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Erlend D. Børve

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Doctoral thesis

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Norwegian University of Science and Technology
Thesis for the Degree of
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Faculty of Information Technology and Electrical
Engineering
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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* (Ph.D.) at the Norwegian University of Science and Technology (NTNU). I have been employed as a Ph.D. candidate at the Department of Mathematical Sciences since June 2019. Of the four years in my contract, one has been devoted to teaching, and one to coursework. The research was supervised by Aslak Buan and co-supervised Steffen Oppermann.

I am grateful to those who has shown guidance and companionship during the last four years. First and foremost, I want to express my deepest gratitude to my supervisor Aslak Buan. He has shown great enthusiasm when introducing me to the topic, persistent encouragement, as well as helpful comments on the content of this thesis. I also gratefully acknowledge the useful discussions I have had with my co-supervisor Steffen Oppermann, and the invaluable proofreading of Eric Hanson and Rune Haugseng.

One of the papers included in this thesis was written in collaboration with Paul Trygsland, a former Ph.D. candidate in the topology group at NTNU. After several informal discussions, we managed to find a project where both of us, an algebraist and a topologist, could contribute. I thank Paul for the many illuminating discussions that followed, and for enthusiastically introducing me to the world of higher categories and simplicial homotopy theory. I am very pleased with the result of our collaboration. I wish him success as a consultant, and fulfillment as a father.

In the winter of 2021, Aslak organised a mini-seminar for his “subgroup” of the Algebra group at NTNU. In addition to Aslak, I thank Eric Hanson and Håvard Terland for their active participation and interesting talks. I thoroughly enjoyed these mini-seminars as a platform for dialogue and dissemination of ideas.

In early 2022, I had the pleasure of spending three lovely months at UVSQ in Versailles. This department was much smaller than NTNU’s, but I appreciated the warm atmosphere. I thank my host Pierre-Guy Plamondon for his generous hospitality and eagerness to collaborate, and Monica García for many useful discussions. The result of our discussions is included in this thesis as a work in progress. I will strive to return the hospitality when Pierre-Guy and Monica are coming to Oslo in March.

My office in Sentralbygg II was shared with Didrik Fosse, whom I met only a few minutes after I first set foot on campus in 2014. Our chairs were so close that I only had to rotate 90 degrees to poke his back and ask a question. Alas, lockdown put that to an end. Little did we know that it would be illegal for the two of us to use the office simultaneously; our shoulders were definitely less than one metre apart. I thank him for a wide range of interesting discussions, both mathematical and non-mathematical, and I give my best wishes to him and his wife Lise.

My last year of employment was spent in Oslo. I thank Aslak for allowing me to take part in the research activities as the Centre of Advanced Study (CAS) at the National Academy of Science and Letters, where I have met several interesting mathematicians working in our field. My office was at Niels Henrik Abels Hus on the campus of the University of Oslo. I am grateful to the Department of Mathematics for their hospitality, and particularly to Yngvar Reichelt for his handling of practical matters.

One year (or more precisely 1680 hours) of my employment at NTNU was devoted to teaching-related activities. I have had the pleasure of lecturing remote students, both prior to and after the outbreak of the COVID-19 pandemic. Despite the challenges posed by lockdown, I was impressed by the determination and perseverance shown by my students. Their unwavering commitment was truly inspiring and served as a reminder of the significance of my role as a lecturer.

Mathematics has long been an addiction of mine, but in May 2021 I got another, namely running. Running is a lot like mathematics; one can achieve a lot through patience and persistence. I thank Magnus Ringerud for including me in the intersection of his cryptography and running circles, and Tjerand Silde for the several hundreds of kilometres we have endured. I also value Petter Bergh's mentorship, although I have come to regard him as a sprinter.

The Department of Mathematical Sciences is a wonderful place to work. The administration is helpful and efficient, thus facilitating productivity. Moreover, I have made lasting friendships with my wonderful colleagues. In addition to those mentioned above I thank Torgeir Aambø, Peter Flydal, Sigurd Gaukstad, Jacob Grevstad, Johanne Haugland, William Hornslien, Eiolf Kaspersen, Sondre Kvamme, Sebastian Martensen, Louis Martini, Kelsey Moran, Endre Rundsveen, Mads Sandøy, Filip Schjerven, Morten Solberg, Laertis Vaso, and others.

I am deeply indebted to my family. They have always provided constant love and encouragement.

Erlend D. Børve,
Oslo, February 2023.

To my parents.

Outline of the thesis

This thesis consists of an introductory chapter and the following three papers, two of which are joint work. The introduction provides a context for the papers summarises their contents.

Paper I:
Two-term silting and τ-cluster morphism categories.
Submitted.

Paper II:
Extension ∞-categories and a theorem of Retakh for exact ∞-categories.
Joint with Paul Trygsland.
Preprint.

Paper III:
Non-crossing partitions and wide subcategories for gentle algebras.
Joint with Pierre-Guy Plamondon.
Work in progress.

Introduction

Outline. Section 1 provides a context for Paper I, Section 2 for Paper II, and Section 3 for Paper III. The last section summarises the content of the papers.

A field k is fixed throughout.

0 Representation theory of algebras and Auslander–Reiten theory

Representation theory of algebras is concerned with classifying modules over associative k-algebras. One aims for a description the category of finitely generated right modules of a k-algebra A, denoted by $\operatorname{mod}(A)$. It suffices to consider algebras up to *Morita equivalence*, which to identify k-algebras whose modules are equivalent as abelian categories. Under the assumption that k is algebraically closed, any k-algebra is Morita equivalent to the path algebra of a quiver with relations. The basic tools of the subject were developed by Auslander and Reiten in the 1970s [Aus74, AR75, AR77a, AR77b, AR78].

1 Two-term silting

Tilting theory [BB80] provides the means for studying the modules over one finite-dimensional algebra in terms of another. The module categories are related by functors expressed in terms of tilting modules. The ideas of tilting theory have been of fundamental importance in derived Morita theory. Indeed, one can classify the algebras that are derived equivalent to a finite-dimensional k-algebra A in terms of the tilting complexes in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(\mathrm{A}))$ [Ric89].

Silting complexes were introduced by Keller and Vossieck to characterise the bounded t-structures of the bounded derived categories $\mathcal{D}_{\mathrm{fd}}(\operatorname{mod}(k\overrightarrow{A_n}))$ [KV88]. The word "silting" is supposedly a portmanteau of "semi" and "tilting;" a tilting complex has no non-zero extensions with itself, whereas a silting complex admits no *positive* self-extensions. Rickard's Morita Theorem can be extended to show that silting complexes induce derived equivalences of non-positive dg (=differental graded) k-algebras [Kel94, Lemma 6.1] [KY14, 5.1].

A significant portion of this thesis is devoted to the study of two-term silting complexes. These are silting complexes in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj(A)})$ of the form

$$\cdots \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots \tag{1}$$

where the subscripts are the negative of the degrees of the components. Let $\mathcal{K}_2^b(\mathrm{proj}(A))$ denote the full subcategory of $\mathcal{K}^b(\mathrm{proj}(A))$ spanned by the two-term complexes. The class of two-term silting objects closely resembles that of tilting modules. For instance, a generalisation of the Brenner–Butler Theorem holds [BZ16, Theorem 1.1] [BZ21, Theorem 2.1].

One can regard the theory of two-term silting as a completion of tilting theory with respect to mutation. Given tilting module T and an indecomposable direct summand U, one can find another tilting module of the form $T' = T/U \oplus U'$ provided that T/U is a faithful module. If so, the tilting module T' is uniquely determined, and is called a *mutation* of T with respect to U. Drawing inspiration from cluster combinatorics [FZ02a, FZ02b, BFZ05, FZ07, BMR$^+$06], Adachi–Iyama–Reiten developed τ-tilting theory as an extension of tilting theory where mutation always can be performed [AIR14]. Since there is a mutation-preserving correspondence between the two-term silting complexes in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj(A)})$ and the support τ-tilting modules in $\mathrm{mod}(A)$ [AIR14, Theorem 3.2], we can indeed regard two-term silting as a "mutationally complete" generalisation of tilting theory.

Mutation is the key ingredient when devising a reduction procedure for two-term silting complexes (or equivalently, support τ-tilting modules). For a (two-term) presilting complex P, corresponding to a τ-rigid pair (M,Q), one can set up a bijection between the following sets [IY18]

- presilting complexes in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj(A)})$ having P as a direct summand,
- presilting complexes in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj(C)})$,

where C is the *τ-tilting reduction* of A with respect to (M,Q) [Jas15]. The bijection restricts to a bijection between the subsets of silting complexes. Jasso shows that $\mathrm{mod}(C)$ sits inside $\mathrm{mod}(A)$

as a *wide subcategory* [Jas15, Proposition 3.6], which is to say that it is closed under extensions, kernels, and cokernels.

2 Exact ∞-categories and extriangulated categories

Quillen introduced exact categories in order to establish a general framework for algebraic K-theory [Qui73]. The n^{th} K-group of an exact category is defined as the $(n+1)^{\text{st}}$ homotopy group of the classifying space of its Q-construction. Given a scheme X, one defines the K-groups of X as those of the exact category of locally free coherent sheaves on X.

Roughly speaking, an *exact category* is an additive category $\mathcal{C}$ equipped with a notion of exactness. Certain maps in $\mathcal{C}$ are *admissible monomorphisms*, and others are *admissible epimorphisms*. Examples include abelian categories, where the notion of exactness is the usual one. Any extension-closed subcategory of an abelian category is exact, and conversely, all exact categories are equivalent to extension-closed subcategories of abelian categories. The latter claim is widely known as the Gabriel–Quillen Embedding Theorem [TT07, Theorem A.7.1].

In some sense, triangulated categories behave a lot like exact categories. They are additive categories, equipped with a set of sequences (the distinguished triangles) that satisfy certain properties. Neeman has developed K-theory for triangulated categories [Nee97a, Nee97b, Nee98a, Nee98b, Nee99, Nee00, Nee05]. One of his seminal results is the Theorem of the Heart, showing that if a triangulated category $\mathcal{T}$ is equipped with a bounded t-structure, then its Neeman K-theory is equivalent to the Quillen K-theory of the heart [Nee98b, Nee99, Nee01].

By specialising to the setting of stable ∞-categories, a well-developed framework for topological enhancements of triangulated categories, Barwick gave a simplified proof of the Theorem of the Heart [Bar15]. If $\mathscr{C}$ is a stable ∞-category, the subcategory $\mathscr{C}_{\geq 0} \subseteq \mathscr{C}$ is the aisle of a t-structure on $\mathscr{C}$, and $\mathscr{C}_0$ is the heart of said t-structure, Barwick constructs equivalences of K-theory spectra [Bar15, Theorem 6.1]

$$K(\mathscr{C}_0) \longrightarrow K(\mathscr{C}_{\geq 0}) \longrightarrow K(\mathscr{C}).$$

Note that $\mathscr{C}_{\geq 0}$ is in general neither the nerve a Quillen exact category nor a stable ∞-category, whence one could not define its K-theory spectrum using the standard theory. It was necessary to broaden the scope.

By adapting Keller's minimal axioms [Kel90] to the setting of ∞-categories, Barwick defines the notion of exact ∞-categories. Classes of examples include nerves of exact categories is a class of examples, as well as stable ∞-categories. The subcategory $\mathscr{C}_{\geq 0} \subseteq \mathscr{C}$ in the paragraph above is also exact, since it is extension-closed in a stable ∞-category. It turns out that a higher version of Gabriel–Quillen Embedding Theorem can be shown; all exact ∞-categories can be embedded into a stable hull [Kle20, Theorem 1] such that the inclusion functor is exact. Thus all exact ∞-categories can be obtained as extension-closed subcategories of some stable ∞-category.

The homotopy category of a stable ∞-category is canonically triangulated [Lur17, Theorem 1.1.2.14], whereas the homotopy category of a nerve of an exact category is exact. In general, the homotopy category of an exact ∞-category will be equipped with some structure that simultaneously generalises exact and triangulated categories.

Indeed, said homotopy categories are extriangulated [NP20]. The definition of extriangulated categories is due to Nakaoka and Palu [NP19]. This simultaneous generalisation of exact and triangulated categories provided a useful unifying framework for the study of reduction and mutation of cotorsion pairs. There is now a wide literature where results and constructions involving exact or triangulated categories are generalised to this setting. For instance, Nakaoka and Palu have joined forces with Iyama to show that the theory of extriangulated categories serves as a powerfully general context for Auslander–Reiten theory [INP18].

An example of an extriangulated category which is neither exact or triangulated is the two-term category of a non-semisimple k-algebra A. This is the extension-closed subcategory $\mathcal{K}_2^b(\mathrm{proj}(A)) \subseteq \mathcal{K}^b(\mathrm{proj}(A))$ spanned by the two-term objects (see Equation (1)). It has almost split extensions (equivalently, it has Auslander–Reiten–Serre duality [INP18, Theorem 3.6]), whence the machinery of Auslander–Reiten theory can be applied to study it. The projective objects in $\mathcal{K}_2^b(\mathrm{proj}(A))$ are the objects concentrated in degree 0 (set $P_1 = 0$ in Equation (1)) and the injective objects are those concentrated in degree -1 (set $P_0 = 0$ in Equation (1)).

Any extension-closed subcategory of a triangulated category has a canonical extriangulated structure. It follows from the existence of a stable hull that all *topological extriangulated categories*, namely those that arise as homotopy category of an exact ∞-category, are extension-closed subcategories of some (topological) triangulated category. The author does not know whether such embeddings can be constructed in general.

3 Gentle algebras

Gentle algebras form a class of algebras whose representation theory is governed by particularly simple combinatorial rules. Since gentle algebras are string algebras, the indecomposable modules fall into two classes, namely string modules and band modules [BR87]. The definition goes back to Assem and Happel [AH81, AH82], and aslo Assem and Skowroński [AS87] introduced them in order to study iterated tilted algebras of type A. Despite the combinatorial nature of the definition, it turns out that gentle algebra enjoy homological properties. Notably, the class of gentle algebras is closed under derived equivalence [SZ03]. One uses the fact that the endomorphism algebra of a rigid module over gentle algebra is gentle [SZ03] to show that τ-tilting reductions of gentle algebras are gentle.

The combinatorics of gentle algebras can be approached geometrically. The tiling algebra of a (partially) triangulated marked surface is gentle, and every gentle algebra arises this way up to isomorphism [BCS21]. This allows for a geometric interpretation of their module categories, where indecomposable string modules are permissible arcs on the surface, and permissible closed curves are band modules. Certain moves between them correspond to irreducible maps, and there is an explicit way of representing Auslander–Reiten sequences. One can also use this characterisation to show that gentle algebras are precisely the endomorphism algebras of a partial cluster-tilting object in the generalised cluster category of a marked surface [BCS21, Proposition 2.8] .

Given a gentle k-algebra A, one can also construct geometric for the triangulated categories $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A))$ and $\mathcal{D}_{\mathrm{fd}}(A)$ [OPS18, APS19, PPP19]. One presents A by a dissection of a marked surface [PPP19, Theorem 4.10]. Every such dissection comes with a unique dual dissection. The indecomposable objects in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A))$ are then given by graded curves, where a *grading* is a certain sequence of integers indexed by the crossings between the curve and the arcs of dual dissection. Koszul duality will then provide the means to geometrically model $\mathcal{D}_{\mathrm{fd}}(A)$. Indeed, we have a triangle equivalence

$$\mathcal{D}_{\mathrm{fd}}(A) \longrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A^!)),$$

where $A^!$ is the Koszul dual of A.

In recent years, gentle algebras have turned up in homological mirror symmetry. In a particular class of partially wrapped Fukaya categories of the marked surface with stops, there are formal generators whose endomorphism rings are graded gentle algebras [HKK17]. We thus have a triangle equivalence

$$\mathcal{F} \longrightarrow \mathcal{D}_{\mathrm{fd}}(A),$$

where $\mathcal{F}$ is the partially wrapped Fukaya category in question, and A is a (trivially graded) gentle algebra of finite global dimension [HKK17, LP20]. The geometric model of $\mathcal{D}_{\mathrm{fd}}(A)$ is thus useful in a mirror symmetry context.

4 Summary of papers

Paper I: Two-term silting and τ-cluster morphism categories

Fix a non-positive proper dg k-algebra A. Let $\mathcal{D}_{\mathrm{fd}}(A)$ be the subcategory of the derived category $\mathcal{D}(A)$ spanned by complexes of finite dimensional total cohomology, and let $\mathrm{per}(A)$ be the thick subcategory generated by A in $\mathcal{D}(A)$. Under these assumptions, we can equip $\mathcal{D}_{\mathrm{fd}}(A)$ with a t-structure whose heart is equivalent to the module category $\mathrm{mod}(H^0A)$. Given a two-term presilting object P in $\mathrm{per}(A)$, we show that the perpendicular category $P^{\perp_{\mathbb{Z}}} \subseteq \mathcal{D}_{\mathrm{fd}}(A)$ is closed under truncation. This makes $P^{\perp_{\mathbb{Z}}}$ a t-exact subcategory of $\mathcal{D}_{\mathrm{fd}}(A)$. Intersecting $P^{\perp_{\mathbb{Z}}}$ with the heart of the t-structure, one obtains a category which is equivalent to the τ-tilting reduction $J(P) \subseteq \mathrm{mod}(H^0A)$

Next, we generalise Jasso's compatibility result [Jas15, Theorem 4.12(b)], relating two-term silting reduction and τ-tilting reduction. More specifically, we show that Buan–Marsh' reduction procedure for support τ-rigid pairs [BM21a, Theorem 3.6] can be linked with two-term silting reduction. For a two-term presilting object $P \in \mathrm{per}(A)$, we have a commutative diagram of bijections

$$\begin{tikzcd}
2\text{-presilt}_P(A) \arrow[r, "H_S"] \arrow[d, "\varphi_P"] & \mathrm{s}\tau\text{-rigid}\,\mathrm{pair}_{H_S(P)}(A) \arrow[d, "\psi_{H_S(P)}"] \\
2_{T_P^+}\text{-presilt}(\mathcal{Z}/[P]) \arrow[r, "H_{T_P^+}"] & \mathrm{s}\tau\text{-rigid}\,\mathrm{pair}(C)
\end{tikzcd}$$

where the left column is the Iyama–Yang silting reduction [IY18, Theorem 3.7], the right column is the reduction map constructed by Buan–Marsh, and the horizontal maps are the correspondences between two-term presilting objects and support τ-rigid objects.

Our main result concerns τ-cluster morphism categories, which we develop purely in terms of two-term silting. In the case of finite dimensional algebras, our construction is equivalent to Buan–Marsh' [BM21a], but the functoriality of silting reduction allows for a more efficient exposition, notably when it comes to establishing the associativity of composition in $\mathfrak{W}_A$.

Finally, we define two-term presilting sequences as an analogue of signed τ-exceptional sequences [BM21b]. It shown that a two-term presilting sequence gives rise to a linearly independent set in the Grothendieck group of $\mathrm{per}(A)$, and that a complete two-term presilting sequence yields an ordered basis.

Paper II: Extension ∞-categories for exact ∞-categories and a theorem of Retakh

Let $\mathscr{C}$ be an exact ∞-category. We define *extension ∞-categories* $\mathscr{E}xt^n_{\mathscr{C}}(\mathscr{C},\mathscr{C})$, whose objects are the n-extensions of $\mathscr{C}$. There are bifibrations

$$e_n\colon \mathscr{E}xt^n_{\mathscr{C}}(\mathscr{C},\mathscr{C}) \longrightarrow \mathscr{C}\times\mathscr{C}$$

whose fibres are n-extensions with fixed endpoints. For $a,b\in\mathscr{C}$, we denote the fibre $e_n^{-1}(b,a)$ by $\mathscr{E}xt^n_{\mathscr{C}}(b,a)$. Our main result asserts that $\{\mathscr{E}xt^n_{\mathscr{C}}(b,a)\}_{n=0}^{\infty}$ forms a spectrum object in the ∞-category of Kan complexes. In the special case where $\mathscr{C}$ is the nerve of a Quillen exact category, one recovers a result of Retakh [Ret86].

Our proof relies heavily on the existence and structure of Klemenc' stable hull [Kle20]. We show that the embedding $\mathscr{C}\hookrightarrow\mathcal{H}^{\mathrm{st}}_{\geq 0}(\mathscr{C})$ induces weak equivalences

$$\mathscr{E}xt^n_{\mathscr{C}}(\mathscr{C},\mathscr{C}) \longrightarrow \mathscr{E}xt^n_{\mathcal{H}^{\mathrm{st}}_{\geq 0}(\mathscr{C})}(\mathscr{C},\mathscr{C}),$$

where the codomain consists of n-extensions with endpoints in $\mathscr{C}$. Using the fact that the n-extension ∞-category $\mathscr{E}xt^n_{\mathcal{H}^{\mathrm{st}}_{\geq 0}(\mathscr{C})}(\mathscr{C},\mathscr{C})$ is equivalent to the ∞-category of maps from objects in $\mathscr{C}$ to an n^{th} suspensions of objects in $\mathscr{C}$, we obtain bifunctors

$$\mathscr{E}xt^n_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C} \longrightarrow \mathrm{Sp}, \tag{2}$$

by restricting the mapping space bifunctor on $\mathcal{H}^{\mathrm{st}}_{\geq 0}(\mathscr{C})$. Here, the ∞-category Sp is that of spectrum objects. The existence of the functor in (2) establishes a strong version of our main theorem.

Nakaoka and Palu have shown that the homotopy category $\mathrm{h}\mathscr{C}$ carries a canonical extriangulation [NP20]. The additive bifunctor of this extriangulated structure is given by

$$\pi_0\,\mathscr{E}xt^1_{\mathscr{C}}\colon (\mathrm{h}\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\mathscr{C} \longrightarrow \mathrm{Ab}.$$

Here, we have applied π_0 to the bifunctor $\mathscr{E}xt^n_{\mathscr{C}}$ in (2), with $n = 1$. For higher values of n, we show that the functors

$$\pi_0\,\mathscr{E}xt^n_{\mathscr{C}}\colon (\mathrm{h}\mathscr{C})^{\mathrm{op}} \times \mathrm{h}\mathscr{C} \longrightarrow \mathrm{Ab}.$$

are naturally isomorphic to the higher extension bifunctors of $\mathrm{h}\mathscr{C}$, as defined by Gorsky–Nakaoka–Palu [GNP21].

Paper III: Wide subcategories and Kreweras complementation for gentle algebras

Let A be a gentle k-algebra. The indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A))$ and $\mathcal{D}_{\mathrm{fd}}(A)$ can then be described in terms of curves on a surface [OPS18, APS19]. The overall aim of this paper is to give geometric interpretations of the results in Paper I, as well as defining Kreweras complements in a more general context.

We give an algorithm that constructs the (co-)Bongartz completion of a two-term presilting object P in terms of the surface model. Using the fact that mapping cylinders are given by resolving crossings of curves [OPS18, Theorem 4.1], the case where P is indecomposable is readily deduced. For the general case, we devise an iterative procedure, where each steps mimics the case where P is indecomposable.

Our next main result holds beyond the restricted setting of gentle algebras. We show that the simple objects in the τ-tilting reduction $J(P)$ of a two-term presilting object P can be acquired by the following algorithm:

1. Construct the Bongartz completion $T_P^+ = P \oplus Q^+$.

2. Let $\mathcal{S}$ be the two-term simple minded collection corresponding to T_P^+, and let $\mathcal{S}_{Q^+}$ be the

subcollection given by the elements in $P^{\perp}$.

3. Then $\mathcal{S}_{Q^+}$ are the simple objects in $J(P)$.

This result allows us to describe Jasso reductions on the surface model, yielding a special case of silting reduction procedure devised by Chang, Jin, and Schroll [CS20, CJS22].

Finally, we give a general definition of the Kreweras complement, which is valid beyond the setting of gentle algebras. It is explicitly shown that the cases of hereditary and representation-finite gentle algebras are special cases of our construction. The latter class has previously been described geometrically by Garver and McConville [GM20] in a different geometric model. We conclude our paper with a geometric interpretation of Kreweras complementation on the surface model of Amiot–Plamondon–Schroll.

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Paper I

Two-term silting and τ-cluster morphism categories

Erlend D. Børve

Submitted

This paper is submitted for publication and is therefore not included.

Paper II

Extension ∞-categories and a theorem of Retakh for exact ∞-categories

Erlend D. Børve and Paul Trygsland

Preprint

This paper will be submitted for publication and is therefore not included.

Paper III

Non-crossing partitions and wide subcategories for gentle algebras

Erlend D. Børve and Pierre-Guy Plamondon

Work in progress

WIDE SUBCATEGORIES AND KREWERAS COMPLEMENTATION FOR GENTLE ALGEBRAS

ERLEND D. BØRVE AND PIERRE-GUY PLAMONDON

Abstract. We give geometric descriptions of Bongartz completion, τ-tilting reduction and Kreweras complementation for gentle algebras.

Contents

0. Introduction and main results

Let A be a gentle k-algebra. The indecomposable objects of $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A))$ and $\mathcal{D}_{\mathrm{fd}}(A)$ can then be described in terms of arcs on a marked surface [OPS18, APS19].

In this note, we give an algorithm that constructs the Bongartz completion of a two-term presilting object P on the surface model. Using the fact that mapping cylinders are given by resolving crossings of curves, the case where P is indecomposable is readily deduced. For the general case, presented here as Theorem 2.8, we devise an iterative procedure, where each steps mimics the case where P is indecomposable.

Our next main result, namely Theorem 3.6, allows us to describe τ-tilting reductions in terms of the surface model. The result obtained corresponds to a specific instance of the silting reduction method developed by Chang, Jin, and Schroll [CS20, CJS22]. Given a two-term presilting complex $P \in \mathrm{per}(A)$, we produce the simple objects in the wide subcategory $J(P) = P^{\perp_{\mathbb{Z}}} \cap \mathrm{mod}(A) \subseteq \mathrm{mod}(A)$ as follows (see Theorem 3.7):

(1) Construct the Bongartz completion $T_P^+ = P \oplus Q^+$.

(2) Let $\mathcal{S}$ be the two-term simple minded collection corresponding to T_P^+, and let $\mathcal{S}_{Q^+}$ be the subcollection given by the elements in $P^\perp$. Then $\mathcal{S}_{Q^+}$ are the simple objects in $J(P)$.

In the case of gentle algebras, both steps can be carried out by drawing arcs on the surface of A. We have thus provided a geometric interpretation of τ-tilting reduction.

Finally, we present a broad definition of the Kreweras complement that extends beyond the scope of gentle algebras. Our construction demonstrates that the established Kreweras complements for hereditary and representation-finite gentle algebras are specific instances of the general definition. The latter has been geometrically defined by Garver and McConville in a different geometric model [GM20]. Lastly, we provide a geometric perspective of Kreweras complementation on a surface.

Notation and conventions. A field k is fixed throughout. For a k-algebra Λ, we denote the category of finitely generated right Λ-modules by $\mathrm{mod}(\Lambda)$. We will refer to right Λ-modules as modules. The category of $\mathrm{mod}(\Lambda)$ spanned by the indecomposable right A-modules will be denoted by $\mathrm{ind}(A)$. Arrows in a quiver and morphisms in a category are both composed from right to left. All subcategories of additive categories will be assumed to be additive.

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1. Preliminaries

1.1. Gentle algebras and geometric models.

Definition 1.1 ([AH81, AH82])**.** A *gentle quiver* is a bound quiver (Q, I) where

(GQ1) each vertex in Q has at most two in-going arrows, and at most two out-going arrows,

(GQ2) the ideal I is generated by paths of length 2,

(GQ3) for each arrow b there is at most one arrow a and at most one arrow c such that $ab \in I$ and $bc \in I$,

(GQ4) for each arrow b there is at most one arrow a and at most one arrow c such that $ab \notin I$ and $bc \notin I$.

The bound path k-algebra of a gentle quiver is a *gentle k-algebra*.

We will not require the path algebra of a gentle quiver to be finite-dimensional. Consequently, we will refrain from using the term *locally gentle* to refer to infinite-dimensional gentle algebras. We will rather emphasise finite-dimensionality whenever necessary.

Gentle k-algebras are string algebras, a class of algebras whose representation theory can be described explicitly by simple combinatorial rules [AS87]. If A is a string algebra , then the set of isomorphism classes of objects in $\mathrm{ind}(A)$ can be partitioned into two subclasses, namely those of string modules and band modules.

Recall that the perfect derived category of a k-algebra A, denoted by $\mathrm{per}(A)$, is the thick closure of the regular module A in the derived category $\mathcal{D}(A)$ of A, and that $\mathcal{D}_{\mathrm{fd}}(A)$ is the full subcategory of $\mathcal{D}(A)$ spanned by the objects with finite-dimensional total cohomology. In this section, we give a brief summary of how curves on a certain marked surface give rise to indecomposable objects in $\mathrm{per}(A)$ and $\mathcal{D}_{\mathrm{fd}}(A)$. The results presented here are drawn from a representation theoretic context [OPS18, APS19, PPP19], but the geometric model we use also bears traits from that of Haiden–Katzarkov–Kontsevich [HKK17, §4.2],

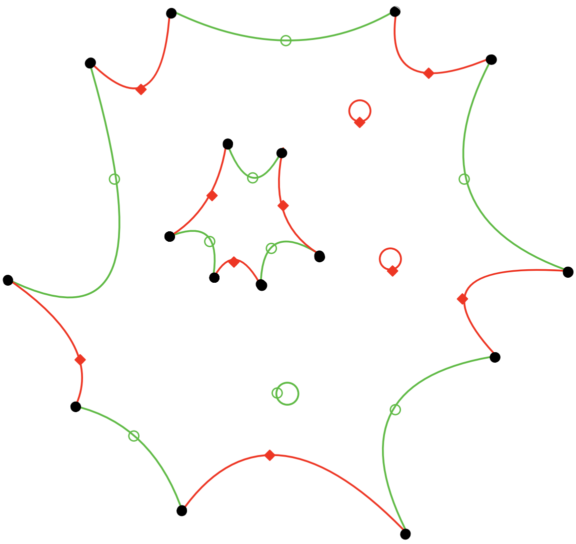


FIGURE 1. A marked surface. It is a topological annulus with one green and two red punctures.

Definition 1.2. Let X be a topological space. A *stratification* of X is a decomposition $\mathscr{S} = \{X_i\}_{i=0}^{d}$, i.e.

$$X = \bigsqcup_{i=0}^{d} X_i$$

into subspaces, subject to the following conditions

(1) the component X_i is either a smooth manifold of dimension i, or empty,

(2) for all j we have that

$$\overline{X_j} \setminus X_j \subseteq \bigcup_{i=0}^{j-1} X_i.$$

The subspace X_j will be called the j^{th} *layer* of the stratification, whereas its closure $\overline{X_j}$ is the *stratum of dimension* j. We refer to the tuple $(X, \mathscr{S})$ as a *statified space*.

Let $\pi_0(X_1)$ be the set of path components in X_1. On this occasion, we refer to the elements $\pi_0(X_1)$ as *boundary segments*.

Definition 1.3. A *marked surface* is a pair $(X, \mathscr{F})$, where $X = X_0 \sqcup X_1 \sqcup X_2$ is a compact orientable stratified surface and $\mathscr{F}$ is a map

$$\mathscr{F} : \pi_0(X_1) \longrightarrow \{\text{green}, \text{red}\}$$

such that if the closures of two components of X_1 share a point in X_0, then they have different colours.

A boundary segment whose closure disjoint from X_0 will be called a *puncture*.

An example is given in Figure 1. The points in of X_0 are marked by solid black points $\bullet$. Apart from colours, we have attached green circles $\circ$ to the green components of X_1, and solid red diamonds $\blacklozenge$ to the red components.

In the following definition and henceforth, curves on marked surfaces will be considered up to isotopy. Thus, two arcs *intersect* only if every choice of isotopic representatives intersect.

Definition 1.4 ([APS19, Definitions 1.8 and 2.4],[PPP19, Definition 3.1])**.** Let X be a marked surface and let $V \subseteq \partial X$ be a subset of the boundary.

(1) A *curve* on a surface X is a smooth parameterised curve $\gamma\colon]\text{-}1,1[\longrightarrow X^\circ$ from the open interval $]\text{-}1,1[$ into the interior of X.

(2) Suppose that V is disjoint from the punctures. Then a *finite* V*-arc* on X is a non-contractible curve γ on X for which both limits $\lim_{x\to 1}\gamma(x)$ and $\lim_{x\to -1}\gamma(x)$ are in V.

(3) An *infinite* V*-arc* X is a curve on X which on at least one end circles around and finally reaches a puncture. We require the circular motion to be counter-clockwise.

(4) If V is the set of green boundary segments (including green punctures), we refer to V-arcs as $\circ$-arcs, whereas $\blacklozenge$-arcs connect red boundary segments.

(5) A *closed curve* is a non-contractible curve whose starting points and ending points are the same puncture of X.

(6) A V*-dissection* is a maximal family of non-intersecting V-arcs. We will assumme that the arcs approach the boundary transversally. We will use the term $\circ$*-dissection* when referring to dissection of $\circ$-arcs, and $\blacklozenge$*-dissection* when it consists solely of $\blacklozenge$-arcs.

(7) The arcs of a dissection Δ and boundary of X bounds a set of polygons on the surface. These are called the *faces* of the dissection. Let $\mathcal{F}(\Delta)$ denote the set of faces of a dissection Δ. We say that Δ is *cellular* if all faces are topological discs.

(8) Let X_1° denote the set of green boundary segments, and $X_1^\blacklozenge$ that of red boundary segments. A $\blacklozenge$-dissection Δ^* is *dual* to a $\circ$-dissection Δ if there exist mutually inverse bijections $X_1^\blacklozenge \longleftrightarrow \mathcal{F}(D)$ and $X_1^\circ \longleftrightarrow \mathcal{F}(\Delta^*)$, denoted $*$ in both directions, such that Δ has an edge joining its vertices $u, v \in X_1^\circ$ and separating its faces $f, g \in \mathcal{F}(\Delta)$ if and only if Δ^* has an edge joining its vertices $f^*, g^* \in X_1^\blacklozenge$ and separating its faces $u^*, v^* \in \mathcal{F}(\Delta^*)$.

A dual dissection of $\circ$-dissections (and $\blacklozenge$-dissections) always exists when each face contains precisely one component of $X_1^\blacklozenge$ [OPS18, Proposition 1.16]. If they exist, they are uniquely determined. Each $\circ$-arc crosses exactly one $\blacklozenge$-arc in the dual dissection [APS19, Proposition 1.12]. We can, and will, assume that these intersections are transversal.

The following result is key when providing geometric models for gentle algebras.

Theorem 1.5 ([PPP19, Theorem 4.10])**.** *Let X be a marked surface equipped with a $\circ$-dissection Δ and a dual $\blacklozenge$-dissection Δ^*. Define a bound quiver (Q_Δ, I_Δ) by*

- *setting the $\circ$-arcs in Δ as the vertices in Q_Δ,*
- *adding an arrow from a to b whenever the $\circ$-arcs a and b share an endpoint v such that b comes immediately after a in the counter-clockwise order around v,*
- *adding a path $\beta\alpha$ in I_Δ whenever α and β are composable arrows corresponding to triples of consecutive edges along a face of Δ.*

Then (Q_Δ, I_Δ) is a gentle quiver. Moreover, this procedure defines a bijection between the following sets:

(1) marked surfaces equipped with a $\circ$-dissection and a dual $\blacklozenge$-dissection,

(2) (possibly infinite-dimensional) gentle algebras up to isomorphism.

These bijections restrict to bijections between

(1) marked surfaces without green punctures, equipped with a $\circ$*-dissection and a dual* ♦*-dissection,*
(2) finite-dimensional gentle algebras up to isomorphism.

Definition 1.6 ([APS19, Definition 2.4])**.** Let (X, Δ) be a marked surface with a $\circ$-dissection Δ. A *graded* $\circ$*-arc* is a tuple (γ, f), where γ is a $\circ$-arc and

$$f : \gamma \cap \Delta^* \longrightarrow \mathbb{Z}$$

is a function whose domain is the totally ordered set of intersection points of γ with the arcs of Δ^*. We require that if p and q are in $\gamma \cap D$ and q is the successor of p, then γ enters a face f enclosed by arcs of Δ via p and leaves it via q. If the the dual boundary segment f^* in this polygon is to the left (resp. right) of γ, then $f(q) = f(p) + 1$ (resp. $f(q) = f(p) - 1$).

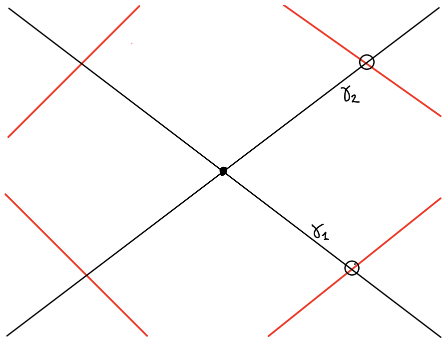
One defines the notion of *graded* ♦*-arc* dually.

Each graded $\circ$-arc gives rise to an indecomposable object in $\mathcal{D}_{\mathrm{fd}}(A)$. For a graded $\circ$-arc (γ, f), let $\gamma \cap \Delta^*$ be the ordered set of crossings with the ♦-arcs in Δ^*. Given a crossing $i \in \gamma \cap \Delta^*$ crossing a certain ♦-arc, let $v(i)$ be the vertex in Q_Δ corresponding to the $\circ$-arc in Δ crossing that ♦-arc. Two consecutive crossings i and j lie on the edge of a red polygon containing a unique green boundary segment. If this green boundary segment is to the left of γ there is a map of projective modules $P_{v(i)} \longrightarrow P_{v(j)}$ given by the composite of the arrows incident to this green boundary segment. Should the green boundary segment be to the right of γ, we have a map $P_{v(j)} \longrightarrow P_{v(i)}$. One constructs an object in $K^{b,-}(\operatorname{proj} A)$, where the underlying $\mathbb{Z}$-graded module is $\bigoplus_{i \in \gamma \cap \Delta^*} P_{v(i)}$ and the components of the complex are these maps. Denote such an object by $P_{(\gamma,f)}$.

Theorem 1.7 ([OPS18, Theorem 2.12])**.** *Let* (X, Δ) *be a marked surface with a* $\circ$*-dissection* Δ *and let* A *the corresponding gentle algebra (see Theorem 1.5). The isomorphism classes of string and band objects in* $\mathcal{D}_{\mathrm{fd}}(A)$ *are respectively represented by*

(1) graded curves (γ, f)*, where* γ *is a finite* $\circ$*-arc on* X *or an infinite* $\circ$*-arc on* X*. The finite* $\circ$*-arcs correspond to the objects in* $\operatorname{per}(A)$*.*
(2) pairs $\big((\gamma, f), M\big)$*, where* (γ, f) *is a graded closed* $\circ$*-curve on* X *and* M *is an isomorphism class of indecomposable* $k[X]$*-modules.*

Definition 1.8 ([OPS18, Definition 3.7 and Remark 3.9])**.** Let (γ_1, f_1) and (γ_2, f_2) be two graded arcs. A *graded orientented intersection* from (γ_1, f_1) to (γ_2, f_2) is an intersection point of the curves γ_1 and γ_2 such that $f_1(r_1) = f_2(r_2)$, where r_i is the encircled crossing of ♦-arcs in Δ^* below.



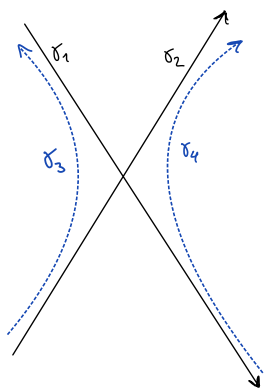
The four ♦-arcs need not be distinct, and the crossing may occur on the boundary of the surface.

Theorem 1.9 ([OPS18, Theorem 3.3])**.** *Let (γ_1, f_1) and (γ_2, f_2) be graded ○-arcs on the marked surface of a gentle algebra A, and let $\mathbf{B}_{1,2}$ be the standard basis of $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(A)}(P_{(\gamma_1,f_1)}, P_{(\gamma_2,f_2)})$. Let $(\gamma_1, f_1) \cap^{\mathrm{gr}}_{\rightarrow} (\gamma_2, f_2)$ be the set of oriented graded intersection points. Then there exists an explicit bijection*

$$(\gamma_1, f_1) \cap^{\mathrm{gr}}_{\rightarrow} (\gamma_2, f_2) \longrightarrow \mathbf{B}_{1,2}.$$

Definition 1.10. A ○-dissection (resp. ♦-dissection) is *admissible* if its arcs do not enclose a subsurface containing boundary segments (nor punctures). A maximal admissible collection of ○-arcs (resp. ♦-arcs) is an admissible ○-dissection (resp. ♦-dissection).

Theorem 1.11 ([OPS18, Theorem 4.1])**.** *Let (γ_1, f_1) and (γ_2, f_2) be two graded arcs with non-empty graded oriented intersection $(\gamma_1, f_1) \cap^{\mathrm{gr}}_{\rightarrow} (\gamma_2, f_2)$ and let $\varphi \colon P_{(\gamma_1,f_1)} \longrightarrow P_{(\gamma_2,f_2)}$ be a standard basis morphism in $\mathcal{D}_{\mathrm{fd}}(A)$ associated to a crossing point X in $(\gamma_1, f_1) \cap^{\mathrm{gr}}_{\rightarrow} (\gamma_2, f_2)$. Then the mapping cone of φ is given by $P_{(\gamma_3,f_3)} \oplus P_{(\gamma_4,f_4)}$, where the graded arcs or closed curves (γ_3, f_3) and (γ_4, f_4) are given by the resolution of the crossing of (γ_1, f_2) and (γ_2, f_2) at X.*



One constructs the mapping cylinder by desuspending the mapping cone. This does not change the underlying curve, but the grading is shifted by -1 in all components.

1.2. **Silting objects.** A k-linear category is *Hom-finite* it its Hom-spaces are finite-dimensional. It is *Krull–Schmidt* if each object can be written as a direct sum of objects with local endomorphism algebras.

Definition 1.12. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ.

(1) An object $P \in \mathcal{C}$ is *presilting* if $\mathcal{C}(P, \Sigma^i P) = 0$ for all $i > 0$.
(2) A presilting object $T \in \mathcal{C}$ *silting* if it, in addition, generates $\mathcal{C}$, i.e. $\mathrm{thick}_{\mathcal{C}}(T) = \mathcal{C}$.
(3) Let S be a silting object in $\mathcal{C}$. An object $T \in \mathcal{C}$ is *two-term with respect to* S (or 2_S-term) if it is (isomorphic to) a cone of a morphism in $\mathrm{add}(S)$. In other words, there exists a triangle in $\mathcal{C}$

$$S_1 \xrightarrow{t} S_0 \longrightarrow T \longrightarrow \Sigma S_1$$

with $S_0, S_1 \in \mathrm{add}(S)$.
(4) Suppose that $\mathcal{C}$ is a Hom-finite and Krull–Schmidt category. An object $X \in \mathcal{C}$ is *basic* if the indecomposable direct summands of its decomposition are pairwise non-isomorphic.

Unless explicitly stated otherwise, it will not be assumed that our (pre)silting objects are basic, nor that we have chosen the unique basic (pre)silting object for each equivalent class. The set of 2_S-term basic presilting objects will be denoted by $2_S\text{-presilt}(\mathcal{C})$, and the subset of 2_S-term basic silting objects by $2_S\text{-silt}(\mathcal{C})$.

The set $2_S\text{-silt}(\mathcal{C})$ admits a partial order $\geq$, where $P \geq Q$ provided that $\mathcal{C}(P, \Sigma^n Q) = 0$ whenever $n > 0$ [AI12, Theorem 2.11 and Proposition 2.14].

If A is a non-positive dg algebra, the regular module A is a silting object in $\mathrm{per}(A)$. In such instances, two-termness is measured with respect to A. We write $2\text{-silt}(A)$ in place of $2_A\text{-silt}(\mathrm{per}(A))$.

Theorem 1.13 ([APS19, Theorem 3.2 and Lemma 3.4])**.** *Let A be the gentle algebra and let (S_A, M, Δ) be its surface model. Any indecomposable summand of a presilting complex in* $\mathrm{per}(A)$ *is isomorphic to an object of the form $P_{(\gamma,f)}$ for some graded $\circ$-arc (γ, f).*

A basic silting object in $\mathrm{per}(A)$ *is isomorphic to a direct sum $\bigoplus_{i=1}^{n} P_{(\gamma_i,f_i)}$, where the set of $\circ$-arcs $\gamma_1, \ldots, \gamma_n$ is an admissible $\circ$-dissection of the surface of A.*

1.3. **Wide subcategories, torsion classes, and semibricks.** Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{W} \subseteq \mathcal{A}$ is *wide* if it is closed under kernels, cokernels and extensions. Equivalently, regarding $\mathcal{A}$ as an exact category, a wide subcategory is an exact subcategory of $\mathcal{A}$ which is abelian. The wide subcategories of a given abelian category $\mathcal{A}$ form a lattice, which we denote by $\mathrm{wide}(\mathcal{A})$.

We will on occasions find it useful to construct wide subcategories using triangulated machinery. Let $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ be the bounded derived category of $\mathcal{A}$. Recall that $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ can be a equipped with a bounded t-structure where the aisle and co-aisle are

$$\mathcal{D}_{\mathrm{fd}}^{\leq 0}(\mathcal{A}) = \{X \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i > 0\},$$
$$\mathcal{D}_{\mathrm{fd}}^{> 0}(\mathcal{A}) = \{X \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i \leq 0\},$$

respectively, where the H^is are the cohomology functors. We have a truncation functor $\sigma^{\leq 0} \colon \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \longrightarrow \mathcal{D}_{\mathrm{fd}}^{\leq 0}(\mathcal{A})$ which is right adjoint to the inclusion $\mathcal{D}_{\mathrm{fd}}^{\leq 0}(\mathcal{A}) \hookrightarrow \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. We say that a subcategory $\mathcal{S} \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ is *t-exact* if $\sigma^{\leq 0}X \in \mathcal{S}$ for all $X \in \mathcal{S}$. The lattice of t-exact subcategories of $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ will be denoted by $\text{t-exact}(\mathcal{A})$.

Theorem 1.14 ([ZC17, Corollary 3.1])**.** *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ be its bounded derived category. We equip $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ with its standard t-structure. The cohomology functor $H^0\colon \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \longrightarrow \mathcal{A}$ induces an isomorphism of lattices*

$$H^0\colon \text{t-exact}(\mathcal{A}) \longrightarrow \text{wide}(\mathcal{A}) \tag{1.A}$$

with inverse $\text{thick}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}\colon \text{wide}(\mathcal{A}) \longrightarrow \text{t-exact}(\mathcal{A})$ sending a wide subcategory $\mathcal{W} \subseteq \mathcal{A}$ to its thick closure in $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.

Let $\mathcal{C}$ be an additive category and $\mathcal{P} \subseteq \mathcal{C}$ an additive subcategory. A morphism $Q \xrightarrow{\beta} Y$ (resp. $X \xrightarrow{\alpha} Q$) is a *right $\mathcal{P}$-approximation* of Y (resp. *left $\mathcal{P}$-approximation* of X) if $Q \in \mathcal{P}$ and the induced morphism $\mathcal{C}(P, \beta)$ (resp. $\mathcal{C}(\alpha, P)$) surjects for all objects $P \in \mathcal{P}$. If every object in $\mathcal{C}$ has a right (resp. left) $\mathcal{P}$-approximation, then $\mathcal{P}$ has the property of being *contravariantly finite* (resp. *covariantly finite*). A *functorially finite subcategory* of $\mathcal{C}$ is both contravariantly and covariantly finite.

Lemma 1.15 ([Eno22, Proposition 4.12])**.** *Let A be an artin algebra. Then a wide subcategory of* $\text{mod}(A)$ *is functorially finite if and only if it is equivalent to a module category of an artin algebra.*

The *rank* of a functorially finite wide subcategory is defined to be the rank of its Grothendieck group.

Let P be a two-term presilting object in $\text{per}(A)$, where A is a finite-dimensional algebra. Jasso's τ-tilting reduction [Jas15] produces a certain wide subcategory $J(P)$ of $\text{mod}(A)$, which can be defined by $J(P) \stackrel{\text{def}}{=} P^{\perp_{\mathbb{Z}}} \cap \text{mod}(A)$. More precisely, the map H^0 in Theorem 1.14 sends the t-exact subcategory $P^{\perp_{\mathbb{Z}}}$ to the τ-tilting reduciton $J(P)$ [Bør21, Corollar 2.12]. A wide subcategory is *τ-perpendicular* if it arises in this manner.

The following result, in conjunction with Lemma 1.15, shows that a τ-perpendicular wide subcategory is functorially finite.

Theorem 1.16 ([Jas15, Theorem 3.8],[DIR$^+$17, Theorem 4.12(b)])**.** *Let A be a finite-dimensional algebra, and let P be a two-term presilting complex in* $\text{per}(A)$*. There exists a finite-dimensional algebra C and an exact equivalence*

$$J(P) \longrightarrow \text{mod}(C).$$

We refer to Buan–Hanson for an example of a functorially finite subcategory which is not τ-perpendicular [BH21, Example 4.9]. If the rank of A is n and P has r indecomposable direct summands up to isomorphism, then the rank of $J(P)$ will be $n - r$ [Jas15, Theorem 3.8].

Recall that a *torsion class* (resp. *torsion-free class*) in an abelian category $\mathcal{A}$ is a subcategory which is closed under factors (resp. submodules) and extensions. Denoting the lattice of torsion classes (resp. torsion-free classes) in $\mathcal{A}$ by $\text{tors}(\mathcal{A})$ (resp. $\text{torf}(\mathcal{A})$), we have mutually inverse order-reversing bijections

$$\begin{tikzcd} \text{tors}(A) \arrow[r, bend right, "(-)^{\perp}"'] & \text{torf}(A) \arrow[l, bend right, "{}^{\perp}(-)"'] \end{tikzcd}$$

that restrict to mutually inverse order-reversing bijections

$$\begin{tikzcd} \text{f-tors}(A) \arrow[rr, bend right, "(-)^{\perp}"'] & & \text{f-torf}(A) \arrow[ll, bend right, "{}^{\perp}(-)"'] \end{tikzcd} \tag{1.B}$$

between the lattices of functorially finite torsion classes and functorially finite torsion-free classes [Sma84]. A *torsion pair* is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$, where $\mathcal{T}$ is a torsion class and and $\mathcal{T}^{\perp} = \mathcal{F}$.

Given an additive subcategory $\mathcal{X} \subseteq \mathrm{mod}(A)$, let $\mathrm{filt}(\mathcal{X})$ be the closure of $\mathcal{X}$ under extensions. The smallest torsion class containing $\mathcal{X}$ will be denoted by $\mathbf{T}(\mathcal{X})$. It can be defined formally as the intersection of all torsion classes containing $\mathcal{X}$. Dually, we can define the smallest torsion-free class $\mathbf{F}(\mathcal{X})$ containing $\mathcal{X}$.

Proposition 1.17 ([MŠ17, Proposition 3.1]). *Let $\mathcal{W} \subseteq \mathrm{mod}(A)$ be a wide subcategory, where A is a finite-dimensional algebra. Then $\mathbf{T}(\mathcal{W}) = \mathrm{filt}\,\mathrm{gen}(\mathcal{W})$. Dually, we have that $\mathbf{F}(\mathcal{W}) = \mathrm{filt}\,\mathrm{sub}(\mathcal{W})$.*

Proposition 1.18 ([MŠ17, Proposition 3.3]). *We have maps of sets*

$$\begin{tikzcd} \mathrm{tors}(A) \arrow[r, bend left, "\alpha"] & \mathrm{wide}(A) \arrow[l, bend left, "\mathbf{T}=\mathrm{filt}\,\mathrm{gen}"] \end{tikzcd}$$

where $\alpha(\mathcal{T}) \overset{def}{=} \{X \in \mathcal{T} \mid \forall (g\colon Y \longrightarrow X) \in \mathcal{T}\colon\ \ker g \in \mathcal{T}\}$. Moreover, the composite $\alpha \circ \mathbf{T}$ is the identity map on $\mathrm{wide}(A)$. In particular, the map α surjects, and $\mathbf{T}$ injects.

Dually, we have maps of sets

$$\begin{tikzcd} \mathrm{torf}(A) \arrow[r, bend left, "\alpha'"] & \mathrm{wide}(A). \arrow[l, bend left, "\mathcal{F}=\mathrm{filt}\,\mathrm{sub}"] \end{tikzcd}$$

where $\alpha'(\mathcal{F}) \overset{def}{=} \{X \in \mathcal{F} \mid \forall (g\colon X \longrightarrow Y) \in \mathcal{F}\colon\ \mathrm{cok}\, g \in \mathcal{F}\}$. The composite $\alpha' \circ \mathbf{F}$ is the identity map on $\mathrm{wide}(A)$, whence the map α' surjects, and $\mathbf{F}$ injects.

The maps $\mathbf{T}$ and $\mathbf{F}$ preserve inclusions, whereas α and α' typically do not.

Definition 1.19. A wide subcategory $\mathcal{W} \subseteq \mathrm{mod}(A)$ is *left finite* (resp. *right finite*) if the torsion class $\mathbf{T}(\mathcal{W})$ is functorially finite (resp. the torsion-free class $\mathbf{F}(\mathcal{W})$ is functorially finite).

Definition 1.20. A *Serre subcategory* is a wide subcategory which is closed under subobjects.

Proposition 1.21 ([GL91, Proposition 5.3]). *If A is an artin algebra and let $\mathcal{W}$ be a wide subcategory of $\mathrm{mod}(A)$. The following assertions are equivalent.*

1. *The wide subcategory $\mathcal{W}$ is a Serre subcategory of $\mathrm{mod}(A)$.*
2. *The wide subcategory $\mathcal{W}$ is generated by simple modules.*
3. *There exists a finitely generated projective A-module Q such that $\mathcal{W} = Q^{\perp}$. In particular, we have that $\mathcal{W} = J(Q)$.*
4. *There exists an idempotent ideal I and an additive equivalence $\mathcal{W} \cong \mathrm{mod}(A/I)$.*

Moreover:

(5) *If* $\mathcal{W} = J(Q)$ *is a Serre subcategory, then the simple objects therein are the simple* A*-modules* S *for which* $\mathrm{Hom}_A(Q, S) = 0$.

We summarise the exposition hitherto in this subsection by copying a diagram of Buan–Hanson [BH21, Remark 4.10]:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{f_L}\text{-wide}(\mathcal{A}) \arrow[dr, phantom, sloped, "\subsetneq"] & & & \\
\mathrm{Serre}(\mathcal{A}) \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \tau\text{-perp-wide}(\mathcal{A}) \arrow[r, phantom, "\subseteq"] & \text{f-wide}(\mathcal{A}) \arrow[r, phantom, "\subseteq"] & \text{wide}(\mathcal{A}) \\
 & \mathrm{f_R}\text{-wide}(A) \arrow[ur, phantom, sloped, "\subsetneq"] & & &
\end{tikzcd} \tag{1.C}$$

Buan–Hanson provide examples justifying that each inclusion may be proper. However, there are some important conditions on the algebra under which some of these posets coincide.

Definition 1.22. The k-algebra A is *τ-tilting finite* the number of isomorphism classes of two-term silting objects is finite.

A k-algebra is τ-tilting finite if and only if $\mathrm{tors}(A) = \text{f-tors}(A)$ [DIJ19, Theorem 3.8]. The following lemma is then immediate.

Lemma 1.23. *If A is τ-tilting finite, then*

$$\mathrm{f_L}\text{-wide}(A) = \mathrm{f_R}\text{-wide}(A) = \tau\text{-perp-wide}(A) = \text{f-wide}(A) = \text{wide}(A).$$

Lemma 1.24 ([IT09, Corollary 2.17])**.** *If A hereditary, then*

$$\mathrm{f_L}\text{-wide}(A) = \mathrm{f_R}\text{-wide}(A) = \tau\text{-perp-wide}(A) = \text{f-wide}(A).$$

Finally, we recall the close connection between wide subcategories and semibricks.

Definition 1.25. Let $\mathcal{A}$ be a k-linear abelian category.

(1) An object $S \in \mathcal{A}$ is a *brick* if $\mathrm{End}_{\mathcal{A}}(S)$ is a division k-algebra.
(2) A set $\mathcal{S}$ of bricks is a *semibrick* provided that $\mathcal{A}(S, S') = 0$ for $S \neq S'$.

We denote the sets of bricks and semibricks in $\mathcal{A}$ by $\mathrm{brick}(\mathcal{A})$ and $\mathrm{sbrick}(\mathcal{A})$, respectively, or simply $\mathrm{brick}(A)$ and $\mathrm{sbrick}(A)$, respectively, if $\mathcal{A} = \mathrm{mod}(A)$ for some k-algebra A.

An object S in an exact category is *simple* if there are no non-split short exact sequences of the form

$$0 \longrightarrow M \longrightarrow S \longrightarrow N \longrightarrow 0$$

Let $\mathrm{simp}(\mathcal{A})$ denote the set of simple objects in $\mathcal{A}$.

Lemma 1.26 ([Rin76, (1.2)])**.** *Let A be finite-dimensional k-algebra. We have mutually inverse bijections*

$$\begin{tikzcd}
\mathrm{sbrick}(A) \arrow[r, bend left, "\mathrm{filt}"] & \mathrm{wide}(A) \arrow[l, bend left, "\mathrm{simp}"]
\end{tikzcd}$$

By transferring the partial ordering on $\mathrm{wide}(A)$ along the bijection simp in Lemma 1.26, we deduce a lattice structure on $\mathrm{sbrick}(A)$.

A semibrick $\mathcal{S}$ is *left finite* (resp. *right finite*) if the corresponding wide subcategory is left finite (resp. right finite). We use the notation $\mathrm{f_L}\text{-}\mathrm{sbrick}(\mathcal{A})$ (resp. $\mathrm{f_R}\text{-}\mathrm{sbrick}(\mathcal{A})$) for this subset of $\mathrm{sbrick}(A)$. A subset of a left finite semibrick need not be left finite [Asa20, Example 3.13].

Lemma 1.27 ([Asa20, Corollary 1.10])**.** *Let A be an artinian k-algebra. If a semibrick $\mathcal{S}$ in $\mathrm{mod}(A)$ is either left or right finite, its cardinality $|\mathcal{S}|$ is bounded above by the rank $|A|$ of A.*

As for wide subcategories, the smallest torsion class (resp. torsion-free class) containing a semibrick $\mathcal{S}$ is given by $\mathbf{T}(\mathcal{S}) \overset{\text{def}}{=} \mathrm{filt}\,\mathrm{gen}(\mathcal{S})$ (resp. $\mathbf{F}(\mathcal{S}) \overset{\text{def}}{=} \mathrm{filt}\,\mathrm{sub}(\mathcal{S})$). We provide lemma for future reference.

Lemma 1.28 ([Asa20, Proposition 1.9])**.** *We have a commutative diagrams of lattice maps*

$$\begin{tikzcd}
 & \mathrm{tors}(A) \\
\mathrm{sbrick}(A) \arrow[ur, "\mathbf{T}"] \arrow[dr, bend left, "\mathrm{filt}"] & \\
 & \mathrm{wide}(A) \arrow[uu, "\mathbf{T}"'] \arrow[ul, bend left, "\mathrm{simp}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{torf}(A) \\
\mathrm{sbrick}(A) \arrow[ur, "\mathbf{F}"] \arrow[dr, bend left, "\mathrm{filt}"] & \\
 & \mathrm{wide}(A) \arrow[uu, "\mathbf{F}"'] \arrow[ul, bend left, "\mathrm{simp}"]
\end{tikzcd}$$

They restrict to commutative diagrams of lattice isomorphisms

$$\begin{tikzcd}
 & \mathrm{f}\text{-}\mathrm{tors}(A) \\
\mathrm{f_L}\text{-}\mathrm{sbrick}(A) \arrow[ur, "\mathbf{T}"] \arrow[dr, bend left, "\mathrm{filt}"] & \\
 & \mathrm{f_L}\text{-}\mathrm{wide}(A) \arrow[uu, "\mathbf{T}"'] \arrow[ul, bend left, "\mathrm{simp}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{f}\text{-}\mathrm{torf}(A) \\
\mathrm{f_R}\text{-}\mathrm{sbrick}(A) \arrow[ur, "\mathbf{F}"] \arrow[dr, bend left, "\mathrm{filt}"] & \\
 & \mathrm{f_R}\text{-}\mathrm{wide}(A) \arrow[uu, "\mathbf{F}"'] \arrow[ul, bend left, "\mathrm{simp}"]
\end{tikzcd}$$

2. Bongartz completion on the surface of a gentle algebra

Let A be a gentle k-algebra. In this subsection, we interpret (co-)Bongartz completion geometrically on the marked surface associated to A. First, we recall the basic properties of (co-)Bongartz completion.

Definition 2.1. Let P be a two-term presilting complex in $\mathrm{per}(A)$.

(1) One constructs the *Bongartz completion* T_P^+ of P by choosing a right $\mathrm{add}(P)$-approximation $P' \xrightarrow{\beta_A} \Sigma A$ of ΣA and then setting $T_P^+ \overset{\text{def}}{=} P \oplus Q^+$, where Q^+ is defined by the triangle

$$A \longrightarrow Q^+ \longrightarrow P' \xrightarrow{\beta_{\Sigma A}} \Sigma A. \tag{2.A}$$

The object Q^+ is called the *Bongartz complement* of P.

(2) Dually, the *co-Bongartz completion* T_P^- of P is constructed by choosing a left $\mathrm{add}(P)$-approximation $A \xrightarrow{\alpha_A} P'$ of A and then setting $T_P^- \overset{\text{def}}{=} P \oplus Q^-$, where Q^- is defined by the triangle

$$A \xrightarrow{\alpha_A} P' \longrightarrow Q^- \longrightarrow \Sigma A. \tag{2.B}$$

The object Q^- is called the *co-Bongartz complement* of P.

The Bongartz and co-Bongartz completions are two-term silting complexes in $\mathrm{per}(A)$ [DF15, §5], [IJY14, Lemma 4.2]. Since the definitions of the Bongartz and co-Bongartz complements depend on the choice of approximations, they are not uniquely determined on the nose. However, they will be well defined if we insist that they be basic. Under the insistence of basicity, the Bongartz and co-Bongartz completions of P become the maximal and minimal objects (with respect to the partial order $\geq$ defined in Section 1.2) of which P is a direct summand, respectively. Importantly, all completions of a two-term presilting complex have the same number of direct summands up to isomorphism. Indeed, if $|P|$ denotes the number of non-isomorphic indecomposable direct summands of a two-term presilting complex P, we have that $|P|$ is a silting complex if and only if $|P| = |A|$ [IJY14, Lemma 4.3]. In particular, we may claim that the Bongartz and co-Bongartz completions have $|A|$ direct summands up to isomorphism.

We include an elementary lemma that will allow us to detect approximations.

Lemma 2.2. *Let $\mathcal{T}$ be a triangulated category and let P be a presilting object in $\mathcal{T}$. Consider the triangle*

$$X \xrightarrow{\alpha} P' \xrightarrow{\beta} Y \longrightarrow \Sigma X.$$

where $P' \in \mathrm{add}(P)$.

(1) The morphism β is a right $\mathrm{add}(P)$-approximation of Y if and only if $\mathcal{T}(P, \Sigma X) = 0$.
(2) Dually, the morphism α is a left $\mathrm{add}(P)$-approximation of Y if and only if $\mathcal{T}(Y, \Sigma P) = 0$.

We now fix a surface model (S_A, Δ) for the gentle algebra A. In so doing, we have fixed a $\circ$-dissection Δ on the surface S, as well as a dual ♦-dissection. It was recalled in Theorem 1.13 that any indecomposable presilting complex can be presented in terms of graded $\circ$-arcs on S. As a first step to achieve the aim of this subsection, we should give a geometric condition characterising the two-term presilting complexes.

Our definition below is a slight adaption of definitions from the literature [PPP19, Definition 3.8] [PPS+18, §4], where a different geometric model has been used.

Definition 2.3. A $\circ$-curve γ is a Δ-*accordion* of the collection of arcs in Δ crossed by γ is connected.

An example of an accordion is shown in Figure 2, where the corresponding two-term presilting complex $P_{(\gamma,f)}$ is

$$P_1 \oplus P_8 \xrightarrow{\begin{pmatrix} cba & gfed \end{pmatrix}} P_4.$$

There may, of course, be arcs in Δ that γ does not cross. These are not shown in Figure 2.

Lemma 2.4. *An indecomposable presilting complex $P_{(\gamma,f)}$ in $\mathrm{per}(A)$ is two-term precisely when γ is a Δ-accordion and the grading f only takes values in $\{-1, 0\}$.*

Proof. It is clear from the construction of string objects in $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}(A))$ that the grading takes values in $\{-1, 0\}$ precisely when $P_{(\gamma,f)}$ is two-term. For $P_{(\gamma,f)}$ to be indecomposable, we need connectedness. □

The geometric procedure of completing presilting complexes thus entails completing an admissible collection of accordions to a maximal one. We will first address the case where the collection consists of

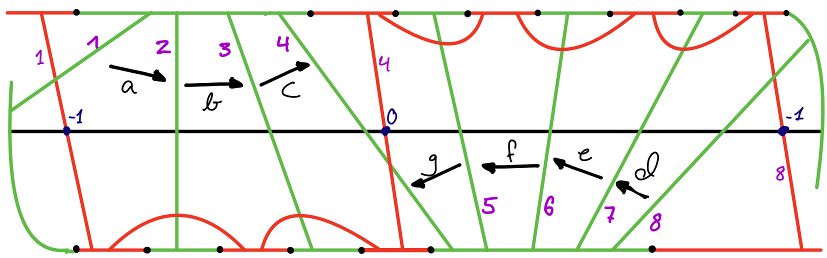


FIGURE 2. An accordion.

one arc. Henceforth, we only concern ourselves with Bongartz completions, noting that our arguments can be dualised to construct co-Bongartz completions.

The following lemma is widely known.

Lemma 2.5. *Let $\mathcal{C}$ be a k-linear Hom-finite triangulated category, and let $X \in \mathcal{C}$. Then* $\mathrm{add}(X)$ *is functorially finite in $\mathcal{C}$. Indeed, given $Y \in \mathcal{C}$, one constructs a right (resp. left)* $\mathrm{add}(X)$*-approximation β of Y by choosing a basis $\mathbf{B} = \{b_1, \ldots, b_t\}$ of $\mathcal{C}(X,Y)$ (resp. $\mathcal{C}(Y,X)$) and setting β equal to*

$$X^{\oplus t} \xrightarrow{(b_i)_i} Y$$

$$(\textit{resp.} \quad Y \xrightarrow{(b_i)_i} X^{\oplus t}).$$

By Theorem 1.9, one can regard the Hom-space $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(A)}(P_{(\gamma_1,f_1)}, \Sigma A)$ as the free k-vector space generated by the set $\bigcup_{i=1}^{n} (\gamma_1, f_1) \cap^{\mathrm{gr}}_{\rightarrow} (\delta_i, -1)$, where $(\delta_i, -1)$ is the graded curve corresponding to the shifted indecomposable projective ΣP_i. One then constructs the mapping cylinder by resolving the crossing using Theorem 1.11.

Lemma 2.6. *Let (γ, f) be an accordion and let R be a presilting object in* $\mathrm{per}(A)$*. Let β_R be a right* $\mathrm{add}(P_{(\gamma,f)})$*-approximation of R and consider the triangle*

$$R \longrightarrow Q \longrightarrow P' \xrightarrow{\beta_R} \Sigma R.$$

Then the presilting object Q is constructed geometrically by resolving all crossings of (γ, f) with curves in R and add them.

If one takes R to be A in Lemma 2.6, one obtains a construction for the Bongartz completion of P. The Bongartz completion of the accordion in Figure 2 is shown in Figure 3.

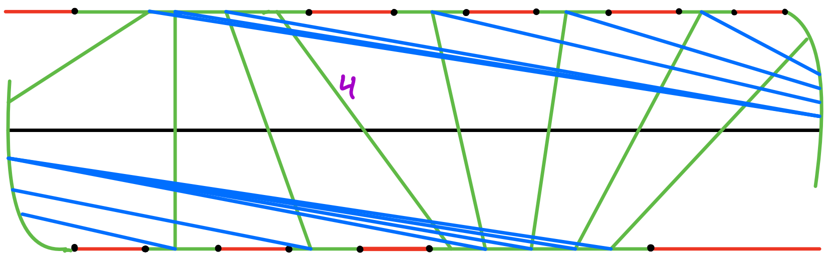


FIGURE 3. The Bongartz completion of the accordion in Figure 2, shown in blue.

We need a lemma to construct Bongartz completions in general.

Lemma 2.7. *Let* $P = \bigoplus_{i=1}^{\ell} P_i$ *be a presilting object in* $\mathrm{per}(A)$. *Consider the triangles*

$$
\begin{array}{rcl}
A \longrightarrow Q_1^+ \longrightarrow P_1' \xrightarrow{\beta_1} \Sigma A \\
Q_1^+ \longrightarrow Q_2^+ \longrightarrow P_2' \xrightarrow{\beta_2} \Sigma Q_1^+ \\
\vdots \\
Q_{\ell-2}^+ \longrightarrow Q_{\ell-1}^+ \longrightarrow P_{\ell-1}' \xrightarrow{\beta_{\ell-1}} \Sigma Q_{\ell-2}^+ \\
Q_{\ell-1}^+ \longrightarrow Q_{\ell}^+ \longrightarrow P_{\ell}' \xrightarrow{\beta_{\ell}} \Sigma Q_{\ell-1}^+
\end{array}
$$

where β_i *is a right* $\mathrm{add}(P_i)$*-approximation. Then* Q_ℓ^+ *is the Bongartz complement of* P.

Proof. The result is immediate for $\ell = 1$. As we have not assumed that the direct summands P_i are indecomposable, it suffices to address the case where $\ell = 2$ to prove the claim in general.

Let $d \geq 2$ and suppose that the claim holds for $\ell < d$. If $\ell = d$, we can apply the induction hypothesis to $P/P_d = \bigoplus_{i=1}^{d-1} P_i$ and deduce that Q_{d-1}^+ is its Bongartz complement. Hence, we have a triangle

$$A \longrightarrow Q_{d-1}^+ \longrightarrow P'_{<d} \xrightarrow{\beta} \Sigma A,$$

where β is a right $\operatorname{add}(P/P_d)$-approximation. By the octahedral axiom, we have a diagram

$$\begin{tikzcd}
 & P'_{<d} \arrow[r, equal] \arrow[d] & P'_{<d} \arrow[d, "\beta"] & \\
Q_d^+ \arrow[r] \arrow[d, equal] & X \arrow[r, "b"] \arrow[d] & \Sigma A \arrow[r] \arrow[d] & \Sigma Q_d^+ \arrow[d, equal] \\
Q_d^+ \arrow[r] & P'_d \arrow[r, "\beta_d"] \arrow[d] & \Sigma Q_{d-1}^+ \arrow[r] \arrow[d] & \Sigma Q_d^+ \\
 & \Sigma P'_{<d} \arrow[r, equal] & \Sigma P'_{<d} &
\end{tikzcd}$$

in which every square commutes and all rows and columns of length four are distinguished triangles. Since X is an extension of objects in $\operatorname{add}(P)$, and P is a presilting object, we have that $X \in \operatorname{add}(P)$. By Lemma 2.2 and the existence of the approximation triangle in the third column above, there are no non-zero morphisms from P/P_d to ΣQ_{d-1}^+. Similarly, having an approximation triangle along the third row, there are no morphisms from P_d to ΣQ_d^+. Since ΣQ_d^+ is an extension of $\Sigma P'_d$ in ΣQ_{d-1}^+, we further deduce that $\operatorname{Hom}_{\operatorname{per}(A)}(P, \Sigma Q_d^+) = 0$. Using Lemma 2.2 again, we find that the map b is a right $\operatorname{add}(P)$-approximation. We may now conclude the inductive step, thus completing the proof. $\square$

We conclude this section with its main theorem.

Theorem 2.8. *Let* $\bigoplus_{i=1}^{\ell} P_{(\gamma_i, f_i)}$ *be a presilting complex in* $\operatorname{per}(A)$*, where each* $P_{(\gamma_i, f_i)}$ *is an indecomposable corresponding to the accordion* γ_i*. Then one constructs the Bongartz completion geometrically as follows:*
Step 1: Construct the Bongartz complement Q_1^+ *of* $P_{(\gamma_1, f_1)}$ *using Lemma 2.6 with* $R = A$*.*
Step i: Construct the mapping cylinder Q_i^+ *of a right* $\operatorname{add}(P_{(\gamma_i, f_i)})$*-approximation of* Q_{i+1}^+*.*
Step ℓ: Return Q_ℓ^+*.*

3. Two-term simple-minded collections and τ-perpendicularity

Definition 3.1. Let k be a field and $\mathcal{D}$ be a k-linear triangulated category with suspension functor Σ. A finite set $\mathcal{S} = \{S_1, ..., S_r\}$ of objects of $\mathcal{D}$ is called a *pre-simple-minded collection* provided that the following three conditions hold:

(SM1) The endomorphism algebras $E_i \overset{\text{def}}{=} \operatorname{End}_{\mathcal{D}}(S_i)$ are all division algebras.

(SM2) We have that

$$\mathcal{D}(S_i, \Sigma^p S_j) = \begin{cases} {}_{E_i}E_i, & p = 0 \text{ and } j = i, \\ 0, & p < 0 \text{ and } j = i, \\ 0, & p \leq 0 \text{ and } j \neq i. \end{cases}$$

A pre-simple-minded collection $\mathcal{S}$ is a *simple-minded collection* if, in addition,

(SM3) The smallest thick subcategory containing $\mathcal{S}$ is all of $\mathcal{D}$.

In the special case where $\mathcal{D}$ is equipped with a t-structure, a simple minded collection $\mathcal{S}$ is *two-term (with respect to this t-structure)* if the cohomology $H^i S$ of each object $S \in \mathcal{S}$ vanishes whenever $i \neq 0, 1$.

Let A be a finite-dimensional algebra. Then the stalk complexes of simple modules in degree zero form a simple-minded collection in $\mathcal{D}_{\mathrm{fd}}(A)$. This is two-term with respect to the standard t-structure. We denote the two-term simple-minded collections in $\mathcal{D}$ by 2-smc$(\mathcal{D})$, or simply by 2-smc(A) if $\mathcal{D} = \mathcal{D}_{\mathrm{fd}}(A)$ for some dg k-algebra A.

Lemma 3.2 ([BY13, Proposition 3.15]). *Let $\mathcal{D}$ be a Hom-finite Krull–Schmidt triangulated category with a simple-minded collection $\{S_1, \dots, S_n\}$. Then the Grothendieck group $\mathrm{K}_0(\mathcal{D})$ of $\mathcal{D}$ is free of rank n.*

Lemma 3.3 ([BY13, Remark 4.11],[GM20, Lemma 8.3]). *Let A be a finite-dimensional algebra and let $\mathcal{D} = \mathcal{D}_{\mathrm{fd}}(A)$. For any two-term simple-minded collection $\mathcal{S}$, all objects in $\mathcal{S}$ are stalk complexes of an indecomposable A-module concentrated in degrees 0 or -1.*

As a modest abuse of terminology, we will say that an object $X \in \mathcal{D}_{\mathrm{fd}}(A)$ *belongs to* $\mathrm{mod}(A)$ if it is a stalk complex concentrated in degree 0. Similarly, it *belongs to* $\Sigma\,\mathrm{mod}(A)$ if it is a stalk complex concentrated in degree -1.

Proposition 3.4 ([Asa20, Theorem 2.3(1)]). *Let A be a finite-dimensional algebra and let $\mathcal{D} = \mathcal{D}_{\mathrm{fd}}(A)$. We have bijections*

$$\begin{tikzcd} & \text{2-smc}(A) \arrow[dl, "-\cap \mathrm{mod}(A)"'] \arrow[dr, "\Sigma^{-1}\left(-\cap\Sigma\,\mathrm{mod}(A)\right)"] & \\ \mathrm{f_L}\text{-sbrick}(A) & & \mathrm{f_R}\text{-sbrick}(A) \end{tikzcd}$$

Proposition 3.5 ([KY14, Theorem 6.1],[SY19, Theorem 1.1]). *We have a bijection*

$$t\colon \text{2-silt}(A) \longrightarrow \text{2-smc}(A).$$

where T is sent to the set of simple objects in the heart of the silting t-structure $(T^{\perp_{>0}}, T^{\perp_{\leq 0}})$. There is also a correspondence between the indecomposable direct summands of T and the elements in tT,

$$t\colon\ \mathrm{ind}\,\mathrm{add}(T) \longrightarrow tT$$

such that

$$\mathrm{Hom}(T_i, \Sigma^p S_j) = \begin{cases} {}_{E_j}E_j, & p = 0 \textit{ and } S_j = tT_i \\ 0, & \text{else} \end{cases}$$

where E_j is the endomorphism k-algebra of S_j.

Theorem 3.6. *Let $P = \bigoplus_{i=1}^r P_i$ be a basic two-term presilting object in* $\mathrm{per}(A)$ *and let $T_P = P \oplus Q^+$ be its Bongartz completion, where $Q^+ = \bigoplus_{i=r+1}^n Q_i^+$. Let $\mathcal{S} = \{S_i\}_{i=1}^n$ be the corresponding two-term simple minded collection, where*

$$\mathrm{Hom}(P_i, \Sigma^p S_j) = \begin{cases} {}_{E_j}E_j, & p = 0,\, j = i \\ 0, & \text{else.} \end{cases} \tag{3.A}$$

$$\mathrm{Hom}(Q_i^+, \Sigma^p S_j) = \begin{cases} {}_{E_j}E_j, & p = 0,\, j = i \\ 0, & \text{else.} \end{cases} \tag{3.B}$$

where E_j is the endomorphism k-algebra of S_j.

(1) *The set $\mathcal{S}_{>r} \overset{def}{=} \{S_i\}_{i=r+1}^n$ consists entirely of objects in* $\mathrm{mod}(A)$.

(2) *The set $\mathcal{S}_{>r}$ is a semibrick in* $\mathrm{mod}(A)$.

(3) *The wide subcategories* $J(P)$ *and* $\operatorname{filt}(\mathcal{S}_{>r})$ *coincide. In particular, we have that* $\operatorname{simp} J(P) = \mathcal{S}_{>r}$.

(4) *The set* $\mathcal{S}_{>r}$ *is a two-term simple-minded collection in* $P^{\perp_{\mathbb{Z}}}$.

Dually, if $T_P^- = P \oplus \bigoplus_{i=r+1}^n Q_i^-$ *is the co-Bongartz completion of* P *and* $\mathcal{S}' = \{S_i'\}_{i=1}^n$ *the corresponding two-term simple minded collection, where*

$$\operatorname{Hom}(P_i, \Sigma^p S_j') = \begin{cases} {}_{E_j'}E_j', & p=0,\ j=i \\ 0, & \text{else.} \end{cases}$$

$$\operatorname{Hom}(Q_i^-, \Sigma^p S_j') = \begin{cases} {}_{E_j'}E_j', & p=0,\ j=i \\ 0, & \text{else.} \end{cases}$$

where E_j' *is the endomorphism* k*-algebra of* S_j'*, then the following dual statement of* (1) *holds:*

(1′) *The set* $\mathcal{S}'_{>r} \overset{def}{=} \{S_i'\}_{i=r+1}^n$ *consists entirely of objects in* $\Sigma \operatorname{mod}(A)$.

Proof. We first prove (1). By Lemma 3.3, the objects of (a subset of) a two-term simple-minded collection either belong to $\operatorname{mod}(A)$ or $\Sigma \operatorname{mod}(A)$. Suppose, on the contrary, that some S_j is in $\Sigma \operatorname{mod}(A)$. There could hence be no non-zero morphism from A to S_j in $\mathcal{D}_{\mathrm{fd}}(A)$, as this would render the 0th cohomology of S_j non-trivial. There is, however, a non-zero morphism from Q^+ to S_j, given by composition of the projection $Q^+ \twoheadrightarrow Q_j^+$ with a non-zero morphism $Q_j^+ \longrightarrow S_j$. Using the triangle (2.A) in Definition 2.1, one induces a non-zero morphism $P \longrightarrow S_j$, which is a contradiction on the definition of $\mathcal{S}$ (specifically (3.A)). One proves (1′) dually.

Since $\mathcal{S}_{>r}$ is a subset of both $\operatorname{mod}(A)$ and a simple-minded collection in $\mathcal{D}_{\mathrm{fd}}(A)$, one checks easily that $\mathcal{S}_{>r}$ is a semibrick in $\operatorname{mod}(A)$, thus proving (2).

Next, we prove (3). By the construction in Proposition 3.5, the objects in $\mathcal{S}$ are precisely the simple objects in the heart of the silting t-structure $((T_P^+)^{\perp_{>0}}, (T_P^+)^{\perp_{\leq 0}})$. The heart is equivalent to $\operatorname{mod}(\operatorname{End}_{\mathcal{D}_{\mathrm{fd}}(A)}(T_P^+))$ [AMY19, Lemma 4.6]. We now use Theorem 1.16 and Proposition 1.21 (4)–(5) to emphasise that $J(P)$ is equivalent to the Serre subcategory of $\operatorname{mod}(\operatorname{End}_{\mathcal{D}_{\mathrm{fd}}(A)}(T_P^+))$, which is generated by the elements in $\mathcal{S}_{>r}$.

The claim in (4) will now be deduced from (3). It is easily checked that $\mathcal{S}_{>r}$ is a pre-simple-minded collection, and (SM3) indeed follows from (3). Recall that the t-structure on $P^{\perp_{\mathbb{Z}}}$ is obtained by restricting that on $\mathcal{D}_{\mathrm{fd}}(A)$. Since the objects in $\mathcal{S}_{>r}$ are two-term with respect to the t-structure on $\mathcal{D}_{\mathrm{fd}}(A)$, they are also two-term with respect to the t-structure on $P^{\perp_{\mathbb{Z}}}$; the cohomology functor of the latter is obtained by restriction of that of the former. □

Theorem 3.6 is inextricably linked with a characterisation of τ-perpendicular subcategories due to Buan and Hanson. They show that they are Serre subcategories of left (or right) finite wide subcategories [BH21, Theorem 1.1]. Thus, any τ-perpendicular wide subcategory is filtered by a semibrick which is a subset of a left finite (or right finite) semibrick. Our result above produces this semibrick explicitly. Indeed, by Theorem 3.6(3), the τ-perpendicular subcategory $J(P)$ is filtered by $\mathcal{S}_{>r}$. As $\mathcal{S}_{>r}$ is a subset of $\mathcal{S} \cap \operatorname{mod}(A)$ and $\mathcal{S}$ is a two-term simple-minded collection, it follows from Proposition 3.4 that $\mathcal{S}_{>r}$ is a subset of a left finite semibrick.

We can now describe τ-tilting reduction geometrically.

Theorem 3.7. *Let A be a gentle algebra and let $P = \bigoplus_{i=1}^{\ell} P_{(\gamma_i, f_i)}$ be a presilting object in* $\mathrm{per}(A)$. *The semibrick that constitutes the simple objects in the τ-tilting reduction $J(P)$ can be constructed geometrically as follows:*

(1) Let $T = P \oplus Q^+$ be the Bongartz completion of P. Use Theorem 2.8 to express it as an admissible $\circ$-dissection D on the surface model of A.

(2) Find D^* be the dual $\circ$-dissection.

(3) The $\blacklozenge$-arcs in D^* that do not cross arcs corresponding to objects in P give the simple objects in $J(P)$.

Using the fact that the class of gentle algebras is closed under derived equivalence [SZ03], one can show that the τ-tilting reduction of a gentle algebra is gentle. Thus, one should be able to find a surface model of τ-tilting reductions. This has been achieved in greater generality by Cheng–Jin–Schroll, who obtain a surface model for arbitrary silting reductions [CJS22, Theorem 3.1].

4. Kreweras complementation

Kreweras complementation is a classical construction in combinatorics [Kre72]. It is an anti-automorphism on the poset of non-crossing partitions. In representation theory, non-crossing partitions correspond to wide subcategories of the path algebra of a (linearly oriented) A_n-quiver [IT09, §3.3]. Describing the Kreweras complement in terms of the Weyl group, one extends the definitions to include all hereditary artin algebras [Rin16]. Garver and McConville have extended Kreweras' definition in another direction, by introducing non-crossing tree partitions as an analogue for gentle algebras of finite representation type [GM20, §5].

We propose a general representation theoretic definition of the Kreweras complement, and propose a geometric description for gentle algebras. The definition will be approached in terms of Galois correspondences, a notion we review presently.

Definition 4.1. Let Y and Z be posets. A *Galois correspondence* between Y and Z a pair

$$Y \xrightarrow{K} Z$$

$$Z \xrightarrow{K'} Y$$

of mutually inverse anti-isomorphisms of posets.

There are several instances of Galois correspondence occurring naturally in representation theory. The correspondence between torsion classes and torsion-free classes in (1.3) is one such example. Since functorially finite torsion classes correspond to functorially finite torsion-free classes, we also have a restricted Galois correspondence between these subposets.

We will now define Kreweras complementation in the general setting. Recall that there is a map

$$\mathrm{tors}(A) \xrightarrow{\alpha} \mathrm{wide}(A).$$

which is a left inverse of the embedding

$$\mathrm{wide}(A) \xhookrightarrow{\mathbf{T} = \mathrm{filt}\,\mathrm{gen}} \mathrm{tors}(A).$$

(see Proposition 1.18). Dually, the map

$$\operatorname{torf}(A) \xrightarrow{\alpha'} \operatorname{wide}(A).$$

is left inverse to

$$\operatorname{wide}(A) \xhookrightarrow{\mathbf{F}=\operatorname{filt}\operatorname{sub}} \operatorname{torf}(A).$$

The following definitions have previously been proposed by Hofmann [Hof22, Definition 3.6].

Definition 4.2. The *right Kreweras complement* on wide(A) is defined by

$$\mathrm{Kr} \overset{\text{def}}{=} \alpha'\big((-)^{\perp}\big) \colon \operatorname{wide}(A) \longrightarrow \operatorname{wide}(A),$$

whereas the *left Kreweras complement* is defined by

$$\mathrm{Kr}' \overset{\text{def}}{=} \alpha\big({}^{\perp}(-)\big) \colon \operatorname{wide}(A) \longrightarrow \operatorname{wide}(A).$$

Since the poset wide(A) is isomorphic to sbrick(A) (by Lemma 1.26), it is also possible to define Kreweras complements on sbrick(A). These will be denoted by Kr and Kr$'$, though we define them by simp $\circ$ Kr $\circ$ filt and simp $\circ$ Kr$'$ $\circ$ filt, respectively.

Proposition 4.3. *(1) The maps* Kr *and* Kr$'$ *restrict to a Galois correspondence between* $\mathrm{f_L}$-wide(A) *and* $\mathrm{f_R}$-wide(A).

(2) Moreover, we have a commutative diagram of bijections

$$\begin{tikzcd}
& 2\text{-smc}(A) \arrow[dl, bend right, "-\cap\operatorname{mod}(A)"'] \arrow[dr, bend left, "\Sigma^{-1}\big(-\cap\Sigma\operatorname{mod}(A)\big)"] & \\
\mathrm{f_L}\text{-sbrick}(A) \arrow[rr, bend right, "\mathrm{Kr}"'] \arrow[d, "\mathrm{filt}"'] & \perp & \mathrm{f_R}\text{-sbrick}(A) \arrow[ll, bend right, "\mathrm{Kr}'"'] \arrow[d, "\mathrm{filt}"] \\
\mathrm{f_L}\text{-wide}(A) \arrow[rr, bend right, "\mathrm{Kr}"'] & \perp & \mathrm{f_R}\text{-wide}(A) \arrow[ll, bend right, "\mathrm{Kr}'"']
\end{tikzcd} \tag{4.A}$$

Proof. The bijectivity of the maps connecting the first and second rows of (4.A) is due to Asai (see Proposition 3.4). We show that Kr is equal to the compositite of $\Sigma^{-1}\big(-\cap\Sigma\operatorname{mod}(A)\big)$ with the inverse of

$- \cap \operatorname{mod}(A)$. Consider the diagram

$$\begin{tikzcd}
\text{f-tors}(A) \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "(-)^{\perp}"'] \arrow[d, shift right, "\alpha"'] & & \text{f-torf}(A) \arrow[ll, bend right, "{}^{\perp}(-)"'] \arrow[d, shift right, "\alpha'"'] \\
\text{f}_{\text{L}}\text{-wide}(A) \arrow[u, shift right, "\mathbf{T}"'] \arrow[rr, bend right, "\mathrm{Kr}"'] & & \text{f}_{\text{R}}\text{-wide}(A) \arrow[u, shift right, "\mathbf{F}"'] \arrow[ll, bend right, "\mathrm{Kr}'"']
\end{tikzcd}$$

whose commutativity has been established by Asai [Asa20, Theorem 2.3(2)]. Since the top row is a Galois correspondence (see (1.B)), and the vertical maps are bijective by Lemma 1.28, it follows from the commutativity of the diagram that the bottom row is a Galois correspondence, as desired. □

With our definition of Kreweras complementation, there is a close connection to the extended κ-map

$$\overline{\kappa}\colon \operatorname{tors}(A)_0 \longrightarrow \operatorname{tors}(A)^0 \tag{4.B}$$

of Barnard–Todorov–Zhu [BTZ21, Definition 4.1.1 and Corollary 4.4.3]. Here, the subposets $\operatorname{tors}(A)_0$ and $\operatorname{tors}(A)^0$ are spanned by the torsion classes of the form $\mathbf{T}(\mathcal{S})$ and ${}^{\perp}\mathcal{S}$, respectively, where $\mathcal{S}$ is a semibrick. The extended κ-map is then an anti-isomorphism sending $\mathbf{T}(\mathcal{S})$ to ${}^{\perp}\mathcal{S}$. We do not include a rigourous treatment of the extended κ-map, but we will make use of the fact that the composite

$$\operatorname{wide}(A) \xrightarrow{\mathbf{T}=\text{filt gen}} \operatorname{tors}(A)_0 \xrightarrow{\quad\overline{\kappa}\quad} \operatorname{tors}(A)^0$$

equals the map $\operatorname{wide}(A) \xrightarrow{(-)^{\perp}} \operatorname{tors}(A)^0$. This yields a commutative square

$$\begin{tikzcd}
\operatorname{tors}(A)_0 \arrow[r, "\overline{\kappa}"] \arrow[d, "\alpha"] & \operatorname{tors}(A)^0 \arrow[d, "\alpha"] \\
\operatorname{wide}(A) \arrow[r, "\mathrm{Kr}'"] & \operatorname{wide}(A)
\end{tikzcd} \tag{4.C}$$

quite similar to the one occurring when A is a finite-dimensional hereditary algebra [BTZ21, Theorem D and §5.3]. In this specical case Kr' is replaced with Ringel's ϵ-map, defined by

$$\epsilon(\mathcal{W}) \overset{\text{def}}{=} {}^{\perp_{0,1}}\mathcal{W} = \{X \in \operatorname{mod}(A) \mid \operatorname{Hom}_A(X, \mathcal{W}) = 0 = \operatorname{Ext}^1_A(X, \mathcal{W})\}.$$

Since the map $\mathbf{T}\colon \operatorname{wide}(A) \longrightarrow \operatorname{tors}(A)_0$ is inverted by α, the map Kr' is the unique map one can insert in the bottom of the diagram (4.C) to render it commutative. We summarise this paragraph with a succinct lemma.

Lemma 4.4. *Let A a finite-dimensional hereditary k-algebra. Then $\mathrm{Kr}'(\mathcal{W}) = \epsilon(\mathcal{W})$ for any wide subcategory $\mathcal{W}$ of $\operatorname{mod}(A)$. Dually, we have that*

$$\mathrm{Kr}(\mathcal{W}) = \mathcal{W}^{\perp_{0,1}} = \{X \in \operatorname{mod}(A) \mid \operatorname{Hom}_A(\mathcal{W}, X) = 0 = \operatorname{Ext}^1_A(\mathcal{W}, X)\}.$$

for any wide subcategory $\mathcal{W}$ *of* $\operatorname{mod}(A)$.

Our generalised definition of Kreweras complementation is inspired by results for representation finite hereditary algebras. Consequently, our definition of the Kreweras complement coincides with the established definition for these algebras. It would be desirable that our definition also coincides with the Kreweras complement that Garver and McConville defined in terms of non-crossing tree partitions [GM20, §5.2].

Proposition 4.5. *Let A be a tiling algebra of a tree embedded in a disc. Then the map*

$$\mathrm{Kr}_{\mathrm{GM}} \colon \mathrm{f}_{\mathrm{R}}\text{-sbrick}(A) \longrightarrow \mathrm{f}_{\mathrm{L}}\text{-sbrick}(A)$$

coincides with $\mathrm{Kr}' \colon \mathrm{f}_{\mathrm{R}}\text{-sbrick}(A) \longrightarrow \mathrm{f}_{\mathrm{L}}\text{-sbrick}(A)$.

Proof. The map $\mathrm{Kr}_{\mathrm{GM}}$ fits into the commutative diagram

$$\begin{tikzcd}
 & \text{2-smc}(A) \arrow[dl, "-\cap \operatorname{mod}(A)"'] \arrow[dr, "\Sigma^{-1}\left(-\cap \Sigma \operatorname{mod}(A)\right)"] & \\
\mathrm{f}_{\mathrm{L}}\text{-sbrick}(A) & & \mathrm{f}_{\mathrm{R}}\text{-sbrick}(A) \arrow[ll, "\mathrm{Kr}_{\mathrm{GM}}"]
\end{tikzcd}$$

It follows immediately that $\mathrm{Kr}_{\mathrm{GM}} = \mathrm{Kr}'$. □

Now that the Kreweras complement has been defined in a manner that is consistent with the literature, we investigate how it interacts with the different types of wide subcategories that appear in (1.C). In that diagram, the largest possibly proper subposet of $\operatorname{wide}(A)$ is $\text{f-wide}(A)$, that of functorially finite wide subcategories.

Proposition 4.6. *Let A be a hereditary finite-dimensional algebra (even artin algebra?). Then the maps* Kr *and* Kr' *restrict to a Galois correspondence*

$$\begin{tikzcd}
\text{f-wide}(A) \arrow[r, bend right, "\mathrm{Kr}"'] \arrow[r, phantom, "\perp"] & \text{f-wide}(A) \arrow[l, bend right, "\mathrm{Kr}'"']
\end{tikzcd}$$

Proof. It suffices to prove that Kr' (and dually Kr) restricts to an anti-automorphism on $\text{f-wide}(A)$.

A result of Enomoto gives a pair of mutually inverse bijections [Eno22, Corollary 4.15] (see also Lemma 1.24)

$$\begin{tikzcd}
\text{f-wide}(A) \arrow[r, bend right, "\mathbf{T}"'] & \text{f-tors}(A). \arrow[l, bend right, "\alpha"']
\end{tikzcd}$$

Further, the extended κ-map restricts to a bijection [BTZ21, Corollary 5.4.3]

$$\text{f-tors}(A) \xrightarrow{\overline{\kappa}} \text{f-tors}(A).$$

Restricting the posets appearing the (4.C) yields a commutative diagram

$$\begin{tikzcd}
\text{f-tors}(A) \arrow[r, "\overline{\kappa}"] \arrow[d, "\alpha"] & \text{f-tors}(A) \arrow[d, "\alpha"] \\
\text{f-wide}(A) \arrow[r, "\mathrm{Kr}'"] & \text{f-wide}(A)
\end{tikzcd}$$

in which we know all maps but Kr' to be bijective. We conclude that Kr' bijects. □

Proposition 4.6 also shows that τ-perpendicularity is preserved whenever A is hereditary; we have that $\text{f-wide}(A) = \tau\text{-perp-wide}(A)$ (Lemma 1.24).

The Kreweras complements do not always form a Galois auto-correspondence on $\mathrm{wide}(A)$. Consider for example the Kronecker algebra, namely the path k-algebra of the quiver $Q = 1 \underset{b}{\overset{a}{\leftleftarrows}} 2$, and the wide subcategory $\mathcal{R}$ of $\mathrm{mod}(kQ)$ containing all regular modules (differently put, the wide subcategory containing all tubes). The poset $\mathrm{wide}(kQ)$ then takes the following form

$$\begin{tikzcd}
 & & & \mathrm{mod}(A) & & & \\
 & & & \mathcal{R} \arrow[u] & & & \\
 & & & \{\mathrm{filt}\, R_I\}_{0 \subsetneq I \subsetneq P^1(k)} \arrow[u, shift left=3] \arrow[u, shift left=1] \arrow[u, phantom, "\cdots"] \arrow[u, shift right=1] \arrow[u, shift right=3] & & & \\
\cdots & \mathrm{add}(M_{-1,-2}) \arrow[uuurr] & \mathrm{add}(M_{0,-1}) \arrow[uuur] & \cdots & \mathrm{add}(M_{1,0}) \arrow[uuul] & \mathrm{add}(M_{2,1}) \arrow[uuull] & \cdots \\
 & & & 0 \arrow[ull] \arrow[ul] \arrow[uu, shift left] \arrow[uu, shift right] \arrow[ur] \arrow[urr] & & &
\end{tikzcd} \tag{4.D}$$

where $M_{i,j}$ is the indecomposable module with dimension vector $(|i|, |j|)$ and R_I is the family of tubes parameterised by the points in I. One can show by hand that $\mathrm{Kr}(\mathcal{R}) = 0$, which is also the Kreweras complement of the entire module category. Interestingly, the Kreweras complements form a Galois correspondence on $\mathrm{wide}(kQ) \setminus \{\mathcal{R}\}$. Indeed, one computes that $\mathrm{Kr}(\mathrm{add}(M_{i,j})) = \mathrm{add}(M_{i-1,j-1})$ (reading right to left along the penultimate row of (4.D)) and that $\mathrm{Kr}(\mathrm{filt}\, R_I) = \mathrm{filt}\, R_{P^1(k)\setminus I}$ (the complementary family of tubes).

In general, Kreweras complementation neither preserves functorial finiteness nor τ-perpendicularity. Our counter-example is typical in the literature [BTZ21, Example 5.4.4] [Asa20, Example 4.5] Consider the algebra path k-algebra of the bound quiver

$$1 \underset{b}{\overset{a}{\leftleftarrows}} 2 \xleftarrow{c} 3,$$

bound by the relation ac. The wide subcategory $\mathrm{add}(I_3)$ is then τ-perpendicular (is it the reduction of $S_2 \oplus \frac{P_1}{P_3}$), but its right Kreweras complement $\mathrm{Kr}(\mathrm{add}(I_3))$ turns out not to be functorially finite. One computes by hand that

$$I_3^{\perp} = \Big\{ \begin{tikzcd} V_1 \arrow[r, shift left, "f_a"] \arrow[r, shift right, "f_b"'] & V_2 \arrow[r, "f_c"] & V_3 \end{tikzcd} \;\Big|\; f_a \text{ is a monomorphism} \Big\}.$$

Then, using the Weak Four Lemma [ML63, p. 14] or otherwise, we deduce that

$$\alpha'\big(I_3^{\perp}\big) \supseteq \Big\{ \begin{tikzcd} V_1 \arrow[r, shift left, "f_a"] \arrow[r, shift right, "f_b"'] & V_2 \arrow[r, "f_c"] & V_3 \end{tikzcd} \;\Big|\; f_a \text{ is an isomorphism} \Big\}.$$

This inclusion is an equality; given a representation $V_1 \overset{f_a}{\underset{f_b}{\rightrightarrows}} V_2 \xrightarrow{f_c} V_3$ in $I_3^{\perp}$, consider the cokernel of the map

$$\begin{tikzcd} V_1 \arrow[r, shift left, "f_a"] \arrow[r, shift right, "f_b"'] \arrow[d, tail, "f_a"] & V_2 \arrow[r, "f_c"] \arrow[d, "1"] & V_3 \arrow[d, "1"] \\ V_2 \arrow[r, shift left, "1"] \arrow[r, shift right, "f_b h_a"'] & V_2 \arrow[r, "f_c"] & V_3, \end{tikzcd}$$

where h_a is a left inverse of f_a. We conclude that $\mathrm{Kr}(\mathrm{add}(I_3))$ is filtered by the semibrick

$$\left(\bigcup_{\lambda \in k} \{R_\lambda\} \right) \cup S_3$$

where R_λ is the module $k \overset{1}{\underset{\lambda}{\rightrightarrows}} k \longrightarrow 0$. If k is an infinite field, then $\mathrm{Kr}(\mathrm{add}(I_3))$ is thus filtered by an infinite semibrick. Since a functorially finite wide subcategory is filtered by a finite semibrick (Lemma 3.2 and [BH21, Theorem 1.1]), the wide subcategory $\mathrm{Kr}(\mathrm{add}(I_3))$ cannot be functorially finite.

Lastly, we prove a result that lets us compute Kreweras complements on the surface.

Theorem 4.7. *Let A be a gentle algebra with surface model (S_A, Δ). Let $\mathscr{A}_2$ be the set of admissible $\circ$-dissections where all arcs are Δ-accordions, and let $\mathscr{A}_2^*$ be the set of the duals of these dissections. Consider the maps*

$$p\colon \mathscr{A}_2 \longrightarrow 2\text{-silt}(A)$$

$$s\colon \mathscr{A}_2^* \longrightarrow 2\text{-smc}(A)$$

mapping such dissections bijectively to two-term silting complexes in $\mathrm{per}(A)$ *and two-term simple-minded collections in $\mathcal{D}_{\mathrm{fd}}(A)$, respectively. For $D \in \mathscr{A}_2$, consider the following decomposition of the dual dissection D^**

$$D^* = D_0^* \sqcup D_1^*$$

where the $\blacklozenge$-arcs in D_0^* have degree 0 at the (unique) crossing with a $\circ$-arc in D, the $\blacklozenge$-arcs in D_0^* have degree 1 at such crossings. Then the map Kr *sends $s(D_0^*)$ to $s(D_1^*)$.*

Proof. Combining Proposition 3.4 and Proposition 3.5 yields a diagram of bijections

$$\begin{tikzcd} & 2\text{-silt}(A) \arrow[dl, bend right, "t_0"'] \arrow[d, "t"] \arrow[dr, bend left, "t_1"] & \\ \mathrm{f_L}\text{-sbrick}(A) & 2\text{-smc}(A) \arrow[l] \arrow[r] & \mathrm{f_R}\text{-sbrick}(A) \end{tikzcd}$$

Let $D = p^{-1}(T)$, where T is an arbitrary object in $2\text{-silt}(A)$. Then $t_0(D) = D_0^*$ and $t_1(D) = D_1^*$. The claim now follows from Proposition 4.3(2). $\square$

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Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway

Email address: erlend.d.borve@ntnu.no

Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, Institut universitaire de France (IUF)

Email address: pierre-guy.plamondon@uvsq.fr

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