{E}\big[\langle\tilde{X}(t),u\rangle\langle\tilde{X}(s),v\rangle\big]$ by

$$\mathrm{e}(t,s,u,v) = |\,\langle(r(t,s) - \tilde{r}(t,s)\mathcal{P}_h)u,v\rangle\,| \leq C\min(t,s)h^s\|u\|_{\mathcal{L}^2(\mathcal{D})}\|v\|_{\mathcal{L}^2(\mathcal{D})}\,.$$

So we have convergence of order $2\beta - \frac{d}{2}$ in $\mathrm{e}(t,s,u,v)$.

6 Conclusion

In this thesis we have analyzed the regularity properties of two generalizations of the stochastic heat equation on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, driven by a Q-Wiener process with $Q = c(\kappa^2 - \Delta)^{-\beta}$ and with initial condition $X(0) = 0$ and zero boundary conditions. We have looked at conditions for the existence of solutions, considered their temporal and spatial smoothness, and looked at their convergence order when approximated using the finite element method. Our results are summarized in the following table.

	$\mathrm{d}X + (\iota^2 - \Delta)^\gamma X \mathrm{d}t = \mathrm{d}W$	$\left(\frac{\mathrm{d}}{\mathrm{d}t} + (\iota^2 - \Delta)^\gamma\right)^\delta X = \dot{W}$
Existence of solutions	$d < 2\beta + 2\gamma$	$d < 2\beta + (4\delta - 2)\gamma$
Total regularity	$\gamma + \beta - \frac{d}{2}$	$(2\delta - 1)\gamma + \beta - \frac{d}{2}$
Relative cost [temp./spatial]	2γ	2γ
Maximal temporal regularity	$\frac{1}{2}$	$\delta - \frac{1}{2}$
Asymptotic covariance operator	$\frac{c}{2}(\iota^2 - \Delta)^{-\delta}(\kappa^2 - \Delta)^{-\beta}$	$\frac{c}{2}\frac{\Gamma\left(\delta - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\gamma)}(\iota^2 - \Delta)^{(1-2\delta)\gamma}(\kappa^2 - \Delta)^{-\beta}$
Temporal correlation decay	$\leq O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h})$	$\leq O(e^{-\alpha(\iota^2+\lambda_1)^\gamma h})$
Spatial correlation decay	$\leq O(e^{-2\pi\iota h}) + O(e^{-2\pi\kappa h})$	$\leq O(e^{-2\pi\iota h}) + O(e^{-2\pi\kappa h})$
Strong convergence order	$\beta - \frac{d}{2}$	$\beta - \frac{d}{2}$

The first and second entry of the table are essentially equivalent; the equations have solutions when the total regularity is > 0. The total regularity can also be interpreted as the maximal spatial regularity. The temporal regularity can be found by dividing the total regularity by the relative cost, which is 2γ in both the cases discussed in this work. The caveat is that we have a hard cap on the temporal regularity; $\frac{1}{2}$ in the case of the space fractional heat equation and $\delta - \frac{1}{2}$ in the case of the space-time fractional heat equation. In other words, every time we apply a spatial derivative we reduce the total regularity of the process by 1, and every time we apply a temporal derivative we reduce the total regularity by the relative cost 2γ.

We see that for both the space fractional heat equation and the space-time fractional heat equation we can freely adjust the total regularity of our solutions by controlling γ, β and δ. However, in the case of the space fractional heat equation we can not freely control the temporal regularity; it is capped at $\frac{1}{2}$ regardless of the parameters. In the case of the space-time fractional heat equation this limitation vanishes; we can get arbitrary temporal regularity by adjusting δ upwards. Another limitation is that in neither case can we adjust the spatial and temporal smoothness independently of each other; if we have spatial regularity s, then the temporal regularity is $\min(\frac{s}{2\gamma}, \delta - \frac{1}{2})$.

In both the case of the space fractional heat equation and the space-time fractional heat equation we can control the decay in temporal correlation by adjusting the parameters ι and γ. Similarly the smallest of the parameters ι and κ control the spatial correlation decay. In all cases the decay is exponential. These results are only upper bounds on the decay and are therefore not very useful for the practical calculation of correlation range, but they give us a qualitative indicator of how the parameters might influence the correlation range of the solutions. We have also calculated the asymptotic spatial covariance operator for both equations. In both cases we inherit the Matern operator $c(\kappa^2 - \Delta)^{-\beta}$, but with some adjustment. In the case where $\kappa = \iota$ the marginal distribution of the solutions are asymptotically Matern.

Finally we considered numerical estimation of our equations using the finite element method. We saw that if we construct partial discretizations for the operators $(\iota^2 - \Delta)^\gamma$ and $(\frac{\mathrm{d}}{\mathrm{d}t} - (\iota^2 - \Delta)^\gamma)^\delta$ using the discrete Laplacian Δ_h we get the same strong convergence order $\beta - \frac{d}{2}$ regardless of the other parameters. Possibly you could get better order of convergence for large values of γ if you directly constructed approximations to the operator $(\iota^2 - \Delta)^\gamma$, instead of using the discrete Laplacian Δ_h, but that is beyond the scope of this thesis. We also considered the rate of convergence of the space-time covariance function in Hilbert-Schmidt norm in the case of the simple heat equation ($\gamma = \delta = 1$) and saw that it was of order $2\beta - \frac{d}{2}$; slightly more than twice the strong convergence order.

A Appendix

A.1 Hilbert-Schmidt operators

Let H and U be separable Hilbert spaces with norms $\|\cdot\|_H$ and $\|\cdot\|_U$ respectively. In this appendix we will consider a class of operators $U \to H$ known as Hilbert-Schmidt operators. These are an important tool in our above analysis of stochastic evolution equations. The treatment here is based on the lecture note by Kovács and Larsson [16].

Definition A.1.1. ***Hilbert-Schmidt operator****. Let $A : U \to H$. Then A is a Hilbert-Schmidt operator if*

$$\sum_k \|Ae_k\|_H^2 < \infty\,,$$

for some orthonormal basis $\{e_k\}_k$ of U. We denote the space of Hilbert-Schmidt operators $U \to H$ by $L_2(U,H)$. In the case where $U = H$ we write $L_2(H) := L_2(H,H)$.

Our definition suggests a natural norm on $L_2(U,H)$, namely

$$\|A\|_{L_2(U,H)} = \sqrt{\sum_k \|Ae_k\|_H^2}\,.$$

This norm is independent of the choice of orthonormal basis. Too see this let $\{f_k\}_k$ be a second orthonormal basis for U. We can then calculate

$$\begin{aligned}
\sum_k \|Ae_k\|_H^2 &= \sum_k \langle Ae_k, Ae_k\rangle_H \\
&= \sum_{k,i,j} \langle e_k, f_i\rangle_U \langle e_k, f_j\rangle_U \langle Af_i, Af_j\rangle_H \\
&= \sum_{i,j} \left\langle \sum_k \langle e_k, f_j\rangle_U e_k, f_i \right\rangle_U \langle Af_i, Af_j\rangle_H \\
&= \sum_{i,j} \langle f_j, f_i\rangle_U \langle Af_i, Af_j\rangle_H \\
&= \sum_i \langle Af_i, Af_i\rangle_H = \sum_k \|Af_k\|_H^2\,.
\end{aligned}$$

Another concept of importance to us is that of the trace.

Definition A.1.2. ***Trace***. *For an orthonormal basis $\{e_k\}_k$ of U and an orthonormal basis $\{f_k\}_k$ of H, the trace of an operator $A : U \to H$ is given by*

$$Tr(A) = \sum_k \langle Ae_k, f_k \rangle_H \ .$$

By a calculation similar to the one we did for the Hilbert-Schmidt norm it is possible to show that the trace is also independent of the choice of orthonormal basis. In addition, using the Cauchy–Bunyakovsky–Schwarz inequality

$$\mathrm{Tr}(A) = \sum_k \langle Ae_k, f_k \rangle_H \leq \sum_k \|Ae_k\|_H = \|A\|^2_{L_2(U,H)} \, .$$

It follows Hilbert-Schmidt operators have finite trace. We now show some properties of the Hilbert-Schmidt norm that will prove useful to us.

Theorem A.1.1. *Let $A \in L_2(U,H)$ and $B \in L(U)$. Then*

1. $\|A\|_{L_2(U,H)} = \|A^*\|_{L_2(H,U)}$
2. $\|A\|^2_{L_2(U,H)} = Tr(A^*A)$
3. $\|AB\|_{L_2(U,H)} \leq \|A\|_{L_2(U,H)} \|B\|_{L(U)}$
4. $\|A\|_{L(U,H)} \leq \|A\|_{L_2(U,H)}$

Proof. **Property 1:** Let $\{e_k\}_k$ be an orthonormal basis for U and let $\{f_k\}_k$ be an orthonormal basis for H. Then

$$\begin{aligned}
\|A\|^2_{L_2(U,H)} &= \sum_k \langle Ae_k, Ae_k \rangle_H \\
&= \sum_{k,j,i} \langle Ae_k, f_i \rangle_H \langle Ae_k, f_j \rangle_H \langle f_i, f_j \rangle_H \\
&= \sum_{k,j} \langle e_k, A^* f_j \rangle^2_U \\
&= \sum_j \|A^* f_j\|^2_U = \|A^*\|^2_{L_2(H,U)} \, .
\end{aligned}$$

Property 2: We see that, since A^*A is an operator $U \to U$

$$\|A\|^2_{L_2(U,H)} = \sum_k \langle Ae_k, Ae_k \rangle_H = \sum_k \langle A^*Ae_k, e_k \rangle_U = \mathrm{Tr}(A^*A) \, .$$

Property 3: By Property 2 we have

$$\begin{aligned}
\|AB\|^2_{L_2(U,H)} = \|B^*A^*\|^2_{L_2(H,U)} &= \sum_k \|B^*A^*e_k\|^2_H \\
&\leq \|B^*\|^2_{L(U)} \|A^*\|^2_{L_2(H,U)} \\
&= \|B\|^2_{L(U)} \|A\|^2_{L_2(U,H)} \, .
\end{aligned}$$

Property 4: Let $u \in U$. Then by Parsevals identity, the Cauchy–Bunyakovsky–Schwarz inequality and Property 1 we have that

$$\begin{aligned}\|Au\|_H^2 &= \sum_k \langle Au, e_k\rangle_H^2 \\ &= \sum_k \langle u, A^* e_k\rangle_U^2 \\ &\le \|u\|_U \sum_k \|A^* e_k\|_U^2 = \|A\|_{L_2(U,H)} \|u\|_U \,.\end{aligned}$$

This shows that $\|A\|_{L(U,H)} \le \|A\|_{L_2(U,H)}$.

□

It is possible to show that the space $L_2(U,H)$ of Hilbert-Schmidt operators forms a separable Hilbert space with the inner product $\langle A, B\rangle_{L_2(U,H)} = \sum_k \langle Ae_k, Be_k\rangle_H$. We will not prove this here; a proof can be found in Kovács and Larsson [16].

A.2 The Laplacian operator and $\dot{H}^s$-spaces

In the above treatment of stochastic evolution equation we specifically consider the stochastic heat equation $\mathrm{d}X(t) - \Delta X(t)\mathrm{d}t = \mathrm{d}W(t)$ with zero boundary conditions, and some generalizations thereof. The common denominator in these generalizations is that the Laplacian operator $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial^2 x_k}$ plays a central role. We are therefore interested in the eigenproperties the Laplacian when defined on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$. The following theorem is adapted from Davies [9].

Theorem A.2.1. ***Eigenproperties of the Laplacian.*** *Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded domain. Let $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial^2 x_k}$ be defined on a subset $D(\Delta)$ of $\mathcal{L}^2(\mathcal{D})$ of sufficiently smooth functions with zero boundary conditions. Then the operator $A := -\Delta$ has a set of orthonormal eigenvectors $\{e_k\}_k$ spanning $\mathcal{L}^2(\mathcal{D})$, with a corresponding non-decreasing sequence of positive eigenvalues $\{\lambda_k\}_k$ satisfying the estimates*

$$C_1 k^{\frac{2}{d}} \le \lambda_k \le C_2 k^{\frac{2}{d}} \,.$$

These estimates in Theorem A.2.1 are known as the Weyl estimates or the Weyl bounds. Theorem A.2.1 allows us to easily define various generalizations of the Laplacian. Let $f : [0,\infty) \to \mathbb{R}$ be a measurable function. If we take $u \in H := \mathcal{L}^2(\mathcal{D})$ we can write $u = \sum_k \langle u, e_k\rangle_H e_k$ since $\{e_k\}_k$ spans H. We can then define

$$f(\Delta)u := \sum_k \langle u, e_k\rangle_{\mathcal{L}^2(\mathcal{D})} f(-\lambda_k) e_k \,. \tag{21}$$

In general it is possible that the sum in Equation 21 does not converge for some elements in $\mathcal{L}^2(\mathcal{D})$. The set of elements $u \in \mathcal{L}^2(\mathcal{D})$ such that Equation 21 converges we define to be the domain of $f(\Delta)$.

$$D(f(\Delta)) := \{u \in \mathcal{L}^2(\mathcal{D}) \;:\; \sum_k \langle u, e_k\rangle_H^2 f(\lambda_k)^2 = \|f(\Delta)u\|_{\mathcal{L}^2(\mathcal{D})}^2 < \infty\} \,.$$

Instead of considering only $u \in H$, we could also consider all formal series in $\{e_k\}_k$, i.e. we define $\mathcal{A} := \{\sum_k a_k e_k : (a_k)_k \subset \mathbb{R}\}$ and $D(f(\Delta)) := \{u \in \mathcal{A} \;:$

$\sum_k a_k^2 f(-\lambda_k)^2 < \infty\}$. The elements $u \in \mathcal{A}$ are not necessarily elements in $H = \mathcal{L}^2(\mathcal{D})$, but can interpreted as generalized functions. For example a simple calculation shows that the delta function centered at $x \in \mathcal{D}$ can be expressed as an element in $\mathcal{A}$ by $\delta_x = \sum_k e_k(x)e_k$. Another example are the non-proper space-time covariance functions discussed in Section 1.2. If we consider a Gaussian random variable X on H with covariance operator $Q = I$, then it has the Karhunen-Loève expansion $X = \sum_k \beta_k e_k$, where $\{\beta_k\}_k$ is a sequence of real standard Gaussian variables. Then $\|X\|^2_{\mathcal{L}^2(\mathcal{D})} = \sum_k \beta_k^2$ diverges almost surely. However we still have that $X \in \mathcal{A}$. The reason this is of interest is that an operator $f(\Delta)$ could possibly "normalize" elements in $\mathcal{A}$. In other words it is possible that $\|f(\Delta)u\|_{\mathcal{L}^2(\mathcal{D})} < \infty$ even when $u \notin H$. For example, assuming $d = 1$, for the Gaussian variable X we have by the three series theorem (see Karr [14]) and the Weyl estimates that

$$\|(-\Delta)^{-\frac{1}{2}} X\|^2_{\mathcal{L}^2(\mathcal{D})} = \sum_k \lambda_k^{-1}\beta_k^2 \leq C \sum_k k^{-2}\beta_k^2 < \infty \text{ a.s.}\,.$$

This motivates the following definition.

Definition A.2.1. ***$\dot{H}^s$-spaces**. Let $\mathcal{A} := \{\sum_k a_k e_k : (a_k)_k \subset \mathbb{R}\}$, the space of formal series in $\{e_k\}_k$. For $s \in \mathbb{R}$ we define*

$$\dot{H}^s = \{u = \sum_k a_k e_k \in \mathcal{A} \;:\; \|(-\Delta)^{\frac{s}{2}} u\|^2_{\mathcal{L}^2(\mathcal{D})} = \sum_k a_k^2 \lambda_k^s < \infty\,\}\,.$$

For $u = \sum_k a_k e_k \in \dot{H}^s$ we also define

$$\|u\|_s := \|(-\Delta)^{\frac{s}{2}} u\|_{\mathcal{L}^2(\mathcal{D})} = \sqrt{\sum_k a_k^2 \lambda_k^s}\,.$$

According to Bolin, Kirchner and Kovacs [5], $\dot{H}^s$ is a Hilbert space for $s \in \mathbb{R}$. This is important sice we frequently use the norm $\|\cdot\|_s$ to measure the smoothness of the solutions to stochastic evolution equations, since $\|u\|_s < \infty$ implies that $(-\Delta)^{\frac{s}{2}} u \in \mathcal{L}^2(\mathcal{D})$. Since Δ is a differential operator of order 2, this loosely implies that u is s times spatially differentiable.

In Sections 3.3 and 4.4 we use delta functions to make sense of pointwise covariances. Assuming the eigenfunctions e_k of Δ to be continuous we can write the delta function δ_x centered at $x \in \mathcal{D}$ as

$$\delta_x = \sum_k \langle \delta_x, e_k \rangle_{\mathcal{L}^2(\mathcal{D})} e_k = \sum_k e_k(x) e_k\,.$$

In practice we can do much better than continuity. The following lemma, taken from Borthwick [7], tells us that the eigenfunctions of the Laplacian are analytic, i.e. infinitely differentiable, on the interior of their domain.

Lemma A.2.1. *For an open, bounded, domain $\mathcal{D} \subset \mathbb{R}^d$. Then the eigenfunctions e_k of $\Delta : D(\Delta) \to \mathcal{L}^2(\mathcal{D})$ are analytic functions.*

In the analysis of covariance we will also need supremum-norm estimates for the eigenfunctions of the Laplacian. The following result is taken from Sogge and Smith [12].

Proposition A.2.1. *For a compact, bounded, domain $\mathcal{D} \subset \mathbb{R}^d$. Then the eigenfunctions e_k of $\Delta : D(\Delta) \to \mathcal{L}^2(\mathcal{D})$ satisfy the estimate*

$$\|e_k\|_{L^\infty(\mathcal{D})} \leq C\lambda_k^{\frac{d-1}{4}} ,$$

where λ_k is the eigenvalue corresponding to e_k.

We can use this and the Weyl estimates to calculate that for almost all $x \in \mathcal{D}$ we have that

$$\|\delta_x\|_s^2 = \sum_k \lambda_k^s e_k(x)^2 \leq C \sum_k \lambda_k^{s+\frac{d-1}{2}} \leq C \sum_k k^{\frac{2}{d}(s+\frac{d-1}{2})} .$$

In order for this to be finite we need $\frac{2}{d}(s + \frac{d-1}{2}) < -1$ or $s < \frac{1}{2} - d$. It follows that for $s < \frac{1}{2} - d$ we have that $\delta_x \in \dot{H}^s$.

Afterword

This thesis is the result of continuous work over the period August 2022 - May 2023. It was written under the guidance of prof. Espen Robstad Jakobsen, with some additional feedback from Artur Jakub Rutkowski and Øyvind Auestad.

Most of Sections 1 and 2 were written during the autumn semester, i.e. August - December, during which I attended a weekly seminar based on the lecture note "Introduction to stochastic partial differential equations" by Kovács and Larsson [16]. For the most part the two sections closely follow the material of this seminar (and thus the lecture note as well).

Section 3 was written in early January, though originally I considered only $A = -\Delta^\gamma$, not $(\iota^2 - \Delta)^\gamma$; I only introduced the parameter ι when I later realized that this would give the asymptotic covariance operator a very nice interpretation. The analysis of the space fractional heat equation in Section 3 is a generalization of the analysis of the simple heat equation in the lecture note by Kovács and Larsson [16]. It is my own work in the sense that I did this generalization myself, though I would not claim it to be novel; a similar analysis probably exists in the literature.

Emboldened by my success in generalizing the results of Kovács and Larsson [16] in Section 3, I attacked the article "Regularity theory for a new class of fractional parabolic stochastic evolution equations" by Kirchner and Willems. This took the rest of January and most of February. My strategy was to use the proofs in the article mainly for inspiration, while trying to reproduce their regularity results in the more concrete framework that I was using. In their article Kirchner and Willems also consider what they call the "family of covariance operators" $\{Q_{Z_\gamma}(s,t)\}_{s,t\in[0,T]}$. I had already defined covariance operator too mean something very specific, so I denoted this family of operators the "space-time covariance function" and adopted the notation $r(s,t)$ for it, a notation more common in statistics. In addition to including some of the covariance results of Kirchner and Willems, I also went back and did a covariance analysis for the equation in Section 3.

Section 5 took most of March and April to write. I have very limited experience with numerical analysis and I had never encountered the finite element method before beginning work on this thesis. I quickly found that the eigenvalue/eigenvector analysis done by David Bolin, Kirchner and Kovács in [5] and [6] was more to my liking than the standard analysis done in many textbooks on the finite element method. Thus I went to their source, the book by Strang and Fix [19]. Strang and Fix work mostly in the purely spatial case and spend only a few pages on the spatio-temporal case, which meant that I had to fill in many gaps myself in order to have a complete analysis. The most notable example of this is Lemma 5.3.1. More general versions of this lemma exists in the literature (see for example Theorem 3.5 in Thomeè [20]), but there is no such result in Strang and Fix. Since I had committed myself to the eigenvector/eigenvalue approach I therefore had to prove this lemma myself. Both the lemma and the proof went through several iterations before I landed on something I was happy with. The rest of the numerical analysis then came fairly easily.

The analysis of the spatial and temporal asymptotics of the space-time covariance function for the two cases was only written during late April and early May. When I wrote the first draft of the conclusion during easter I wanted to write that the correlation range was controlled by the parameters ι and κ, in analogy with the

Matern fields. I realized that I had no formal argument for why was the case, only intuition. This was the motivation behind looking at the rate of decay in spatial and temporal covariance. It is the most experimental and perhaps the most novel of the work that I have done.

The rest of May was spent mainly on polishing already existing content. The defence was (is to be at the time of writing) on 31.05.23.

- *S.K. Furset*

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