A comparison of different procedures of handling the effects of close and distant topographic masses in gravimetric geoid computations with the classical and recent formulae

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Abstract

Different procedures for considering the effects of topographic masses in Stokes-Helmert scheme of geoid determination are reviewed. Classical integral formulae use a planar approximation of the geoid and a limited area of integration, and thus account only for the local contributions of topographic effects. In the other hand, the spherical harmonic representation of the topographic effects normally only includes the long-wavelength information. Another description of the effects of topographic masses given by Martinec and Vaníček (1994a, b) uses near and far-zone integration areas and a spherical approximation of the geoid. Finally, two formulae, that combine short and long-wavelength contributions, are presented for topographical effects. These recent formulae imply that the integral formulae for determining the topographic effects may have some numerical problems in representing global information (for truncated integration domain). On the other hand, a representation of the effects by a set of spherical harmonic coefficients of the topography to, say, degree and order 360 leads to omission of significant short-wavelength information. All above-mentioned procedures were used for computation and comparisons in a test area in Iran with the maximum elevation of 3053 m. The results of these comparisons show that the Martinec and Vaníček (1994a, b) integral formulae and the recent combined formulae presented by Sjöberg and Nahavandchi (1999) and Nahavandchi (2000) are in good agreement with each other. These formulae use a spherical approximation of the geoid, contrary to the classical formulae which use a planar approximation. Only the combined formulae include, however, all wavelength constituents and are recommended for precise geoid determination. Further, the gravimetric geoidal heights were computed applying these different procedures of handling the topographic effects. The results were then compared at Global Positioning System (GPS)-leveling stations in Iran. The standard deviation of the fit with the combined formulae is the best among the other methods and is equal to ± 10.1 cm.

Key words: Direct topographic effect, Indirect topographic effect, Geoidal height, Helmert condensation

1 Introduction

The geoid is frequently determined from ground gravity data by the well-known Stokes formula. This integral formula is the solution of an exterior type boundary value problem, which implies that masses exterior to the geoid are not permitted in the formulation of this problem. This is achieved mathematically by removing the effect of external masses and replace them by the effect of additional masses below the geoid [direct topographic effect (DTE) on gravity]. However, it should be noted that the DTE on gravity anomaly equals the sum of the DTE on gravity and the so-called indirect effect of the topography on gravity or the secondary indirect topographic effect (SITE) on the geoidal height. Thereafter, as Helmert's reduction is used, the corrected ground gravity anomaly (Helmert anomaly) must be continued downward to the geoid (downward continuation) prior to perform Stokes's integration. The effect of removed masses is then restored after applying Stokes's integral [primary indirect topographic effect (PITE) on the geoidal height]. These procedure follows the principals described in Vaníček and Martinec (1994), Martinec (1998) or Nahavandchi and Sjöberg (1998).

Recognizing that a valid solution to geoid determination would occur only if there were no masses outside the geoid, Helmert suggested that the masses outside the geoid could be condensed as a surface layer directly at the the reference sphere in a spherical approximation of the geoid. In this study, Helmert's second condensation method is used that preserves the Earth mass, for which the Helmert model of the Earth has the same mass as the real Earth. A discussion of some attributes of Helmert's second method of condensation may be found in Heiskanen and Moritz (1967), Wichiencharoen (1982), Heck (1992), Martinec et al. (1993), Vaníček et al. (1995) and Nahavandchi and Sjöberg (1998).

Two different formulae for the remove-restore problem were presented by Moritz (1980) and Vaníček and Kleusberg (1987). Moritz (1968, 1980) examined the role of the topography to show a relationship between Helmert's condensation reduction and the approximate solution of the Molodenskij boundary value problem. He derived the DTE referred to the geoid. Vaníček and Kleusberg (1987) derived the DTE referred to Earth's surface, which means that the ground gravity anomalies corrected with their formula still need a downward continuation correction to be used in Stokes's integral. These two classical formulae are limited to the second power of elevation H and suffer from the planar approximation of the geoid.

Sjöberg (1994) suggested a spherical harmonic representation of the topographic effects. This approach was implemented by Sjöberg (1995, 1996) to the second power of elevation H and by Nahavandchi and Sjöberg (1998) to the third power of elevation H, where the DTE was derived at the surface of the Earth, respectively.

Another description of the Stokes-Helmert method for geoid determination was given by Vaníček and Martinec (1994). The specific problem of determining the DTE and the PITE were treated by Martinec and Vaníček (1994a, b), who pointed out that the classical formulae may severely be biased because of the planar approximation of the geoid in the derivations.

Later, Nahavandchi (1998a, b), Sjöberg and Nahavandchi (1999), Sjöberg (2000) and Nahavandchi (2000) argued that the DTE and the PITE were composed of both short (local effects) and long-wavelength (global effects) contributions. This implies that the integral formulae for determining the topographic effects (using a limited spherical cap around computation points) may have some problems in representing the long-wavelength contributions. On the other hand, a representation by the set of spherical harmonic coefficients of the topography omits significant short-wavelength information, as in the practice it is limited to a maximum degree of 360 in this study. They derived two formulae for handling the DTE and the PITE with a combination of the integral formulae and the set of spherical harmonic coefficients of Earth's topography.

In this paper, all above-mentioned formulae for topographic effects will be computed in the test area with maximum elevation of 3053 m. The differences will be compared and discussed. Finally, the results of gravimetric geoid heights computed with different topographic effects will be compared to the geoidal heights derived at GPS-leveling stations.

2 Topographic effects in gravimetric geoid determination

The formulae used for topographic effects here are based on a constant topographic density. These formulae can also be generalized to a laterally variable density simply by putting it inside the surface integrals of the DTE and the PITE (see also Martinec 1998) . In addition to topographic effects, geoid determination by Stokes's formula also requires that the gravity anomalies, Δg, must refer to the geoid. For satisfying this condition, the gravity anomalies available on Earth's surface have to be reduced to the geoid. This reduction is called a downward continuation.

The Stokes integration with the Helmert anomaly and considering the PITE on geoid is realized by the formula (Heiskanen and Moritz 1967, p. 324)

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi) \Delta g^{\mathrm{H}^*} d\sigma + \delta N_{\mathrm{I}}^* \tag{1}$$

where N is the geoid height, Δg^{H^*} is the ground free-air gravity anomaly (Δg) corrected for the DTE (resulting in Helmert's anomaly Δg^H at the Earth's surface) and then reduced to the geoid (i.e., downward-continued to the geoid), γ is normal gravity, $S(\psi)$ is Stokes's function, ψ is the spherical distance between the computation and running points, σ is the unit sphere, R is the mean radius of the Earth, and $\delta N_{\mathrm{I}}^{*}$ is the PITE on the geoid. In this study, the Helmert second condensation method was used to remove the effect of external masses and replace them by the effect of additional masses below the geoid. The Helmert anomaly Δg^{H} at Earth's surface can therefore be expressed via

$$\Delta g^{\mathrm{H}} = \Delta g + \delta\Delta g_{\mathrm{dir}} \tag{2}$$

where Δg is the ground free-air gravity anomaly and $\delta\Delta g_{\mathrm{dir}}$ is the DTE on gravity anomaly determined at Earth's surface. In this section, different formulae for correcting the effects of topographic masses in gravimetric geoid computations within the Stokes-Helmert scheme are presented, and the downward continuation problem will be discussed in Sect. 3.

The SITE on the geoidal height is usually two orders of magnitude smaller than the DTE. Nahavandchi (1998b) computed this effect at 23 GPS-leveling stations in Sweden with the mean value of less than 0.7 cm. This term is neglected in this study. Also, the geoid atmospheric effect (Sjöberg and Nahavandchi 2000) and other corrections to Helmert's anomalies on the Earth's surface (see e.g. Vaníček et al. 1999), are not studied here.

2.1 DTE to gravity anomaly

2.1.1 DTE with the classical integral formulae

Moritz (1980) derived a formula for the removing of the effect of topographic masses. This correction should be added to the ground free-air gravity anomalies in Stokes's formula. This formula which uses the planar approximation of the geoid, is expressed as (Moritz 1980)

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{M}^*}(H_P) = \frac{\mu R^2}{2} \int\int_{\sigma} \frac{(H - H_P)^2}{\ell_0^3} d\sigma \tag{3}$$

where μ=$G\rho$, G is the universal gravitational constant, ρ is the constant density of topography, H and H_P are the orthometric heights of the running and computation points, respectively, and the spatial distance $\ell_0 = R\sqrt{2(1-\cos\psi)} = 2R\sin\frac{\psi}{2}$.
The topographic effect $\delta\Delta g_{\mathrm{dir}}^{\mathrm{M}^*}$ is related to the points on the geoid. This formula assumes that the gravity anomalies in a downward continuation integral are linearly proportional to the topographical heights according to the so-called Pellinen assumption (Moritz 1968, 1980). Hence, the resulting Moritz topographic effect also involves the effect of the downward continuation of gravity anomalies. This effect is, however, described only approximately since the linear relationship between gravity anomalies and topographical heights coresponds to the reality only roughly (see e.g. Heiskanen and Moritz 1967).

Vaníček and Kleusberg (1987) approximated the geoid by a horizontal plane and the constant topographic density ρ was also used in their derivations. Their formula for DTE determination at the point P, at Earth's surface, can be approximated as follows (Vaníček and Kleusberg 1987)

$$\delta\Delta g_{\mathrm{dir}}^{\mathrm{VK}}(H_P) = \frac{\mu R^2}{2} \int\int_\sigma \frac{H^2 - H_P^2}{\ell_0^3} d\sigma \tag{4}$$

In a strict sense, Eqs. (3) and (4) can only be used for the far-zone integration area, where $\ell_0 \gg H$, and the effect of the near zone and the Bouguer shell (which cannot be derived in the planar model) are completely missing (Martinec and Vaníček 1994a; Nahavandchi 2000). It should be mentioned that the power series of height H used in the integration is limited to the second order. In addition, Eqs. (3) and (4) also suffer from other approximations. The most important one is that the slope of the topography must be within 45°. This limitation was pointed out by, e.g., Heck (1992), Martinec and Vaníček (1994a) and Sjöberg and Nahavandchi (1999).

The DTE used by Moritz (1980), and Vaníček and Kleusberg (1987) may significantly be different. One notes that Eq. (3) is always a positive quantity while Eq. (4) may be both positive and negative. Wang and Rapp (1990) and Nahavandchi (1998a) compared these two methods. They obtained large differences in the DTE on gravity and geoid. These differences are larger with complexity of the topography. They proposed that Vaníček and Kleusberg's free-air gravity anomaly should not be used in the Stokes formula. The difference was also explained by Martinec et al. (1993) as being due to the fact that while Vaníček and Kleusberg's results refer to the Earth surface, Moritz's results refer to the geoid.

2.1.2 DTE represented by the spherical harmonic expansion

Sjöberg (1994, 1995) developed the DTE in spherical harmonics to power H^2, and Nahavandchi and Sjöberg (1998) extended this approach to power H^3. The DTE on gravity with the spherical harmonic representation is (Nahavandchi and Sjöberg 1998)

$$\begin{aligned}
\delta\Delta g_{\mathrm{dir}}^{\mathrm{NS}}(H_P) &\doteq \frac{\pi\mu}{2R}\left[5H_P^2 + 3\overline{H_p^2} + 2\sum_{n,m}^{M'} n(H^2)_{nm} Y_{nm}(P)\right] \\
&+ \frac{\pi\mu}{2R^2}\left[\frac{28}{3}H_P^3 + \frac{9}{2}\overline{H_P^2}H_P - \frac{1}{2}\overline{H_P^3}\right. \\
&+ H_P \sum_{n,m}^{M'} n(2n+9)(H^2)_{nm} Y_{nm}(P) \\
&- \left.\frac{1}{3}\sum_{n,m}^{M'} n(2n+7)(H^3)_{nm} Y_{nm}(P)\right]
\end{aligned} \tag{5}$$

where Y_{nm} are fully normalized spherical harmonics obeying the following rule

$$\frac{1}{4\pi}\iint_{\sigma} Y_{nm}Y_{n'm'}d\sigma = \begin{cases} 1 & \text{if} \quad n = n' \quad \text{and} \quad m = m' \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and

$$(H^v)_{nm} = \frac{1}{4\pi}\iint_{\sigma} H_P^v Y_{nm} d\sigma \ ; \ v = 2, 3, \tag{7}$$

$$H_P^v = \sum_{n,m}^{M'} (H^v)_{nm} Y_{nm}(P), \tag{8}$$

$$\overline{H_P^v} = \sum_{n,m}^{M'} \frac{1}{2n+1} (H^v)_{nm} Y_{nm}(P). \tag{9}$$

In Eq. (5), M' is the maximum degree and order of height coefficients in a spherical harmonic expansion. Rewriting the formula in Eq. (5) for the point P at Earth's surface to the second power of elevation H, one obtains (Nahavandchi 2000)

$$\delta\Delta g_{\rm dir}^{\rm NS}(H_P) \quad \doteq \quad \frac{-2\pi\mu}{R} \sum_{n,m}^{M'} \left(\frac{R}{r}\right)^{n+1} \frac{(n+2)(n+1)}{2n+1} (H^2)_{nm} Y_{nm}(P). \tag{10}$$

These spherical harmonic representations [Eqs. (5) or (10)] of the DTE are simple for practical computations. They are also free from the problems encountered in the integral formulae, such as the singularity at the computation point. However, the harmonic expansion series of H^2 (and H^3) will only include the long-wavelength constitutents for M' =360. To incorporate all significant contributions of both short and long-wavelength constitutents, an expansion in spherical harmonics of H^2 (and H^3) to very high degrees should be required, which is practically difficult and ruins the simplicity of this method. Nahavandchi and Sjöberg (1998) showed that the dominant part of the power series in Eq. (5) is the second power of elevation H. For example, the contribution from the harmonic expansion series H^3 on the geoid is within 9 cm in the Himalayas. Later, Nahavandchi (1999) showed that the contribution from the harmonic expansion series H^4 and H^5 can safely be neglected for M'=360 (also see Sun and Sjöberg 2001).

2.1.3 DTE presented by the integral formula of Martinec and Vaníček (1994a)

The specific problem on determining the DTE was also treated by Martinec and Vaníček (1994a), who pointed out that the classical formulae may severely be biased because of the planar model of the geoid used in their derivations. To solve this problem, the spherical approximation of the geoid was used, but the effect was still considered only locally as a result of a limited integration area. Martinec and Vaníček (1994a) divided the integration area(full spatial angle) (σ) into a near zone (σ_1) and a far zone (σ_2) resulting in:

$$\begin{aligned}\delta\Delta g^{\mathrm{MV}}_{\mathrm{dir}}(H_P) &= -\frac{4\pi\mu}{R}H_p^2 + \frac{\mu R^2}{2}\iint_{\sigma_1}\frac{H_P^2-H^2}{\ell^3}\Big(1-\frac{3H_P^2}{\ell^2}\Big)d\sigma \\ &\quad +\frac{\mu R^2}{2}\iint_{\sigma_2}\frac{H_P^2-H^2}{\ell^3}\Big(1-3\sin^2\frac{\psi}{2}\Big)d\sigma \qquad (11)\end{aligned}$$

where the spatial distance $\ell=\sqrt{r^2+R^2-2rR\cos\psi)}$.

The above formula produces a relative error of 3×10^{-3} for the spherical approximation of the geoid, which in turn causes an error in geoidal heights of 6 mm at most. Also, a planar approximation of distances (not to be confused with the planar approximation of the geoid) is used in this formula which produces another error which is of the same order of magnitude as the error of the spherical approximation of the geoid. This error is acceptable for the precise determination of the regional geoid. Note that in Eq. (11) ℓ is used instead of ℓ_0 that is used in the classical integral formulae (3) and (4). Contrary to Eqs. (3) and (4), the near-zone effect and the Bouguer shell are also included in this formula. It is obvious that both of these effects are significant and must be considered in precise geoid determination.

2.1.4 DTE with combination of an integral formula and the spherical harmonic expansion

The spherical harmonic representations of the DTE [Eqs. (5) or (10)] are simple for practical computations. However, the harmonic expansion series of H^2 (and H^3) will include only the long-wavelength constitutents for M'=360. On the other hand, integral formulae are computed locally and include the short and in the most cases also the medium-wavelength constitutents (depending on the cap size).

A combination of local contributions and long-wavelength information was firstly proposed by Nahavandchi (1998a). Later, Nahavandchi (1998b), Sjöberg (2000) and Nahavandchi (2000) derived the direct gravitational effect of the topography at a topographic surface point P to the second power of H with a combination of the integral formula and the spherical harmonic expansion as (Nahavandchi 2000)

$$\begin{aligned}\delta\Delta g^{\mathrm{new}}_{\mathrm{dir}}(H_P) &= -\frac{4\pi\mu}{R}H_p^2 - \frac{3\mu}{8}\iint_{\sigma}\frac{H^2-H_P^2}{\ell_0}d\sigma \\ &\quad +\frac{\mu R^2}{2}\iint_{\sigma}\frac{H_P^2-H^2}{\ell^3}\Big(1-\frac{3H_P^2}{\ell^2}\Big)d\sigma \qquad (12)\end{aligned}$$

or

$$\delta\Delta g^{\mathrm{new}}_{\mathrm{dir}}(H_P) = -\frac{5\pi\mu}{2R}H_p^2 - \frac{3\pi\mu}{2R}\overline{H_P^2} + \frac{\mu R^2}{2}\iint_{\sigma}\frac{H_P^2-H^2}{\ell^3}\Big(1-\frac{3H_P^2}{\ell^2}\Big)d\sigma \qquad (13)$$

Equation (13) uses the spherical model of the geoid and contrary to Eq. (11) can include long-wavelength constituents (if Eq. (11) uses a limited integration area). The effect of the Bouguer shell is also included. It is also free from the singularity problems and topography limitations (Sect. 2.1.1) in classical integral formulae as it uses ℓ instead of ℓ_0. The same relative errors as in Eq. (11) are produced in Eq. (13).

2.2 Primary indirect topographic effect

2.2.1 PITE with the classical integral formula

The classical formula for determining the PITE on the geoid for Helmert's second condensation method with mass preservation is (Wichiencharoen 1982)

$$\delta N_{\mathrm{I}}^{\mathrm{classic}^*}(P') = \frac{-\pi\mu H_{P'}^2}{\gamma} - \frac{\mu R^2}{6\gamma}\iint_\sigma \frac{H^3 - H_{P'}^3}{\ell_0^3} d\sigma \tag{14}$$

with the same notations as before. This formula uses the planar approximation of the geoid and assumes the constant topographic density. Martinec and Vaníček (1994b) and Sjöberg and Nahavandchi (1999) showed that the PITE determined on the basis of the planar approximation of the geoid differs significantly from that resulting from the spherical model of the geoid. They obtained differences up to a 0.5 m.

2.2.2 PITE represented by the spherical harmonic expansion

The spherical harmonic representation of the PITE can be shown to the third power of topographic height in the point P' on the geoid as (Nahavandchi and Sjöberg 1998)

$$\delta N_{\mathrm{I}}^{\mathrm{NS}^*}(P') = -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\frac{n-1}{2n+1}H_n^2(P') + \frac{2\pi\mu}{3R\gamma}\sum_{n=0}^{\infty}\frac{n(n-1)}{2n+1}H_n^3(P') \tag{15}$$

where

$$H_n^\nu(P') = \frac{2n+1}{4\pi}\iint_\sigma H^\nu P_n(\cos\psi) d\sigma;\ \nu = 2, 3 \tag{16}$$

where $P_n(\cos\psi)$ is the Legendre polynomial. Again, this formula is very simple for practical computations but with the global height information available in this study, it represents only the long-wavelength constitutents.

2.2.3 PITE represented by the integral formula of Martinec and Vaníček (1994b)

The PITE derived by Martinec and Vaníček (1994b) is based on the spherical approximation of the geoid. However, they considered this effect only locally as a result of a limited integration area (spherical cap). The PITE in the point P' on the geoid is (Martinec and Vaníček 1994b)

$$\begin{aligned}\delta N_{\rm I}^{\rm MV^*}(P') &= -\frac{2\pi\mu}{\gamma}H_{P'}^2 + \frac{\mu R^2}{\gamma}\iint_\sigma \Big[2\frac{(\ell_0^2+H^2)^{0.5}-(\ell_0^2+H_{P'}^2)^{0.5}}{R} \\ &+ \ln\frac{\frac{\ell_0}{2R}+H+(\ell_0^2+H^2)^{0.5}}{\frac{\ell_0}{2R}+H_{P'}+(\ell_0^2+H_{P'}^2)^{0.5}} - \frac{H-H_{P'}}{\ell_0}\Big]d\sigma \end{aligned} \tag{17}$$

The spherical approximation of the geoid in this formula produces again an relative error of 3×10^{-3} in the geoidal heights. On the other hand, a planar approximation of distances is used in this formula which produces an error of the same order of magnitude as the error due to the spherical approximation of the geoid.

2.2.4 PITE with combination of the integral formula and the spherical harmonic expansion

The classical formula [Eq. (14)] is not practical for numerical evaluations, as it requires an integration over surface of the whole Earth to include long-wavelength contributions. It also suffers from the planar approximation of the geoid (Martinec and Vaníček 1994b, Sjöberg and Nahavandchi 1999). On the other hand, the spherical harmonic representation of the PITE [Eq. (15)] needs a very high maximum degree of expansion, to consider all short-wavelength information. A suitable compromise may therefore be of the form (Sjöberg and Nahavandchi 1999)

$$\begin{aligned}\Delta\delta N_{\rm I}(P') &= \delta N_{\rm I}^{\rm classic^*} - \delta N_{\rm I}^{\rm new^*} = -\frac{3\pi\mu}{\gamma}H_{P'}^2 - \frac{3R\mu}{4\gamma} \\ &\times \iint_\sigma \frac{H^2-H_{P'}^2}{\ell_0}d\sigma - \frac{\mu}{8\gamma}\iint_\sigma \frac{H^3-H_{P'}^3}{\ell_0}d\sigma \end{aligned} \tag{18}$$

or

$$\delta N_{\rm I}^{\rm new^*} = \delta N_{\rm I}^{\rm classic^*} - \Delta\delta N_{\rm I} \tag{19}$$

where $\Delta\delta N_{\rm I}$ in spectral form is approximated as

$$\Delta\delta N_{\rm I}(P') = -\frac{3\pi\mu}{\gamma}\overline{H_{P'}^2} + \frac{\pi\mu}{2R\gamma}(H_{P'}^3 - \overline{H_{P'}^3}) \tag{20}$$

Equation (19) includes the integral part for short-wavelength constituents and the spherical harmonic representation to consider the long-wavelength information. It produces the same relative errors as Eq. (17).

3 Numerical investigations

3.1 Data sources

A test area of size $2^\circ \times 2^\circ$ in Iran is chosen. It is limited by latitudes 31° N and 33° N and longitudes 54° E and 56° E. The topography in this area varies from 785 to 3053

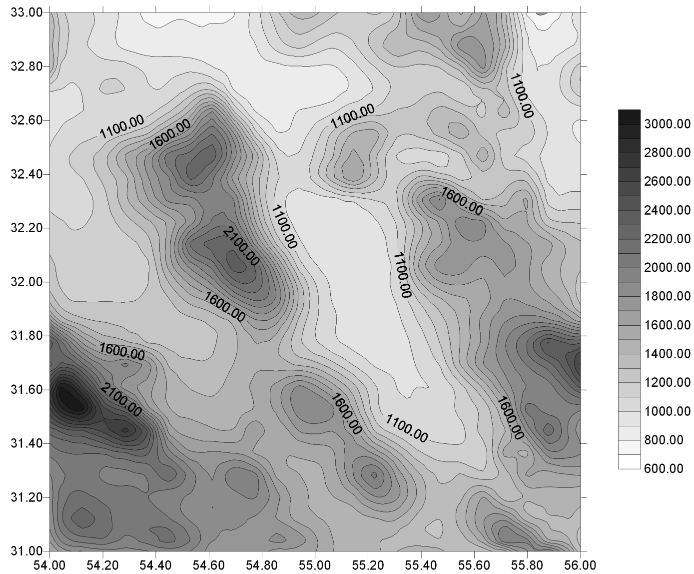

Figure 1: Presentation of topography in the test area. Contour interval =100 m

metres, shown in Fig. 1. The height spherical harmonic coefficients $(H^2)_{nm}$ and $(H^3)_{nm}$ are determined from Eqs. (7) and (8) using global topography. For this, a $30' \times 30'$ Digital Terrain Model (DTM) is generated by averaging the Geophysical Exploration Technology (GETECH) $5' \times 5'$ DTM (GETECH 1995a), using area weighting. Since the interest is in continental elevation coefficients and one aims to evaluate the effect of the masses above the geoid, the heights below sea level are all set to zero. The spherical harmonic coefficients are computed to degree and order 360. The parametr $\mu = G\rho$ is computed using $G = 6.673 \times 10^{-11} m^3\ kg^{-1}\ s^{-2}$ and $\rho = 2670\ kg/m^3$. The values of R=6371 km, and $\gamma = 9.81\ m/s^2$. are also used in computations. In all the integral equations in this study a $2' \times 2'$ DTM produced in National Cartographic Center of Iran is used. It should be mentioned that this DTM is not adequate for computing the local contributions of topographic effects in practice and only give an insight in the medium-wavelength constituents. A denser DTM is in preparation. Height data in all integral equations are extended to $6°$ from the computation point.

3.2 Computations of the DTE with different formulae

The DTE is computed in the test area with the classical integral formulae of Moritz (1980) [Eq. (3)] and Vaníček and Kleusberg (1987) [Eq. (4)], the spherical harmonic formula of Nahavandchi and Sjöberg (1998) [Eq. (5)], the integral formula of Martinec and Vaníček (1994a) [Eq. (11)], and Nahavandchi (2000) combined formula [Eq. (13)]. Table 1 shows the statistics of the results of the computations with the above-mentioned formulae. To give further insight into the differences, the results of the computations of the DTE are

Table 1: Statistics of the direct topographic effect on gravity with different formulae in the test region in mGal.

	Min	Max	Mean	SD
Classical Moritz formula	0.28	6.46	0.97	0.72
Classical Vaníček and Kleusberg formula	-33.61	8.05	0.004	5.11
Spherical harmonic approach of Nahavandchi and Sjöberg	-2.53	3.10	0.01	1.33
Integral formula of Martinec and Vaníček	-45.23	8.03	-0.71	6.96
Combined formula of Nahavandchi	-27.90	7.48	0.13	4.72

plotted. Figures 2-6 depict the DTE obtained with the different formulae mentioned above. Results in Table 1 and Figs. 2-6 show that different procedures for computation of the DTE result in significant differences.

It should be noted that Fig. 5 [integral formula of Martinec and Vaníček (1994a)] and Fig. 6 (combined formula) are similar in shape with minor differences in magnitude. The absolute maximum difference of 5.71 mGal was computed for these two procedures. Figure 3 [classical integral formula of Vaníček and Kleusberg (1987)] is similar in shape with Figs. 5 and 6 but with larger differences in magnitude. The absolute maximum difference of 18.82 mGal was computed for the differences between the combined formula [Nahavandchi (2000)] and the classical integral formula of Vaníček and Kleusberg (1987).

Of course, there are several reasons for these differences. For example, the Moritz (1980) formula is not really comparable to the other expressions in this study as it contains a combination of two different effects. This formula for DTE, however, will be used in the next step of computaions, which is geoid height determination with different methods of handling the topographic corrections. The Vaníček and Kleusberg's (1987) DTE refers to the point on Earth's surface while the Moritz (1980) formula refers to the point on the geoid, which justify the large differences between these two formulae. The spherical harmonic representation of the DTE will include only the long-wavelength information in this study (for M'=360), and most of short-wavelength information, which is included in the other formulae, is missing. This is the main reason for large differences between this method and the other ones. Martinec and Vaníček (1994a) integral formula and Nahavandchi (2000) combined formula for the DTE are in good agreement with each other. These two formulae use the spherical approximation of the geoid, contrary to the classical formulae which use the planar model. There are some minor differences between these two formulae, however, which originate from the exclusion of some parts of the long-wavelength constituents in Martinec and Vaníček (1994a) integral formula, which are included in the combined formula.

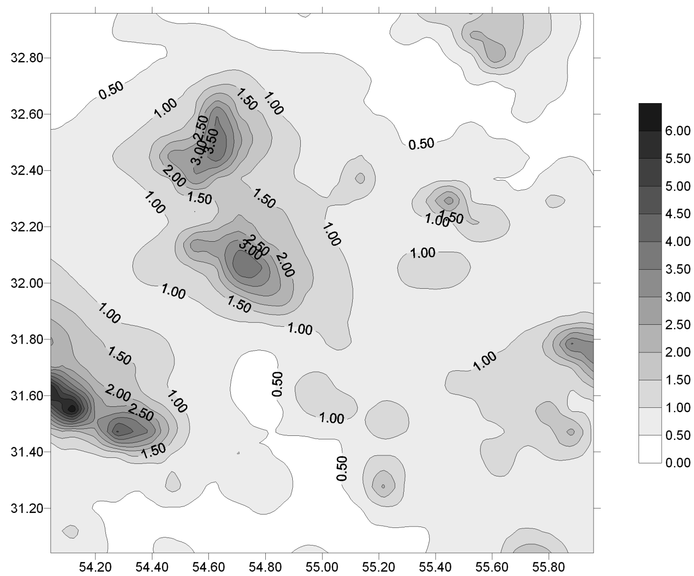

Figure 2: The direct topographic correction computed by the classical integral formula of Moritz (1980). Contour interval = 0.5 mGal

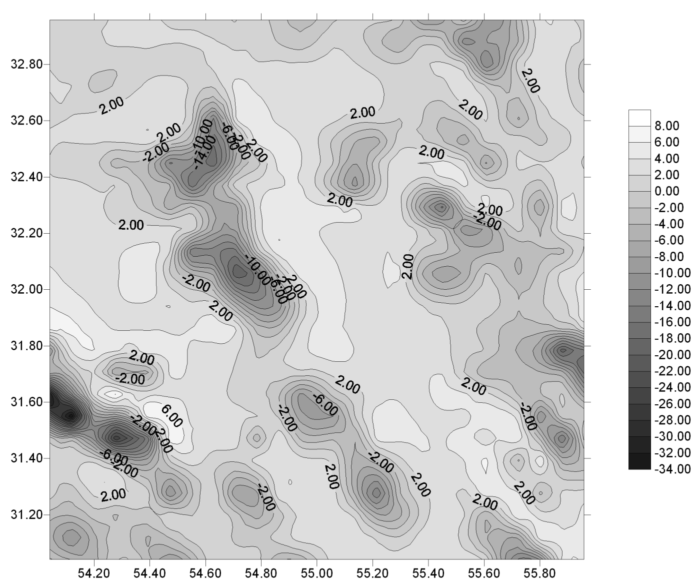

Figure 3: The direct topographic correction computed by the classical integral formula of Vanicek and Kleusberg (1987). Contour interval = 2 mGal

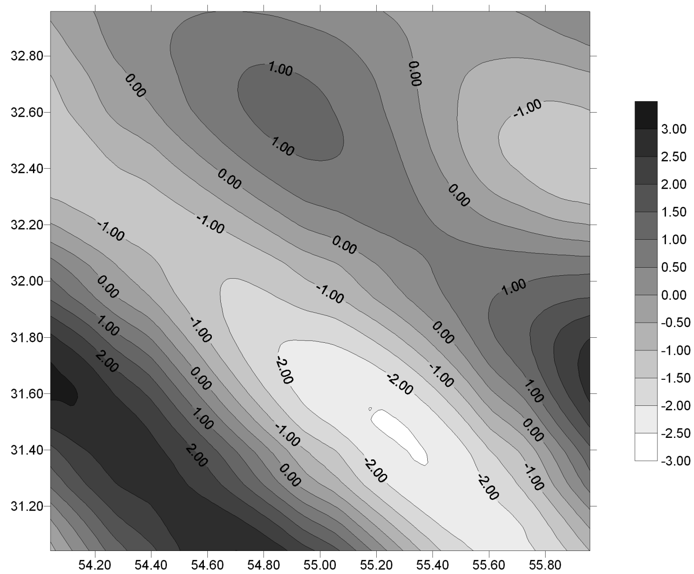

Figure 4: The direct topographic correction computed by the spherical harmonic approach of Nahavandchi and Sjoberg (1998). Contour interval = 0.5 mGal

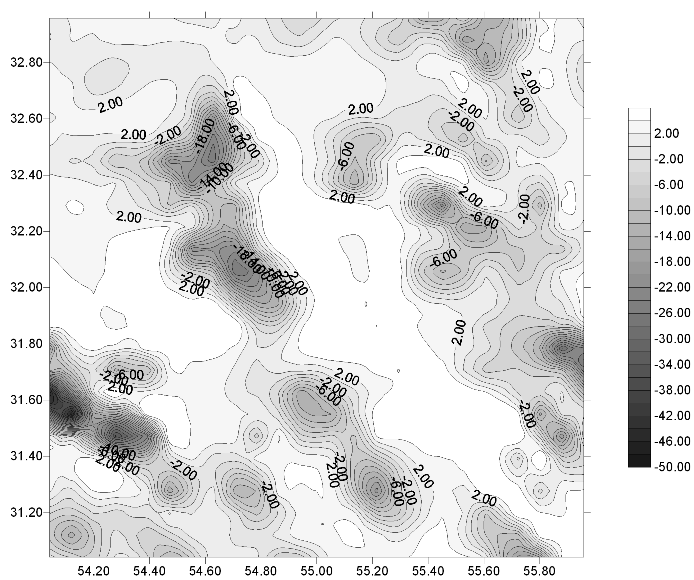

Figure 5: The direct topographic correction computed by the integral l formula of martinec and Vanicek (1994a). Contour interval = 2 mGal

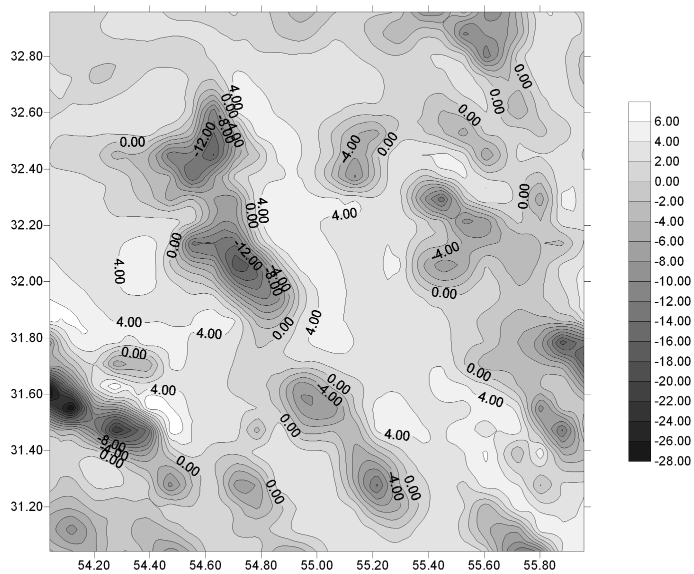

Figure 6: The direct topographic correction computed by the combined formula of Nahavandchi (2000). Contour interval = 2 mGal

Nahavandchi (2000) compared the combined formula [Eq. (13)] with the integral formula of Martinec and Vaníček (1994a) in a test area in Sweden with the maximum elevation of 1147 m. The maximum difference between these two formulae reached 2.31 μGal. In that study, however, Martinec and Vaníček (1994a) formula was integrated up to 20° from computation points using the GETECH $2.5' \times 2.5'$ DTM (GETECH 1995b). Further out the global $30' \times 30'$ DTM (GETECH 1995a) was used. This justifies the belief that some parts of the global information might be missing in the results from the integral formula of Martinec and Vaníček (1994a)(depending to the cap size of integration area).

3.3 Computations of the PITE with different formulae

The PITE is computed in the test area with the classical integral formula [Eq. (14)], the spherical harmonic formula of Nahavandchi and Sjöberg (1998) [Eq. (15)], the integral formula of Martinec and Vaníček (1994b) [Eq. (17)], and Sjöberg and Nahavandchi (1999) combined formula [Eq. (19)]. Table 2 shows the statistics of the results of the computations with the above-mentioned formulae. Again, the results of the computations of the PITE on the geoid height with the different formulae mentioned above are plotted. The results are shown in Figs. 7-10. Table 2 and Figs. 7-10 present the differences between different procedures for computation of the PITE. The same explanations as in case of the direct topographic effect can be repeated here. Figure 9 [integral formula of Martinec and Vaníček (1994b)] and Fig. 10 [combined formula of Sjöberg and Nahavandchi (1999)] are similar in shape but with minor differences in magnitude. The absolute maximum

Table 2: Statistics of the primary indirect topographic effect on the geoidal height with different formulas in the test region in cm.

	Min	Max	Mean	SD
Classical integral formula	-48.69	-3.68	-12.69	7.14
Spherical harmonic approach of Nahavandchi and Sjöberg	-11.48	6.31	-0.04	4.35
Integral formula of Martinec and Vaníček	-34.47	6.23	-0.83	6.29
Combined formula of Sjöberg and Nahavandchi	-36.89	8.08	-0.91	7.11

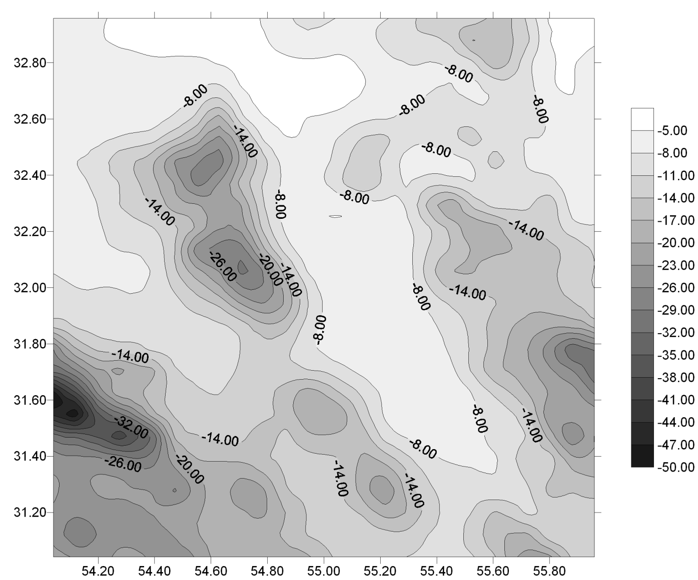

Figure 7: The primary indirect topographic correction computed by the classical integral formula . Contour interval = 3 cm

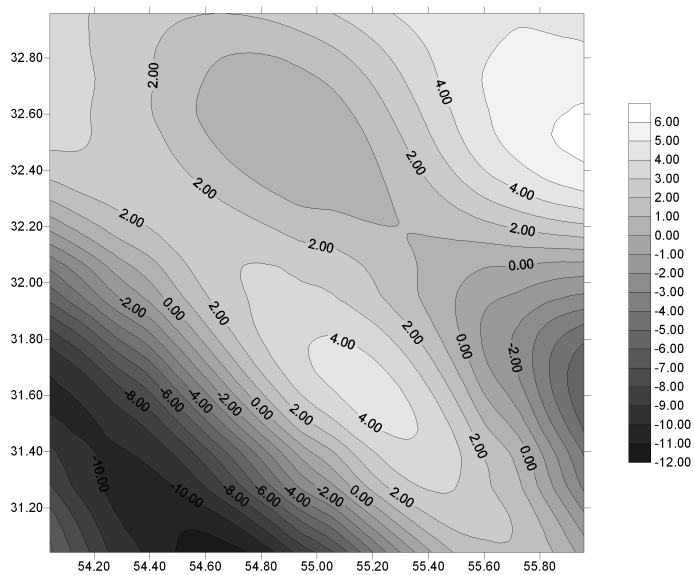

Figure 8: The primary indirect topographic correction computed by the spherical harmonic approach of Nahavandchi and Sjoberg (1998). Contour interval = 1 cm

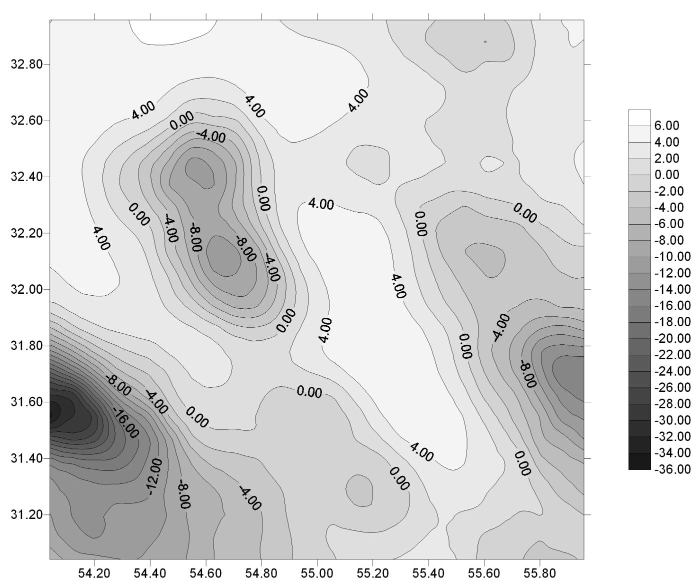

Figure 9: The primary indirect topographic correction computed by the integral formula of Martinec and Vanicek (1994b). Contour interval = 2 cm

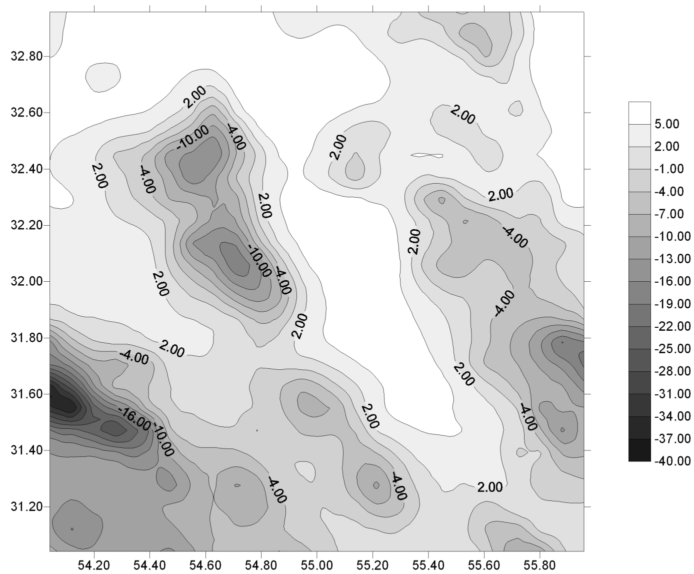

Figure 10: The primary indirect topographic correction computed by the combined formula of Sjoberg and Nahavandchi (1999). Contour interval = 3 cm

difference of 4.94 cm is computed between these two methods. Figure 7 (classical integral formula) is similar in shape with Figs. 9 and 10, but with larger differences. The absolute maximum difference of 11.91 cm was computed between the classical and the combined formulae.

While the classical integral formula suffers from the planar approximation of the geoid, the Martinec and Vaníček (1994b) integral formula and Sjöberg and Nahavandchi (1999) combined formula use the spherical model. The spherical harmonic representation of the PITE only includes the long-wavelength constituents in this study due to the use of M'=360, while the other integral formulae only include the short-wavelength information (due to the use of the integration area). Sjöberg and Nahavandchi (1999) combined formula include both short and long-wavelength information, contrary to Martinec and Vaníček (1994b) integral formula which only include the local contributions. But results from these two formulae are in good agreement with each other in comparison with the other methods. Sjöberg and Nahavandchi (1999) computed the differences between these two formulae in the test area in Sweden with the maximum elevation of 1051 m. They obtained the absolute maximum difference of 0.71 cm between these two formulae. It should be noted, however, that the integral formula of Martinec and Vaníček (1994b) was integrated up to 20° from computation points using the GETECH $2.5' \times 2.5'$ DTM (GETECH 1995b). The global $30' \times 30'$ DTM (GETECH 1995a) was used outside the 20° cap. This again justifies the belief that they might be some parts of the long-wavelength constituents, which are missing in the integral formula of Martinec and Vaníček (1994b) (due to the choice of integration area).

To see how the different formulae for the DTE and the PITE work in different test areas, a flatter test area in Iran was chosen. The heights in this area vary from 625 to 1537 m. The same computations as above were carried out in this second test area. The results of comparisons between different formulae were the same as in the first test area. The differences were smaller and smoother, however, that shows the expected dependence of the topographical effects on elevations.

3.4 Comparisons

In order to obtain further insight into how the methods differ, and which model is better suited to describe the "height reference surface" of the national height reference system, 7 GPS-leveling stations were used as an external source to obtain the geoid heights. These stations belong to National Cartographic Center of Iran. The elevations of the GPS stations vary from 1431 to 1798 m. The accuracy of the ellipsoidal heights (h) of these stations is of the order of few centimetres. Iran is using the orthometric height system. The GPS-leveling geoidal heights in these 7 stations are computed with the well known formula

$$N \doteq h - H. \tag{21}$$

This formula is, however, only valid if orthometric height refers to the geoid where the "height reference surface" normally does not coincide with the geoid, which is the case in this study too. For the numerical investigation of different methods of handling the effects of topographic masses, the gravimetric geoid heights at these 7 GPS-leveling stations are also computed. Thereafter, the gravimetric results were compared with the GPS-leveling geoid heights. This will help to understand which method of topographic effects computations is better suited to describe the used "height reference surface".

For computing the gravimetric geoid heights, Stokes's formula in Eq. (1) with the least-squares modification of Stokes's kernel is used according to Nahavandchi and Sjöberg (2001a). Short-wavelength part of the geoid height was computed through Stokes's integration up to 6° from computation points and long-to-medium-wavelength part was computed from the global gravity geopotential model EGM96 (Lemoine et al. 1997). It is important to notice, however, that the EGM96 is based on free-air gravity anomalies and the used model here is Helmert's second method of condensation. The differences are normally very small but they might have larger values in mountainous areas. The terrestrial gravity anomalies in Stokes's integral are in $110'' \times 160''$ geographical cells and taken from National Cartographic Center of Iran. The interested readers are referred to Nahavandchi (1998b) and Nahavandchi and Sjöberg (2001a) for the procedures and formulae used in the modification of Stokes's formula.

To compute the gravimetric geoid height with the Stokes-Helmert scheme for geoid determination [see Eq. (1)], Helmert's anomaly at the geoid (Δg^{H^*}) is needed. The Moritz's (1980) DTE formula already includes the effect of the downward continuation to the geoid, while Vaníček and Kleusberg's (1987), Martinec and Vaníček's (1994a), spherical harmonic approach, and Nahavandchi's (2000) DTE formulae refer to a point at the ground level [see Eq. (2)]. Therefore, a downward continuation procedure must be implemented in these methods to reduce the Δg^H from topography to the geoid resulting in Δg^{H^*}. The Poisson integral formula with the same procedure carried out in Vaníček et al. (1996), Nahavandchi (1998c) and Nahavandchi and Sjöberg (2001b) was used.

The modified Poisson formula can be written as (see e.g. Vaníček et al. 1996; Nahavandchi and Sjöberg 2001b)

$$\Delta g^H = \frac{R}{4\pi} \iint_{\sigma_0} \Delta g^{H^*} K^M(r, \psi, R, \psi_0) d\sigma + \delta g_T + \Delta g_M^H \tag{22}$$

where δg_T is the truncation error, Δg_M^H are the low-degree spherical harmonics of the gravity anomaly, $K^M(r, \psi, R, \psi_0)$ is the modified Poisson kernel and σ_0 denotes the integration domain within a spherical cap of radius ψ_0. The truncation error is minimized following the Molodenskij technique to reduce potential errors coming from the employed global gravity model. The minimization is carried out in the sense of minimizing the upper bound of the absolute value of the truncation error by subtracting from Poisson's kernel an appropriately selected linear combination of spherical harmonic functions taken to degree and order M. The interested reader is referred to Vaníček et al. (1996) and Nahavandchi and Sjöberg (2001b) for the complete explanation of the above-mentioned modified Poisson formula to derive unknown Δg^{H^*} from given the Helmert gravity anomaly Δg^H.

Different formulae to determine the topographic effects and downward continuation problem are catagorized in the following 5 procedures. The classical integral formulae of Moritz with the DTE in Eq. (3) (which also includes the downward continuation procedure) and the PITE in Eq. (14) is the first method. Second, the classical integral formulae of Vaníček and Kleusberg with the DTE in Eq. (4) and downward continuation procedure in Eq. (22) and the PITE in Eq. (14) are used. Thereafter, the spherical harmonic approach was employed with the DTE in Eq. (5) and the downward continuation procedure in Eq. (22) and the PITE in Eq. (15). The integral formulae of Martinec and Vaníček with the DTE in Eq. (11) and the downward continuation procedure in Eq. (22) and the PITE in Eq. (17) is the fourth method. Finally, the combined formulae with the DTE in Eq. (13) and the downward continuation procedure of Eq. (22) and the PITE in Eq. (19) are used. Thereafter, the gravimetric geoid heights (with different correction procedures mentioned above) are computed at 7 GPS-leveling stations and the statistics of differences between the gravimetric and the GPS-leveling geoid heights are shown in Table 3. Table 3 shows that the gravimetric geoid heights agree better with the

Table 3: Statistics of the differences between gravimetric and 7 GPS-leveling stations' geoid heights with different procedures of handling the topographic corrections. Units in metres.

	Min	Max	Mean	SD
Classical Moritz formulae	-0.191	1.301	0.560	0.462
Classical Vaníček and Kleusberg formulae	-0.202	1.292	0.583	0.421
Spherical harmonic approaches of Nahavandchi and Sjöberg	-0.238	1.322	0.622	0.518
Integral formulae of Martinec and Vaníček	-0.118	1.221	0.562	0.381
Combined formulae of Nahavandchi and Sjöberg	-0.131	1.118	0.521	0.322

GPS-leveling geoid heights when the integral formulae of Martinec and Vaníček and the combined formulae of Nahavandchi and Sjöberg are used, compared to the other methods, with the latter formula as the best. This justifies the belief that the combined formulae used in this study for handling the effects of topographic masses include all wavelengths and are better suited to describe the height reference surfaces like the geoid.

In addition, a fitting process of the gravimetric and GPS-leveling geoid was conducted. The geoid change ΔN can be written in geographical coordinates as (Heiskanen and Moritz 1967):

$$N_{\mathrm{Grav}} - N_{\mathrm{GPS}} = \Delta N = \cos\phi\cos\lambda\Delta X + \cos\phi\sin\lambda\Delta Y + \sin\phi\Delta Z + kR \qquad (23)$$

where ϕ and λ are the geographical coordinates, ΔX, ΔY, ΔZ are the three translations and k is the scale factor. Equation (23) represents a very useful regression formula, which can be used for fitting a regional gravimetric geoid to the GPS-leveling stations. Table 4 shows the statistics of the differences, after fitting, between gravimetric and GPS-leveling geoid. Results of Table 4 shows that, after regression, the gravimetric geoid heights computed with the topographic effects of the combined formulae of Nahavandchi and Sjöberg still improve the fit of the gravimetric geoid to GPS-leveling stations, significantly. The standard deviation of the fit after regression with this method is computed as ± 10.1 cm compared to ± 12.3 cm with the integral formulae of Martinec and Vaníček, the second best method among the other methods. It should be mentioned, however, that these computations should be carried out in test areas with more available GPS-leveling stations.

4 Discussion and conclusions

The DTE and the PITE with different methods and different approximations are discussed. Classical integral formulae use the planar approximation of the geoid while the

Table 4: Statistics of the differences between GPS-leveling and gravimetric geoid heights with different procedures of handling the topographic corrections after fitting to 7 GPS stations. Units in metres.

	Min	Max	Mean	SD
Classical Moritz formulae	-0.514	0.435	0.000	0.192
Classical Vaníček and Kleusberg formulae	-0.483	0.466	0.000	0.181
Spherical harmonic approaches of Nahavandchi and Sjöberg	-0.557	0.429	0.000	0.238
Integral formulae of Martinec and Vaníček	-0.361	0.281	0.000	0.123
Combined formulae of Nahavandchi and Sjöberg	-0.252	0.312	0.000	0.101

recent formulae use the spherical approximation of the geoid. The spherical harmonic representation of the topographic effects only include the long-wavelength information with available maximum degree M'=360 used in this study, while pure integral formulae only include the local contributions depending on the integration area. The combined formulae of the PITE and the DTE derived by Sjöberg and Nahavandchi (1999) and Nahavandchi (2000), respectively, include all the significant information. As the important part of the topographic effects are the local contributions, the results of these formulae are in good agreement with the Martinec and Vaníček (1994a, b) integral formulae, which also use the spherical model of the geoid, but do not include the whole long-wavelength contributions. It can be stated that the combined formulae model better the long-wavelength constituents with respect to the procedure described in Martinec and Vaníček (1994a, b). It should be noted that the above-mentioned results should also be tested in other test areas.

The aim of this study was to show the differences between different procedures of handling the effect of topographic masses in precise geoid determination. It is shown that significant differences between different methods exist, which were expected. It is also concluded that the effects of distant topographic masses can not be neglected in precise geoid computations. It means that the long-wavelength contributions of these effects, which are included in the combined formulae, represent better the reality. To justify this belief, geoidal heights were computed applying different topographic effects. The gravimetric geoid heights were then compared with the 7 GPS-leveling geoid heights. The results of these comparisons prove the belief that the gravimetric geoid height computations corrected for topographic effects with the combined formulae work better with GPS-leveling data. The standard deviation of the fit (after the regression procedures) is determined to be equal to $\pm$10.1 cm for the topographic effects of combined formulae, while it is equal to $\pm$19.2 cm for the classical method. Finally, the use of the combined formulae of DTE [Eq. (13)] and PITE [Eq. (19)] are suggested for a precise geoid determination. The results of gravimetric geoid height comparisons with GPS-leveling height data over Sweden also recommended the use of the combined formulae (Nahavandchi and Sjöberg 2001a, b).

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