Use of global geopotential models for comparison of gravimetric geoid estimators by the modification of Stokes's formula

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Summary. – Considering today struggles towards the 1-cm geoid, in an attempt to study the efficiency of some geoidal height estimators, Molodensky et al. (1962), Wong and Gore (1969), Vincent and Marsh (1974), Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) modification models are numerically evaluated. These estimators combine a Global Geopotential Model (GGM) with the regional gravity data convolved with Stokes's kernel. The geoidal heights are computed in a test area using the above-mentioned geoidal height estimators. Next, geoidal heights are computed in the test area only using the geopotential coefficients. This geoid model is considered as "reference geoid model" in this study. The geoid heights computed with five estimators have then been compared with this reference model. It is shown that the different procedures to modify the original Stokes's formula result in different geoidal heights. The results of comparisons show that the least-squares and Vanicek and Kleusberg (1987) estimators are in better agreement with the "reference geoid model" than the other estimators in this study. They use the spheroidal-type kernel in the model and, therefore, the truncation error in these two models reduces significantly.

Keywords: geoid, EGM, Stokes's formula, modification.

1. – INTRODUCTION

The solution to the boundary value problem can be solved by Stokes's well-known formula for the anomalous potential and through Bruns's formula, the geoidal height can be obtained. The geoid has been held by many as a fundamental reference surface of geodesy, and its precise determination has been and still is the center of discussions for many geodesists.

This paper deals with numerical comparisons of some models of gravimetric geoidal height estimators. These estimators modify the original Stokes formula in different ways. The idea of this study is to show that the different geoid estimators result in different geoidal heights. Finding suitable models for geoidal height determination are also investigated. Earlier GPS-leveling data were used to compare between different geoid estimators (Nahavandchi 1998; Nahavandchi and Sjöberg 2001). It was shown that the least-squares estimator (Sjöberg, 1984) agrees best with the GPS-leveling derived geoidal heights among the other modification procedures. In this study, a GGM is used as other source of information to compare between different geoid estimators.

The original method to modify Stokes's formula was presented by Molodensky et al. (1962). The main idea in this method is to reduce the truncation error committed by limiting the area of integration under Stokes's integral to a spherical cap around the computation point. Another model, the modified Wong and Gore (1969), employs a residual field and a modified Stokes's kernel. This estimator corresponds to high degree reference gravity field and kernel modification. Vincent and Marsh (1974) model estimate geoidal heights in a slightly different way, which is the third model used in this study. The principle in this method is to use a high degree reference gravity field in Stokes's formula, implying a localized gravity field, but no kernel modification, and then adding the long-wavelength contributions from geopotential coefficients. Another estimator presented by Sjöberg (1984) reduces the impact of the errors stemming from truncation, erroneous terrestrial gravity data and potential harmonics in a least-squares sense. Similar to Molodensky et al. (1962), who make a modification to the spherical Stokes kernel, Vanicek and Kleusberg (1987) make a modification to the spheroidal Stokes kernel. This is the fifth geoid estimator in this study.

Many authors have investigated different procedures of the modification of Stokes kernel. A geoid model with the parameters chosen according to Molodensky et al. (1962) and Meissl (1971) has been studied in Jekeli (1980) and (1981), with the terrestrial anomaly error omitted. Despotakis (1987) used Sjöberg's least-squares estimator to compute the geoidal heights at laser tracking stations. It was shown that the least-squares model was the most accurate technique for geoid undulation computations, whenever the error anomaly degree variances, due to the terrestrial gravity anomalies and erroneous potential coefficients, were properly selected. Recently, Featherstone et al. (1998) studied the modification of Stokes's formula. A Meissl-modified Vanicek and Kleusberg kernel was proposed, which makes the truncation error converging to zero faster. Later, Vanicek and Featherstone (1998) showed that the use of a spheroidal and modified spheroidal kernels were preferable in real practice.

2. – THE “REFERENCE GEOID MODEL” COMPUTED FROM GEOPOTENTIAL MODEL ALONE

In modern methods of determining the geoidal undulations, the long-to-medium wavelength components are frequently obtained from a global geopotential model in the modified Stokes formula. The short-wavelength information is then computed from Stokes's integral. The geoidal heights (N) can be determined from EGM96 geopotential coefficients (Lemoine et al., 1997) using spherical harmonic representation by the following expansion that is complete to degree M' (=360 in this study) (Heiskanen and Moritz 1967, Chaps. 1 and 2)

$$N(R,\varphi,\lambda) = \frac{GM_3}{R\gamma}\sum_{n=0}^{M'}\left(\frac{a_1}{R}\right)^n \sum_{m=0}^{n}\left[\left(\frac{GM_1}{GM_3}C_{nm} - \frac{GM_2}{GM_3}\left(\frac{a_2}{a_1}\right)^n C'_{nm}\right)\cos m\lambda + \right.$$
$$\left. + \frac{GM_1}{GM_3}S_{nm}\sin m\lambda\right]P_{nm}(\sin\varphi) - \frac{1}{\gamma}(W_0 - U_0) \qquad (1)$$

where

R= the mean geoid radius

(φ, λ)= the spherical latitude and longitude of the computation point

γ = the normal gravity at the ellipsoid to which the geoidal height N will refer

a_1= equipotential scale factor of EGM96 (6378.1363 km)

a_2= equipotential radius of GRS-80 (6378.137 km)

GM_1= gravity-mass constant of EGM96 ($3.986004415\times 10^{14}$ m^3/s^2)

GM_2= gravity-mass constant of GRS-80 ($3.986005000\times 10^{14}$ m^3/s^2)

GM_3= best estimate of gravity-mass constant for the Earth ($3.986004418\times 10^{14}$ m^3/s^2)

W_0=adopted gravity potential on the geoid (62636856.88 m^2/s^2)

U_0=defined normal gravity potential on the ellipsoid (62636860.85 m^2/s^2)

C_{nm} and S_{nm}= fully normalized geopotential coefficients of degree n and order m of EGM96 in the non-tidal system (C_{00}=1.0; S_{00}=0.0; C_{1m}=S_{1m}=0.0; C_{20}=non-tidal; S_{20}=0.0)

C'_{nm} and S'_{nm}= fully normalized normal potential coefficients of degree n and order m of GRS-80 in the non-tidal system (C'_{00}=1.0; S'_{00}=0.0; C'_{1m}=S'_{1m}=0.0; C'_{20}=non-tidal; S'_{20}=0.0). We have also made use of the fact that S'_{nm}=0.0 for all n and m.

P_{nm}= fully normalized Legendre functions.

This model will be considered as "reference geoid model" for comparison of the different geoid estimators. Also, as here, the evaluation point lies inside Earth's masses, the geoidal height computed from Eq. (1) is inaccurate (see e.g. Nahavandchi and Sjöberg, 1998; Rapp, 1997). Therefore, the effects of topographic masses must be considered.

However, as this study aims at comparison of some geoid estimators, and not the precise geoidal height determination, the topographic corrections are not included in the reference model. The effect of topographic masses is also disregarded in the geoid estimators, and the study is kept to this level of accuracy. Other corrections are also excluded in both the reference model and the geoid estimators.

In the gravimetric geoid estimators below, the degree of kernel modification is considered equal to the degree and order of the global geopotential model. This was examined in Nahavandchi (1998) and Nahavandchi and Sjöberg (2001), who combined a global geopotential model to maximum degree and order with regional gravity data using the spheroidal Stokes kernel.

3. – GRAVIMETRIC GEOID ESTIMATORS

3.1. – THE MOLODENSKY MODIFICATION

The well-known Molodensky truncation theory (Molodensky et al., 1962) is the base of current notations of modifications by combining terrestrial gravity information with a set of geopotential coefficients. They aimed at minimizing the upper bound of the truncation error.

Assuming a cap of integration σ_0 with geocentric angle ψ_0 around the computation point, an estimator N_1 of the geoidal height N that combines the Stokes integral with a global geopotential model can be written as (Molodensky et al., 1962)

$$N_1 = \frac{c}{2\pi} \iint_{\sigma_0} S_M(\psi)\, \Delta g d\sigma + c \sum_{n=2}^{M} s_n \Delta g_n \quad (2)$$

where

$S_M(\psi)$ = the spheroidal Stokes kernel = $S(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k p_k(\cos\psi)$

$s_0, s_1, s_2, \ldots, s_M$= modification parameters,

$S(\psi)$ = the spherical Stokes kernel= $\sum_{k=2}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi)$

$P_k(\cos\psi)$ = k-th Legendre's polynomial,

ψ = spherical distance between computation and running points,

Δg = the terrestrial gravity anomaly at the geoid level derived from the observed magnitude of gravity at the Earth's surface,

Δg_n = n-th Laplace harmonic of Δg determined from potential coefficients,

$c = R/2\gamma$,

M = degree of the global geopotential model and degree of kernel modification in the geoidal height estimators.

Different choices of the modification parameters s_n lead to different estimations of the geoidal heights. The modification parameters can be determined from the system of linear equation

$$\sum_{r=2}^{M} a_{kr} s_r = h_k \qquad k = 2, 3, \cdots, M \tag{3}$$

were in accordance with Molodensky's method (Molodensky et al., 1962):

$$a_{kr} = \frac{2r+1}{2}\frac{2k+1}{2} e_{kr} \tag{4}$$

and

$$h_k = \frac{2k+1}{2} Q_k \tag{5}$$

Here Paul's function (Paul, 1973)

$$e_{kn}(\psi_0) = \int_{\psi=\psi_0}^{\pi} P_n(\cos\psi)\, P_k(\cos\psi) \sin\psi \, d\psi \tag{6}$$

and

$$Q_k(\psi_0) = \int_{\psi=\psi_0}^{\pi} S(\psi)\, P_k(\cos\psi) \sin\psi \, d\psi \tag{7}$$

are the Molodensky truncation coefficients.

3.2. – SJÖBERG'S BIASED LEAST-SQUARES MODIFICATION

Sjöberg (1984) proposes least-squares modification of Stokes's formula, which reduces the truncation error, erroneous terrestrial gravity data and potential harmonic errors in a least-squares sense. Referring to the Eq. (3), the modification parameters in the least-squares model are computed as below (Sjöberg, 1984):

$$a_{kr} = (\sigma_r^2 + dc_r)\delta_{kr} - \frac{2k+1}{2}\sigma_r^2 e_{kr} - \frac{2r+1}{2}\sigma_k^2 e_{rk} + \frac{2k+1}{2}\frac{2r+1}{2}\sum_{n=2}^{M} e_{nk} e_{nr}(\sigma_n^2 + c_n) \tag{8}$$

and

$$h_k = \frac{2\sigma_k^2}{k-1} - Q_k \sigma_k^2 + \frac{2k+1}{2}\sum_{n=2}^{m}\left(Q_n e_{nk}(\sigma_n^2 + c_n) - \frac{2}{n-1} e_{nk}\sigma_n^2\right) \tag{9}$$

where

$$c_n = \frac{1}{4\pi}\iint_{\sigma} \Delta g_n^2 \, d\sigma \tag{10}$$

and σ_n^2 is the n-th gravity anomaly error degree variance, dc_n is the expected mean square error of Δg_n and δ_{kr} is Kronecker's delta symbol. The gravity anomaly degree variance c_n can be computed from EGM96 as

$$c_n = \frac{(GM_1)^2}{a_1^4}(n-1)^2 \sum_{m=0}^{n}(C_{nm}^2 + S_{nm}^2) \tag{11}$$

The gravity anomaly error degree variance, due to erroneous potential coefficients is computed from

$$dc_n = \frac{(GM_1)^2}{a_1^4}(n-1)^2 \sum_{m=0}^{n}\left(\delta_{C_{nm}}^2 + \delta_{S_{nm}}^2\right) \tag{12}$$

where δ_{Cnm} and δ_{Snm} are the standard deviations of potential coefficients taken from EGM96. The error degree variances for the terrestrial gravity anomalies (σ_n^2) can be estimated from knowledge of an error degree covariance function. One covariance function is, for example, given by (Sjöberg, 1986)

$$C(\psi) = c_1\left[\frac{1-\Omega}{\sqrt{1-2\Omega\cos\psi+\Omega^2}} - (1-\Omega) - (1-\Omega)\Omega\cos\psi\right] \tag{13}$$

where σ_n^2 are expressed from

$$\sigma_n^2 = c_1(1-\Omega)\Omega^n \tag{14}$$

The parameters c_1 and Ω are determined from knowledge of the error variance $C(0)$ and the correlation length ξ; the value of the argument for which $C(\psi)$ has decreased to half of its value at ψ=0 (Moritz, 1980). The value of $C(0)$=10 mGal^2 and a correlation length of 0.1° are used in this study.

Both Molodensky and the least-squares models use the original Pizzetti reference field. This was also pointed in Jekeli (1981) and Sjöberg (1984). Below some other models that use higher than second-degree reference field will be investigated.

3.3. – WONG AND GORE (1967) MODIFICATION

The modified Wong and Gore (1967) method employs a high-degree residual field and spheroidal Stokes kernel. This model is

$$N_2 = \frac{c}{2\pi}\iint_{\sigma_0} S_M(\psi)\,\Delta g^M d\sigma + c\sum_{n=2}^{M} s_n \Delta g_n \tag{15}$$

where Δg^{M} are the residual terrestrial gravity anomalies which have been reduced by the corresponding spherical harmonic of degree and order M of the global geopotential model as:

$$\Delta g^M = \Delta g - \sum_{n=2}^{M}\Delta g_n \tag{16}$$

Wong and Gore (1969) model choice of modification parameters is $s_n = 2/n - 1$ in Eq. (15). The term modified means that high-degree reference gravity field and kernel modification are combined in this model.

The use of a higher-degree reference field in Stokes's integral in this estimator results to the subtraction of the long-wavelength contribution of gravity anomalies (computed from a global EGM) from the terrestrial gravity anomalies. This subtraction is a time consuming work, which must be done for each computation point (especially for large values of M).

3.4. – VINCENT AND MARSH (1974) MODIFICATION

Vincent and Marsh (1974) choice of the modification parameters are also $s_n = 2/n - 1$. However, this method corresponds to a high-degree reference gravity field with no kernel modification, resulting in:

$$N_3 = \frac{c}{2\pi} \iint_{\sigma_0} S(\psi)\, \Delta g^M d\sigma + c \sum_{n=2}^{M} s_n \Delta g_n \tag{17}$$

This model uses the original spherical Stokes integration kernel.

3.5. – VANICEK AND KLEUSBERG (1987) MODIFICATION

Following Molodensky et al. (1962) who used a modification to the spherical Stokes kernel, Vanicek and Kleusberg (1987) made a modification to the spheroidal Stokes kernel resulting in

$$N_4 = \frac{c}{2\pi} \iint_{\sigma_0} S_M^s(\psi)\, \Delta g^M d\sigma + c \sum_{n=2}^{M} s_n \Delta g_n \tag{18}$$

The modification parameters, s_n, were determined from the system of linear equations

$$\sum_{n=2}^{M} \frac{2n+1}{2} e_{kn}(\psi_0) s_n(\psi_0) = Q_K^M(\psi_0) \tag{19}$$

and the Vanicek and Kleusberg (or spheroidal Molodensky) truncation coefficients are evaluated from

$$Q_K^M(\psi_0) = Q_k(\psi_0) - \sum_{j=2}^{M} \frac{2j+1}{j-1} e_{kj}(\psi_0) \tag{20}$$

The Molodensky-modified spheroidal Stokes function is

$$S_M^s(\psi) = S_{M+1}(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k P_k(\cos\psi) \tag{21}$$

where

$$S_{M+1}(\psi) = \sum_{k=M+1}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi) = S(\psi) - \sum_{k=2}^{M} \frac{2k+1}{k-1} P_k(\cos\psi) \tag{22}$$

These five geoidal height estimators are investigated in this study. These geoid estimators combine, in different ways, the global geopotential model with Stokes's integral. Different types of modification parameters are used in this study. Different Stokes’s kernels including spherical, spheroidal and modified spheroidal kernels are chosen in the estimators. The use of both higher-degree reference field and the original Pizzetti reference field is also obvious in different geoid estimators.

In real practice, it is supposed that the terrestrial gravity anomalies be used in the integral-part of the geoidal height estimators. In this study, however, the terrestrial gravity data were not used. As the idea of this study is a comparison of the different geoidal height estimators with a "reference geoid model" [geoid derived from EGM96 with Eq. (1)], the gravity anomalies in the integral-part of geoid estimators are also computed from the EGM96 coefficients. The gravity anomalies are derived from the following formula (Heiskanen and Moritz, 1967)

$$\Delta g^F(r,\varphi,\lambda) = \frac{GM_3}{r^2}\sum_{n=0}^{M'}(n-1)\left(\frac{a_1}{R}\right)^n \sum_{m=0}^{n}\left[\left(\frac{GM_1}{GM_3}C_{nm} - \frac{GM_2}{GM_3}\left(\frac{a_2}{a_1}\right)^n C'_{nm}\right)\cos m\lambda + \right. \\ \left. + \frac{GM_1}{GM_3}S_{nm}\sin m\lambda\right] P_{nm}(\sin\varphi) + \frac{2}{r}(W_0 - U_0) \qquad (23)$$

where r is the geocentric distance to point of interest. To have a better consistency with the real situation, the geopotential coefficients used for the computation of the gravity anomalies (in the integral-part of the geoid estimators) are infected with a white noise, of mean zero and standard error of σ_0. This standard error equals to the maximum standard error of the original EGM96 geopotential coefficients. Note that in the "reference geoid model" the potential coefficients are noise-free and are not infected with the above-mentioned random errors. It should be, however, noted that the noise on coefficients is giving strong correlations in the spatial gravity anomalies errors computed from those coefficients. In reality the noise characteristics of gravity anomalies will be different. However, as this is used for all gravimetric geoid estimators, and this study aims in comparison not the geoidal height computations, this level of accuracy might be sufficient.

It is obvious that the "reference geoid model" only includes long-wavelength information and the short-wavelength constituents, which are derived from local contributions, are missing in the reference model. It is the reason that the terrestrial gravity anomalies (which include the short-wavelength information) are not used in the geoidal height estimators too. This means that for comparison sake, only long-wavelength constituents are used. Also, all corrections (the most important one, the topographic corrections) are not included in this study, as the idea of this work is only a comparison between different geoid estimators not the geoidal height computations.

With the use of EGM96-derived gravity anomalies infected with the white-noise, the error anomaly degree variances in the least-squares estimator must then be computed by

$$dc_n = \frac{(GM_1)^2}{a_1^4}(n-1)^2\sigma_0^2(2n+1) \qquad (24)$$

instead of Eq. (12).

Also, as the terrestrial gravity anomalies are replaced with EGM96-derived gravity anomalies, the error anomaly degree variances for the terrestrial gravity anomalies have to be obtained from

$$\sigma_n^2 = dc_n \qquad (25)$$

instead of Eq. (14).

These error degree variances are not very realistic, but they may be sufficient for this comparison. The selection of the σ_n and dc_n models is critical for the least-squares estimator, whereas the c_n model does not play an important role (see also Despotakis, 1987). But, whenever these models are properly selected, the optimal solution is achieved (see also Despotakis, 1987).

4. – NUMERICAL INVESTIGATIONS

Five geoid estimators, Molodensky et al. (1962), Wong and Gore (1969), Vincent and Marsh (1974), Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) are numerically investigated. A test area of $5^\circ \times 5^\circ$ is chosen. It is delimited by latitudes 50° N and 55° N, and longitudes 30° E and 35°E, located in Iran. All computation points in this study are the center of the cells with the size of 30' × 30'. EGM96 model is the global geopotential model used in this study. The mean geoid radius R is selected to 6371000 m.

First, the geoidal heights are computed from EGM96 using Eq. (1). The degree and order of expansion are complete to 360. These geoidal heights are considered as the "reference geoid model". This model is used for comparison of the geoid estimators.

Next, geoidal heights are computed with five geoid estimators. Instead of the terrestrial gravity anomalies, the gravity anomalies determined from EGM96 are used in the integral-part of the estimators. Eq. (23) is used to compute these gravity anomalies. In the geoid estimators, the geopotential coefficients to degree and order of 360 are used for the determination of Δg in the integral-part. The long-wavelength part of the geoid estimators is computed by the coefficients to only degree and order 60. To be more consistent with the real situation, random numbers are added to the geopotential coefficients, derived from a white-noise process. Thereafter, the geoidal heights are computed in the test area with the five estimators. The integration area in geoid estimators is limited to 6° from computation points (see Nahavandchi, 1998; Nahavandchi and Sjöberg, 2001). The Molodensky truncation coefficient, Q_n and the e_{ki} coefficients are estimated according to Hagawara (1976) and Paul (1973), respectively.

Table 1 shows the statistics of differences between the geoidal heights estimated from five geoid estimators and the "reference geoid model". The smallest mean value and standard deviation of differences are obtained with the least-squares model, computed to -8 cm and ±29 cm, respectively. The next smallest standard deviation of differences is computed with Vanicek and Kleusberg (1987) model, equal to ±32 cm. Table 1 reveals the fact that different results with different geoid estimators are expected in real practice. Table 1 also shows that both Pizzetti reference field and the higher than second-degree reference field provide good results compared to the "reference geoid model". The least-squares model uses the spheroidal Stokes kernel, while the modified spheroidal Stokes kernel is used in Vanicek and Kleusberg (1987) model.

In order to obtain further insight into the comparison between the two reference fields, the Molodensky et al. (1962) and Sjöberg's least-squares (1984) models are evaluated using a residual gravity field Δg^M [see Eq. (16)] instead of Δg [see Eq. (2)]. Thereafter, the geoidal heights have been compared with the "reference geoid model". Table 2 shows the statistics of differences with the reference model. Results of Table 2 show no significant differences between the two estimators and "reference model", whether the Pizzetti or higher than second-degree reference fields are used (compare the results of Tables 1 and 2). Actually, the standard deviations of differences are smaller when second-degree reference field is used. It is computed to ±33 cm and ±44 cm in least-squares and Molodensky models, respectively, when the higher degree reference field is used.

Table 1. The statistics of differences on geoid between different geoidal height estimators and the "reference geoid model". Units in metres

	Molodensky et al.	Wong and Gore	Vincent and Marsh	L-S	Vanicek and Kleusberg
Min	-0.81	-0.79	-0.86	-0.65	-0.78
Max	1.16	1.14	1.17	1.02	1.07
Mean	-0.14	-0.13	-0.15	-0.08	-0.11
SD	0.43	0.40	0.45	0.29	0.32

Table 2. The statistics of differences on geoid between Molodensky et al. and the least-squares estimators with the "reference geoid model", using the higher than second-degree reference field. Units in metres

	Molodensky et al.	L-S
Min	-0.87	-0.89
Max	1.17	1.15
Mean	-0.19	0.16
SD	0.44	0.33

In the next experiment, the Wong and Gore model [Eq. (15)] is computed using the second-degree reference field. It means that Δg^M in this model are replaced with the Δg. Thereafter, the "reference geoid model" is used for the comparison. Table 3 shows the statistics of differences. Surprisingly, the same results are mostly obtained in the Wong and Gore model with the second degree and higher than second-degree reference fields. This means that the computation labor is saved if one uses the Pizzetti field in the Wong and Gore model, and same accuracy can be expected. However, this should be tested in other areas. In the next test, Vanicek and Kleusberg (1987) model [Eq. (18)] is computed using the Pizzetti reference field. Table 4 shows the differences between this model and the "reference model". The results of Table 4 show large differences between this geoid estimator and the "reference model", when the second-degree reference field is used. The use of the modified spheroidal Stokes kernel is the reason. When integration of gravity anomalies Δg with the modified Stokes kernel is used, the long wavelength frequency components must be subtracted from Δg as the modified spheroidal kernel is no longer blind to the low frequencies. It means that in this type of kernel, the residual gravity field must be used. This is not the case for the original spheroidal kernel.

Table 3. The statistics of differences on geoid between Wong and Gore model and the "reference geoid model", using the Pizzetti reference field. Units in metres

	Wong and Gore
Min	-0.85
Max	1.22
Mean	-0.15
SD	0.43

Further, Vincent and Marsh model is computed with the original gravity field Δg instead of Δg^M in Eq. (17). Table 5 shows the statistics of differences between this model and the "reference geoid model". The standard deviation of differences is computed to ±78 cm using the Pizzetti field versus ±45 cm using the residual gravity field. Also, mean of differences are computed to -325 cm versus -15 cm using higher degree reference field. The reason for these differences is the use of the original spherical Stokes function in this geoid estimator. It means that the long wavelength frequency components must always be subtracted from Δg in this model.

Table 4. The statistics of differences on geoid between Vanicek and Kleusberg model and the "reference geoid model", using the Pizzetti reference field. Units in metres

	Vanicek and Kleusberg
Min	-1.18
Max	1.92
Mean	0.62
SD	0.58

Table 5. The statistics of differences on geoid between Vincent and Marsh model and the "reference geoid model", using the Pizzetti reference field. Units in metres

	Vincent and Marsh
Min	-3.25
Max	2.19
Mean	-3.08
SD	0.78

The overall results show that the least-squares and Vanicek and Kleusberg geoid estimators present the best agreement with the "reference geoid model" in this study. To see how good these two geoid estimators are in agreement with each other, both models are computed in the test area and compared. Table 6 shows the results of this comparison. A mean difference of -6 cm and a standard deviation of ±5 cm are resulted.

Table 6. The statistics of differences on geoid between the least-squares and Vanicek and Kleusberg models. Units in metres

Min	-0.13
Max	0.11
Mean	-0.06
SD	0.05

All computations above are implemented in other test area in Iran with a size of $2^{\circ} \times 2^{\circ}$. The same results are mostly obtained. However, these computations must be tested in other areas.

4.1. – TRUNCATION ERROR IN GEOID ESTIMATORS

The Stokes integration, in practice, is performed over a truncated spherical cap. As the Stokes kernels is non-zero in the region outside the integration cap, the effects of the gravity anomalies in these zones cause the truncation error. In this section, the truncation errors in Vincent and Marsh (1974), Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) geoid estimators are evaluated. These three estimators use the spherical, spheroidal and modified spheroidal Stokes's kernels. They also use both the second degree and higher degree reference fields.

It can be shown that the truncation error in spectral form in Vincent and Marsh (1974) model is

$$\delta N_{\mathrm{VM}}(r,\varphi,\lambda) = c\sum_{n=M+1}^{\infty} Q_n(\psi_0)\,\Delta g_n \tag{26}$$

The truncation error is a function of the truncation coefficients.

Sjöberg's least-squares (1984) estimator holds the following truncation error as:

$$\delta N_{\mathrm{LS}}(r,\varphi,\lambda) = c\sum_{n=M+1}^{\infty} Q_{Mn}(\psi_0)\,\Delta g_n \tag{27}$$

where

$$Q_{Mn}(\psi_0) = Q_k(\psi_0) - \sum_{K=2}^{M}\frac{2k+1}{2}s_k e_{nk}(\psi_0) \tag{28}$$

It is obvious that the least-squares estimator is biased for all harmonics from degree two and up. It can be shown that the spheroidal kernel tapers off to zero for smaller truncation radii ψ_0. Moreover, $\|S_M(\psi)\| \le \|S(\psi)\|$ for $\psi<\psi_0$, where $\|\bullet\|$ indicates the "norm" operator. Therefore, it can be considered that the impact of the truncation error is reduced more in the least-squares estimator than in Vincent and Marsh (1974) model.

The modified spheroidal Stokes kernel is used in Vanicek and Kleusberg (1987) geoid estimator. The corresponding truncation error is

$$\delta N_{\mathrm{VK}}(r,\varphi,\lambda) = c\sum_{n=M+1}^{\infty} Q_n^M(\psi_0)\,\Delta g_n \tag{29}$$

It can be shown that $\left\|S_M^s(\psi)\right\|\left\|\Delta g^M\right\| \le \|S(\psi)\|\left\|\Delta g^M\right\|$ for $\psi<\psi_0$. Therefore, the modification to the spheroidal Stokes kernel reduces the truncation error compared to Vincent and Marsh (1974) geoidal height estimator.

To evaluate these errors numerically, the truncation errors are computed in the test area according to Eqs. (26)-(29). The maximum degree of expansion 360 is used in this investigation. The maximum value of truncation error in Vincent and Marsh (1974) model is computed to 15.2 cm. It is computed to 7.11 cm and 6.18 cm in Vanicek and Kleusberg (1987) and Sjöberg's least-squares (1984) models, respectively. It is again shown that the least-squares estimator, which uses a spheroidal kernel, is a good model as far as the truncation errors are concerned. The Vanicek and Kleusberg (1987) model, which uses a modified spheroidal kernel, mostly gives the same results.

4.2. – TERRESTRIAL GRAVITY DATA

Similar steps are done in the test area but with the terrestrial gravity anomalies used in the geoid estimators. Thereafter, the geoidal heights are compared with the "reference geoid model". The terrestrial gravity data are in 110" × 160" cells. The center of the cells is the computation points for the geoidal heights.

It is important to note that the corrections required in gravimetric geoidal height computations (the most important one, the topographical corrections) are not considered in the geoid estimators, as the study is limited to the comparison of different geoid estimators not the geoidal height computations. On the other hand, the geoid estimation in this section using the terrestrial gravity anomaly data in the integral part of the estimators includes short-wavelength information, while it is disregarded in the "reference geoid model". Again, as this study only aims on comparison of different geoid estimators, this level of accuracy seems suitable. The situation is considered the same for all estimators and this will not ruin the nature of the comparison.

The statistics of the differences between the geoidal heights computed from Molodensky et al. (1962), Wong and Gore (1969), Vincent and Marsh (1974), Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) models (with the terrestrial gravity data in the integral-part of the estimators) and the "reference geoid model" are shown in the Table 7. As it was expected the differences enlarge as the terrestrial gravity data are used in the geoid estimators. These represent the local contributions, which are not included in the "reference geoid model". However, as it is mentioned, this is only a measure to see how different geoid estimators work in different situations compared to a global gravity model. This may help, for example, to see the differences between geoid estimators and to finally suggest an optimal estimator. One may expect that the situation is the same in the real practice. Again, the least-squares estimator provides the smallest standard error of differences with the "reference model". Standard deviation of differences is computed to ±38 cm. The second smallest standard deviation is computed with the Vanicek and Kleusberg (1987) model, as it was expected.

Table 7. The statistics of differences on geoid between different geoidal height estimators and the "reference geoid model", using the terrestrial gravity data. Units in metres

	Molodensky et al.	Wong and Gore	Vincent and Marsh	L-S	Vanicek and Kleusberg
Min	-1.08	-0.99	-1.16	-0.86	-0.98
Max	1.41	1.33	1.43	1.21	1.27
Mean	-0.16	-0.15	-0.18	-0.10	-0.13
SD	0.54	0.51	0.57	0.38	0.43

5. – DISCUSSION AND CONCLUSIONS

To study the efficiency of some different geoid estimators, Molodensky et al. (1962), Wong and Gore (1969), Vincent and Marsh (1974), Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) models are numerically studied. The above geoid estimators are compared with a "reference geoid model" determined from EGM96 geopotential coefficients.

Note that the short-wavelength information is missing in the "reference model", as a global geopotential model is used to derive the reference geoid. For the comparison sake, the geopotential coefficients are used to determine the gravity anomalies in the integral-part of the different geoid estimators too. The "reference geoid model" might not be precise and realistic, but it is sufficient for the comparison of the different geoidal height estimators in this study.

Different modification parameters, different Stokes kernels (spherical, spheroidal, modified spheroidal) and different reference fields (Pizzetti versus the higher than second -degree) are used. These different parameters are all included in the five geoid estimators mentioned above.

The Sjöberg's least-squares (1984) and Vanicek and Kleusberg (1987) estimators were in best agreement with the "reference geoid model". The former model uses the Pizzetti type reference field and the spheroidal Stokes kernel, where the latter uses the higher than second-degree reference field and modified spheroidal Stokes kernel.

It is shown that the use of Pizzetti reference field and spheroidal Stokes kernel mostly provide the same results as with the use of a higher than second degree field and modified spheroidal Stokes kernel. This means that the former procedure results to the same accuracy as the latter with less computational labor in this study. These results must however be tested in other areas.

One can also conclude that the modification of original Stokes's formula is still an open investigation when 1-cm geoid is desired. However, the results of this study show that the least-squares and Vanicek and Kleusberg models can be considered as two suitable estimators for geoidal height determination. The former reduces the truncation error, erroneous terrestrial gravity data and potential harmonic errors in a least-squares sense.

The truncation errors are numerically evaluated in the geoid estimators. It is numerically shown that these errors reduce significantly in the spheroidal kernel than in the spherical one.

The same results with the terrestrial gravity data are obtained in an attempt to compare the five geoidal height estimators with the "reference geoid model". The corrections to the gravimetric geoid estimators (e.g. topographical corrections) are of no interest in this study.

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