The Quest for a Precise Geoidal Height Model

H. Nahavandchi

Department of Geomatics, Norwegian University of Science and Technology, N-7491, Trondheim, Norway
hossein.nahavandchi@ntnu.no
Kart og Plan, Vol. 64(1), pp. 46-56. P. O. Box 5003, N-1432 Ås. ISSN 0047-3278

Abstract

The geoid is defined as an equipotential surface of the Earth's actual gravity field inside the topographical masses on land (in most cases) and more or less coinciding with mean sea level at sea. Over the last decade, there has been an increased interest in the determination of the geoid. This is mainly due to the demands for height transformation from users of GPS. Physical heights in geodesy are referred to the geoid. The knowledge of the geoidal height is thus necessary for transforming the ellipsoidal to physical heights and vice versa. The geoid represents a vertical datum for heights used in many countries. The improved knowledge of the geoid model can contribute to many other applications of Earth studies like ocean circulation, climate, post-glacial rebound, plate tectonics and mantle convection studies. This article reviews some recent developments within geoidal height determination and its theoretical limitations.

Key Words: Stokes's formula, geoid, Helmert condensation, Topographical corrections

Introduction

The geoid is the equipotential surface of the Earth's gravity field more or less coinciding with mean sea level and is used as the vertical datum for heights. Civil engineers use it as the reference surface for elevations while oceanographers use it for studies of ocean circulation, currents, and tides. It is also valuable to geophysicists for geodynamics studies, geophysical interpretation of the Earth's crust, and prospecting. These types of applications require knowledge of the geoid with a precision of ±1-10 cm. To obtain such high accuracy levels for the computation of the geoid, terrestrial gravity observations as well as satellite observations in a combination with a global Earth model are applied. The new technologies of satellite altimetry and satellite positioning have placed a high demand on precise geoid determination research.

An accurate solution of the boundary-value problem in physical geodesy has usually been found using Stokes's well-known formula (1849) for the anomalous gravity potential, with the geoidal height calculated through Bruns's formula. The solution is given by:

$$N(R,\varphi,\lambda) = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\psi)\Delta g(R,\varphi,\lambda)\mathrm{d}\sigma \tag{1}$$

where N is the geoidal height computed at a point on the geoid with latitude φ and longitude λ and mean radius R, ψ is the spherical distance between computation and running points, Δg is the gravity anomaly on the geoid, γ is normal gravity on the ellipsoid, σ is the unit sphere and $S(\psi)$ is the original Stokes's formula. Stokes's formula stipulates the relation between the geoidal height (gravitational potential) at a single point on the geoid and the gravity anomalies on the entire geoid. It can be seen that Stokes's formula is a rigorous formula for computing the geoidal height from globally and continuously distributed gravity anomalies in a spherical approximation.

At present, a homogenous coverage of high-resolution gravity all over the Earth is hard to come by and, at the same time, the gravity data are available at discrete points. This promotes the restriction of the integration area of Stokes's integral, where a homogenous and relatively high-resolution gravity anomaly data set can be found at our disposal in conjunction to the global geopotential model. Such an approach reduces the impact of the spherical approximation to Stokes's formula, the reason being that most of the geoid's power is contained in the low-frequency spectrum.

The application of Stokes's formula for the computation of the geoid requires that there are no masses outside the geoid. This is achieved by mathematically removing the external masses or shifting them inside the geoid. The masses are then restored after applying Stokes's integral. One of the most usual methods for this remove-restore problem is the Stokes-Helmert Scheme. One usually uses the Helmert condensation method that preserves the mass of the Earth, for which the Helmert-model Earth has the same mass as the real Earth. For discussion of this point see Wichiencharoen (1982). Recent studies (e.g. Martinec et al. 1993; Sjöberg 1994; Vanicek et al. 1995; Nahavandchi and Sjöberg 1998) have concentrated on Helmert's second condensation method, wherein the topographical masses are condensed into a surface layer with density ρ equal to $\rho_0 H$, ρ_0 being the density of the topographical masses and H the topographic height. The effect of topography in precise geoid determination was also discussed by many other researchers (Sideris 1990; Tziavos et al. 1992; Forsberg 1994; Martinec and Vanicek 1994a, b; Nahavandchi and Sjöberg 1998; Sjöberg and Nahavandchi 1999; Nahavandchi 2000 etc.).

The gravimetric geoid determination has essentially employed the original Stokes's formula. The incomplete global coverage and availability of accurate gravity measurements have precluded an exact determination of the geoid using Stokes's formula. Instead, an approximate solution is used in practice, where only gravity data in and close to the computation area are used. This truncation causes an error in the computed geoid height, called the truncation error. This error can be reduced by introducing a modification to the Stokes kernel. The lack of a global coverage of gravity data can be compensated by a combination of terrestrial gravity with a global geopotential model; in essence the long-wavelength geoid height contributions will be determined from a geopotential model and the short-wavelength information from terrestrial gravity and topographic data. The modification of Stokes's formula, originating with Molodensky, aimed at reducing the upper bound of the truncation error committed by limiting the area of integration under the Stokes's integral to a spherical cap around the computation point (Molodensky et al. 1962).

Applications

The geoid and its corresponding gravity field are highly valuable quantities in several geoscientific disciplines. In geodesy, following applications can be listed for precise geoidal heights:

- inertial navigation,
- orbit determination,
- global unification of vertical datums,

- and probably the most important application, i.e. leveling by GPS, where geometric GPS- derived heights are transformed into physical heights through the geoid.

For oceanography a precise geoid serves as equipotential surface. Altimetry provides the geometric sea-surface. From differences between both surfaces the ocean circulation can be determined at several scales. Since there is circulation in the ocean there clearly is a separation between the ocean surface and the geoid. This separation is called sea surface topography or dynamic height. The dynamic height and its slope play an important role in the study of ocean circulation. The success of ocean circulation studies that use geoidal heights as one component of the analysis clearly depends, in part, on the accuracy of the determination of the geoid. Ocean Circulation and Transport Between North Atlantic and the Arctic Sea (OCTAS) is a project funded by Norwegian Research Council including 8 national and international partners. One goal of this project is to determine an accurate geoid in the Fram Strait and the adjacent seas. Together with the results from the on-going EU-funded project GOCINA, where in a similar approach an accurate geoid is determined for the region between Greenland and the UK, this will create a platform for validation of future GOCE Level 2 data and higher order scientific products. The new and accurate geoid is used together with an accurate Mean Sea surface to determine the Mean Dynamic Topography which in turn will result in valuable information on ocean circulation and mass and heat transport and finally the ocean role in climate.

An accurate geoid and gravity field is used by geophysicists for several purposes:
- modeling phenomena of the oceanic lithosphere (ridges, trenches, and seamounts),
- the continental lithosphere (mountain ranges, roots…),
- large-scale mantle effects (convection…),
- the determination of the global ice balance.

Geoidal height computation formulas

Removing the effect of external masses to the geoid (topography + atmosphere) or reducing them inside the geoid (direct effect) is a requirement of Stokes's formula. The effects of masses are then restored after applying Stokes's integral (primary indirect effect). There is also another indirect effect resulting from a free-air correction of gravity from geoid to co-geoid, i.e. secondary indirect effect. Stokes's formula also requires that gravity anomalies must refer to the geoid (downward continuation).

The combined idea of Stokes-Helmert is the most precise procedure in geoid computations (Vanicek and Martinec 1994, Nahavandchi and Sjöberg 2001). This idea can be realized by the following formula (see e.g., Heiskanaen and Moritz 1967, Nahavandchi and Sjöberg 2001):

$$\begin{aligned} N(R,\varphi,\lambda) &= N_1(R,\varphi,\lambda) + N_2(R,\varphi,\lambda) + \delta N_{\text{total}}^{\text{atmosphere}}(R,\varphi,\lambda) \\ &+ \delta N_{\text{secondary indirect}}^{\text{topography}}(R,\varphi,\lambda) + \delta N_{\text{indirect}}^{\text{topography}}(R,\varphi,\lambda) \end{aligned} \quad (2)$$

where N is the final geoidal height computed at a point on the geoid, $\delta N_{\text{total}}^{\text{atmosphere}}(R,\varphi,\lambda)$ is the total effect (direct+indirect) of atmospheric masses, $\delta N_{\text{secondary indirect}}^{\text{topography}}(R,\varphi,\lambda)$ is the secondary indirect effect due to the topography, $\delta N_{\text{indirect}}^{\text{topography}}(R,\varphi,\lambda)$ is the primary indirect effect on geoid due to topography, $N_1(R,\varphi,\lambda)$ is the short-wavelength contribution to the geoid computed from Stokes's integral and $N_2(R,\varphi,\lambda)$ is the long-wavelength contribution to the final geoid determined from a global geopotential model. The short-wavelength part of the geoid can be determined from

$$N_1(R,\varphi,\lambda) = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^{Modified}(\psi)\Delta g^{H^*}(R,\varphi,\lambda)\mathrm{d}\sigma \tag{3}$$

where

$$\Delta g^H(r,\varphi,\lambda) = \frac{R}{4\pi}\iint_{\sigma} K(r,\psi,R)\Delta g^{H^*}(R,\varphi,\lambda)\mathrm{d}\sigma \tag{4}$$

and

$$\Delta g^H(r,\varphi,\lambda) = \Delta g_{\text{free -air}}(r,\varphi,\lambda) + \delta\Delta g^{\text{topography}}_{\text{direct}}(r,\varphi,\lambda) + \varepsilon_{\text{ellipsoidal correction}}(R,\varphi,\lambda) \tag{5}$$

where $\Delta g_{\text{free -air}}$ is computed ($\Delta g_{\text{free -air}} = g - \gamma + 0.3086H$, where H is in meters and the gravity units are mGal) from the observed terrestrial gravity anomalies (g) corrected for the free-air correction ($0.3086H$), $\delta\Delta g^{\text{topography}}_{\text{direct}}$ is the direct topographical correction to the observed gravity anomalies, $\varepsilon_{\text{ellipsoidal correction}}$ is the ellipsoidal correction to the terrestrial gravity anomalies due to the assumption of a spherical surface, Δg^{H^*} is the corrected terrestrial gravity anomalies for the topography and ellipticity and downward continued to the geoid using Poisson's integral in Eq. (4), where $K(r,\psi,R)$ is the Poisson kernel. Here r is the geocentric distance, usually computed from a spherical approximation $r=R+H$, and $S^{Modified}(\psi)$ is a modification to the original Stokes's formula $S(\psi)$. For example, one such modification can be written as (see below for notation definitions):

$$S_N = S(\psi) - \sum_{\mathrm{n}=2}^{\mathrm{N}} \frac{2n+1}{2} s'_k P_n(\cos\psi) \tag{6}$$

Many researchers have investigated the theoretical basis for the reduction of the truncation errors using modified integration kernels over a spherical cap of integration, σ_0, with geocentric angle, ψ_0, instead the whole Earth σ. These include Molodensky et al. (1962); de Witte (1966); Wong and Gore (1969); Meisel (1971); Vincent and Marsh (1974); Sjöberg (1984); Vanicek and Kluesberg (1987); Sjöberg (1991); Vanicek and Sjöberg (1991); Neyman et al. (1996); Martinec (1998).

There are different procedures for the computation of the long-wavelength part of the final geoid. One of them can be expressed as (Vanicek and Sjöberg 1991, Sjöberg 1991)

$$N_2(R,\varphi,\lambda) = \frac{R}{2\gamma}\sum_{n=2}^{M}(Q_{Nn} + s'_n)\Delta g_n + \delta N^{\text{topography}}_{\text{direct}}(R,\varphi,\lambda) \tag{7}$$

where

$$\delta N_{\text{direct}}^{\text{topography}}(R,\varphi,\lambda) = -\frac{2\pi\mu}{\gamma}\sum_{n,m}\frac{n+2}{2n+1}(H^2)_{nm}Y_{nm}(P) - \frac{2\pi\mu}{R\gamma}\sum_{n,m}\frac{(n+2)(n+1)}{3(2n+1)}(H^3)_{nm}Y_{nm}(P) \quad (8)$$

and

$$Q_{Nn} = Q_n - \sum_{\text{k=2}}^{\text{N}}\frac{2k+1}{2}s'_k e_{nk} \quad (9)$$

where

$$Q_n = \int_{\psi_0}^{\pi} S(\psi)P_n(\cos\psi)\sin\psi \mathrm{d}\psi \quad (10)$$

are the Molodenski truncation coefficients and

$$e_{nk} = \int_{\psi_0}^{\pi} P_n(\cos\psi)P_k(\cos\psi)\sin\psi \mathrm{d}\psi \quad (11)$$

are the Paul (1973) function, P_n (cos ψ) are the Legendre polynomial and s'_k are the modification parameters, N is degree of kernel modification and M is degree of a global geoid model. Here $\mu = G\rho_0$ where G is the universal gravitational constant, and

$$(H^{\upsilon})nm = \frac{1}{4\pi}\iint_{\sigma} H_P^{\upsilon}Y_{nm}d\sigma, \qquad \upsilon = 2,3 \quad (12)$$

and

$$H_P^{\upsilon} = \sum_{n,m}(H^{\upsilon})_{nm}Y_{nm}(P) \quad (13)$$

where Y_{nm} are fully normalized spherical harmonics obeying the following rule:

$$\frac{1}{4\pi}\iint_{\sigma} Y_{nm}Y_{n'm'}d\sigma = \begin{cases} 1 & \text{if } n=n' \text{ and } m=m' \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

Corrections

The most important and probably problematic part of any geoid computations is the corrections. Topographic and atmospheric corrections as well as downward continuation correction are the requirements of the Stokes-Helmert procedure. The ellipsoidal correction to the observed gravity anomalies arises from spherical approximations and neglecting the terms of order e^2 (e being first eccentricity of the reference ellipsoid). However, there are different formulas for each correction terms to the geoid computations, and unfortunately, they provide different results to the final geoidal height (see for example Nahavandchi 2003). Recent investigations by author showed differences in different formulas for the topographic corrections (Nahavandchi 2003). For the sake of completeness, we provide some of the correction formulas here.

The direct topographical effect on gravity can be expressed by (Nahavandchi 2001):

$$\delta\Delta g_{\text{direct}}^{\text{topography}}(r,\varphi,\lambda) = -\frac{5\pi\mu}{2R}H_P^2 - \frac{3\pi\mu}{2R}\overline{H}_P^2 + \frac{\mu R^2}{2}\iint_{\sigma_0}\frac{H_P^2 - H^2}{\ell^3}\left(1 - \frac{3H_P^2}{\ell^2}\right)d\sigma \tag{15}$$

where H and H_p are the heights of the running and computation points, respectively, and $\ell = \sqrt{(r_P^2 + r^2 - 2r_P r\cos\psi)}$, $r_P = R + H_P$ and

$$\overline{H}_P^2 = \sum_{n,m}\frac{1}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{16}$$

The indirect (primary) effect on geoid is derived as (Sjöberg and Nahavandchi 1999):

$$\delta N_{\text{indirect}}^{\text{topography}}(R,\varphi,\lambda) = -\frac{\pi\mu}{\gamma}H_P^2 + \frac{3\pi\mu}{\gamma}\overline{H}_P^2 - \frac{\pi\mu}{2R\gamma}(H_P^3 - \overline{H}_P^3) - \frac{\mu R^2}{6\gamma}\iint_{\sigma_0}\frac{H^3 - H_P^3}{\ell_0^3}d\sigma \tag{17}$$

where

$$\overline{H}_P^3 = \sum_{n,m}\frac{1}{2n+1}(H^3)_{nm}Y_{nm}(P) \tag{18}$$

and

$$\ell_0 = R\sqrt{2(1-\cos\psi)} = 2R\sin\frac{\psi}{2} \tag{19}$$

The total atmospheric effect (direct + indirect) to the modified Stokes formula, implying the combination with geopotential coefficient, is derived as (Sjöberg and Nahavandchi 2000):

$$\begin{aligned}\delta N_{\text{total}}^{\text{atmosphere}}(R,\varphi,\lambda) &= \frac{2\pi R\rho^0 G}{\gamma}\sum_{n=0}^{1}(s_n' + Q_{Nn})\frac{n+2}{2n+1}H_n(P)\\ &-\frac{2\pi R\rho^0 G}{\gamma}\sum_{n=2}^{M}\left(\frac{2}{n-1} - s_n' + Q_{Nn}\right)H_n(P)\\ &-\frac{2\pi R\rho^0 G}{\gamma}\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - \frac{n+2}{2n+1}Q_{Nn}\right)H_n(P)\end{aligned} \tag{20}$$

where ρ^0 is the density of the atmosphere at the radius of sea level and

$$H_n(P) = \sum_{n,m}H_{nm}Y_{nm}(P) \tag{21}$$

The secondary indirect effect on the geoid can be computed from (Nahavandchi and Sjöberg 1998):

$$\delta N^{\text{topography}}_{\text{secondary}\quad\text{indirect}} = \frac{4\pi\mu}{\gamma}\sum_{n,m}\frac{n+2}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P)$$

$$-\frac{\pi\mu}{R\gamma}H_P\sum_{n,m}\frac{4n^2+2n+3}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) \tag{22}$$

$$+\frac{2\pi\mu}{3R\gamma}\sum_{n,m}\frac{2n^2-8n-3}{(2n+1)(n-1)}(H^3)_{nm}Y_{nm}(P)$$

The ellipsoidal correction to the terrestrial gravity anomalies can be evaluated by the formula given in Moritz (1980):

$$\varepsilon_{\text{ellipsoidal correction}}(R,\varphi,\lambda) = e^2\Delta g^1 \tag{23}$$

where

$$\Delta g^1 = \frac{1}{R}\sum_{n,m}(G_{nm}\cos m\lambda + H_{nm}\sin m\lambda)P_{nm}(\sin\varphi) \tag{24}$$

and G_{nm} and H_{nm} are defined in Moritz (1980) and P_{nm} are fully normalized Legendre functions.

To complete this part, downward continuation problem is presented. In order to obtain the boundary values in Stokes's formula, the gravity anomalies at the topography should be reduced onto the geoid [see Eqs. (3)- (5)]. This reduction is downward continuation. The Poisson formula in Eq. (4) is used for this process. However, this procedure is very unstable due to the masses between the topography and geoid and the irregularity of the density distribution. To solve the Poisson's integral, the Poisson kernel is modified. The correction due to the truncation error in Poisson integral is computed and a division to the low- and high-frequency parts to the Poisson's integral are made.

The Poisson kernel in Eq. (4) only needs to be integrated in a spherical area σ_0 over a small spherical cap ψ_0 instead of over the whole Earth. Again, the truncation error is reduced using Molodensky's truncation modification technique. Also, the low-degree spherical harmonics of the Helmert anomaly Δg_M^H are subtracted from the gravity anomalies Δg^H at the surface of the Earth. Not presenting all details Eq. (4), therefore, can be rewritten as:

$$\Delta g^H(r,\varphi,\lambda) = \frac{R}{4\pi}\iint_{\sigma_0} K^M(r,\psi,R,\psi_0)\Delta g^{H^*}(R,\varphi,\lambda)\mathrm{d}\sigma + dg \tag{25}$$

where

$$dg = \delta g_T + \Delta g_M^H(r,\varphi,\lambda) \tag{26}$$

where δg_T is the truncation error and Δg_M^H are the low-degree spherical harmonics of gravity anomaly. Here the modified Poisson's kernel can be written as:

$$K^{M}(r,\psi,R,\psi_0) = K(r,\psi,R) - \sum_{n=0}^{N}\frac{2n+1}{2}s_n(r,R,\psi_0)P_n(\cos\psi) \tag{27}$$

where $s_n(r,R,\psi_0)$ are the unknown coefficients to be computed from Molodensky procedure. This procedure was presented in detail in Vanicek et al. (1995), Nahavadchi (1998) and Nahavandchi and Sjöberg (2001).

Numerical investigations to Stokes-Helmert scheme

Geoid determination has been under investigation by many scientists in different countries. The author has also investigated different aspects of geoid determination with Stokes-Helmert scheme and its related corrections as well as other models. However, the aim of this study was not these types of computations but to test the effect of different type of data sets and improvements in accuracy of the resulted geoidal height. It should be mentioned that the results provided here are only preliminary and further investigations are in process. Iran is used as a test area. Two different gravity data sets were used, the old and a newer data set, with better measurement accuracy in the new gravity observations. Roughly speaking, we computed the geoidal heights over Iran with the theory presented above. We used two mentioned gravity data sets for computations, the old data set and the old plus new data set, and two series of geoidal heights were computed for each point. It was expected that the use of old plus new data set would improve the accuracy dramatically. 39 GPS-levelling stations were used as an external source of information for sake of comparisons. Both gravimetric geoidal heights are compared with GPS-levelling data and results are shown in Table 1.

Table 1. The statistics of differences between GPS-levelling-derived geoidal heights and gravimetric geoidal heights at 39 stations. Units are in meter

	Min	Max	Mean	Standard Deviations
Old gravity data	-0.181	1.78	0.621	0.520
Old + new gravity data	-0.175	1.651	0.591	0.508

As it can be seen from Table 1, the inclusion of new and more accurate gravity observations did not improve the accuracy significantly, as it was expected. However, these computations should be tested in other areas with other data sets. But, as a preliminary result one can say that the theory of geoidal height determination probably needs a closer look, especially an investigation in different procedures for topographical corrections are suggested. A closer look in the modification process of Stokes's kernel will probably provide interesting results.

Two other geoidal height models

Nahavandchi (2002) presented two other models for geoidal height computations. Those models did not use the terrestrial gravity observations but a spherical harmonic representation of the geopotential, topographic corrections and the height-anomaly-geoidal height difference were used. The models did not have the special complications present in the Stokes-Helmert scheme and were somehow easy to compute. Some special correction terms were however applied. Very short-wavelength part of the geoidal heights might be missing in this procedure, as the terrestrial gravity anomalies did not use in both models, but a dense DTM was used in the second model. The expectation was very less accurate results for the geoidal heights compared to the Stokes-Helmert scheme.

The first model

This model employs geopotential coefficients for the geoidal height computations with the assumption that the external harmonic series expansion is convergent on the Brillouin sphere. However, a bias for external harmonic series when applied at the geoid within topographic masses is expected. This bias can be estimated by removing the topographic ($\delta N_{\text{total}}^{\text{topography}}$) and atmospheric ($\delta N_{\text{total}}^{\text{atmosphere-Geop}}$) masses (such that we can now continue the external harmonic series of the geopotential downwards to the geoid- they are now harmonic between the geoid and the topography). The realization is shown by the following formula (Heiskanen and Moritz 1967; see also Nahavandchi 2002):

$$N(R,\phi,\lambda) = \frac{GM_3}{R\gamma}\sum_{n=0}^{M}\left(\frac{a_1}{R}\right)^n \sum_{m=0}^{n}\left[\left(\frac{GM_1}{GM_3}C_{nm} - \frac{GM_2}{GM_3}\left(\frac{a_2}{a_1}\right)^n C'_{nm}\right)\cos m\lambda + \frac{GM_1}{GM_3}S_{nm}\sin m\lambda\right]P_{nm}(\sin\phi) - \frac{1}{\gamma}\left(W_0 - U_0\right) + \delta N_{\text{total}}^{\text{topography}} + \delta N_{\text{total}}^{\text{atmosphere-Geop}} \tag{28}$$

where

$$\delta N_{\text{total}}^{\text{topography}} = -\frac{2\pi\mu}{\gamma}\sum_{n,m}(H^2)_{nm}Y_{nm}(P) - \frac{4\pi\mu}{3R\gamma}\sum_{n,m}(H^3)_{nm}Y_{nm}(P) \tag{29}$$

and

$$\delta N_{\text{total}}^{\text{atmosphere-Geop}} = \frac{2\pi\rho^0 G}{\gamma}\sum_{n,m}(H^2)_{nm}Y_{nm}(P) \tag{30}$$

where a_1 is the equipotential scale factor of EGM96 (6378.1363 km)
a_2 is the equipotential radius of GRS-80 (6378.137 km)
GM_1 is the gravity-mass constant of EGM96 (3.986004415×10^{14} m^{3}s^{-2})
GM_2 is the gravity-mass constant of GRS-80 (3.986005000×10^{14} m^{3}s^{-2})
GM_3 is the best estimate of gravity-mass constant for the Earth (3.986004418×10^{14} m^{3}s^{-2})
W_0 is adopted gravity potential on the geoid (62636856.88 m^{2}s^{-2})
U_0 is defined normal gravity potential on the ellipsoid (62636860.8 m^{2}s^{-2})
C_{nm}, S_{nm} are fully normalized geopotential coefficients of EGM96 in non-tidal system
C'_{nm} are fully normalized normal potential coefficients of GRS-80 in non-tidal system ($S'_{nm}=0$).

The second model

The second model is based on the fact that the geoid is computed through height anomaly ζ_0 and the height anomaly is computed by the geopotential coefficients at the topography not onto the geoid. Therefore, one can ignore the bias due to the non-harmonicity in the previous formula. However, some special corrections are used in this procedure too to compute the geoidal heights through height anomaly. This procedure is realized as (see Heiskanen and Moritz 1967; Sjöberg 1995; Rapp 1997; and Nahavandchi 2002):

$$N(R,\phi,\lambda)=\zeta_0(r,\phi,\lambda)+\frac{\partial\zeta}{\partial r}H+\frac{\partial\zeta}{\partial\gamma}\frac{\partial\gamma}{\partial H}H+\frac{(\Delta g_{\text{Bouguer}})}{\bar{\gamma}}H+\frac{H_P^2}{2\bar{\gamma}}\left(\frac{\partial\Delta g_{\text{Free-air}}}{\partial \text{H}}\right) \tag{31}$$

where

$$\zeta_0=\frac{GM_3}{r\gamma}\sum_{n=0}^{M}\left(\frac{a_1}{r}\right)^n\sum_{m=0}^{n}\left[\left(\frac{GM_1}{GM_3}C_{nm}-\frac{GM_2}{GM_3}\left(\frac{a_2}{a_1}\right)^n C'_{nm}\right)\cos m\lambda+\frac{GM_1}{GM_3}S_{nm}\sin m\lambda\right]\times$$
$$\times P_{nm}(\sin\phi)-\frac{1}{\gamma}\left(W_0-U_0\right)+\delta N_{\text{total}}^{\text{atmosphere-Geop}} \tag{32}$$

and

$$\Delta g_{\text{Bouguer}}(\phi,\lambda)=\Delta g_{\text{Free-air}}(\phi,\lambda)-0.1119H(\phi,\lambda) \tag{33}$$

where H is in meters and the gravity units are mGal and

$$\Delta g_{\text{Free-air}}(r,\phi,\lambda)=\frac{GM_3}{r^2}\sum_{n=0}^{M}(n-1)\left(\frac{a_1}{r}\right)^n\sum_{m=0}^{n}\left[\left(\frac{GM_1}{GM_3}C_{nm}-\frac{GM_2}{GM_3}\left(\frac{a_2}{a_1}\right)^n C'_{nm}\right)\cos m\lambda+\frac{GM_1}{GM_3}S_{nm}\sin m\lambda\right]$$
$$\times P_{nm}(\sin\phi)+\frac{2}{r}\left(W_0-U_0\right) \tag{34}$$

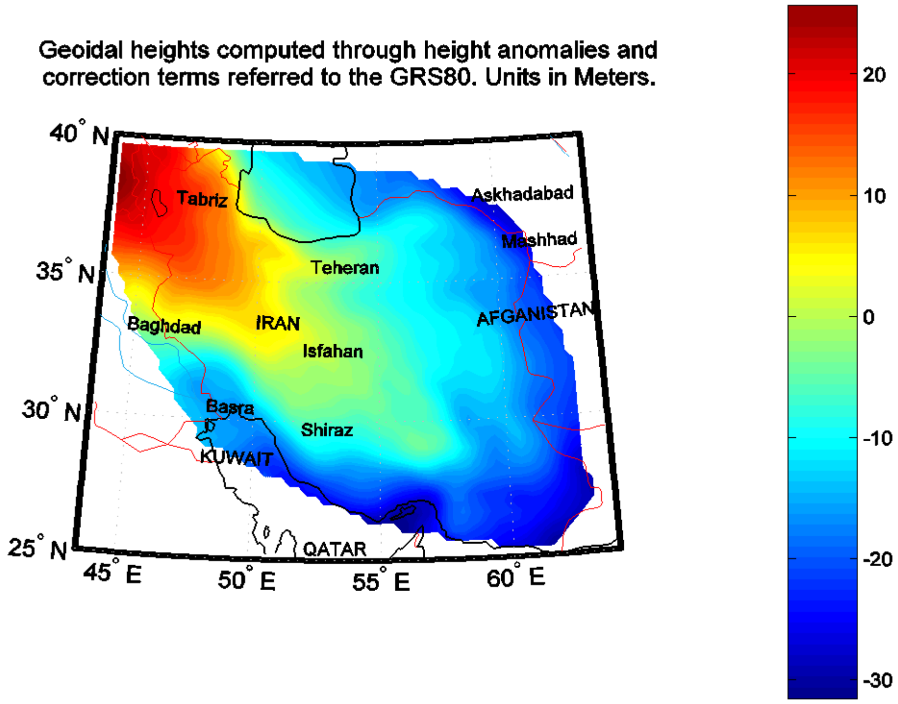
Numerical investigations to the two geoid models

The two geoid models were tested in Iran and at the 39 GPS-levelling stations mentioned in previous section. The results are shown in Table 2.

Table 2. The Statistics of differences between GPS-leveling-derived Geoidal heights and the two other geoid models in meters

	Min	Max	Mean	Standard Deviations
Geoid model 1	-0.356	1.983	0.752	0.614
Geoid model 2	-0.275	1.811	0.653	0.578

Surprisingly, very good results are achieved, considering not using the terrestrial gravity data. This work is still under investigations and the preliminary results are only shown here. This procedure must be tested in other test areas with different data sets. These results again justify that a closer look to the geoid computations theory must be carried out. To finish this part the second geoidal height models computed over Iran is depicted in below.



Conclusions

The last two decades there has been an increased interest in the gravimetric determination of the geoid. The need for good and accurate models of the geoid has been driven principally by the demands of different users, especially GPS users, who must transform GPS-derived ellipsoidal heights to the heights in local vertical datums. Some different theories for geoid computations are here outlined and some numerical computations are implemented. The result of computations shows different level of accuracy in different geoid models. Also, it opens the door for necessary theoretical comparisons and probably improvements in the theory of the geoidal height models. The geoidal height models have recently been improved significantly by many researchers in different countries; however, a validation procedure could be a very useful tool to understand better the differences and to improve the models, if necessary. In summary some recommendations are listed below:

- The geoidal height determination, especially topographic corrections and modification procedure is still an open investigation.
- A validation procedure must be applied.
- The use of a synthetic Earth model is useful.
- A mixed non-linear Gravimetric-GPS Boundary Value Problem must be investigated.
- A mixed non-linear Gravimetric-Altimetry Boundary Value Problem must be investigated.

References

Forsberg R (1994) Terrian effects in geoid computations. In: Lecture Notes, International School for the determination and use of the geoid, IGeS: 101-134.

Heiskanen WA, Moritz H (1967) Physical Geodesy. W H Freeman and Company, San Francisco.

Martinec Z (1998) Boundary-value problems for gravimetric determination of a precise geoid. Springer-Verlag, Berlin Heidelberg, Germany.
Martinec Z, Vanicek P (1994a) Direct topographical effect of Helmert's condensation for a spherical approximation of the geoid. Manuscripta Geodaetica 19: 257-268.
Martinec Z, Vanicek P (1994b) The indirect effect of topography in the Stokes-Helmert technique for a spherical approximation of the geoid. Manuscripta Geodaetica 19: 213-219.
Martinec Z, Matyska C, Grafarend EW, Vanicek P (1993) On the Helmert's 2nd condensation method. Manuscripta Geodaetica 18: 417-421.
Meissl P (1971) Preparation for the numerical evaluation of second order Molodensky-type formulas. Rep 163, Department of Geodetic Science, The Ohio State University, Columbus.
Molodensky MS, Eremeev VF, Yurkina MI (1962) Methods for study of the external gravitational field and figure of the Earth. Office of Technical Services, Department of Commerce, Washington D.C.
Moritz H (1980) Advanced physical geodesy. Herbert Wichman Verlag, Karlsruhe
Nahavandchi H (1998c) On some methods of downward continuation of mean free-air gravity anomaly. IGeS Bulletin 8: 1-17.
Nahavandchi H (2000) The direct topographical correction in gravimetric geoid determination by the Stokes-Helmert method. Journal of Geodesy 74: 488-496.
Nahavandchi H (2002) Two different methods of geoidal height determinations using a spherical harmonic representation of the geopotential, topographic corrections and the height anomaly-geoidal height difference. Journal of Geodesy 76: 345-352.
Nahavandchi H (2003) A comparison of the classical and recent formulae of handling the effects of close and distant topographic masses in gravimetric geoid computations. IGeS Submitted.
Nahavandchi H, Sjöberg LE (1998) Terrain correction to power H3 in gravimetric geoid determination. Journal of Geodesy 72: 124-135.
Nahavandchi H, Sjöberg LE (2001) Precise geoid determination over Sweden using the Stokes-Helmert method and improved topographic corrections. Journal of Geodesy 75: 74-88.
Nahavandchi H, Sjöberg LE (2001b) Two different views of topographical and downward continuation corrections in the Stokes-Helmert approach to geoid computations, Journal of Geodesy 74: 816-822.
Neyman YM, Li J, Liu Q (1996) Modification of Stokes and Vening Meinesz's formulas for the inner zone of arbitrary shape by minimization of upper bound truncation errors. Journal of Geodesy 70: 410-418.
Paul NK (1973) A method of evaluating the truncation error coefficients for geoidal height. Bulletin Geodesique 110: 413-425.
Rapp RH (1997) Use of potential coefficient models for geoid undulation determinations using a spherical harmonic representation of the height anomaly/geoid undulation difference. Journal of Geodesy 71: 282-289.
Sideris MG (1990) Rigorous gravimetric terrain modeling using Molodenski's operator. Manuscripta Geodaetica 15: 97-106.
Sjöberg LE (1984) Least squares modification of Stokes's and Vening Meinez' formulas by accounting for errors of truncation, potential coefficients and gravity data, Department of Geodesy, University of Uppsala, No. 27, Uppsals, Sweden.
Sjöberg LE (1991) Refined least-squares modification of Stokes's formula. Manuscripta Geodaetica 16: 367-375.
Sjöberg LE (1994) On the total terrain effects in geoid and quasigeoid determinations using Helmert second condensation method. Division of Geodesy, Rep 36, Royal Institute of Technology, Stockholm.
Sjöberg LE (1995) On the quasigeoid to geoid separation. Manuscripta Geodaetica 20: 182-192.
Sjöberg LE, Nahavandchi H (1999) On the indirect effect in the Stokes-Helmert method of geoid determination. Journal of Geodesy 73: 87-93
Sjöberg LE, Nahavandchi H (2000) The atmospheric geoid effects in Stokes formula. Geophysical Journal International 140: 95-100.
Stokes GG (1849) On the variation of gravity on the surface of the Earth. Trans Camb Phil Soc 8: 672-695.
Tziavos IN, Sideris MG, Schwarz KP (1992) A study of the contribution of various gravimetric data types on the estimation of gravity field parametersin the mountains. Journal of Geophysical Research 97: 8843-8852.
Vanicek P, Kleusberg A (1987) The canadian geoid-Stokesian approach, Manuscripta Geodaetica 12: 86-98.
Vanicek P, Sjöberg LE (1991) Reformulation of Stokes'stheory for higher than second degree reference field and modification of integration kernels, Journal of Geophysical Research 96: 6529-6539.
Vanicek P, Martinec Z (1994) The Stokes-Helmert scheme for the evaluation of a precise geoid. Manuscripta Geodaetica 19: 119-128.
Vanicek P, Najafi M, Martinec Z, Harrie L, Sjöberg LE (1995) Higher-order reference field in the generalized Stokes-Helmert scheme for geoid computation. Journal of Geodesy 70: 176-182.

Vincent S, Marsh J (1974) Gravimetric global geoid. In Veis G (ed) Use of artificial satellites for geodesy and geodynamics. Proceedings International Symposium, National Technical University, Athens.

Wichiencharoen C (1982) The indirect effects on the computation of geoid undulations. Rep 336, Department of Geodetic Science, The Ohio State University, Columbus.

de Witte L (1966) Truncation errors in the Stokes's and Vening Meinesz's formulae for different order spherical harmonic gravity terms. Aerospace Report TR-669 (SG 230-35)-5.

Wong L, Gore R (1969) Accuracy of geoid heights from the modified Stokes's kernels, Geophysical Journal Research Astronomocal Society 18: 81-91.