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Abstract Homotopy Theory of Smooth Manifolds

Bachelor's thesis in BMAT
Supervisor: Gereon Quick
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Abstract

In this thesis we present the construction of a homotopy theory for smooth manifolds in a way which mimics Morel and Voevodsky's construction of the $\mathbb{A}^1$-homotopy theory for schemes. This is based on the works of Dugger, Isaksen, Jardine, Morel, Quillen, Voevodsky and many others.

Acknowledgements

Firstly, I want to thank my supervisor Gereon for his feedback and explanations, and for helping me understand the theory. I would not have gotten this far without his aid.
I also want to thank everyone who has gone through the trouble of reading this and giving feedback. A special thanks goes to my friend Håkon, who allegedly was up until 03:30 on a Sunday reading this thesis and pointing out my grammatical errors. My friend Maximilian also deserves an extra thanks, for letting me read his (at the time work in progress) thesis when I needed inspiration and for giving me his opinions on how a mathematical text should be formatted.

Contents

Chapter I

Introduction

Compared to topological spaces smooth manifolds are rather rigid objects. This makes homotopical constructions in the category of smooth manifolds, denoted $\mathbf{Man}^{\infty}$, much more difficult or even impossible. Thus, the task of creating a homotopy theory for smooth manifolds becomes quite challenging. Our goal for this thesis is to present the construction of an abstract homotopy theory for smooth manifolds. We will use the theory of model categories, simplicial sets, Grothendieck topologies and descent, amongst other things, to achieve this.

We now give a quick overview of the construction. We want the abstract homotopy theory to mimic the homotopy theory of topological spaces. However, given the vast difference in the structures of topological spaces and smooth manifolds, in terms of their defining properties, we cannot simply start constructing and expect to get something remotely close to the homotopy theory of topological spaces. We have to somehow alter the category of smooth manifolds in a way which sufficiently resembles the category of topological spaces. The problem with smooth manifolds is that we cannot make any and all constructions and still expect to get a smooth manifold. To be specific, the category $\mathbf{Man}^{\infty}$ is not both complete and cocomplete. The way we fix is this is by embedding $\mathbf{Man}^{\infty}$ into the category of presheaves on $\mathbf{Man}^{\infty}$, $\mathbf{Pre}(\mathbf{Man}^{\infty})$, which we know is complete and cocomplete.
The problem we now face is that presheaves "forget" geometry, as in certain colimits and limits which already exist in $\mathbf{Man}^{\infty}$ are altered when going to $\mathbf{Pre}(\mathbf{Man}^{\infty})$. An example of this is taking the disjoint union of two n-dimensional manifolds, $M_1 \coprod M_2$. We can show that the Yoneda embeddings

$$r(M_1 \coprod M_2) := \mathrm{Hom}_{\mathbf{Man}^{\infty}}(-, M_1 \coprod M_2)$$

and

$$rM_1 \coprod rM_2 := \mathrm{Hom}_{\mathbf{Man}^{\infty}}(-, M_1) \coprod \mathrm{Hom}_{\mathbf{Man}^{\infty}}(-, M_2)$$

behave wildly different by considering maps from $rS^0 := \mathrm{Hom}_{\mathbf{Man}^{\infty}}(-, S^0)$ into each object. We have

$$\mathrm{Hom}_{\mathbf{Pre}(\mathbf{Man}^{\infty})}(rS^0, r(M_1 \coprod M_2)) \cong r(M_1 \coprod M_2)(S^0)$$

by the Yoneda lemma, and this corresponds exactly to picking out two points in the disjoint union.

On the other hand,

$$\mathrm{Hom}_{\mathbf{Pre(Man^\infty)}}(rS^0, rM_1 \coprod rM_2) \cong (rM_1 \coprod rM_2)(S^0)$$

corresponds to picking out four points.
We try to fix this by taking into account local data. To do this we give $\mathbf{Man}^\infty$ a Grothendieck topology and consider sheaves on $\mathbf{Man}^\infty$. There is again another problem with this. Model category theory tells us that in certain cases we can induce a model structure on functor categories when our target category is a nice enough model category. The category of sheaves on $\mathbf{Man}^\infty$ is a functor category that lands in **Set**, but there are very few interesting model structures on **Set**. Furthermore, they don't really have a strong connection to the model structures of **Top**, something we want in our final model structure. It turns out that the correct notion to consider is that of simplicial presheaves on $\mathbf{Man}^\infty$, denoted $\mathbf{sPre}(\mathbf{Man}^\infty)$, which is the category of contravariant functors $\mathbf{Man}^\infty \to \mathbf{sSet}$. We will see that this is a step in the right direction, but still not enough as the object which corresponds to $\mathbb{R}$ in $\mathbf{sPre}(\mathbf{Man}^\infty)$ is not contractible. To try and fix this, we recall our Grothendieck topology on $\mathbf{Man}^\infty$ and bring Čech descent into the picture, which we will think of as a generalization of the sheaf condition.
This is still not enough, but has at least brought the geometry of $\mathbf{Man}^\infty$ back into the picture. All that is left to do is to contract $\mathbb{R}$. Our abstract homotopy theory is then ready once we have done this.

Overview

This thesis consists of three chapters:

Ch. 2 introduces the basics of model categories. We define model categories, left and right homotopies and the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$. Furthermore, we explain that this category is equivalent to the localization of $\mathcal{C}$ at the class of weak equivalences and prove Whiteheads theorem for model categories. Lastly, we define Quillen adjunctions and equivalences, along with derived functors.

Ch. 3 introduces the basics of simplicial sets. This chapter start with the basic definitions and aims to build up our intuition via examples and properties. We then start working our way towards the Kan-Quillen model structure, while taking a quick informal detour into simplicial homotopy groups. We end the chapter with simplicial model categories.

Ch. 4 is the chapter where everything falls in place. Here we expand on what was said in the introduction using the theory from chapters 2 and 3, while introducing and explaining the remaining theory. At every step of the way we aim to explain the problem we face and how we aim to solve it.

The reader will notice that chapters 2 and 3 have their own motivation section. This is done because we feel that the topics discussed in the respective chapters are interesting enough to be studied on their own. We therefore try to motivate them without mention of the homotopy theory of smooth manifolds.

Lastly, this thesis assumes familiarity with basic category theory, homological algebra and basic algebraic topology. It is also a benefit to know the definition of a smooth manifold, but this is not a must.

Regarding the Difficulty of the Theory

As the theory used in this thesis becomes increasingly more difficult to understand, I have had no choice other than to not include the proofs of many propositions and lemmas. Instead I have opted to tell a "story" with the focus being geometric intuition. This is especially evident in chapter 4, section 2, where the entirety of Bousfield localization had to be "blackboxed". This is not to say that there are no proofs provided. Many of the proofs in chapter 2 are my own, some of which I spent countless hours on (for example parts of theorem II.3.10). This is mostly due to my main source on model categories ([Bal21]) being a reference book for those already familiar with model categories, something I found out a little too late. I have however gone through other sources, such as [Hov99] and [Hir03], and checked that my proofs on model categories are correct. Furthermore, the proof of theorem III.4.1 is my own attempt to fill in missing details of the proof found in [GJ09], and the proof of proposition IV.1.2 is my own.
I also admit that there is still a lot of theory which I don't completely understand, but I am nonetheless proud of this thesis and the progress that I have made in a single semester. I have also learned a lot of new and interesting mathematics, which is maybe more important than the thesis itself.

Chapter II

Model Categories

II.1 Motivation

One of the categories studied in homological algebra is the derived category $\mathcal{D}(\mathcal{A})$ of some abelian category $\mathcal{A}$. This category is often constructed using equivalence classes of *roofs* between chain complexes and it is possible to show that this construction inverts quasi-isomorphisms. In fact, this is one of the main reasons we care about the derived category: We would like to study chain complexes up to homology and forcing quasi-isomorphisms to become isomorphisms is one way of doing this.

Another example of a similar idea is studying topological spaces up to some algebraic invariant. Common algebraic invariants include (co)homology (with various coefficients) and homotopy groups. It is well known that none of these algebraic invariants can differentiate between weakly equivalent spaces, i.e. spaces $X, Y \in \mathbf{Top}$ such that there exists a map $f : X \to Y$ that induces isomorphisms on all homotopy groups. One could thus argue that we are actually studying the category of topological spaces up to weak equivalence.

More generally, let's say that we are working in some category $\mathcal{C}$ and we are given a class of morphisms W which we would like to invert. We could form the *localization* of $\mathcal{C}$ with respect to all morphisms in W, which (if exists) is a new category $\mathcal{C}[W^{-1}]$ along with a functor $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which satisfies certain universal properties[1]. In this new category, all morphisms in W are sent to isomorphisms in $\mathcal{C}[W^{-1}]$ and one way of constructing this category is seen in definition 1.2.1 in [Hov99][2]. However we see that for an arbitrary category, the resulting morphisms (often called *zig-zags*) may be too long and as such our Hom-sets may not actually be sets. This means that $\mathcal{C}[W^{-1}]$ may in some cases not even be a category. Another problem is that the zig-zags are notoriously difficult to describe. There are of course nice and simple cases, such as the derived category, but even there one must first assume that the collection of equivalence classes of roofs between any two chain complexes is a set. How do we work our way around this? One way could be to impose restrictions on our class W: We can for example set rules for how morphisms in W should interact with other classes of morphisms. This is exactly what is done when defining model categories, and we will see that the properties of model categories lead to a lot of interesting and powerful theory.

[1] Depending on the author, the universal properties might be taken as definition or simply as consequences from a different definition. See f.ex. [Hov99] definition 1.2.1 and lemma 1.2.2.

[2] This is more or less the same process as when constructing the derived category.

In addition, we will see that model categories gives a nice framework for abstracting homotopy theory to categories which at first seem unsuitable for such notions.

II.2 Definitions and Basic Properties

We start our journey through model category theory with some basic definitions.

Definition II.2.1 (Retracts)**.** A morphism $f : A \to B$ in a category $\mathcal{C}$ is a *retract* if and only if there exists a commutative diagram

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_A}{\frown} & & \\ A & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \\ & & \underset{\mathrm{id}_B}{\smile} & & \end{array} \tag{II.1}$$

One sees that such retracts of morphisms aim to mimic topological retracts. For those that know about the arrow category, retracts of morphisms in $\mathcal{C}$ are retracts in the arrow category of $\mathcal{C}$.

Definition II.2.2 (Model Category)**.** A *model category* is a category $\mathcal{C}$ with three distinguished classes of morphisms:

- Weak Equivalences - $W_{\mathcal{C}}$
- Fibrations - $\mathrm{Fib}_{\mathcal{C}}$
- Cofibrations - $\mathrm{Cof}_{\mathcal{C}}$

which are all closed under composition. A morphism which is both a weak equivalence and a fibration (resp. cofibration) is called an *acyclic fibration* (resp. *acyclic cofibration*). The three classes and the category $\mathcal{C}$ must satisfy the following axioms:

MC1: $\mathcal{C}$ has all small limits and colimits. Thus, $\mathcal{C}$ has both an initial and terminal object, denoted $\emptyset$ and $*$ respectively.

MC2: If f, g are two composable morphisms and if two of f, g, gf are weak equivalences, then so is the third. This is known as the 2-of-3 property.

MC3: $W_{\mathcal{C}}$, $\mathrm{Fib}_{\mathcal{C}}$ and $\mathrm{Cof}_{\mathcal{C}}$ are closed under retracts.

MC4: Given the solid part of a commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{II.2}$$

a lift, i.e. the dotted arrow h making everything commute, exists either when i is a cofibration and p is an acyclic fibration or i is an acyclic cofibration and p is a fibration.

MC5: Each morphism $f \in \mathcal{C}$ can be factored as $f = pi$ where either

1. i is a cofibration and p is an acyclic fibration
   or
2. i is an acyclic cofibration and p is a fibration

Remark II.2.3. We also assume that model categories admit *functorial factorizations*, i.e. that the factorization in **MC5** is functorial. See definition 1.1.1 in [Hov99]. Also, some authors prefer *trivial* instead of acyclic. Lastly, a *model structure* on a category $\mathcal{C}$ is the data of weak equivalences, cofibrations and fibrations satisfying all model category axioms except for possibly **MC1**.

It is important to get comfortable with the idea of lifts in diagrams of the form II.2, as they will play a major role in the theory going forwards. In fact, many model category structures are given by defining a class of morphisms (often acyclic cofibrations/fibrations) and then generating the other class using lifting properties, see example II.2.12. We will also encounter this idea in the next chapter.

Definition II.2.4. Given the solid part of a commutative diagram of the form II.2, we say i has the *left lifting property* (LLP) with respect to p if a lift exists. Similarly, p has the *right lifting property* (RLP) with respect to i if a lift exists.

Definition II.2.5. Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$. Then X is:

- *fibrant* if the map $X \to *$ is a fibration
- *cofibrant* if the map $\emptyset \to X$ is a cofibration
- *bifibrant* if it is both fibrant and cofibrant

Using the factorization provided by **MC5** we can factor the map $X \to *$ as $X \to Y \to *$ where we can choose $Y \to *$ to be a fibration, making Y a fibrant object and forcing $X \to Y$ to be an acyclic cofibration. Similarly for $\emptyset \to X$. Functorial factorization gives us a canonical choice of Y and maps $X \to Y, Y \to *$. This leads us to define the following:

Definition II.2.6. Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$.

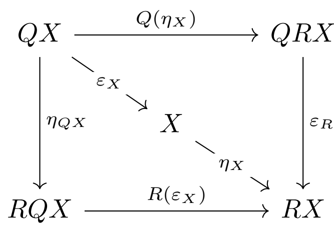
- A *fibrant replacement* of X is a fibrant object RX with an acyclic cofibration $X \to RX$
- A *cofibrant replacement* of X is a cofibrant object QX with an acyclic fibration $QX \to X$

Note that many such replacements may exist, but by the above discussion we more or less always choose it to be the objects and maps provided by the functorial factorization. The topologically minded reader might want to compare cofibrant replacement with the concept of CW-approximation.

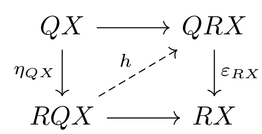
Proposition II.2.7. *The fibrant replacement of a cofibrant replacement RQX is still cofibrant. Dually for the cofibrant replacement of a fibrant replacement QRX. We thus have bifibrant replacements and there is a weak equivalence $RQX \to QRX$*

Proof. We first show that RQX is cofibrant (that is $\iff \emptyset \to RQX$ is a cofibration). We have by definition an acyclic cofibration $QX \to RQX$ and QX is cofibrant so the unique map $\emptyset \to QX$ is also a cofibration. Composing these two results in the unique map $\emptyset \to RQX$ and since all three classes of morphisms in $\mathcal{C}$ are closed under composition, $\emptyset \to RQX$ is a cofibration. Thus, the fibrant replacement of a cofibrant replacement is still cofibrant. Similarly for QRX.

The proof of there existing a weak equivalence $RQX \to QRX$ is not too difficult, however it requires one to really think of fibrant/cofibrant replacements as functors (we can do this by functorial factorization). Thus we have natural transformations $Q \underset{\varepsilon}{\Rightarrow} \mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}} \underset{\eta}{\Rightarrow} R$. This gives us a diagram of the form



By naturality of η (resp. ε), the lower (resp. the upper) triangle commutes. So the entire diagram commutes. This implies that the diagram below commutes and since $\eta_{QX} \in W_{\mathcal{C}} \cap \mathrm{Cof}_{\mathcal{C}}$ and $\varepsilon_{RX} \in W_{\mathcal{C}} \cap \mathrm{Fib}_{\mathcal{C}}$ we have a lift $h : RQX \to QRX$ by **MC4**. Note that ε_{RX} and η_{QX} are precisely the maps from an object to its (co)fibrant replacement as in definition II.2.6, so they really are weak equivalences.



By commutativity, $\varepsilon_{RX} Q(\eta_X) = \eta_X \varepsilon_X$ and both $\eta_{RX}, \eta_X \varepsilon_X \in W_{\mathcal{C}}$ so by 2-of-3 (**MC2**) $Q(\eta_X)$ is also a weak equivalence. Then again by 2-of-3, $h : RQX \to QRX$ is also a weak equivalence for $\eta_{QX} \in W_{\mathcal{C}}$. ■

The next step is to try and understand how these three classes of morphisms interact with each other. To do this we need a lemma.

Lemma II.2.8 (Retract Argument)**.** *Suppose we have a factorization $f = pi$ in a category $\mathcal{C}$ and that f has the LLP with respect to p. Then f is a retract of i. Dually, we have that if f has the RLP with respect to i them f is a retract of p*

The proof of the retract argument is found on page 5 in [Hov99]. Now we state an important connection between the different classes of morphisms.

Proposition II.2.9. *In a model category $\mathcal{C}$, a map is a cofibration (acyclic cofibration) if and only if it has the LLP with respect to all acyclic fibrations (fibrations). The dual statement holds for fibrations (acyclic fibrations) and acyclic cofibrations (cofibrations).*

Proof. We prove that a map f is a cofibration if and only if it has the LLP to all acyclic fibrations. The other cases are similar.
$\underline{\Rightarrow}$: By **MC4** (i.e. the lifting axiom) a lift exist for any acyclic fibration p.
$\underline{\Leftarrow}$: Suppose f has the LLP with respect to all acyclic fibrations. Factor $f = pi$ such that i is a cofibration and p is an acyclic fibration. Thus f has the LLP with respect to p and by the retract argument (Lemma II.2.8) we have that f is a retract of i. Since i is a cofibration, we have by **MC3** (closed under retracts) that f is also a cofibration. ■

As immediate corollaries we get

Corollary II.2.10. *The class of (acyclic) fibrations (resp. (acyclic) cofibrations) is closed under pullbacks (resp. pushouts).*

Corollary II.2.11. *A map $f : X \to Y$ is an isomorphism $\iff f \in W_{\mathcal{C}} \cap Fib_{\mathcal{C}} \cap Cof_{\mathcal{C}}$, i.e. f is in all three classes of morphisms.*

The proof of the first corollary follows from the universal property of pullbacks together with proposition II.2.9, while the second follows from choosing the correct square to find a lift in, together with proposition II.2.9. We end this section with an example:

Example II.2.12. An example of a model structure on a (hopefully) well known category would be the *Quillen* model structure on **Top**. In this model structure we define $W_{\mathbf{Top}}$ to be the class of (topological) weak equivalences, $\mathrm{Fib}_{\mathbf{Top}}$ to be the class of *Serre fibrations* and $\mathrm{Cof}_{\mathbf{Top}}$ to be the class of all maps that have the LLP with respect to all acyclic fibrations. We denote this model category as $\mathbf{Top}_{Quillen}$.

For an extensive collection of model structures on various categories, we refer the reader to [Bal21]. With these basic definitions and properties in hand, we move on towards our main goal of constructing $\mathcal{C}[W^{-1}]$.

II.3 The Homotopy Category

The general construction of $\mathcal{C}[W^{-1}]$, as seen in [Hov99], involves forming the free category $F(\mathcal{C}, W^{-1})$ and then quotienting out by some relations on the the morphisms. This construction is not the prettiest and as mentioned before, it might not even result in an actual category. So our plan will be to cook up a different category $\mathrm{Ho}(\mathcal{C})$ where weak equivalences become isomorphisms, and then show that this is equivalent to $\mathcal{C}[W^{-1}]$. Along the way we will see that $\mathrm{Ho}(\mathcal{C})$ is indeed an actual category and the equivalence between $\mathcal{C}[W^{-1}]$ and $\mathrm{Ho}(\mathcal{C})$ gives an explicit description of the zig-zags in $\mathcal{C}[W^{-1}]$. As the name suggests, we will first have to establish the idea of homotopy in a model category.

Definition II.3.1 (Cylinder Object)**.** Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$. A *cylinder object* $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal map $\nabla_X : X \coprod X \to X$ as

$$\nabla_X : X \coprod X \underset{(i_0, i_1)}{\to} \mathrm{Cyl}(X) \underset{p}{\to} X$$

where the map $(i_0, i_1) : X \coprod X \underset{(i_0, i_1)}{\to} \mathrm{Cyl}(X)$ is a cofibration and $p : \mathrm{Cyl}(X) \to X$ is a weak equivalence. Such factorizations always exist by **MC5**, where p is chosen to be an acyclic fibration.

Remark II.3.2. Some sources differentiate between cylinder objects and *good* cylinder objects. The difference here is that such sources don't require that (i_0, i_1) is a cofibration for cylinder objects and define good cylinder ojects to be cylinder objects where (i_0, i_1) is a cofibration. We will always assume (i_0, i_1) is a cofibration.

Definition II.3.3 (Left Homotopy)**.** Let $f, g : X \to Y$ be morphisms in a model category $\mathcal{C}$. A *left homotopy* is a morphism $\eta : \mathrm{Cyl}(X) \to Y$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{i_0} & \mathrm{Cyl}(X) & \xleftarrow{i_1} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array} \tag{II.3}$$

We often write $\eta : f \sim_L g$.

Intuitively $\mathrm{Cyl}(X)$ is a generalization of the topological cylinder $X \times [0, 1]$. For any topological space $X \in \mathbf{Top}$ we have a continuous map $X \coprod X \to X \times [0, 1]$ given by the two inclusions $X \hookrightarrow X \times \{0\}$ and $X \hookrightarrow X \times \{1\}$. Furthermore we have the projection $X \times [0, 1] \to X$ and these two maps compose to the codiagonal map in **Top**. We also have that X and $X \times [0, 1]$ are homotopy equivalent spaces. A homotopy $H : X \times [0, 1] \to Y$ between two continuous maps f and g by definition satisfies $H(-, 0) = f, H(-, 1) = g$, which is precisely what diagram II.3 encodes.

Definition II.3.4 (Path Object)**.** Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$. A *path object* $\mathrm{Path}(X)$ for X is a factorization of the diagonal map $\Delta_X : X \to X \prod X$ as

$$\Delta_X : X \underset{s}{\to} \mathrm{Path}(X) \underset{(d_0,d_1)}{\to} X \prod X$$

where $s \in W_{\mathcal{C}}$ and (d_0, d_1) a fibration. By **MC5**, such factorizations always exist by letting s be an acyclic cofibration.

Remark II.3.5. Just as for cylinder objects, some authors differentiate between *good* path objects and path objects, the difference being whether or not (d_0, d_1) is a fibration. We will stick to the above definition, so (d_0, d_1) is a fibration.

Definition II.3.6 (Right Homotopy)**.** Let $f, g : X \to Y$ be morphisms in a model category $\mathcal{C}$. A *right homotopy* is a morphism $\varepsilon : X \to \mathrm{Path}(Y)$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f}\swarrow & \downarrow{\scriptstyle \varepsilon} & \searrow{\scriptstyle g} & \\ Y & \xleftarrow[d_0]{} & \mathrm{Path}(Y) & \xrightarrow[d_1]{} & Y \end{array} \tag{II.4}$$

We often write $\varepsilon : f \sim_R g$.

Just as for $\mathrm{Cyl}(X)$, $\mathrm{Path}(Y)$ aims to generalize path spaces in **Top**. For a topological space Y we define $\mathrm{Path}(Y) = \{\gamma : [0, 1] \to Y | \gamma \text{ is continuous}\}$. Given a homotopy H between two continuous maps f, g, we can fix an $x \in X$ and consider the path given by $\varepsilon(x) = H(x, -) : [0, 1] \to Y$. So if we define $d_i(\gamma) = \gamma(i),\ i = 0, 1$ we get that for all $x \in X,\ (d_0 \circ \varepsilon)(x) = d_0(H(x, -)) = H(x, 0) = f(x)$

and similarly for $(d_1 \circ \varepsilon)(x) = d_1(H(x,-)) = H(x,1) = g(x)$. This is what diagram II.4 encodes.

One important thing to note is that a cylinder object for an $A \in \mathcal{C}$ is the same as a path object for the same object in the dual category. Similarly, left homotopies correspond to right homotopies in the dual category. Thus it suffices to prove statements for cylinder objects and left homotopies and the dual statement will then hold for path objects and right homotopies.
We now define the notion of homotopy and homotopy equivalence in a model category. As one would expect, these definitions borrow heavily from the topological notion of homotopy.

Definition II.3.7. A pair of morphisms $f, g : X \to Y$ in a model category are *homotopic* if they are both left and right homotopic. Denote this as $f \sim g$.
A morphism $f : X \to Y$ in a model category is a *homotopy equivalence* if there exists a morphism $h : Y \to X$ such that $fh \sim \mathrm{id}_Y$ and $hf \sim \mathrm{id}_X$. We then say that f has a *homotopy inverse* h.

The obvious question to ask oneself is "Does homotopy determine an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for all objects $A, B \in \mathcal{C}$?". The (perhaps not too surprising) answer to this, is that it depends on the model structure on $\mathcal{C}$. More specifically, it depends on the cofibrant and fibrant objects.

Lemma II.3.8. *Let $\mathcal{C}$ be a model category and $X, Y \in \mathcal{C}$. Then being left (resp. right) homotopic is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for X cofibrant and Y fibrant. Moreover, the two notions coincide in such cases.*

Proof of lemma II.3.8 follows from proposition 1.2.5 in [Hov99][3]. The important takeaway from lemma II.3.8 is that for bifibrant objects, being homotopic is an equivalence relation on the Hom-sets. This will become important later when we define $\mathrm{Ho}(\mathcal{C})$.

The next important theorem we are going to state and prove, will be a generalization of Whitehead's theorem to an arbitrary model category. We will see that this theorem plays an important role when forming the homotopy category and in some sense, this theorem is exactly what is needed to show that $\mathcal{C}[W^{-1}]$ and $\mathrm{Ho}(\mathcal{C})$ are equivalent categories. We begin with an important lemma.

Lemma II.3.9. *Both $i_0, i_1 : A \to \mathrm{Cyl}(A)$ are acyclic cofibrations when A is cofibrant.*

Proof. Recall that $A \xrightarrow{i_j} \mathrm{Cyl}(A) \to A,\ j = 0, 1$ composes to id_A, and since id_A is an isomorphism, it is in all three classes (see corollary II.2.11). By definition, $\mathrm{Cyl}(A) \to A$ is a weak equivalence. Then by the 2-of-3 axiom (**MC2**) we have that $i_j \in W_{\mathcal{C}}$. We must show that i_j is a cofibration.
The diagram below is a pushout diagram, and A being cofibrant implies that in_j is also a cofibration (corollary II.2.10).

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \mathrm{in}_0} \\ A & \xrightarrow{\mathrm{in}_1} & A \coprod A \end{array}$$

We know that i_j factors as $A \underset{\mathrm{in}_j}{\longrightarrow} A \coprod A \underset{(i_0, i_1)}{\longrightarrow} \mathrm{Cyl}(A)$, which is the composition of two cofibrations. Thus i_0 and i_1 are acyclic cofibrations. ■

[3] In fact, lemma II.3.8 is almost identical to corollary 1.2.6 in [Hov99], which is why we don't bother with proving it here.

We are now ready for Whitehead's generalization for model categories.

Theorem II.3.10 (Whitehead for Model Categories)**.** *Let $\mathcal{C}$ be a model category and denote by $\mathcal{C}_{cf}$ the full subcategory of bifibrant objects. Then a morphism in $\mathcal{C}_{cf}$ is a weak equivalence $\iff$ It is a homotopy equivalence.*

Proof. By lemma II.3.8, in $\mathcal{C}_{cf}$ left and right homotopies coincide. So we need only consider left or right homotopies.
$\underline{\Rightarrow:}$ Let $f : A \to B$ be a weak equivalence in $\mathcal{C}_{cf}$. Factor $f : A \xrightarrow{i} C \xrightarrow{p} B$ with i being an acyclic cofibration and p being a fibration. By 2-of-3 axiom we have that p is also a weak equivalence. A being bifibrant implies that $A \to *$ is a fibration and so the diagram below admits a lift $r : C \to A$ by **MC4**.

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle i}\downarrow & \overset{r}{\nearrow} & \downarrow \\ C & \longrightarrow & * \end{array}$$

By commutativity, $ri = \mathrm{id}_A$ and so $ri \sim \mathrm{id}_A$. For ir, consider diagram II.5 below.
Here we have $s : C \to \mathrm{Path}(C)$, thus by definition $(d_0, d_1) \circ s = (\mathrm{id}_C, \mathrm{id}_C)$ so $(d_0, d_1) \circ si = (i, i)$. We also have that $(ir, \mathrm{id}_C) \circ i = (iri, i) = (i, i)$ so diagram II.5 commutes. Again, by definition (d_0, d_1) is a fibration and we have assumed that i is an acyclic cofibration. Then by **MC4** we have a lift $h : C \to \mathrm{Path}(C)$.

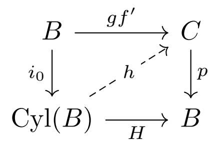
$$\begin{array}{ccc} A & \xrightarrow{si} & \mathrm{Path}(C) \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle (d_0,d_1)} \\ C & \underset{(ir,\mathrm{id}_C)}{\longrightarrow} & C \prod C \end{array} \tag{II.5}$$

Unwrapping diagram II.5 we get that the diagram below commutes, which shows that h is a right homotopy between ir and id_C.

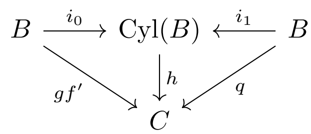
$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle ir}\swarrow & \downarrow{\scriptstyle h} & \searrow{\scriptstyle \mathrm{id}_C} & \\ C & \underset{d_0}{\longleftarrow} & \mathrm{Path}(C) & \underset{d_1}{\longrightarrow} & C \end{array}$$

By the discussion at the start of the proof, we have that ir and id_C are homotopic and so we have shown that $i : A \to C$ is a homotopy equivalence. Similarly, we can show that $p : C \to B$ is also a homotopy equivalence. Since homotopy is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}_{cf}}(X, Y)$ for all $X, Y \in \mathcal{C}_{cf}$, we get that f is also a homotopy equivalence.
$\underline{\Leftarrow:}$ Suppose $f : A \to B$ is a homotopy equivalence. Then there exists an $f' : B \to A$ such that $ff' \sim \mathrm{id}_B$ and $f'f \sim \mathrm{id}_A$. Let H be the homotopy between ff' and id_B. Factor f as $f : A \xrightarrow{g} C \xrightarrow{p} B$, with g being an acyclic cofibration and p a fibration. It follows that $C \in \mathcal{C}_{cf}$ and so by above, g is also a homotopy equivalence (for it is a weak equivalence between bifibrant objects). The solid part of the diagram below commutes and by lemma II.3.9, i_0 is an acyclic

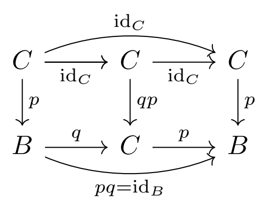
cofibration and so there exists a lift $h : \mathrm{Cyl}(B) \to C$ by **MC4**.



Define $q := hi_1 : B \to \mathrm{Cyl}(B) \to C$. Then $pq = phi_1 = Hi_1 = \mathrm{id}_B$ and the diagram below commutes, which shows that $gf' \sim q$.



Recall that g was also a homotopy equivalence, which implies there exists a homotopy inverse g'. Then $p \sim pgg' \sim fg' \implies qp \sim (gf')(fg') \sim g(f'f)g' \sim gg' \sim \mathrm{id}_C$. We have essentially shown that p is a homotopy equivalence with q as the homotopy inverse. Let $K : \mathrm{Cyl}(C) \to C$ be the homotopy between id_C and qp. $Ki_0 = \mathrm{id}_C$ a weak equivalence and since i_0 is also a weak equivalence (lemma II.3.9), we have by 2-of-3 axiom that K is also a weak equivalence. Thus, $Ki_1 = qp$ is also a weak equivalence. The diagram below shows that p is a retract of qp and thus p is also a weak equivalence.



Since $f = gp$ and both g, p are weak equivalences, we have that f is also a weak equivalence. ■

With Whitehead's theorem in place, we can finally define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$. It is of course possible to define $\mathrm{Ho}(\mathcal{C})$ before Whitehead's theorem is established, but defining it after makes it very clear why we would care about it. Namely, Whitehead's theorem shows that when restricting to the subcategory of bifibrant objects $\mathcal{C}_{cf}$ and then quotienting out the homotopy equivalence relation, we are in fact giving the weak equivalences inverses. So in a way $\mathrm{Ho}(\mathcal{C})$ is exactly the category we are after and we got to it without the mess that is $\mathcal{C}[W^{-1}]$. We make this more precise below.

Definition II.3.11 (The Homotopy Category)**.** Let $\mathcal{C}$ be a model category. Then its *homotopy category*, up to equivalence of categories, is the the category $\mathrm{Ho}(\mathcal{C})$ whose

- Objects are the objects of $\mathcal{C}_{cf}$, i.e. bifibrant objects.
- Morphisms are homotopy classes of $\mathcal{C}$.

This is very similar to the construction of the topological homotopy category **hTop**, where we leave the objects be but define $\mathrm{Hom}_{\mathbf{hTop}}(X, Y) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)/(f \sim g)$ where $\sim$ denotes topological homotopy. Indeed, $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)/(f \sim g)$ for all $A, B \in \mathcal{C}_{cf}$. Since $\mathcal{C}_{cf}$ is a full subcategory, it doesn't matter if we write $\mathrm{Hom}_{\mathcal{C}}(A,B)/(f \sim g)$ or $\mathrm{Hom}_{\mathcal{C}_{cf}}(A,B)/(f \sim g)$.

Recall that our goal was to find a nice and simple description of $\mathcal{C}[W^{-1}]$. So far we have $\mathrm{Ho}(\mathcal{C})$ and Whitehead's theorem, which by the above discussion gives some connection to $\mathcal{C}[W^{-1}]$. The next theorem is maybe the most crucial theorem in the basics of model categories[4] and finally gives us our sought after description of $\mathcal{C}[W^{-1}]$. More importantly, it says that $\mathcal{C}[W^{-1}]$ is in fact a category if $\mathcal{C}$ is a model category. We state the theorem without proof, however it is not too complicated. It requires only some lemmas regarding subcategories of $\mathcal{C}[W^{-1}]$ and the canonical functors $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ and $\delta : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$.

Theorem II.3.12. *Let $\mathcal{C}$ be a model category and $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ the canonical functor. The following hold:*

- *The inclusion $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}$ induces an equivalence of categories between $\mathrm{Ho}(\mathcal{C})$ and $\mathcal{C}[W^{-1}]$. More specifically, it is given by $\mathrm{Ho}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}_{cf}[W^{-1}] \to \mathcal{C}[W^{-1}]$ where the first arrow is an isomorphism of categories.*
- *Let Q, R be the cofibrant and fibrant replacement functors. Then there are natural isomorphisms $\mathrm{Hom}_{\mathcal{C}}(QRX, QRY)/(f \sim g) \cong \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(\gamma X, \gamma Y) \cong \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)/(f \sim g)$, where $f \sim g$ denotes homotopy.*
- *$\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ identifies left or right homotopic morphisms.*
- *$f : X \to Y$ a morphism in $\mathcal{C}$ such that $\gamma(f)$ an isomorphism, then f is a weak equivalence.*

Remark II.3.13. Point 2 of theorem II.3.12 tells us that zig-zags of maximum length 3 are needed in $\mathcal{C}[W^{-1}]$. More specifically, the zig-zags are of the form $X \leftarrow QX \to RY \leftarrow Y$.
Another thing to note is that for model categories, $\mathcal{C}[W^{-1}]$ always exists and is unique up to equivalence of categories.

Theorem II.3.12 essentially wraps up our story about $\mathcal{C}[W^{-1}]$. Along the way we introduced homotopy theoretic notions and saw that these concepts were crucial in forming $\mathcal{C}[W^{-1}]$. Using these notions one can define a homotopy theory for various categories and study said categories from a homotopic viewpoint. The next step in our journey will be to compare the homotopy categories of model categories and in turn compare the resulting homotopy theories.

II.4 Quillen Adjunctions and Equivalences

The point of this section will be to introduce functors between two model categories such that we can compare their resulting homotopy categories in a meaningful way. We will not go too deep in to the theory of this section, but for those interested [Hov99] and [Hir03] are good texts that cover this in depth. Additionally, [Bal21] is a good general reference text for model categories.

Definition II.4.1. Suppose $\mathcal{C}, \mathcal{D}$ are model categories. A pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ (F being the left adjoint) is a *Quillen adjunction* if the following equivalent conditions hold:

- F preserves cofibrations and acyclic cofibrations.

[4]Some authors go as far as to call this the fundamental theorem of model categories, f.ex. [Bal21] and [Hov99]

- U preserves fibrations and acyclic fibrations.
- F preserves cofibrations and U preserves fibrations.
- F preserves acyclic cofibrations and U preserves acyclic fibrations.

We then say F is a *left Quillen functor* and U is a *right Quillen functor*.

A proof of these 4 conditions being equivalent can be found in [Hir03] proposition 8.5.3. Also, the name Quillen comes from Daniel Quillen, the mathematician who is credited with developing and introducing model category theory during the 1960's.

Definition II.4.2. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction between model categories.

- The *(total) left derived functor* of F is the composition $\mathbb{F} : \mathrm{Ho}(\mathcal{C}) \overset{\mathrm{Ho}(Q)}{\to} \mathrm{Ho}(\mathcal{C}) \overset{\mathrm{Ho}(F)}{\to} \mathrm{Ho}(\mathcal{D})$, where Q is the cofibrant replacement functor.
- The *(total) right derived functor* of U is the composition $\mathbb{U} : \mathrm{Ho}(\mathcal{D}) \overset{\mathrm{Ho}(R)}{\to} \mathrm{Ho}(\mathcal{D}) \overset{\mathrm{Ho}(U)}{\to} \mathrm{Ho}(\mathcal{C})$, where R is the fibrant replacement functor.

Of course, $\mathrm{Ho}(R), \mathrm{Ho}(F), ...$ are the induced functors on the homotopy categories.

Remark II.4.3. Notice how the left and right derived functors depend partly on the replacement functors Q and R. Without assuming functorial factorization, we would have to make a choice of functorial (co)fibrant replacement and so the derived functors would not only depend on the model structure, but also on the replacement functor. To make our lives easier we just assume we have a functorial factorization, which in turn gives us a canonical choice for replacement functors.
Also, the reason we call them total right/left derived functors is because there is a different notion of derived functors, denoted $\mathbb{L}F$ and $\mathbb{R}U$. These are defined by cofibrant (resp. fibrant) replacement followed by F (resp. U).

Example II.4.4. It is possible to regard taking limits and colimits over a diagram $\mathcal{D}$ in a category $\mathcal{C}$ as functors. More specifically, limits (resp. colimits) are right (left) adjoint to the constant functor $\mathcal{D} \to \mathbf{Fun}(\mathcal{D}, \mathcal{C})$. We will later see that in certain cases the category $\mathbf{Fun}(\mathcal{D}, \mathcal{C})$ inherits a nice model structure when $\mathcal{C}$ is a model category, and so we may define derived functors of limit and colimit. Of these there are two we care about: The homotopy limit and the homotopy colimit. These are respectively the right derived functor of limit, $\mathbb{R}\lim$, and the left derived of colimit, $\mathbb{L}\,\mathrm{colim}$. We denote these holim and hocolim.

Definition II.4.5 (Quillen Equivalence)**.** Let $\mathcal{C}, \mathcal{D}$ be model categories with Quillen adjunctions $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$. Then we say $\mathcal{C}$ and $\mathcal{D}$ are *Quillen equivalent*, written $\mathcal{C} \sim_Q \mathcal{D}$, if the derived functors $\mathbb{F} : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{U}$ are equivalences of categories.

Intuitively, Quillen equivalent categories encode the same homotopy theories but we can think of them as coming from different viewpoints.

Chapter III

Simplicial Sets

III.1 Motivation

Combinatorial objects are often prefered objects to study, as their combinatorial structure make them especially easy to describe and nice to work with. Furthermore, some combinatorial objects provide good enough approximations of regular objects to justify the extensive study of them. This is especially evident in algebraic topology, where CW-complexes are examples of such objects. We know by the cellular approximation theorem that they provide excellent approximations of regular spaces and are generally much easier to work with.

Recall that CW-complexes are built by gluing disks together along their boundaries. We can loosen this gluing, by gluing together topological simplices along their faces. Furthermore, we can also allow for the collapsing of simplices to lower dimensions. In fact, we can detach ourselves from topological spaces completely and just consider the combinatorial interactions between the different dimensional simplices. We do this by defining sets of n-simplices, $n \in \mathbb{N}$, and defining maps from this set to the next and previous dimension. Of course, these maps have to satisfy certain properties that mimic that of boundary maps of topological simplices. So in a way, these sets and maps between them describe topological spaces without having to work in the category of topological spaces.

What we have described above is the underlying idea behind simplicial sets and can be used to study topological spaces without having to work in **Top**. Simplicial sets are often described very categorically and this allows one to transfer the idea of topological space to other categories. It turns out that our informal introduction above is not completely fallacious, as there is a very nice model structure on the category of simplicial sets which shows that there is a close relation between topological spaces and simplicial sets. This relation will be our main goal in this chapter, but we will along the way introduce other interesting ideas and concepts.

III.2 Definitions and Basic Properties

We start of by defining the elementary objects and categories we will be working with in this section. Denote by Δ the *simplex category*, i.e. the category whose objects are finite totally ordered sets $[n] = \{0, 1, ..., n\}$ and morphisms being order preserving maps. Of the morphisms in Δ, there are two in particular which are of interest:

- *Coface maps* $d^i : [n-1] \to [n]$ for $n > 0,\ 0 \le i \le n$, that is the injection whose image leaves out $i \in [n]$.
- *Codegeneracy maps* $s^i : [n+1] \to [n]$ for $n > 0,\ 0 \le i \le n$, that is the surjection such that $\sigma_i(i) = \sigma_i(i+1) = i$.

These maps satisfy the *cosimplicial identities*:

$$\begin{aligned} d^j d^i &= d^i d^{j-1},\ i < j \\ s^j d^i &= d^i s^{j-1},\ i < j \\ s^j d^j &= 1 = s^j d^{j+1} \\ s^j d^i &= d^{i-1} s^j,\ i > j+1 \\ s^j s^i &= s^i s^{j+1},\ i \le j \end{aligned}$$

and it can be shown that they generate the morphisms in Δ.

Definition III.2.1. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$. Collect these into a category whose objects are simplicial sets and morphisms are natural transformations between them, denoted **sSet**.

Clearly the category **sSet** is nothing more than the category of presheaves on Δ, $\mathbf{Pre}(\Delta)$. Knowing this already gives us a lot a of information about **sSet**. For example, every presheaf category is both complete and cocomplete. Furthermore, a morphism in a preasheaf category is an epimorphism (resp. monomorphism) exactly when it is an object-wise surjection (resp. injection). Both of these facts will come in handy later.
We often write X_n instead of $X([n])$ for a simplicial set X. The elements of X_n are called *n-simplices* and X_n is thus called the *set of n-simplices*. As such, we can form simplicial sets completely combinatorially. This is done exactly how one suspects, namely determine what each set of n-simplices X_n should be and then specify the appropriate *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracy maps* $s_i : X_n \to X_{n+1}$. These maps should of course satisfy the *simplicial identities*, which are dual to the cosimplicial identities. Simplicial sets are often drawn pictorially as follows:

$$... \substack{\to \\ \to \\ \to \\ \to} X_2 \rightrightarrows X_1 \to X_0$$

Here the rightwards arrows denote the face maps and degeneracies are usually not drawn, to avoid making a mess.

One important simplicial set is the *standard (simplicial) n-simplex* $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. By the (contravariant) Yoneda lemma, we have that for any $X \in \mathbf{sSet}$,

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) = \mathrm{Nat}(\mathrm{Hom}_\Delta(-, [n]), X) \cong X_n$$

where Nat denotes the collection of all natural transformations between two functors. In this case it is the Hom-set in **sSet**. The standard simplicial n-simplices will play a major role in the homotopy theory of simplicial sets, just as topological n-simplices do in the homotopy theory of topological spaces. There is in fact a very close relation between simplicial and topological n-simplices.
Let Δ_n denote the standard topological n-simplex. There is a covariant functor $\Delta \to \mathbf{Top}$ by $[n] \mapsto \Delta_n$ and for a morphism $\phi : [n] \to [m]$ we get $\phi_*(x_0, ..., x_n) = (\sum_{j\in\phi^{-1}(0)} x_j, ..., \sum_{j\in\phi^{-1}(m)} x_j)$. We use this to construct geometric objects from our simplicial sets, which at first glance seemed highly categorical.

Definition III.2.2 (Geometric Realization)**.** For $X \in \mathbf{sSet}$, the *geometric realization* of X is defined as the space

$$|X| = \coprod_n (X_n \times \Delta_n)/\sim$$

where if $(x, u) \in X_m \times \Delta_n$ then $(\phi^*(x), u) \sim (x, \phi_*(u))$ for $\phi : [n] \to [m]$. Here ϕ^* is the map induced by X and ϕ_* is the map described above. Here X_n is given the discrete topology.

Indeed, associating a simplicial set to its geometric realization extends to a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$. We call this the geometric realization functor. Intuitively one could think of geometric realization as having one topological n-simplex Δ_n for each simplicial n-simplex $x \in X_n$ and then using the face and degeneracy maps to glue these simplices together. Often, the degeneracy maps collapse higher dimensional simplices and the face maps glue them together.

Remark III.2.3. There is a more categorical construction of $|-|$. First we define it on only the standard simplicial n-simplices, so $\Delta^n \mapsto \Delta_n$. Then using the co-Yoneda lemma[1], we know that every simplicial set X is a colimit of standard simplicial n-simplices: $X \cong \underset{\Delta^n \to X \text{in} \Delta\downarrow X}{\text{colim}} \Delta^n$. Here $\Delta \downarrow X$ is the *simplex category of* X. Then we define geometric realization as follows:

$$|X| = \underset{\Delta^n \to X \text{in} \Delta\downarrow X}{\text{colim}} |\Delta^n|$$

For a more thorough explanation, see page 7 of [GJ09].

Example III.2.4. Let $k \geq 0$. Then $\Delta^0_k = \text{Hom}_\Delta([k], [0])$ has one element, namely the constant map. $\Delta^1_k = \text{Hom}_\Delta([k], [1])$ has $k + 2$ maps: The two constant maps and then the other maps are determined by which $i \in [k]$, $i \neq 0$ is the first to be sent to 1. Of the latter there are obviously k such maps. The two maps $[0] \to [1]$ induce injections $\Delta^0_k \to \Delta^1_k$ for all k by $c \mapsto c_i$, $i = 0, 1$, where c_i denotes the constant map from $[k]$ to $i \in [1]$, and c is the unique constant map in Δ^0_k. These injections combine to form two morphisms $\Delta^0 \to \Delta^1$, denoted δ_0 and δ_1. The image of δ_i is the constant map at i, $i = 0, 1$ and the union of these form the functor we denote as $\partial\Delta^1$. More specifically, $\partial\Delta^1([k]) = \{c_0, c_1\} \subseteq \text{Hom}_\Delta([k], [1]) = \Delta^1_k$. Define the simplicial circle, also denoted S^1, as the functor $\Delta^1/\partial\Delta^1$. The quotient here is taken objectwise, i.e. $S^1([k]) = \frac{\text{Hom}_\Delta([k],[1])}{\{c_0,c_1\}}$. It is possible to show that the geometric realization of the simplicial circle is in fact homeomorphic to the topological circle, however this computation is quite long. The computation of this does show how face and degeneracy maps play together: All simplices of dimension ≥ 2 are collapsed to either 0- or 1-simplices, whereas the face maps on the 1-simplices are responsible for gluing their boundaries together.

[1] See theorem 7.1 in [Lan10].

Example III.2.5. Unsurprisingly, $|\Delta^n| \cong \Delta_n$. We will later define the functors $\partial\Delta^n$ and Λ^n_k properly, and again we have (unsurpisingly) that $|\partial\Delta^n| \cong \partial\Delta_n$ and $|\Lambda^n_k|$ is homeomorphic to the topological n-horn, also denoted Λ^n_k.

Definition III.2.6 (Singular Complex)**.** For $Y \in \mathbf{Top}$, define $\mathrm{Sing}(-) : \mathbf{Top} \to \mathbf{sSet}$ by setting $\mathrm{Sing}(Y)([n]) = \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, Y)$.

$\mathrm{Sing}(Y)$ really is a simplicial set, as for all $[n] \in \Delta$ we get a set $\mathrm{Sing}(Y)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, Y)$ and for a map $f : [n] \to [m]$ we get the induced map $f_* : \mathrm{Hom}_{\mathbf{Top}}(\Delta_m, Y) \to \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, Y)$ by precomposing $g \in \mathrm{Hom}_{\mathbf{Top}}(\Delta_m, Y)$ with the induced map $f^* : \Delta_n \to \Delta_m$. Functoriality of $\mathrm{Sing}(Y)$ follows easily. We also see that $\mathrm{Sing}(-)$ itself is a functor and has some nice properties.

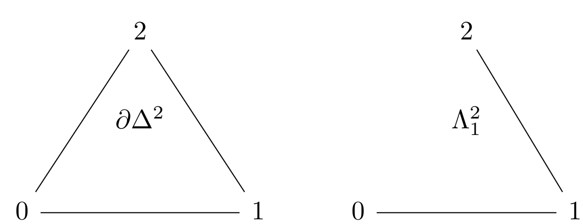
Proposition III.2.7. *The realization functor* $|-|$ *is left adjoint to the singular functor* $\mathrm{Sing}(-)$.

Proof. See proof of lemma 2.2.9 in [Dun+07]. ■

One important consequence of this adjunction is that we now know that $|-|$ preserves all colimits and we already know that **sSet** is cocomplete.

Definition III.2.8. Define the *boundary* of Δ^n as the sub-simplicial set $\partial\Delta^n \subseteq \Delta^n$ generated by the faces of Δ^n, i.e. generated by $d_i(\mathrm{id}_{[n]})$ for $i = 0, 1, ..., n$.
Similarly, define the *kth horn* of Δ^n as the sub-simplicial set $\Lambda^n \subseteq \Delta^n$ generated by all faces of Δ^n except the kth, i.e. generated by $d_i(\mathrm{id}_{[n]})$ for $i \neq k$.

Example III.2.9. We give an explicit calculation of the boundary of the simplicial 2-simplex $\partial\Delta^2$. By definition this is generated by the maps $d_i(\mathrm{id}_{[2]})$. Recall that d_i is the induced map $\Delta^n_k \to \Delta^n_{k-1}$ and is given by pre-composition by d^i. So $d_i(\mathrm{id}_{[2]}) = d^i : [1] \to [2]$. Define the maps $\delta^i_k : \mathrm{Hom}_\Delta([k], [1]) \to \mathrm{Hom}_\Delta([k], [2])$ by $f \mapsto d_i f$. Then $\partial\Delta^2_k = \cup^2_{i=0}\mathrm{Im}\delta^i_k$.
For the horn Λ^2_1 we take the union of all images of δ^i_k except the image of δ^1_k. Of course, this degreewise union can be done because **sSet** is a presheaf category and in such categories unions are taken objectwise. The figure below gives a pictorial representation of this.



For Δ^2, imagine $\partial\Delta^2$ as above but filled in. Here we can identify δ^i with the line/face that does not include i. So δ^0 would be the line from 1 to 2, etc.

III.3 Towards a Model Structure on sSet

This section aims to introduce the different classes of morphisms in **sSet** which will together form a model structure on simplicial sets, known as the *Kan-Quillen* model structure. The model structure is also sometimes referred to as the classical model structure on **sSet**. We start this journey with the fibrations.

Definition III.3.1 (Kan Fibration)**.** A map $p : X \to Y$ of simplicial sets is a *(Kan) fibration* if for every commutative diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a lift making everything commute.

Recall that **sSet** is both complete and cocomplete, so there exists a terminal object. This terminal object, denoted $*$, is $\Delta^0 = \mathrm{Hom}_\Delta(-, [0])$.

Definition III.3.2 (Kan Complex)**.** A simplicial set $X \in$ **sSet** is a *Kan Complex* if the unique morphism $X \to *$ is a Kan fibration.

Thus, Kan complexes are precisely the fibrant objects in the Kan-Quillen model structure. It possible to show that for all topological spaces $A \in$ **Top**, the simplicial set $\mathrm{Sing}(A)$ is a Kan complex. The proof relies on $|\Lambda^n_k|$ being a (strong deformation) retract of Δ_n.
For a Kan complex Y we have the following diagram below, which admits a lift (dotted line).

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & * \end{array}$$

It is common practice to drop the $*$, resulting in the diagram below. Intuitively this diagram tells us that Kan complexes are nice enough objects that given a map $\Lambda^n_k \to Y$, they admit a "filler" $\Delta^n \to Y$ that agrees on the subcomplex Λ^n_k.

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

We move on to the weak equivalences in **sSet**. There are two classical ways of describing these, one harder than the other. We will stick to the easier one.

Definition III.3.3 (Weak Equivalences)**.** A morphism $f : X \to Y$ in **sSet** is a *weak equivalence* if $|f| : |X| \to |Y|$ is a topological weak equivalence. That is, $\pi_n(|f|) : \pi_n(|X|, x_0) \to \pi_n(|Y|, |f|(x_0))$ is an isomorphism for all n.

It is possible to define weak equivalences in **sSet** without going through **Top**, but this requires a lot more theory. We will however give a quick informal rundown on the steps to take to define weak equivalences without **Top**[2].

First step is to define *simplicial homotopy.* This is very similar to the way we define left homotopy in model categories, see definition II.3.3, but we also need to define homotopy relative to

[2] I admit that I don't know enough about this construction, but found it interesting enough to include it in this thesis.

a subspace. Essentially if $L \subseteq K$ is a sub-simplicial set of $K \in \mathbf{sSet}$ and if two maps $f, g : K \to Y$ agree on L, i.e. $\alpha := f|_L = g|_L$ then the compositions

$$L \times \Delta^1 \xrightarrow{i \times 1} K \times \Delta^1 \xrightarrow{h} Y$$

and

$$L \times \Delta^1 \xrightarrow{pr_L} L \xrightarrow{\alpha} Y$$

should be equal. Here pr_L is the projection onto L, i is the inclusion $L \hookrightarrow K$ and $h : K \times \Delta^1 \to Y$ is the simplicial homotopy. Of course, $K \times \Delta^1$ acts as a cylinder object here. If indeed the two compositions above agree, then we say that f and g are *homotopic (rel. L)*. With this in hand, we can move on and show when exactly this determines an equivalence relation on the Hom-sets. The specifics of this are not too important, just remember that it is an equivalence relation when we need it to be (in this case at least).
Now we define simplicial homotopy groups. Intuitively, these are exactly like topological homotopy groups but we define them only for Kan complexes. The reason why will be apparent later.

Definition III.3.4 (Simplicial Homotopy Groups)**.** Let $X \in \mathbf{sSet}$ be a Kan complex and $v \in X_0$ a vertex of X. Define $\pi_n(X, v)$ to be the set of homotopy classes of maps $\alpha : \Delta^n \to X$ that fit into the commutative diagram below

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow v \\ \partial\Delta^n & \longrightarrow & \Delta^0 = * \end{array} \tag{III.1}$$

Here we have identified the vertex $v \in X_0$ with the unique map $\Delta^0 \to X$ which corresponds to v by the Yoneda lemma.

It turns out that the simplicial homotopy groups really are groups. Even better, for $n \geq 2$ they are abelian groups. The obvious question to ask now is "What is the group operation?". The group operation in simplicial homotopy groups are a bit more complicated than those in topological homotopy groups and is the main reason we define homotopy groups for Kan complexes. It turns out that the lifting/filling property of Kan complexes allows one to define a multiplication of maps which fit into diagram III.1 and is independent of representatives of homotopy classes. We will not write this down, as it requires alternative descriptions of $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, Y)$ in terms of tuples of simplices of Y, something we won't be needing at all in this text.
Once all of this is in place, we define weak equivalences to be maps $f : X \to Y$ of Kan complexes that induce isomorphisms on all simplicial homotopy groups $\pi_n(X, v) \to \pi_n(Y, f(v))$. On $n = 0$ we only require a bijection. How do we extend this notion to all simplicial sets? The obvious answer would be to find a fibrant replacement functor, which is exactly what Kan's Ex^∞ functor does[3]. Unfortunately this is far beyond the scope of this text. It turns out however, that for Kan complexes X there is an isomorphism of groups $\pi_n(X, v) \cong \pi_n(|X|, v)$ and so a map of Kan complexes f is a weak equivalence if and only if its geometric realization $|f|$ is a topological weak equivalence. This is our refined motivation for definition III.3.3. For a more thorough explanation of simplicial homotopy groups, see [GJ09] chapters I.6 and I.7.

[3]This is the extent of my knowledge on the Ex^∞ functor.

Lastly we cover the class of maps which will become our cofibrations. As mentioned in example II.2.12, most model structures are given by specifying the weak equivalences, either the fibrations or the cofibrations and then generating the last class by the correct lifting property. Unsurprisingly this often results in difficult and unsatisfactory descriptions of the class. This is not exactly the case here. We start by characterizing acyclic Kan fibrations, i.e. maps which are both weak equivalences and Kan fibrations.

Proposition III.3.5. *A morphism of simplicial sets $f : X \to Y$ is both a weak equivalence and a Kan fibrations $\iff$ f has the RLP with respect to all inclusions $\partial\Delta^n \subseteq \Delta^n,\ n \geq 0$.*

Proof of the above proposition is identical to the proof of theorem I.11.2 in [GJ09].
An important takeaway is that the class of maps which have the LLP with respect to acyclic Kan fibrations contain the inclusions $\partial\Delta^n \hookrightarrow \Delta^n,\ n \geq 0$. Thus we define cofibrations in **sSet** to be inclusions of simplicial sets, $A \hookrightarrow B$. Note that this definition of cofibrations implies that every simplicial set is cofibrant, as $\emptyset \hookrightarrow X$ is always an inclusion in **sSet**.

III.4 The Kan-Quillen Model Structure

We now move on to the main theorem of this chapter. Once this theorem is established, all our powerful machinery from the previous chapter comes into play.

Theorem III.4.1. *There is a model structure on* **sSet***, called the Kan-Quillen model structure, where the three classes of morphisms are given as follows:*

- $W_{\mathbf{sSet}}$ *is the class of simplicial weak equivalences.*
- $Fib_{\mathbf{sSet}}$ *is the class of Kan fibrations*
- $Cof_{\mathbf{sSet}}$ *is the class of monomorphisms, i.e. degreewise injections.*

It is common to denote **sSet** with the Kan-Quillen model structure as $\mathbf{sSet}_{\text{Quillen}}$. We will not bother with this.
The entire proof of the above theorem will not be shown here, as **MC5** is particularly difficult to show. More specifically, it requires a small object argument[4] and certain lemmas and propositions which we will not cover here. We will however prove **MC1-MC3** and refer the reader to page 62 and 63 in [GJ09] for a proof of the remaining two axioms.

Proof. Clearly all three classes of morphisms are closed under composition.
MC1: In section III.2 we said that **sSet** is nothing more than $\mathbf{Pre}(\Delta)$, the presheaf category over the simplex category. Thus we know that **sSet** is both complete and cocomplete, for every presheaf category is.
MC2: This follows immediately from the fact that topological weak equivalences satisfy the 2-of-3 property.

[4]The small object argument is a somewhat set-theory heavy construction which may be applied in a category after certain conditions are met. The argument is used to construct the factorization seen in **MC5**. See chapter 2.4 in [Bal21] and chapters 2.1. to 2.1.2 in [Hov99].

MC3: Let $g : X \to Y$ be a retract of $f : A \to B$. Assume first that f is a weak equivalence. Then applying $|-|$ followed by $\pi_n(-)$ to diagram II.1 gives us the following (basepoints omitted):

$$\begin{array}{ccccc} \pi_n(|X|) & \xrightarrow{r_*} & \pi_n(|A|) & \xrightarrow{t_*} & \pi_n(|X|) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} \\ \pi_n(|Y|) & \xrightarrow{i_*} & \pi_n(|B|) & \xrightarrow{j_*} & \pi_n(|Y|) \end{array} \qquad t_* r_* = \mathrm{id}_{\pi_n(|X|)}, \quad j_* i_* = \mathrm{id}_{\pi_n(|Y|)}$$

A subscript $*$ denotes the induced map from $\pi_n(|-|)$. One can check that the composition $h = t_* f_*^{-1} i_*$ is an inverse for g_*, implying that g is a simplicial weak equivalence.

Now assume f is a Kan fibration and assume the left box of the diagram below commutes. Since g is a retraction of f we then have that the entire diagram commutes.

$$\begin{array}{ccccccc} \Lambda^n_k & \longrightarrow & X & \longrightarrow & A & \longrightarrow & X \\ \big\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \Delta^n & \longrightarrow & Y & \longrightarrow & B & \longrightarrow & Y \end{array} \qquad (X \to A \to X) = 1, \quad (Y \to B \to Y) = 1$$

Since the diagram commutes and f is a Kan fibration, we have a lift $h : \Delta^n \to A$. Composing this with the map $A \to X$ we get our required lift $k : \Delta^n \to X$. Commutativity of the diagram follows from the definition k.

Lastly, assume f is a cofibration. Then f is a degreewise injection in **Set** and we get the following commutative diagram in **Set**:

$$\begin{array}{ccccc} X_n & \xrightarrow{r_n} & A_n & \xrightarrow{t_n} & X_n \\ \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} \\ Y_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & Y_n \end{array} \qquad t_n r_n = 1, \quad j_n i_n = 1$$

Since $1 = t_n r_n$ is an injection, r_n must also be an injection. Then $f_n r_n = i_n g_n$ is an injection and so g_n is also an injection. Thus g is a cofibration. ■

Since every simplicial set is cofibrant we immediately know what the bifibrant objects are: namely Kan complexes. Thus the objects of Ho(**sSet**) are Kan complexes, while morphisms are homotopy classes of morphism.

Remark III.4.2. It is worth noting that $\mathbf{sSet}_{\mathrm{Quillen}}$ is a *cofibrantly generated* model category, meaning that we can specify a set of generating cofibrations and acyclic cofibrations that in turn generate the rest of the cofibrations and acyclic cofibrations. In this case, the set of boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ is our set of generating cofibrations (see proposition III.3.5) while the set of horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ is our set of generating acyclic cofibrations (see definition III.3.1).

Recall that $|-|$ and $\mathrm{Sing}(-)$ were an adjoint pair of functors. It is possible to show that this is a Quillen adjunction and that the geometric realization of a Kan fibration is a Serre fibration. Furthermore, this adjoint pair sets up a Quillen equivalence between $\mathbf{sSet}_{\text{Quillen}}$ and $\mathbf{Top}_{\text{Quillen}}$ (recall the latter model structure from example II.2.12).

III.5 Simplicial Model Categories

Recall that for topological spaces $X, Y \in \mathbf{Top}$, we can give $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ the *compact-open* topology, making each Hom-set in **Top** a topological space. We would like a similar notion in **sSet**.

Definition III.5.1 (Simplicial Mapping Space)**.** Let $X, Y \in \mathbf{sSet}$. The *simplicial mapping space* $\mathrm{Map}(X, Y)$ is the simplicial set given by $\mathrm{Map}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$.
For a map $\theta : [m] \to [n]$ define $\theta^* : \mathrm{Map}(X, Y)_n \to \mathrm{Map}(X, Y)_m$ by

$$(f : X \times \Delta^n \to Y) \mapsto (X \times \Delta^m \overset{1\times\theta}{\to} X \times \Delta^n \overset{f}{\to} Y)$$

The simplicial mapping space is also known as the (simplicial) function complex in some books, for example [GJ09]. We will see that the simplicial mapping space will play an important role in the theory going forwards (although most of its uses will be in the background), so we will state some basic facts about it.
Firstly, there is an *evaluation map* $\mathrm{ev} : X \times \mathrm{Map}(X, Y) \to Y$ by $(x, f : X \times \Delta^n \to Y) \mapsto f(x, \mathrm{id}_{[n]})$. It is possible to show that ev is natural in both X and Y[5]. The next proposition hints at $\mathrm{Map}(X, -)$ being a right adjoint functor.

Proposition III.5.2 (Exponential Law)**.** *ev_* : $\mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{Map}(X, Y)) \to \mathrm{Hom}_{\mathbf{sSet}}(X \times K, Y)$ defined by sending $g : K \to \mathrm{Map}(X, Y)$ to the composite $X \times K \overset{1\times g}{\to} X \times \mathrm{Map}(X, Y) \overset{ev}{\to} Y$ is a bijection which is natural in K, X and Y*

Proof. The inverse of ev_* is given by $(g : X \times K \to Y) \mapsto (g_* : K \to \mathrm{Map}(X, Y))$ where for $x \in K_n$, $g_*(x)$ is the composite

$$X \times \Delta^n \overset{1\times i_x}{\to} X \times K \overset{g}{\to} Y$$

where we have identified i_x with x by the Yoneda lemma. ■

The following proposition turns out to be quite important but unfortunately we will have to skip the proof of this, as it is a bit too difficult for the author to completely understand and also requires theory which we have not covered.

Proposition III.5.3. *Suppose $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \to Y$ is a Kan fibration. Then the map $\mathrm{Map}(L, X) \overset{(i^*,p_*)}{\to} \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$, induced from the diagram below, is a fibration.*

$$\begin{array}{ccc} \mathrm{Map}(L, X) & \xrightarrow{p_*} & \mathrm{Map}(L, Y) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathrm{Map}(K, X) & \xrightarrow{p_*} & \mathrm{Map}(K, Y) \end{array}$$

[5] See page 20 of [GJ09].

Of course, $\mathrm{Map}(K,X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L,Y)$ denotes the pullback along $\mathrm{Map}(K,Y)$. The proof of proposition III.5.3 can be found on page 21 of [GJ09].
So why exactly did we define $\mathrm{Map}(X,Y)$ and dump two seemingly random facts about it (i.e. proposition III.5.2 and III.5.3) onto the readers lap? It is because the simplicial mapping space allows us to give model categories even more structure, namely by "enriching" a model category $\mathcal{C}$ by **sSet**!

Definition III.5.4 (Simplicial Category)**.** A category $\mathcal{C}$ is a *simplicial category* if there is a mapping space functor $\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ such that for $A, B \in \mathcal{C}$:

1) $\mathrm{Map}_{\mathcal{C}}(A,B)_0 = \mathrm{Hom}_{\mathcal{C}}(A,B)$
2) $\mathrm{Map}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{sSet}$ has a left adjoin $A \otimes - : \mathbf{sSet} \to \mathcal{C}$ that is associative, as in

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L$$

3) $\mathrm{Map}_{\mathcal{C}}(-,B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ has left adjoint $\mathrm{Hom}_{\mathcal{C}}(-,B) : \mathbf{sSet} \to \mathcal{C}^{\mathrm{op}}$

Beware that $\otimes$ in the definition above is not necessarily the normal tensor product of modules. This is just notation which has lingered around since Quillen's time.

Definition III.5.5 (Simplicial Model Category)**.** Say $\mathcal{C}$ is both a model category and a simplicial category. Then we say $\mathcal{C}$ is a *simplicial model category* if the following holds:

SMC: Suppose $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration.
Then $\mathrm{Map}_{\mathcal{C}}(B,X) \overset{(i^*,p_*)}{\to} \mathrm{Map}_{\mathcal{C}}(A,X) \times_{\mathrm{Map}_{\mathcal{C}}(A,Y)} \mathrm{Map}_{\mathcal{C}}(B,Y)$ is a Kan fibration, which is acyclic if i or p is.

So propositions III.5.2 and III.5.3 essentially prove the following:

Theorem III.5.6. **sSet** *is a simplicial model category.*

Of course, there are many simplicial model categories different from **sSet**. In fact, the next couple of important model categories we will encounter are also simplicial model categories. Because of this we will differentiate between the various mapping spaces by a subscript, denoting which category this belongs to. For example, the simplicial mapping space will be denoted $\mathrm{Map}_{\mathbf{sSet}}$.

Chapter IV

Towards the Local Model Structure on $\mathbf{sPre}(\mathbf{Man}^\infty)$

IV.1 Simplicial Presheaves and their Model Structure

Say $\mathcal{C}$ is a model category, $\mathcal{D}$ a small category and consider the functor category $\mathbf{Fun}(\mathcal{D}, \mathcal{C})$. Can this new category be given model structure from $\mathcal{C}$? Yes, there are two obvious candidates: The *projective* and *injective* model structures. We will mainly be focusing on the projective model structure, as the fibrant objects (which will be important later) have a much nicer description in this model structure. However it can be shown that in the case we care about, the projective and injective structures are Quillen equivalent (See theorem 3.2.1 c) in [Dug98]).
First off, some notation and terminology. For a category $\mathcal{C}$, $\mathrm{Mor}(\mathcal{C})$ denotes the class of all morphisms in $\mathcal{C}$. Also, we say that a functor (or natural transformation) F has an objectwise property P if for all objects A, $F(A)$ has property P.

Definition IV.1.1 (Projective Model Structure)**.** Let $\mathcal{C}$ be a model category and define the following model structure on $\mathbf{Fun}(\mathcal{D}, \mathcal{C})$:

- $W_{\mathbf{Fun}(\mathcal{D},\mathcal{C})} = \{f \in \mathrm{Mor}(\mathbf{Fun}(\mathcal{D}, \mathcal{C})) | f \text{ is an objectwise weak equivalence in } \mathcal{C}\}$
- $\mathrm{Fib}_{\mathbf{Fun}(\mathcal{D},\mathcal{C})} = \{f \in \mathrm{Mor}(\mathbf{Fun}(\mathcal{D}, \mathcal{C})) | f \text{ is an objectwise fibration in } \mathcal{C}\}$
- $\mathrm{Cof}_{\mathbf{Fun}(\mathcal{D},\mathcal{C})} = \{f \in \mathrm{Mor}(\mathbf{Fun}(\mathcal{D}, \mathcal{C})) | f \text{ has the LLP with respect to acyclic fibrations in } \mathbf{Fun}(\mathcal{D}, \mathcal{C})\}$

This is known as the *projective model structure*[1], denoted $\mathbf{Fun}(\mathcal{D}, \mathcal{C})_{\mathrm{proj}}$.

Of course, we have to show that the above classes determine a valid model structure. One of the cases where we are guaranteed the existence of this structure is when $\mathcal{C}$ is a cofibrantly generated model category (See remark III.4.2).

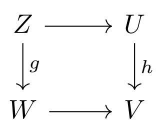
Proposition IV.1.2. *Let $\mathcal{C}$ be a cofibrantly generated model category. Then the projective model structure on $\mathbf{Fun}(\mathcal{D}, \mathcal{C})$ makes it a model category.*

[1] The only difference between injective and projective is that in the injective model we choose objectwise cofibrations instead.

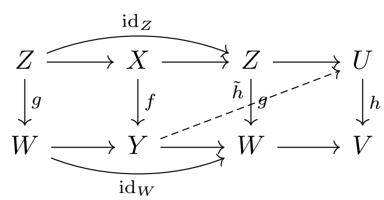
Proof. Just as we did with theorem III.4.1, we will only show **MC1** to **MC3**. Clearly all three classes are closed under composition.
MC1: As $\mathcal{C}$ is a model category, it is both complete and cocomplete. It follows that $\mathbf{Fun}(\mathcal{D},\mathcal{C})$ is also, by calculating (co)limits objectwise.
MC2: Let $f : X \to Y$ and $g : Y \to Z$.

- If $f, g \in W_{\mathbf{Fun}(\mathcal{D},\mathcal{C})}$ then $gf \in W_{\mathbf{Fun}(\mathcal{D},\mathcal{C})}$ by closure in $\mathcal{C}$.
- Let $f, gf \in W_{\mathbf{Fun}(\mathcal{D},\mathcal{C})}$. For all $A \in \mathcal{D},\ f_A : X(A) \to Y(A)$ and $gf_A : X(A) \to Z(A)$ are weak equivalences. Then certainly $g_A : Y(A) \to Z(A)$ is a weak equivalence by 2-of-3 in $\mathcal{C}$, so g is a weak equivalence in $\mathbf{Fun}(\mathcal{D},\mathcal{C})$.
- The case $g, gf \in W_{\mathbf{Fun}(\mathcal{D},\mathcal{C})}$ is similar to the previous case.

MC3: Let $A \in \mathcal{D},\ g : Z \to W$ be a retract of $f : X \to Y$. Then one can check that if f is a weak equivalence (resp. fibration) then g is also a weak equivalence (fibration) by drawing the retract diagram II.1 for A and noticing that we are now just staring at a retract in $\mathcal{C}$.
The case for f a cofibration is a little bit more tedious. Let $h : U \to v$ be an acyclic fibration in $\mathbf{Fun}(\mathcal{D},\mathcal{C})$. Say the square below commutes



Then the diagram below commutes and the necessary lift is the composite $W \to Y \xrightarrow{\tilde{h}} U$.



Thus g is also a cofibration. ■

We will now turn our attention to the case $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{sSet})$, where $\mathcal{C}^{\mathrm{op}}$ is a small category. This category is called the category of *simplicial presheaves on* $\mathcal{C}$ and is important enough to be denoted $\mathbf{sPre}(\mathcal{C})$. Recall that **sSet** is cofibrantly generated and so by the above proposition, $\mathbf{sPre}(\mathcal{C})$ admits the projective model structure. We gather some facts about $\mathbf{sPre}(\mathcal{C})$[2]:

a) $\mathbf{sPre}(\mathcal{C})$ is a simplicial model category with $\mathrm{Map}_{\mathbf{sPre}(\mathcal{C})}(X,Y)_n = \mathrm{Hom}_{\mathbf{sPre}(\mathcal{C})}(X \times \Delta^n, Y)$, where Δ^n is the functor sending maps $f : A \to B$ to id_{Δ^n}. This is found as proposition 2.22 in [Jar16].

b) Any presheaf $F \in \mathbf{Pre}(\mathcal{C})$ lets us construct a *simplicially constant presheaf* sF. This is the object in $\mathbf{sPre}(\mathcal{C})$ that has F at every dimension and with face/degeneracy maps being the identity. Importantly, this gives us an embedding of $\mathcal{C}$ into $\mathbf{sPre}(\mathcal{C})$:

[2] See see remark 3.2.4 in [Dug98]

First embed $\mathcal{C}$ into $\mathbf{Pre}(\mathcal{C})$ via the Yoneda embedding, i.e. $A \in \mathcal{C}$ is sent to $rA := \mathrm{Hom}_{\mathcal{C}}(-, A)$ in $\mathbf{Pre}(\mathcal{C})$. Then $s(rA)$ is the following simplicial set:

$$...\substack{\to\\\to\\\to} \mathrm{Hom}_{\mathcal{C}}(-, A) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, A)$$

where every arrow is the identity. Often for simplicity's sake, we just write sA instead of $s(rA)$.

c) Homotopy limits and colimits (recall these from example II.4.4) are computed objectwise. Given a diagram $D : \mathcal{I} \to \mathbf{sPre}(\mathcal{C})$, then $\operatorname{hocolim} D$ is weakly equivalent to the functor $X \mapsto \operatorname{hocolim}_\alpha D_\alpha(X)$, $\alpha \in \mathcal{I}$ and $X \in \mathcal{C}$, where $\operatorname{hocolim}_\alpha D_\alpha(X)$ is taken in $\mathbf{sSet}$. This follows in part from $\mathbf{sPre}(\mathcal{C})$ being a simplicial model category.

d) In the projective model structure of $\mathbf{sPre}(\mathcal{C})$, representables (for example rX for $X \in \mathcal{C}$) are cofibrant objects and fibrant objects are those that are objectwise fibrant in $\mathbf{sSet}$.

And now we finally restrict ourselves to the category we actually care about. Letting $\mathcal{C}^{\mathrm{op}} = (\mathbf{Man}^\infty)^{\mathrm{op}}$ we get the category of *simplicial presheaves on smooth manifolds* $\mathbf{sPre}(\mathbf{Man}^\infty)$. This category has all the nice properties that we discussed above. For example, it is complete and cocomplete (something which $\mathbf{Man}^\infty$ is not), and it has a nice model structure which in some sense is related to the Quillen model structure on topological spaces. There is however a slight issue concerning a certain contractible manifold.

Recall that our goal was to construct a homotopy theory for smooth manifolds. As such it would make sense for us to want the smooth manifold $\mathbb{R}$ to be contractible, or at least weakly equivalent to a point. It turns out that presheaves, even simplicial presheaves, forget the underlying geometry of $\mathbf{Man}^\infty$. We illustrate this with an example.

Example IV.1.3. We will show that $\mathbb{R}$ in $\mathbf{sPre}(\mathbf{Man}^\infty)$ is not contractible. The way we will go about this is by showing that the embeddings of $\mathbb{R}, * \in \mathbf{Man}^\infty$, denoted $s\mathbb{R}$ and $s*$ respectively, are not weakly equivalent in the projective model structure of $\mathbf{sPre}(\mathbf{Man}^\infty)$.

We first recall how the objects $s\mathbb{R}$ and $s*$ look like:

$$s\mathbb{R} = [...\substack{\to\\\to\\\to} \mathrm{Hom}_{\mathbf{Man}^\infty}(-, \mathbb{R}) \rightrightarrows \mathrm{Hom}_{\mathbf{Man}^\infty}(-, \mathbb{R}) \to \mathrm{Hom}_{\mathbf{Man}^\infty}(-, \mathbb{R})]$$
$$s* = [...\substack{\to\\\to\\\to} \mathrm{Hom}_{\mathbf{Man}^\infty}(-, *) \rightrightarrows \mathrm{Hom}_{\mathbf{Man}^\infty}(-, *) \to \mathrm{Hom}_{\mathbf{Man}^\infty}(-, *)]$$

A weak equivalence in $\mathbf{sPre}(\mathbf{Man}^\infty)_{\mathrm{proj}}$ is a morphism f that is objectwise a weak equivalence, i.e. for all $M \in \mathbf{Man}^\infty$, $f(M)$ should be a weak equivalence in $\mathbf{sSet}$. This happens exactly when the geometric realization $|f(M)|$ is a topological weak equivalence. Thus we calculate the geometric realizations of $s*(M)$ and $s\mathbb{R}(M)$ where $M \in \mathbf{Man}^\infty$. We use the formula given by definition III.2.2.

Since $*$ is terminal in $\mathbf{Man}^\infty$ we have that for all $M \in \mathbf{Man}^\infty$, $s*(M) = [...\substack{\to\\\to\\\to} * \rightrightarrows * \to *]$. All face and degeneracy maps are the identity here. Let $u \in \Delta_n$ and $\Phi : [n] \to [m]$ for $m < n$. Then $(*, \Phi^*(u)) \sim (\Phi_*(*), u) \sim (\mathrm{id}_*(*), u) = (*, u)$ which implies that all higher dimensional topological simplices Δ_n are collapsed to lower dimensional simplices Δ_m. This shows that for all $M \in \mathbf{Man}^\infty$, $|s*(M)| = * \in \mathbf{Top}$.

We also have that $s\mathbb{R}(M) = [...\substack{\to\\\to\\\to} \mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R}) \rightrightarrows \mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R}) \to \mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R})]$. By the same calculation as for $s*(M)$ we get that $|s\mathbb{R}(M)| \cong \mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R})$. Importantly, $\mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R})$ has the discrete topology. It follows that if M has more than 1 element (which it does if $M = *$) then the only continuous paths $\gamma : [0, 1] \to \mathrm{Hom}_{\mathbf{Man}^\infty}(M, \mathbb{R})$ are constant paths.

If not, then taking preimages of $\{\gamma(0)\}$ and $\mathrm{Im}(\gamma) \setminus \gamma(0)$ gives us two open and disjoint sets which cover $[0,1]$, which is impossible as $[0,1]$ is connected.
Thus there exists an M such that the cardinality of $\pi_0(|s\mathbb{R}(M)|)$ is greater than 1, but for all $M \in \mathbf{Man}^\infty$ the cardinality of $\pi_0(|s*(M)|)$ is 1. Thus $s*$ and $s\mathbb{R}$ cannot be weakly equivalent (even by a zig-zag of weak equivalences) in $\mathbf{sPre}(\mathbf{Man}^\infty)_{\mathrm{proj}}$.

IV.2 Grothendieck Topologies and Čech Descent

Why would we expect presheaves to remember geometry and local conditions? All we have done is collected functors $(\mathbf{Man}^\infty)^{\mathrm{op}} \to \mathbf{sSet}$ and given them a model structure. Nowhere have we specified how these functors should act on local data. It turns out that the correct notion we are looking for are closer to sheaves. More specifically, we want to introduce the idea of descent to our category.
To do this we have to first discuss Grothendieck topologies. The idea here is to give each object $U \in \mathcal{C}$ a covering by other objects U_α in a way which mimics topological coverings.

Definition IV.2.1 (Grothendieck Topology)**.** A *Grothendieck topology* on a category $\mathcal{C}$ consists of families of *covers* $\{\Phi_\alpha : U_\alpha \to U\},\ U \in \mathcal{C}$ such that:

1) If $\{\Phi_\alpha : U_\alpha \to U\}$ is a cover and $\Psi : V \to U$ is a morphism in $\mathcal{C}$, then $\{V \times_U U_\alpha \to V\}$ is a cover for V.

2) $\{\Phi_\alpha : U_\alpha \to U\}$ is a cover and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}_\beta$ is a cover for all U_α, then $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U\}_{\alpha,\beta}$ is a cover for U.

3) $\{\mathrm{id} : U \to U\}$ is a cover.

A category with a Grothendieck topology is called a *site*.

Remark IV.2.2. What we have defined above is what is commonly known as a *Grothendieck pre-topology*. For our purposes it is enough to only consider pretopologies, but for the curious readers a discussion of Grothendieck topologies can be found in section 3.1 of [Jar16].

Example IV.2.3. Let X be a topological space and consider the poset category of open sets in X, $O(X) = \{U \subset X | U \text{ is open in } X\}$. Then $O(X)$ with covers given by open covers is a site:
Clearly axioms 2 and 3 are satisfied. Any morphism $V \to X$ in $O(X)$ is by definition the inclusion, so the pullback is nothing more than the intersection of open sets. Thus axiom 1 is satisfied.

The reason we care about Grothendieck topologies is because it gives us a nice framework to define descent on. This is especially obvious when dealing with topological categories such as $\mathbf{Top}, \mathbf{Man}$ and $\mathbf{Man}^\infty$, where we can make each into a site by letting the covers be given by open coverings of open subspaces/submanifolds. See example b) on page 113 of [LM92] for an in depth explanation.
The idea of descent will be very similar to the sheaf condition (see page 67 of [LM92]) and in some sense can be seen as a generalization of sheaves. We will be defining descent, more specifically Čech descent, only for the cases we care about but a more general definition can be found in the appendix of [DHI04].

To begin with, let $U = \{U_\alpha \to X\}$ be a cover of $X \in \mathbf{Man}^\infty$ and let $U_{\alpha_0,\ldots,\alpha_n} = \cap_{i=0}^n U_{\alpha_i}$. Then given a simplicial presheaf F we can form the diagram

$$\prod_{\alpha_0} F(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_0,\alpha_1} F(U_{\alpha_0,\alpha_1}) \substack{\to\\\to\\\to} \cdots \tag{IV.1}$$

and the inclusions $U_{\alpha_0,\ldots\alpha_n} \hookrightarrow X$ gives us maps $F(X) \to \prod_{\alpha_0,\ldots,\alpha_n} F(U_{\alpha_0,\ldots\alpha_n})$. We can collect these maps into one giant commutative diagram:

$$\begin{array}{ccccccc} F_0(U) & \rightrightarrows & F_1(U) & \substack{\to\\\to\\\to} & F_2(U) & \substack{\to\\\to\\\to\\\to} & \cdots \\ \uparrow & \nearrow & & \nearrow & & \nearrow & \\ F(X) & & & & & & \end{array} \tag{IV.2}$$

where $F_n(U) = \prod_{\alpha_0,\ldots,\alpha_n} F(U_{\alpha_0,\ldots\alpha_n})$. This entire diagram is in **sSet**. We can take the homotopy limit of diagram IV.1 by using first the fibrant replacement, denoted R, and then taking the formal limit. Taking the fibrant replacement of $F(X)$ gives another diagram like IV.2, where every object is replaced by fibrant objects. By the universal property of limits, there is a unique map $RF(X) \to \operatorname{holim}_n F_n(U)$ and by definition of R there is a weak equivalence $F(X) \to RF(X)$. So there is a map $F(X) \to \operatorname{holim}_n F_n(U)$.

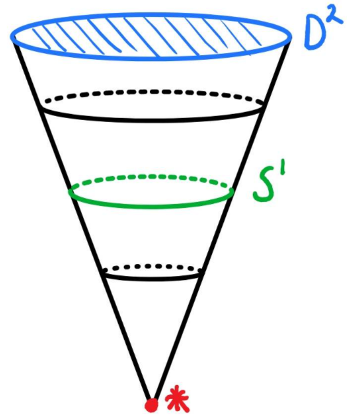
Definition IV.2.4 (Čech Descent)**.** A simplicial presheaf $F \in \mathbf{sPre}(\mathbf{Man}^\infty)$ is said to satisfy *Čech descent* if the map $F(X) \to \operatorname{holim}_n F_n(U)$ given above is a weak equivalence for all open covers $U = \{U_\alpha \to X\}$.

So what is the idea behind Čech descent? Let us momentarily take a step away from all the rigor. To each cover $U = \{U_\alpha \to X\}$ we can associate a simplicial set known as the *Čech complex* $\check{C}U$ given by

$$[n] \mapsto \coprod_{\alpha_0,\ldots\alpha_n} U_{\alpha_0,\ldots,\alpha_n}$$

with face maps given by inclusions $U_{ij} \hookrightarrow U_i$ and degeneracies given by $U_i \to U_{ii}$. We also know that colimits in topology can often be regarded as types of gluing, for example the pushout in **Top**, and as such we might speculate that $\operatorname{colim}_n \check{C}U_n$ is a space approximating X to some degree. Maybe we can think of this approximation as gluing the open sets together along their intersections, looking at higher and higher intersections to "refine" our gluing. Thus the idea here is to try and recover information of a space X by patching together a bunch of local data. We have already established that we want to work in simplicial presheaves, so we essentially push this idea into $\mathbf{sPre}(\mathbf{Man}^\infty)$ as above. Thus, $F \in \mathbf{sPre}(\mathbf{Man}^\infty)$ satisfying descent is similar to the condition that F allows us to reconstruct X by stitching together small patches of X. Furthermore this should hold for all open covers U. This can be thought of as a generalization of sheaves, for those that are familiar with them.

Why do we choose homotopy limit instead of limit? One way of motivating this is by looking at the topological pushout versus the homotopy pushout. Let us try to construct S^2 by gluing the boundary of D^2 together. Categorically, this is given by the pushout of $* \leftarrow S^1 \hookrightarrow D^2$. Up to homotopy D^2 is just a point, so replacing D^2 with $*$ gives a pushout of $* \leftarrow S^1 \to *$, which is the same as collapsing S^1 to a single point. The problem here is that S^2 is not contractible, which shows that regular pushouts and homotopy are not on speaking terms. We mend their relationship by introducing the homotopy pushout, which for a diagram of the form $Y \xleftarrow{f} X \xrightarrow{g} Z$ is the space given by gluing Y to the top of the cylinder $X \times I$ via f and Z to the bottom via g. Here X is identified by the middle slice. Pictorially we have the following:



Essentially, what we want is a notion of limit and colimit which play well with homotopy. It turns out (unsurprisingly) that holim and hocolim accomplish this.

IV.3 The Local Model Structure

The next step will be to incorporate Čech descent into $\mathbf{sPre}(\mathbf{Man}^\infty)$ in a way which takes into account the topology on $\mathbf{Man}^\infty$. The way we will be doing this is, via a powerful piece of machinery known as *Bousfield localization*. We will unfortunately have to blackbox Bousfield localization, but the curious reader might want to consider taking a look at [Hir03] and [Bal21] for more information. We will however explain the idea behind it.

The name Bousfield localization might seem a bit misleading, as it is slightly different from the idea of localization we introduced in section I. Say we are given a model category $\mathcal{C}$ with a class of weak equivalences $W_\mathcal{C}$. We might run into scenarios (such as now) where we are not satisfied with the class $W_\mathcal{C}$ and want to add more morphisms. Obviously, just adding new morphisms to $W_\mathcal{C}$ is not the wisest decision as it will probably mess up our current model structure. This is where Bousfield localization comes in. Given that certain technical assumptions are met, Bousfield localization will expand the class of weak equivalences and keep the class of cofibrations unchanged. Of course, this comes at the expense of our fibrations (see proposition II.2.9). Thus, in the Bousfield localized model structure we have:

- More weak equivalences
- Unchanged cofibrations
- Fewer fibrations

We will refer to the new classes of morphisms as *local* morphisms. So the new weak equivalences are called local weak equivalences. Similarly for fibrations and cofibrations. Furthermore, the non-local classes will be refered to as *global*.
One important (perhaps obvious) thing to note is that every global weak equivalence is also a local weak equivalence. The same goes for cofibrations, as these are unchanged by localization. The local fibrations however are the morphisms that satisfy RLP with respect to all local acyclic cofibrations. As "acyclic" is the only part of acyclic cofibration which is changed, we have that local fibrations are global fibrations which satisfy some extra condition.

Let $X \in \mathbf{Man}^\infty$ and $U = \{U_\alpha \to X\}$ be an open covering of X. Recall that $r(-)$ was the Yoneda embedding of $\mathbf{Man}^\infty$ into $\mathbf{Pre}(\mathbf{Man}^\infty)$ and that $s(-)$ was our embedding of $\mathbf{Pre}(\mathbf{Man}^\infty)$ into $\mathbf{sPre}(\mathbf{Man}^\infty)$, by constructing a simplicial presheaf which has $r(-)$ in every dimension. We can then construct a simplicial presheaf from U, denoted $\check{\mathbf{U}}$, by taking the coproduct of $r(U_{\alpha_0,\ldots\alpha_n})$ over all $\alpha_0, \ldots\alpha_n$ and using the inclusions to create the face/degeneracy maps. More specifically, $\check{\mathbf{U}}_n = \coprod_{\alpha_0,\ldots\alpha_n} r(U_{\alpha_0,\ldots,\alpha_n})$. The inclusions $U_{\alpha_0,\ldots,\alpha_n} \hookrightarrow X$ give maps $r(U_{\alpha_0,\ldots,\alpha_n}) \to r(X)$, which in turn gives a map $\check{\mathbf{U}} \to s(X)$ in $\mathbf{sPre}(\mathbf{Man}^\infty)$.

Definition IV.3.1 (Local Model Structure)**.** The *local model structure* on $\mathbf{sPre}(\mathbf{Man}^\infty)$ is the model structure obtained by localizing $\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{proj}}$ at the set of all maps $\check{\mathbf{U}} \to s(X)$ described above. The localization is done via Bousfield localization and the model structure is denoted $\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{loc}}$.

This model structure is commonly known as the *projective Čech model structure*, but is sometimes also called the projective local or Bousfield-Kan Čech model structure.
Intuitively we can think of this category as a homotopy theory of $\mathbf{Man}^\infty$ where the Čech complex of an open cover U approximates X up to weak equivalence, which was more or less exactly what was said when we discussed the idea behind Čech descent in the previous section[3].

The next proposition, which we unfortunately are unable to prove in this thesis, gives us a total characterization of the fibrant objects in $\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{loc}}$. Importantly, it shows that our somewhat lengthy discussion of Čech descent was justified.

Proposition IV.3.2. *The local fibrant objects $F \in \mathbf{sPre}(\mathbf{Man}^\infty)$ are the objectwise fibrant objects which satisfy Čech descent.*

Recall that when forming the homotopy category of a model category $\mathcal{C}$, we restrict to the bifibrant objects and quotient out by the homotopy relation (which is indeed an equivalence relation for bifibrant objects by lemma II.3.8). This means that every object in $\text{Ho}(\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{loc}})$ satisfies descent. Furthermore we know that every object in a model category is weakly equivalent to a bifibrant object. Geometrically we can interpret these two facts together as the property that every space X can be approximated (up to weak equivalence) by an object $\tilde{X}$ that in turn can be recovered by patching together local data. This property is quite satisfactory, as far as geometry is concerned.
We end this section by an alternative characterization of the fibrant objects in the local model structure. This characterization is not really needed going forwards, but it is nonetheless interesting and might give some much need intuition.

[3]This intuition is more or less stolen from [Dug98] page 24.

Definition IV.3.3. The *stalk in dimension n* of a simplicial presheaf $F \in \mathbf{sPre}(\mathbf{Man}^\infty)$ is the following:

$$p_n = \operatorname*{colim}_{k \to \infty} F(B_k^n)$$

where $B_k^n \subset \mathbb{R}$ is the n-dimensional ball of dimension $\frac{1}{k}$.
A map of simplicial presheaves $F \to G$ is a *stalkwise weak equivalence* if $p_n(F) \to p_n(G)$ are weak equivalences for all $n \in \mathbb{N}$.

Stalks at first glance seem to reflect the topology of $\mathbf{Man}^\infty$ much better than Čech descent does, as stalks are more or less considering behaviour at a point. So why didn't we consider stalks instead of descent? It turns out that we could have gone either route and still ended up at the same place.

Proposition IV.3.4. *The class of local weak equivalences is equal to the class of stalkwise weak equivalences.*

The proof of this is way outside the scope of this text, but it is a nice fact to know. We refer the interested reader to page 31 and 32 in [Dug98] for a lengthier discussion about stalks.

IV.4 The $\mathbb{R}$-Local Model Structure

It seems that Bousfield localization, Grothendieck topologies and Čech descent have fixed our problem of presheaves forgetting the underlying geometry. This is at least partly true. Unfortunately we still have that $\mathbb{R}$ is not contractible in $\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{loc}}$, something which is proven in exercise 3.3.4 in [Dug98][4]. So how do we fix this? We again turn to our trusted friend, Bousfield localization.

Definition IV.4.1. The $\mathbb{R}$-*local model structure* on $\mathbf{sPre}(\mathbf{Man}^\infty)$, denoted $\mathbf{sPre}(\mathbf{Man}^\infty)_{\mathbb{R}}$, is the model structure obtained by localizing $\mathbf{sPre}(\mathbf{Man}^\infty)_{\text{loc}}$ at the set of maps

$$\{s(X \times \mathbb{R}) \to sX\}_{X \in \mathbf{Man}^\infty}$$

where the maps are induced by the projection $X \times \mathbb{R} \to X$.

This definition essentially wraps up our construction of the homotopy theory of smooth manifolds. We end this thesis with a somewhat surprising proposition.

Proposition IV.4.2. *The model categories* $\mathbf{sPre}(\boldsymbol{Man})_{\mathbb{R}}$ *and* $\mathbf{Top}_{Quillen}$ *are Quillen equivalent.*

The proof of the proposition requires theory that goes beyond the scope of this thesis. We refer the reader to section 3.4 of [Dug98] for a more in depth discussion of the $\mathbb{R}$-local model structure on **Man**.

Remark IV.4.3. The above proposition, as stated, concerns topological manifolds and not smooth manifolds. This is not really a problem, as we could simply replace $\mathbf{Man}^\infty$ with **Man** at every step of our construction and instead build a homotopy theory for topological manifolds. The only reason we chose smooth manifolds is because the author thinks they are more interesting than just topological manifolds.

[4] I tried multiple times to find an alternative proof of this fact, but was unable too. Also, note that this exercise requires a lot more theory than I have bothered to include in this thesis. Lastly, I was not able to complete the exercise.

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