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Highly Composite Numbers

Master's thesis in mathematics, MLREAL
Supervisor: Andrii Bondarenko
December 2022

Master's thesis

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Abstract

The main result of this thesis is to show that there are only finitely many integers n such that both n and $d(n)$ are highly composite numbers at the same time, where $d(n)$ is the divisor function. Bertrand's postulate [4] is used many times throughout the thesis and allows us to write a proof that is as simple (and as short) as possible. This thesis is meant to solve the open problem from the "On-Line Encyclopedia of Integer Sequences" (OEIS): A189394 [3].

The main idea for solving the problem comes from the comment in A189394; n will contain many primes with exponent 1 when n is a large highly composite number. This implies that $d(n)$ contains a lot of factors of 2. We then estimate the factor 2^{β_1} in $d(n)$ in terms of the largest prime in $d(n)$ from above and from below to give us a contradiction when n is large enough. We end by finding a list of all highly composite n such that $d(n)$ is also highly composite.

Abstrakt

Hovedresultatet for denne oppgaven er å vise at det kun finnes endelig mange tall n slik at både n og $d(n)$ er "antiprimtall", hvor $d(n)$ er divisorfunksjonen. Gjennom hele oppgaven blir Bertrands postulat [4] brukt mange ganger. Dette har gjort at bevisene kan skrives så enkelt som mulig. Oppgaven skal løse det åpne problemet fra "On-Line Encyclopedia of Integer Sequences" (OEIS): A189394 [3].

Hovedidéen for hvordan vi løser problemet kommer fra kommentaren i A189394; Når n er et stort antiprimtall, vil n inneholde mange primtall med eksponent 1. Det vil si at $d(n)$ inneholder mange faktorer av 2. Vi estimerer faktoren 2^{β_1} i $d(n)$ nedenfra og ovenfra i forhold til den største primtallsfaktoren i $d(n)$ for å få en motsigelse når n er stor nok. Vi avslutter med å finne alle antiprimtall n slik at $d(n)$ også er et antiprimtall.

Chebyshev said it
and I'll say it again,
There's always a prime
between n and $2n$.

- Bertrand's postulate

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Contents

1 Preliminaries

Definition 1.1. The divisor function $d : \mathbb{N} \to \mathbb{N}$ is defined to be the number of positive divisors of a given positive integer n. It is usually represented as the sum over the divisors of n

$$d(n) = \sum_{d|n} 1.$$

In most cases this definition does not say much on how large $d(n)$ is in terms of the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. Counting the number of divisors of each of the prime powers we can see that $p_k^{\alpha_k}$ has $1, p_k^1, p_k^2, \dots p_k^{\alpha_k}$ as its divisors: a total of $\alpha_k + 1$ of them. Since $d(n)$ is multiplicative we may then write

$$d(n) = \prod_{j=1}^{r} (\alpha_j + 1).$$

That is, the number of divisors of a given number does not depend on its prime factors, only on the exponents in the prime powers.

Definition 1.2. A positive number n is said to be *highly composite* if the number of divisors of n is greater than the number of divisors of every number smaller than n:

$$n \text{ is } \textit{highly composite} \text{ if for every } m < n \text{ we have } d(m) < d(n).$$

Building on this definition, we can see that the sequence of highly composite numbers is the sequence of numbers n where $d(n)$ obtains a new maximum. It is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is a highly composite number then n consists of the smallest r primes with certain exponents α_k. Using larger primes with the same exponents yields a larger number with the same number of divisors. Furthermore, the exponents must be in decreasing order $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r$. If not, then rearranging the exponents in a decreasing order also yields a smaller number with the same number of divisors. We also have that $\alpha_r = 1$ for all highly composite n, except for the small cases when $n = 1, 4, 36$, which we will see as a consequence of the first lemma.

2 Lemmas

In this section we estimate the factor 2^{β_1} in $d(n)$ from above using lemma 2.2, and from below using the other lemmas. The first lemma is almost identical to Erdös' lemma 2 in [1], but here it is written and proven to hold for all $n > 50400$.

Lemma 2.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be a highly composite number greater than* 50400. *Then*

$$\alpha_j = 1 \textit{ for all } p_j \in \left[\frac{p_r + 1}{2}, p_r + 1\right]. \tag{1}$$

Proof. Take any $p_j \in [\frac{p_r+1}{2}, p_r + 1]$ and assume $\alpha_j \geq 2$. Then $\alpha_{j-2} \geq \alpha_{j-1} \geq \alpha_j \geq 2$ as mentioned in the introduction. Consider the number

$$n' = n \cdot \frac{p_{r+1} p_{r+2}}{p_j p_{j-1} p_{j-2}}.$$

Clearly, $p_{r+1}p_{r+2}$ contributes to a factor of 4 in $d(n')$, and $d(\frac{n}{p_j p_{j-1} p_{j-2}})$ is at least $d(n) \cdot \left(\frac{2}{3}\right)^3$. So

$$d(n') \geq d(n) \cdot 4 \cdot \left(\frac{2}{3}\right)^3 > d(n).$$

Our goal is to show that $n' < n$ giving us a contradiction to n being highly composite. Bertrand's postulate [4] gives us that $p_{k+1} < 2p_k$. So

$$p_{r+1}p_{r+2} < 8p_r^2, \text{ and } \frac{1}{p_j p_{j-1} p_{j-2}} < \frac{8}{p_j^3}$$

Since $p_j \in [\frac{p_r+1}{2}, p_r + 1]$ we also have $p_j \geq \frac{p_r+1}{2}$, and so

$$\frac{1}{p_j^3} \leq \frac{1}{(\frac{p_r+1}{2})^3} = \frac{8}{(p_r+1)^3} < \frac{8}{p_r^3}.$$

Thus

$$n' = n\frac{p_{r+1}p_{r+2}}{p_j p_{j-1} p_{j-2}} < n\frac{512}{p_r},$$

which is less than n whenever $p_r > 512$. This means we have a contradiction whenever the largest prime p_r in n is larger than 512. We check $p_r \leq 512$ in appendix A.1 to get that every $n > 50400$ satisfies the lemma. □

Remark. *For $n > 54000$ we get that the last two exponents α_{r-1} and α_r must be 1 by $\pi(p_r + 1) - \pi(\frac{p_r+1}{2}) \geq 2$ for $p_r \geq 11 = R_2$, where R_k are the so called Ramanujan primes [4]. Checking a list of highly composite numbers [2] when $n \leq 50400$ gives us that $\alpha_r = 1$ always, except for small cases when $n = 1, 4$ or 36.*
Also, if we write $d(n) = 2^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$ for n highly composite, we have that

$$\beta_1 \geq \pi(p_r + 1) - \pi\left(\frac{p_r + 1}{2}\right) \tag{2}$$

where $\pi(x) = \sum_{p \leq x} 1$ is the prime counting function. This is because each prime $p_j \in [\frac{p_r+1}{2}, p_r + 1]$ of n contributes to a factor of 2 in $d(n)$.

The next lemma gives us the necessary upper bound for 2^{β_1} in $d(n)$.

Lemma 2.2. *Suppose that $d(n) \geq 12$ is a highly composite number and write $d(n) = 2^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$. Then*

$$2^{\beta_1} < 8q_s^2.$$

Remark. *Note that we don't assume n to be highly composite.*

Proof. We begin by choosing the smallest $1 \leq h \leq \beta_1$ such that

$$2q_s 2^{\beta_1 - h} < 2^{\beta_1}.$$

In other words h is the smallest integer such that $2^h > 2q_s$ holds.
If we cannot find $h \leq \beta_1$, we set $h = \beta_1$ so $h - 1 > \frac{\beta_1 - 1}{2}$ for $\beta_1 \geq 2$. Then $2^{h-1} \leq 2q_s$ and we skip the next step.

If $h \leq \beta_1$, consider

$$m = d(n) q_{s+1} 2^{-h}.$$

Then $m < d(n) 2q_s 2^{-h}$ by Bertrand's postulate [4], which again is less than $d(n)$ by construction of h.
Then certainly we must have $d(m) < d(d(n))$ since $d(n)$ is highly composite. This yields

$$2(\beta_1 - h + 1) < \beta_1 + 1 \text{ or } h > \frac{\beta_1 + 1}{2}$$

That is, whenever $2^h > 2q_s$ we have $h > \frac{\beta_1+1}{2}$. Since h was chosen to be the smallest number with the property that $2^h > 2q_s$ we then have $2q_s \geq 2^{h-1}$ so

$$\begin{aligned}
2q_s &\geq 2^{h-1} > 2^{\frac{\beta_1+1}{2}-1} \\
4q_s^2 &> 2^{\beta_1 - 1} \\
8q_s^2 &> 2^{\beta_1}
\end{aligned}$$

□

The next lemma tells us that the largest prime in $d(n)$ is bounded above by the largest prime in n.

Lemma 2.3. *Write n and $d(n)$ as earlier and suppose that both n and $d(n)$ are highly composite. Then if $p_r \geq 13$ we have*

$$q_s < p_r. \tag{3}$$

Our proof is quite simple and is a just consequence of lemma 2.2.

Proof. We need only to prove that $p_r > \alpha_1 + 1$ for the multiplication formula for $d(n)$ gives us that $\alpha_1 + 1 \geq q_s$. We also have that the exponents in n are decreasing $\alpha_1 \geq \cdots \geq \alpha_r$ if $\alpha_1 + 1$ is not prime. We now apply lemma 2.2, but this time on n highly composite, giving us

$$\begin{aligned}
8p_r^2 &> 2^{\alpha_1}, \text{ so} \\
p_r &> 2^{\frac{\alpha_1 - 3}{2}} \geq \alpha_1 + 1, \text{ if } \alpha_1 \geq 12
\end{aligned}$$

Meaning for $p_r \geq 13$ we have $p_r > \alpha_1 + 1 \geq q_s$ and we are done. The result also holds for some smaller p_r, but it is not needed here. □

The next lemma gives us a bound for $\pi(p_r + 1) - \pi\left(\frac{p_r+1}{2}\right)$ from Ramanujan's proof of Bertrand's postulate. The proof can be found in [4].

Lemma 2.4.

$$\pi(x) - \pi(\frac{1}{2}x) > \frac{1}{\log x}(\frac{1}{6}x - 3\sqrt{x}), \textit{ for } x > 300. \tag{4}$$

From these lemmas we move on to the main result.

3 Main result

Theorem. *There are only finitely many integers n such that both n and $d(n)$ are highly composite.*

Proof. Suppose that both n and $d(n)$ are highly composite, and write them as earlier. Since $d(n)$ is highly composite we have the upper bound from lemma 2.2 for 2^{β_1}

$$2^{\beta_1} < 8q_s^2.$$

Now, assuming $p_r \geq 2164$, the following calculation gives us a lower bound on 2^{β_1}:

$$\begin{aligned} 2^{\beta_1} &\geq 2^{\pi(p_r+1)-\pi\left(\frac{p_r+1}{2}\right)} && \text{(lemma 2.1)} \\ &> 2^{\frac{1}{\log(p_r+1)}\left(\frac{1}{6}(p_r+1)-3\sqrt{p_r+1}\right)} && \text{(lemma 2.4)} \\ &> 8p_r^2 \\ &> 8q_s^2 && \text{(lemma 2.3)} \end{aligned} \tag{5}$$

Where we have used the fact that line 3 holds whenever $p_r \geq 2164$. This gives us the contradiction.
In appendix A.2 we use a search on all primes $p_r \leq 2164$ to find that

$$2^{\pi(p_r+1)-\pi\left(\frac{p_r+1}{2}\right)} \geq 8p_r^2$$

also holds for all primes $181 \leq p_r < 2164$. This means there are no highly composite numbers n with largest prime $p_r \geq 181$ such that $d(n)$ is also highly composite. $\square$

We end the main part of this thesis with table 1; A list of all the highly composite numbers n such that $d(n)$ is also highly composite. This table is generated by checking if $d(n)$ is highly composite for every highly composite number n with largest prime $p_r \leq 181$. To guarantee that $p_r \leq 181$, we use the bound obtained by Erdös in his proof of "lemma 1" [1]. He obtains that $n < p_r^{\pi(p_r)}$, meaning $n < 181^{41} \leq 10^{556}$ for $p_r \leq 181$. We do this in appendix A.3.

n	$d(n)$	n	$d(n)$
1	1	55440	120
2	2	277200	180
6	4	720720	240
12	6	3603600	360
60	12	61261200	720
360	24	2205403200	1680
1260	36	293318625600	5040
2520	48	6746328388800	10080
5040	60	195643523275200	20160

Table 1: The cases where both n and $d(n)$ are highly composite

A Appendix

A.1 Small cases in lemma 2.1

Here p_r is the largest prime in n, p_j is the smallest prime greater than or equal to $\frac{p_r+1}{2}$. The last column is the factor we multiply n by in the proof. These are the only cases when this factor is ≥ 1 for $p_r \leq 512$.

p_r	p_j	$\frac{p_{r+1}p_{r+2}}{p_j p_{j-1} p_{j-2}}$
7	5	4.76667
11	7	2.10476
13	7	3.07619
17	11	1.13506
19	11	1.73247

We manually check the cases $p_r \in [11, 19]$ to satisfy (1). A simple check against a list [2] of highly composite numbers yields 12 numbers with $p_r = 7$ as its largest prime factor. Of these, only $n = 50400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$ and $n = 25200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$ do not satisfy (1) and proving our lemma. For the smaller primes $p_r = 2, 3, 5$ only a handful of highly composite numbers satisfies (1), namely $n = 2, 6, 60, 120$ and 240, but is not needed here.
Below is the Python program used for this search.

```
p = list of primes < 600
table = [["p_r","p_j","p_r+1 p_r+2/p_j p_j-1 p_j-2"]]

for r in range(2,len(p)-2):
    j=0
    while p[j] < int((p[r]+1)/2):
        j+=1
    K = p[r+1]*p[r+2]/(p[j]*p[j-1]*p[j-2])
    if (K >= 1):
        table.append([p[r],p[j],K])
print(table)
```

A.2 Lowering the largest prime in eq. (5)

Here we skip the lower bound by Ramanujan [4] of $\pi(x) - \pi(\frac{x}{2})$ to check directly when

$$2^{\pi(p_r+1)-\pi\left(\frac{p_r+1}{2}\right)} \geq 8p_r^2, \tag{6}$$

for all primes p_r up to 2200. The Python program for this search is written below and gives us that (6) holds whenever $p_r \geq 181$.

```
primes = list of primes < 2200
table = [["p_r", "8p_r^2", "2^(pi(p_r+1)-pi((p_r+1)/2))"]]
def pi(x):
    return "number of primes less than x"
def pi2(x):
    return 2^(pi(x)-pi(x/2))
def f(x):
    return 8x^2
```

```
for p in primes:
    if f(p) < pi2(p+1):
        table.append(p, [f(p), pi2(p+1)])
print(table)
```

A.3 Generating all highly composite n and $d(n)$

We need only to check all highly composite $n \leq 10^{556}$ as explained earlier. The 12500-th highly composite number is greater than 10^{560} by direct evaluation from Flammenkamp's list ("HCN.gz", [2]). We also find that $d(n) \leq 10^{100}$ by direct calculation of the 12500-th highly composite number n. The list `hcn` in the program below is the list of all highly composite numbers less than 10^{100} and we check if $d(n)$ is in this list to generate our table. This gives us table 1, which is same numbers as in A189394 [3]!

```
hcn = list of h.c.n. < 10^100
dhcn = []
def genlist(lst):
    x = [i for i in lst]
    for l in x:
        if "^" in l:
            k = l.split("^")
            i = [int(k[0])]*int(k[1])
            lst.remove(l)
            [lst.append(o) for o in i]
    lst = [int(i) for i in lst]
    return lst
f = open("HCN.gz")
lines = f.readlines()
for line in lines:
    line = line.replace("\n", "").split(" ")
    line = line[3:]
    line = genlist(line)
    dn = math.prod([i+1 for i in line])
    if dn in hcn: dhcn.append([dn,line])
print(dhcn)
```

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