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Detection of Gravitational Waves Using the Moon

Master's thesis in Physics
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ABSTRACT

The purpose of this master's thesis is to calculate the seismic response of the Moon to resonance with gravitational waves in general relativity and Brans-Dicke gravity. It has been suggested to place seismometers on the surface of the Moon to listen for gravitational wave signals. The response of three different models for the Moon has been considered.

The Post Minkowskian theory is considered in short and concluded to be a highly useful tool for further study into areas of interest with strong curvature of space-time. The theory is not useful for our purposes of the Moon's response to gravitational waves, but a valuable method for future work involving any region with a strong gravitational field.

Other theories of gravity than Einstein's general theory of relativity can add observable polarisation of the produced gravitational waves. To measure these polarisations are therefore of high interest if one wishes to look for theories of gravity beyond general relativity. One such theory is scalar-tensor theory of Brans-Dicke. An additional scalar polarisation is produced in this theory and we have in this thesis calculated the response of a potential Moon detector to this new scalar polarisation.

We find that a scalar polarised gravitational wave will excite a distinguishable signal from the plus- and cross-polarisations from general relativity. It is determined that given a gravitational signal of strength h_0, the three models in question would produce a signal of magnitude $\approx h_0 \times 10^{11}$ cm. The detection of gravitational waves using the Moon's normal modes is therefore determined as possible given seismometers with a high enough sensitivity.

SAMMENDRAG

Formålet med denne masteroppgaven er å beregne den seismiske responensen til Månen ved respons av dens normal modus med gravitasjonsbølger i generell relativitetsteori og Brans-Dicke-gravitasjon. Det har blitt foreslått å plassere seismometre på månens overflate for å lytte etter gravitasjonsbølgesignaler. Responsen til tre forskjellige modeller for månen har blitt vurdert.
Post Minkowskian teori er kort vurdert og konkludert med å være en svært nyttig verktøy for å studere områder av intresse med sterk krumning av romtid. Teorien er ikke nyttig for å studere Månens respons til gravitasjonsbølger, men en viktig metode for framtidig arbeid som involverer områder med sterke gravitasjonsfelt.
Andre teorier om gravitasjon enn Einsteins generelle relativitetsteori kan legge til observerbare polariseringer av de produserte gravitasjonsbølgene. Å måle disse polarisasjonene er derfor av stor interesse hvis man ønsker å lete etter alternative gravitasjonsteorier utover generell relativitetsteori. En slik teori er skaler-tensor teorien Brans-Dicke. En ekstra skalarpolarisering produseres i denne teorien, og vi har i denne oppgaven beregnet responsen til en potensiell Månedetektor på denne nye skalarpolariseringen.
Vi har funnet at en skalarpolarisert gravitasjonsbølge vil eksitere en unik modus som vil kunne skilles fra moduser eksitert fra pluss- og krysspolarisasjonen fra generell relativitetsteori. Det er fastslått at gitt en gravitasjonsbølge med amplitude h_0, forutsier de tre nevnte modellene et signal med størrelse $\approx h_0 \times 10^{11}$ cm. Vi konkluderer derfor med at det er mulig å oppdage gravitasjonsbølger ved å bruke egenmodusene til Månen, forutsatt at seismometrene har tilstrekkelig høy sensitivitet.

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CHAPTER
ONE

INTRODUCTION

1.1 Lunar gravitational wave detection and Brans-Dicke gravity

The year is 1916 and one year has passed since Einstein published the general theory of relativity. A theory which will turn out to have a profound and enduring impact on our understanding of gravity. Einstein's new theory also predicts a new phenomenon, gravitational waves. These waves are ripples in spacetime travelling at the speed of light throughout the Universe. It would take almost 100 years before Einstein's predictions of gravitational waves would be detected. The collaboration of LIGO managed to detect an extremely tiny signal from two orbiting merging black holes [1]. With this detection, a new era in astronomy has begun and many detectors have been proposed to further study what we might learn from this new and interesting signal. In this thesis we will look deeper into one of these detectors and study the response of the normal modes of the Moon to gravitational waves.

The idea of using the Moon as a detector for gravitational waves is not new. Weber [2] was the first to come up with the detector concept of a resonant bar detector and A. Ben-Menahem [3] suggested gravitational waves could be detected on Earth using Weber's resonant bar idea. The Apollo 17 mission brought seismometers that attempted to detect gravitational waves on the surface however they ran into a technical problem and the data were useless [4]. There have also been attempts at spherical detectors of much smaller scales than the Earth and Moon such as MiniGRAIL [5], however, they remained unsuccessful in the attempt of detecting gravitational waves.

The main goal of this thesis is to study the resonance response of the Moon to gravitational waves given three different Moon models. This detection of gravitational waves using the normal modes of the Moon could become a reality in the next decades. It has the potential to be a valuable detector working in parallel with new gravitational wave detectors such as LISA [6] and the Einstein Telescope [7]. The Moon's response is dependent on the interior geology. Accurate measurements, therefore, require further knowledge of the geology of the Moon's interior. The sensitive frequency range of this detector is directly determined by the geology. The detector concept could therefore prove to be more sensitive in a certain range of the frequency spectrum than other planned detectors. The dependence on the Moon's eigenfrequencies led to the analysis of multiple models for the Moon. It will be of interest to see the displacement we can expect to get at resonance, what normal modes of the Moon we expect the gravitational wave will excite, and for what frequency one might expect to detect gravitational waves. It will also be interesting to see the frequency range this detector is most sensitive to and the general behaviour of the response as a function of the frequency of gravitational waves. In conclusion, this thesis will derive an expression for the resonance response of the Moon predicted by three models in general relativity and Brans-Dicke gravity [8]. We will also elaborate on the response of the Moon to a gravitational wave in Brans-Dicke gravity with the frequency range of the gravitational wave.

1.2 Structure of the thesis

The thesis is effectively split into two parts. The first part has no mention of the Moon and focuses on general relativity and Brans-Dicke gravity with the goal of deriving the polarisation and source terms of gravitational waves for the two theories. In Chapter 2 we consider the post-Minkowskian approximation. In Chapter 3 we will consider the Landau-Lifshitz formulation of general relativity and Brans-Dicke gravity. With this formulation as a starting point, we will derive the polarisation tensor in general relativity and Brans-Dicke gravity. The second part of the thesis starts of the theory of elasticity and it is in this part we encounter the normal modes of a spherical body for the first time. From this, we derive the expression for the response of the Moon to a gravitational wave for both toroidal and spheroidal oscillations. We will see how one can greatly simplify the expressions arising from a quadrupole-quadrupole moment. In Chapter Five we present and discuss our numerical integration of the predicted response from the Earth and Moon models. We begin by considering the response in general relativity before we continue to Brans-Dicke gravity.

1.3 Notation

In this thesis, we will use the following conventions, the mainly positive spacetime metric such that for flat space the metric is

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1). \tag{1.1}$$

Four vectors are written with Greek letter indices and spatial vectors with Latin letter indices. We assume Einstein's summation convention. The four gradient is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1.2}$$

with double indices meaning the double derivative

$$\partial_{\mu\nu} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}. \tag{1.3}$$

For a cross product between two vectors a_α and b_β we will use,

$$c_i = \epsilon_{ijk} a_j b_k, \tag{1.4}$$

where ϵ_{ijk} is the Levi-Civita tensor. For the differentiation for vectors or tensors, we will at times employ the notation

$$\partial_\mu u_\nu = u_{\nu,\mu}. \tag{1.5}$$

For the curl of a vector a_α we use,

$$\epsilon_{ijk} a_{k,j}. \tag{1.6}$$

We will encounter vectors and functions with many indices, especially in Chapter 4. We have therefore chosen the notation as follows: We set indices describing four and three vectors to the above-mentioned notation. Other indices representing some property of the variable are to be inside the parenthesis unless otherwise specified. For the radial spherical unit vector, we then have the notation $\mathbf{e}_i^{(r)}$. The r in parenthesis represents that it is the radial unit vector and the indices i represent the vector nature. For the Legendre polynomials and other functions which have indices we put them into the parenthesis. For the Legendre polynomial for example with l and m we then have

$$P^{(ml)}(x) = P_l^m(x) \tag{1.7}$$

where P_l^m is the standard notation. Given spherical bodies are of key interest in the thesis we will use a spherical coordinate system defined in the standard mathematical notation.

$$\hat{e}_i^{(r)} = \sin\theta\cos\phi\hat{e}_i^{(x)} + \sin\theta\sin\phi\hat{e}_i^{(y)} + \cos\theta\hat{e}_i^{(z)}, \tag{1.8a}$$

$$\hat{e}_i^{(\theta)} = \cos\theta\cos\phi\hat{e}_i^{(x)} + \cos\theta\sin\phi\hat{e}_i^{(y)} - \sin\theta\hat{e}_i^{(z)}, \tag{1.8b}$$

$$\hat{e}_i^{(\phi)} = -\sin\phi\hat{e}_i^{(x)} + \cos\phi\hat{e}_i^{(y)}. \tag{1.8c}$$

1.3.1 Expression summary

The thesis will include expressions that have rather complicated forms. We, therefore, summarize some expressions here.

- $g^{\alpha\beta}$ Metric tensor
- $\mathfrak{g}^{\alpha\beta}$ Gothic metric tensor
- $H^{\alpha\mu\beta\nu}$ Landau-Lifshitz tensor density
- $h^{\alpha\beta}$ Gravitational potentials
- $\mathcal{I}^{jk}$ Multipole moment (in Chapter 3)
- $\mathcal{I}^{jk}$ Stress tensor(in Chapter 4)
- $\mathcal{E}^{jk}$ Polarisation tensor
- $\epsilon\epsilon^{jk}$ Strain tensor
- $j_l(x)$ Bessel function
- $P^{(ml)}(x)$ Associated Legendre polynomial

CHAPTER

TWO

THE POST MINKOWSKIAN THEORY

2.1 Formulation of Post-Minkowskian Theory

Post-Minkowskian theory, or as we shall call it, Post-Minkowskian approximation is a procedure to approximate the behaviour of phenomena described by general relativity. This approximation procedure excels for weak gravitational fields, but can for other purposes than what will be used in this thesis adequately describe stronger fields as well. In the Post-Minkowskian approximation, the gravitational field is expanded in powers of the gravitational constant G. To make the best use of Post-Minkowskian approximation it is useful to rewrite the field equations in a form more suitable for the approximation procedure that will follow. We will therefore begin with an introduction to the Landau-Lifshitz formulation of general relativity. From this, we will arrive at a wave equation for a soon-to-be-introduced gravitational potential $h^{\alpha\beta}$. It is the goal of this section to derive an approximate expression for this gravitational potential in powers of c^{-n}. This chapter is heavily inspired by chapter 6 of reference [9].

2.1.1 Landau-Lifshitz formulation of general relativity

To implement the post-Minkowskian theory on Einstein's field equation it is useful to express the field equations in the Landau-Lifshitz formulation. This is an equivalent formulation to Einstein's theory, although it is not equivalent in its usefulness. It will be useful for our purposes. In the Landau-Lifshitz framework, one works with the Gothic metric instead of the standard metric tensor which is defined as

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}. \tag{2.1}$$

Here $g^{\alpha\beta}$ is the inverse metric and $\sqrt{-g}$ is the square root of the metric determinant. To get the field equations in this formalism we will require the tensor density built from the Gothic metric,

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}. \tag{2.2}$$

We mention that $H^{\alpha\mu\beta\nu}$ is a tensor density. Tensor densities are objects which differ from tensors only by the multiplication of a determinant of the metric. We wish to connect this new tensor density to the Einstein tensor such that we can rewrite the field equations. The tensor density $H^{\alpha\mu\beta\nu}$ satisfies the general identity,

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL}, \tag{2.3}$$

where $G^{\alpha\beta}$ is the Einstein tensor known from the standard field equations and $t^{\alpha\beta}_{LL}$ takes on the complicated form

$$\begin{aligned} -(g)t^{\alpha\beta}_{LL} =& \frac{c^4}{16\pi G}\bigg(\partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\ &- g^{\alpha\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\ &+ \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma}\bigg). \end{aligned} \tag{2.4}$$

We now have a connection between the new tensor density $H^{\alpha\mu\beta\nu}$ and the Einstein tensor $G^{\alpha\beta}$. With (2.3) we can write the field equations with $H^{\alpha\beta\mu\nu}$ giving us,

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL}), \tag{2.5}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor of the matter distribution. This does not seem to have gotten us into a more favorable position than before in regard to the approximation procedure. It is however at this point that we can use the harmonic gauge conditions.

2.1.2 Harmonic gauge conditions

For the field equations (2.5) we have not considered the harmonic coordinate conditions,

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0, \tag{2.6}$$

which comes from our freedom of choosing a coordinate system to describe the inherent physics. We make a short motivation of why this statement holds. We make the assumption that we have some coordinate system such that $\partial_\beta g^{\alpha\beta} \neq 0$, and then pick a new coordinate system described by the coordinates x'^μ which is connected to the old coordinate system x^α by $x'^\mu = f^\mu(x^\alpha)$. The Gothic metric in this new coordinate system is now,

$$\partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \partial_{\nu'}\sqrt{-g'}g^{\mu'\nu'} + \sqrt{-g'}\partial_{\nu'} g^{\mu'\nu'}. \tag{2.7}$$

Using the identity, $\Gamma^\sigma_{\sigma\beta} = \frac{1}{\sqrt{-g}}\partial_\beta\sqrt{-g}$ [9] in the first term we get,

$$\partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \sqrt{-g'}\left[\Gamma^\sigma_{\sigma\nu'}g^{\mu'\nu'} + \partial_{\nu'} g^{\mu'\nu'}\right]. \tag{2.8}$$

Relabeling $\nu' \to \beta$ and expanding $g^{\mu'\beta} = \partial_\alpha f^\mu g^{\alpha\beta}$ by the transformation rule of the metric under a coordinate transformation we get,

$$\partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \sqrt{-g'}\left[\Gamma^\sigma_{\sigma\beta}g^{\alpha\beta}\partial_\alpha f^\mu + \partial_\beta(g^{\alpha\beta}\partial_\alpha f^\mu)\right], \tag{2.9}$$

where the term in the bracket parenthesis is just the curved spacetime d'Alembertian $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ applied on f^μ. We then get at last,

$$\partial_{\nu'}\mathfrak{g}^{\mu'\nu'} = \sqrt{-g'}\Box f^\mu. \tag{2.10}$$

If we were to choose four harmonic functions f^μ, then $\Box f^\mu = 0$ and one can therefore always find a coordinate system where the harmonic coordinate condition on the Gothic metric holds. We proceed to rewrite the equations of motion by a new potential $h^{\alpha\beta}$ which is constructed of the Minkowski metric and the Gothic metric tensor,

$$h^{\alpha\beta} = \eta^{\mu\nu} - \mathfrak{g}^{\alpha\beta}. \tag{2.11}$$

The harmonic coordinate condition implies then that,

$$\partial_\beta h^{\alpha\beta} = 0. \tag{2.12}$$

From now we will call these the harmonic gauge condition. We can write the earlier equation of motion (2.5) in terms of the new potential instead of $\mathfrak{g}^{\alpha\beta}$,

$$\partial_{\mu\nu}H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta} - \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu}. \tag{2.13}$$

If we rewrite by moving the $-\Box h^{\alpha\beta}$ on the l.h.s and $\partial_{\mu\nu}H^{\alpha\mu\beta\nu}$ to the r.h.s we get the wave equation,

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{2.14}$$

where

$$\tau^{\alpha\beta} = (-g)(T^{\alpha\beta} + \tau^{\alpha\beta}_{LL} + \tau^{\alpha\beta}_{H}), \tag{2.15}$$

and we defined $\tau^{\alpha\beta}_H$ as the last three terms of (2.13). Explicitly we have that,

$$(-g)\tau^{\alpha\beta}_H = \frac{c^4}{16\pi G}\left(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\right). \tag{2.16}$$

From the harmonic condition, we then get that $\tau^{\alpha\beta}$ satisfies

$$\partial_\beta\tau^{\alpha\beta} = 0. \tag{2.17}$$

It is worthwhile to mention that this is an exact formulation of general relativity and no approximation has taken place yet. We have formulated the field equations in this way because they are especially easy to work with when we are to derive an approximated expression for gravitational waves at a later stage. The equations of (2.14) are called the relaxed Einstein equations. What separates them from the general Einstein equations of general relativity is the gauge conditions. It is the combination of the relaxed Einstein equations and the gauge condition which are equivalent to the usual formulation of general relativity and not the relaxed equation alone.

2.1.3 Solution to the wave equation

We are now interested in the solution to the wave equation (2.14). We will find the solution by making the assumption that we can write the potential as an expansion in powers of the gravitational constant. The form of the potential is then

$$h^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2k_2^{\alpha\beta} + \dots \tag{2.18}$$

This expansion in powers of G is the Post-Minkowskian expansion. Such an expansion in its current form looks like an asymptotic expansion. We say this is an asymptotic expansion since we are expanding in powers of the gravitational constant G, a value with dimensions. It is however a representation or placeholder expansion parameter and the real expansion parameter depends on the problem at hand and should be dimensionless. Assuming we have such an expansion to the analytical potential, then mathematically we require for this expansion to converge that,

$$g_{\alpha\beta}(x) - g^{(n)}_{\alpha\beta}(x) = O(G^{n+1}), \tag{2.19}$$

when x is in the domain of the manifold and $g^{(n)}_{\alpha\beta}(x)$ is the sum of n terms in the expansion. The procedure to find $h^{\alpha\beta}_n(x)$ is by iteration. One starts by setting $h^{\alpha\beta}_0 = 0$ which implies that $g^0_{\alpha\beta} = \eta_{\alpha\beta}$. From the Minkowski metric, we can construct the terms of the effective energy-momentum tensor $T^{\alpha\beta}, \tau^{\alpha\beta}_{LL}$ and $\tau^{\alpha\beta}_H$ where $\tau^{\alpha\beta}_{LL}[h_0]$ and $\tau^{\alpha\beta}_H[h_0]$ are zero. This is the zeroth iteration and we move on to the first iteration. In the first iteration, one takes the effective energy-momentum tensor of the zeroth iteration and inserts it into the wave equation,

$$\Box h^{\alpha\beta} = -\frac{16\pi}{c^4}\tau^{\alpha\beta}_0. \tag{2.20}$$

We should be able to integrate this wave equation, at least in principle since the source term is known. For the second iteration, one follows the same procedure. One then takes the effective energy-momentum tensor resulting from the first iteration and inserts it into the wave equation for the second order term for the potential $h^{\alpha\beta}_2$. Doing n iterations of this procedure we end up with the potential

$$h^{\alpha\beta}_n = Gk^{\alpha\beta}_a + G^2 k^{\alpha\beta}_2 + \dots \tag{2.21}$$

We must at the end also invoke the gauge condition of

$$\partial_\beta h^{\alpha\beta}_n = 0. \tag{2.22}$$

In the procedure, we have not used the gauge condition until after the iterations. We could not have invoked it after every iteration as trying this one will realize leads to contradictions. It is therefore important to only do the iterative procedure on the relaxed Einstein equations and leave the gauge condition for the very end. This procedure is quite demanding calculation-wise. We, therefore, refer to Chapter 7 of [9] for further details on simplifications. We will not require the procedure here for our further studies of the Moon. This procedure is however extremely useful when studying places with strong curvature of spacetime. It could be a valuable tool for anyone studying the production of gravitational waves through binaries and are interested in the spacetime close to the rotating astrophysical objects. It also can sufficiently describe spacetime at a safe distance to neutron stars and could be of interest for further study into the observation of continuous gravitational wave observations.

2.1.3.1 Formal solution of the wave equation

The formal solution of the wave equation takes on the form

$$h^{\alpha\beta}(x) = \frac{4G}{c}\int G(x,x')\tau^{\alpha\beta}(x')d^4x' \tag{2.23}$$

where x describes the field point, x' is the source point and $G(x,x')$ is the retarded Green function. For (2.23) to satisfy the wave equation the Green function must satisfy the following equation when acted upon by the Minkowski operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$,

$$\Box G(x,x') = -4\pi\delta(x-x'), \tag{2.24}$$

and is given as

$$G(x,x') = \frac{\delta(ct-ct'-|x-x'|)}{|x-x'|}. \tag{2.25}$$

If we consider a general function ψ satisfying the wave equation

$$\Box\psi(x) = -4\pi\mu(x), \tag{2.26}$$

where μ is the source function, then a general solution to this equation is

$$\psi(x) = \int \frac{\delta(ct-ct'-|x-x'|)}{x-x'}\mu(x')d^4x' = \int \frac{\mu(t-|x-x'|/c, x')}{|x-x'|}d^4x'. \tag{2.27}$$

2.1.4 Near zone and wave zone

To simplify the calculations in the Post-Minkowskian approximation procedure we will split the integration domain into two zones, a near zone and a wave zone. We call the near zone $\mathcal{N}$ and the wave zone $\mathcal{W}$. We relate the zones to the wave equation by the properties

$$t_c = \text{characteristic time scale of a source}, \tag{2.28}$$

$$\omega_c = \frac{2\pi}{t_c} = \text{characteristic frequency of the source}, \tag{2.29}$$

$$\lambda_c = \frac{2\pi c}{\omega_c} = ct_c = \text{characteristic wavelength of the radiation from the source.} \tag{2.30}$$

The characteristic time scale t_c is the time for there to be a noticeable change in the system of interest. For us, this is the source. From the characteristic wavelength we define the near- and wave zone

$$\text{near zone}: \quad r \ll \lambda_c \tag{2.31}$$

$$\text{wave zone}: \quad r \gg \lambda_c \tag{2.32}$$

To do the integral (2.27) we divide the integral domain into a near zone domain $\mathcal{N}(x)$ and a wave zone domain $\mathcal{W}(x)$. The boundary between these two domains is called $\mathcal{R}$, where $\mathcal{R}$ is of the same order of magnitude as λ_c. We can therefore write the wavefunction as

$$\psi(x) = \psi_{\mathcal{N}}(x) + \psi_{\mathcal{W}}(x). \tag{2.33}$$

We wish to study signals of gravitational waves far away from the sources, and so we are only interested in the case of the field point being in the wave zone and the source point laying in the near zone. We, therefore, assume $\psi(x) \simeq \psi_{\mathcal{N}}(x)$ where ψ is the wavefunction describing the exact solution of a wave equation while $\psi_{\mathcal{N}}$

is the approximated wave integrated in the near zone. We shall only include the integration over the near zone in what follows. We can rewrite the integrand as

$$\frac{\mu(t-|x-x'|/c,x')}{|x-x'|} = \int \frac{\mu(t-|x-x'|/c,y)}{|x-x'|}\delta(y-x')d^3y \tag{2.34}$$

$$= \int g(x,x',y)\delta(y-x')d^3y \tag{2.35}$$

We now expand $g(x,x',y)$ in terms of x'. We know x' is small given that we are in the near zone. A Taylor expansion around $x'=0$ gives

$$g(x,x',y) = g(x,0,y) + \frac{\partial g}{\partial x'^j}x'^j + \frac{1}{2}\frac{\partial^2 g}{\partial x'^j \partial x'^k}x'^j x'^k + \dots \tag{2.36}$$

Since $g(x,x',y)$ only depends on $|x-x'|$, then $\partial g/\partial x'^j = -\partial g/\partial x^j$, such that we can rewrite the Taylor expansion as

$$g(x,x',y) = g(x,0,y) - \frac{\partial g}{\partial x^j}x'^j + \frac{1}{2}\frac{\partial^2 g}{\partial x^j \partial x^k}x'^j x'^k + \dots \tag{2.37}$$

Given that we differentiate with respect to x instead of x' we can set $x'=0$ before taking the derivative as the derivative is evaluated at $x'=0$ regardless. This has made the g function independent of x' and a sole function of $|x|=r$ and y. The Taylor expansion simplifies to

$$g(x,x',y) = g(x,0,y) - \frac{\partial g(r,0,y)}{\partial x^j}x'^j + \frac{1}{2}\frac{\partial^2 g(r,0,y)}{\partial x^j \partial x^k}x'^j x'^k + \dots \tag{2.38}$$

which we can write as a sum,

$$g(x,x',y) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L g(r,0,y), \tag{2.39}$$

here L is shorthanded for the indices $L = j_1 j_2 \dots j_l$. If we insert this back into our earlier expression for the wave equation we get

$$\psi_{\mathcal{N}}(t,x) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{r}\int_{\mathcal{M}}\mu(t-r/c,x')x'^L d^3x'\right). \tag{2.40}$$

This is our expression for the wave function in the near zone which we will take into use a number of times later on in this thesis. Here $\mathcal{M}$ is the surface of constant time bounded by the boundary $\mathcal{R}$ between the near- and wave zone.

2.2 The gravitational wave potential

In this section, we will implement the ideas of section 2.1. We begin by recalling the main aspects of the last section: We found in the Landau-Lifshitz formulation of general relativity that the field equations take the form of a wave equation with the potential $h^{\alpha\beta}$ and the effective energy-momentum tensor $\tau^{\alpha\beta}$ as the source term. We had to include the harmonic gauge condition as well for the new field equations to stay equivalent to the standard field equations of general relativity. For the approximation procedure, we began with an effective energy-momentum tensor equal to the energy-momentum tensor $T^{\alpha\beta}$ formed from the matter distribution. This is inserted into the wave equation which then has to be solved. Solving the wave equation leads us to a new metric tensor $g^{\alpha\beta}$ which we use to construct a new effective energy-momentum tensor. This effective energy-momentum tensor goes again into a wave equation we must solve and one continues this procedure until one has reached the required accuracy. We will in this section find an expression for the gravitational potential in the near zone,

2.2.1 Wave zone gravitational potential

We recall from (2.40) that the gravitational potential can be written as

$$h^{\alpha\beta}_{\mathcal{N}}(t,x) = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{\alpha\beta}(\tau,x')x'^L d^3x'\right], \tag{2.41}$$

when the field point is taken in the wave zone and the source point is taken in the near zone, where τ is here the retarded time. We will employ the conservation equation $\partial_\beta\tau^{\alpha\beta} = 0$ to simplify the first terms of our expansion. We will require the identities,

$$\tau^{0j} = \partial_0(\tau^{00}x^j) + \partial_k(\tau^{0k}x^j), \tag{2.42a}$$

$$\tau^{jk} = \frac{1}{2}\partial_{00}(\tau^{00}x^jx^k) + \frac{1}{2}\partial_p(2\tau^{p(j}x^{k)} - \partial_q\tau^{pq}x^jx^k). \tag{2.42b}$$

We also define the notation of the multiple moments,

$$\mathcal{I}(\tau) = \int_{\mathcal{M}} c^{-2}\tau^{00}(\tau,\mathbf{x})d^3x, \tag{2.43a}$$

$$\mathcal{I}^j(\tau) = \int_{\mathcal{M}} c^{-2}\tau^{00}(\tau,\mathbf{x})x^jd^3x, \tag{2.43b}$$

$$\mathcal{I}^{jk}(\tau) = \int_{\mathcal{M}} c^{-2}\tau^{00}(\tau,\mathbf{x})x^jx^kd^3x. \tag{2.43c}$$

Restricting our focus on the h^{00} component of the potentials and writing the term for $l = 0$ by itself while keeping the sum for $l \geq 1$ we have that,

$$h^{00} = \frac{4G}{c^4 r}\int_{\mathcal{M}}\tau^{00}d^3x' + \frac{4G}{c^4}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{00}(\tau,x')x'^L d^3x'\right].$$

We define the $l = 0$ term as the monopole moment,

$$M_0 = \int c^{-2}\tau^{00}(\tau,x)d^3x, \tag{2.44}$$

which approximately satisfies a conservation law. If we take the time derivative of M_0 and use $\partial_0\tau^{00} = \partial_i\tau^{0i}$ then we can rewrite (2.44) into a surface integral over $\partial\mathcal{M}$ which is small by the near zone assumption. The $l = 1$ contribution includes,

$$M_0R_0^j = \int_{\mathcal{M}} c^{-2}\tau^{00}(\tau,\mathbf{x})x^jd^3x, \tag{2.45}$$

where R_0^j is the position of the center of mass in the domain $\mathcal{M}$. It turns out that this $l = 1$ contribution vanishes and the reason for this is that,

$$\frac{d}{d\tau}\int_{\mathcal{M}} c^{-2}\tau^{00}x^j d^3x = P^j + \text{surface integral}, \tag{2.46}$$

where,

$$P^j = \int_{\mathcal{M}} c^{-1}\tau^{0j}(\tau, x)d^3x, \tag{2.47}$$

is the near zone momentum and we get a surface integral by using (2.42a). We can choose a reference frame where $P^j = 0$, in the rest frame of the system. We might also set the centre of mass position R_0^j to zero by placing the centre of mass at the spatial origin of the harmonic coordinates. We can therefore set the whole $l = 1$ term to zero for h^{00}. For h^{0j} we get for the $l = 0$ term,

$$h^{0j}{}_{\{l=0\}} = \frac{4G}{c^4}\frac{1}{r}\int_{\mathcal{M}} \tau^{0j}(\tau, x')d^3x', \tag{2.48}$$

which we rewrite to

$$\frac{4G}{c^4}\frac{1}{r}\left(\int_{\mathcal{M}} \partial_0\tau^{00}(\tau, x')x^j d^3x' + \int_{\mathcal{M}} \partial_k\tau^{0k}(\tau, x')x^j d^3x'\right) \tag{2.49}$$

by the use of the identity (2.42a). This is equivalent to

$$\frac{4G}{c^3 r}\dot{\mathcal{I}}^j + \text{surface term}. \tag{2.50}$$

Since $\dot{\mathcal{I}}^j = P^j$ + surface integral we can remove this contribution by our choice of reference frame. The $l = 1$ contribution is

$$h^{0j}{}_{\{l=1\}} = -\frac{4G}{c^4}\partial_0\left[\frac{1}{r}\int_{\mathcal{M}} \tau^{0j}(\tau, x')x'^0 d^3x'\right] - \frac{4G}{c^4}\partial_k\left[\frac{1}{r}\int_{\mathcal{M}} \tau^{0j}(\tau, x')x'^k d^3x'\right]. \tag{2.51}$$

We will focus on the second term, which we rewrite as,

$$\frac{4G}{c^4}\frac{1}{r^2}\int_{\mathcal{M}} \tau^{0j}(\tau, x')x'^k d^3x'. \tag{2.52}$$

Dropping the prime and using the identity (2.42b) we get,

$$\frac{4G}{c^4}\frac{1}{r^2}\int_{\mathcal{M}} \tau^{0j}x^k d^3x = \frac{2G}{c^4}\frac{1}{r^2}\int_{\mathcal{M}} \left(\dot{\mathcal{I}}^{jk} - \epsilon^{mjk}J_0^m\right) d^3x, \tag{2.53}$$

where J_0^m is the angular momentum in the near zone. Finally studying the h^{jk} term and more specifically the $l = 0$ part of the expression we can simplify to

$$h^{jk}_{\{l=0\}} = \frac{2G}{c^4 r}\ddot{\mathcal{I}}^{jk} + \text{surface terms}, \tag{2.54}$$

where the identity,

$$\tau^{jk} = \frac{1}{2}\partial_{00}(\tau^{00}x^j x^k) + \frac{1}{2}\partial_p(2\tau^{p(j)}x^{k)} - \partial_q\tau^{pq}x^j x^k), \tag{2.55}$$

was used. Summarizing what we have found for all components of the gravitational potential we have found that the components of the potential can be simplified to,

$$h_{\mathcal{N}}^{00} = \frac{4GM_0}{c^2 r} + \frac{4G}{c^2} \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{\mathcal{I}^L(\tau)}{r} \right], \tag{2.56a}$$

$$\begin{aligned} h_{\mathcal{N}}^{0j} = &-\frac{2G}{c^3} \frac{(n \times J_0)^j}{r^2} - \frac{2G}{c^3} \partial_k \left[\frac{\dot{\mathcal{I}}^{jk}(\tau)}{r} \right] \\ &+ \frac{4G}{c^4} \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \tau^{0j}(\tau, x') x'^L d^3x' \right], \end{aligned} \tag{2.56b}$$

$$h_{\mathcal{N}}^{jk} = \frac{2G}{c^4} \frac{\ddot{\mathcal{I}}^{jk}(\tau)}{r} + \frac{4G}{c^4} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \tau^{jk}(\tau, x') x'^L d^3x' \right], \tag{2.56c}$$

where differentiation, here written with a dot, is with respect to the retarded time $\tau = t - r/c$. With these equations, we could, given that we know the effective energy-momentum tensor calculate the gravitational potential from a source described by $\tau^{\alpha\beta}$. We shall for further purposes require only the general shape of the potentials of (2.56) in the zone where we make the assumption that r is so big only the terms proportional to r^{-1} will be included.

CHAPTER

THREE

GRAVITATIONAL WAVES IN GENERAL RELATIVITY AND BRANS-DICKE GRAVITY

3.1 Gravitational field and polarization in general relativity

Before we proceed to examine gravitational waves in Brans-Dicke gravity we review gravitational waves in general relativity. We will introduce concepts and approximations that will continue to be useful once we go to the more complicated theories. This chapter is inspired by Chapters 11 and 13 of [9].

3.1.1 Decomposition of the gravitational potential

We continue to consider the far-away wave zone in which the gravitational potentials $h^{\alpha\beta}$ from (2.56) take on the forms,

$$h^{00} = \frac{4GM}{c^2R} + \frac{G}{c^4R}C(\tau, N), \tag{3.1a}$$

$$h^{0j} = \frac{G}{c^4R}D^j(\tau, N), \tag{3.1b}$$

$$h^{jk} = \frac{G}{c^4R}A^{jk}(\tau, N). \tag{3.1c}$$

It is interesting to note that these potentials satisfy the wave equation $\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta}/c^4$ given that $\tau^{\alpha\beta}$ falls of at least as fast as R^{-2}. We now decompose the components of the gravitational potential into irreducible components. Using the harmonic gauge condition we can remove four degrees of freedom greatly simplifying the final expression of our gravitational wave. We split D^j and A^{jk} into longitudinal and transversal components as such:

$$D^j = DN^j + D^j_T, \tag{3.2}$$

$$A^{jk} = \frac{1}{3}\delta^{jk}A + \left(N^jN^k - \frac{1}{3}\delta^{jk}\right)B + N^jA^k_T + N^kA^j_T + A^{jk}_{TT}. \tag{3.3}$$

We have split D^j into one longitudinal part with direction N^j and two transverse parts. We therefore have $N_jD^j_T = 0$. Similarly, but slightly more complicated for

A^{jk}, we split A^{jk} into a trace part A, a longitudinal tracefree part B, a longitudinal-transverse part A_T^j and a transverse-traceless part A_{TT}^{jk}. So we must necessarily have that $N_j A_T^j = 0$ and $N_j A_{TT}^{jk} = 0$. We now proceed with the harmonic condition.

3.1.2 Harmonic gauge condition on the polarization

The next goal is to put a restriction on the longitudinal and transverse components of the gravitational wave by using the harmonic gauge condition. We can write the harmonic gauge condition as,

$$\frac{1}{c}\partial_\tau h^{00} + \partial_k h^{0k} = 0, \tag{3.4a}$$

$$\frac{1}{c}\partial_\tau h^{0j} + \partial_k h^{jk} = 0. \tag{3.4b}$$

We can simplify these relations by the use of the identity,

$$\partial_j h^{\alpha\beta} = -\frac{1}{c} N_j \partial_\tau h^{\alpha\beta}, \tag{3.5}$$

which is true whenever we neglect terms that are $O(R^{-2})$. We arrive at this identity by observing from (3.1) that the three components of consideration under the differentiation $\partial_j h^{\alpha\beta}$ are $\partial_j R^{-1}$, $\partial_j \tau$ and $\partial_j \mathbf{N}$. Of these three terms only $\partial_j \tau$ results in a term of $O(R^{-1})$ that is $\partial_j \tau = \partial_j (t - R/c) = -N_j/c$. Using (3.5) for (3.4a) and (3.4b) we then have,

$$\partial_\tau (h^{00} - h^{0k} N_k) = 0, \tag{3.6a}$$

$$\partial_\tau (h^{0j} - h^{jk} N_k) = 0. \tag{3.6b}$$

Inserting our expansions (3.1a), (3.1b) and (3.1c) into (3.6a) and (3.6b) we get equations for the undetermined $C(\tau, N)$, $D^j(\tau, N)$ and $A^{jk}(\tau, N)$,

$$\partial_\tau \left(\frac{4GM}{c^2 R} + \frac{G}{c^4 R} C - \frac{G}{c^4 R} (D N^k + D_T^k) N_k \right) = 0, \tag{3.7}$$

and

$$\partial_\tau \left(\frac{G}{c^4 R} (D N^j + D_T^j) - \left(\frac{G}{c^4 R} \frac{1}{3} \delta^{jk} A + \left(N^j N^k - \frac{1}{3} \delta^{jk} \right) B + \right.\right.$$
$$\left.\left. + N^j A_T^k + N^k A_T^j + A_{TT}^{jk} \right) N_k \right) = 0. \tag{3.8}$$

Equation (3.7) can be simplified to,

$$\partial_\tau C = \partial_\tau D, \tag{3.9}$$

so C and D are equal up to an integration constant. Equation (3.8) can be simplified as well giving us the two equations

$$\partial_\tau D = \partial_\tau \left(\frac{1}{3} A + \frac{2}{3} B \right), \tag{3.10}$$

$$\partial_\tau D_T^j = \partial_\tau A_T^j, \tag{3.11}$$

and so D^j_T and A^j_T are also equal up to an integration constant. We can set these constants equal to zero because a τ dependence in C would correspond to an unphysical shift in the total gravitational mass M and a τ dependence in D^j would fall off as R^{-2} instead of R^{-1}. So we write,

$$C = D, \tag{3.12a}$$

$$D = \frac{1}{3}A + \frac{2}{3}B, \tag{3.12b}$$

$$D^j_T = A^j_T. \tag{3.12c}$$

We have successfully removed four redundant components of the gravitational potential. In this section, we began with ten components before we reduced the number of relevant components to six. We move forward and simplify again, now taking advantage of the wave zone.

3.1.3 The Transverse-Traceless gauge

As mentioned, we can simplify our expression in the far away zone even further, removing another four redundant components. We do this by requiring the harmonic gauge condition $\partial_\beta h^{\alpha\beta}$ to hold. Let us first discuss how the gravitational potential transform under a gauge transformation.

3.1.3.1 Gauge transformation of $h^{\alpha\beta}$

We assume that the metric of an area of spacetime can be adequately described by the Minkowski metric and an additional small deviation compared to the flat spacetime. Even though we have made this simplification of the metric, the physics should still stay invariant of a Lorentz transformation. The coordinate will then transform under a Lorentz transformation as,

$$x'^\mu = x^\mu + \zeta^\mu \tag{3.13}$$

the metric will therefore transform as,

$$g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu(\eta_{\nu\gamma}\zeta^\gamma) - \partial_\nu(\eta_{\mu\gamma}\zeta^\gamma) + O(\epsilon^2). \tag{3.14}$$

We are interested in how the gravitational potential transforms under a gauge transformation. We must therefore first write the metric in terms of the gravitational potential $h_{\alpha\beta}$. We know from our definition of $h_{\alpha\beta}$ that

$$g_{\alpha\beta} = \sqrt{-\mathfrak{g}}(\eta_{\alpha\beta} + h_{\alpha\beta}). \tag{3.15}$$

Since $h^{\alpha\beta}$ is of order G we can expand the $\sqrt{-\mathfrak{g}}$ term in powers of $h^{\alpha\beta}$ and only keep the first order terms. The expansion of the determinant is,

$$\sqrt{-\mathfrak{g}} = \left(1 - \frac{1}{2}h^\mu{}_\mu + O(h^2)\right) \tag{3.16}$$

and so the metric is,

$$g_{\mu\nu} = \left(1 - \frac{1}{2}h^\gamma{}_\gamma + O(h^2)\right)(\eta_{\mu\nu} + h_{\mu\nu}) = \eta_{\mu\nu} + h_{\mu\nu} - \frac{1}{2}h^\gamma{}_\gamma\eta_{\mu\nu} + O(h^2). \tag{3.17}$$

From (3.14) and (3.17) we see that the gravitational potential transform according to the following rule,

$$h^{\alpha\beta} \to h^{\alpha\beta} - \partial^\alpha \zeta^\beta - \partial^\beta \zeta^\alpha + (\partial_\mu \zeta^\mu)\eta^{\alpha\beta}, \tag{3.18}$$

under a gauge transformation.

3.1.3.2 Simplification of $h^{\alpha\beta}$

It follows from the gauge transformation derived in the last section, (3.18) that the harmonic gauge condition under a coordinate transformation is now,

$$\begin{aligned}\partial_\beta h^{\alpha\beta} &\to \partial_\beta h^{\alpha\beta} - \partial_\beta \partial^\alpha \zeta^\beta - \partial_\beta \partial^\beta \zeta^\alpha + \partial_\beta (\partial_\mu \zeta^\mu)\eta^{\alpha\beta} \\ &= \partial_\beta h^{\alpha\beta} - \partial^\alpha (\partial_\beta \zeta^\beta) - \Box \zeta^\alpha + \partial^\alpha (\partial_\mu \zeta^\mu) \\ &= \partial_\beta h^{\alpha\beta} - \Box \zeta^\alpha. \end{aligned} \tag{3.19}$$

For the harmonic gauge condition to hold we must have that $\Box\zeta^\alpha = 0$. We now employ the approximation that the field point is such far away from we only need to consider terms up to $O(R^{-1})$. We split ζ into a time- and spatial-dependent part. We then have for ζ that,

$$\zeta^0 = \frac{G}{c^3 R}\alpha(\tau, N) + O(R^{-2}), \tag{3.20a}$$

$$\zeta^j = \frac{G}{c^3 R}\beta^j(\tau, N) + O(R^{-2}). \tag{3.20b}$$

We can also split the β^j into a transverse and longitudinal piece

$$\beta^j = \beta N^j + \beta^j_T. \tag{3.21}$$

We now have all the tools we require to simplify the gravitational potentials. Using (3.1a), (3.12a) and (3.18) we get for h^{00},

$$h^{00} \to \frac{4GM}{c^2 R} + \frac{G}{c^4 R}\frac{1}{3}(A + 2B) - 2\partial^0 \xi^0 + (\partial_\mu \xi^\mu)\eta^{00}. \tag{3.22}$$

Taking now use of (3.20) and (3.21) we conclude that the potential h^{00} transform as,

$$h^{00} \to= \frac{4GM}{c^2 R} + \frac{G}{c^4 R}\frac{1}{3}(A + 2B) + \frac{G}{c^4 R}\partial_\tau \alpha + \frac{G}{c^4 R}\partial_j \beta^j. \tag{3.23}$$

Using the differentiation rule for the wave zone in the last term and remembering that $\beta^j_T N_j = 0$ we get that the potentials transform like,

$$h^{00} \to \frac{4GM}{c^2 R} + \frac{G}{c^4 R}\frac{1}{3}(A + 2B) + \frac{G}{c^4 R}\partial_\tau \alpha - \frac{G}{c^4 R}\partial_\tau \beta. \tag{3.24}$$

We can do the same for h^{0j},

$$\begin{aligned} h^{0j} &\to \frac{G}{c^4 R}\left(\frac{1}{3}(A + 2B)N^j + A^j_T\right) - \frac{G}{c^3 R}\partial^0 \beta^j - \frac{G}{c^3 R}\partial^j \alpha, \\ &= \frac{G}{c^4 R}\left(\frac{1}{3}(A + 2B)N^j + A^j_T\right) + \frac{G}{c^4 R}\partial_\tau \beta^j + \frac{G}{c^4 R}\partial_\tau \alpha N^j, \end{aligned}$$

and finally h^{jk},

$$\begin{aligned} h^{jk} &\to h^{jk} - \partial^j\beta^k - \partial^k\beta^j + (\partial_0\alpha + \partial_i\beta^i)\delta^{jk} \\ &= h^{jk} + \partial_\tau\beta N^k N^j + \partial_\tau\beta N^j N^k + \partial_\tau\alpha\delta^{jk} - \partial_\tau\beta N^i N_i \delta^{jk}, \\ &= h^{jk} + 2\partial_\tau\beta N^k N^j + \partial_\tau\alpha\delta^{jk} - \partial_\tau\beta\delta^{jk}, \\ &= \frac{1}{3}\delta^{jk}A + \left(N^j N^k - \frac{1}{3}\delta^{jk}\right)B + N^j A_T^k + N^k A_T^j + A_{TT}^{jk}, \\ &\quad + 2\partial_\tau\beta N^k N^j + (\partial_\tau\alpha - \partial_\tau\beta)\delta^{jk}. \end{aligned}$$

This results in the transformation rules,

$$\begin{aligned} \frac{1}{3}(A+2B) &\to \frac{1}{3}(A+2B) + \partial_\tau\alpha + \partial_\tau\beta, \\ A_T^j &\to A_T^j + \partial_\tau\beta_T^j \\ \frac{1}{3}(A+B) &\to \frac{1}{3}(A+B) + \partial_\tau\alpha - \partial_\tau\beta \\ B &\to B + 2\partial_\tau\beta, \end{aligned}$$

which then results in the changes by the gauge transformation

$$\begin{aligned} A &\to A + 3\partial_\tau\alpha - \partial_\tau\beta, \\ B &\to B + 2\partial_\tau\beta, \\ A_T^j &\to A_T^j + \partial_\tau\beta_T^j, \\ A_{TT}^{jk} &\to A_{TT}^{jk}. \end{aligned}$$

Since we can arbitrarily choose α and β as long as $h^{\alpha\beta}$ satisfy the wave equation through ζ we can choose an α and β so that A, B and A_T^j can be set to zero. In this way, all radiative degrees of freedom of the gravitational wave are contained in A_{TT}^{jk}. We have therefore shown that by the harmonic gauge condition, we can greatly simplify the expression for the gravitational potentials. Inserting into the expansion of (3.1) the simplifications of $C = D$, $D = 1/3A + 2/3B$, $D_T^j = A_T^j$ and $A = B = A_T^j = 0$ we get,

$$h^{00} = \frac{4GM}{c^2 R}, \tag{3.25a}$$

$$h^{0j} = 0, \tag{3.25b}$$

$$h^{jk} = \frac{G}{c^4 R} A_{TT}^{jk}(\tau, \mathbf{N}). \tag{3.25c}$$

The gravitational potentials now have a very simple form which we will take great use of.

3.2 The polarization tensor for General relativity

We wish to create a transverse-traceless projector. We start forming a projection operator on the subspace orthogonal to $\mathbf{N}$. This can be written as

$$P_k^j = \delta_k^j - N^j N_k, \tag{3.26}$$

which we can then use to create the transverse-traceless projector

$$(TT)^{jk}_{pq} = P^j_p P^k_q - \frac{1}{2} P^{jk} P_{pq}. \tag{3.27}$$

To describe the polarisation of the gravitational wave we require a coordinate system. We choose the following,

$$\mathbf{N} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta], \tag{3.28a}$$

$$\vartheta = [\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta] \tag{3.28b}$$

$$\varphi = [-\sin\phi, \cos\phi, 0], \tag{3.28c}$$

Using the identity $\delta^{jk} = N^j N^k + \vartheta^j\vartheta^k + \varphi^j\varphi^k$, the projection operator can be written as

$$P^{jk} = \vartheta^j\vartheta^k + \varphi^j\varphi^k, \tag{3.29}$$

and our transverse-traceless operator becomes

$$(TT)^{jk}_{pq} = (\vartheta^j\vartheta_p + \varphi^j\varphi_p)(\vartheta^k\vartheta_q + \varphi^k\varphi_q) - \frac{1}{2}(\vartheta^j\vartheta^k + \varphi^j\varphi^k)(\vartheta_p\vartheta_q + \varphi_p\varphi_q). \tag{3.30}$$

We want to decompose A^{jk}_{TT} into two main parts described by the vectors ϑ_i and φ_j that are left invariant when acted upon by the transverse-traceless operator. We are therefore interested in what combination of ϑ and φ remains unchanged when acted upon by our projection operator. Let us look at $(TT)^{jk}_{pq}\vartheta_j\vartheta_k$ for example,

$$\begin{aligned}(TT)^{jk}_{pq}\vartheta_j\vartheta_k &= \left((\vartheta^j\vartheta_p + \varphi^j\varphi_p)(\vartheta^k\vartheta_q + \varphi^k\varphi_q) - \frac{1}{2}(\vartheta^j\vartheta^k + \varphi^j\varphi^k)(\vartheta_p\vartheta_q + \varphi_p\varphi_q)\right)(\vartheta_j\vartheta_k)\\ &= \vartheta_p\vartheta_q - \frac{1}{2}\vartheta_p\vartheta_q - \frac{1}{2}\varphi_p\varphi_q,\\ &= \frac{1}{2}(\vartheta_p\vartheta_q - \varphi_p\varphi_q),\end{aligned}$$

and let us look at $\varphi_j\varphi_k$ as well,

$$\begin{aligned}(TT)^{jk}_{pq}\varphi_j\varphi_k &= \left((\vartheta^j\vartheta_p + \varphi^j\varphi_p)(\vartheta^k\vartheta_q + \varphi^k\varphi_q) - \frac{1}{2}(\vartheta^j\vartheta^k + \varphi^j\varphi^k)(\vartheta_p\vartheta_q + \varphi_p\varphi_q)\right)(\varphi_j\varphi_k)\\ &= \varphi_p\varphi_q - \frac{1}{2}\vartheta_p\vartheta_q - \frac{1}{2}\varphi_p\varphi_q\\ &= \frac{1}{2}(\vartheta_p\vartheta_q - \varphi_p\varphi_q)\end{aligned}$$

Thus a tensor $A^{jk}_{+} = A_{+}(\vartheta^j\vartheta^k - \varphi^j\varphi^k)$ is left unchanged when acted upon by the transverse traceless-operator. We do not show it here, but similarly for $\vartheta^j\vartheta^k - \varphi^j\varphi^k$ the combination $\vartheta_j\varphi_k + \varphi_k\vartheta_j$ stays unchanged when acted upon by the transverse-traceless operator. We can therefore write the transverse traceless part of our potential as

$$A^{jk}_{TT} = A_{+}(\vartheta^j\vartheta^k - \varphi^j\varphi^k) + A_{\times}(\vartheta^j\varphi^k + \varphi^k\vartheta^j). \tag{3.31}$$

Having started with ten components we have managed to reduce the number of relevant components for a gravitational wave in general relativity to just two components. A plus component A_{+} and a cross component $A_{\times}$. We proceed to show how these polarisations affect spacetime.

3.3 Geodesic deviation

The gravitational wave will slightly deviate the displacement vector between test particles in a detector. We obtain the details from the equation of geodesic deviation,

$$\frac{D^2\xi^\alpha}{ds^2} = -R^\alpha{}_{\beta\gamma\delta}u^\beta\xi^\gamma u^\delta. \tag{3.32}$$

It is possible to simplify this equation in the low-velocity limit, i.e. $u^\alpha = (-c, 0)$,

$$\frac{D^2\xi^\alpha}{ds^2} = -c^2 R^\alpha{}_{0\gamma 0}\xi^\gamma. \tag{3.33}$$

Another simplification can also be made. We can change the covariant derivative D/ds to d/dt if we assume that the test masses are slowly moving,

$$\frac{d^2\xi_j}{dt^2} = -c^2 R_{0j0k}\xi^k. \tag{3.34}$$

We proceed to calculate the Riemann tensor R_{0j0k}. Writing it out as Christoffel symbols we get,

$$\begin{aligned} R_{0j0k} &= g_{00}R^0{}_{j0k} \\ &= g_{00}\left[\partial_0\Gamma^0{}_{kj} - \partial_k\Gamma^0{}_{0j} + \Gamma^0{}_{0\lambda}\Gamma^\lambda{}_{kj} - \Gamma^0{}_{k\lambda}\Gamma^\lambda{}_{0j}\right]. \end{aligned}$$

Using now the definition of the Christoffel symbols,

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}), \tag{3.35}$$

we find two different terms to first order in $h^{\alpha\beta}$. From (3.17) we have that $g^{\alpha\beta}$ is defined as $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} - \frac{1}{2}h\eta^{\alpha\beta}$ such that the derivative of the Christoffel symbol written in terms of the gravitational potential is,

$$\begin{aligned} \partial_e\Gamma^a{}_{bc} &= \partial_e\left(\frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})\right) \\ &= \frac{\partial_e}{2}\left((\eta^{ad} + h^{ad} - \frac{1}{2}h\eta^{ad})(h_{db,c} - \frac{1}{2}(h\eta_{db})_{,c} + h_{dc,b} - \frac{1}{2}(h\eta_{dc})_{,b}\right. \\ &\quad \left. - h_{bc,d} + \frac{1}{2}(h\eta_{bc})_{,d})\right) \\ &= \frac{1}{2}\partial_e(h^a{}_{b,c} + h^a{}_{c,b} - h_{bc}{}^{,a} - \frac{1}{2}\eta^a{}_b h_{,c} - \frac{1}{2}\eta^a{}_c h_{,b} + \frac{1}{2}\eta_{bc}h^{,a}) + \mathcal{O}(h^2). \end{aligned}$$

We can insert this expression back into the Riemann tensor which now becomes

$$\begin{aligned} R_{0j0k} &= g_{00}\left[\frac{1}{2}\partial_0(h^0{}_{k,j} + h^0{}_{j,k} - h_{kj}{}^{,0} - \frac{1}{2}\eta^0{}_k h_{,j} - \frac{1}{2}\eta^0{}_j h_{,k} + \frac{1}{2}\eta_{kj}h^{,0}))\right. \\ &\quad \left. + \frac{1}{2}\partial_k(h^0{}_{0,j} + h^0{}_{j,0} - \frac{1}{2}\eta^0{}_0 h_{,j} - \frac{1}{2}\eta^0{}_j h_{,0} + \frac{1}{2}\eta_{0j}h^{,0}\right] \\ &= -\left[\frac{1}{2}(\partial_{0j}h_{0k} + \partial_{0k}h_{0j} - \partial_{00}h_{kj} + \partial_{kj}h_{00} + \partial_{k0}h_{0j} - \right. \\ &\quad \left. - \partial_{k0}h_{0j} - \frac{1}{2}\partial_{00}h\delta_{jk} + \frac{1}{2}\partial_{kj}h)\right] \\ &= -\left[\frac{1}{2}(\partial_{0j}h_{0k} + \partial_{0k}h_{0j} - \partial_{00}h_{kj} + \partial_{kj}h_{00} - \frac{1}{2}\partial_{00}h\delta_{jk} + \frac{1}{2}\partial_{kj}h)\right] \end{aligned} \tag{3.36}$$

Having an expression for the simplified Riemann tensor in terms of the gravitational potential we can now get an explicit expression for A^{jk}_{TT} in (3.25). By the use of (3.36) and (3.34) we now get,

$$c^2 R_{0j0k} = -\frac{G}{2c^4 R}\frac{\partial^2}{\partial\tau^2}A^{jk}_{TT}(\tau,\mathbf{N}), \tag{3.37}$$

If we integrate the equation of geodesic deviation,

$$\frac{d^2\xi_j}{dt^2} = -c^2 R_{0j0k}\xi^k = \frac{G}{2c^4 R}\frac{\partial^2}{\partial\tau^2}A^{jk}_{TT}(\tau,\mathbf{N})\xi_k, \tag{3.38}$$

to first order in displacement we get for $\xi(t)$,

$$\xi_j(t) = \xi_j(0) + \frac{G}{2c^4 R}A^{jk}_{TT}(\tau,\mathbf{N})\xi_k(0). \tag{3.39}$$

3.3.1 Gravitational wave on a ring of particles

The transverse-traceless part of the potential A^{jk}_{TT} takes on a simple form if we imagine a gravitational wave traveling purely in the z-direction such that $\mathbf{N} = (0,0,1)$. This implies that $\theta = 0$ and we are free to choose $\phi = 0$. The transverse-traceless part of the potential is then in matrix form,

$$A^{jk}_{TT} = \begin{bmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.40}$$

Let us imagine a ring of particles and ask how a gravitational wave effects this ring as it passes through. The displacement for one particle from the center of a ring is given by the solution to the geodesic equation,

$$\xi^j(t) = \xi^j(0) + \frac{G}{2c^4 R}A^{jk}_{TT}\xi_k(0). \tag{3.41}$$

After inserting for A^{jk}_{TT} this leads to,

$$x(t) = x_0 + \frac{G}{2c^4 R}(A_+ x_0 + A_\times y_0), \tag{3.42}$$

$$y(t) = y_0 + \frac{G}{2c^4 R}(A_\times x_0 - A_+ y_0). \tag{3.43}$$

where to be clear, the positions in the Cartesian coordinates are here assumed to be in the plane orthogonal to the momentum vector of the gravitational wave. This shows how we get an elliptical polarization. The time evolution of the plus polarisation over one period is shown in figure 3.3.1 and the time evolution for the cross polarisation is shown in figure 3.3.2.

3.4 Introduction to Brans-Dicke gravity

We give a short introduction to Brans-Dicke gravity and derive expressions which will be useful for section 3.8. This section is based on section 13.5.1 of reference [9]. We will begin with the Brans-Dicke action before we proceed to derive the field equations. Inspired by the Landa-Lifshitz formalism of general relativity we will perform a similar procedure for the new field equations in Brans-Dicke. This will provide an excellent starting point to approximate the gravitational potential for this theory in the wave zone.

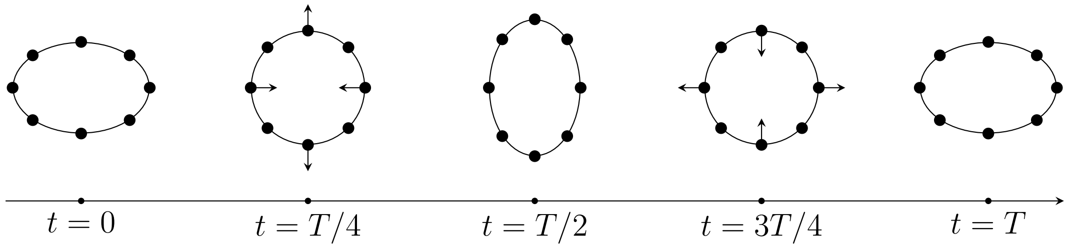

Figure 3.3.1: Time evolution of the plus polarisation of a gravitational wave on a ring of particles over the period T. The arrows represent the movement of the particles.

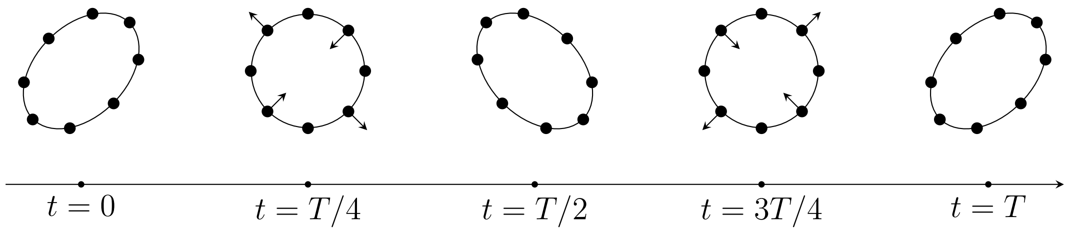

Figure 3.3.2: Time evolution of the cross polarisation of a gravitational wave on a ring of particles over the period T. The arrows represent the movement of the particles.

3.4.1 The Brans-Dicke action

In Brans-Dicke gravity an additional field is introduced. We no longer have just a metric field $g_{\mu\nu}$ as in general relativity but an additional scalar field which we call ϕ. The physical effect of this new field is understood in the gravitational constant which no longer is constant but is instead dependent on the spacetime position. We begin with the action for Brans-Dicke gravity [9] which is

$$S_g = \frac{c^3}{16\pi G_0} \int \sqrt{-g} \left(\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right) d^4x, \tag{3.44}$$

and the matter part of the action is,

$$S_m = \int \sqrt{-g} \mathcal{L}(m, g_{\alpha\beta}) d^4x, \tag{3.45}$$

where m includes all the matter variables and ω is a parameter to be determined by experiments. We observe that the matter part of the Lagrangian does not include the scalar field ϕ so the scalar field does not couple directly to any matter variable. We now wish to derive the field equations and connect these field equations with the field equations from general relativity. The reason behind this will be clear shortly. We could vary the metric and scalar field in the action and retrieve the field equations in the standard way, however, this leads to some complicated and not useful equations for our purpose. We use instead the auxiliary metric $\tilde{g}_{\alpha\beta}$ which is connected to the familiar metric $g_{\alpha\beta}$ through the relation,

$$g_{\alpha\beta} = (\phi/\phi_0) \tilde{g}_{\alpha\beta}, \tag{3.46}$$

where ϕ_0 is just some constant which we can choose at convenience later. The motivation for this change lies in the fact that using the auxiliary metric instead of the usual metric leads to field equations more optimal for our purposes of the Minkowskian approximation as this will remove the coupling of the new Ricci scalar with the scalar field. Moving on we wish to rewrite the action in terms of our new auxiliary metric. We, therefore, require to rewrite the Ricci scalar R in terms of only the auxiliary metric, ϕ_0 and ϕ. This is a simple but long calculation which result turns out to be

$$\begin{aligned} R =&(\phi_0/\phi)\left\{\tilde{R}+6\left[\partial_\alpha\left(\tilde{g}^{\alpha\beta}\frac{1}{2}\partial_\beta\ln\phi\right)+\tilde{\Gamma}^\alpha_{\alpha\mu}\left(\tilde{g}^{\mu\beta}\frac{1}{2}\partial_\beta\ln\phi\right)\right]\right. \\ &\left.-6\left(\frac{1}{2}\partial_\alpha\ln\phi\right)\left(\tilde{g}^{\alpha\beta}\frac{1}{2}\partial_\beta\ln\phi\right)\right\}. \end{aligned}$$

The term,

$$\partial_\alpha\left(\tilde{g}^{\alpha\beta}\frac{1}{2}\partial_\beta\ln\phi\right)+\tilde{\Gamma}^\alpha_{\alpha\mu}\left(\tilde{g}^{\mu\beta}\frac{1}{2}\partial_\beta\ln\phi\right), \tag{3.47}$$

is a total derivative,$\tilde{\nabla}_\alpha\left(1/2\tilde{g}^{\mu\beta}\partial_\beta\ln\phi\right)$, and can be removed by Gauss theorem in four dimensions. We insert our expression for the Ricci scalar into the gravitational action (3.44) which then takes the form,

$$S_g=\frac{c^3}{16\pi\tilde{G}}\int\sqrt{-\tilde{g}}\left[\tilde{R}-\frac{2\omega(\phi)+3}{2\phi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-\frac{\phi_0V(\phi)}{\phi^2}\right]d^4x, \tag{3.48}$$

where $\tilde{G}=G_0/\phi_0$. We have up until this point not made a comment on ϕ_0. We see here that it has a physical interpretation as the change on the bare gravitational constant G_0 with the measured constant $\tilde{G}$. The Ricci scalar is now alone in the action which is what we wished to accomplish by introducing the auxiliary metric. The matter action will necessarily also change,

$$S_m=\int\sqrt{-\tilde{g}}(\phi/\phi_0)^2\mathcal{L}(m,\phi,\tilde{g}_{\alpha\beta})d^4x. \tag{3.49}$$

We can now vary the action with respect to the auxiliary metric $\tilde{g}_{\alpha\beta}$ and the scalar field ϕ. For the variation w.r.t $\tilde{g}_{\alpha\beta}$ we obtain the field equations,

$$\tilde{G}_{\alpha\beta}-\frac{1}{2}\tilde{\Theta}_{\alpha\beta}=\frac{8\pi\tilde{G}}{c^4}\tilde{T}_{\alpha\beta}, \tag{3.50}$$

where $\tilde{G}_{\alpha\beta}$ is the Einstein tensor in terms of the auxiliary metric, $\tilde{\Theta}_{\alpha\beta}$ is given by

$$\tilde{\Theta}_{\alpha\beta}=\frac{2\omega+3}{\phi^2}\left(\partial_\alpha\phi\partial\beta\phi-\frac{1}{2}\tilde{g}_{\alpha\beta}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)-\frac{\phi_0V(\phi)}{\phi^2}\tilde{g}_{\alpha\beta}, \tag{3.51}$$

and $\tilde{T}_{\alpha\beta}$ is the auxiliary energy momentum tensor from the variation of $\tilde{g}_{\alpha\beta}$ on S_m. Varying the scalar field gives us another field equation,

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi+\tilde{F}=\frac{8\pi\tilde{G}}{c^4}\frac{\phi}{2\omega+3}\tilde{g}^{\alpha\beta}\tilde{T}_{\alpha\beta}. \tag{3.52}$$

Here $\tilde{\nabla}_\alpha$ is the covariant derivate defined in terms of the auxiliary metric and,

$$\tilde{F} = \frac{1}{2}\frac{d}{d\phi}\left[\ln\left(\frac{2\omega+3}{\phi^2}\right)\right]\tilde{g}^{\alpha\beta}\partial_\alpha\partial_\beta - \frac{\phi^2/\phi_0}{2\omega+3}\frac{d}{d\phi}\left(\frac{V(\phi)}{\phi^2}\right). \tag{3.53}$$

Equations (3.50) and (3.52) are the field equations of scalar-tensor gravity and were the goal of this section. In their current form, they are perfectly correct, however, we shall rewrite them to better suit our approximation scheme.

3.4.2 Landau-Lifshitz formulation of Brans Dicke

Motivated by the Landau-Lifshitz formulation used for general relativity back in chapter 2 we introduce a gothic metric for the auxiliary metric,

$$\tilde{\mathfrak{g}}^{\alpha\beta} = \sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}. \tag{3.54}$$

We create a tensor $\tilde{H}^{\alpha\mu\beta\nu}$ by the gothic metric in the same way as equation (2.2). Using the identity of (2.3) with the auxiliary tensor $\tilde{H}^{\alpha\mu\beta\nu}$ we arrive at,

$$\partial_{\mu\nu}\tilde{H}^{\alpha\mu\beta\nu} = \frac{16\pi\tilde{G}}{c^4}(-\tilde{g})\left(\tilde{T}^{\alpha\beta} + \tilde{t}^{\alpha\beta}_\phi + \tilde{t}^{\alpha\beta}_{LL}\right), \tag{3.55}$$

where $\tilde{t}^{\alpha\beta}_{LL}$ is defined in the same way as (2.4) with the replacement of the standard metric with the auxiliary metric and $\tilde{t}^{\alpha\beta}_\phi$ is defined in terms of (3.51) as,

$$\tilde{t}^{\alpha\beta}_\phi = \frac{c^4}{16\pi\tilde{G}}\tilde{\Theta}^{\alpha\beta}. \tag{3.56}$$

We now wish to create an analog of equation (2.14) for the gravitational potential $\tilde{h}^{\alpha\beta}$ defined by,

$$\tilde{h}^{\alpha\beta} = \eta^{\alpha\beta} - \tilde{\mathfrak{g}}^{\alpha\beta}. \tag{3.57}$$

We begin with the identity

$$\partial_{\mu\nu}\tilde{H}^{\alpha\mu\beta\nu} = -\Box\tilde{h}^{\alpha\beta} - \frac{16\pi\tilde{G}}{c^4}(-\tilde{g})\tilde{t}^{\alpha\beta}_H, \tag{3.58}$$

where $\tilde{t}^{\alpha\beta}_H$ is defined in the same way as (2.16). We also use the conformal harmonic gauge condition,

$$\partial_\beta\tilde{h}^{\alpha\beta} = 0, \tag{3.59}$$

which follows from the harmonic gauge condition so that we can rewrite (3.55) to the wave equation

$$\Box\tilde{h}^{\alpha\beta} = -\frac{16\pi\tilde{G}}{c^4}\tilde{\tau}^{\alpha\beta}, \tag{3.60}$$

where $\tilde{\tau}^{\alpha\beta}$ is the effective energy momentum tensor which consists of

$$\tilde{\tau}^{\alpha\beta} = (-\tilde{g})(\tilde{T}^{\alpha\beta} + \tilde{t}^{\alpha\beta}_\phi + \tilde{t}^{\alpha\beta}_{LL} + \tilde{t}^{\alpha\beta}_H). \tag{3.61}$$

It is at this point we will employ the solution of the wave equation as the expansion over multipoles from equation (2.40) with $\psi \to \tilde{h}^{\alpha\beta}$ and $\mu \to \frac{4\tilde{G}}{c^4}\tilde{\tau}^{\alpha\beta}$. It follows then that the solution to the wave equation for the auxiliary gravitational potentials are given by (2.56). We therefore get for the auxiliary gravitational potential,

$$\tilde{h}^{00} = \frac{4\tilde{G}\tilde{M}}{c^2R} + \frac{2\tilde{G}}{c^4R}\ddot{\tilde{\mathcal{I}}}^{jk}N_jN_k + O(c^{-5}), \tag{3.62a}$$

$$\tilde{h}^{0j} = \frac{2\tilde{G}}{c^4R}\dddot{\tilde{\mathcal{I}}}^{jk}N_k + O(c^{-5}) + O(R^{-2}), \tag{3.62b}$$

$$\tilde{h}^{jk} = \frac{2\tilde{G}}{c^4R}\ddot{\tilde{\mathcal{I}}}^{jk} + O(c^{-5}). \tag{3.62c}$$

We managed to rewrite the first field equation of the Brans-Dicke gravity into a wave equation and find an approximate expression in the wave-zone with the additional restriction to only keep terms up to $O(R^{-2})$. We can do a similar thing for the field equation for the scalar field. For this we will require the identity,

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi = \frac{1}{\sqrt{-\tilde{g}}}\partial_\alpha(\tilde{\mathfrak{g}}\partial_\beta\phi) = \frac{1}{\sqrt{-\tilde{g}}}(\Box\phi - \tilde{h}^{\alpha\beta}\partial_{\alpha\beta}\phi). \tag{3.63}$$

Inserting (3.63) into the field equation of (3.52) results in the wave equation,

$$\Box\phi = -\frac{8\pi\tilde{G}}{c^4}\tau_S, \tag{3.64}$$

with the source term,

$$\begin{aligned}\tau_S = &- \sqrt{-\tilde{g}}\frac{\phi}{2\omega+3}\tilde{T} + \frac{c^4}{16\pi\tilde{G}}\frac{d}{d\phi}\left[\ln\left(\frac{2\omega+3}{\phi^2}\right)\right](\eta^{\alpha\beta} - \tilde{h}^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi \\ &- \frac{c^4}{8\pi\tilde{G}}\left(\tilde{h}^{\alpha\beta}\partial_{\alpha\beta}\phi + \sqrt{-\tilde{g}}\frac{\phi^2/\phi_0}{2\omega+3}\frac{d}{d\phi}\left(\frac{U(\phi)}{\phi^2}\right)\right).\end{aligned} \tag{3.65}$$

It is possible to greatly simplify this expression using the post-Minkowskian approximation and expanding $\omega(\phi)$ in powers of c^{-2}. There is a thorough analysis in [9], however as the details are not very relevant for our purposes we simply state the result. For additional information and insights see section 13.5 of [9]. By the post Minkowskian approximation and the expansion of $\omega(\phi)$ we have that the scalar energy momentum τ_S tensor is equal to $\zeta\phi_0\tilde{\tau}^{00}/(1-\zeta)$ to first order in a post-Newtonian expansion and satisfies,

$$c^{-2}\tau_S = \frac{\zeta\phi_0}{1-\zeta}\left(c^{-2}\tilde{\tau}^{00} - \frac{1}{c^2}\mu + O(c^{-3})\right), \tag{3.66}$$

where,

$$\mu = \rho^*\left(v^2 + \left(\frac{1}{2} + 2\lambda\right)U\right) + 3p - \frac{1}{4\pi Gc^2}\left(\frac{7}{4} - 3\zeta + \lambda\right)\nabla^2U^2. \tag{3.67}$$

Here ρ^* is the rescaled mass density, $\mathbf{v}$ is the velocity field, p is the pressure, λ is a function of the ω parameter from the Brans-Dicke action and U is the Newtonian gravitational potential written out explicitly in terms of our variables,

$$U = \frac{\tilde{G}}{1-\zeta}\int\frac{\rho^*(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'. \tag{3.68}$$

Now we can again use (2.40) to solve the wave equation for our scalar field (3.64). The solution is then,

$$\phi = \phi_0 + \frac{\tilde{G}}{c^2 R}\left[\mathcal{I}_S - \frac{1}{c}\dot{\mathcal{I}}_S^j N_j + \frac{1}{2c^2}\ddot{\mathcal{I}}_S^{jk} N_j N_k + O(c^{-3})\right]. \tag{3.69}$$

In reference [9] there is a sign error in the $\frac{1}{c}\dot{\mathcal{I}}_S^j N_j$ part of the above expression, this error has been confirmed by the author. We can in the same manner as for the earlier multipole moments define a scalar multipole moment as

$$\mathcal{I}_S = \int_{\mathcal{M}} c^{-2}\tau_S(\tau, \mathbf{x}) d^3x, \tag{3.70a}$$

$$\mathcal{I}_S^j = \int_{\mathcal{M}} c^{-2}\tau_S(\tau, \mathbf{x}) x^j d^3x, \tag{3.70b}$$

$$\mathcal{I}_S^{jk} = \int_{\mathcal{M}} c^{-2}\tau_S(\tau, \mathbf{x}) x^j x^k d^3x. \tag{3.70c}$$

Another error was found in (3.70) where the above equations were missing a factor of c^{-2}. This error has also been confirmed by the author. It is useful for notational purposes to also define multipole moments associated with μ,

$$\mathcal{E}(\tau) = \int_{\mathcal{M}} \mu(\tau, \mathbf{x}) d^3x, \tag{3.71a}$$

$$\mathcal{E}^j(\tau) = \int_{\mathcal{M}} \mu(\tau, \mathbf{x}) x^j d^3x, \tag{3.71b}$$

$$\mathcal{E}^{jk}(\tau) = \int_{\mathcal{M}} \mu(\tau, \mathbf{x}) x^j x^k d^3x. \tag{3.71c}$$

Inserting now (3.67) into (3.70) we can get an expression for the scalar multipole moments in terms of the new multipole moments $\mathcal{E}$,

$$\mathcal{I}_S = \frac{\zeta\phi_0}{1-\zeta}\left[\tilde{M} - \frac{1}{c^2}\mathcal{E} + O(c^{-3})\right] \tag{3.72a}$$

$$\mathcal{I}_S^j = \frac{\zeta\phi_0}{1-\zeta}\left[-\frac{1}{c^2}\mathcal{E}^j + O(c^{-3})\right] \tag{3.72b}$$

$$\mathcal{I}_S^{jk} = \frac{\zeta\phi_0}{1-\zeta}\left[\tilde{\mathcal{I}}^{jk} - \frac{1}{c^2}\mathcal{E}^{jk} + O(c^{-3})\right] \tag{3.72c}$$

We define a combination $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{E}(\tau) + \frac{1}{c}\dot{\mathcal{E}}^j(\tau) N_j - \frac{1}{2}\ddot{I}^{jk}(\tau) N_j N_k, \tag{3.73}$$

and arrive at a new expression for ϕ/ϕ_0,

$$\phi/\phi_0 = 1 + \frac{2\zeta G}{c^2 R}\left[M - \frac{1}{c^2}\mathcal{A}(\tau, \mathbf{N}) + O(c^{-3})\right]. \tag{3.74}$$

This is the final result of this section and will be used at the end of this chapter to get an expression for the source terms of the scalar polarisation of gravitational waves in Brans-Dicke gravity.

3.5 Polarization of GW in alternative gravity

We wish to find an expression for the polarization of gravitational waves in alternative theories of gravity. Alternative theories of gravity are a broad term including an enormous variety of different theories. We, therefore, restrict ourselves quite a lot: We will mean scalar-vector-tensor gravity for alternative gravity and most specifically for the theory of Brans and Dicke. We will comment on scalar-vector-tensor gravity, but with a main focus on scalar-tensor gravity. Our approach is using Post-Minkowskian theory to expand an exact formulation of general relativity to an approximation scheme suitable for weak gravitational fields. However, first, we require an expression for the polarisation for a general theory. If we allow the source to be time-dependent and to emit gravitational waves then the stationary potentials in the far-away zone are supplemented by,

$$\Delta h^{\alpha\beta} = \frac{G}{c^4 R} A^{\alpha\beta}(\tau, N) \tag{3.75}$$

We decompose this into irreducible pieces in the same way as before in 3.1,

$$\Delta h^{00} = \frac{G}{c^4 R} C(\tau, \mathbf{N}) \tag{3.76}$$

$$\Delta h^{0j} = \frac{G}{c^4 R} D^j(\tau, \mathbf{N}) \tag{3.77}$$

$$\Delta h^{jk} = \frac{G}{c^4 R} A^{jk}(\tau, \mathbf{N}) \tag{3.78}$$

We summarize how h^{00}, h^{0j} and h^{jk} transforme under a gauge transformation,

$$\Delta h'^{00} = \Delta h^{00} + \frac{G}{c^4 R}(\partial_\tau \alpha + \partial_\tau \beta) \tag{3.79}$$

$$\Delta h'^{0j} = \Delta h^{0j} + \frac{G}{c^4 R}\partial_\tau(\alpha + \beta)N^j + \frac{G}{c^4 R}\partial_\tau \beta_T^j \tag{3.80}$$

$$\Delta h'^{jk} = \Delta h^{jk} + \frac{G}{c^4 R}\left[\frac{1}{3}\partial_\tau(3\alpha - \beta)\delta^{jk} + \left(N^j N^k - \frac{1}{3}\delta^{jk}\right) 2\partial_\tau \beta + 2N^j \partial_\tau \beta_T^k\right] \tag{3.81}$$

Comparing this with the expression for A^{jk} in equation (3.8) we summarize how the irreducible pieces of $\Delta h^{\alpha\beta}$ transform under a gauge transformation.

$$\begin{aligned} C' &= C + \partial_\tau(\alpha + \beta) \\ D' &= D + \partial_\tau(\alpha + \beta) \\ D_T^{j'} &= D_T^j + \partial_\tau \beta_T^j \\ A' &= A + \partial_\tau(3\alpha - \beta) \\ B' &= B + 2\partial_\tau \beta \\ A'^j &= A_T^j + \partial_\tau \beta_T^j \\ A'^{jk}_{TT} &= A^{jk}_{TT}. \end{aligned} \tag{3.82}$$

We have the freedom to choose α, β and β_T^j. This freedom means that we for example can choose to set C to zero, however then A and B are no longer zero. No matter what choices we make we end up with six independent degrees of freedom. Thus a more complicated expansion of the gravitational potential has appeared for the more general theory compared to general relativity.

3.6 Geodesic deviation revisited

What is the physical meaning of the six remaining degrees of freedom? We expect two of these to arise as the h_+ and $h_\times$ modes from general relativity, however, how do the remaining four polarizations affect test particles in a detector? For this we require to revisit the equation of geodesic deviation,

$$\frac{d^2\xi_j}{dt^2} = -c^2 R_{0j0k}\xi^k. \tag{3.83}$$

We will proceed in the same manner as for general relativity in section 3.3 only now we have six independent helicity states instead of two. Similarly for general relativity, we can write the linearized Riemann tensor as

$$R_{0j0k} = -\left[\frac{1}{2}(\partial_{0j}h_{0k} + \partial_{0k}h_{0j} - \partial_{00}h_{kj} + \partial_{kj}h_{00} - \frac{1}{2}\partial_{00}h\delta_{jk} + \frac{1}{2}\partial_{kj}h)\right]. \tag{3.84}$$

We now continue to find a connection between the linearized Riemann tensor and our irreducible representation of $\Delta h^{\alpha\beta}$. Inserting our expansion of the gravitational potentials (3.1) into (3.84) including (3.82) we get,

$$c^2 R_{0j0k} = -\frac{G}{2c^4 R}\frac{\partial^2}{\partial\tau^2}\mathcal{E}^{jk}(\tau,\mathbf{N}), \tag{3.85}$$

where,

$$\mathcal{E}^{jk} = (\delta^{jk} - N^j N^k)A_S + N^j N^k A_L + 2N^{(j}A_V^{k)} + A_{TT}^{jk}, \tag{3.86}$$

and A_S, A_L and A_V are,

$$A_S = -\frac{1}{6}(A + 2B - 3C), \tag{3.87a}$$

$$A_L = \frac{1}{3}(A + 2B + 3C - 6D), \tag{3.87b}$$

$$A_V^k = A_T^k - D_T^k. \tag{3.87c}$$

Here the subscript S stands for scalar, L for longitudinal and V for vectorial. In the same way as for general relativity we can integrate the equation of geodesic deviation,

$$\frac{d^2\xi_j}{dt^2} = -c^2 R_{0j0k}\xi^k = \frac{G}{2c^4 R}\frac{\partial^2}{\partial\tau^2}\mathcal{E}^{jk}(\tau,\mathbf{N})\xi^k, \tag{3.88}$$

to first order in displacement and we end up with

$$\xi_j(t) = \xi_j(0) + \frac{G}{2c^4 R}\mathcal{E}^{jk}(\tau,\mathbf{N})\xi^k(0). \tag{3.89}$$

We now use a vector basis $(\mathbf{N}, \vartheta, \varphi)$ defined in equation (3.28) to easier describe the wave modes. We define,

$$\begin{aligned} A_{V1} &= \vartheta_k A_V^k, \\ A_{V2} &= \varphi_k A_V^k, \\ A_+ &= \frac{1}{2}(\vartheta_j\vartheta_k - \varphi_j\varphi_k)A_{TT}^{jk}, \\ A_\times &= \frac{1}{2}(\vartheta_j\varphi_k - \varphi_j\vartheta_k)A_{TT}^{jk}. \end{aligned}$$

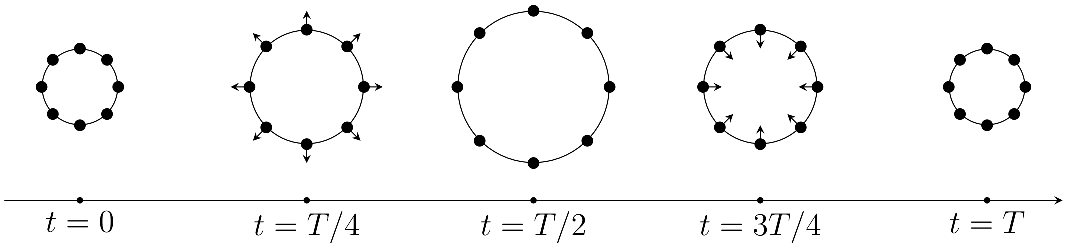

Figure 3.6.1: Time evolution of the scalar polarisation of a gravitational wave on a ring of particles over the period T. The arrows represent the movement of the particles.

Inserting this into $\mathcal{E}^{jk}$ we get a new expression for the polarisation tensor,

$$\begin{aligned}\mathcal{E}^{jk} =& A_S(\vartheta_j\vartheta_k + \varphi_j\varphi_k) + A_L N^j N^k + 2A_{V1}N^{(j}\vartheta^{k)} \\ &+ 2A_{V2}N^{(j}\varphi^{k)} + A_+(\vartheta^j\vartheta^k - \varphi^j\varphi^k) + A_\times(\vartheta^j\varphi^k + \varphi^j\vartheta^k). \end{aligned} \tag{3.90}$$

We see that the gravitational potential is described by six independent components, $A_S, A_L, A_{V_1}, A_{V_2}, A_+$ and $A_\times$. If the wave travels in the z-direction then $\vartheta = 0$. We may also choose $\varphi = 0$ and the polarisation tensor $\mathcal{E}^{jk}$ can now be displayed in the simple matrix form of

$$\mathcal{E}_{jk} = \begin{bmatrix} A_S + A_+ & A_\times & A_{V_1} \\ A_\times & A_S - A_+ & A_{V_2} \\ A_{V_1} & A_{V_2} & A_L \end{bmatrix} \tag{3.91}$$

Comparing with the polarisation tensor from general relativity (3.40) where it was also assumed a direction of the gravitational wave in the $+z$ direction shows that we have arrived at an additional four components. Detecting any additional polarisations would be a possible way of looking for theories of gravity beyond general relativity. In this simple form, we can also discuss how each polarisation component would affect spacetime. For our purposes, we will only focus on the plus, cross, and scalar part of the polarisation tensor. However adding a vector field in addition to the scalar field added in Brans-Dicke gravity would result in the vector components of A_{V_1}, A_{V_2} and the longitudinal component A_L.

3.7 Source of Brans-Dicke polarisation

We have derived the polarisation tensor for gravitational waves in Brans-Dicke gravity. There is however more to the scalar-tensor theory than a change of the polarisation. In this section, we will derive an expression for the scalar polarisation given the monopole, dipole and quadrupole moment of a source and we will observe that the effect of adding a scalar potential to our Lagrangian leads to additional terms for the three polarisations in this new theory. The derivation in this section is based on section 13.5.5 of [9]. We begin by writing out the gravitational potential in Brans-Dicke theory. This is from (3.57) and (3.46) given as,

$$h^{\alpha\beta} = (\phi/\phi_0)\tilde{h}^{\alpha\beta} + (1 - \phi/\phi_0)\eta^{\alpha\beta}. \tag{3.92}$$

where ϕ/ϕ_0 is given from (3.69) as,

$$\phi/\phi_0 = 1 + \frac{2\zeta G}{c^2 R}\left[M - \frac{1}{c^2}\mathcal{A}(\tau, \mathbf{N}) + O(c^{-3})\right]. \tag{3.93}$$

We recall the expressions of h^{00}, h^{0j} and h^{jk} in the wave zone (2.56). We have now a more complicated effective energy-momentum tensor $\tilde{\tau}$ and we use (2.56) for $\tilde{h}^{00}, \tilde{h}^{0j}$ and $\tilde{h}^{ij}$ with $\tilde{\tau}$ of (3.61) with the approximation used for $\tilde{\tau}_S$ in (3.66). We also make the assumption of $r \gg \lambda_c$ such that we can neglect all terms which are of at least $O(R^{-2})$. Inserting this into (3.92) we arrive at the following expression for the gravitational potential,

$$h^{00} = \frac{2(2-3\zeta)GM}{c^2 R} + \frac{G}{c^4 R}\left[2(1-\zeta)\ddot{I}^{jk}N_j N_k + 2\zeta\mathcal{A}(\tau, \mathbf{N}) + O(c^{-1}))\right], \tag{3.94a}$$

$$h^{0j} = \frac{G}{c^4 R} + \left[2(1-\zeta)\ddot{I}^{jk}N_k + O(c^{-1})\right], \tag{3.94b}$$

$$h^{jk} = \frac{2\zeta GM}{c^2 R}\delta^{jk} + \frac{G}{c^4 R}\left[2(1-\zeta)\ddot{I}^{jk} - 2\zeta\mathcal{A}\delta^{jk} + O(c^{-1})\right]. \tag{3.94c}$$

If we now compare with the general decomposition (3.1) we get for each of the components,

$$A = 2(1-\zeta)\ddot{I}^{pp} - 6\zeta\mathcal{A}, \tag{3.95a}$$

$$B = 3(1-\zeta)\ddot{I}^{\langle jk\rangle}N_j N_k, \tag{3.95b}$$

$$C = 2(1-\zeta)\ddot{I}^{\langle jk\rangle}N_j N_k + \frac{2}{3}(1-\zeta)\ddot{I}^{pp} + 2\zeta\mathcal{A} \tag{3.95c}$$

$$D = 2(1-\zeta)\ddot{I}^{\langle jk\rangle}N_j N_k + \frac{2}{3}(1-\zeta)\ddot{I}^{pp} \tag{3.95d}$$

$$A_T^j = 2(1-\zeta)P^j{}_p\ddot{I}^{\langle pk\rangle}N_k, \tag{3.95e}$$

$$D_T^j = 2(1-\zeta)P^j{}_p\ddot{I}^{\langle pk\rangle}N_k, \tag{3.95f}$$

$$A_{TT}^{jk} = 2(1-\zeta)\ddot{I}_{TT}^{jk}. \tag{3.95g}$$

Here $I^{\langle jk\rangle} = I^{jk} - \frac{1}{3}I^{pp}$. By the use of equation (3.87) we then get the different components of the polarisation vector as

$$A_S = 2\zeta\mathcal{A}, \tag{3.96a}$$

$$A_L = 0, \tag{3.96b}$$

$$A_V^j = 0, \tag{3.96c}$$

$$A_{TT}^{jk} = 2(1-\zeta)\ddot{I}_{TT}^{jk}. \tag{3.96d}$$

Making the assumption of a gravitational wave traveling in the $+z$-direction then the N_j vector from (3.28) simplifies to $(0,0,1)$. Inserting our simplified components (3.96) into our expression of the gravitational potential in (3.94c) we get that the known polarisations of h_+ and $h_\times$ from general relativity take the slightly different form in Brans-Dicke gravity,

$$h_+ = \frac{G}{rc^4}(1-\zeta)(\ddot{I}^{xx}(\tau) - \ddot{I}^{yy}(\tau)), \tag{3.97}$$

$$h_\times = \frac{2G}{rc^4}(1-\zeta)\ddot{I}^{xy}(\tau), \tag{3.98}$$

Inserting for $\mathcal{A}$ and approximating $\mu \simeq \rho$ in the low-velocity limit we get for the new scalar polarisation,

$$h_S = \frac{2G}{rc^2}\zeta\left[M(\tau) + \frac{1}{c}\dot{D}^z(\tau) - \frac{1}{2c^2}\ddot{I}^{zz}(\tau)\right]. \tag{3.99}$$

Here the x- and y directions on the quadrupole moment comes from the plus- and cross-polarisation comes from the chosen $z-$direction of the gravitational wave. Similarly for the scalar polarisation only here is the $z-$ direction on the dipole and quadrupole moment instead. We note one of the main differences between Brans-Dicke and general relativity: We cannot remove the monopole and dipole term as the scalar field does not satisfy a conservation law. Therefore, we have a contribution from the monopole and dipole moment of the gravitational wave source, which do not produce gravitational waves in general relativity. Letting $\zeta \to 0$ we observe that we recover general relativity as a limit as is expected.

CHAPTER

FOUR

GRAVITATIONAL WAVE RESONANCE IN SPHERICAL MODELS

In this chapter we will derive two expressions for the response of a spherical body to a gravitational wave, a toroidal response and a spheroidal response. With these expressions, we will calculate the displacement in an Earth model and finally Moon models. For this, we will require some fundamental background from the theory of infinitesimal elasticity and make simplifying assumptions for the spherical Earth and Moon body. Finally, we will need the help of spherical Bessel functions with relations before we can arrive at our final equations for the oscillations.

4.1 Infinitesimal theory of elasticity

4.1.1 Lamé parameters and the stress-strain relation

We explain in this section the basics of infinitesimal elasticity. We follow the same procedure as Sections 1.1, 1.2 and 1.3 from reference [10]. The goal is to relate the stress tensor $\mathcal{I}_{ij}$ to the strain tensor ϵ_{ij} and the Lamé parameters which describe the properties of the elastic medium. We begin by defining an elastic body. An elastic body is a body in which the relative position of internal parts of the body may move when subject to external body or surface forces and the body returns to its initial state after the forces are removed. For any body in consideration, we will ignore any rotation and motion of the entire body. Cauchy's strain tensor ϵ_{ij} describes infinitesimal strains and is defined as

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{4.1}$$

The interpretation of the strain tensor is as follows. Given some line element $dr_i = n_i dr$ with direction n_i, where $|\mathbf{n}| = 1$, we get under some displacement $dr \to dr'$ that

$$n_i \epsilon_{ij} n_j = \frac{dr' - dr}{dr}. \tag{4.2}$$

Or in other words, the strain tensor components give the change in position per unit length in the direction of n_i. To give a more physical understanding of the strain tensor we can look at what the different components of the tensor represents when we contract the tensor $n_i n_j$ and ϵ_{ij}. If n_i is in the form

$$n_i = l\hat{e}_i^1 + m\hat{e}_i^2 + n\hat{e}_i^3 \tag{4.3}$$

then $n_i n_j \epsilon_{ij} = \epsilon$ can be written

$$\epsilon = l^2\epsilon_{11} + m^2\epsilon_{22} + n^2\epsilon_{33} + 2mn\epsilon_{23} + 2nl\epsilon_{31} + 2lm\epsilon_{12}. \tag{4.4}$$

The interpretation of the elements should now be clear. Taking the ϵ_{11} element as an example we see that it is the extension of a line element which in the unstrained case lies parallel to the $\hat{e}_i^1$ direction. The diagonal elements are called normal strains while the off diagonal elements are called shearing strains. The off diagonal elements, say ϵ_{12} for example, is the decrease in the angle between two line element which were parallel to the unit vectors $\hat{e}_i^1$ and $\hat{e}_i^2$ before the deformation. To summarize, the strain tensor describes the deformation of a material from an applied stress. We should therefore be able to find some relation between the stress tensor and our strain tensor. Let us begin by making the simple assumption that the deforming process will take place without any changes in temperature. We also assume that the deformation is sufficiently slow such that the kinetic energy of the particles can be neglected. We now assume that we already have a deformed body and that we superimpose an additional displacement δu_i. The work done by surface tractions and body forces F_i is then

$$\delta W = \int_S (dS_i \mathcal{I}_{ij})\delta u_j + \int_V \rho(F_i \delta u_i) dV = \int_V \left[(\mathcal{I}_{ij}\delta u_i)_{,j} + \rho F_i \delta u_i\right] dV. \tag{4.5}$$

We can recast the divergence of the stress tensor and displacement δu_i by using the equation of equilibrium $\mathcal{I}_{ij,j} = -\rho F_i$ such that

$$(\mathcal{I}_{ij}\delta u_i)_{,j} + \rho F_i \delta u_i = (\mathcal{I}_{ij}\delta u_i)_{,j} - \mathcal{I}_{ij,j}\delta u_i = \mathcal{I}_{ij}\delta u_{i,j} = \mathcal{I}_{ij}\delta\epsilon_{ij}. \tag{4.6}$$

Inserting back into our expression for δW we get that the work can be expressed as the contraction of the stress tensor with the strain tensor

$$\delta W = \int_V \mathcal{I}_{ij}\delta\epsilon_{ij} dV. \tag{4.7}$$

From (4.7) we see that $\delta\epsilon_{ij}$ is the increment of the strain by the extra displacement δu_i. If we now set our focus on a small volume element dV, then we get

$$\delta W = \mathcal{I}_{ij}\delta\epsilon_{ij}. \tag{4.8}$$

The total work is the integral over W. Since we assumed an elastic material our path in the integral is independent of the final work, the integral must therefore be a total differential. We can then write

$$dW = \mathcal{I}_{ij}\delta\epsilon_{ij}. \tag{4.9}$$

This can be written with the stress tensor alone as

$$\mathcal{I}_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}. \tag{4.10}$$

We now rename W as the strain energy density. We assume W is only dependent on the strain ϵ_{ij} and that the strain tensor components are sufficiently small such that we can Taylor expand W in powers of strain tensor components and later throw away the higher order terms. We expand W as

$$W = a_0 + b_{ij}\epsilon_{ij} + C_{ijkl}\epsilon_{ij}\epsilon_{kl} + O(\epsilon^3). \tag{4.11}$$

Setting $a_0 = 0$ by normalizing the energy such that $\epsilon_{ij} = 0$ at zero energy and requiring that the energy must be in a minimum state at $\epsilon_{ij} = 0$, leading to $b_{ij} = 0$. This results in $\mathcal{I}(\epsilon_{ij})$ becoming a quadratic equation in ϵ_{ij},

$$\mathcal{I}_{ij} = C_{ijkl}\epsilon_{kl}. \tag{4.12}$$

This is a generalization of Hooke's law. We are however not quite done yet. We shall now make some assumptions to simplify C_{ijkl} in such a way to reach a stress strain relation for an isotropic medium. The C_{ijkl} tensor contains 3^4 components in its most general form, however not all of these components need to necessarily be independent. Should we have a homogeneous material then the components are constant. Also since the stress tensor and the strain tensor are symmetric we have

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{4.13}$$

By the additional relation $C_{ijkl} = C_{klij}$ we are left with just 21 independent components. The isotropic assumption imply we can write the tensor in the general form

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{4.14}$$

This follows from the fact that the tensor components retain their value by a rotation transformation. By (4.13) we see that $\nu = 0$ and we are left with

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{4.15}$$

Inserting this into (4.12) we get the famous stress strain relation for an isotropic material

$$\mathcal{I}_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}). \tag{4.16}$$

Here λ and μ are known as Lamé parameters and are determined for each material from experiments.

4.1.2 Cauchy's equation of motion

We consider a continuous distribution of matter in motion with respect to some inertial reference frame. We assume there exist a boundary which we name S such that the density field $\rho(\mathbf{r}, t)$ and velocity field $v_i(\mathbf{r}, t)$ is defined inside a volume V which is enclosed by S. An element ρdV can be totally described by its velocity vector in a given fixed reference frame. We call this small element ρdV a particle. We focus our interest on the total change of a field A which describe some property of the particle when it moves on a certain path in a time interval dt. We introduce a covariant derivative D/Dt. This should be understood in the same way we introduce the covariant derivative in general relativity. If one would measure some change in the field A at a fixed point, then $\partial_t A$ would suffice.

However since one could make a measurement while being in a reference system which is moving, we have to add a term which describes the spatial change in A along the reference frames trajectory over the time dt. This covariant derivative is therefore defined as

$$\frac{DA}{Dt} := \frac{\partial A}{\partial t} + v_i \frac{\partial A}{\partial x_j}. \tag{4.17}$$

The total mass, linear momentum $P_i(t)$ and angular momentum $M_k(t)$ is then

$$m = \int_V \rho dV, \quad P_i(t) = \int_V v_i \rho dV, \quad M_k(t) = \int_V (\epsilon_{ijk} r_i v_j) \rho dV. \tag{4.18}$$

Assuming that the total mass of the system is conserved, Λ is the total force applied and Γ is the total torque applied about the origin of the inertial reference frame, then the following equations will hold in the volume V,

$$\frac{Dm}{Dt} = 0, \tag{4.19a}$$

$$\frac{D}{Dt} P_i(t) = \Lambda_i, \tag{4.19b}$$

$$\frac{D}{Dt} M_i(t) = \Gamma_i. \tag{4.19c}$$

The first equation restates the conservation of mass, the second equation is the Newton Euler principle of linear momentum and the final equation is the Newton-Euler principle of angular momentum. The total applied force Λ_i can be split into two parts. A body force term ρF_i and a surface force term $n_i \mathcal{I}^{ij}$ which acts on the boundary S. We can therefore write the total force acting on the continuous medium as

$$\Lambda_i = \int_S (\mathcal{I}_{ij} dS_i) + \int_V \rho F_i dV = \int_V (\mathcal{I}_{ij,j} + \rho F_i) dV, \tag{4.20}$$

where we used Gauss's theorem to change from a boundary integral to a volume integral. We can split the total torque in the exact same way

$$\Gamma_k = \int_S \epsilon_{ijk} r_i (dS_a \mathcal{I}_{aj}) + \int_V \epsilon_{ijk} r_i \rho F_j dV \tag{4.21}$$

If we apply the covariant derivative to equation (4.19a) we get

$$\frac{Dm}{dt} = \frac{D}{Dt} \int_V \rho dV = \int_V \left[\frac{D\rho}{Dt} + \rho v_{i,i} \right] dV = \int_V \left[\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} \right] dV = 0. \tag{4.22}$$

If we instead now consider the covariant derivative of ρv_i then we can use (4.19b) and arrive at

$$\begin{aligned} \Lambda_i &= \frac{D}{Dt} \int_V v_i \rho dV = \int_V \left[\frac{D(\rho v_i)}{Dt} + \rho v_i v_{j,j} \right] dV \\ &= \int_V \left[\rho \frac{Dv_i}{Dt} + v_i \left(\frac{D\rho}{Dt} + \rho v_{j,j} \right) \right] dV = \int_V \left(\rho \frac{Dv_i}{Dt} \right) dV \end{aligned}$$

Combining this with equation (4.20) we arrive at Euler's equation of motion,

$$\mathcal{I}_{ij,j} + \rho F_i = \rho \frac{Dv_i}{Dt}. \tag{4.23}$$

This is a more general version of the equation of motion that will be the basis of the calculations which follow. We make the assumption that the velocity field $v_i(\mathbf{r}, t)$ is given by

$$v_i = \frac{Du_i}{Dt} = \frac{\partial u}{\partial t} + v_i u_{j,j} \simeq \frac{\partial u}{\partial t}. \tag{4.24}$$

This approximation holds as long as $|\frac{\partial u_i}{\partial x_j}| \ll 1$, and we then have

$$\frac{Dv_i}{Dt} \simeq \frac{\partial^2 u_i}{\partial t^2}. \tag{4.25}$$

Inserting (4.25) into (4.23) and writing the displacement as the Fourier transformed displacement

$$u_i(t) = \int_{-\infty}^{\infty} u_i(\omega) e^{i\omega t} d\omega, \tag{4.26}$$

we get the equation,

$$\mathcal{I}_{ij,j} + \rho F_i + \rho\omega^2 u_i = 0. \tag{4.27}$$

This is Cauchy's equation of motion for the infinitesimal theory of elasticity and will be the equation used to describe the oscillations of the Earth and Moon in the rest of the thesis. Additional assumptions are a spherical Earth and Moon and the spherical coordinate system (r, θ, ϕ) will be used with the mathematical notation with the origin at the center of the sphere. The models which will be considered are described by a density ρ and the two Lame parameters λ and μ with the gravitational acceleration g determined from the density.

4.2 Hansen vectors

To describe the oscillations of spherical models we will first discuss the Hansen vectors in spherical coordinates. Let us begin by considering the motion of a homogenous isotropic elastic solid subjected to the body force distribution $F^i(\mathbf{r}, t)$. The displacement on this body is described by the Navier equation,

$$\alpha^2 (u_{i,i})_{,j} - \beta^2 \epsilon_{jak} (\epsilon_{abc} u_{c,b})_{,k} + F_j = \partial_{tt} u_j. \tag{4.28}$$

Introducing the Fourier transform of the displacement $u^j(\mathbf{r}, t)$ and the force per unit mass $F^j(\mathbf{r}, t)$,

$$u^j(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} u^j(\mathbf{r}, t) e^{-i\omega t} dt, \tag{4.29a}$$

$$F^j(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} F^j(\mathbf{r}, t) e^{-i\omega t} dt. \tag{4.29b}$$

Inserting the Fourier transformed expressions into the equation of motion results in a new equation of motion,

$$\alpha^2 (u_{i,i})_{,j} - \beta^2 \epsilon_{jak} (\epsilon_{abc} u_{c,b})_{,k} + F_j + \omega^2 u_j = 0. \tag{4.30}$$

We can now split the displacement u_j into two parts which we denote as $u_j^{(\alpha)}$ and $u_j^{(\beta)}$ such that $u_j = u_j^{(\alpha)} + u_j^{(\beta)}$. We make the assumptions that $\epsilon_{abc}u_{c,b}^{(\alpha)} = 0$ and $u_{i,i}^{(\beta)} = 0$. If we let $F_j = 0$ then we can now split the equation of motion into two equations,

$$(\partial_{ii} + \omega^2/\alpha^2)u_j^{(\alpha)} = 0, \tag{4.31a}$$

$$(\partial_{ii} + \omega^2/\beta^2)u_j^{(\beta)} = 0. \tag{4.31b}$$

For the following we define for notation $k^{(\alpha)} = \omega/\alpha$ and $k^{(\beta)} = \omega/\beta$. Now we wish to introduce the Hansen vectors. We consider the curvilinear coordinates (q_1, q_2, q_3). We also introduce a vector a_i which is orthogonal to the surface created by $q_1 =$ constant. We are interested in a spherical coordinate system. The Hansen vectors can be a valuable representation for other types of coordinate systems as well however we will stick with the spherical coordinate system. For the spherical coordinate system, we then have that $a_i = r\hat{e}_i^{(r)}$. We can now define the Hansen vectors in a spherical coordinate system. The three vectors M_i, N_i, L_i are defined from the three potentials Ψ, χ, Φ as,

$$M_i = \epsilon_{ibc}\Psi a_{c,b} \tag{4.32a}$$

$$N_i = \frac{1}{k^{(c)}}\epsilon_{iak}(\epsilon_{abc}\chi a_{c,b})_{,k} \tag{4.32b}$$

$$L_i = \frac{1}{k^{(c)}}\Phi_{,i}, \tag{4.32c}$$

where the potentials are solutions to the scalar Helmholtz equation. We do not distinguish between the potentials and assume that the potential can be separated into the form $\Psi = f(r)Y(\theta, \phi)$. Inserting this into (4.32) we get,

$$M_i = fC_i, \tag{4.33a}$$

$$k^{(c)}N_i = \left[(k^{(c)})^2(rf) + \frac{\partial^2}{\partial(rf)r^2}\right]P_i + \frac{1}{r}\frac{\partial(fr)}{\partial r}B_i, \tag{4.33b}$$

$$k^{(c)}L_i = \left(\frac{\partial f}{\partial r}\right)P_i + \frac{1}{r}fB_i \tag{4.33c}$$

where C_i, P_i and B_i are vector surface harmonics and vectors that depend only on θ and ϕ. They are defined in spherical coordinates as,

$$\sqrt{l(l+1)}C_i^{(ml)}(\theta, \phi) = \epsilon_{ijk}(r_{k,j}Y(\theta, \phi)), \tag{4.34a}$$

$$P_i^{(ml)} = \hat{e}_i^{(r)}Y^{(ml)}(\theta, \phi), \tag{4.34b}$$

$$\sqrt{l(l+1)}B_i^{(ml)}(\theta, \phi) = rY_{,i}^{(ml)}(\theta, \phi). \tag{4.34c}$$

Here $P^{(ml)}(\theta, \phi)$ is the associated Legendre polynomial. The Hansen vectors are linearly independent and a general displacement can therefore be written as some linear combination of the Hansen vectors. As the Hansen vector themselves are a combination of the vector surface harmonics C_i, P_i, B_i we can write a general displacement in the form of,

$$\begin{aligned} u_i(\mathbf{r}) = U(r)P_i^{(ml)}(\theta, \phi) + V(r)\sqrt{l(l+1)}B_i^{(ml)}(\theta, \phi) \\ + W(r)\sqrt{l(l+1)}C_i^{(ml)}(\theta, \phi), \end{aligned} \tag{4.35}$$

where $U(r), V(r)$ and $W(r)$ are some functions of r. Inserting (4.35) into our earlier equations of (4.31a) and (4.31b) we will get equations for $U(r), V(r)$ and $W(r)$.

4.3 Radially inhomogeneous self-gravitating spherical model

It is practical to employ vectors that reflect the spherical symmetry. We can in general write the displacement as a linear combination of the three Hansen vectors,

$$u_i(\mathbf{r}, t) = \sum_{\sigma,m,l} \alpha^{(ml)} M_i^{(ml)} + \beta^{(ml)} N_i^{(ml)} + \gamma^{(ml)} L_i^{(ml)}, \tag{4.36}$$

where $\alpha^{(ml)}$, $\beta^{(ml)}$ and $\gamma^{(ml)}$ are some constants. Oscillations in which the displacement only consists of a sum of M_i vectors are defined as toroidal oscillations while oscillations where the displacement only consists of N_i and L_i vectors are defined as spheroidal oscillations. The Hansen vectors takes on specific forms depending on the symmetries of the system. We will use the form respecting the spherical symmetry of the Earth and Moon bodies.

4.3.1 Assumptions

Let us consider some assumptions to simplify our system. In general we can say that a realistic spherical body will possess a certain amount of initial stress. This is additional stress to any disturbing additional force and would have to be taken into account for a realistic model. We therefore make a separation of a disturbed model with initial stresses and a disturbed model with initial stress and an additional stress. We make seven assumptions on our system.

1. We neglect small quantities of second or higher order.

2. The initial stress $\mathcal{I}_{jk}^{(i)}$ of the model is in equilibrium and totally described by hydrostatic pressure,

$$\mathcal{I}_{jk}^{(i)} = -p^{(0)}\delta_{jk}, \qquad \text{where,} \qquad \partial_j p^{(0)} = -g^{(0)}\rho^{(0)}\hat{e}_j^{(r)}. \tag{4.37}$$

3. A small element carries its initial stress with it under a displacement and may get additional stresses depending on the compression and distortions it is affected by under the displacement. In other words $p^{(0)}(\mathbf{r}) = p^{(0)}(\mathbf{r} - \mathbf{u})$.

4. The stress can in a strained state be separated into an initial equilibrium stress part and an elastic stress part. The initial stress is then, by assumption 3,

$$\mathcal{I}_{jk}^{(i)}(\mathbf{r}) = -p^{(0)}(\mathbf{r} - \mathbf{u})\delta_{jk}. \tag{4.38}$$

While the elastic stress follows the usual stress-strain relation

$$\mathcal{I}_{jk}^{(e)}(\mathbf{u}) = \lambda\delta_{jk}u_{i,i} + \mu(u_{j,k} + u_{k,j}). \tag{4.39}$$

5. The only body force is the force of gravity which we derive from the gravitational potential Ψ:
$$F_j = \partial_j \Psi. \tag{4.40}$$

6. The values of ρ and Ψ are assumed to be split into an equilibrium part $\rho^{(0)}$ and $\Psi^{(0)}$, and a small perturbation $\rho^{(1)}$ and $\Psi^{(1)}$.

7. The quantities ρ, λ, μ, g and Ψ are independent of any angles θ and ϕ and only radial dependent.

With these assumptions, we can simplify our equations of motion to an easier solvable form. We do this in the following section.

4.3.2 Simplified differential equations

It is the goal of this section to arrive at a simplified version of the differential equation (4.27). We begin by employing some of our assumptions to get the first term in equation (4.27) in an simpler form,

$$\mathcal{I}_{jk,j} = \mathcal{I}^{(i)}_{jk,j} + \mathcal{I}^{(e)}_{jk,j}. \tag{4.41}$$

Using our assumptions for the initial stress (4.38) and the identity $(f\delta_{jk})_{,j} = f_k$ we can simplify the divergence of the initial stress tensor as

$$\mathcal{I}^{(i)}_{jk,j} = (-(p^{(0)} + g^{(0)}\rho^{(0)}u_r)\delta_{jk})_{,j} = -(p^{(0)} + g^{(0)}\rho^{(0)}u_r)_{,k}, \tag{4.42}$$

where u_r is not a vector but the radial component of the displacement. For the elastic stress $\mathcal{I}^{(e)}_{jk}$ we can write out the divergence of the stress tensor in spherical coordinates using (4.39) as

$$\mathcal{I}^{(e)}_{jk,j} = (\lambda u_{j,j})_{,k} + \mu(u_{k,jj} + (u_{j,j})_{,k} + \frac{d\mu}{dr}\left(2\frac{\partial u_k}{\partial r} + \epsilon_{kab}\hat{e}^{(r)}_a(\epsilon_{bcd}u_{d,c})\right), \tag{4.43}$$

where we in the last term used that

$$\hat{e}^{(r)}_j(u_{j,k} + u_{k,j}) = 2\frac{\partial u_k}{\partial r} + \epsilon_{kab}\hat{e}^{(r)}_a(\epsilon_{bcd}u_{d,c}). \tag{4.44}$$

Having applied our assumptions to the first term of (4.27) we move on to the second term. We start by applying our assumptions to the density which by the continuity equation satisfies

$$\rho(\mathbf{r}) = \rho^{(0)}(\mathbf{r} - \mathbf{u}) \cdot (1 - u_{i,i}) = \rho^{(0)}(\mathbf{r}) - (\rho^{(0)}u_i)_{,i}. \tag{4.45}$$

With the assumptions on the gravitational potential and using the identity $(fV_j)_{,j} = fV_{j,j} + f_{,j}V_j$ to get the first term, we can write

$$\begin{aligned}\rho F_j &= \left(\rho^{(0)} - \frac{d\rho^{(0)}}{dr}u_r - \rho^{(0)}u_{k,k}\right)(\psi_{,j} - g^{(0)}\hat{e}^{(r)}_j),\\ &= \rho^{(0)}\psi_{,j} + g^{(0)}\left(\frac{d\rho^{(0)}}{dr}u_r + \rho^{(0)}u_{k,k} - \rho^{(0)}\right)\hat{e}^{(r)}_j. \end{aligned} \tag{4.46}$$

Adding (4.42),(4.43) and (4.46) together we arrive at our differential equation subject to our assumptions,

$$\mu(u_{k,jj} + (u_{j,j})_{,k}) + \frac{d\mu}{dr}\left(2\frac{\partial u_k}{\partial r} + \epsilon_{kab}\hat{e}_a^{(r)}(\epsilon_{bcd}u_{d,c})\right) + (\lambda u_{i,i})_{,k}$$
$$+ \rho_0((\psi - g^{(0)}u_r)_{,k} + g_0\hat{e}_k^r u_{i,i}) + \omega^2\rho^{(0)}u_k = 0. \qquad (4.47)$$

Let us now consider the gravitational potential. The potential Ψ obeys the Poisson equation,

$$\Psi_{,jj} = -4\pi G\rho. \qquad (4.48)$$

Inserting our expansion of the potential $\Psi = \Psi^{(0)} + \psi$ into (4.48) we conclude that the perturbation of the gravitational potential also satisfies a similar equation,

$$\psi_{,ii} = -4\pi G\rho' = 4\pi G(\rho^{(0)}u_j)_{,j}. \qquad (4.49)$$

Equation (4.47) and (4.49) are our differential equation system which we will have to solve under boundary conditions which we will discuss next.

4.3.3 Boundary conditions

We must also include boundary conditions which our system must satisfy. The ultimate goal is to set up a system of differential equations which can be solved numerically. We will do this in the next sections for first toroidal- and then spheroidal oscillations. The differential equation from the last section should be solved under the boundary conditions

1. The solution is well defined at the origin.
2. The stresses vanish at the deformed surfaces and stay continuous at an internal deformed surface of discontinuity.
3. The displacements are continuous at an internal surface of discontinuity, with the exception of a solid-liquid interface where only the radial displacement is continuous.
4. The gravitational potential and its radial derivative are continuous at the deformed surface of the earth at an internal deformed surface of discontinuity.

Mathematically these boundary conditions can be written more concise. If we imagine there is a surface of discontinuity at $r = c$ and look at the stress close to this surface in a strained state we get

$$\mathcal{I}_{jk}(c + u_r) = \mathcal{I}_{jk}(c) + u_r\left(\frac{\partial \mathcal{I}_{jk}}{\partial r}\right)_{r=c} = \mathcal{I}_{jk}^{(e)}(c) - p_0(c)\delta_{jk}. \qquad (4.50)$$

The additional elastic stress at c and at $c + u_r$ is equal to the first order in u_r. We also see that a small element of the medium carries its initial stress with it when it moves from one place to another. Boundary condition 4 regards only the gravitational potential. Mathematically it says that

$$\Psi^{(1)} = \Psi^{(2)} \quad \text{and} \quad \frac{d\Psi^{(1)}}{dr} = \frac{d\Psi^{(2)}}{dr} \quad \text{at } r = c + u_r. \qquad (4.51)$$

Index 1 and 2 represents Ψ being evaluated at opposite points of the surface of discontinuity. We expand Ψ into its equilibrium value and its perturbation and neglect any value to higher than linear order. The boundary condition can then be written,

$$\psi^{(1)} + \Psi^{(01)} + u_r \frac{d\Psi^{(01)}}{dr} = \psi^{(2)} + \Psi^{(02)} + u_r \frac{d\Psi^{(02)}}{dr}, \tag{4.52a}$$

$$\frac{d\psi^{(1)}}{dr} + \frac{d\Psi^{(01)}}{dr} + u_r \frac{d^2\Psi^{(01)}}{dr^2} = \frac{d\psi^{(2)}}{dr} + \frac{d\Psi^{(02)}}{dr} + u_r \frac{d^2\Psi^{(02)}}{dr^2} \tag{4.52b}$$

If we just look at the unstrained state then $\psi = 0$ and $u_r = 0$ and,

$$\Psi^{(01)} = \Psi^{(02)}, \quad \frac{\Psi^{(01)}}{dr} = \frac{d\Psi^{(02)}}{dr}, \quad r = c. \tag{4.53}$$

Taking use of the Poisson equation we see that

$$\frac{d^2\Psi^{(0)}}{dr^2} + \frac{2}{r}\frac{d\Psi^{(0)}}{dr} = -4\pi G\rho^{(0)}. \tag{4.54}$$

From (4.52) and (4.52) we can now set restrictions on the deviation ψ,

$$\psi^{(1)} = \psi^{(2)}, \qquad \dot{\psi}^{(1)} - 4\pi G\rho^{(01)} u_r = \dot{\psi}^{(2)} - 4\pi G\rho^{(02)} u_r. \tag{4.55}$$

At the free surface we then must have that,

$$\psi = \psi^{(e)}, \quad \dot{\psi} - 4\pi G\rho^{(0)} u_r = \dot{\psi}^{(e)}, \tag{4.56}$$

where ψ_e is the gravitational potential outside the spherical model.

4.4 Toroidal Oscillations

4.4.1 Differential equation system for numerical integration

We begin our discussion of oscillations with toroidal oscillations. It turns out that spheroidal oscillations will give a much bigger contribution to the oscillations from gravitational wave interactions, but we give a discussion on toroidal oscillations regardless. This is firstly for the sake that toroidal oscillations are mathematically simpler, but still similar enough so that ideas can be grasped more quickly when we arrive at the spheroidal case. Secondly to compare the results we will present in chapter five to other sources and check the validity of our model. For purely toroidal oscillations u_r and $u_{\mu,\mu}$ both vanish. The equation of motion (4.47) then simplifies to

$$\mu \mathbf{u}_{j,ii} + \frac{d\mu}{dr}\left(2\frac{\partial u_j}{\partial r} + \epsilon_{jab}\hat{e}_a^{(r)}(\epsilon_{bcd}u_{d,c})\right) + \omega^2\rho^{(0)}u_j = 0. \tag{4.57}$$

A radially inhomogenous model has parameters which only depends on the radius. With this in mind we make the ansatz that the displacement takes on the form

$$u_j = \sum_{\sigma,m,l} y_1(r)\sqrt{l(l+1)}C_j^{(\sigma ml)}(\theta,\phi), \qquad \sigma = c, s. \tag{4.58}$$

σ represents the sum over the core and mantle. We split the displacement into a sum over the core and mantle to make sure that the solution is indeed well defined at both the origin and in all parts of the mantle. We insert our ansatz into (4.47) and arrive at a new differential equation for y_1,

$$\mu\left(\frac{d^2y_1}{dr^2}+\frac{2}{r}\frac{dy_1}{dr}\right)+\frac{d\mu}{dr}\left(\frac{dy_1}{dr}-\frac{y_1}{r}\right)+\omega^2\rho^{(0)}y_1-\frac{l(l+1)}{r^2}\mu y_1=0. \tag{4.59}$$

By boundary condition 2 we must have that the stresses vanish at the core-mantle boundary. We therefore define a function y_2 such that this function is zero at the boundary. We write the stress in the radial direction and defining y_2 as

$$\hat{e}_i^{(r)}\mathcal{I}_{ij}=\sum_{\sigma,m,l}y_2(r)\sqrt{l(l+1)}\mathbf{C}_j^{(\sigma ml)}(\theta,\phi), \qquad \sigma=c,s. \tag{4.60}$$

We find y_2 more explicitly by the insertion of (4.58) into $\mathcal{I}_{ij}(\mathbf{u})$ to be

$$y_2=\mu\left(\frac{dy_1}{dr}-\frac{y_1}{r}\right). \tag{4.61}$$

Boundary condition 2 requires the following conditions on y_2,

$$y_2=0 \qquad \text{at} \quad r=b \quad \text{and} \quad r=a. \tag{4.62}$$

Our only problem now is that the differential of the second Lamé parameter causes problems for the numerical integration. This problem arrives from the fact that we do not have an analytic expression for the parameter. We instead have a set of data points. We can however rewrite our equations to avoid this problem. We use the equivalent system of differential equations

$$\begin{aligned}\frac{dy_1}{dr}&=\frac{y_1}{r}+\frac{y_2}{\mu},\\ \frac{dy_2}{dr}&=\left(\frac{l^2+l-2}{r^2}\mu-\omega^2\rho^{(0)}\right)y_1-\frac{3}{r}y_2.\end{aligned} \tag{4.63}$$

This is our final result for this subsection. With this differential system together with the boundary conditions we can find the toroidal eigenfrequencies for our Earth and Moon models by numerical integration.

4.4.2 Induced toroidal motion from gravitational waves

The displacement of the mode ${}_mT_l$ given the n'th eigenfrequency from a force distribution $f_i^{(n)}(\mathbf{r}^{(0)})$ and surface stresses $T_{ij}^{(0)}(\mathbf{u})$ to a radially heterogeneous, anelastic self-gravitating Earth or Moon model is given as,

$$\begin{aligned}\mathbf{u}_i^{(nml)}(\mathbf{r},t)=&\int_V\mathcal{G}_{ji}^{(nml)}(\mathbf{r}|\mathbf{r}^{(0)},t)f_j^{(n)}(\mathbf{r}^{(0)})d^3\mathbf{r}^{(0)}\\ &+\int_S\mathcal{G}_{ji}^{(nml)}(\mathbf{r}|\mathbf{r}^{(0)},t)[n_k^{(0)}\cdot T_{kj}^{(0)}(\mathbf{u})]dS(\mathbf{r}^{(0)}).\end{aligned} \tag{4.64}$$

Here $\mathcal{G}^{(nml)}$ is the toroidal Green tensor taken from section 6.3.3.2 of [3], which is given as the product over the eigenvectors, a normalisation and the time-dependent effect,

$$\mathcal{G}_{ij}^{(nml)}(\mathbf{r}|\mathbf{r}^{(0)},t) = M_i^{*(nlm)}(\mathbf{r})M_j^{(nlm)}(\mathbf{r}^{(0)})\bar{g}^{(n)}(t)(\Lambda_T^{(nml)})^{-1}. \tag{4.65}$$

The time dependent effect $\bar{g}(t)$, the normalization $\Lambda^{(nml)}$ and the vector $M_i^{(nml)}$ are given by

$$\bar{g}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(\omega)}{\omega_n^2-\omega^2+i\omega\omega_n/Q_n}e^{(i\omega t)}d\omega, \tag{4.66}$$

$$\Lambda_T^{(nml)} = 4\pi\frac{l(l+1)}{2l+1}\int_0^a [y_{1n}^T]^2\rho_0(r)r^2dr, \tag{4.67}$$

$$M_i^{(nml)} = y_{1n}^T(r)\sqrt{l(l+1)}\mathbf{C}_i^{(lm)}(\theta,\phi). \tag{4.68}$$

Here ω is the frequency of the gravitational wave, $g(\omega)$ is the source function, and Q_n is the dissipation parameter. The source function $g(\omega)$ deserves additional comments. This is the Fourier transform of the time-dependent part of the source of interest. If we are observing a source that produces gravitational waves which are an infinite monochromatic wave, then $g(t) = e^{i\omega_{gw}t}$ and $g(\omega) = \delta(\omega-\omega_{gw})$. Given some model, we showed that we can find the eigenfrequency $\omega_n = \omega_{nlm}$ and $y_{1n}^T(r)$. We now proceed to simplify our expression for the displacement so that we can calculate its value. We begin by studying the Lagrangian density of an elastic continuum. The Lagrangian with displacement u_i, density ρ, stress tensor T^{ij} and strain tensor E^{ij} is in non-relativistic solid mechanics given by

$$L = \frac{1}{2}\rho\left(\frac{\partial u^i}{\partial t}\right)\left(\frac{\partial u_i}{\partial t}\right) - \frac{1}{2}(T^{ij}E_{ij}). \tag{4.69}$$

We extend the strain tensor by the metric perturbation tensor h^{ij} such that $E^{ij} \to E^{ij}+h^{ij}$. The stress tensor extends to $T^{ij} \to T - \mu h^{ij}$, where μ is the second Lamé parameter. The new Lagrangian is by the replacement of E^{ij} with $E^{ij}+h^{ij}$,

$$\mathcal{L} = L - \frac{1}{2}(h^{ij}T_{ij}). \tag{4.70}$$

The metric perturbation tensor h^{ij} obeys the equations

$$\nabla^2 h^{ij} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}h^{ij} = 0, \qquad \nabla_j h^{ij} = 0. \tag{4.71}$$

We can now find the equations of motion by using the Lagrangian density and arrive at,

$$\partial_t(\rho\dot{u}^i) = \nabla_j T^{ij} - \nabla_j(\mu h^{ij}). \tag{4.72}$$

The driving force from the plane gravitational wave is then

$$f^{(n)j}(r) = -\nabla_i(\mu h^{ij}) = -\frac{\partial\mu}{\partial r}\hat{e}_i^{(r)}h^{ij}. \tag{4.73}$$

We assume the incoming gravitational wave is of the form

$$h_{ij} = h_0\mathcal{E}_{ij}e^{(-ik_ir^i)}, \tag{4.74}$$

where k_i is the wave vector written out in terms of the direction of the wave p_i as,

$$k_i = \frac{\omega_{gw}}{c} p_i, \tag{4.75}$$

of the gravitational wave, ϵ_{ij} is the polarization tensor and h_0 is the dimensionless source amplitude parameter. Combining everything we can rewrite (4.82) to

$$u_j^{(nml)}(\mathbf{r}, t) = -\int_V \dot{\mu}(r)\mathcal{G}_{ij}^{(nml)}(\mathbf{r}|\mathbf{r}^{(0)}, t)(\hat{e}_a^{(r)} h^{ai}) d^3 r^{(0)} + \mu(R)\int_S \mathcal{G}_{ij}^{(nml)}(\mathbf{r}|\mathbf{R}, t)(\hat{e}_a^{(r)} h^{ai}) dS^{(0)}(\mathbf{R}).$$

We can simplify this expression to

$$\mathbf{u}_j^{(nml)}(\mathbf{r}, t) = h_0(\Lambda^{(nml)})^{-1}\bar{g}(t) y_{1n}(r)\sqrt{l(l+1)} C_j^{*(lm)}(\theta, \phi) F_T \tag{4.76}$$

where

$$F_T = -\int_0^a \frac{d\mu}{dr} y_{1n}(r) r^2 dr \left[\mathcal{E}^{ij}\sqrt{l(l+1)}\int_0^{2\pi}\int_0^{\pi} \hat{e}_i^{(r)} C_j^{(lm)} e^{-ik_a r^a}\sin\theta d\theta d\phi\right] + R^2\mu(R) y_{1n}(R)\left[\mathcal{E}^{ij}\sqrt{l(l+1)}\int_0^{2\pi}\int_0^{\pi} \hat{e}_i^{(r)} C_j^{(lm)} e^{-ik_a R^a}\sin\theta d\theta d\phi\right]. \tag{4.77}$$

It is with this equation we will calculate the response of our models to gravitational waves. A long calculation shows that we can write our expression for F_T as a sum which we will show to quickly converge. We show this calculation in appendix A. This gives us a much easier calculation of the toroidal displacement. The integral for F_T is expanded in terms of the Wigner symbols. Our expansion for F_S is then from appendix A,

$$F_T = -8\pi i \sum_{l_1=0}^{\infty}\sum_{m_l=-l_1}^{l_1} (2l_1+1) i^{-l_1} Y_{l_1}^{*m_l}(e, \lambda)\begin{pmatrix} l_1 & l & 1 \\ 0 & 0 & 0 \end{pmatrix} H_{lm}^{l_1 m_1}(\lambda, e, \nu) \int_0^a \dot{\mu} y_{1n}^T(r) j_{l1}(kr) r^2 dr + 8\pi i \mu(a) a^2 y_{1n}^T(R)\sum_{l_1=0}^{\infty}\sum_{m_l=-l_1}^{l_1}(2l_1+1) i^{-l_1} j_{l_1}(ka) Y_{l_1}^{*m_1}(e, \lambda)\begin{pmatrix} l_1 & l & 1 \\ 0 & 0 & 0 \end{pmatrix} H_{lm}^{l_1 m_1}(\lambda, e, \nu). \tag{4.78}$$

Here (λ, e, ν) are angles that depend on the entrance angle of the gravitational wave. The additional $H_{lm}^{l_1 m_1}(\lambda, e, \nu)$ function introduced above comes from the polarisation tensor in (4.77). Since we have rewritten the integral as a sum, we are now contracting the polarisation tensor with the tensors of $T^{0,\pm1,\pm2}$ from Appendix A. Given a coordinate system where the origin is in the centre of the sphere model and the Cartesian coordinates are described by x, y, z, we define the angles λ and e through the momentum vector $\mathbf{p}$ of the gravitational wave in the Cartesian coordinate system such that,

$$p_i = (\sin e\cos\lambda, \sin e\sin\lambda, \cos e). \tag{4.79}$$

This defines the angles of e and λ, however ν is not mentioned yet. The angle ν is the angle that the plane of the momentum vector $\mathbf{p}_i$ makes with the z-axis. The form of $H_{lm}^{l_1m_1}(\lambda, e, \nu)$ is therefore given by,

$$H_{lm}^{l_1m_1}(\lambda, e, \nu) = \sum_{k=-2}^{k=2} A_k \sin^{2-k}\left(\frac{e}{2}\right)\cos^{2+k}\left(\frac{e}{2}\right)e^{ik\lambda+2i\nu}. \tag{4.80}$$

Given that we have assumed a spherical non-rotating body, one can always rotate the coordinate system in such a way as to consider an especially simple combination of entrance angles. $H_{lm}^{l_1m_1}$ then takes on the simplified form of,

$$H_{lm}^{l_1m_1} = \frac{i}{2\sqrt{2}}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} l_1 & l & 1 \\ m_1 & m-1 & -1 \end{pmatrix}. \tag{4.81}$$

when $e = 0, \lambda = 0$ and $\nu = \pi/4$, which corresponds to the gravitational wave momentum vector p_i being aligned with the $\hat{e}_i^z$ unit vector. Starting with equation (4.76) we now have all the required ingredients to calculate the toroidal displacement at a radius r, time t, and angles θ and ϕ for a radially heterogeneous, anelastic self-gravitating, spherical model by calculating and inserting (4.68), (4.67) and (4.78) into (4.76).

4.5 Spheroidal Oscillations

4.5.1 Differential equation system for y_1 and y_3

This section is based on section 6.3.4 of reference [10]. We recall how spheroidal oscillations were defined as displacements described by the $\mathbf{N}$ and $\mathbf{L}$ Hansen vectors only. In spheroidal coordinates, these vectors are made up of the vector surface harmonics $\mathbf{P}$ and $\mathbf{B}$ vectors and a general displacement can be written as

$$u_j(\mathbf{r}) = \sum_{\sigma,m,l}\left(y_{1n}(r)P_j^{(\sigma ml)}(\theta,\phi) + y_{3n}(r)\sqrt{l(l+1)}B_j^{(\sigma ml)}(\theta,\phi)\right). \tag{4.82}$$

We recall that the vectors $\mathbf{P}$ and $\mathbf{B}$ take on a specificly simple form in spherical coordinates,

$$P_i^{(ml)}(\theta,\phi) = \hat{\mathbf{e}}_i^{(r)}Y_l^m, \tag{4.83a}$$

$$\sqrt{l(l+1)}B_i^{(ml)}(\theta,\phi) = r\nabla_i Y_l^m, \tag{4.83b}$$

where $Y^{(ml)} = P^{(ml)}(\cos\theta)e^{im\phi}$ in which $P^{(ml)}(x)$ is the associated Legendre polynomial. In the same way as for the toroidal case we have made the assumption that we can split the displacement into a radial scalar function part and an angle dependent vector part. We also assume that the gravitational perturbation can be written as

$$\psi(\mathbf{r}) = \sum_{\sigma,m,l} y_{5n}(r)Y_l^{(\sigma)m}(\theta,\phi). \tag{4.84}$$

Our procedure is as follows. We insert (4.82) and (4.84) into the differential equations (4.47) and (4.49). In return we get a differential equation system for y_1, y_3 and y_5. Rewriting the system so that no parameter takes on a derivative and we have a system close to ready for numerical integration. A tedious, but not too complicated calculation of inserting (4.82) into (4.47) and setting the coefficients of $P_i^{(ml)}$ and $B_i^{(ml)}$ to zero leads to the system

$$\begin{aligned}
&\mu\left(2\frac{dX}{dr}-\frac{l(l+1)}{r}Z\right)+\frac{d(\lambda X)}{dr}+2\dot{\mu}\frac{dy_1}{dr}\\
&+\rho^{(0)}\left(\frac{dy_5}{dr}-4\pi G\rho^{(0)}y_1+g^{(0)}\left(X-\frac{dy_1}{dr}+\frac{2}{r}y_1+\omega^2 y_1\right)\right)=0,\\
&(\lambda+2\mu)\frac{X}{r}-\frac{d}{dr}(\mu Z)-\mu\frac{Z}{r}+2\dot{\mu}\left(\frac{dy_3}{dr}+Z\right)\\
&+\rho_0\left(\frac{1}{r}(y_5-g^{(0)}y_1)+\omega^2 y_3\right)=0.
\end{aligned} \tag{4.85}$$

where

$$X=\frac{dy_1}{dr}+\frac{2}{r}y_1-\frac{l(l+1)}{r}y_3, \tag{4.86}$$

$$Z=\frac{1}{r}(y_1-y_3)-\frac{dy_3}{dr}. \tag{4.87}$$

To accommodate for the boundary conditions and to make a system of first order differential equations instead of two second order differential equations we calculate the elastic stresses. Using (4.82) and the elastic stress definition we can write the stress tensor contracted with the radial unit vector as

$$\hat{e}_\mu^{(r)}\mathcal{I}_{(e)}^{ij}=\sum_{\sigma,m,l}\left(y_2(r)P_j^{\sigma ml}(\theta,\phi)+y_4(r)\sqrt{l(l+1)}B_j^{(\sigma ml)}(\theta,\phi)\right), \tag{4.88}$$

where y_2 and y_4 are given by

$$y_2=\lambda X+2\mu\frac{dy_1}{dr}=(\lambda+2\mu)\frac{dy_1}{dr}+\frac{2\lambda}{r}y_1-\lambda\frac{l(l+1)}{r}y_3, \tag{4.89}$$

$$y_4=\mu\left(Z+2\frac{dy_3}{dr}\right)=\mu\left(\frac{1}{r}(y_1-y_3)+\frac{dy_3}{dr}\right). \tag{4.90}$$

We now insert (4.84) into (4.49) to get a differential equation for y_5, the result is

$$\frac{d^2y_5}{dr^2}+\frac{2}{r}\frac{dy_5}{dr}-\frac{l(l+1)}{r^2}y_5=4\pi G(\rho^{(0)}X+\dot{\rho}^{(0)}y_1). \tag{4.91}$$

If we take the value of ψ outside the boundary of the spherical model then by $\nabla^2\psi_e=0$ we can write the boundary condition as

$$\frac{dy_5}{dr}-4\pi G\rho^{(0)}y_1=-\frac{l+1}{r}y_5 \quad \text{at} \quad r=R. \tag{4.92}$$

We define a y_6 as

$$y_6=\frac{dy_5}{dr}-4\pi G\rho^{(0)}y_1. \tag{4.93}$$

We can summarize the boundary conditions as

$$y_2=0, \qquad y_4=0, \qquad y_6+\frac{l+1}{r}y_5=0 \quad \text{at} \quad r=R. \tag{4.94}$$

We now treat each $y_1, y_2, ..., y_6$ as independent variables. With this we can create the first-order differential equation system,

$$\frac{dy_1}{dr} = -\frac{2\lambda}{(\lambda+2\mu)r}y_1 + \frac{1}{\lambda+2\mu}y_2 + \frac{l(l+1)\lambda}{(\lambda+2\mu)r}y_3, \tag{4.95a}$$

$$\begin{aligned}\frac{dy_2}{dr} &= \left[-\omega^2\rho^{(0)} - 4\frac{g^{(0)}\rho_0}{r} + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}\right]y_1 - \frac{4\mu}{(\lambda+2\mu)r}y_2, \\ &+ \frac{l(l+1)}{r}\left[g^{(0)}\rho^{(0)} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r}\right]y_3 + \frac{l(l+1)}{r}y_4 - \rho^{0)}y_6\end{aligned} \tag{4.95b}$$

$$\frac{dy_3}{dr} = -\frac{1}{r}y_1 + \frac{1}{r}y_3 + \frac{1}{\mu}y_4 \tag{4.95c}$$

$$\begin{aligned}\frac{dy_4}{dr} &= \left[\frac{g^{(0)}\rho^{(0)}}{r} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}\right]y_1 - \frac{\lambda}{(\lambda+2\mu)r}y_2, \\ &+ \left(-\omega^2\rho^{(0)} + ((2l^2+2l-1)\lambda + 2(l^2+l-1)\mu)\frac{2\mu}{(\lambda+2\mu)r^2}\right)y_3, \\ &- \frac{3}{r}y_4 - \frac{\rho_0}{r}y_5\end{aligned} \tag{4.95d}$$

$$\frac{dy_5}{dr} = 4\pi G\rho^{(0)}y_1 + y_6, \tag{4.95e}$$

$$\frac{dy_6}{dr} = -4\pi\frac{l(l+1)}{r}G\rho^{(0)}y_3 + \frac{l(l+1)}{r^2}y_5 - \frac{2}{r}y_6. \tag{4.95f}$$

This system is to be used in the mantle. In the core on the other hand we have

$$\mu = 0, \qquad y_2 = \lambda X, \qquad y_4 = 0. \tag{4.96}$$

And we get a simpler differential equation system,

$$\begin{aligned}\frac{dy_1}{dr} &= -\frac{2}{r}y_1 + \frac{1}{\lambda}y_2 + \frac{l(l+1)}{r}y_3, \\ \frac{dy_2}{dr} &= -\left(\omega^2\rho^{(0)} + \frac{4g^{(0)}\rho^{(0)}}{r}\right)y_1 + \frac{l(l+1)}{r}g^{(0)}\rho^{(0)}y_3 - \rho^{(0)}y_6, \\ \frac{dy_5}{dr} &= 4\pi G\rho^{(0)}y_1 + y_6 \\ \frac{dy_6}{dr} &= -4\pi\frac{l(l+1)}{r}G\rho^{(0)}y_3 + \frac{l(l+1)}{r^2}y_5 - \frac{2}{r}y_6.\end{aligned}$$

The paramter function y_4 is zero in the core, but the parameter funtion y_3 is not. The expression for y_3 in the core comes from (4.95d) using the conditions of (4.96) so that y_3 satisfies,

$$y_3 = \frac{1}{\omega^2 r}\left(g_0 y_1 - \frac{1}{\rho_0}y_2 - y_5\right). \tag{4.97}$$

We now have close to everything we need to proceed with the numerical integration. With a spherical-core-mantle model, we can now employ these two differential equation systems starting in the core and continuing throughout the mantle. The details of the numerical procedure in finding the eigenfrequencies and the radial functions $y_1, y_2, ..., y_6$ will be explained in detail at the end of this chapter.

4.5.2 Induced spheroidal motion from gravitational waves

We employ the same tactic as before with the toroidal Green tensor now switched with the spheroidal Green tensor taken from section 6.3.4.1 of [3],

$$\mathcal{G}_{ij}^{(nml)}(\mathbf{r}|\mathbf{r}^{(0)},t) = Q_i^{*(nml)}(\mathbf{r})Q_j^{(nml)}(\mathbf{r}^{(0)})\bar{g}(t)(\Lambda_S^{(nml)})^{-1}, \tag{4.98}$$

where the vector Q_i is given as

$$Q_i^{(nml)}(\mathbf{r}) = y_{1n}(r)P_i^{(lm)}(\theta,\phi) + y_{3n}(r)\sqrt{l(l+1)}B_i^{(lm)}(\theta,\phi). \tag{4.99}$$

And the normalization $\Lambda_S^{(nml)}$ is given by

$$\Lambda_S^{(nml)} = \frac{4\pi}{2l+1}\int_0^R \left((y_{1n})^2 + l(l+1)(y_{3n})^2\right)\rho^{(0)}(r)r^2dr. \tag{4.100}$$

Using (4.98) and inserting (4.99) and (4.100) we get a similar expression to (4.76) for the induced displacement from the spheroidal oscillations

$$u_j^{(nml)}(\mathbf{r},t) = h_0(\Lambda_S^{nml})^{-1}\bar{g}(t)Q_j^{*(nml)}(\mathbf{r})(F_{S_1}+F_{S_2}), \tag{4.101}$$

where F_{S_1} and F_{S_2} are given by

$$\begin{aligned} F_{S_1} = &-\int_0^R \dot{\mu}y_{1n}(r)r^2dr\left(\mathcal{E}^{ij}\int_0^{2\pi}\int_0^{\pi}\hat{e}_i^{(r)}P_j^{(lm)}e^{-ik_ar^a}\sin\theta d\theta d\phi\right) + \\ &+ R^2\mu(a)y_{1n}(R)\left(\mathcal{E}^{ij}\int_0^{2\pi}\int_0^{\pi}\hat{e}_i^{(r)}P_j^{(lm)}e^{-ik_aR^a}\sin\theta d\theta d\phi\right), \end{aligned} \tag{4.102a}$$

$$\begin{aligned} F_{S_2} = &-\int_0^R \dot{\mu}y_{1n}(r)r^2dr\left(\mathcal{E}^{ij}\int_0^{2\pi}\int_0^{\pi}\hat{e}_i^{(r)}\sqrt{l(l+1)}B_j^{(lm)}e^{-ik_ar^a}\sin\theta d\theta d\phi\right) + \\ &+ R^2\mu(a)y_{1n}(R)\left(\mathcal{E}^{ij}\int_0^{2\pi}\int_0^{\pi}\hat{e}_i^{(r)}\sqrt{l(l+1)}B_j^{(lm)}e^{-ik_aR^a}\sin\theta d\theta d\phi\right). \end{aligned} \tag{4.102b}$$

We wish to again simplify our expressions for F_{S_1} and F_{S_2}. To do this we will study the quadrupole moment of a continuous mass distribution. With respect to a given coordinate system this is given as the volume integral

$$\mathcal{D}_{ij} = \int_V (3r_ir_j)\rho(r)dV. \tag{4.103}$$

Given a displacement such that $r_i \to r_i + u_i$ then to first order we get the change of quadrupole moment as

$$\delta\mathcal{D}_{ij} = \int_V 3(r_iu_j + u_ir_j)\rho dV. \tag{4.104}$$

If we consider only a single spheroidal mode, then we can represent the displacement contribution to $\delta\mathcal{D}_{\mu\nu}$ by

$$u_j^{(nml)} = y_{1n}P_j^{ml} + y_{3n}\sqrt{l(l+1)}B_j^{ml}. \tag{4.105}$$

Inserting (4.105) into (4.104) we can rewrite the total quadruple moment for a mode into a P vector and B vector part such that

$$\delta\mathcal{D}^{(P)}_{ij} = 3\int_0^R y_{1n}(r)\rho^{(0)}(r)r^2dr \int_0^{2\pi}\int_0^{\pi}\left(\hat{e}^{(r)}_i P^{(ml)}_j + P^{(ml)}_i \hat{e}^{(r)}_j\right)\sin\theta d\theta d\phi, \tag{4.106a}$$

$$\delta\mathcal{D}^{(B)}_{ij} = 3\int_0^R y_{3n}(r)\rho^{(0)}(r)r^2dr \int_0^{2\pi}\int_0^{\pi}\left(\sqrt{l(l+1)}(\hat{e}^{(r)}_i B^{(ml)}_j + B^{(ml)}_i \hat{e}^{(r)}_j)\right)\sin\theta d\theta d\phi. \tag{4.106b}$$

Neglecting the phase factor we can recast the amplitude coefficients in terms of the quadruple moment changes

$$F_{S_1} = \frac{R^2\mu(R)y_{1n}(R) - \int_0^R \dot{\mu} y_{1n}(r)r^2dr}{3\int_0^R \rho_0 y_{1n}(r)r^3dr}(\mathcal{E}^{ij}\delta\mathcal{D}^{(P)}_{ij}), \tag{4.107}$$

$$F_{S_2} = \frac{R^2\mu(R)y_{3n}(R) - \int_0^R \dot{\mu} y_{3n}(r)r^2dr}{3\int_0^R \rho_0 y_{3n}(r)r^3dr}(\mathcal{E}^{ij}\delta\mathcal{D}^{(B)}_{ij}). \tag{4.108}$$

We arrive at a new interpretation of the amplitude coefficients. They are the quadrupole-quadrupole interaction between the gravitational wave and the elastic spherical body. It is worthwhile to rewrite the integrals of the change in the quadrupole moment. From appendix B we have that the integral

$$I^{(P)}_{ab}(\mathbf{k}) = \sqrt{l(l+1)}\int_0^{2\pi}\int_0^{\pi} \hat{e}^{(r)}_a P^{(ml)}_b(\theta,\phi)e^{-ik_ir^i}\sin\theta d\theta d\phi, \tag{4.109}$$

can be written as

$$\begin{aligned} I^{(P)}_{ab}(\mathbf{k}) =& 4\pi\sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1}(2l_1+1)i^{-l_1}j_{l_1}(kr)Y^{*m_1}_{l_1}(e,\lambda)\begin{pmatrix} l_1 & l_2 & 2\\ 0 & 0 & 0\end{pmatrix}\cdot \\ &\cdot\left(\sum_{j=-2}^{2}\Gamma^{(j)}\begin{pmatrix} l_1 & l & 2\\ m_1 & m & j\end{pmatrix}\right). \end{aligned} \tag{4.110}$$

And the integral

$$I^{(B)}_{ab}(\mathbf{k}) = \sqrt{l(l+1)}\int_0^{2\pi}\int_0^{\pi} \hat{e}^{(r)}_a B^{(ml)}_b(\theta\phi)e^{-ik_ir^i}\sin\theta d\theta d\phi, \tag{4.111}$$

can be written as

$$\begin{aligned} I^{(B)}_{ab}(\mathbf{k}) = 4\pi\sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1}(2l_1+1)i^{-l_1}j_{l_1}(kr)Y^{*m_1}_{l_1}(e,\lambda)\begin{pmatrix} l_1 & l & 0\\ 0 & 0 & 0\end{pmatrix}\cdot \\ \cdot\left(\sum_{j=-2}^{2}D^{(j)}\Gamma^{(j)}_{ab}\right). \end{aligned} \tag{4.112}$$

If the sums converge quickly enough then this would be a useful expansion of the integrals. The integrals take a particular simple form at $k=0$ as then the Bessel function turns into $j_{l_1}(0) = \delta_{l_1,0}$. Forcing $l_1 = 0$ for a nonzero result puts strict restrictions on the Wigner symbols in $D^{(j)}$ and the integrals take on the much simpler forms

$$I_{ab}^{(P)}(\mathbf{k}=0) = \frac{4\pi}{5}\delta_{l,2}(\Gamma^{(0)}\delta_{m,0} - \Gamma^{(1)}\delta_{m,-1} + \Gamma^{(-1)}\delta_{m,1} + \Gamma^{(2)}\delta_{m,-2} + \Gamma^{(-2)}\delta_{m,2}),$$

$$I_{ab}^{(B)}(\mathbf{k}=0) = \frac{12\pi}{5}\delta_{l,2}(\Gamma^{(0)}\delta_{m,0} - \Gamma^{(1)}\delta_{m,-1} + \Gamma^{(-1)}\delta_{m,1} + \Gamma^{(2)}\delta_{m,-2} + \Gamma^{(-2)}\delta_{m,2}).$$

If we insert this into the equation for the change in the quadrupole moment (4.106) we get a tensor of Kronecker delta's which we will later contract with the polarisation tensor,

$$\delta\mathcal{D}_P + \delta\mathcal{D}_B = \frac{24\pi}{5\sqrt{6}}\delta_{l,2}\int_0^R (y_{1n} + 3y_{3n})\rho_0(r)r^2 dr \times \Delta \tag{4.114}$$

where

$$\Delta = \begin{pmatrix} \delta_{m,2} + \delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m,0} & i\delta_{m,2} - i\delta_{m,-2} & \delta_{m,1} + \delta_{m,-1} \\ i\delta_{m,2} - i\delta_{m,-2} & -\delta_{m,2} - \delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m,0} & -i\delta_{m,-1} + i\delta_{m,1} \\ \delta_{m,1} + \delta_{m,-1} & -i\delta_{m,-1} + i\delta_{m,1} & \frac{2}{3}\sqrt{6}\delta_{m,0} \end{pmatrix}.$$

This is a very interesting result because even though the wave number for our gravitational wave is not zero it will be close enough to zero for it to create resonance in the Moon such that the expression above will be the main contribution to the change in quadrupole moment. This is because the argument of the Bessel function kr is much smaller than one, i.e. $kr \ll 1$. If we study the approximation of the Bessel function $j_l(x)$ when $x \ll 0$ we have that,

$$j_{l_1}(x) \approx \frac{1}{(2l_1+1)!!}(x)^{l_1}, \tag{4.115}$$

From (4.110) and (4.112) we see that if we insert this approximation for $j_{l_1}(kr)$ then for each term in the sum we would add to the total expression something proportional to $(kr)^{l_1}$. If $kr \ll 1$ then only the first few terms, i.e. $l_1 = 0$ would be sufficient to calculate a good approximation for the displacement. The total displacement would then be to a very good approximation described by (4.114). We can therefore use the simple expression (4.114) to analyse how different gravitational wave polarisation of different theories of gravity will excite the same modes up to a certain accuracy given a small enough resonance frequency. The quadrupole tensor is to be contracted with the polarisation tensor of the gravitational wave. We can therefore see very quickly that first, $l = 2$ is the only l which will give any contribution because of the Kronecker delta $\delta_{l,2}$ in front of the integral. Secondly, if we let the polarisation tensor be of the form of 3.91 then the scalar polarization will give a particularly interesting contribution. The contraction of (4.114) with 3.91 will give,

$$h_s(\delta_{m,2} + \delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m,0}) + h_s(-\delta_{m,2} - \delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m,0}) = -2h_s\sqrt{\frac{2}{3}} \tag{4.116}$$

Doing the same for the standard polarisation from general relativity results in a slightly higher polarisation. It is therefore slightly more difficult to detect a scalar polarized gravitational wave from the moon than the other polarizations.

4.6 Numerical integration

To find the response in the radially symmetric Earth and Moon models we must begin by solving the differential equations system of (4.63) for toroidal oscillations or (4.95) for spheroidal oscillations. We begin by making all state parameters $\rho(r), \lambda(r), \mu(r), g(r)$, the radius r, the gravitational constant G and eigenfrequency ω dimensionless by introducing the dimensionless parameters,

$$\tilde{r} = \frac{r}{R} \qquad \tilde{\mu}(r) = \frac{\mu(r)}{\mu_r}, \qquad \tilde{\rho}(r) = \frac{\rho(r)}{\rho_r},$$
$$\tilde{g}(r) = \frac{R\rho_r}{\mu_r} g(r), \qquad \tilde{\lambda}(r) = \frac{\lambda(r)}{\mu_r}, \qquad \tilde{G} = \frac{R^2\rho_r^2}{\mu_r} G, \qquad \tilde{\omega} = R\sqrt{\frac{\rho_r}{\mu_r}}\omega.$$

We also introduce the augmented $\tilde{y}_2$, $\tilde{y}_4$, $\tilde{y}_5$ and $\tilde{y}_6$

$$\tilde{y}_2 = \frac{R}{\mu_r} y_2, \qquad \tilde{y}_4 = \frac{R}{\mu_r} y_4,$$
$$\tilde{y}_5 = \frac{\mu_r}{R} y_5, \qquad \tilde{y}_6 = \frac{\mu_r}{R^2} y_6.$$

The new differential equation systems are now ready for numerical integration.

4.6.1 Toroidal oscillations

For toroidal oscillations we have the starting point of equations (4.63). Given how toroidal oscillations give a zero displacement in the core we only consider the mantle part of the integration. We must do one final step to ensure the differential equation system is dimensionless. We define

$$\hat{y}_2 = \frac{R}{\mu_r} y_2. \tag{4.117}$$

With this definition we get the exact same system as (4.63) with the dimensionless parameters and replacing y_2 with $\hat{y}_2$. We now must solve this system with the boundary conditions $y_2(r_c/R) = y_2(1) = 0$. We do this by making a guess for y_1, set the initial condition of $y_2 = 0$ at the core-mantle boundary, and an additional guess for the eigenfrequency ω. We proceed to solve the system using Runge Kutta 4 method [11] from r_c/R to 1. Varying the eigenfrequency until we satisfy the second boundary conditions leads us to the correct eigenfrequency. We finish by normalizing y_1 to be 1 at the surface.

4.6.2 Spheroidal oscillations

Our procedure is as follows. We begin by choosing an arbitrary value for the dimensionless eigenfrequency $\tilde{\omega}$. We proceed to split our system into two parts: A core part and a mantle part. The differential equation system varies slightly for the different areas of the Earth model. This means that we will perform the total integration in two steps. We begin the integration in the core at radius $r = 0$

with arbitrary starting values to $\tilde{y}_2$ and $\tilde{y}_6$ while all other starting values are set to zero. We perform the integration from r = 0 to the core-mantle boundary. We then use the final values of y_1, $\tilde{y}_2$, $\tilde{y}_5$ and $\tilde{y}_6$ as initial conditions for the second integration over the mantle. For y_3 we choose an arbitrary starting value at the core-mantle boundary, while for $\tilde{y}_4$ we set the starting value to be zero as $\tilde{y}_4(r)$ is zero throughout the core. The boundary conditions,

$$\begin{aligned} y_2(R) &= 0, \\ y_4(R) &= 0, \\ y_6(R) + \frac{l(l+1)}{R} y_5(R) &= 0, \end{aligned}$$

must also be satisfied. We wish to constrain the system by varying only the eigenfrequency ω, however, we must also choose three arbitrary initial conditions. It is therefore not too arbitrary what we choose these initial conditions to be, they have to be somehow correlated. To achieve all boundary conditions simultaneously one could make three sets of initial conditions, creating three sets of linearly independent solutions. We construct a matrix like follows,

$$\begin{bmatrix} y_2^{(1)}(R) & y_2^{(2)}(R) & y_2^{(3)}(R) \\ y_4^{(1)}(R) & y_4^{(2)}(R) & y_4^{(3)}(R) \\ y_6^{(1)}(R) + \frac{l(l+1)}{R} y_5^{(1)}(R) & y_6^{(2)}(R) + \frac{l(l+1)}{R} y_5^{(2)}(R) & y_6^{(3)}(R) + \frac{l(l+1)}{R} y_5^{(3)}(R) \end{bmatrix} = BC$$

The boundary conditions can then be written at the mantle boundary as

$$BC \times \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.118}$$

This system only has a solution if the determinant is zero. We therefore vary the eigenfrequency until the determinant changes sign. Then we perform Newtons method to arrive at a sufficient enough accuracy for the eigenfrequency.

To determine the seismic response from a gravitational wave we also need values for y_1 and y_3. We, therefore, need to determine the coefficients A, B and C. Numerically this is straightforward. We choose nine arbitrary initial values from the eigenfrequency calculation, and run the differential equation system three times, once for each set of initial conditions. We the perform Gaussian elimination on the resulting matrix and given the accuracy of the integration the 3x3 element of the matrix should approach zero compared to the other nonzero elements. We can then set this element to zero, effectively choosing C as equal to 1, and find the resulting values of A and B by simple algebra. Since we have now arrived at a set of initial conditions which satisfy all boundary conditions simultaneously we have arrive at the goal of finding the parameter functions y_1, y_2,..., y_6. It is worthwile to mention here the detail of setting $C = 1$ or equivalently, set the $(3, 3)$ element of the matrix to exactly zero. To get a nontrivial solution of $(0, 0, 0)$ we need the determinant of the matrix to be zero. It should therefore be sufficient to set this element to zero as long as we realise that this nontrivial solution is just an approximation and that changing this value from a nonzero number to a zero number should only be allowed if the number is $\ll 1$.

4.7 The plus-, cross-, and scalar polarisation from a simple pulsar model

It would be of interest to consider one type of astrophysical object in which we could predict the response in a Moon detector. For this, we follow the paper of [12] to derive an expression for the plus, cross, and scalar polarisation produced from a simple pulsar model. The model in question needs to have some sort of asymmetry to get a time-dependent dipole and quadrupole moment. We assume a spherical neutron star with a "mountain" that can be approximated as a point mass. We say "mountain" in quotation marks as it is not a real mountain, but some asymmetry on the neutron star. The size of the asymmetry is assumed to be much smaller than the size of the neutron star. The mass density of the mountain can then be assumed to take the simple expression of,

$$\rho = m\delta(x - R)\delta(y)\delta(z), \tag{4.119}$$

where we here assumed the point mass to be at the surface at the coordinate of $x = R$, $y = 0$ and $z = 0$. The dipole moment is by the simple expression of the density easy to calculate. It is then,

$$D_i = (mR, 0, 0). \tag{4.120}$$

The quadrupole moment is similarly easy to calculate and we get,

$$I^{ij} = \begin{bmatrix} \frac{2}{3}mR^2 & 0 & 0 \\ 0 & -\frac{1}{3}mR^2 & 0 \\ 0 & 0 & -\frac{1}{3}mR^2 \end{bmatrix}. \tag{4.121}$$

The expression for the dipole and quadrupole moments are in the source frame. To transform them into the wave frame so that they can be inserted into (3.97), (3.98) and (3.99) we have to transform the dipole and quadrupole moments into an inertial frame and then into the wave zone. Let the matrix S be a matrix transformation from the source frame to an inertial frame and let W be a matrix transformation from an inertial frame to the wave zone. If ϑ is the angle between the angular momentum vector and the direction of travel for the gravitational wave then the S matrix has the form,

$$S = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix}. \tag{4.122}$$

If $\varphi(t)$ is the instantaneous rotational phase of the neutron star, then the W matrix takes the form,

$$W = \begin{bmatrix} \cos\varphi(t) & -\sin\varphi(t) & 0 \\ \sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.123}$$

Making the assumption that the rotational phase is varying slowly allows us to Taylor expand $\varphi(t)$,

$$\varphi(t) = \varphi(0) + 2\pi \sum_{l=0}^{\infty} f^{(l)}(0)\frac{t^{l+1}}{(l+1)!}. \tag{4.124}$$

where $f^{(l)}(0)$ is the l'th derivative of $\varphi(t)$ evaluated at $t = 0$. Using now equation (3.97), (3.98) and (3.99) we get an expression for each polarisation in terms of the properties of the pulsar. Let us consider the plus and cross polarisation first, which by insertion into (3.97) and 3.98 becomes,

$$h_+(t) = h_0 \frac{1 + \cos^2 \vartheta}{2} \cos 2\varphi(t), \tag{4.125a}$$

$$h_\times = h_0 \cos \vartheta \sin 2\varphi(t), \tag{4.125b}$$

where,

$$h_0 = \frac{16\pi^2 G}{c^4}(1 - \zeta)(I^{xx} - I^{yy})\frac{f_0^2}{r}, \tag{4.126}$$

and I^{xx}, I^{yy} is taken in the source frame and f_0 is the spin frequency of the star. Inserting for I^{xx} and I^{yy}, we then get,

$$h_0 = \frac{16\pi^2 G m}{3c^4}(1 - \zeta)\frac{f_0^2 R^2}{r}, \tag{4.127}$$

Assuming the simple form of the dipole moment from (4.120), then for the scalar polarisation we get with (3.99),

$$h_S(t) = -\frac{4\pi G}{rc^3}\zeta \left(mRf_0 \sin \vartheta \sin \varphi(t) + \frac{2\pi}{3c} mR^2 f_0^2 \sin^2 \vartheta \cos 2\varphi(t) \right). \tag{4.128}$$

Given values for the parameters for the pulsar one could now determine the polarisation response by the potential Moon detector through (4.76) or (4.101).

CHAPTER

FIVE

EARTH AND MOON RESPONSE TO GRAVITATIONAL WAVES

We have now arrived at the results of this thesis. We begin with an analysis of the Jefferys-Bullen A' Earth Model [13] before we continue to a similar analysis on three Moon models taken from [14]. All models assume a spherical symmetry with a liquid core and a solid mantle. It is additionally assumed that the core and mantle are described completely by the radius R, core radius r_c, density $\rho(r)$, and the first and second Lamé parameters $\lambda(r)$ and $\mu(r)$. We will begin with the Earth model making a comparison to known tabulated values. We will proceed to study the different Moon models and make comparisons on the eigenfrequencies and parameter functions $y_n(r)$ which are highly relevant for the calculation of the response of the Moon to gravitational waves. We will then calculate the expected displacement of the Moon predicted by different models to a gravitational wave signal. Lastly, we will study the response over a frequency spectrum and make comments on the frequency range that our models predict to be most sensitive for a potential gravitational wave detector on the Moon.

5.1 Earth response

5.1.1 Earth model

The Jeffreys-Bullen A' Model is used to determine the eigenfrequencies of the Earth. The model is described by the density $\rho(r)$, the radius of the core and mantle R and r_c, and the two Lamé parameters $\lambda(r)$ and $\mu(r)$. The parameters are shown as a function of radius in figure 5.1.1. The data points are taken from pages (1057-1059) of [10]. For the following analysis of this chapter we use a linear interpolation between each data point. The core-mantle boundary is located at $r = 3470$km. In figure 5.1.1 this is easily seen for the second Lamé parameter $\mu(r)$. This parameter is exactly zero in the core. We also note how the gravitational acceleration does not follow the usual relation for the gravitational acceleration,

$$g_0(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0(x) x^2 dx, \tag{5.1}$$

where $g_0(r)$ is completely determined from the integral over the density $\rho(r)$. We stated introductory that all models used are completely described by $R, r_c, \rho(r), \mu(r)$

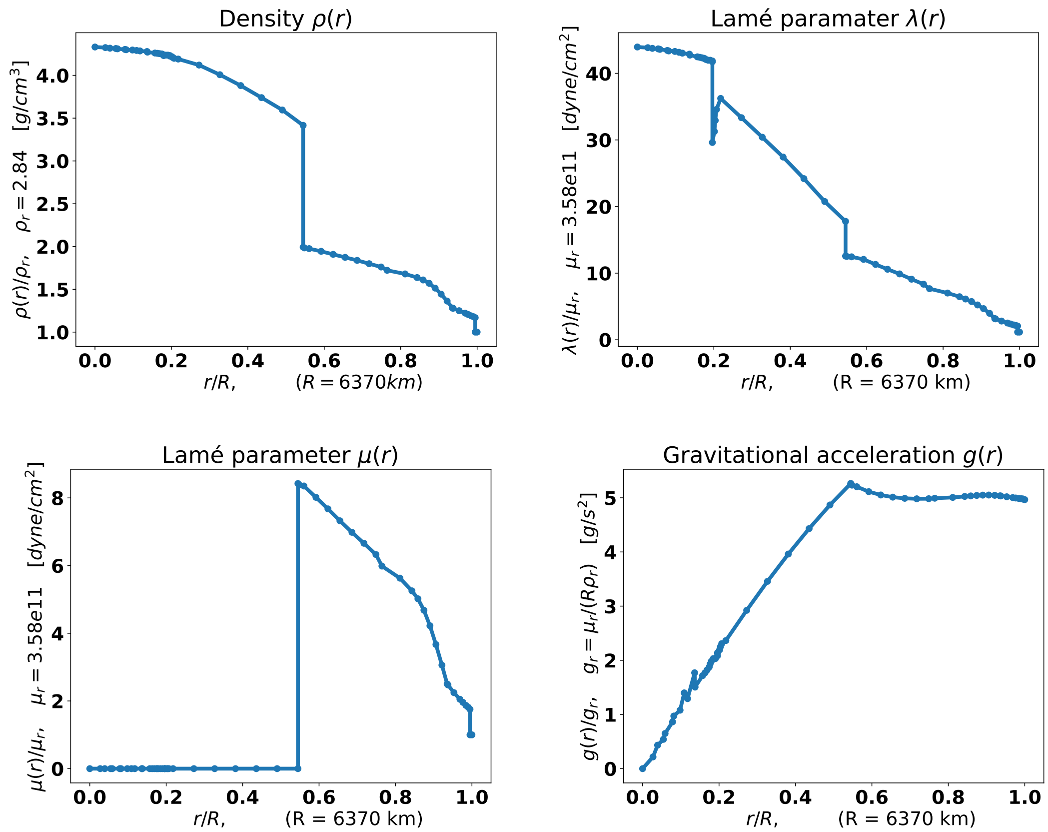

Figure 5.1.1: The parameters describing the Jeffreys-Bullen A' Earth model. The density is shown in the top-left plot, the first Lamé parameter is shown in the top-right plot, the second Lamé parameter is shown in the bottom-left plot and the gravitational attraction is shown in the bottom-right plot.

and $\lambda(r)$. Here, $g(r)$ is not mentioned. The gravitational acceleration plotted in figure 5.1.1 is taken from the same table as the values for ρ, λ and μ. It is for simplicity kept as is for our discussion and analysis.

5.1.2 Eigenfrequencies for Jeffreys-Bullen A' Earth model

The four first numerically calculated eigenfrequencies of the Jeffreys-Bullen A' Earth model are presented in Table 5.1.1, where the numerical procedure was described in Section 4.5.1 for the toroidal eigenfrequencies and in Section 4.5.2 for spheroidal oscillations. For the toroidal eigenfrequencies, we used the initial

n	Earth Model Toroidal	Earth Model Spheroidal
0	0.00241 s^{-1} (4.315)	0.00205 s^{-1} (3.682)
1	0.00834 s^{-1} (14.967)	0.00432 s^{-1} (7.766)
2	0.0142 s^{-1} (25.508)	0.00691 s^{-1} (12.392)
3	0.0206 s^{-1} (36.899)	0.0108 s^{-1} (19.431)

Table 5.1.1: The first four eigenfrequencies for the Jeffreys-Bullen A' Earth model with the numerical dimensionless eigenfrequencies in parentheses.

condition $y_0 = 1$. For the spheroidal eigenfrequencies, we recall that we needed three sets of three initial conditions and that these sets had to be different or in some sense linearly independent. These were chosen to be,

$$\text{set one of initial conditions} = (\tilde{y}_2^{(1)}, \tilde{y}_3^{(1)}, \tilde{y}_6^{(1)}) = (0.4, 0.2, 0.1), \tag{5.2a}$$

$$\text{set two of initial conditions} = (\tilde{y}_2^{(2)}, \tilde{y}_3^{(2)}, \tilde{y}_6^{(2)}) = (0.1, 0.1, 0.2), \tag{5.2b}$$

$$\text{set three of initial conditions} = (\tilde{y}_2^{(3)}, \tilde{y}_3^{(3)}, \tilde{y}_6^{(3)}) = (0.2, 0.2, 0.3). \tag{5.2c}$$

To go from the dimensionless eigenfrequency to the value without parentheses in the table one has to multiply the dimensionless eigenfrequency by the reference value introduced in section 4.5 such that the differential equation system became dimensionless. For the first model, this reference value for the frequency was $\omega_r = \frac{1}{R}\sqrt{\frac{\mu_r}{\rho_r}} \simeq 0.000557$. We will employ the first eigenfrequency to calculate the displacement from a gravitational wave. We could have chosen any of the eigenfrequencies, however, as the response will look similar for any of the infinite choices of eigenfrequencies we restrict ourselves to the first. We will later in this chapter investigate the response behaviour for varying frequency and will show that the amplitude of the response is reduced for higher frequencies. In figure 5.1.2 we observe the first parameter functions for the Earth model. We can see how the boundary condition is indeed satisfied and that the behavior of y_1 means the displacement grows approximately linearly with the distance from the core.

5.1.3 Displacement for Earth model in general relativity

In this section, we will present the displacement of the gravitational wave signal for the Earth predicted by the model. We will require the eigenfrequencies from the

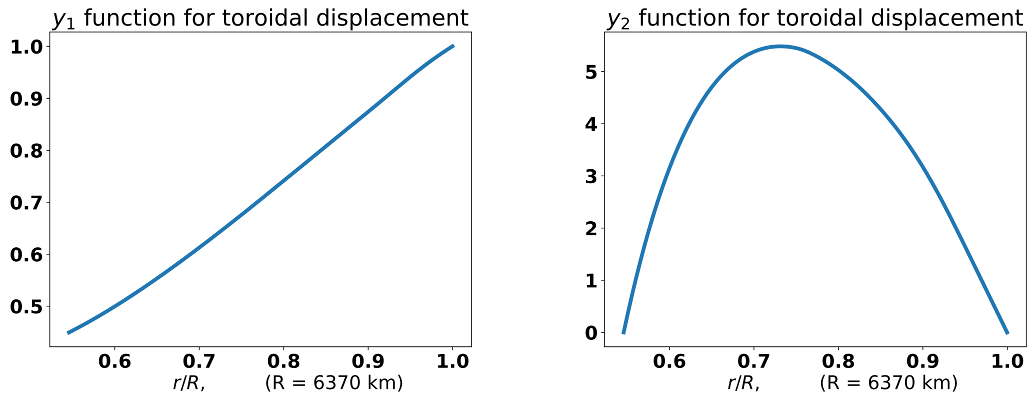

Figure 5.1.2: The normalised parameter functions y_1 and y_2 of the first toroidal eigenfrequency $\omega_0 \simeq 0.00241 s^{-1}$ for the Earth model.

last section to determine the parameter functions y_n. For the current discussion, we restrict our analysis to only study the displacement of the model at resonance, i.e. $\omega_n = \omega_{gw}$, where ω_{gw} is the frequency of the gravitational wave. We have for any calculation on the Earth model chosen the quality factor of $Q_0 = 600$ where this is the same value used in [3]. There could potentially be a difference in the quality factor of the toroidal and spheroidal oscillations. We assume however that both oscillations can be described by the chosen value for the quality factor in this thesis for simplicity.

5.1.3.1 Toroidal displacement at resonance

From the last section, we note that the first eigenfrequency is in the millihertz range. The total displacement, as seen explicitly from(4.78), depends on kr via the Bessel function $j_l(kr)$. We can use an expansion of the Bessel function to simplify the expression of the total displacement. As the argument of the Bessel function will be of the order of $\omega R/c \approx 10^{-5}$ we can safely employ the approximation

$$j_{l_1}(x) \approx \frac{1}{(2l_1+1)!!}(x)^{l_1}. \tag{5.3}$$

Since the dependence grows with powers of $\omega R/c$, the biggest contribution comes from $l_1 = 0$. This term turns out to be zero however[1] and we must continue with the $l_1 = 1$ term to get a nonzero contribution. With this in mind, there are additional ways to simplify the expression of F_T further. We also observe that for the Wigner symbols with all lower components with zeroes, we must require that $l + l_1$ is odd to get a nonzero contribution. With $l_1 = 1$ we must then have $l = 2$. The only component of F_T which depends on m is $H_{lm}^{l_1 m_1}(e, \lambda, \nu)$. It is difficult at first glance to observe for which m $H_{lm}^{l_1 m_1}(e, \lambda, \nu)$ gives a nonzero result as we must take the angles e, λ, ν into consideration as well. If we consider a gravitational wave travelling in the $z-$direction giving us the familiar polarisation tensor from chapter 3, more specifically (3.40) then the angles are $e = 0, \lambda = 0$, and $\nu = \pi/4$.

[1] See appendix A for more details.

$H_{lm}^{l_1m_1}$ now simplifies to

$$H_{2m}^{1m_1} = \frac{i}{2\sqrt{2}}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} 1 & 2 & 1 \\ m_1 & m-1 & -1 \end{pmatrix}. \tag{5.4}$$

We now must have that $m_1 + m - 1 - 1 = 0$ and $m \neq -l$ or $m = l+1$. Let us consider the contribution from $l_1 = 1$. This implies that $l = 2$. For m we must then see if any of the combinations of $-1 \leq m_1 \leq 1$ and $-2 \leq m \leq 2$ satisfies $m_1 + m - 1 - 1 = 0$. For $l = 2$ we quickly realize that $m = 2$ and $m = 1$ are the only candidates. However, since the associated Legendre polynomial P_2^1 vanishes we get that $m = 2$ is the only combination that survives for $l = 2$. This simplifies our expression for F_T even further to

$$F_T = \frac{8\pi}{5\sqrt{6}}\int_0^R \dot{\mu} y_{1n}^T(r) k r^3 dr - \frac{8\pi i}{5\sqrt{6}}\mu(R) k R^3 y_{1n}^T(R). \tag{5.5}$$

We will now calculate the displacement for the toroidal oscillations from a gravitational wave with the momentum vector determined by the angles $e = 0, \lambda = 0$, and $\nu = \pi/4$. As discussed it is sufficient to stick to $l_1 = 1$. We let $l = 2$ and $m = 2$ and use (4.76) to calculate the displacement. We leave the parameter function $y_1(r)$ and the vector $\mathbf{C}(\theta, \phi)$ out of the discussion for now since $y_1(r)$ is normalized to 1 at the surface and we are mainly interested in the size of the displacement and not where on the sphere the displacement takes place for now. Assuming that the incoming gravitational wave is a monochromatic wave with a signal lasting the time 2τ, the source function can be written as

$$g(t) = (H(t+\tau) - H(t-\tau))e^{i\omega_{gw}t}, \tag{5.6}$$

where $H(t)$ is the Heaviside step function. The effect of the eigenvibration becomes by (4.67),

$$\bar{g}(t) = \frac{g(t)}{(\omega_0 - \omega_n)^2 + \omega_n^2/(4Q_n)}. \tag{5.7}$$

We assume an amplitude h_0 of the gravitational wave and a signal with the length of the first eigenperiode. We define this as

$$\xi_T(t) = h_0(\Lambda_T^{022})^{-1} F_T \bar{g}(t). \tag{5.8}$$

At the surface of the Earth model, the displacement is then,

$$\xi_T\left(t = \frac{\pi}{2\omega_0}\right) = h_0 \times 2.6 \times 10^5 \text{cm}, \tag{5.9}$$

where Λ_T^{022} and F_T are calculated using the normalized parameter function $y_1(r)$. The time was chosen to get the maximum displacement. The displacement for the toroidal mode ${}_mT_l$, where $m = l = 2$ given the discussion from the last section, at resonance for the first toroidal eigenfrequency is plotted in figure 5.1.3 at the surface with the same time chosen such that the displacement is maximum. The plots in figure 5.1.3 deserve a comment. We have assumed that the angles λ, e, ν defined back in Chapter 4 for the momentum vector of the incoming wave are chosen such that the function of $H_{lm}^{l_1m_1}$ takes on a particularly simple form. In

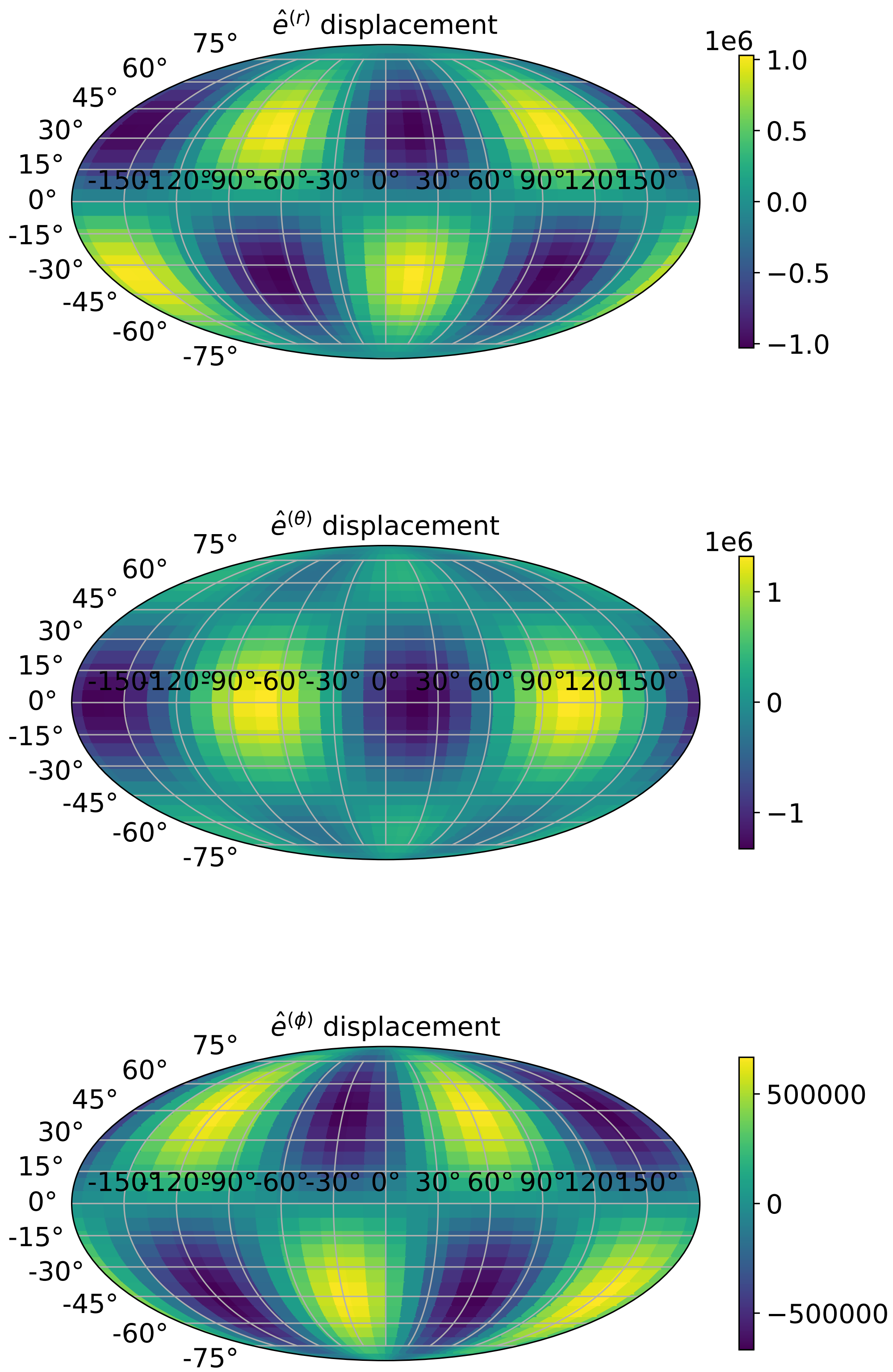

Figure 5.1.3: The ${}_2T_2$ displacement per unit strain (i.e $h_0 = 1$) of the Earth model to a gravitational wave at resonance with the first eigenfrequency in units of cm. The top figure is the displacement in the $\hat{e}^{(r)}$ direction, the middle figure in the $\hat{e}^{(\theta)}$ direction and the bottom figure in the $\hat{e}^{(\phi)}$ direction.

this simple form, the main contribution of the oscillations comes only from the T_{22} mode. If instead a gravitational wave is described by other sets of angles, then it can excite other modes. The radial displacement in the top figure 5.1.3 is the displacement in the direction of the unit vector $\hat{e}_r$. The yellow colour represents a positive displacement out of the sphere while the blue colour is the negative displacement and points into the sphere. The middle and bottom plot in Figure 5.1.3 shows the displacement in the direction of $\hat{e}_i^{(\theta)}$ and $\hat{e}_i^{(\phi)}$ respectively. For the middle figure of Figure 5.1.3 we recall that a positive displacement in yellow points in the positive z-direction, in other words, up on the page. A yellow colour then represents a displacement pointing upwards in the middle figure and a negative blue colour represents a displacement pointing downwards. For the bottom figure, a yellow positive displacement points to the right while a negative blue colour points to the left. We hope this will avoid confusion for the spheroidal plots following this with the same notations. To summarise, gravitational wave resonance with the first eigenfrequency of the Earth, predicted by the first model is a small displacement given by $\approx h_0 \times 10^5$cm. Given a common source such as the binary black holes detected by LIGO [1] then we have $h_0 \approx 10^{-21}$. One can then expect a displacement of $\approx 10^{-16}$cm.

5.1.3.2 Spheroidal displacement at resonance

For the spheroidal oscillations of the Earth model we use equation (4.101) to find the displacement. We again ignore the vector part and the dimensionless h_0 to get an idea of the response of the Earth body to a gravitational wave at the first resonance frequency. The parameter functions for the first spheroidal eigenfrequency in table 5.1.1 are plotted in figure 5.1.4. We observe that the boundary conditions are satisfied with $y_2(R) = -1.6 \times 10^{-6}$, $y_4(R) = -1.1 \times 10^{-5}$ and $y_6 + 6y_5 = 1.4 \times 10^{-5}$. With the parameter functions, we can now find the displacement from the spheroidal oscillations from gravitational wave resonance. We recall from Chapter 4 that the $l = 2$ mode gives the main contribution when the argument of the Bessel function kR satisfies $kR \ll 1$, or $\frac{\omega_0 R}{c} \ll 1$. This is satisfied for the first eigenfrequency so we assume that the total quadrupole moment can be approximated to $\delta\mathcal{D}_P$ and $\delta\mathcal{D}_B$ from (4.114). We then have for a monochromatic gravitational wave with amplitude h_0 ignoring the vector part $Q_i^{22}(\theta, \phi)$ the response, similar to the definition for the toroidal case, defined as,

$$\xi_S(t) = h_0(\Lambda_S^{022})^{-1}(F_{S_1} + F_{S_2})\bar{g}(t), \tag{5.10}$$

Choosing the time with the maximum response we get,

$$\xi_S\left(t = \frac{\pi}{2\omega_0}\right) = h_0 \times 4.915 \times 10^9 \text{cm}. \tag{5.11}$$

This is a much bigger response than the toroidal oscillations. This is not unexpected since for the toroidal case we had to go to the term of $l_1 = 1$ in the expansion of the integral (6.19), while for the spheroidal case of the expansion (6.33) and (6.33) gives a nonzero result for the $l_1 = 0$ term. Figure 5.1.5 is the plot of the displacement in the $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ directions for the spheroidal mode ${}_2S_2$ with the amplitude of the gravitational wave $h_0 = 1$. We see that the displacement $u \approx h_0 \times 10^8$ cm. If one had a source with $h_0 = 10^{-21}$ then this would produce

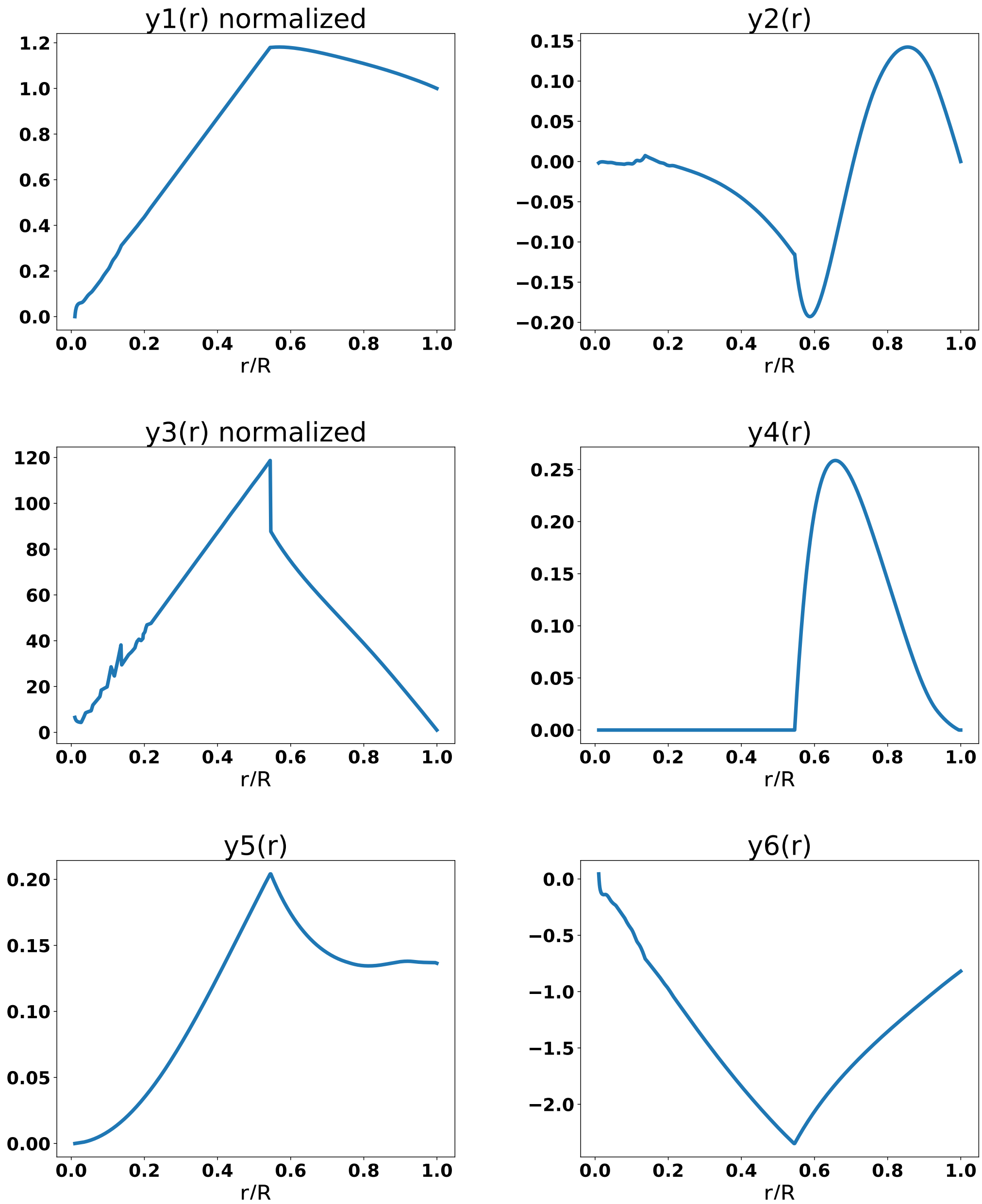

Figure 5.1.4: The spheroidal parameter functions for the first eigenfrequency of the Jeffreys-Bullen A' Earth model.

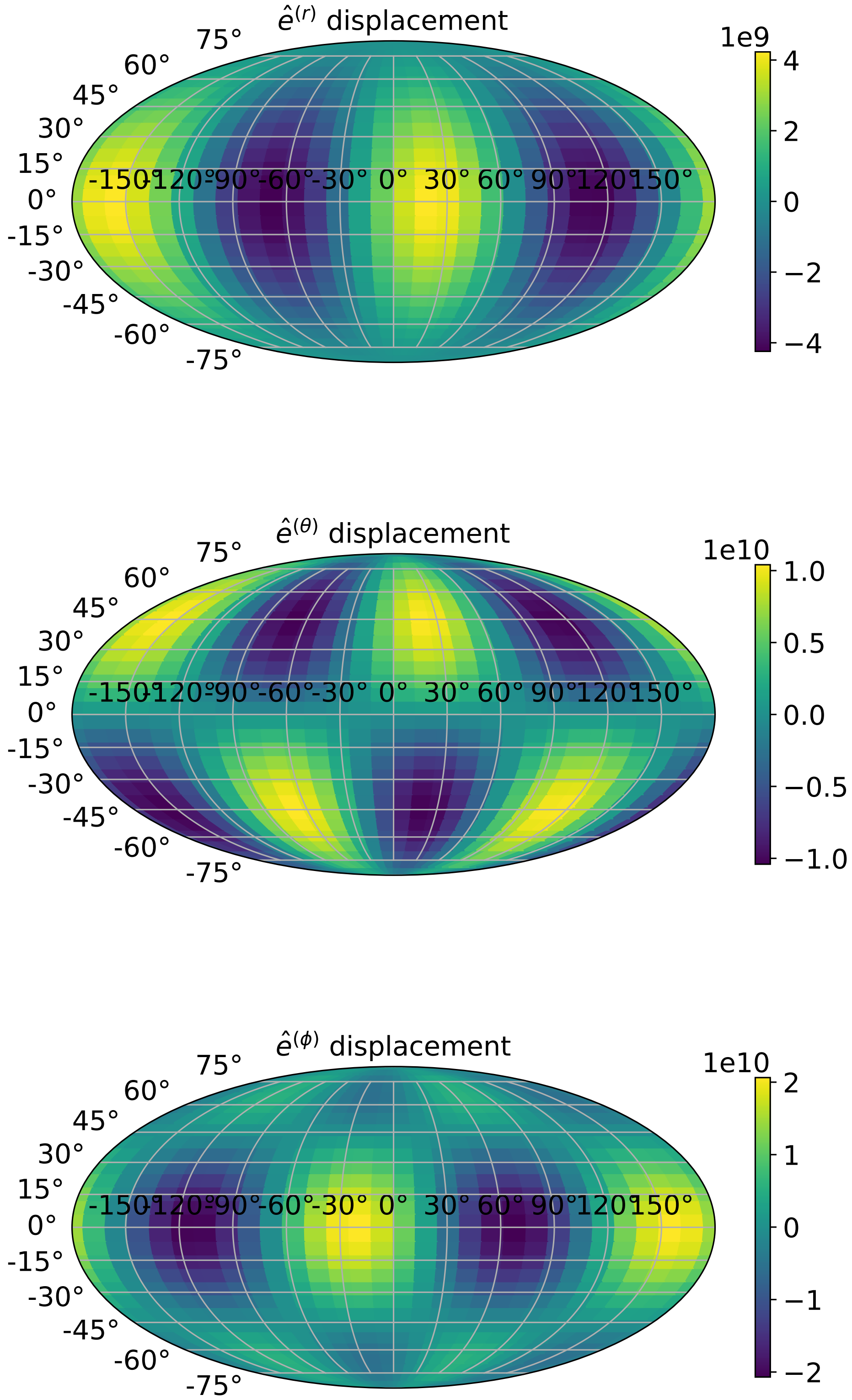

Figure 5.1.5: The ${}_2S_2$ displacement of Earth model per unit strain in units of cm. The top figure is the displacement in the $\hat{e}^{(r)}$ direction, the middle figure is the displacement in the $\hat{e}^{(\theta)}$ direction and the bottom figure in the $\hat{e}^{(\phi)}$ direction.

a displacement on Earth of order $\approx 10^{-13}$ cm. A significantly higher contribution than the toroidal case. This is also in agreement with our approximation scheme of having to go to $l_1 = 1$ for the toroidal case while $l_1 = 0$ gave a nonzero result for the spheroidal case.

5.1.4 Displacement for Earth in Brans-Dicke

From Chapter 4 we know that the main contribution of oscillations from scalar polarised gravitational waves is from the $l = 2$, $m = 0$ spheroidal mode, i.e. ${}_0S_2$. We assume a monochromatic source and plot the displacement in figure 5.1.6. We proceed to also calculate the displacement given a monochromatic gravitational wave with amplitude h_0 ignoring the vector dependent part $Q_i^{(ml)}(\theta, \phi)$ of the displacement. We arrive at,

$$\xi_S\left(t = \frac{\pi}{2\omega_0}\right) = h_0 \times 3.956 \times 10^9\text{cm}, \tag{5.12}$$

which is of the same order as the response from the cross and plus polarisation as expected, however slightly lower. The ${}_0S_2$ mode excited from the scalar polarisation is very different from the ${}_2S_2$ mode excited from the plus- and cross polarisation. We do observe from equation (4.114) that this mode can be exited in general relativity and no does not need Brans-Dicke to explain a potential excitation of this form. What is new is the fact that if we are aware of where the gravitational wave is coming from and rotate the coordinate system such that the polarisation tensor takes on the specifically easy form of (3.91) then one would have evidence for theories of gravity beyond general relativity.

5.2 Moon response

5.2.1 Moon models

Three different Moon models are studied to analyze the response from gravitational waves. The three models are taken from reference [14]. We will summarize the key characteristics of each model in this section. We recall that all models are described by the density $\rho(r)$, the Lamé parameters $\lambda(r)$ and $\mu(r)$, and gravitational acceleration $g(r)$. The parameterization of the models is based on polynomial C_1 Bézier curves. The advantage of this method is a non-regular spaced discretion of the models and no prior constraint on layer thickness or location of seismic discontinuities. The values used in the models are median values of the model's ensembles. For in-depth details of the construction of the models, we refer to reference [14]. The parameters of the models are shown in figure 5.2.1. Red represents the first model, blue is the second model, and green is the third model. We can see some similarities immediately. All models predict to a certain approximation a similar behavior of all parameters in most of the mantle, while in the core the parameters for all models depart. Moreover, the models have a slight disagreement on the core radius. The first Lamé parameter $\lambda(r)$ is especially interesting as it completely splits as we reach the core. The models determine the eigenfrequencies of the Moon's body. For our purpose of determining the displacement by a gravitational wave, we study the components of (4.58) and (4.82). We see that it

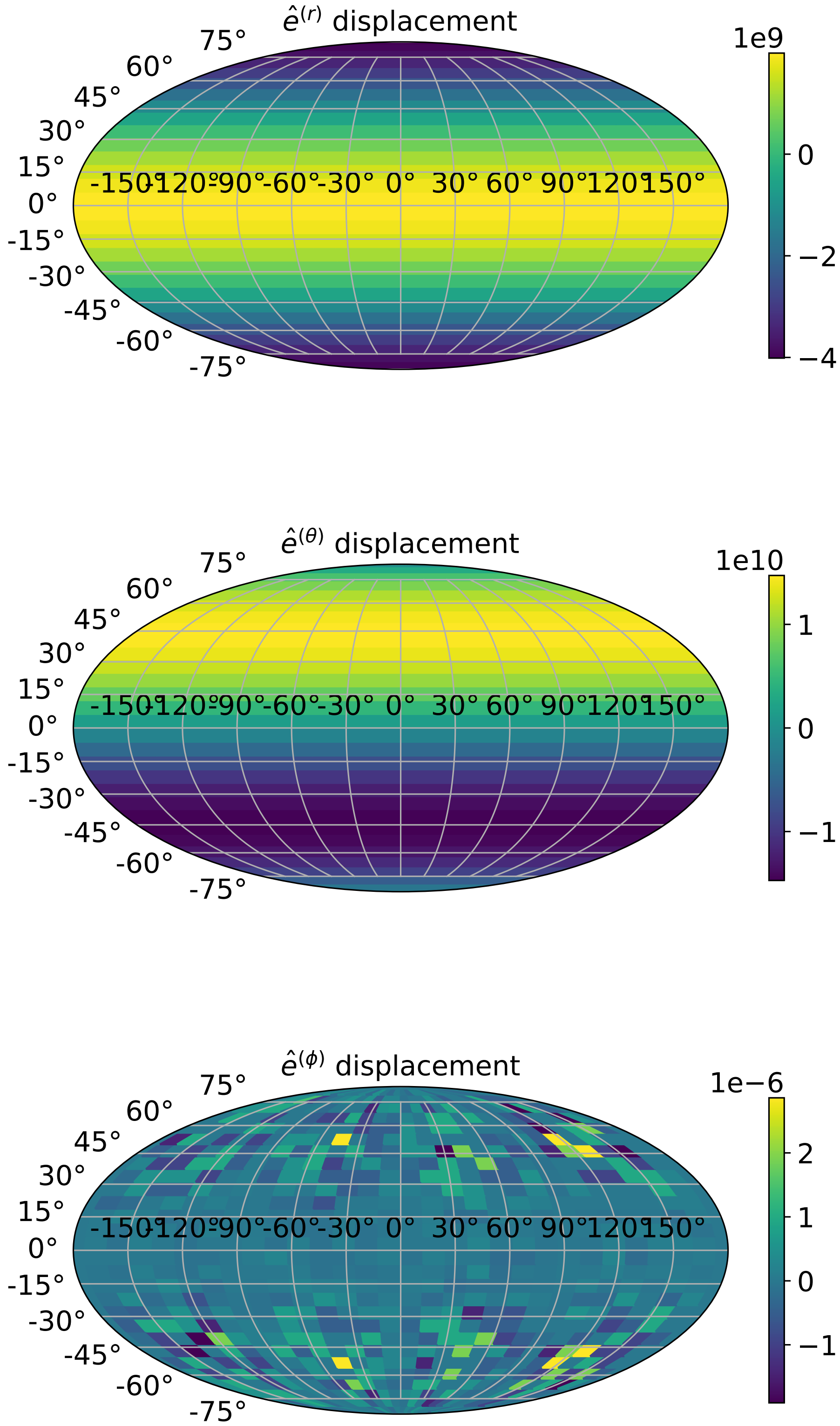

Figure 5.1.6: The ${}_0S_2$ displacement of Earth model per unit strain in units of cm by a scalar polarised gravitational wave. The top figure is the displacement in the $\hat{e}^{(r)}$ direction, the middle figure is the displacement in the $\hat{e}^{(\theta)}$ direction and the bottom figure in the $\hat{e}^{(\phi)}$ direction.

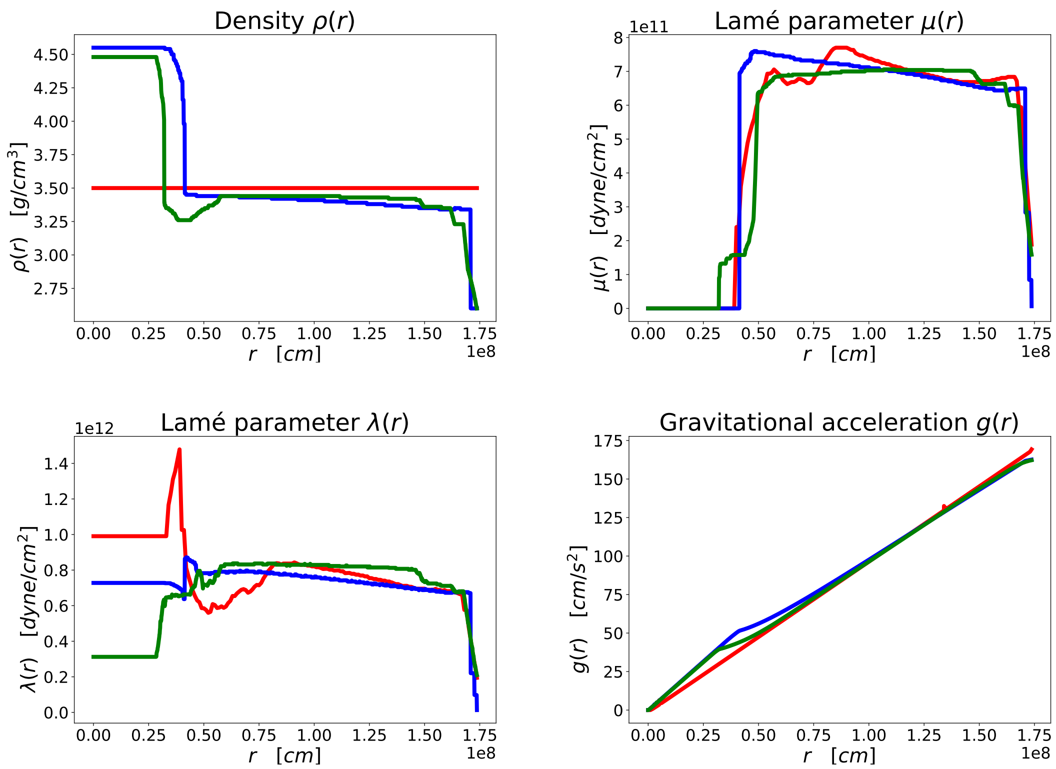

Figure 5.2.1: The parameters $\rho(r)$, $\mu(r)$, $\lambda(r)$ and $g(r)$ of the three Moon models. Model 1 in red, model 2 in blue and model 3 in green.

is the parameter functions y_1 for toroidal oscillations and y_1 and y_3 for spheroidal oscillations with the eigenfrequencies which will directly affect the response the models will predict. We will first consider the predicted eigenfrequencies by the models.

5.2.2 Eigenfrequencies of Moon models

By the procedure of section 4.4, we can find the eigenfrequencies of the Moon models. Given that it is possible to measure the eigenfrequency for different modes of the moon it is interesting to examine if one would get a measurable difference for the models. We would expect that this difference is not too big for the toroidal eigenfrequencies as the oscillation only takes place in the mantle and crust. Given that our models mostly agree in the mantle and the mantle radius is much bigger than the crust radius we should expect not too much of a difference here. For spheroidal oscillations on the other hand we expect to see a bigger difference. Here the whole model must be taken into account and the core would give some effect on the eigenfrequencies. We recall that our procedure to find an eigenfrequency of a model is to vary a test frequency until our constructed determinant which depends on this frequency and the boundary conditions becomes zero. We summarize the eigenfrequencies in Table 5.2.1. Here the number without a parenthesis is the eigenfrequency in seconds while the numbers in parenthesis are the dimensionless eigenfrequency which we include these number for convenience to reproduce results. To go from the dimensionless value to the real eigenfrequency one has to

n	**M1:Toroidal**	**M1:Spheroidal**
0	0.00624 s^{-1} (4.672)	0.00665 s^{-1} (4.797)
1	0.0182 s^{-1} (13.648)	0.0116 s^{-1} (8.696)
2	0.0268 s^{-1} (20.071)	0.0184 s^{-1} (13.795)
3	0.0354 s^{-1} (26.508)	0.0250 s^{-1} (18.707)
n	**M2:Toroidal**	**M2:Spheroidal**
0	0.00637 s^{-1} (22.144)	0.00660 s^{-1} (22.939)
1	0.0185 s^{-1} (64.407)	0.0119 s^{-1} (41.264)
2	0.0276 s^{-1} (95.788)	0.0181 s^{-1} (62.891)
3	0.0369 s^{-1} (128.106)	0.0251 s^{-1} (87.166)
n	**M3:Toroidal**	**M3:Spheroidal**
0	0.00636 s^{-1} (4.469)	0.00635 s^{-1} (4.625)
1	0.0183 s^{-1} (12.896)	0.0119 s^{-1} (8.381)
2	0.0256 s^{-1} (18.022)	0.0169 s^{-1} (11.861)
3	0.0310 s^{-1} (21.780)	0.0261 s^{-1} (18.330)

Table 5.2.1: The first four spheroidal eigenfrequencies of the Moon models

multiply the dimensionless value by the reference value which was chosen for each model. This is why the dimensionless value for model 2 is much bigger than for model 1 and model 3. The reference value for the model one is, $\omega_r \simeq 0.00134$ s^{-1}, model two $\omega_r \simeq 0.000288$ s^{-1} and for model three $\omega_r \simeq 0.00142$ s^{-1}. It is interesting to note the level of agreement for the eigenfrequencies for all models. With this motivation, we make a quick analysis of how strongly our parameters influence the eigenfrequency. We take Moon model 1 into consideration for this. We make a simpler model which is easier to adjust than the original. This new simple model is constructed in the following way. We restrict ourselves to only three different areas: The core, the mantle and the crust and let the parameters be constant in these areas. We then take the constant values to be the same as the $r = 0$ for the core, the average for the mantle and the same as $r = R$ for the crust. The new simplified model now labeled as M1* is plotted with the original model in figure 5.2.2

We will use this simplified model to adjust the core-mantle boundary to study

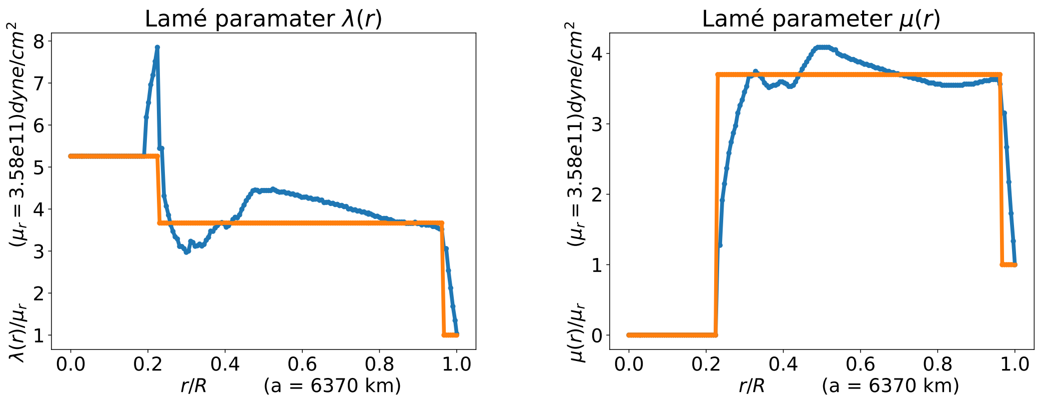

Figure 5.2.2: Comparison of the Lamé parameters in the Moon model 1 in blue and in the simplified Moon model 1* in orange. The left figure shows the comparison of the first Lamé parameter $\lambda(r)$, and the right figure shows the comparison of the second Lamé parameter $\mu(r)$

how the eigenfrequency varies with the boundary radius. It is interesting to see how the eigenfrequencies of this new simplified model compare to the original M1 model. The first four eigenfrequencies of the M1* model with the same core radius as the M1 model are summarised in table 5.2.2. Comparing the eigenfrequencies of the M1 and M1* models we see that the agreement of the simplified M1* model with the original model M1 is high. It should therefore be a reasonable starting point to adjust the core-mantle boundary. This variation is plotted in Figure 5.2.3. We observe a change as we move the boundary. It is interesting to note that the eigenfrequency stays relatively stable with small changes to the core-boundary radius. Even making a core radius at twice the size of the M1 model keeps the frequency from disagreeing substantially from our original value. We note how the core-mantle boundary affects the spheroidal frequency in a much higher degree than the toroidal frequency. To check if the suggested eigenfrequencies indeed are eigenfrequencies of the system we can plot the parameter functions $y_1, y_2, ..., y_6$, and see if they obey the boundary conditions which we recall to be $y_2 = 0$, $y_4 = 0$ and $y_6 + l(l+1)y_5 = 0$ at the surface $r = R$. It will also be useful to plot y_1

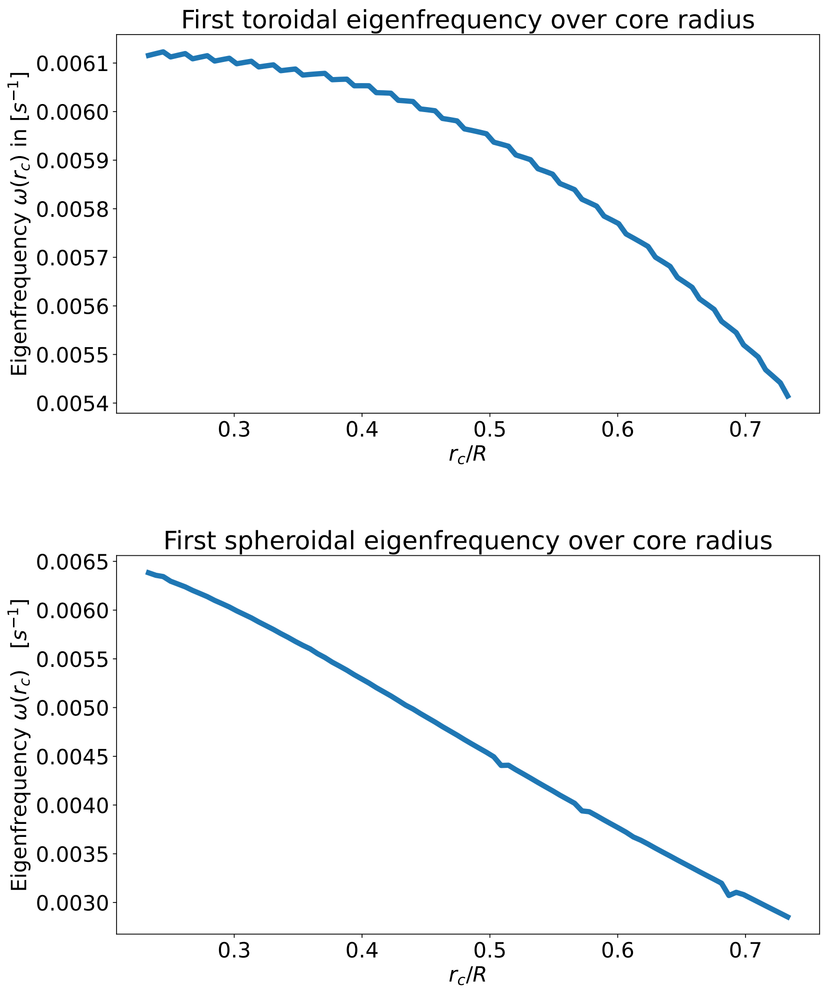

Figure 5.2.3: The variation of the first toroidal eigenfrequency (left) and the first spheroidal eigenfrequency (right) in s^{-1} on the core radius r_c for the M1* model

n	Model 1 Toroidal	Model 1 Spheroidal
0	0.00624 s^{-1} (4.672)	0.00665 s^{-1} (4.797)
1	0.0182 s^{-1} (13.648)	0.0116 s^{-1} (8.696)
2	0.0268 s^{-1} (20.071)	0.0184 s^{-1} (13.795)
3	0.0354 s^{-1} (26.508)	0.0250 s^{-1} (18.707)
n	**Model 1* Toroidal**	**Model 1* Spheroidal**
0	0.00613 s^{-1} (4.588)	0.00646 s^{-1} (4.838)
1	0.0182 s^{-1} (13.624)	0.0114 s^{-1} (8.559)
2	0.0268 s^{-1} (20.099)	0.0184 s^{-1} (13.781)
3	0.0356 s^{-1} (26.645)	0.0247 s^{-1} (18.524)

Table 5.2.2: Eigenfrequencies of the M1 and M1* models

and y_3 since they are of importance in the calculation of the total displacement response to the gravitational waves. One can see a small deviation where the dif-

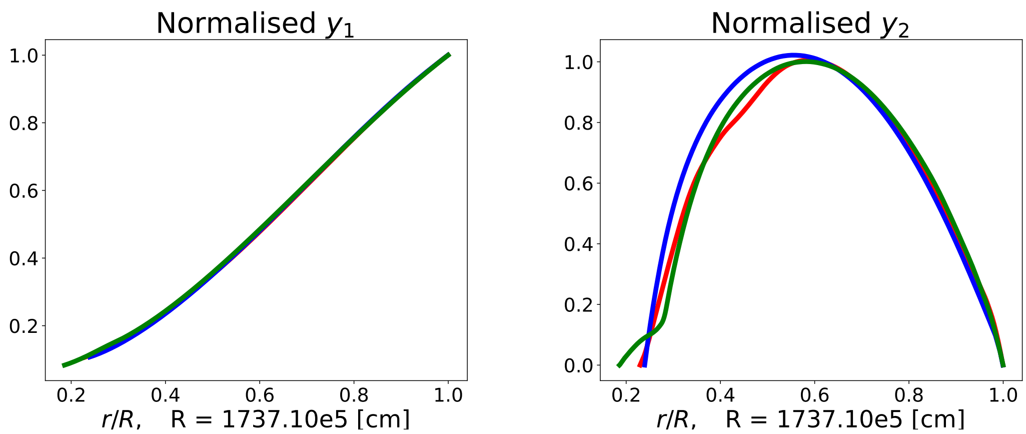

Figure 5.2.4: Comparison of the normalised toroidal parameter functions y_1 and y_2 for the first eigenfrequency of the three main Moon models considered. Moon model one in red, model two in blue, and model three in green.

ferent y_1 and y_2 functions start. This is because the core-mantle boundary has a slightly different value for the three models. In figure 5.2.4 we plotted the parameter functions for the three models for the first toroidal eigenfrequency keeping the colour coding of red for model 1, blue for model 2, and green for model 3. In figure 5.2.5 we plotted the parameter functions for the three models for the first spheroidal eigenfrequency keeping again the same colour code. We observe how the boundary conditions are indeed satisfied. We note that these functions have been normalized so that their behaviour can be clearly compared. The y_1 functions have been normalized so that their value is one at the surface, while y_2 have been normalized by their value at$r = R/2$, as their value at the surface is optimally zero making this an obviously bad normalization. From the right plot in figure 5.2.4 we see that the boundary conditions seem to be satisfied with $y_2 = 0$ for all models. These values are indeed very close to zero as $y_2^{(1)}(R) \approx 0.0002$, $y_2^{(2)}(R) \approx -7.5 \times 10^{-5}$ and $y_2^{(3)}(R) \approx 0.0002$ Figure 5.2.5 contains the parameter functions for the first eigenfrequency of the three Moon models. The functions y_1, y_3, y_5 and y_6 have been normalised at the surface of the Moon, while y_2 and y_4 are normalized at $r = R/2$ such that $y_2(R/2) = y_4(R/2) = 1$. We observe an agreement on the general behaviour of all the functions, although we also observe some deviations. The parameter function y_3 stands out, especially as the y_3 function for Moon model 3 looks very different from its Moon model 1 and 2 counterparts. This is mostly due to numerical sensitivity. It is interesting to compare if the boundary conditions for the three models agree with the plots. The values for the boundary conditions for the three models are summarised in table 5.2.3. Here BC1 corresponds to $y_2 = 0$, BC2 to $y_4 = 0$ and BC3 to $y_6 + l(l+1)y_5 = 0$. We observe that most predominantly for model 2 the boundary conditions are only loosely satisfied. For model 3 it is closer to the required boundary conditions, but still not nearly as close as for model 1. Increasing the accuracy of the eigenfrequency for Moon model 3 for example leads to a very small change in the value for the eigenfrequency, but the parameter functions are very sensitive to even a small change in the eigenfrequency.

	Model 1	**Model 2**	**Model 3**
BC1	0.000490	0.248127	-0.07140
BC2	-0.000340	-0.242149	0.032954
BC3	0.000336	-0.005468	-0.002932

Table 5.2.3: The boundary conditions for the first spheroidal eigenfrequencies constructed by y_2, y_4, y_5 and y_6 from figure 5.2.5

5.2.3 Displacement for the Moon

This section will present data on the displacement of three Moon models from our numerical integration procedure explained in Chapter Four. We assume a quality factor for all Moon models calculations for the first eigenfrequency $Q_0 = 3300$ taken from [14]. Studying the equation for the displacement (4.76) and (4.101) we observe that the vector part of these expressions does not directly depend on the models at the surface. We wish to compare the response of the models, so for this comparison, we use (5.8) for the toroidal response and (5.10) for the spheroidal case which is as we recall just the response of (4.76) and (4.101) without the vectorial part and the amplitude of the gravitational wave. For the spheroidal case. With the use of ξ_T and ξ_S we will compare the response of the different Models. We begin with the consideration of the toroidal response. By the argument of section 5.1.3 we know that the ${}_2T_2$ mode is of special interest and therefore the only mode we will consider. Figure 5.2.6 includes the toroidal displacement for the ${}_2T_2$ mode of Moon model One in spherical coordinates. The shape of the oscillation

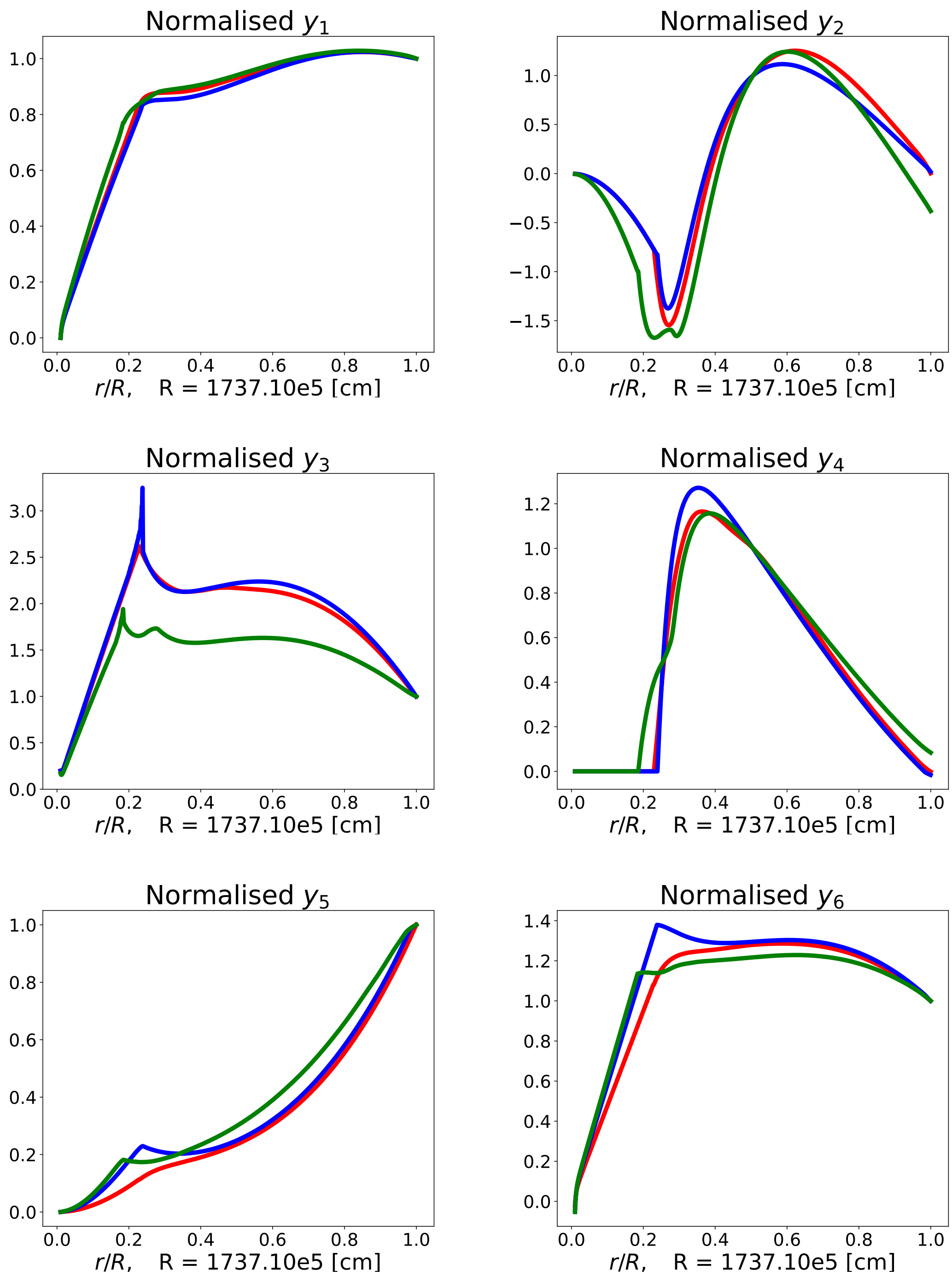

Figure 5.2.5: Comparison of the normalised spheroidal parameter functions y_1, y_2, y_3, y_4, y_5 and y_6 for the first eigenfrequency of the three main Moon models considered. Moon model one in red, model two in blue, and model three in green.

is to be expected from the standard ${}_2T_2$ oscillations. It is more interesting for our purposes of studying a potential measurement of gravitational wave resonance on the Moon to compare the displacement value for each of the models. We include the plots for the first model to give a visual idea of the predicted response but since the second and third Moon models will produce a similar shape response we do not plot these. The differences between the models are captured by ξ_T and ξ_S and their values determined by our numerical integrations are summarised in Table 5.2.4 The toroidal displacement shown in figure 5.2.6 shows a higher response than for the Earth model by a factor of two orders of magnitude. Comparing the toroidal values of model 2 and model 3 there is a certain agreement with Model 2 having a higher value of 1.530×10^7cm compared to 7.127×10^6cm for Model 1 and 7.520×10^6cm for Model 2. It is interesting to note this difference as comparing just Model 1 and Model 3 the difference is small. Looking back at the model parameters plot of figure 5.2.1 we do observe that the second model stands out compared to the other two models close to the surface, particularly for the second Lamé parameter $\mu(r)$. The second Lamé parameter is two orders lower than the value of the parameter for model 1 and model 3. This might be the source of this difference as observing that the parameter functions of the models agree quite well and we believe will have a negligible effect on the final displacement.

We proceed to the displacement from the spheroidal oscillations. By the same arguments as used for the Earth model, we are mainly interested in the ${}_2S_2$ and ${}_{-2}S_2$ mode. The ${}_{-2}S_2$ mode will give a similar contribution to the ${}_2S_2$ mode so we restrict our attention to the ${}_2S_2$ mode which is plotted in figure 5.2.7 for Moon model 1. Comparing the three Moon models we see in table 5.2.4 that the three models agree quite well on the value for ξ_S for the ${}_2S_2$ mode. We do observe the same trend as for the toroidal mode in that the response of model 2 is greater than the other models with model 3 greater than the model 1 response. The difference seems to be lower in the spheroidal case. We suspect this arises from the spheroidal dependence on the parameter values of the core and not just the mantle as in the toroidal case.

The last mode for discussion is the mode excited by the scalar polarised gravitational wave ${}_0S_2$. The response for the first model is plotted in spherical coordinates in figure 5.2.8. Comparing the values of ξ_S for the ${}_0S_2$ mode there is a similar pattern for the ${}_2S_2$ mode in that the greatest response is predicted by model 2, followed by model 3 with the weakest response from model 1. We also observe a smaller contribution for this mode in the same model as the ${}_2S_2$ mode. This is in agreement with the final discussion from Section 4.5. This difference is however not so large and the response is of a similar order. From our analysis, we therefore conclude that all the models agree on the order of magnitude for the displacement from gravitational wave resonance on the Moon. If one could measure the plus- and cross-polarisation state of a gravitational wave then one should also be able to detect the scalar polarisation given that it exists.

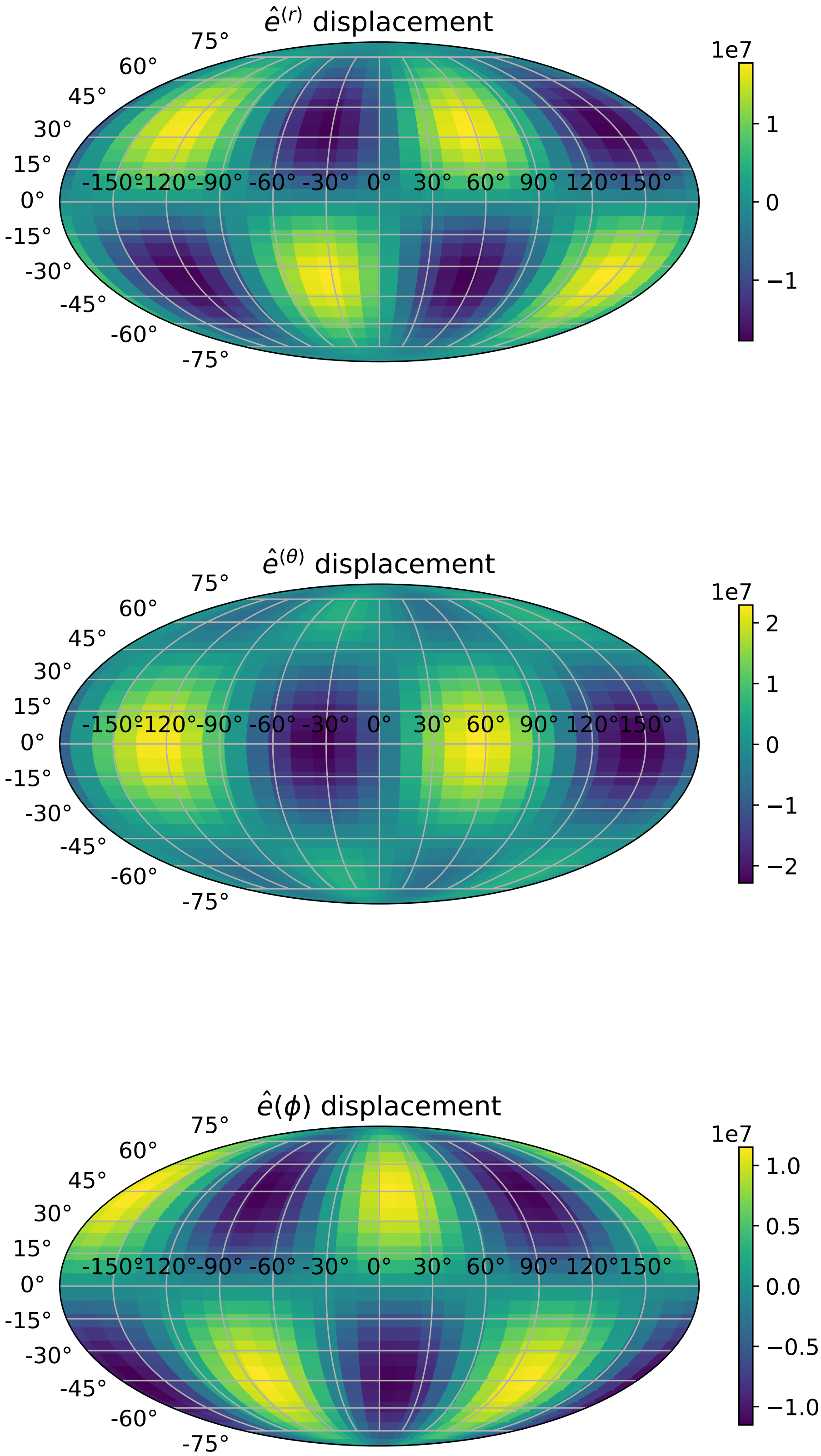

Figure 5.2.6: The ${}_2T_2$ displacement per unit strain $h_0 = 1$ of Moon model 1 to a plus and cross polarised gravitational wave in units of cm. The top figure is the displacement in the $\hat{e}^{(r)}$ direction, the middle figure in the $\hat{e}^{(\theta)}$ direction and the bottom figure in the $\hat{e}^{(\phi)}$ direction.

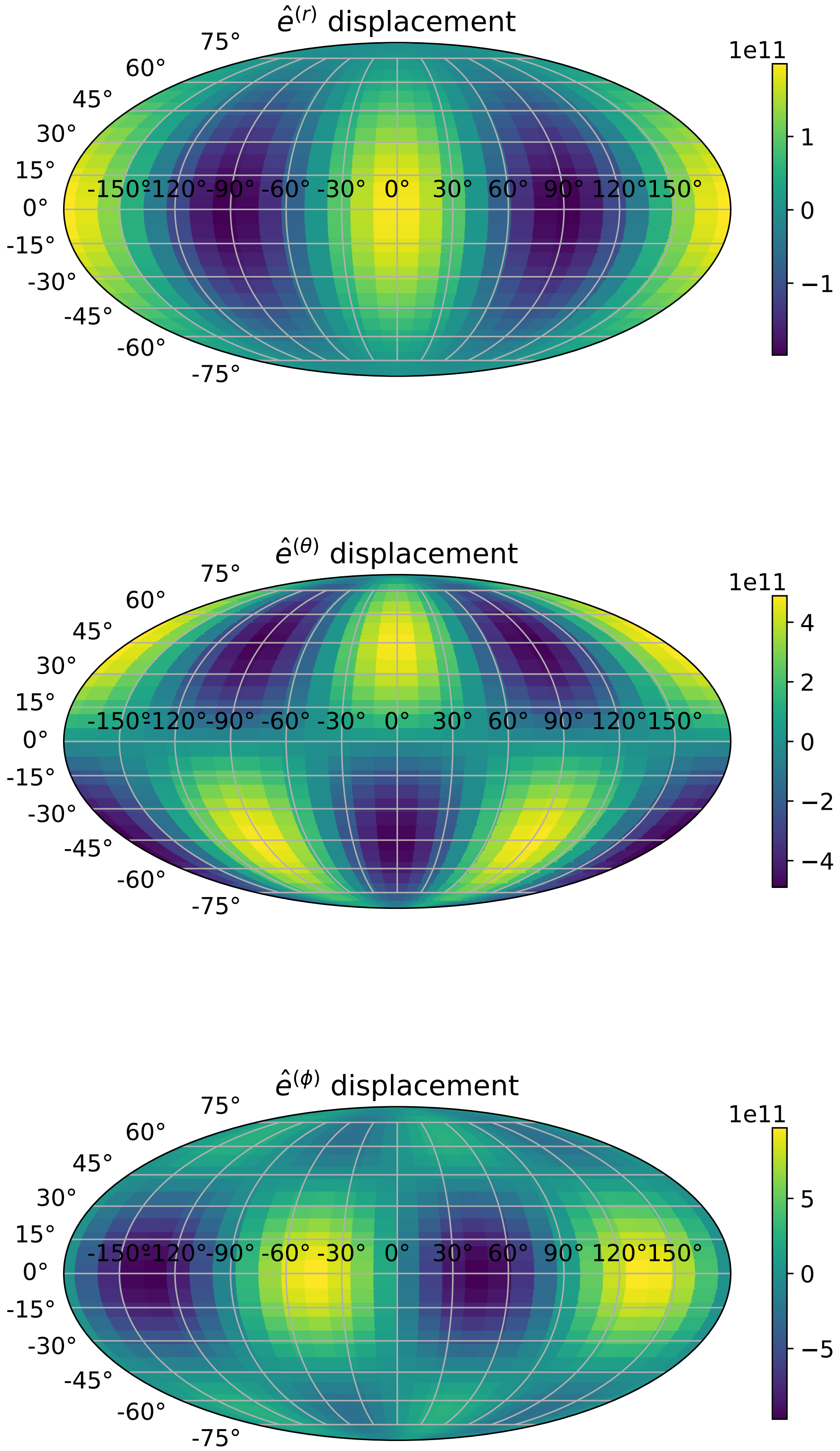

Figure 5.2.7: The ${}_2S_2$ displacement of Moon model 1 to a plus- and cross-polarised gravitational wave in units of cm. The top figure shows the displacement in the $\hat{e}^{(r)}$ direction, the middle figure shows the displacement in the $\hat{e}^{(\theta)}$ direction and the bottom figure shows the displacement in the $\hat{e}^{(\phi)}$ direction.

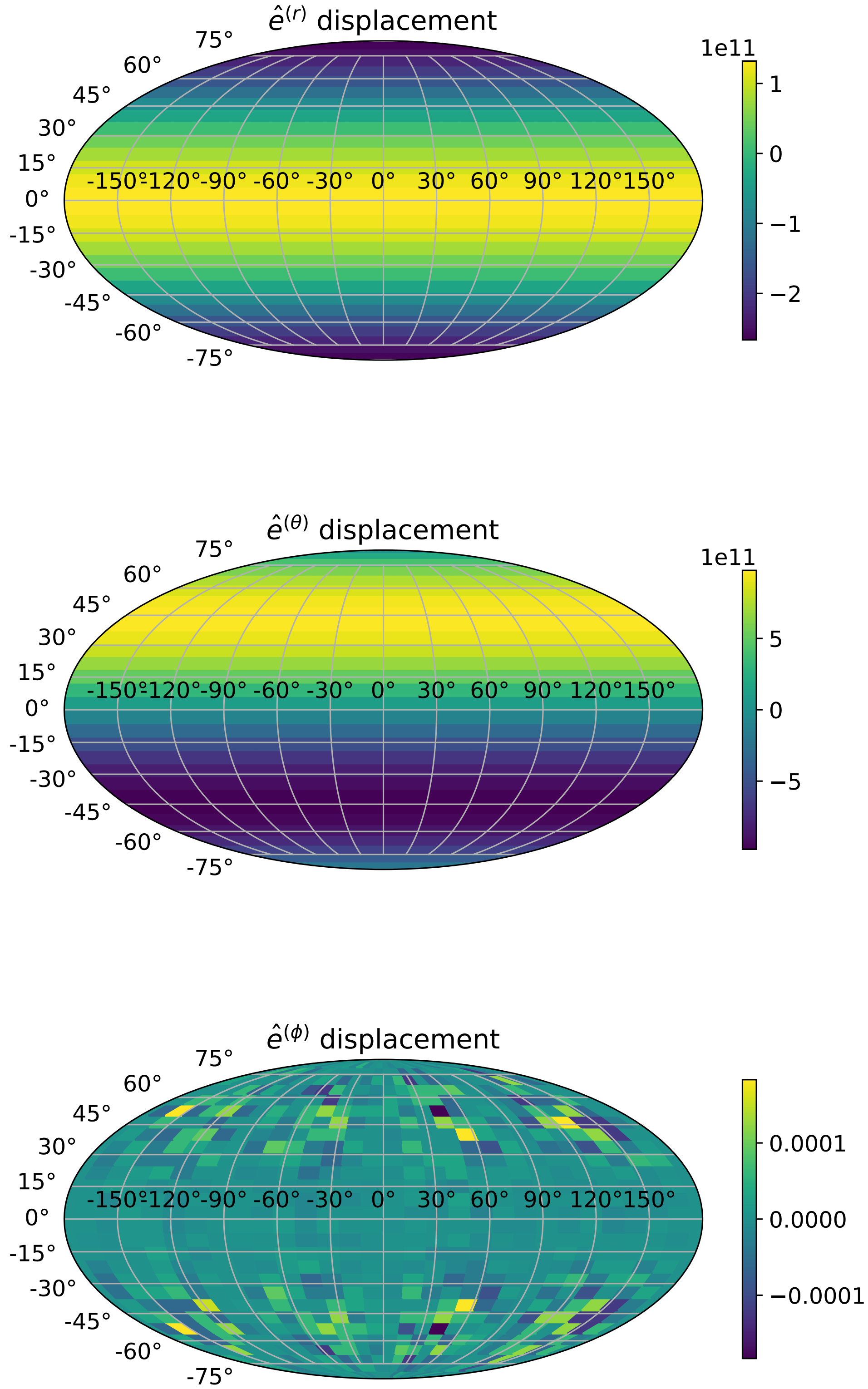

Figure 5.2.8: The ${}_0S_2$ displacement of Moon model 1 to a scalar polarised gravitational wave in cm. The top model shows the displacement in the $\hat{e}^{(r)}$ direction, the middle model shows the displacement in the $\hat{e}^{(\theta)}$ direction and the bottom model shows the displacement in the $\hat{e}^{(\phi)}$ direction.

Mode	Model One	Model Two	Model Three
${}_2T_2$	7.127×10^6cm	1.530×10^7cm	7.520×10^6cm
${}_2S_2$	3.261×10^{11}cm	5.642×10^{11}cm	4.814×10^{11}cm
${}_0S_2$	2.663×10^{11}cm	4.606×10^{11}cm	3.931×10^{11}cm

Table 5.2.4: ξ_T/h_0 and ξ_S/h_0 for the different Moon models of the ${}_2T_2$, ${}_2S_2$ and ${}_0S_2$ mode.

5.3 Total response over frequency

We have up to this point kept our focus mainly on the first eigenfrequency of the Moon models. Even though we have presented the first four eigenfrequencies we have calculated only the response of the Moon from this first eigenfrequency. It is interesting to see if the response changes as we move to other frequencies for the incoming gravitational wave. Since we can not guarantee that the frequency of the signal we hope to detect will exactly match the eigenfrequency of the Moon or any other spherical body. If we again assume a gravitational wave with the incoming vector $\mathbf{p} = (0, 0, 1)$, then the expression for the main contribution from spheroidal oscillations is

$$u_i^{(nml)}(\mathbf{r}, t) = h_0(\Lambda^{(nml)})^{-1}\bar{g}(t)Q_i^{*(nml)}(\mathbf{r})(F_{S_1} + F_{S_2}). \tag{5.13}$$

We now completely ignore h_0 considering only the response per unit strain. We are also not interested in the vectorial part of the expression, but in its magnitude and so the vector Q_i is ignored as well. We are left with $\Lambda, F_{S_1}, F_{S_2}$ and $\bar{g}(t)$. We define $\xi^{nml}(t) = (\Lambda^{(nml)})^{-1}\bar{g}(t)(F_{S_1} + F_{S_2})$. We are interested in the total response and we must therefore sum over the number of eigenfrequencies. We restrict the analysis to $l = 2$ and $m = 2$,

$$\xi_{tot}(t) = \sum_{n=0}^{\infty} \xi^{n22}(t). \tag{5.14}$$

We choose t such that the response is maximum. We make the assumption that the response from all the eigenfrequencies adds constructively. If we proceed to assume a signal of a finite monochromatic wave of $2T_0$ where T_0 is the eigenperiod for the first eigenfrequency then we have that the source effect on the eigenvibration is

$$\bar{g}(t) = \frac{1}{(\omega_n - \omega_{gw})^2 + i\omega_n^2/(4Q_n)}(H(t + 2T_0) - H(t - 2T_0)), \tag{5.15}$$

where the sinusoidal time dependence was omitted. We plot the gravitational response per unit strain as a function of the frequency of the incoming gravitational wave in figure 5.3.1. We choose again a quality factor of $Q_0 = 3300$ for the first eigenfrequency. For the other eigenfrequencies however we assume a behaviour of $Q_n \propto \omega_n^{-1}$. We have included the first 37 eigenfrequencies of the Moon model.

We observe a general trend of decreasing response at higher resonances and the distance between each resonance point decreasing as we move to higher frequencies in Figure 5.3.1. At frequencies smaller than the first eigenfrequency we

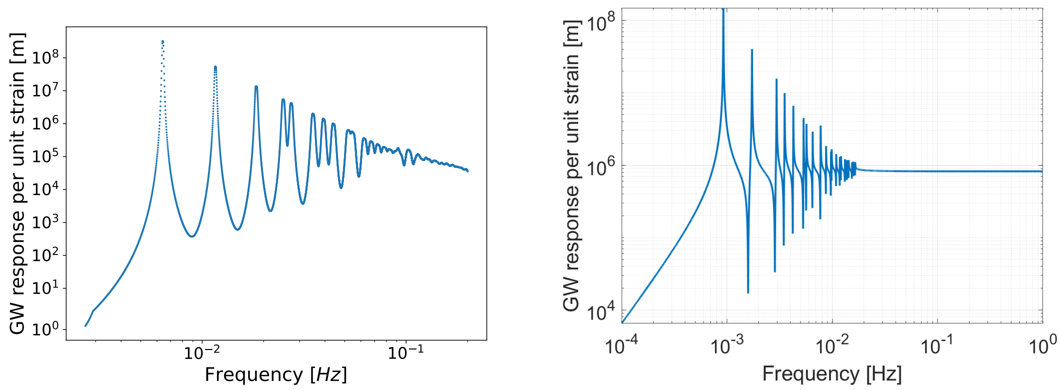

Figure 5.3.1: Comparison of the gravitational wave response between Moon model 1 and figure taken from [15]. The left figure is the response predicted by Moon model 1 having taken the first 37 eigenfrequencies into account. The right figure is the predicted response per unit strain taken from [15] where the first 22 eigenfrequencies are considered.

observe a tiny response. This implies that the detectability of a gravitational wave signal with a frequency less than millihertz frequency is unlikely from the normal modes predicted by the first Moon model. We also note on the trend of becoming broadband detector for higher frequencies. In figure 5.3.1 we observe at frequencies close to $\approx 10^{-1}$Hz a difficulty to distinguish peaks. The resonance detector behaviour transitions into a continuous response with a slow tendency for a lower response at even higher frequencies than what is plotted. The reason for the decrease in magnitude comes from the quality factor Q_n at higher n as we have in our model assumed that it evolves as $1/f^n$, where f^n is the n'th eigenfrequency, which eventually dies out as the frequency grows. We have also plotted for higher frequencies than what we have included eigenfrequencies. This means that for a certain frequency, we no longer have resonance in the calculations even though there will be on the Moon. This is a limitation from the numerical procedure.

We also wish to comment on a figure taken from [15] as it served as inspiration for the creation of 5.3.1 a. Firstly we note that two figures do agree to a certain extent for the degree of response. They do however differ when it comes to many of the eigenfrequencies and their general shape. Reference [15] uses a different source effect on the eigenvibration given by,

$$\bar{g}(t) = \frac{-\omega_{gw}^2}{\omega_n^2 - \omega_{gw}^2 + i\omega_n^2/Q_n}. \tag{5.16}$$

For the eigenfrequencies we seem to differ in the model from reference [15]. The first eigenfrequency for the first Moon model lies at $\omega_0 \approx 0.006$ Hz. The article has its first eigenfrequency at $\omega_0 \approx 0.001$ Hz however, which means we end up with a disagreement on the specific values of the eigenfrequencies. Most of the resonance part of the response for Harms et al. appears in the millihertz range which for the Moon model is in the 10^{-2}Hz range instead with only one eigenfrequency in the millihertz range. We also do not use the same quality factors for the eigenfrequencies. This will have an effect on the response amplitude and might

give an explanation on why we differ by one order for the first eigenfrequency.
We wish to make one remark when we discuss the figure. We must remember that we assumed in Chapter 4 when deriving the expression used for F_{S_1} and F_{S_2} that the frequency of the gravitational wave was $\simeq 0$ so that we could completely ignore all terms of the sum over the Wigner symbols except for the $l = 0$ term. As we move to higher frequencies these other terms become more and more relevant and should be included as we move to frequencies close to say 1 Hz. This is because the $l = 1$ term in (6.25) and (6.33) from the appendix becomes relevant as $\omega_{gw} R/c$ becomes large enough.

CHAPTER

SIX

DISCUSSIONS AND CONCLUSION

In this last chapter, we will give a summary of our results and make comments on any future work.

6.1 Summary

We have now seen how the Earth and Moon respond to gravitational waves in general relativity and Brans-Dicke gravity. Our calculations predict a maximum response of approximately $h_0 \times 10^{11}$ cm given a dimensionless amplitude of the gravitational wave h_0. It has been found that the response of spheroidal displacements is significantly higher than the toroidal displacement. This is due to the dominant first term in the series expansion of the integral for the displacement (4.102).

It has been determined that scalar polarised gravitational waves will excite a distinguishable normal mode from the modes excited by the plus- and cross-polarised gravitational waves. The displacement for the scalar polarisation is shown to be of the same order of magnitude as the displacement predicted by gravitational waves in general relativity. Possible detection of scalar polarised gravitational is therefore feasibly given that such a signal exists in nature and is of the same order of magnitude as the plus- and cross-polarisation of general relativity.

An analysis of the eigenfrequency dependence on the core radius of three current interior Moon models was performed [14]. It was found that the dependence is low and a significant change in the core radius would be required for a significant change to the eigenfrequencies.

It has also been shown that a spherical body response will depend on the frequency of the incoming gravitational wave. A gravitational wave with a frequency close to the first few eigenfrequencies gives a significantly higher response than the eigenfrequencies to the higher-order normal modes. We also note that the eigenfrequencies become more difficult to distinguish as we go to higher normal modes. The signal response of the Moon is predicted to possess resonance behaviour for lower frequencies and broadband behaviour for higher frequencies.

We conclude that using the Moon's normal modes as a detector for gravitational waves is determined probable and most sensitive in the gravitational wave frequency band of $\simeq 0.00665 \pm 0.0004$ Hz. It was also shown that a Moon detector would be a possible way to also look for theories of gravity beyond general relativity such as the scalar-tensor theory of Brans-Dicke.

6.2 Discussion and Future Work

Our knowledge of the interior of the Moon is for the moment limited. The placement of Lunar gravitational wave antennas on the Moon, therefore, has the potential to improve the accuracy of our current models. This is of great interest for both geology on the Moon and the potential LGWA detector [15]. An accurate model is of key interest for the purposes of calculating the response from gravitational waves and would provide more accurate predictions of the response. We do comment however that by our analysis of the model parameters, any small deviation from the three models used here will not produce a significantly higher response in orders of magnitude from the interaction with gravitational waves. It is perhaps still of interest to have a highly accurate model if one wishes to be able to put some restrictions on the parameters of a source.
We do, however, not see the accuracy of a Moon model as a big restriction for the possible detection of gravitational waves. With time the models of the Moon's interior will improve drastically. One could also compare the data with LISA given that these detectors will share some overlap of observational frequency sensitivity and the gravitational wave data of LISA could potentially help get even more accurate data for the interior of the Moon.

The dependence of the parameter functions on the different Lamé parameters could be of interest. The placement of seismometers on the Moon has the potential to greatly improve the models. Another comment is the time frame of scientific experiments on the Moon [16]. If we wish to detect gravitational waves using the normal modes on the Moon, then, as suggested in the paper [15] multiple locations on the Moon with seismometers are optimal. However, as the seismometers will be sensitive to seismic noise it is important to have locations in which other experiments or commercial projects do not interfere with the detection. It should therefore be taken into consideration which projects for Moon astronomy have the highest priority and if perhaps the LGWA detector should have early priority. The Moon is very quiet compared to Earth, but large projects which plan on using equipment or construction with the byproduct of seismic noise could be destructive for the potential deployment and future of an LGWA sensor on the Moon. It is of interest for future work to see if other projects on the Moon will interfere with a potential array of seismometers on the Moon. One project worth mentioning is the detection of the 21-cm line in cosmology and a potential detector on the far side of the Moon [17] as it is argued that the quiet nature of the dark side of the Moon is unique for the probing of the dark ages in cosmology.
It is crucial to have sensitive enough equipment for the detection of gravitational waves using the Moon. One possibility is cryogenic superconducting inertial sensors detailed in reference [18] as is also mentioned in reference [15]. The detector equipment is outside the scope of this thesis, but a mention of possible equipment for the realisation of a Lunar Gravitational wave antenna is worthwhile.

If time would have permitted it, we would have added a section on potential astrophysical objects which are of interest to the Moon detector. Reference [12], a paper on pulsars was a source of early inspiration for the thesis. A study of pulsars with potential for the detection of Brans-Dicke polarisations could provide interesting results and a restriction on certain parameters of the pulsars in question. We also include that other astrophysical objects could be of interest, however binary identical systems such as two orbiting black holes have a suppressed dipole emission [19]. One, therefore needs to look for more eccentric sources to hopefully detect the scalar polarisation.

We have in this thesis not taken rotation into account. The orbital period of the Moon could provide a unique opportunity for the measurements. Another comment is the effect of the eigenfrequencies on a rotating body. According to [3] we will get a degenerate number of eigenfrequencies given that the spherical body has a rotational frequency Ω that satisfies,

$$\frac{2\Omega^2}{4\pi G\rho} < 1. \tag{6.1}$$

This could be of interest for later research as this would have an effect on the differential equations system explained in Chapter 4 and used for the results in Chapter 5.

Certain sources such as [20] state that there are only five modes that are excited by gravitational waves in a spherical detector. We have shown that this is true as a first approximation for our simple expression of (4.114). It is interesting to consider if one could in theory have a spherical detector for which more modes are relevant. If one has to include higher order terms of the series of (4.110), then other modes would also give a response. This is determined by the combination of the frequency of the gravitational wave, the radius of the spherical detector, and the speed of light, $\omega_{gw}R/c$. If this combination could approach unity then one would observe more excited modes. For very big astrophysical objects this could be realized given high enough frequencies. This then requires that the eigenfrequencies of the mentioned object are much higher than what has been considered in this thesis. The theory and method used have made the assumptions of only a spherical body with a liquid core and solid mantle described by the density and the two Lamé parameters. One could therefore in theory study other spherical astrophysical objects for their response to gravitational wave resonance.

APPENDIX

A - F_T sum expansion

In this appendix, we will derive the expansion of F_T in terms of the Wigner symbols. The derivation is taken from appendix B from reference [3]. The Wigner symbols take a very complicated form when written out, but is calculation wise not very difficult. We use the following convention for the Wigner symbols,

$$\begin{aligned}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =& W \sum_n \frac{(-1)^{l_1-l_2-m_3+n}}{n!} \\ & \times \left(\frac{\Pi_{i=1}^3 (l_i+m_i)!(l_i-m_i)!}{(l_1+l_2-l_3-n)!(l_1-m_1-n)!(l_2+m_2-n)!} \right) \\ & \times (l_3-l_2+m_1+n)!(l_3-l_1-m_2+n)!,\end{aligned}$$

where,

$$W = \sqrt{\left(\frac{\Pi_{i=1}^3 (2p-2l_i)!}{(2p+1)!} \right)} \delta_{m_1+m_2-m_3}, \tag{6.3}$$

and $2p = l_1 + l_2 + l_3$. The sum goes over positive values for n where the sum stop when the denominator of the expression becomes negative. It is therefore not an infinite sum and consists of a finite number of terms. The Wigner symbols possess some very useful symmetries and rules to quickly determine if it is nonzero. The symbols are non-zero only when

$$m_1 + m_2 + m_3 = 0, \tag{6.4}$$

and,

$$|l_a - l_b| \leq l_c \leq |l_a + l_b|, \text{ where, } \quad a, b, c = 1, 2, 3. \tag{6.5}$$

Equations (6.4) and (6.5) are of great help in quickly determining if the Wigner symbols are zero. Especially if the Wigner symbols are included in a sum which will be the case for us. To arrive at an expansion of F_T includes rewriting the integral

$$I_{jk}^{(C)}(\mathbf{k}) = \sqrt{l(l+1)} \int_0^{2\pi} \int_0^{\pi} \hat{e}_j^{(r)} C_k^{(ml)} e^{-ik_i r^i} \sin\theta d\theta d\phi. \tag{6.6}$$

We begin by writing the $C_k^{(ml)}$ vector in spherical coordinates,

$$\sqrt{l(l+1)} C_j^{(ml)} = \left(\hat{e}_j^{(\theta)} \frac{1}{\sin\theta} \partial_\phi - \hat{e}_j^{(\phi)} \partial_\theta \right) Y_l^m. \tag{6.7}$$

Here we wish to make it clear that in Y_l^m, m and l are not tensor indices, but are related to the degree and order of the associated Legandre polynomials through,

$$Y_l^m = \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}. \tag{6.8}$$

We can write this vector in Cartesian components in the following way

$$\sqrt{l(l+1)} C_j^{(ml)} = a_j^{(0)} Y_l^m + a_j^{(1)} Y_l^{m+1} + a_j^{(2)} Y_l^{m-1}. \tag{6.9}$$

Where $a_j^{(0)}$, $a_j^{(1)}$ and $a_j^{(2)}$ is a combination of Cartesian unit vectors and given by

$$a_j^{(0)} = -im\hat{e}_j^{(z)}, \tag{6.10}$$
$$a_j^{(1)} = \frac{i}{2}\sqrt{(l-m)(l+m+1)}(\hat{e}_j^{(x)} - i\hat{e}_j^{(y)}), \tag{6.11}$$
$$a_j^{(2)} = \frac{i}{2}\sqrt{(l+m)(l-m+1)}(\hat{e}_j^{(x)} + i\hat{e}_j^{(y)}). \tag{6.12}$$

We now use the expansion of the exponential in terms of the Bessel function and Y_l^m,

$$e^{-ik_ir^i} = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (2l_1+1) i^{-l_1} j_{l_1}(kr) Y_{l_1}^{m_1}(\theta,\phi) Y_{l_1}^{*m_1}(e,\lambda). \tag{6.13}$$

If we now insert (6.9) and (6.13) into (6.6) then we get a new expression for the integral,

$$I_{jk}^{(C)}(\mathbf{k}) = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (2l_1+1) i^{-l_1} j_{l_1}(kr) Y_{l_1}^{*m_1}(e,\lambda) \left(A_j^{(0)} a_k^{(0)} + A_j^{(1)} a_k^{(1)} + A_j^{(2)} a_k^{(2)} \right), \tag{6.14}$$

where,

$$A_j^{(0)} = \int_0^{2\pi} \int_0^{\pi} \hat{e}_\alpha^{(r)} Y_{l_1}^{m_1} Y_l^m \sin\theta d\theta d\phi, \tag{6.15}$$
$$A_j^{(1)} = \int_0^{2\pi} \int_0^{\pi} \hat{e}_\alpha^{(r)} Y_{l_1}^{m_1} Y_l^{m+1} \sin\theta d\theta d\phi, \tag{6.16}$$
$$A_j^{(2)} = \int_0^{2\pi} \int_0^{\pi} \hat{e}_\alpha^{(r)} Y_{l_1}^{m_1} Y_l^{m-1} \sin\theta d\theta d\phi. \tag{6.17}$$

To progress we require the identity,

$$\int_0^{2\pi} \int_0^{\pi} Y_{l_1}^{m_1} Y_{l_2}^{m_2} Y_{l_3}^{m_3} = 4\pi \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{6.18}$$

Using this identity back into (6.14) we arrive at the result

$$\begin{aligned} I_{jk}^{(C)}(\mathbf{k}) =& 4\pi \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (2l_1+1) i^{-l_1} j_{l_1}(kr) Y_{l_1}^{*m_1}(e,\lambda) \begin{pmatrix} l_1 & l_2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ & \times \left(\gamma_j^{(1)} \hat{e}_k^{(z)} + \frac{(\gamma_j^{(2)} - \gamma_j^{(3)})}{\sqrt{2}} \hat{e}_k^{(x)} - \frac{(\gamma_j^{(2)} + \gamma_j^{(3)})}{\sqrt{2}} i\hat{e}_k^{(y)} \right). \end{aligned} \tag{6.19}$$

The $\gamma_j^{(i)}$ components is a linear combination of $a_j^{(i)}$ components which is a linear combination of the unit Cartesian vectors. If we write it all out then this becomes

$$\gamma_j^{(1)}\hat{e}_k^{(z)} + \frac{(\gamma_j^{(2)} - \gamma_j^{(3)})}{\sqrt{2}}\hat{e}_k^{(x)} - \frac{(\gamma_j^{(2)} + \gamma_j^{(3)})}{\sqrt{2}} i\hat{e}_k^{(y)} = A^{(0)}T^{(0)} + \frac{1}{2}A^{(1)}T^{(1)} + \frac{1}{2}A^{(-1)}T^{(-1)} - A^{(2)}T^{(2)} + A^{(-2)}T^{(-2)}. \quad (6.20)$$

Where $A^{(k)}$ contains the Wigner symbols and coefficients from the $\gamma_j^{(i)}$. The $T^{(k)}$ and $A^{(k)}$ are given as

$$T^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad T^{(\pm 1)} = \begin{pmatrix} 0 & 0 & \mp i \\ 0 & 0 & 1 \\ \mp i & 1 & 0 \end{pmatrix}, \qquad T^{(\pm 1)} = \begin{pmatrix} \mp i & 1 & 0 \\ 1 & \pm i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and,

$$\begin{aligned} A^{(0)} = -m \begin{pmatrix} l_1 & l_2 & 1 \\ m_1 & m_2 & 0 \end{pmatrix} - \\ - \frac{1}{2\sqrt{2}}\sqrt{(l+m)(l-m+1)} \begin{pmatrix} l_1 & l_2 & 1 \\ m_1 & m_2 - 1 & 1 \end{pmatrix} + \\ + \frac{1}{2\sqrt{2}}\sqrt{(l-m)(l+m+1)} \begin{pmatrix} l_1 & l_2 & 1 \\ m_1 & m_2 + 1 & -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} A^{(1)} = -2\frac{m}{\sqrt{2}} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m-1 & -1 \end{pmatrix} - \\ -\sqrt{(l+m)(l-m+1)} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m-1 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} A^{(-1)} = -2\frac{m}{\sqrt{2}} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m+1 & 1 \end{pmatrix} + \\ +\sqrt{(l-m)(l+m+1)} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m+1 & 0 \end{pmatrix}, \end{aligned}$$

$$A^{(2)} = \frac{1}{2\sqrt{2}}\sqrt{(l+m)(l-m+1)} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m-1 & -1 \end{pmatrix},$$

$$A^{(-2)} = \frac{1}{2\sqrt{2}}\sqrt{(l-m)(l+m+1)} \begin{pmatrix} l_1 & l & 1 \\ m_1 & m+1 & 1 \end{pmatrix}.$$

Inserting (6.20) into (6.19) is the result of this appendix and the expansion we were interested in.

B - F_{S_1} and F_{S_2} expansion

In this appendix, taken from appendix C of [3], we will simplify the integrals

$$I_{jk}^{(P)}(\mathbf{k}) = \sqrt{l(l+1)} \int_0^{2\pi} \int_0^{\pi} \hat{e}_j^{(r)} P_k^{(ml)}(\theta,\phi) e^{-ik_i r^i} \sin\theta d\theta d\phi, \tag{6.23a}$$

$$I_{jk}^{(B)}(\mathbf{k}) = \sqrt{l(l+1)} \int_0^{2\pi} \int_0^{\pi} \hat{e}_j^{(r)} B_k^{(ml)}(\theta\phi) e^{-ik_i r^i} \sin\theta d\theta d\phi. \tag{6.23b}$$

We do one integral at the time. We begin with the P integral. In spherical coordinates $P_k^{(ml)}(\theta,\phi)$ takes on the form,

$$P_j^{(ml)} = \hat{e}_j^{(r)} Y_l^m(\theta,\phi), \tag{6.24}$$

We apply the expansion of the exponential (6.13). We then get the product of $\hat{e}_j^r \hat{e}_k^r$ which written in terms of Certesian unit coordinates and the associated Legandre polynomial is

$$\begin{aligned}
\hat{e}_j^{(r)}\hat{e}_k^{(r)} =& \frac{1}{3}\delta_{jk} + \frac{1}{3}Y_2^0(\hat{e}_j^{(x)}\hat{e}_k^{(x)} - \hat{e}_j^{(y)}\hat{e}_k^{(y)} + 2\hat{e}_j^{(z)}\hat{e}_k^{(z)}) \\
&+ \frac{1}{\sqrt{6}}Y_2^1\left(\hat{e}_j^{(z)}(\hat{e}_k^{(x)} - i\hat{e}_k^{(y)}) + \hat{e}_k^{(z)}(\hat{e}_j^{(x)} - i\hat{e}_j^{(y)})\right) \\
&- \frac{1}{\sqrt{6}}Y_2^{-1}\left(\hat{e}_j^{(z)}(\hat{e}_k^{(x)} + i\hat{e}_k^{(y)}) + \hat{e}_k^{(z)}(\hat{e}_j^{(x)} + i\hat{e}_j^{(y)})\right) \\
&+ \frac{1}{\sqrt{6}}Y_2^2(\hat{e}_j^{(x)} - i\hat{e}_j^{(y)})(\hat{e}_k^{(x)} - i\hat{e}_k^{(y)}) + \frac{1}{\sqrt{6}}Y_2^{-2}(\hat{e}_j^{(x)} + i\hat{e}_j^{(y)})(\hat{e}_k^{(x)} + i\hat{e}_k^{(y)}).
\end{aligned}$$

We employ the identity (6.18) to get

$$\begin{aligned}
I_{ik}^{(P)}(\mathbf{k}) =& 4\pi \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} (2l_1+1) i^{-l_1} j_{l_1}(kr) Y_{l_1}^{*m_1}(e,\lambda) \begin{pmatrix} l_1 & l_2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \\
&\cdot \left(\sum_{j=-2}^{2} \Gamma^{(j)} \begin{pmatrix} l_1 & l & 2 \\ m_1 & m & j \end{pmatrix} \right)
\end{aligned} \tag{6.25}$$

Where the $\Gamma^{(j)}$ is a combination of cartesian unit vectors,

$$\Gamma^{(0)} = \frac{1}{3}(-\hat{e}_j^{(x)}\hat{e}_k^{(x)} - \hat{e}_j^{(y)}\hat{e}_k^{(y)} + 2\hat{e}_j^{(z)}\hat{e}_k^{(z)}), \tag{6.26}$$

$$\Gamma^{(1)} = \frac{1}{\sqrt{6}}\left(\hat{e}_j^{(z)}(\hat{e}_k^{(x)} - i\hat{e}_k^{(y)}) + (\hat{e}_j^{(x)} - i\hat{e}_j^{(y)})\hat{e}_k^{(z)}\right), \tag{6.27}$$

$$\Gamma^{(-1)} = \frac{1}{\sqrt{6}}\left(\hat{e}_j^{(z)}(\hat{e}_k^{(x)} + i\hat{e}_k^{(y)}) + (\hat{e}_j^{(x)} + i\hat{e}_j^{(y)})\hat{e}_k^{(z)}\right), \tag{6.28}$$

$$\Gamma^{(2)} = \frac{1}{\sqrt{6}}(\hat{e}_j^{(x)} - i\hat{e}_j^{(y)})(\hat{e}_k^{(x)} - i\hat{e}_k^{(y)}), \tag{6.29}$$

$$\Gamma^{(-2)} = \frac{1}{\sqrt{6}}(\hat{e}_j^{(x)} + i\hat{e}_j^{(y)})(\hat{e}_k^{(x)} + i\hat{e}_k^{(y)}). \tag{6.30}$$

We have now arrived at a simpler expression to evaluate for the P vector integral in our displacement relation. We move on to write the B vector integral as a sum over Wigner symbols as well. We begin with the relation

$$\sqrt{l(l+1)}B_j^{(ml)} = \epsilon_{ikj}\hat{e}_i^{(r)}C_k^{(ml)}. \tag{6.31}$$

Then we get with the expansion of $B_\nu^{(ml)}$ in terms of (6.9),

$$\begin{aligned}\sqrt{l(l+1)}\hat{e}_\mu^{(r)}B_\nu^{(ml)} =& \hat{e}_\mu^{(r)}(\epsilon_{\alpha\beta\nu}\hat{e}_\alpha^{(r)}a_\beta^{(0)})Y_l^m + \hat{e}_\mu^{(r)}(\epsilon_{\alpha\beta\nu}\hat{e}_\alpha^{(r)}a_\beta^{(1)})Y_l^{m+1}. \\ &+ \hat{e}_\mu^{(r)}(\epsilon_{\alpha\beta\nu}\hat{e}_\alpha^{(r)}a_\beta^{(-1)})Y_l^{m-1}\end{aligned} \tag{6.32a}$$

We have simplified our calculation to studying the products $\hat{e}_\mu^{(r)}(\epsilon_{\alpha\beta\nu}\hat{e}_\alpha^{(r)}a_\beta^{(i)})$. If we use the representation from before with Γ then we get five different types of tensors,

1. Tensors which are zero,
2. Antisymmetric tensors which will give zero when contracted with the polarisation tensor,
3. Tensors which has symmetric parts,
4. Nonsymmetric tensors,
5. Symmetric tensors.

We can therefore remove the tensors of the type 1) and 2) and the antisymmetric parts of 3) and 4). We keep the rest and the integral over the B vector can therefore be written as

$$\begin{aligned}I_{ik}^{(B)}(\mathbf{k}) = 4\pi \sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1}(2l_1+1)i^{-l_1}j_{l_1}(kr)Y_{l_1}^{*m_1}(e,\lambda)\begin{pmatrix} l_1 & l & 0 \\ 0 & 0 & 0\end{pmatrix}\cdot \\ \cdot\left(\sum_{j=-2}^{2}D^{(j)}\Gamma_{ik}^{(j)}\right).\end{aligned} \tag{6.33}$$

Where the $D^{(j)}$ is given as

$$\begin{aligned}
D^{(0)} &= -\frac{3}{2\sqrt{6}}\sqrt{(l-m)(l+m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m+1 & -1 \end{pmatrix} \\
&\quad -\frac{3}{2\sqrt{6}}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m-1 & 1 \end{pmatrix}, \\
D^{(1)} &= -\frac{m}{2}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m & 1 \end{pmatrix} - \frac{3}{2\sqrt{6}}\sqrt{(l-m)(l+m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m+1 & 0 \end{pmatrix} \\
&\quad -\frac{1}{2}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m-1 & 2 \end{pmatrix}, \\
D^{(-1)} &= -\frac{m}{2}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m & -1 \end{pmatrix} + \frac{3}{2\sqrt{6}}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m-1 & 0 \end{pmatrix} \\
&\quad +\frac{1}{2}\sqrt{(l-m)(l+m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m+1 & -2 \end{pmatrix}, \\
D^{(2)} &= -m\begin{pmatrix} l_1 & l & 2 \\ m_1 & m & 2 \end{pmatrix} - \frac{1}{2}\sqrt{(l-m)(l+m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m+1 & 1 \end{pmatrix}, \\
D^{(-2)} &= m\begin{pmatrix} l_1 & l & 2 \\ m_1 & m & -2 \end{pmatrix} - \frac{1}{2}\sqrt{(l+m)(l-m+1)}\begin{pmatrix} l_1 & l & 2 \\ m_1 & m-1 & -1 \end{pmatrix},
\end{aligned}$$

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