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Extreme values of the argument of the Riemann zeta function

Master's thesis in MSMNFMA
Supervisor: Kristian Seip
May 2023

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Abstract

We prove an explicit Ω-result for the argument of the Riemann zeta function. We also exhibit large values of the argument of the Dedekind zeta function of a cyclotomic field, resulting in improved Ω-results for either the argument of the Riemann zeta function or the argument of Dirichlet L-functions.

Sammendrag

Vi beviser et eksplisitt Ω-resultat for argumentet til Riemann's zeta funksjon. Deretter finner vi store verdier av argumentet til Dedekind zeta funksjonen til en syklotomisk kropp. Dette resulterer i forbedrete Ω-resultat for enten argumentet til Riemann zeta funksjonen eller argumentet til Dirichlet L-funksjoner.

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Markus Valås Hagen, May 2023.

Introduction

Since Riemann's highly influential 1859-paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*[1], the Riemann zeta function has been under intensive study. Defined as a simple Dirichlet series for complex $s = \sigma + it$ with $\text{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

one could wonder how this function can be an interesting object. The easy explanation is that $\zeta(s)$ is intimately linked with the distribution of the prime numbers. The first sight of such a relation dates back to Euler. In 1737 he noted that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.^2$$

The relation holds for $\text{Re}(s) > 1$, and the right hand side is rightfully named the Euler product of the zeta function. Riemann did not only analytically extend this function to more or less the whole complex plane, except a pole in $s = 1$. He also found a deeper and more explicit relationship between the zeta function and the primes. He sketched a proof and it was first made rigorous by von Mangoldt in 1895. We need to introduce three functions to state this beautiful formula. First, $\Lambda(n)$ is called the von-Mangoldt function: it is defined to be $\log p$ if $n = p^k$ where $k \in \mathbb{N}$, i.e. a prime exponent, and 0 otherwise. Furthermore we define the Chebyshev function ψ and the closely related ψ_0 as

$$\psi(x) = \sum_{n \le x} \Lambda(n) \qquad \psi_0(x) = \lim_{h \to 0} \frac{1}{2} \left(\psi(x+h) - \psi(x-h) \right).$$

Then we have the following:

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}).$$

On the left hand side we are essentially counting primes using a weight, and on the right hand side we have terms we understand very well, except the sum over ρ. This is a sum over

[1]English: *On the number of primes less than a given quantity.*

[2]From here p will always denote a prime, and we will drop the extra term "prime".

the *non-trivial* zeroes of the Riemann zeta function. The phrasing *non-trivial* zero suggest the existence of trivial zeroes. This is the case for the negative even integers, where we have $\zeta(-2n) = 0$. The remaining zeroes are said to be non-trivial and are on the whole, a mystery. However, they are all believed to be on the critical line $\frac{1}{2} + it$. This is known as the Riemann Hypothesis (RH). The hypothesis has resisted any proof, although many routes of attack has been engineered and tried in combat. So far we know that at least 41% of the non-trivial zeroes lie on the critical line [7].

The zeroes of the Riemann zeta function seem to be out of reach for now. However, the nature of the critical line of the Riemann zeta function can reveal itself in other ways. One central question in this regard is how the function grows on the critical line. The Lindelöf hypothesis (LH) asserts that

$$\zeta\left(\frac{1}{2} + it\right) = O\left(t^{\varepsilon}\right) \qquad \text{for any } \varepsilon > 0.$$

LH is still unresolved. The best result to date in this direction is due to Bourgain [6], with $\frac{13}{84} + \varepsilon$ in the exponent. On RH it is known that

$$\zeta\left(\frac{1}{2} + it\right) \ll \exp\left((C + o(1))\frac{\log t}{\log\log t}\right)$$

for some $C > 0$. This result is old and dates back to Littlewood in the 1920's. Hundred years later the magnitude has not been improved, and the research has been focused towards getting the best possible constant. The best constant known as of today is $C = \frac{\log 2}{2}$ [9].

If we are to believe Farmer, Gonek and Hughes [14], the conditional bound due to Littlewood is not very sharp. They conjectured that

$$\max_{0 \le t \le T}\left|\zeta\left(\frac{1}{2} + it\right)\right| = \exp\left((1 + o(1))\sqrt{\frac{1}{2}\log T \log\log T}\right).$$

Results that produce lower bounds for an infinite sequence of points are known as Ω-results. More specifically, we write $f(x) = \Omega(g(x))$ if $\limsup_{x\to\infty} |f(x)|/g(x) > 0$. In this thesis we are concerned about such results for the Riemann zeta function. We will make an Ω-result for the argument of the Riemann zeta function explicit. Furthermore, we extend a result from [2] to the arguments of Dirichlet L-functions and the Riemann zeta function.

We will proceed as follows.

1. In Chapter 1 we set the scene. We introduce the objects we will be concerned about.
2. In Chapter 2 we introduce the resonance method and survey its history. We shall attempt at giving an informal explanation of how Aistleitner and Bondarenko–Seip improved the resonance method.
3. In Chapter 3 we prove an explicit Ω-result for $S(t)$.

4. In Chapter 4 we shall look at the resonance method applied to Dedekind zeta functions $\zeta_K(s)$ of cyclotomic fields $K = \mathbb{Q}(\exp(2\pi i/q))$. It turns out $\zeta_K(s)$ will produce rather large values. This leads to an interesting dichotomy regarding the size of Riemann's zeta function vs Dirichlet L-functions.

Contents

Notation and conventions

If not stated otherwise, k, ℓ, m, n denotes natural number, and p will always denote a prime. A sum of the form $\sum_{n\leq x}$ is to be understood as starting from $n=1$. Similarly, $\sum_{n\geq 1}$ should be interpreted as $\sum_{n=1}^{\infty}$. For ease of notation, we will sometimes write a double sum $\sum_m \sum_n$ as $\sum_{m,n}$.

The **Fourier transform** $\widehat{f}(\xi)$ of $f \in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}.$$

The **relation** $f \ll g$ (resp. $\widetilde{f} \gg \widetilde{g}$) mean that there exists a universal constant C (resp. $\widetilde{C}$) such that $|f(x)| \leq Cg(x)$ (resp. $|\widetilde{f}(x)| \geq \widetilde{C}\widetilde{g}(x)$) for all sufficiently large x. The two relations $f(x) = O(g(x))$ and $f \ll g$ are synonymous. Finally $f(x) = o(g(x))$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

The **relation** $f(x) = \Omega(g(x))$ means that $\limsup_{x\to\infty} |f(x)|/g(x) > 0$. Furthermore we have $f(x) = \Omega_{\pm}(g(x)) \iff \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$ and $\liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0$.

Chapter 1

Preliminaries on $\zeta(s)$, $S(t)$, $L(s,\chi)$ and $\zeta_K(s)$

In this chapter we survey the objects we will be working with in this thesis. We first introduce the Riemann zeta function and its argument, $S(t)$[1]. Then we take a quick look at Ω-results for $S(t)$. Finally we introduce two other L-functions that will be central in Chapter 4.

1.1 $\zeta(s)$ and $S(t)$

Throughout this thesis we let $s = \sigma + it$ be a complex number. For $\mathrm{Re}(s) > 1$ we define the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The product definition is known as the Euler product definition of the Riemann zeta function. It can be seen to be equal to the the first series definition for $\mathrm{Re}(s) > 1$, by a simple argument realizing $(1-p^{-s})^{-1}$ as a geometric series. The series $\sum_{n\geq 1} n^{-s}$ converges uniformly on all compact subsets of the half-plane $\mathrm{Re}(s) > 1$, and hence defines an analytic function in that half-plane. Except a pole in $s = 1$, we can analytically extend the function into the whole complex plane (see [25] for details). The analytic extension obey a functional equation:

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}. \tag{1.1}$$

Here $\Gamma(s)$ is the Gamma function, defined by the infinite product[2]

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^{\infty}\left(1 + \frac{s}{n}\right)^{-1} e^{\frac{s}{n}}.$$

[1] Actually we have been slightly lying thus far. In fact, $S(t)$ is defined to be π times the argument. We will soon give the correct definition.

[2] The γ in the exponent $e^{-\gamma s}$ is known as the Euler-Mascheroni constant, defined by $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right)$.

The Gamma function is holomorphic except its poles at $z = 0, -1, -2, \ldots$.

So what can we say about the zeroes of this mysterious function $\zeta(s)$? By the symmetry of the functional equation it is enough to look at the zeroes that have imaginary part bigger than 0. From the Euler product representation of ζ, valid for $\mathrm{Re}(s) > 1$, one can see that there are no zeroes for $\mathrm{Re}(s) > 1$. The poles of the Γ-function alongside the functional equation implies in turn that the only zeroes of $\zeta(s)$ for $\mathrm{Re}(s) < 0$ are at $s = -2, -4, -6, \ldots$. On the line $\mathrm{Re}(s) = 1$, $\zeta(s)$ also has no zeroes, but this is relatively deep fact compared to the earlier mentioned facts about the zeroes. Some would even say that this is the statement of the prime number theorem. Why is this so?

If $\pi(x)$ counts the number of primes p in the range $1 \leq p \leq x$, then the statement of the prime number theorem is that[3] $\pi(x) \sim x/\log x$. We can also state this equivalently in terms of the Chebyshev function defined in the introduction: $\psi(x) \sim x$. The definite integral from 0 to x of the Chebyshev function is known as $\psi_1(x)$:

$$\psi_1(x) = \int_0^x \psi(t)\,\mathrm{d}t.$$

In terms of ψ_1, $\psi(x) \sim x$ reads $\psi_1(x) \sim x^2/2$. We have an explicit formula for $\psi_1(x)$ similar to the one for $\psi_0(x)$ that was mentioned in the introduction:

$$\psi_1(x) = \frac{x^2}{2} + \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} + \text{smaller error terms}.$$

The sum over the non-trivial zeroes ρ, converges uniformly. Hence, if one divide both sides by $x^2/2$ and take the limit as $x \to \infty$, this allows us to pull the limit inside the sum. The fact that ζ has no zeroes $\rho = \beta + i\gamma$ with $\beta = 1$ then gives that $\psi_1(x) \sim x^2/2$. Going back to the topic of zeroes of ζ, we are then left with studying the zeroes of $\zeta(s)$ for $0 < \mathrm{Re}(s) < 1$. This is often referred to as the critical strip. All the zeroes inside the critical strip are called non-trivial.

In contrast to the horizontal distribution of the zeroes, the vertical distribution of the zeroes is much better understood. To elaborate more on this, we start by defining Riemann's ξ-function as

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The functional equation (1.1) then takes the short elegant form $\xi(s) = \xi(1-s)$. From a general complex analytical viewpoint, ξ is a very nice function as it is entire of order 1. Its zeroes are precisely the non-trivial zeroes of ζ-function (with same position, i.e. inside the critical strip we have $\zeta(s) = 0 \iff \xi(s) = 0$). Thus if we want to study the non-trivial zeroes of ζ we could just as well study the zeroes of ξ. The number of zeroes $\rho = \beta + i\gamma$[4] of ξ with $0 \leq \gamma \leq T$ is defined to be $N(T)$. Let R be rectangle with vertices at $2, 2+iT, -1+iT$ and -1. The argument principle from complex analysis then gives

$$N(T) = \oint_R \frac{\xi'}{\xi}(s)\,\mathrm{d}s.$$

[3]Here $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

[4]Not to be confused with the Euler-Mascheroni constant γ.

A tedious calculation including Stirling's formula for the Γ-factor of $\xi(s)$, gives that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + S(T) + \frac{7}{8} + O(T^{-1}).$$

Here the term $S(T)$ appears from the ζ-factor in the definition of ξ, and it is the most mysterious one. More explicitly, if we let L be the line starting at 2, going to $2+iT$, then finally to $1/2+iT$, $S(T)$ is defined as

$$S(T) = \frac{1}{\pi}\int_L \frac{\zeta'}{\zeta}(s)\,\mathrm{d}s.$$

Equivalently $S(t)$ can be defined by

$$S(T) = \frac{1}{\pi}\arg\zeta(1/2+iT)$$

where the argument is obtained by continuous variation along L and with $\arg\zeta(2)=0$.

Von Mangoldt [27] proved that $S(t) = O(\log t)$ in 1905, and since then no one has been able to improve the magnitude in this upper bound. Although this is sufficient to determine the main term of $N(T)$, the finer structure in the distribution of zeroes, tends to be hidden in the $S(T)$-term. As an example, it is relatively easy to deduce from the formula for $N(T)$ that the vertical distance $\gamma_{n+1}-\gamma_n$ between consecutive zeroes $\rho_n = \beta_n + i\gamma_n$ and $\rho_{n+1} = \beta_{n+1}+i\gamma_{n+1}$ is $O(1)$. Better bounds on $S(t)$ have been proven to produce better bounds on the distance between consecutive zeroes. Assuming RH one has $S(t) = O(\log t/\log\log t)$. The same proof as for the fact that $\gamma_{n+1}-\gamma_n = O(1)$, will with this new bound give

$$\gamma_{n+1}-\gamma_n = O\left(\frac{1}{\log\log\gamma_n}\right).$$

It belongs to this discussion to mention that one can, without assuming RH, get better bounds then $\gamma_{n+1}-\gamma_n = O(1)$. With clever usage of the Borel-Carathéodory lemma and Hadamard's three-circle theorem, it is possible to prove that $O(1/\log\log\log\gamma_n)$. For details, we refer to [25, Thm 9.12.].

The example above shows the possibility $S(t)$ has to control the finer structure of the distribution of zeta zeroes. Thus it has become a function many brilliant mathematicians have studied over the last centuries. Its true size is of course on of the questions that researchers have been pondering about. Farmer, Gonek and Hughes [14] conjectured that

$$\limsup_{t\to\infty}\frac{S(t)}{\sqrt{\log t\log\log t}} = \frac{1}{\pi\sqrt{2}}.$$

If one is to believe this conjecture, the bounds we have for $S(t)$ are far away from the actual truth. Even the conditional bound (that is, assuming RH) mentioned above miss by quite a lot.

1.2 Ω-results for $S(t)$

Since we seem to be bounding $S(t)$ by way too much, we could instead go the opposite way and ask how large values of $S(t)$ we can exhibit. Can we then get closer to the conjectural maximum of Farmer–Gonek–Hughes? Tsang [26] proved in the 80s that

$$S(t) = \Omega_{\pm}\left(\left(\frac{\log t}{\log\log t}\right)^{1/3}\right).$$

As far as the author is aware, this is still the best unconditional Ω-result for $S(t)$ to this date. The approach of Tsang originated from works of Selberg[22] using high moments to detect large values of $S(t)$.

The best conditional result is much more recent, and is due to Bondarenko and Seip [4]. In 2018 they proved that there exists a constant $\mathscr{C}_\beta$ such that

$$\max_{T^\beta \le t \le T} |S(t)| \ge \mathscr{C}_\beta \sqrt{\frac{\log T \log\log\log T}{\log\log T}}. \tag{1.2}$$

They obtain this result by means of the resonance method. This method was initiated independently by Soundararajan and Hilberdink [17, 23] in 2008 and 2009 respectively. We will survey the work of Soundararajan in the next chapter.

The result (1.2) has been generalized and extended. An extension is due to Chirre and Mahatab [10]. Still assuming RH, they were able to show that large values of both signs occur, i.e. they improved the Ω-result to a $\Omega_\pm$-result. They use the same resonator as Bondarenko–Seip, but use a different convolution formula. Assuming GRH, Xiao and Yang [29] has generalized the result of Chirre–Mahatab to Rankin–Selberg L-functions of holomorphic cusp forms. Our contribution to this list of generalizations and extensions is two-folded. We provide a constant $\mathscr{C}_\beta$ for (1.2), and we generalize the result to also hold for Dedekind zeta functions of cyclotomic fields. In this case we get a much larger constant, of the form $\mathscr{C}_\beta\sqrt{\varphi(q)}$[5]. The latter give us the opportunity to deduce an interesting dichotomy. Like our predecessors we keep assuming (G)RH.

1.3 More L-functions

The Riemann zeta function has generalizations in several directions, going into both algebraic and algebro-geometric aspects to mention two of them. In general, generalizations of the Riemann zeta function fall under the term "L-functions", and we will now introduce two new L-functions that we will be working with in Chapter 4.

[5] $\varphi(q)$ denotes Euler's totient function counting the number of positive integers a with $\gcd(a, q) = 1$.

Dirichlet L-functions

Dirichlet L-functions were introduced in 1837 by Dirichlet in [13]. Many mathematicians agree that this marks the beginning of analytic number theory[6]. Although Dirichlet came before Riemann, it is still right to say that the class of Dirichlet L-functions is a generalization of the Riemann zeta function, because the Riemann zeta function falls out as a special case. Dirichlet's goal with introducing these functions was to study primes in arithmetic progressions: $a, a+q, a+2q, a+3q, \ldots$ with $\gcd(a,q)=1$. Moreover he wanted to prove that there were an infinitude of them. To do this he used what is now known today as Dirichlet characters. They are multiplicative group homomorphisms $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$. We extend χ to all of $\mathbb{Z}$ by declaring $\chi(k)=0$ whenever $\gcd(k,n)>1$. In this case we say that a χ is a (Dirichlet) character modulo[7] q. There are $\varphi(q)$ distinct Dirichlet characters modulo q.

To each Dirichlet character we can associate a Dirichlet L-function, $L(s,\chi)$. It is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\frac{\chi(p)}{p^s}}$$

for $\mathrm{Re}(s)>1$. The trivial character χ_0, i.e. the one sending everything to 1, has more or less the Riemann zeta function as its Dirichlet L-function:

$$L(s,\chi_0) = \prod_{p|q} \left(1-\frac{1}{p^s}\right)\zeta(s).$$

This justifies the assertion from earlier about Dirichlet L-functions being generalizations of the Riemann zeta function.

All Dirichlet L-functions modulo q interact in an interesting way together. The characters obey a certain orthogonality relation:

$$\sum_{\chi} \chi(a)\overline{\chi(b)} = \begin{cases} \varphi(q) & a \equiv b \pmod{q} \\ 0 & \text{otherwise.} \end{cases}$$

Here the sum is taken over all Dirichlet characters modulo q. This relation is what allows us to tap into the prime distribution in arithmetic progressions $\{a+nq\}_n$. As an example of this principle in action, let us multiply the Dirichlet series for $L(s,\chi)$ with $\overline{\chi(a)}$ and sum over all χ:

$$\sum_{\chi} \chi(a) \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\chi} \overline{\chi(a)}\chi(n) = \sum_{\substack{n\geq 1,\\ n\equiv a \pmod{q}}} \frac{1}{n^s}.$$

Now we just sum over the arithmetic progression $\{a+kq\}_k$ on the right. An application of this idea to the logarithm of the Euler product, is the starting point of most proofs for the fact that there are infinitely many primes of the form $a+nq$.

[6]In fact, one of the most classical books on the subject, Multiplicative Number Theory by Harold Davenport, starts with: *Analytic number theory may be said to begin with the work of Dirichlet, and in particular with Dirichlet's memoir of 1837 on the existence of primes in a given arithemtic progression.*

[7]We remark that q does not denote a prime here.

We shall not go into any more depth about the theory of Dirichlet L-functions, but mention that the function enjoys many of the same properties as the Riemann zeta function. Most importantly, we have analytic continuation into the rest of the complex plane (with a pole in $s = 1$ if χ is trivial), as well as a functional equation.

Dedekind zeta functions

A number field K is a finite field extension of the rational numbers. An example of one such is $\mathbb{Q}(i)$: this is the set of all numbers of the form $a + bi$ with $a, b \in \mathbb{Q}$. To each number field K, we associate a ring $\mathcal{O}_K$ called the ring of integers of K. If $K = \mathbb{Q}$, then $\mathcal{O}_K = \mathbb{Z}$. If $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$, i.e. the Gaussian integers. In general, $\mathcal{O}_K$ is constructed in such a way that it mimics the relation between $\mathbb{Z}$ and $\mathbb{Q}$. In this way, we study the integers in a field extension K by looking at $\mathcal{O}_K$. The study of $\mathcal{O}_K$ is at the heart of algebraic number theory.

The ring $\mathbb{Z}$ is a unique factorization domain. For other ring of integers though, this may not be the case. It turns out however, that if one goes to the level of ideals, one recovers unique factorization. That is, if we have some non-zero ideal in $\mathcal{O}_K$, it admits a unique factorization into prime ideals. This is one of the very pleasant features of $\mathcal{O}_K$. Prime ideals thus become the "correct" generalization of prime numbers, when we are working with $\mathcal{O}_K$. We remark in the case $K = \mathbb{Q}$, that the non-zero prime ideals correspond to precisely the prime numbers.

The Dedekind zeta function $\zeta_K(s)$ of a number field K is "the Riemann zeta function" for the number field K. It is used to study the distribution of prime ideals in $\mathcal{O}_K$, just like the Riemann zeta function is used to study the distribution of primes in $\mathbb{Z}$. The norm of a non-zero ideal I in $\mathcal{O}_K$ is defined as

$$\|I\| := |\mathcal{O}_K/I|.$$

The norm can be proven to be finite, and it enjoys the property of being multiplicative:

$$\|IJ\| = \|I\|\|J\|.$$

This property give us an Euler-product for the Dedekind zeta function. Without further ado, we define

$$\zeta_K(s) := \sum_I \frac{1}{\|I\|^s} = \prod_P \frac{1}{1 - \|P\|^{-s}}.$$

The sum is over all non-zero ideals, and the product is over all non-zero prime ideals.

Like the Riemann zeta function, the Dedekind zeta function of a number field K can be analytically extended to the whole complex plane except a pole in $s = 1$. Furthermore we have a similar functional equation to that of the Riemann zeta function. This was first proven by Hecke [16] in 1917. His proof mimics Riemann's classical proof for the functional equation of the Riemann zeta function using Poisson's summation formula. The proof now however is done using multidimensional Fourier analysis. An elegant recast of Hecke's argument was offered by Tate [24][8] in the 50's in his PhD thesis[9]. Tate did also use Fourier analysis, but in a much more abstract setting.

Dedekind zeta functions can often be factorized into other L-functions. This falls under the theory of the Artin L-function. We will be concerned about one such factorization in this thesis, namely the factorization of cyclotomic Dedekind zeta function into Dirichlet L-functions. Although such a factorization hold for all cyclotomic fields, we will here only consider the case $K = \mathbb{Q}(\exp(2\pi i/q))$ with q prime. We have in this case that (see [5, Chapter 5, Section 2, Eq. (2.10)])

$$\zeta_K(s) = G(s) \prod_{\chi \bmod q} L(s,\chi). \tag{1.3}$$

Here $G(s)$ is a finite product over ramified primes, more precisely because q is prime we have

$$G(s) = \prod_{\mathfrak{p}|q\mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = \left(1 - \frac{1}{q^s}\right)^{-1}.$$

Taking the logarithm of (1.3) we get the following formula we will use in Chapter 4:

$$\sum_{\chi} \log L(1/2 + i(t+u), \chi) - \log\left(1 - \frac{1}{q^s}\right) = \log \zeta_K(1/2 + i(t+u)). \tag{1.4}$$

To give a proof of (1.3) would lead us too far astray, because we would need to introduce a certain amount of algebraic number theory. The interested reader may find a nice exposition of the proof in [20, p. 137-139.]

[8] A version of Tate's thesis can be found in Cassels & Frölich [8].

[9] Tate's approach was independently discovered by Iwasawa around the same time. It generalizes to many more L-functions, and the theory has gotten the fitting name Iwasawa–Tate-theory.

Chapter 2

A primer on the resonance method

2.1 The resonance method - a historical survey

Soundararajan's resonance method

The resonance method is perhaps easiest motivated by means of Soundararajan's approach. The method revolves around choosing a function, $|R(t)|^2$, called a *resonator*, that "picks out" large values (along vertical lines) of some function that we are interested in[1]. Soundararajan's desire was to pick out large values of the Riemann zeta function on the critical line. Clearly we have

$$\left|\int_T^{2T} \zeta(1/2+it)|R(t)|^2 \,\mathrm{d}t\right| \leq \max_{T\leq t\leq 2T} |\zeta(1/2+it)| \int_T^{2T} |R(t)|^2 \,\mathrm{d}t \tag{2.1}$$

and thus

$$\max_{T\leq t\leq 2T} |\zeta(1/2+it)| \geq \frac{\left|\int_T^{2T} \zeta(1/2+it)|R(t)|^2 \,\mathrm{d}t\right|}{\int_T^{2T} |R(t)|^2 \,\mathrm{d}t}. \tag{2.2}$$

A good choice for a resonator is as a Dirichlet polynomial,

$$R(t) = \sum_{1\leq n\leq N} \frac{r(n)}{n^{it}}$$

where $r(n) \in \mathbb{C}$. This keep things computable, and it interacts reasonably well with the Riemann zeta function. The reason for the latter is that the Riemann zeta function can be approximated quite well inside the critical strip, by truncating its Dirichlet series definition that was earlier valid only for $\mathrm{Re}(s) > 1$. More specifically, we have

$$\zeta(1/2+it) = \sum_{n\leq T} n^{-1/2-it} + O(T^{-1/2})$$

[1] The name *resonance method* comes from the idea that $|R(t)|^2$ should resonate well with large values of the choosen function

for $T \leq t \leq 2T$. To compute the second moment of the resonator, i.e. the denominator of (2.1), we require that $N \leq T^{1-\varepsilon}$. This last requirement is perhaps easiest understood in light of the Montgomery–Vaughan mean value theorem. It states that

$$\int_T^{2T} \left| \sum_{n \leq N} \frac{r(n)}{n^{it}} \right|^2 \mathrm{d}t = (T + O(N)) \sum_{n \leq N} |r(n)|^2. \tag{2.3}$$

The idea of the proof of this theorem is to expand the sum inside the integral and differ between the *diagonal* terms and the *off-diagonal* terms. Expanding the sum reveals these two categories:

$$\begin{aligned} \int_T^{2T} \left| \sum_{n \leq N} \frac{r(n)}{n^{it}} \right|^2 \mathrm{d}t &= \int_T^{2T} \sum_{m \leq N} \sum_{n \leq N} r(m)\overline{r(n)} \left(\frac{m}{n}\right)^{-it} \mathrm{d}t \\ &= \underbrace{\int_T^{2T} \sum_{m \leq N} |r(m)|^2 \,\mathrm{d}t}_{\text{diagonal terms}} + \underbrace{\int_T^{2T} \sum_{m,n \leq N, m \neq n} r(m)\overline{r(n)} \left(\frac{m}{n}\right)^{-it} \mathrm{d}t}_{\text{off-diagonal terms}}. \end{aligned}$$

The diagonal terms are those terms from the double sum where $m = n$, and the off-diagonal terms consists of the remaining ones. The diagonal term above clearly evaluates to $T \sum_{n \leq N} |r(n)|^2$. The off-diagonal terms on the other hand can be a bit cumbersome to deal with. What (2.3) tells us, is that if we keep $N \ll T^{1-\varepsilon}$, the off-diagonal terms stays under control — they become lower order error terms.

Let us turn back to (2.2). This is more or less how Soundararajan's setup is. There is a small difference - he also has a smoothening factor in the integrand. This is to avoid possible sharp cutoffs in both endpoints of the integration intervals. Let ϕ denote a smooth, non-negative function with compact support in $[1, 2]$ and such that $\phi(y) = 1$ for $5/4 \leq y \leq 7/4$. Then Soundararajan instead considers the fraction

$$\frac{\left| \int_{-\infty}^{\infty} \zeta(1/2 + it) |R(t)|^2 \phi(t/T) \,\mathrm{d}t \right|}{\int_{-\infty}^{\infty} |R(t)|^2 \phi(t/T) \,\mathrm{d}t} \leq \max_{T \leq t \leq 2T} |\zeta(1/2 + it)|. \tag{2.4}$$

Going through some simple analysis, one finds that the fraction on the left hand side is bigger than or equal to

$$(1 + o(1)) \frac{\left| \sum_{mk \leq N} r(m)\overline{r(mk)}/\sqrt{k} \right|}{\sum_{n \leq N} |r(n)|^2}. \tag{2.5}$$

This is precisely the diagonal terms one would end up with if one calculated the numerator and denominator of (2.4). Soundararajan quite elegantly compute the supremum of (2.5) taken over all possible coefficients $\{r(n)\}_{n=1}^N$. He proves that

$$\max_r \frac{\left| \sum_{mk \leq N} r(m)\overline{r(mk)}/\sqrt{k} \right|}{\sum_{n \leq N} |r(n)|^2} = \exp\left(\sqrt{\frac{\log N}{\log\log N}} + O\left(\frac{\sqrt{\log N}}{\log\log N} \right) \right), \tag{2.6}$$

and so going back to the Riemann zeta function one deduce for sufficiently large T that

$$\max_{T\le t\le 2T}|\zeta(1/2+it)|\ge \exp\left((1+o(1))\sqrt{\frac{\log T}{\log\log T}}\right).$$

The beginning of the long resonator method

When $N\le T^{1-\varepsilon}$, Soundararajan's theorem is optimal. Because he is able to compute the asymptotic of (2.6), there is nothing left to improve in the current setup. To improve upon Soundararajan's method one has to introduce novel ideas into the very setup itself. Recall that we took our resonator $|R(t)|^2$ to be

$$R(t)=\sum_{1\le n\le N}\frac{r(n)}{n^{it}}$$

with the constraint $N\le T^{1-\varepsilon}$. To improve upon Soundararajan's method we have to get rid of this constraint in one way or another. This is roughly what the *long resonance method* sets out to do. As the name suggest we consider a longer resonator than usual — we will allow terms $r(n)n^{-it}$ that go past the restriction $n\le N$ that we had earlier. In light of (2.3) we cannot treat the off-diagonals like we did earlier. It required some new ideas to figure out how to deal with a longer resonator. The sort of discrete optimization problem that we end up with in this case is also different to the one of Soundararajan, i.e. (2.6).

It is probably correct to say that the modern development of the long resonator method started with Hilberdink and Aistleitner. In [17], Hilberdink rediscovers a connection between lower bounds of the maximum of $|\zeta(\sigma+it)|$ and gcd-sums:

$$\sum_{k,\ell=1}^{N}\frac{\gcd(n_k,n_\ell)^{2\sigma}}{(n_kn_\ell)^\sigma}.$$

Here $n_1,\dots,n_N$ are arbitrary natural numbers. These kind of sums are also often called Gál sums because the first systematic study was initiated by Gál[15] by determining the asymptotics in the case $\sigma=1$. This connection allowed Hilberdink to prove an Ω-result for $\zeta(\sigma+it)$ for fixed $1/2<\sigma<1$, which we shall not state here. Some years later Aistleitner pointed out that (see his remark [1, p. 479]) Voronin had given more or less the same proof in 1988 [28]. Sadly his paper appears to have gone unnoticed. It has six citations according to MathSciNet and all of them are from 2016 or later, and it is probably thanks to Aistleitner that it surfaced again.

The idea of Hilberdink closely resembles that of Soundararajan in the beginning. He considers the ratio

$$\frac{\int_0^T|\zeta(\sigma+it)|^2|A(t)|^2\,\mathrm{d}t}{\int_0^T|A(t)|^2\,\mathrm{d}t} \tag{2.7}$$

which as we have seen before provides a lower bound for $\max_{t\in[0,T]}|\zeta(\sigma+it)|$. Soundararajan's resonator $|R(t)|^2$ is interchanged with an Euler product,

$$A(t):=\prod_{r=1}^{M}(1+p_r^{it})=\sum_{k\le N}b_k^{it}$$

where $p_1, p_2, \ldots$ denotes the primes in ascending order, and $b_1, b_2, \ldots$ are the numbers formed by multiplying these, i.e. of the form $p_1^{\beta_1} \cdots p_M^{\beta_M}$ with $\beta_i \in \{0, 1\}$. Like Soundararajan, Hilberdink approximates $\zeta(\sigma + it)$ by a Dirichlet polynomial:

$$\zeta(\sigma + it) = \sum_{n \le t} \frac{1}{n^{\sigma + it}} + O(t^{-\sigma}).$$

Because Hilberdink instead considers the second moment of ζ instead of the first moment like Soundararajan did, he ends up with a different optimization problem. Under suitable conditions one has

$$\begin{aligned} &\int_0^T |\zeta(\sigma + it)|^2 |A(t)|^2 \,\mathrm{d}t \\ &= \sum_{k,\ell \le N} \sum_{m,n \le T} \frac{1}{(mn)^\sigma} \int_{\max\{m,n\}}^T \left(\frac{mb_k}{nb_\ell} \right)^{it} \mathrm{d}t + \text{lower order error terms.} \end{aligned} \tag{2.8}$$

Like before we do a diagonal/off-diagonal analysis. The diagonal are those terms where $mb_k = nb_\ell$. From elementary theory of diophantine equations, we know that the solutions to the equation $mb_k = nb_\ell$ are given by

$$m = \frac{jb_\ell}{\gcd(b_k, b_\ell)} \qquad n = \frac{jb_k}{\gcd(b_k, b_\ell)}.$$

Under certain conditions on M, $b_k \le T^\varepsilon$ for any $\varepsilon > 0$, and in this case one can guarantee a solution to $mb_k = nb_\ell$ in the range $1 \le m, n \le T^\varepsilon$. This in turn implies that the diagonal $mb_k = nb_\ell$ of (2.8) is

$$\gg T \sum_{k,\ell \le N} \frac{(\gcd(b_k, b_\ell))^{2\sigma}}{(b_k b_\ell)^\sigma}.$$

The off-diagonal is luckily negligable, and for this it is important that the fractions $(mb_k)/(nb_\ell)$ stay bounded away from 1 when $mb_k \ne nb_\ell$. For the calculation of the second moment of $A(t)$, i.e. the denominator of (2.7), it is also crucical that b_k/b_ℓ stay bounded away from 1.

This is where Aistleitner [1] enters the scene. He was able to employ a much longer resonator than Hilberdink (taking a larger M than Hilberdink could), by introducing some novel ideas. As mentioned in the paragraph above, it is important to have control over the ratios $mb_k/(nb_\ell)$ and b_k/b_ℓ. Increasing the value of M in Hilberdink's setup would cause problems regarding this. To get around this problem Aistleitner "glue" together those pairs (b_k, b_ℓ) where b_k/b_ℓ is close to 1 (but not 1!). One can show that such terms would not contribute much to the gcd-sum after all. This gluing or "sparsification" process is done as follows. Let B be the set of all the $b_1, b_2, \ldots$. We partition this set into new sets B_j for $j = 1, \ldots, K$ defined by

$$B_j := B \cap \left((1 + T^{-1})^{j-1}, (1 + T^{-1})^j \right].$$

Observe that if b_k and b_ℓ are in the same B_j then their ratio are necessarily very close to 1. From each non-empty B_j, we choose one representative d_j, namely the least one:

$$d_j := \min B_j.$$

In place of the previous $A(t)$ one now considers

$$A(t) := \sum_{k=1}^{K} d_k^{it}.$$

This, alongside some other ideas, was enough to improve upon Hilberdink's result, and improve the magnitude in his Ω-result. Although we have not seen it now, we remark that Aistleinter's method hinged on the positivity of the coefficients of the Dirichlet polynomial that he used to approximate the Riemann zeta function. This has since stayed as one of the shortcomings of the long resonator method — Soundararajan's resonance method does not suffer from this.

The long resonator method ala Bondarenko–Seip

In this subsection we shall give a short exposition of the long resonator method ala Bondarenko–Seip as done in their breakthrough paper [3]. At the time the preprint of [1] was announced, one knew what the optimal upper bound for

$$\sum_{k,\ell=1}^{N} a_k a_\ell \frac{\gcd(n_k, n_\ell)^{2\sigma}}{(n_k n_\ell)^\sigma}$$

was in the case $\sigma \in (1/2, 1)$. Here $\sum_{k=1}^{N} a_k^2 \leq 1$. The bound was of the form

$$\exp\left(\frac{c_\sigma (\log N)^{1-\sigma}}{(\log\log N)^\sigma}\right)$$

for some constant c_σ. The correct magnitude in the case $\sigma = 1/2$ was however still not known. One knew that it would not be far away from the pattern seen for $\sigma \in (1/2, 1)$ - at most we missing with a factor of $\sqrt{\log\log\log N}$ in the exponent. Bondarenko and Seip showed in [4] that this extra factor was in fact necessary. Building on the work of Aistleinter, introducing some new ideas on their own, they deduced as a consequence that

$$\max_{T^{1/2} \leq t \leq T} |\zeta(1/2 + it)| \geq \exp\left((1/\sqrt{2} + o(1))\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right). \tag{2.9}$$

Bondarenko–Seip did go back to considering the first moment of ζ as Soundararajan, instead of the second moment like Aistleitner and Hilberdink. This avoids a certain rather technical part of Aistleitner's paper. They also developed further on Aistleinter's gluing/sparsification idea, introducing weights to each d_k in the $A(t)$. It is important to note here that Bondarenko and Seip choose a completely different $A(t)$. We should denote their resonator by $R(t)$. We will take a closer look at it in a few paragraphs.

As alluded to in the previous section, a certain positivity criterion on some coefficients will be crucial for this approach to work out. This again comes from the fact that we are dealing with a much longer resonator than usual, and thus we can't handle the off-diagonals in the same way Soundararajan did. The crucial observation to get around this, is that we only really need a lower bound for the numerator in (2.5). Expanding the numerator of (2.5) and applying the usual approximation

$$\zeta(1/2+it) = \sum_{k \le T} \frac{1}{n^{1/2+it}} + O(T^{-1/2}),$$

we get that

$$\begin{aligned}\int_{-\infty}^{\infty} \zeta(1/2+it)|R(t)|^2 \phi(t/T)\,\mathrm{d}t &= \sum_{k\le T} \sum_{m,n\le N} \frac{r(m)\overline{r(n)}}{k^{1/2}} \int_{-\infty}^{\infty} \left(\frac{km}{n}\right)^{-it} \Phi(t/T)\,\mathrm{d}t \\ &= T \sum_{k\le T} \sum_{m,n\le N} \frac{r(m)\overline{r(n)}}{k^{1/2}} \widehat{\phi}(T\log(km/n)). \end{aligned} \tag{2.10}$$

As we have seen before, Soundararajan chooses ϕ to be a bump function supported on $[1,2]$. Such a function cannot have a real Fourier transform, let alone positive, because it is not symmetric about 0. Because Soundararajan's resonator is shorter than the one we will employ, he can essentially ignore the off-diagonals and hence the choice of Φ doesn't really matter in this setting (as long as it has sufficiently nice decay). It follows that (2.10) equals

$$(1+o(1))T\widehat{\phi}(0) \sum_{mk\le N} \frac{r(m)\overline{r(mk)}}{\sqrt{k}}.$$

Since we will now be dealing with a longer resonator, we cannot do this transition. Instead we will have to lower bound (2.10), and for that we need a function ϕ with positive Fourier transform. Let $\phi(t) = \Phi(t) := \exp(-t^2/2)$. It will turn out that this is a good smoothening function for this setup. It is its own Fourier transform up to scaling (and is thus positive), and has good decay.

Since Φ does not vanish outside $[1,2]$ like our earlier bump function ϕ did, it is not obvious that this version of the resonance method should catch large values on the interval $[T,2T]$. In fact, this turns out to not be the case — we do not have enough decay in our weight Φ on such a "short" interval. If we instead settle on the interval $[T^\beta, T]$ for some fixed $0 \le \beta < 1$, we will be fine. This discussion has lead us to consider the problem of making the following fraction as big as possible

$$\frac{\left|\int_{T^\beta}^{T} \zeta(1/2+it)|R(t)|^2 \Phi(t\log T/T)\,\mathrm{d}t\right|}{\int_{T^\beta}^{T} |R(t)|^2 \Phi(t\log T/T)\,\mathrm{d}t}. \tag{2.11}$$

The new factor of $\log T$ in the weight is there for decay reasons.

Similarly to how Aistleitner obtained his resonator by "sparsifying" the resonator of Hilberdink, we can think of the resonator of Bondarenko–Seip as a sparsified version of

$$\sum_{m\in\mathscr{M}} \frac{f(n)}{n^{it}}.$$

Here $\mathscr{M}$ is choosen to be a set that nearly maximizes

$$\sum_{m,n\in\mathscr{M}} \frac{\gcd(m,n)}{\sqrt{mn}}.$$

Observe that we now also have weights $f(n)$. We shall not define those, as they serve no purpose for the discussion at hand. The only property they have that will be important for the discussion is that they are multiplicative, i.e.

$$f(mn) = f(m)f(n).$$

The actual resonator $|R(t)|^2$ of Bondarenko–Seip is now defined by

$$R(t) \coloneqq \sum_{n\in\mathscr{M}'} \frac{r(n)}{n^{it}}.$$

Here the $r(n)$ are defined by

$$r(n) \coloneqq \left(\sum_{\substack{n\in\mathscr{M}, \\ 1-T^{-1}(\log T)^2 \le m/n \le 1+T^{-1}(\log T)^2}} f(m)^2 \right)^{1/2}. \tag{2.12}$$

This probably looks rather unmotivated right now, but we will see soon the reason for this definition. The set $\mathscr{M}'$ is a set that has been obtained from another set $\mathscr{M}$ by Aistleitner's "gluing/sparsification"-idea. There is a condition on the size of $\mathscr{M}$ to make the proof of Bondarenko–Seip work, but it is not important for the discussion at hand. Let us now look at how one could go about bounding the numerator of (2.11) from below. Expanding the numerator we have

$$\frac{T}{\log T} \sum_{k\le T} \sum_{m,n\in\mathscr{M}'} \frac{r(m)\overline{r(n)}}{k^{1/2}} \widehat{\Phi}\left(\frac{T}{\log T} \log(km/n) \right). \tag{2.13}$$

All the terms are positive, so we get a lower bound for (2.13) by restricting to any subsum. A natural subsum to consider would be the diagonal $km = n$. If the sum $\sum_{m,n\in\mathscr{M}'}$ in (2.13) was instead $\sum_{m,n\in\mathscr{M}}$, we could have done this. The construction of $\mathscr{M}$ will guarantee the existence of sufficiently many solutions to the equation $km = n$. However, because we are dealing with our sparsified set $\mathscr{M}'$, this is not the case anymore. It turns out that if we instead restrict to the set $|km/n - 1| \le \frac{3}{T}$, then we get all our desired solutions. Furthermore one can use Cauchy–Schwarz to get

$$\sum_{\substack{m',n'\in\mathscr{M}', \\ |km'/n'-1| \le \frac{3}{T}}} r(m')r(n') \ge \sum_{\substack{m,n\in\mathscr{M}, \\ km=n}} f(m)f(n). \tag{2.14}$$

The latter inequality is the reason why the coefficients $r(n)$ are defined the way they are in (2.12). They are sort of designed to be applicable to this application of Cauchy–Schwarz.

We shall look at this precise inequality later with slightly other coefficients $r(n)$. For the application of Cauchy–Schwarz in this case we thus refer the reader to the end of proof of Theorem 1 in [3]. Piecing together all the tricks above we get the following lower bound:

$$\begin{aligned}&\sum_{k\leq T}\sum_{m,n\in\mathscr{M}'}\frac{r(m)\overline{r(n)}}{k^{1/2}}\widehat{\Phi}\left(\frac{T}{\log T}\log(km/n)\right)\\ \geq&(\widehat{\Phi}(0)+o(1))\sum_{k\leq T,m,n\in\mathscr{M},km=n}\frac{f(m)f(n)}{k^{1/2}}.\end{aligned} \tag{2.15}$$

This leave us with one unanswered question, and that is how the latter sum in (2.15) relates to large GCD-sums. We have completely ignored the construction of $\mathscr{M}$ and the coefficients $f(n)$ thus far. The construction/definition of both those will be given in the subsequent chapters. One property that we will see the extremal set $\mathscr{M}$ has, is begin divisor closed. This means that $d\in\mathscr{M}$ if $d\mid m$ for some $m\in\mathscr{M}$. The condition $km=n$ with $n\in\mathscr{M}$ thus implies $k\in\mathscr{M}$ as well. As mentioned before, the coefficients $f(n)$ are defined in such a way that they are multiplicative, i.e. $f(mn)=f(m)f(n)$. Putting these two facts together, and assuming we can remove the restriction $k\leq T$ in (2.15), we have the following chain of equalities:

$$\begin{aligned}\sum_{k,m,n\in\mathscr{M},km=n}\frac{f(m)f(n)}{k^{1/2}}&=\sum_{n\in\mathscr{M}}f(n)\sum_{m\mid n}f(m)\left(\frac{n}{m}\right)^{-1/2}\\&=\sum_{n\in\mathscr{M}}\frac{f(n)}{\sqrt{n}}\sum_{m\mid n}f(m)\sqrt{m}.\end{aligned}$$

The latter double sum relates to GCD-sums in the following way:

$$\sum_{m,n\in\mathscr{M}}f(n)f(m)\frac{\gcd(m,n)}{\sqrt{mn}}\geq\sum_{n\in\mathscr{M}}\frac{f(n)}{\sqrt{n}}\sum_{m\in\mathscr{M},m\mid n}f(m)\sqrt{m}. \tag{2.16}$$

This inequality turns out to not be too lossy. Thus a set that makes the GCD-sum

$$\sum_{m,n\in\mathscr{M}}f(n)f(m)\frac{\gcd(m,n)}{\sqrt{mn}}$$

large, will also make the right hand side of (2.16) large.

We have in the preceding paragraphs looked only at the numerator of (2.11). As discussed in the previous subsection, the sparsification of $\mathscr{M}$ is also important to be able to compute the denominator of (2.11). We shall not give an informal explanation of how one would go about this, as we have done with the numerator. But do not worry — we get back to the denominator in Lemma 3.1.2.

Optimizing the Bondarenko–Seip-approach

The long resonator method reached its best result so far regarding maximizing $|\zeta(1/2+it)|$ with a paper of de la Bretèche and Tenenbaum [12]. There they computed the asymptotic size of

$$\sup_{|\mathcal{M}|=N} \sum_{m,n\in\mathcal{M}} \frac{\gcd(m,n)}{\sqrt{mn}}.$$

Although the correct magnitude was already known, this put a definite end to which constant one should have in front. This improved[2] the $1/\sqrt{2}$ in (2.9) to $\sqrt{2}$. Funnily enough de la Bretèche–Tenenbaum did go back to the second moment of ζ again. Thus we have reason to believe that the next Ω-result for ζ should use the first moment of ζ.

2.2 The resonance method applied to $S(t)$

We will now sketch how Bondarenko and Seip obtained large values of $S(t)$ in [4]. We shall keep our informal style from the earlier sections, and stress that the details will appear in Chapter 3. They proved that there is a constant C_β such that for sufficiently large T, we obtain large values of size at least

$$C_\beta \sqrt{\frac{\log T \log\log\log T}{\log\log T}}$$

on the interval $[T^\beta, T]$. The starting point of their proof is the following convolution formula due to Selberg. It seems to first have appeared in the more general form in a paper of Tsang [26].

Lemma 2.2.1. *Let $1/2 \le \sigma < 1$ and let $K(x+iy)$ be an analytic function in the horizontal strip $\sigma - 2 \le y \le 0$. Suppose that K is such that*

$$V(x) := \max_{\sigma-2\le y\le 0} |K(x+iy)| = O\left(|x|^{-1}\log^{-2}|x|\right).$$

Then we have for every real $t \neq 0$ that

$$\begin{aligned}\int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))K(u)\,\mathrm{d}u = &\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\widehat{K}(\log n)n^{-\sigma-it} \\ &+ 2\pi\sum_{\beta>\sigma}\int_0^{\beta-\sigma} K(\gamma - t - i\alpha)\,\mathrm{d}\alpha + O(V(t)).\end{aligned}$$

Here the second sum is over all the non-trivial zeroes $\rho = \beta + i\gamma$ with $\beta > \sigma$.

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[2] Between the two papers [3] and [12], Bondarenko–Seip improved the constant from $1/\sqrt{2}$ to 1 in [4]. They did this by using a convolution formula associated to $|\zeta(1/2+it)|$ instead of approximating $\zeta(1/2+it)$ by $\sum_{n\le T} n^{-1/2-it}$. de la Bretèche and Tenenbaum did also use a convolution formula, but for $|\zeta(1/2+it)|^2$ instead of $|\zeta(1/2+it)|$. The latter convolution formula makes the gcd-sums appear more naturally than it did in the Bondarenko–Seip setup

Proof. See [26, Lemma 5]. □

Assuming RH and choosing $\sigma = 1/2$, we see that the second sum in Lemma 2.2.1 vanishes. Taking imaginary parts and assuming that $K(u)$ is real valued for $u \in \mathbb{R}$, we obtain a convolution formula for $S(t)$:

$$\int_{-\infty}^{\infty} S(t+u)K(u)\,\mathrm{d}u = \frac{1}{\pi}\mathrm{Im}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}\widehat{K}(\log n)n^{-1/2-it} + O(V(t)). \tag{2.17}$$

We have here extended the definition of $S(t)$ to the whole of $\mathbb{R}$ by declaring it to be an odd function. To obtain large values of $S(t)$ we will as before integrate (in the variable t) against a resonator.

Integrating (2.17) against the resonator $|R(t)|^2$, where as before

$$R(t) := \sum_{n\in\mathscr{M}'}\frac{r(n)}{n^{it}},$$

we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2\Phi(t/T)\,\mathrm{d}t \\
&= \frac{1}{\pi}\mathrm{Im}\int_{-\infty}^{\infty}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{\log k}\widehat{K}(\log k)k^{-1/2-it}|R(t)|^2\Phi(t/T)\,\mathrm{d}t + \underbrace{O\left(\int_{-\infty}^{\infty} V(t)|R(t)|^2\Phi(t/T)\,\mathrm{d}t\right)}_{:=E} \\
&= \frac{1}{\pi}\mathrm{Im}\sum_{k=2}^{\infty}\sum_{n,m\in\mathscr{M}}\frac{\Lambda(k)r(m)r(n)}{k^{1/2}\log k}\widehat{K}(\log k)\int_{-\infty}^{\infty}\Phi(t/T)\left(\frac{kn}{m}\right)^{-it}\mathrm{d}t + E \\
&= \frac{T}{\pi}\mathrm{Im}\sum_{k=2}^{\infty}\sum_{n,m\in\mathscr{M}}\frac{\Lambda(k)r(m)r(n)}{k^{1/2}\log k}\widehat{K}(\log k)\widehat{\Phi}(T\log(kn/m)) + E.
\end{aligned} \tag{2.18}$$

$\Phi(t)$ still denotes the Gaussian, $\exp(-t^2/2)$. The change of summation and integration above is justified by absolute convergence as long as K is chosen to be sufficiently nice in this regard. We will end up with choosing

$$K(t) = -(\log\log T)^2 t\Phi(t\log\log T).$$

It may very well be the case that there are other perfectly fine choices for K, but this one suffices. It enjoys three properties: sufficiently good decay, the function is odd, and the imaginary part of the Fourier transform is positive. The weight of $\log\log T$ is there more or less for technical reasons.

Some computing will reveal that the left hand side side of (2.18) is localized where $T^\beta \leq$

$|t| \leq T \log T$. Using basic properties of K, it follows that

$$\begin{aligned}
&\int_{T^\beta \leq |t| \leq T\log T} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2\Phi(t/T)\,\mathrm{d}u\,\mathrm{d}t \\
&\ll \left(\max_{T^\beta/2 \leq t \leq 2T\log T} |S(t)|\right) \int_{T^\beta \leq |t| \leq T\log T} |R(t)|^2\Phi(t/T)\,\mathrm{d}t \\
&\ll \left(\max_{T^\beta/2 \leq 2T\log T} |S(t)|\right) \int_{-\infty}^{\infty} |R(t)|^2\Phi(t/T)\,\mathrm{d}t. \qquad (2.19)
\end{aligned}$$

Observe that we have extended the interval by dividing by 2 in the lower bound and multiplying by 2 in the upper. This is for technical reasons, and have no impact on the final result. We now obtain just as in (2.4), the following inequality

$$\max_{T^\beta/2 \leq t \leq 2T\log T} |S(t)| \gg \frac{\int_{T^\beta \leq |t| \leq T\log T} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2\Phi(t/T)\,\mathrm{d}u\,\mathrm{d}t}{\int_{-\infty}^{\infty} |R(t)|^2\Phi(t/T)\,\mathrm{d}t}.$$

We need to extend the integration range in t to the whole real line to be able to utilize the positivity of the Fourier transform of Φ. This will only induce small error terms. The ratio we want to maximize is thus

$$\frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2\Phi(t/T)\,\mathrm{d}u\,\mathrm{d}t}{\int_{-\infty}^{\infty} |R(t)|^2\Phi(t/T)\,\mathrm{d}t}. \qquad (2.20)$$

The denominator has the following upper bound:

$$\int_{-\infty}^{\infty} |R(t)|^2\Phi(t/T)\,\mathrm{d}t \ll T \sum_{n\in\mathscr{M}'} r(n)^2.$$

We prove this bound in Lemma 3.1.2. We now insert (2.18) into (2.20), and use the bound for the denominator to get

$$\begin{aligned}
&\frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2\Phi(t/T)\,\mathrm{d}u\,\mathrm{d}t}{\int_{-\infty}^{\infty} |R(t)|^2\Phi(t/T)\,\mathrm{d}t} \\
&\gg \frac{\mathrm{Im}\sum_{k=2}^{\infty}\sum_{m,n\in\mathscr{M}'} \frac{\Lambda(k)r(m)r(n)}{k^{1/2}\log k}\widehat{K}(\log k)\widehat{\Phi}(T\log(kn/m))}{\sum_{n\in\mathscr{M}'} r(n)^2} + O\left(\frac{\int_{-\infty}^{\infty} V(t)|R(t)|^2\Phi(t/T)\,\mathrm{d}t}{\sum_{n\in\mathscr{M}'} r(n)^2}\right).
\end{aligned}$$

Since all the terms in the numerator in the latter sum are positive we can restrict to any subsum. For this we proceed in the following way:

$$\begin{aligned}
&\mathrm{Im}\sum_{k=2}^{\infty}\sum_{m,n\in\mathscr{M}'} \frac{\Lambda(k)r(m)r(n)}{k^{1/2}\log k}\widehat{K}(\log k)\widehat{\Phi}(T\log(kn/m)) \\
&\geq \left(\min_{p\in P} \mathrm{Im}\widehat{K}(\log p)\right) \mathrm{Im}\sum_{p\in P}\sum_{m,n\in\mathscr{M}'} \frac{r(m)r(n)}{k^{1/2}}\widehat{\Phi}(T\log(pn/m)) \\
&\geq \left(\min_{p\in P} \mathrm{Im}\widehat{K}(\log p)\right) \mathrm{Im}\sum_{p\in P}\sum_{\substack{m,n\in\mathscr{M}',\\ |pm/n-1|\leq 3/T}} \frac{r(m)r(n)}{k^{1/2}}\widehat{\Phi}(T\log(pn/m)).
\end{aligned}$$

Here P is a certain interval of primes (see the beginning of Section 3.1 for definition). Now we are in a position to use (2.14). This give us a way to relate the chain of inequalities above to GCD-sums like we did towards the end of the previous section.

This is more or less the way we will proceed in the next chapter. There is only a small difference: we will sparsify the set $\mathscr{M}$ in a (slightly) different way. This results in a marginally larger constant in Theorem 3.0.1 compared to what the "old" sparsification process would give.

Chapter 3

An explicit Ω-result for $S(t)$

The goal for this chapter is to prove the following theorem.

Theorem 3.0.1. *Assume RH and let* $0 < \beta \le 1$ *be fixed. Then for any constant* $c < \sqrt{1-\beta}$ *we have*

$$\max_{T^\beta \le t \le T} |S(t)| \ge 0.02929c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}.$$

We will use the following notation for the iterated logarithm $\log_2 x := \log\log x, \log_3 x := \log\log\log x$. We declare $\Phi(x) := \exp(-x^2/2)$.

3.1 Some explicit estimates

Set $N = [T^\kappa]$ where $\kappa < 1 - \beta$ and $0 < \beta \le 1$. Let $0 < \gamma < 1$ and let P be the set of all primes p such that

$$e \log N \log_2 N < p \le \log N \exp((\log_2 N)^\gamma) \log_2 N.$$

We partition P into sets P_k,

$$P_k := P \cap (e^k \log N \log_2 N, e^{k+1} \log N \log_2 N]$$

for $k = 1, 2, \dots, [(\log_2 N)^\gamma]$. We define a multiplicative function f supported on the set of square-free numbers on primes p by:

$$f(p) := \begin{cases} \sqrt{\dfrac{\log N \log_2 N}{\log_3 N}} \dfrac{1}{\sqrt{p}(\log p - \log_2 N - \log_3 N)} & p \in P \\ 0 & \text{otherwise.} \end{cases}$$

Fix now $1 < a < \frac{1}{\gamma}$. For k in the same range as above, we let M_k be the set of those integers having at least $\frac{a \log N}{k^2 \log_3 N}$ prime divisors in P_k. From these sets we define

$$\mathscr{M} := \operatorname{supp}(f) \Big\backslash \bigcup_{k=1}^{[(\log_2 N)^\gamma]} M_k.$$

To define our resonator,

$$R(t) := \sum_{n \in \mathscr{M}'} \frac{r(n)}{n^{it}},$$

we have to sparsify our set[1] $\mathscr{M}$. Let m_1 be the smallest element of $\mathscr{M}$, then we choose m_2 to be the smallest element of $\mathscr{M}$ such that $m_2 > m_1(1+T^{-1})$. We continue like this, taking m_{i+1} to be the smallest element of $\mathscr{M}$ such that $m_{i+1} > m_i(1+T^{-1})$. Then we define the set $\mathscr{M}'$ to consist of such m_i. The resonator coefficients are now defined as the local ℓ^2-average:

$$r(m_i) := \left(\sum_{n \in \mathscr{M}, m_i \le n \le m_i(1+T^{-1})} f(n)^2 \right)^{1/2}.$$

We gather some properties of the resonator in the following lemma.

Lemma 3.1.1.

1. $|\mathscr{M}'| \le |\mathscr{M}| \le N$.
2. $\sum_{m' \in \mathscr{M}'} r(m')^2 = \sum_{n \in \mathscr{M}} f(n)^2$.
3. $|R(0)|^2 \le T^\kappa \sum_{n \in \mathscr{M}} f(n)^2$.
4. $$\frac{1}{\sum_{i \in \mathbb{N}} f(i)^2} \sum_{n \in \mathscr{M}} f(n)^2 \sum_{p|n} \frac{1}{f(p)\sqrt{p}} \ge (\gamma + o(1)) \sqrt{\frac{\log N \log_3 N}{\log_2 N}}.$$
5. *Let $\varepsilon > 0$. Then for sufficiently large T,*
$$\sum_{m,n \in \mathscr{M}, mk=n} f(m)f(n) \le \sum_{m',n' \in \mathscr{M}', |km'/n'-1| \le (1+\varepsilon)/T} r(m')r(n').$$

Proof. The first inequality in the first point is clear. The second inequality is proven in [3]. The second point follows from the definition of $r(m')$ and $\mathscr{M}'$. The third point follows from the second point and Cauchy–Schwarz. The fourth point is Lemma 4 in [4]. Alternatively, we can also obtain it by letting $q = 2$ in Lemma 4.3.4. For the fifth point, fix $\varepsilon > 0$ and let $m' \in \mathscr{M}'$. Consider the set $J(m') := [m', m'(1+T^{-1})]$. If $mk = n$ with $m \in J(m'), J(n')$ we then have

$$\left| \frac{m'k}{n'} - 1 \right| \le \frac{1}{T} + O\left(\frac{1}{T^2}\right). \tag{3.1}$$

Using the definition of r and Cauchy–Schwarz we arrive at

$$\sum_{m,n \in \mathscr{M}, mk=n, m \in J(m'), n \in J(n')} f(m)f(n) \le r(m')r(n').$$

Summing over all m', n' with $|km'/n' - 1| \le \frac{1+\varepsilon}{T}$, (3.1) and the definition of $\mathscr{M}'$ yields the desired conclusion for sufficiently large T. □

[1] I want to thank Winston Heap for telling me about this other possible way to sparsify $\mathscr{M}$.

Next we have to bound what's essentially the second moment of our resonator.

Lemma 3.1.2. *We have*

$$\int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \le (c_1 + o(1))T \sum_{n\in\mathscr{M}} f(n)^2.$$

Here $c_1 = 5.009\sqrt{2\pi}$.

Proof. The proof is a classical diagonal/off-diagonal analysis. Expanding the sum inside the integral gives

$$\begin{aligned}\int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t &= T \sum_{n,m\in\mathscr{M}'} r(m)r(n)\widehat{\Phi}\left(T\log\frac{m}{n}\right)\\ &= T\sqrt{2\pi} \sum_{n,m\in\mathscr{M}'} r(m)r(n)\Phi\left(T\log\frac{m}{n}\right).\end{aligned}$$

The diagonal is simply

$$T\sqrt{2\pi}\sum_{m\in\mathscr{M}'} r(n)^2,$$

so let us turn to the off-diagonals. Using $a^2+b^2 \ge 2ab$ for $a, b > 0$ the off-diagonal is

$$\begin{aligned}&\sqrt{2\pi}T \sum_{\substack{m,n\in\mathscr{M}',\\ m\neq n}} r(m)r(n)\Phi\left(T\log\frac{m}{n}\right)\\ \le&\sqrt{2\pi}T \sum_{\substack{1\le j,\ell\le|\mathscr{M}'|,\\ j\neq \ell}} r(m_j)r(n_\ell)\Phi\left(T(|\ell-j|-1)\log(1+T^{-1})\right)\\ \le&\sqrt{2\pi}T \sum_{\substack{1\le j,\ell\le|\mathscr{M}'|,\\ j\neq \ell}} r(m_j)^2\Phi\left(T(|\ell-j|-1)\log(1+T^{-1})\right).\end{aligned} \tag{3.2}$$

In the first transition above we used that for $i > j$:

$$\frac{m_i}{m_j} \ge (1+T^{-1})^{i-j-1}.$$

We need a way to deal with the sum over ℓ in (3.2). To this end, let $\Phi_T(t) := \Phi(t\sqrt{1-T^{-1}})$. Because T is sufficiently large,

$$(T\log(1+T^{-1}))^2 = 1 - \frac{1}{T} + \frac{11}{12T^2} + O\left(\frac{1}{T^3}\right).$$

Using this and $|\mathscr{M}'| \le N \le T^\kappa$, we then get

$$\begin{aligned}&\Phi\left(T\log(1+T^{-1})(|\ell-j|-1)\right)\\ =&\exp\left(-\frac{(|\ell-j|-1)^2}{2}\left(1-\frac{1}{T}+\frac{11}{12T^2}+O\left(\frac{1}{T^3}\right)\right)\right)\\ \le&\exp\left(-\frac{(|\ell-j|-1)^2}{2}\left(1-\frac{1}{T}\right)\right)\exp\left(\frac{(|\ell-j|-1)^2}{2}O\left(\frac{1}{T^3}\right)\right)\\ =&(1+o(1))\Phi_T\left(|\ell-j|-1\right).\end{aligned} \tag{3.3}$$

The $1/T^2$-term in the inequality above vanishes since $\exp(-\frac{11}{12T^2}\frac{(|\ell-j|-1)^2}{2}) \leq 1$. Using (3.3), we derive by means of Euler's summation formula that

$$\sum_{\substack{1\leq \ell \leq |\mathscr{M}'| \\ \ell \neq j}} \Phi_T\left(|\ell-j|-1\right) \leq 2 \sum_{n=0}^{\lceil |\mathscr{M}'|/2 \rceil} \Phi_T\left(n\right) \leq (1+o(1)) \underbrace{\left(2+2\Phi(1)+2\int_1^\infty \Phi(t)\,\mathrm{d}t\right)}_{<4.009}.$$

With the estimate above, we go back to (3.2), and see that the off-diagonals are bounded by

$$\leq (1+o(1))4.009T\sqrt{2\pi}\sum_{m\in\mathscr{M}'} r(m)^2.$$

Using Lemma 3.1.1, we conclude by adding the diagonal and off-diagonal estimates together that

$$\int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right)\,\mathrm{d}t \leq T(1+o(1))5.009\sqrt{2\pi}\sum_{n\in\mathscr{M}} f(n)^2.$$

□

Lemma 3.1.3. *Assume*

$$G(t) \coloneqq \sum_{n\geq 2} \frac{\Lambda(n)a_n}{\log n} n^{-1/2-it}$$

is absolutely convergent and that $a_n \geq 0$. Then with $N = [T^\kappa]$,

$$\int_{-\infty}^{\infty} G(t)|R(t)|^2 \Phi\left(\frac{t}{T}\right)\,\mathrm{d}t \geq (c_2+o(1))T\sqrt{\kappa}\sqrt{\frac{\log T \log_3 T}{\log_2 T}}\left(\min_{p\in P} a_p\right)\sum_{n\in\mathscr{M}} f(n)^2.$$

Here

$$c_2 = \sqrt{2\pi/e}.$$

Proof. By absolute convergence we can change the order of summation and integration.

$$\begin{aligned}
&\int_{-\infty}^{\infty} G(t)|R(t)|^2 \Phi\left(\frac{t}{T}\right)\,\mathrm{d}t \\
=&\sqrt{2\pi}T\sum_{m',n'\in\mathscr{M}'}\sum_{k\geq 2} \frac{\Lambda(k)a_k r(m')r(n')}{k^{1/2}\log n}\widehat{\Phi}\left(T\log\frac{km'}{n'}\right) \\
\geq&\sqrt{2\pi}T\left(\min_{p\in P} a_p\right)\sum_{m',n'\in\mathscr{M}'}\sum_{p\in P}\frac{r(m')r(n')}{\sqrt{p}}\Phi\left(T\log\frac{km'}{n'}\right). \qquad (3.4)
\end{aligned}$$

In the inequality we restricted to the set P, which we may because all the terms in the sum are positive. Now we restrict to the pairs (m',n') such that $|\frac{m'p}{n'}-1| \leq \frac{1+\varepsilon}{T}$. By Lemma 3.1.1, we then have

$$\sum_{\substack{m,n\in\mathscr{M}, \\ mp=n}} f(m)f(n) \leq \sum_{m',n'\in\mathscr{M}', |\frac{m'p}{n'}-1|\leq(1+\varepsilon)/T} r(m')r(n').$$

Furthermore
$$-\frac{1}{T}(1+o(1)) \leq \log\frac{m'p}{n'} \leq \frac{1}{T}(1+o(1))$$
since we can make ε arbitrarily small. Putting together the two latter facts we deduce that
$$\begin{aligned}\sum_{p\in P}\sum_{m',n'\in\mathscr{M}'}\frac{r(m')r(n')}{\sqrt{p}}\Phi\left(T\log\frac{km'}{n'}\right) &\geq (1+o(1))\Phi(1)\sum_{p\in P}\sum_{\substack{m,n\in\mathscr{M}\\ mp=n}}\frac{f(m)f(n)}{\sqrt{p}}\\ &= (1+o(1))\Phi(1)\sum_{n\in\mathscr{M}}f(n)^2\sum_{p\in P, p|n}\frac{1}{f(p)\sqrt{p}}.\end{aligned}$$
Using (3.4) and Lemma 3.1.1, we finally arrive at the desired conclusion
$$\int_{-\infty}^{\infty}G(t)|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t \geq (1+o(1))\sqrt{\frac{2\pi}{e}}\left(\min_{p\in P}a_p\right)T\sqrt{\kappa}\sqrt{\frac{\log T\log_3 T}{\log_2 T}}\sum_{n\in\mathscr{M}}f(n)^2.$$
□

Lemma 3.1.4. *Let* $K(t) \coloneqq -(\log_2 T)^2 t\Phi((\log_2 T)t)$. *Then*
$$\min_{p\in P}\operatorname{Im}\widehat{K}(\log p) \geq c_3 + o(1),$$
where $c_3 = \sqrt{2\pi/e}$.

Proof. We know
$$\operatorname{Im}\widehat{K}(\xi) = \sqrt{2\pi}(\log_2 T)^{-1}\xi\Phi(\xi/\log_2 T).$$
Because $N = [T^\kappa]$, we get that
$$\begin{aligned}\min_{p\in P}\log(p)\Phi(\log p/\log_2 T) &\geq (1+\log_2 N+\log_3 N)\Phi\left(\frac{\log\left(e^{(\log_2 N)^\gamma}\log N\log_2 N\right)}{\log_2 T}\right)\\ &\geq \log_2 N(1+o(1))\Phi\left((1+o(1))\frac{\log_2 N}{\log_2 T}\right)\\ &= \log_2 T(1+o(1))(e^{-1/2}+o(1)).\end{aligned}$$
□

3.2 Proof of Theorem 3.0.1

We closely follow [4, Section 5]. Let $K(t) := -(\log_2 T)^2 t\Phi((\log_2 T)t)$ and assume that $\kappa < 1-\beta$. Using Cauchy–Schwarz and a classical bound of Selberg [22] on the second moment of $S(t)$ we have
$$\begin{aligned}\int_{|t|\leq T^\beta}\int_{-\infty}^{\infty}|S(t+u)K(u)|\,\mathrm{d}u\mathrm{d}t &\ll T^\beta + \int_{|t|\leq T^\beta}\int_{|u|\leq T^\beta}|S(t+u)K(u)|\,\mathrm{d}u\mathrm{d}t\\ &\ll T^\beta + \int_{|t|\leq T^\beta}|S(t)|\,\mathrm{d}t \ll T^\beta\sqrt{\log_2 T}.\end{aligned}$$

This in addition to the rapid decay of $\Phi(t)$ yields

$$\left| \int_{|t|<T^\beta} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right.$$
$$\left. + \int_{|t|>T\log T} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| \leq D_1 T \sum_{n\in\mathscr{M}} f(n)^2, \tag{3.5}$$

for some positive constant D_1. We also have for some $D_2 > 0$ that

$$\left| \int_{T^\beta \leq |t| \leq T\log T} \int_{|u+t|<T^\beta/2} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right.$$
$$\left. + \int_{T^\beta \leq |t| \leq T\log T} \int_{|u+t|>2T\log T} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| \leq D_2 T \sum_{n\in\mathscr{M}} f(n)^2. \tag{3.6}$$

Using $\int_{-\infty}^{\infty} |K(u)|\,\mathrm{d}u = 2$, (3.5), (3.6) and Lemma 3.1.2,

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| \\
\leq &\left| \int_{T^\beta < |t| \leq T\log T} \int_{T^\beta/2 \leq |u+t| \leq 2T\log T} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| + D_3 T \sum_{n\in\mathscr{M}} f(n)^2 \\
\leq &2\left(\max_{T^\beta/2 \leq t \leq 2T\log T} |S(t)| \right) T(c_1 + o(1)) \sum_{n\in\mathscr{M}} f(n)^2 + D_3 T \sum_{n\in\mathscr{M}} f(n)^2
\end{aligned} \tag{3.7}$$

for some $D_3 > 0$. On the other hand [4, Equation 9] and Lemma 3.1.3 implies that

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t+u)K(u)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| \\
= &\left| \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \mathrm{Im} \sum_{n\geq 2} \frac{\Lambda(n)}{\log n} \widehat{K}(\log n) n^{-1/2-it} + O(V(t)) \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}u\mathrm{d}t \right| \\
\geq &\frac{1}{\pi}(c_2 + o(1)) T \sqrt{\kappa} \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \left(\min_{p\in P} \mathrm{Im} \widehat{K}(\log p) \right) \sum_{n\in\mathscr{M}} f(n)^2 \\
&- \left| O\left(\int_{-\infty}^{\infty} V(t)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \right) \right|.
\end{aligned} \tag{3.8}$$

Finally Lemma 3.1.4, together with (3.7), (3.8), and using the explicit expression for $V(t)$ (see [4]) yields

$$\max_{T^\beta/2 \leq t \leq 2T\log T} |S(t)| \geq \left(\frac{c_2 c_3}{2\pi c_1} + o(1) \right) \sqrt{\kappa} \sqrt{\frac{\log T \log_3 T}{\log_2 T}}.$$

Changing T to $\frac{T}{2\log T}$ and making β slightly smaller we obtain the desired range $T^\beta \leq t \leq T$.

Chapter 4

The resonance method in cyclotomic fields

4.1 Introduction

As we discussed in Chapter 2, the positivity of the coefficients when using the *long* resonator method is very important. For the Riemann zeta function, we of course have no problems with this positivity since the Dirichlet series has positive coefficients. However, we shall not look far before we find other L-functions that do not satisfy this criterion. Dirichlet L-functions are examples of such functions.

Any non-principal (i.e. not associated to the trivial character) Dirichlet L-function will have some non-positive coefficients. As an explicit example: for $q = 3$, we have a Dirichlet L-function $L(s, \chi)$ defined by $\chi(1) = 1, \chi(2) = -1$. Trying to go through the same steps as we did in Chapter 2, one would end up with the problem of optimizing the resonator coefficients such that the following is maximal

$$\frac{\operatorname{Im} \sum_{k=2}^{\infty} \sum_{m,n \in \mathscr{M}'} \frac{\chi(k)\Lambda(k) r(m) r(n)}{k^{1/2} \log k} \widehat{K}(\log k) \widehat{\Phi}(T \log(kn/m))}{\sum_{n \in \mathscr{M}'} r(n)^2}.$$

Observe the appearance of $\chi(k)$ here. Because it is not necessarily positive, we cannot lower bound the sum by a desireable subsum, as we did before. This is one of the fundamental shortcomings of the long resonance method — it is not able to deal with Dirichlet series whose coefficients are not all ≥ 0. We still do not know how to surpass this shortcoming directly. However, there is a method that sort of bypass this problem, albeit in an indirect way. The idea can essentially be summarized in a few sentences. Assume we are given two Dirichlet series L_1 and L_2 with not necessary real non-negative coefficients. Then it still may be the case that the newly formed Dirichlet series $L_1 L_2$ have only positive coefficients, and thus is admissible for the resonance method. One can then exhibit a large value of $L_1 L_2$, and if it is sufficiently big it will give a *non-trivial* large value of L_1 or L_2.

This idea first appeared in [2], where the author together with Bondarenko, Darbar, Heap and Seip, exhibited large values of the Dedekind zeta function of a cyclotomic field $K = \mathbb{Q}(\omega_q)$.

There we proved that for $q \ll (\log\log T)^A$ and T sufficiently large we have

$$\max_{t\in[0,T]} |\zeta_K(1/2+it)| \geq \exp\left((1+o(1))\sqrt{\varphi(q)}\sqrt{\frac{\log T \log_3 T}{\log_2 T}}\right). \tag{4.1}$$

The main reason of the gain of a factor of $\sqrt{\varphi(q)}$ is due to a slightly new resonator. It utilizes crucially that the resonator essentialy only have to be supported on primes $p \equiv 1 \pmod q$. In light of the factorization (1.3), and thanks to the large constant $\sqrt{\varphi(q)}$, (4.1) implies a dichotomy where at least one of the following two is true:

- We have an improved Ω-result for at least one Dirichlet L-function $L(s,\chi)$ with χ non-principal satisifes

  $$\max_{t\in[0,T]} |L(1/2+it,\chi)| \geq \exp\left(c\sqrt{\frac{\log T \log_3 T}{\log_2 T}}\right)$$

  for some constant c.

- We can exhibit even larger values of $|\zeta(1/2+it)|$ than before. As an extreme case, if it turns out that the Soundararajan Ω-result is optimal, i.e. $|L(1/2+it,\chi)|$ is smaller than $\exp(c\sqrt{\log T/\log_2 T})$ on $[0,T]$ for all non-principal χ modulo some prime $q \sim \frac{\log_3 T}{4c^2}$ then

  $$\max_{t\in[0,T]} |\zeta(1/2+it)| \geq \exp\left(\left(\frac{1}{4c}+o(1)\right)\sqrt{\frac{\log T}{\log_2 T}}\log_3 T\right).$$

The theorem nor its proof gives any clue about which of these two that happen, but it is perhaps more likely that Dirichlet L-function of non-prinicipal Dirichlet character exhibit large values of Bondarenko–Seip level. The goal of this chapter is to prove a result similar to this for the argument of the Riemann zeta function and the argument of Dirichlet L-functions.

Our main theorem in this chapter is the following.

Theorem 4.1.1. *Assume GRH. Let* $K = \mathbb{Q}(\exp(2\pi i/q))$ *for a prime* q. *Let* $S_K(T) = \frac{1}{\pi}\operatorname{Im}\log\zeta_K(1/2+it)$. *Let* $A>0$ *be any constant and* c *any constant less than* $\sqrt{1-\beta}$ *where* $0<\beta\leq 1$. *Suppose that*

$$q-1=\varphi(q)\leq(\log\log(T^c))^A.$$

Then we have

$$\max_{T^\beta\leq t\leq T} |S_K(T)| \geq 0.02929c\sqrt{q-1}\sqrt{\frac{\log T\log_3 T}{\log_2 T}}.$$

Even though we let q be prime in the statement above, a similar result should hold for non-prime q as well. Taking q prime however makes the argument more streamlined. In particular we only need to deal with Dirichlet L-functions of primitive characters.

Just like before we also now get a dichotomy. The author is not aware of any Ω-results for the argument of Dirichlet L-functions (let us denote these by $S_\chi(t)$). On RH however, it is known that $S(t) = \Omega_\pm(\sqrt{\log t/\log\log t})$, so a reasonable guess is that one could also prove $S_\chi(t) = \Omega_\pm(\sqrt{\log t/\log\log t})$. In the extreme (but possible) event that this turns out to be optimal, i.e. an upper bound for $|S_\chi(t)|$ is of the same magnitude, we obtain very large values of $S(t)$. More specifically we could choose $q \approx \log_3 T$ in Theorem 4.1.1 to get

$$\max_{T^\beta \le t \le T} |S(t)| \gg \sqrt{\frac{\log T}{\log_2 T}} \log_3 T.$$

The chapter is split up into four additional sections. In the next section we find a convolution formula for ζ_K. In Section 3 we construct our resonator and prove a GCD-type inequality. In Section 4 we provide some explicit estimates regarding the resonator and the kernel in the convolution formula. The actual proof of Theorem 4.1.1 is carried out in Section 5.

4.2 The setup

We again follow the setup of Bondarenko–Seip, using a similar formula to the convolution formula of Tsang. Before we state this formula, we declare the standard conventions for the definition of $S_K(t)$.

When t is not an ordinate of a zero of ζ_K, $\log\zeta_K(\sigma+it)$ is obtained by continuous variation along the two following line segments: $[2, 2+it]$ and $[2+it, \sigma+it]$. For such t we define

$$S_K(t) := \frac{1}{\pi}\mathrm{Im}\log\zeta_K(1/2+it).$$

If t is the ordinate of a ζ_K-zero, we let

$$S_K(t) := \lim_{\varepsilon\to 0} \frac{S(t+\varepsilon)+S(t-\varepsilon)}{2}.$$

We extend the definition of $S_K(t)$ to the whole real line so that $S_K(t)$ is an odd function.

Lemma 4.2.1. *Assume GRH. Let χ be a primitive Dirichlet character modulo q. Let $K(x+iy)$ be an analytic function in the horizontal strip $-\frac{3}{2} \le y \le 0$ that satisfy*

$$V(x) := \max_{-\frac{3}{2}\le y\le 0} |K(x+iy)| = O\left(\frac{1}{|x|\log^2|x|}\right)$$

when $|x| \to \infty$. Then for every $t \neq 0$ we have

$$\int_{-\infty}^{\infty} \log L(1/2+i(t+u),\chi)K(u)\,\mathrm{d}u = \sum_{n\ge 2} \frac{\Lambda(n)\chi(n)\widehat{K}(\log n)}{n^{1/2+it}\log n} + O(V(t)).$$

Proof. We refer the reader to [26, Lemma 5], whose proof strategy also applies in this setting. In [26, Eq. (2.13)], we rather use the estimate

$$\int_{\sigma}^{2} |\log L(\alpha + it, \chi)| \,\mathrm{d}\alpha = O(\log(q|t|)).$$

□

Summing over all Dirichlet characters mod q, and then using formula (1.4) yields

$$\begin{aligned}&\int_{-\infty}^{\infty} \log \zeta_K(1/2 + i(t+u))K(u)\,\mathrm{d}u \\ =&(1 + O(q^{-1/2}))\varphi(q) \sum_{\substack{n\geq 2,\\ n\equiv 1 \pmod q}} \frac{\Lambda(n)\widehat{K}(\log n)}{n^{1/2+it}\log n} + O(\varphi(q)V(t)).\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u = (1 + O(q^{-1/2}))\frac{\varphi(q)}{\pi}\mathrm{Im} \sum_{\substack{n\geq 2,\\ n\equiv 1 \pmod q}} \frac{\Lambda(n)\widehat{K}(\log n)}{n^{1/2+it}\log n} + O(\varphi(q)V(t)) \tag{4.2}$$

whenever $K(u)$ is real-valued for real arguments u.

4.3 The resonator

We devote this section to the resonator and associated estimates. The construction is strongly inspired by [4], and there are really only three differences. The first one is that we now only consider primes p that are 1 mod q. This falls out naturally from the congruence condition in (4.2). This brings us over to the second difference, which is that we take larger primes (now multiplied by a factor of $\varphi(q)$). The reason that we can take larger primes is essentialy because the condition $p \equiv 1 \pmod q$ gives some extra room for more primes. Finally, we slightly alter the resonator coefficients. The extra $(-\log\varphi(q))$ is needed because we take larger primes, but the really important change is the factor of $\sqrt{\varphi(q)}$ in front of the resonator coefficients $f(n)$. This comes from the congruence condition $p \equiv 1 \pmod q$, and is needed for certain sums over primes (see the displayed equation under (4.6)) to stay sufficiently small. Ideally one would like to take $\varphi(q)^{\theta}$ with θ as small as possible, but $\theta = 1/2$ seems to be a natural limit with the current method.

Throughout this section we shall need the following theorem several times.

Theorem 4.3.1. (Siegel–Walfisz theorem)
Let $\pi_{q,a}(x)$ be the number of primes $1 < p \leq x$ in the arithmetic sequence $\{a + nq\}_n$. Let $A > 0$ be given. Suppose that $q \leq (\log x)^A$ and assume $\gcd(a, q) = 1$*. Then*

$$\pi_{q,a}(x) = (1 + o(1))\frac{x}{\varphi(q)\log x}.$$

Proof. See [21, Corollary 11.21]. □

We now fix q and fix N a sufficiently large integer. Furthermore assume $q-1 \leq (\log\log N)^A$ for some fixed $A > 0$. Also let $0 < \gamma < 1$ and $1 < a < \frac{1}{\gamma}$. In particular $a\gamma < 1$.

Let P_q be the set of all primes p such that the following is true.

1. $p \equiv 1 \pmod{q}$,
2. $\varphi(q)e \log N \log_2 N < p \leq \varphi(q) \log N \exp\left((\log_2 N)^\gamma\right) \log_2 N$.

We then define a multiplicative function f that is supported on squarefrees and such that on primes p we have

$$f(p) = \begin{cases} \sqrt{\varphi(q)}\sqrt{\dfrac{\log N \log_2 N}{\log_3 N}} \dfrac{1}{\sqrt{p}(\log p - \log_2 N - \log_3 N - \log\varphi(q))} & p \in P_q \\ 0 & \text{otherwise.} \end{cases}$$

We will now partition P_q into $[(\log_2 N)^\gamma]$ sets $P_{q,k}$:

$$P_{q,k} := P_q \cap (\varphi(q)e^k \log N \log_2 N, \varphi(q)e^{k+1} \log N \log_2 N].$$

Here $k = 1, \ldots, [(\log_2 N)^\gamma]$. We now define $M_{q,k}$ to be the set of integers that has at least $\frac{a \log N}{k^2 \log_3 N}$ prime divisors in $P_{q,k}$. Furthermore we let $M'_{q,k}$ be the set of integers from $M_{q,k}$ that only have divisors from $P_{q,k}$. We then finally let

$$\mathscr{M}_q := \operatorname{supp}(f) \Big\backslash \bigcup_{k=1}^{[(\log_2 N)^\gamma]} M_{q,k}.$$

Lemma 4.3.2. *We have $|\mathscr{M}_q| \leq N$ depending on a and γ.*

Proof. Let $n \in \mathscr{M}_q$. Then for any k, there are at most

$$\binom{|P_{q,k}|}{1} + \binom{|P_{q,k}|}{2} + \cdots + \binom{|P_{q,k}|}{\left[\frac{a \log N}{k^2 \log_3 N}\right]}$$

ways to pick out primes from $P_{q,k}$. Hence

$$|\mathscr{M}_q| \leq \prod_{k=1}^{[(\log_2 N)^\gamma]} \sum_{j=1}^{\left[\frac{a \log N}{k^2 \log_3 N}\right]} \binom{|P_{q,k}|}{j}.$$

We then derive an estimate for $|P_{q,k}|$. Let $\pi_{q,1}(x)$ be the number of primes $p \leq x$ such that $p \equiv 1 \pmod q$. By the Siegel–Walfisz theorem we have for sufficiently large N that

$$\begin{aligned}
|P_{q,k}| &= \pi_{q,1}\left(\varphi(q)e^{k+1}\log N\log_2 N\right) - \pi_{q,1}\left(\varphi(q)e^{k}\log N\log_2 N\right)\\
&\leq 1.1\frac{\varphi(q)e^{k+1}\log N\log_2 N}{\varphi(q)\log(\varphi(q)e^{k+1}\log N\log_2 N)} - 0.9\frac{\varphi(q)e^{k}\log N\log_2 N}{\varphi(q)\log(\varphi(q)e^{k}\log N\log_2 N)}\\
&= e^k\log N\log_2 N\left(\frac{1.1e}{\log(\varphi(q)e^{k+1}\log N\log_2 N)} - \frac{0.9}{\log(\varphi(q)e^{k}\log N\log_2 N)}\right)\\
&\leq e^k\log N\log_2 N\left(\frac{1.1e}{\log(\varphi(q)e^{k}\log N\log_2 N)} - \frac{0.9}{\log(\varphi(q)e^{k}\log N\log_2 N)}\right)\\
&= e^k\log N\frac{1.1e-0.9}{1+\frac{k}{\log_2 N}+\frac{\log\varphi(q)}{\log_2 N}+\frac{\log_3 N}{\log_2 N}}\\
&\leq e^k\log N(1.1e-0.9) \leq e^{k+1}\log N.
\end{aligned}$$

Thus the rest of the calculation of the cardinality of $\mathscr{M}_q$ will be identical to that of $\mathscr{M}$ in [3, Proof of Lemma 2.]. □

Before we go to the main result about our resonator we will define a few more sets and prove one lemma. Fix $0 < \alpha < 1$. Then let $L_{q,k}$ be the set of integers in $\operatorname{supp}(f)$ that have at most $\frac{\alpha\log N}{k^2\log_3 N}$ prime divisors in $P_{q,k}$ for $k = 1, \ldots, [(\log_2 N)^\gamma]$. Furthermore let $L'_{q,k}$ be the integers from $L_{q,k}$ that only have divisors in $P_{q,k}$. We then define

$$\mathscr{L}_q = \mathscr{M}_q \Big\backslash \bigcup_{k=1}^{[(\log_2 N)^\gamma]} L_{q,k}.$$

In other words $\mathscr{L}_q$ is the set of integers from $\mathscr{M}_q$ that have at least $\frac{\alpha\log N}{k^2\log_3 N}$ divisors in $P_{q,k}$ for $k = 1, \ldots, [(\log_2 N)^\gamma]$.

The main result of this section will hinge a lot on the fact that $\mathscr{L}_q$ is non-empty.

Lemma 4.3.3. *$\mathscr{L}_q$ is non-empty, depending on α.*

Proof. It is enough to prove that

$$|P_{q,k}| \geq \frac{\alpha\log N}{k^2\log_3 N}.$$

By the Siegel–Walfisz theorem we have for sufficiently large N,

$$\begin{aligned}
|P_{q,k}| &\geq 0.9\frac{\varphi(q)e^{k+1}\log N\log_2 N}{\varphi(q)\log(\varphi(q)e^{k+1}\log N\log_2 N)} - 1.1\frac{\varphi(q)e^{k}\log N\log_2 N}{\varphi(q)\log(\varphi(q)e^{k}\log N\log_2 N)}\\
&\geq e^k\log N\log_2 N\left(\frac{0.9e-1.1}{\log(\varphi(q)e^{k+1}\log N\log_2 N)}\right)\\
&= e^k\log N\frac{0.9e-1.1}{1+\frac{k+1}{\log_2 N}+\frac{\log\varphi(q)}{\log_2 N}+\frac{\log_3 N}{\log_2 N}}.
\end{aligned}$$

Now for sufficiently large N,

$$1+\frac{k+1}{\log_2 N}+\frac{\log\varphi(q)}{\log_2 N}+\frac{\log_3 N}{\log_2 N}\leq 3.1$$

for all k. Here we used $k\leq\log_2 N$, as well as the bound on $\varphi(q)$. Hence we conclude that

$$|P_{q,k}|\geq 0.02e^k\log N.$$

Thus it is enough to show that

$$\frac{\alpha\log N}{k^2\log_3 N}\leq 0.02e^k\log N,$$

but this is equivalent to

$$\log_3 Ne^k 0.02\geq\frac{\alpha}{k^2}$$

which is true for large enough N. □

We now turn to the main result of this section.

Lemma 4.3.4. *We have*

$$\frac{1}{\sum_{i\in\mathbb{N}}f(i)^2}\sum_{n\in\mathscr{M}_q}f(n)^2\sum_{\substack{p|n,\\ p\equiv 1\pmod q}}\frac{1}{f(p)\sqrt{p}}\geq(\gamma+o(1))\frac{1}{\sqrt{\varphi(q)}}\sqrt{\frac{\log N\log_3 N}{\log_2 N}}. \tag{4.3}$$

Proof. Assume that

$$\frac{1}{\sum_{i\in\mathbb{N}}f(i)^2}\sum_{n\notin\mathscr{L}_q}f(n)^2=o(1). \tag{4.4}$$

In that case the left hand side of (4.3) would be bigger than (or equal to)

$$\begin{aligned}\min_{n\in\mathscr{L}_q}\sum_{\substack{p|n,\\ p\equiv 1\pmod q}}\frac{1}{f(p)\sqrt{p}}&=\min_{n\in\mathscr{L}_q}\sum_{p|n}\frac{1}{f(p)\sqrt{p}}\\ &\geq(1-o(1))\sum_{k=1}^{[(\log_2 N)^\gamma]}\frac{\alpha\log N}{k^2\log_3 N}\min_{p\in P_{q,k}}\frac{1}{f(p)\sqrt{p}}\\ &\geq(1-o(1))\sum_{k=1}^{[(\log_2 N)^\gamma]}\frac{\alpha\log N}{k^2\log_3 N}k\sqrt{\frac{\log_3 N}{\log N\log_2 N}}\frac{1}{\sqrt{\varphi(q)}}\\ &\geq(1-o(1))\alpha\gamma\frac{1}{\sqrt{\varphi(q)}}\sqrt{\frac{\log N\log_3 N}{\log_2 N}}.\end{aligned}$$

Here the first equality follows from the fact that all divisors of n necessarily are $n\equiv 1\pmod q$. Furthermore there exists a minimal element by Lemma 4.3.3. The calculation above implies (4.3) if we choose α arbitrarily close to 1. We are left with proving (4.4). To this end, observe that

$$\mathscr{L}_q = \mathrm{supp}(f) \Big\backslash \bigcup_{k=1}^{[(\log_2 N)^\gamma]} (M_{q,k} \cup L_{q,k}),$$

and so it is enough to prove the two following equalities

$$\frac{1}{\sum_{i\in\mathbb{N}} f(i)^2} \sum_{k=1}^{[(\log_2 N)^\gamma]} \sum_{n\in L_{q,k}} f(n)^2 = o(1),$$
$$\frac{1}{\sum_{i\in\mathbb{N}} f(i)^2} \sum_{k=1}^{[(\log_2 N)^\gamma]} \sum_{n\in M_{q,k}} f(n)^2 = o(1). \tag{4.5}$$

We start with proving the second of these. To this end, fix k. Then by the definition of $M_{q,k}$ and $M'_{q,k}$, as well as using f is multiplicative, we have

$$\begin{aligned}\frac{1}{\sum_{i\in\mathbb{N}} f(i)^2} \sum_{n\in M_{q,k}} f(n)^2 &= \frac{1}{\prod_p (1+f(p)^2)} \left(\prod_{j=1, j\neq k}^{[(\log_2 N)^\gamma]} \prod_{p\in P_{q,j}} (1+f(p)^2) \right) \sum_{n\in M'_{q,k}} f(n)^2 \\ &= \frac{1}{\prod_{p\in P_{q,k}} (1+f(p)^2)} \sum_{n\in M'_{q,k}} f(n)^2.\end{aligned}$$

To bound this we shall use a snazzy trick: Rankin's trick usually refers to some version of the following observation: for any $\alpha > 0$ we have

$$\sum_{n>X} f(n) \le X^{-\alpha} \sum_{n>X} f(n) n^\alpha \le X^{-\alpha} \sum_{n=1}^{\infty} f(n) n^\alpha.$$

In this case, recall that $M'_{q,k}$ is the set of integers that have only prime divisors in $P_{q,k}$, and at least $\frac{a \log N}{k^2 \log_3 N}$ of those. Because $b > 1$ we thus get

$$\sum_{n\in M'_{q,k}} f(n)^2 \le b^{-\frac{a\log N}{k^2 \log_3 N}} \prod_{p\in P_{q,k}} (1 + b f(p)^2).$$

This in turn implies that

$$\frac{1}{\prod_{p\in P_{q,k}} (1+f(p)^2)} \sum_{n\in M'_{q,k}} f(n)^2 \le b^{-\frac{a\log N}{k^2\log_3 N}} \exp\left(\sum_{p\in P_{q,k}} (b-1) f(p)^2 \right). \tag{4.6}$$

Thus we have to estimate the sum over primes inside the exponential. Using Mertens theorem (see for example [19, Theorem 1.1.]) we have

$$
\begin{aligned}
&\sum_{p\in P_{q,k}} f(p)^2\\
=&\frac{\log N\log_2 N}{\log_3 N}\varphi(q)\sum_{p\in P_{q,k}}\frac{1}{p(\log p-\log_2 N-\log_3 N-\log\varphi(q))^2}\\
\leq&\frac{\log N\log_2 N}{k^2\log_3 N}\varphi(q)\sum_{p\in P_{q,k}}\frac{1}{p}\\
=&(1+o(1))\frac{\log N\log_2 N}{k^2\log_3 N}\frac{\varphi(q)}{\varphi(q)}\underbrace{\left(\log_2(\varphi(q)e^{k+1}\log N\log_2 N)-\log_2(\varphi(q)e^{k}\log N\log_2 N)\right)}_{\sim(\log_2 N)^{-1}}\\
=&(1+o(1))\frac{\log N}{k^2\log_3 N}. \qquad (4.7)
\end{aligned}
$$

Here we used $\varphi(q)\leq(\log_2 N)^A$ in the application of Mertens theorem. Inserting this into (4.6) we arrive at

$$
\begin{aligned}
\frac{1}{\prod_{p\in P_{q,k}}(1+f(p)^2)}\sum_{n\in M'_{q,k}} f(n)^2 &\leq \exp\left(-\frac{a\log N}{k^2\log_3 N}\log b+(b-1)\frac{\log N}{k^2\log_3 N}\right)\\
&=\exp\left(\frac{\log N}{k^2\log_3 N}\left((b-1)-a\log b\right)\right).
\end{aligned}
$$

Thus we can conclude that

$$\frac{1}{\sum_{i\in\mathbb{N}} f(i)^2}\sum_{k=1}^{[(\log_2 N)^\gamma]}\sum_{n\in M_{q,k}} f(n)^2\leq(\log_2 N)^\gamma\exp\left(\frac{\log N}{\log_3 N}(b-1-a\log b)\right).$$

We have $b-1-a\log b<0$ if we choose b sufficiently close to 1. Choosing b this way settles the equation on the second line in (4.5).

We now turn to the case of $L_{q,k}$, i.e. the upper equation in (4.5). Let $b<1$. Following the same lines as for $M_{q,k}$ we also, using Rankin's trick, arrive at

$$\frac{1}{\prod_{p\in P_{q,k}}(1+f(p)^2)}\sum_{n\in L'_{q,k}} f(n)^2\leq b^{-\frac{\alpha\log N}{k^2\log_3 N}}\exp\left(\sum_{p\in P_{q,k}}(b-1)f(p)^2\right).$$

Now because $b<1$ we must seek an upper bound for the sum over primes. Again by Mertens, we get

$$\begin{aligned}
&\sum_{p\in P_{q,k}} f(p)^2 \\
=&\frac{\log N \log_2 N}{\log_3 N} \sum_{p\in P_{q,k}} \varphi(q)\frac{1}{p(\log p - \log_2 N - \log_3 N - \log \varphi(q))^2} \\
\geq&\frac{\log N \log_2 N}{(k+1)^2 \log_3 N}\varphi(q) \sum_{p\in P_{q,k}} \frac{1}{p} \\
=&(1+o(1))\frac{\log N \log_2 N}{k^2 \log_3}\frac{\varphi(q)}{\varphi(q)} \underbrace{\left(\log_2(\varphi(q)e^{k+1}\log N \log_2 N) - \log_2(\varphi(q)e^{k}\log N \log_2 N)\right)}_{\sim(\log_2 N)^{-1}} \\
=&(1+o(1))\frac{\log N}{k^2 \log_3 N}.
\end{aligned} \tag{4.8}$$

Just as before we then get

$$\begin{aligned}
\frac{1}{\prod_{p\in P_{q,k}}(1+f(p)^2)} \sum_{n\in M'_{q,k}} f(n)^2 &\leq \exp\left(-\frac{\alpha \log N}{k^2 \log_3 N}\log b + (b-1)\frac{e \log N}{k^2 \log_3 N}\right) \\
&= \exp\left(\frac{\log N}{k^2 \log_3 N}\left((b-1) - \alpha \log b\right)\right)
\end{aligned}$$

and we finish this case like we did for the other case using $(b-1)-\alpha \log b < 0$ for b sufficiently close to 1. $\square$

Let us now fix our resonator. Sparsifying our set just like in Chapter 3, we end up with a sparsified set $\mathscr{M}'_q$. We then define our resonator to be

$$R(t) := \sum_{m\in \mathscr{M}'_q} \frac{r(m)}{m^{it}},$$

where

$$r(m) := \left(\sum_{n\in \mathscr{M}_q, m\leq n\leq m(1+T^{-1})} f(n)^2\right)^{1/2}.$$

4.4 Some estimates

We start this section by stating some estimates that we will need in what follows. In most of the cases their proof will be more or less identical to the proofs in Chapter 3, and will thus be omitted. Like before we set $N = [T^\kappa]$ where $\kappa < 1-\beta$ and $0 < \beta \leq 1$.

Next we have to bound what is essentially the second moment of our resonator.

Lemma 4.4.1. *We have*

$$\int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) \, \mathrm{d}t \leq (c_1 + o(1))T \sum_{n \in \mathscr{M}_q} f(n)^2.$$

Here $c_1 = 5.009\sqrt{2\pi}$.

Proof. The proof is more or less identical to that of Lemma 3.1.2, and is thus omitted. □

Lemma 4.4.2. *Assume*

$$G(t) := \sum_{\substack{n \geq 2, \\ n \equiv 1 \pmod q}} \frac{\Lambda(n) a_n}{\log n} n^{-1/2-it}$$

is absolutely convergent and that $a_n \geq 0$*. Then with* $N = [T^\kappa]$*,*

$$\int_{-\infty}^{\infty} G(t)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \, \mathrm{d}t \geq (c_2 + o(1))T \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \left(\min_{p \in P_q} a_p\right) \sum_{n \in \mathscr{M}_q} f(n)^2.$$

Here

$$c_2 = \sqrt{2\pi/e}.$$

Proof. Again the proof is very similar to that of Lemma 3.1.3, and we thus omit it. □

Lemma 4.4.3. *Let* $K(t) = -(\log_2 T)t\Phi((\log_2 T)t)$*. Then*

$$\min_{p \in P_q} \operatorname{Im} \widehat{K}(\log p) \geq c_3 + o(1)$$

where $c_3 = \sqrt{2\pi/e}$*.*

Proof. We have

$$\operatorname{Im} \widehat{K}(\log p) = \sqrt{2\pi}(\log_2 T)^{-1} \xi \Phi(\xi / \log_2 T).$$

Using $\varphi(q) \leq (\log\log N)^A$ and $N = [T^\kappa]$ we get that

$$\begin{aligned}
&\min_{p \in P_q} \log(p)\Phi(\log p / \log_2 T) \\
\geq &(\log \varphi(q) + 1 + \log_2 N + \log_3 N)\Phi\left(\frac{\log\left(\varphi(q) e^{(\log_2 N)^\gamma} \log N \log_2 N\right)}{\log_2 T}\right) \\
\geq &\log_2 N(1 + o(1))\Phi\left((1 + o(1))\frac{\log_2 N}{\log_2 T}\right) \\
= &\log_2 T(1 + o(1))(e^{-1/2} + o(1)).
\end{aligned}$$

□

4.5 Proof of Theorem 4.1.1

We will now combine the preceding sections to provide a proof of our main theorem.

Proof of Theorem 4.1.1. Throughout we assume T is sufficiently large. We choose the same kernel as Bondarenko–Seip,

$$K(t) = -(\log_2 T)^2 t\Phi((\log_2 T)t).$$

The Fourier transform is

$$\widehat{K}(\xi) = i\sqrt{2\pi}(\log_2 T)^{-1}\xi\Phi(\xi/\log_2 T).$$

Observe that K satisfies the assumptions of Lemma 4.2.1. Let $S_\chi(t)$ be the argument of the assoicated Dirichlet L-function $L(s,\chi)$, i.e. $S_\chi(t) = \frac{1}{\pi}\operatorname{Im}\log L(1/2+it,\chi)$. Since we are working modulo a prime q, all Dirichlet characters are primitive, so we can use [11, Chapter 16, Equation 2.] which gives $S_\chi(t) = O(\log(qt))$. Then

$$S_K(t) = O(\varphi(q)\log qt) = O((\log T)^2)$$

by the assumption on the size of $\varphi(q)$. This implies in particular that

$$\int_{-2T^\beta}^{2T^\beta} |S_K(t)|^2 \,\mathrm{d}t \ll T^\beta(\log T)^4.$$

Thus we find by Cauchy–Schwarz that

$$\begin{aligned}\int_{-T^\beta}^{T^\beta}\int_{-\infty}^{\infty} |S_K(t)K(u)| \,\mathrm{d}u\mathrm{d}t &\ll T^\beta + \int_{-T^\beta}^{T^\beta}\int_{|u|\le T^\beta} |S_K(t+u)K(u)| \,\mathrm{d}u\mathrm{d}t \\ &\ll T^\beta + \int_{-2T^\beta}^{2T^\beta} |S_K(t)| \,\mathrm{d}t \ll T^\beta + T^\beta(\log T)^2 \ll T^\beta(\log T)^2.\end{aligned}$$

Furthermore we have by the rapid decay of the Gaussian that

$$\int_{|t|>T\log T}\int_{-\infty}^{\infty} |S_K(t+u)K(u)| \,\mathrm{d}u |R(t)|^2\Phi\left(\frac{t}{T}\right) \mathrm{d}t \ll o(1)R(0)^2 \ll o(1)T^\kappa \sum_{n\in\mathscr{M}_q} f(n)^2,$$

where we in the last bound used

$$|R(0)|^2 \le 3T^\kappa \sum_{n\in\mathscr{M}_q} f(n)^2, \tag{4.9}$$

which follows from Cauchy–Schwarz and $N = [T^\kappa]$. Putting the estimates so far together along with the trivial bound $|R(t)|^2 \le |R(0)|^2$, we find for some constant D_1 that

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\right| \\ \le &\left|\int_{T^\beta\le t\le T\log T}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi^\lambda\left(\frac{t}{T}\right)\mathrm{d}t\right| \\ &+ D_1T^{\kappa+\beta}(\log T)^2\sum_{n\in\mathscr{M}_q} f(n)^2.\end{aligned} \tag{4.10}$$

Taking $\kappa < 1-\beta$ one sees from (4.10) that

$$\begin{aligned}&\left|\int_{T^\beta\le|t|\le T\log T}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right|\\ \ge&\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right| - O(T)\sum_{n\in\mathscr{M}_q} f(n)^2. \qquad (4.11)\end{aligned}$$

When $|u|\le T\log T$, we have by Lemma 4.4.1 that

$$\begin{aligned}&\left|\int_{T^\beta/2\le|t|\le 2T\log T}\int_{|u|\le T\log T} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi^\lambda\left(\frac{t}{T}\right)\,\mathrm{d}t\right|\\ \le&(1+o(1))2c_1T\left(\max_{T^\beta/2\le t\le 2T\log T}|S_K(t)|\right)\sum_{n\in\mathscr{M}_q} f(n)^2. \qquad (4.12)\end{aligned}$$

Thus by (4.11) and (4.12) we arrive at

$$\max_{T^\beta/2\le t\le 2T\log T}|S_K(T)| \ge \frac{\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right|}{(2c_1+o(1))T\sum_{n\in\mathscr{M}_q} f(n)^2} - \frac{D_5}{c_1+o(1)}. \qquad (4.13)$$

Let

$$G(t) := \sum_{\substack{n\ge 2,\\ n\equiv 1 \pmod q}} \frac{\Lambda(n)\widehat{K}(\log n)}{\pi\log n} n^{-1/2-it}.$$

Invoking (4.2) gives for any $\varepsilon > 0$ that

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_K(t+u)K(u)\,\mathrm{d}u|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right|\\ \ge&\left|(1+O(q^{-1/2}))\varphi(q)\mathrm{Im}\int_{-\infty}^{\infty} G(t)|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right| - D_6\varphi(q)T^{\kappa+\varepsilon}\sum_{n\in\mathscr{M}_q} f(n)^2\\ \ge&\left|(1+O(q^{-1/2}))\varphi(q)\mathrm{Im}\int_{-\infty}^{\infty} G(t)|R(t)|^2\Phi\left(\frac{t}{T}\right)\,\mathrm{d}t\right| - D_6T^{\kappa+2\varepsilon}\sum_{n\in\mathscr{M}_q} f(n)^2. \qquad (4.14)\end{aligned}$$

Here we used an explicit bound on V, Lemma 4.4.1 and the bound on q. Using Lemma 4.4.2 and 4.4.3, and the assumption on the bound on $\varphi(q)$, we get from (4.13), for any $c < \sqrt{\varphi(q)(1-\beta)}$, that

$$\max_{T^\beta/2\le t\le 2T\log T}|S_K(t)| \ge c\left(\frac{c_2c_3}{2\pi c_1}+o(1)\right)\sqrt{\frac{\log T\log_3 T}{\log_2 T}}$$

Adjusting T and β appropriately we get the desired range. □

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