Doctoral theses at NTNU, 2023:113

Johanne Haugland

Subcategory structures, Grothendieck groups and higher homological algebra

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of Philosophiae Doctor
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of Science and Technology

Johanne Haugland

Subcategory structures, Grothendieck groups and higher homological algebra

Thesis for the Degree of Philosophiae Doctor

Trondheim, April 2023

Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the Degree of Philosophiae Doctor

Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

ISBN 978-82-326-5599-1 (printed ver.)
ISBN 978-82-326-5310-2 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2023:113

Printed by NTNU Grafisk senter

CONTENTS

Acknowledgements

Introduction

I. The Grothendieck group of an n-exangulated category
Appl. Categ. Structures 29 (2021), no. 3, 431-446.

II. Auslander–Reiten triangles and Grothendieck groups of triangulated categories
Algebr. Represent. Theory 25 (2022), 1379-1387.

III. The role of gentle algebras in higher homological algebra
Forum Math. 34 (2022), no. 5, 1255-1275.

IV. Higher Koszul duality and connections with n-hereditary algebras

V. The category of extensions and a characterisation of n-exangulated functors

VI. A characterisation of higher torsion classes

Appendix: Norwegian translations

ACKNOWLEDGEMENTS

First I would like to thank my supervisor Petter Andreas Bergh for his continuous support and many interesting and enjoyable discussions during my years as a PhD-student. I moreover thank all members of the algebra group as well as the department as a whole, including administrative and technical staff, for a stimulating atmosphere and excellent working conditions.

I next want to express my gratitude for having been part of the project "IDUN – from PhD to professor" at the IE-faculty. In particular, I wish to thank Sibylle Schroll for being a great mentor and collaborator.

In the fall semester of 2020, I participated in the Junior Trimester Program "New Trends in Representation Theory" at the Hausdorff Research Institute for Mathematics (HIM) in Bonn. I extend my thanks to HIM for inviting me, and for their extraordinary efforts in organising an exiting trimester program during a global pandemic.

I spent parts of the fall semester of 2022 visiting the Aarhus homological algebra group. I would like to thank Karin M. Jacobsen and Peter Jørgensen for inviting me and the Department of Mathematics at Aarhus University for their hospitality. Additionally, I thank the project "Pure mathematics in Norway" funded by the Trond Mohn Foundation for economical support covering both my stay in Denmark and the expenses related to inviting two researchers from Aarhus to Trondheim in February 2023.

There are many fantastic colleagues, collaborators and friends — with a significant overlap between these groups — who would have deserved to be mentioned by name. To avoid making this a very long text, I instead wish to extend my sincere thanks to all of you. You made the journey a lot of fun and are an important reason for why I love my job.

I would finally like to express my gratitude to my family. A particular thanks to Bjørn for believing in me, for sharing my enthusiasm for research and for following me and Tora on several travels for conferences and research stays.

Johanne Haugland
Trondheim, January 2023

INTRODUCTION

Representation theory of finite dimensional algebras constitutes an overall framework for this PhD-thesis. A fundamental idea in representation theory is to investigate abstract patterns that allow us to provide structure to complicated mathematical systems. An important approach towards understanding the mathematical objects known as *algebras*, is to investigate how an algebra acts on the universe it is part of. More precisely, we study categories one can associate to an algebra in a natural way, for instance module categories and derived categories.

As the categories in question usually are of a complex nature, it is often not feasible to explore them on a global level. An underlying idea in this thesis is to instead solve problems locally, before changing perspective to deduce what the local answers reveal about the global situation. This involves studying smaller pieces, or more precisely *subcategories*, of our original categories. These subcategories or "building blocks" are often more tractable than the ambient category itself.

The thesis consists of the following six papers:

[H1] J. Haugland, *The Grothendieck group of an n-exangulated category*, Appl. Categ. Structures 29 (2021), no. 3, 431-446.

[H2] J. Haugland, *Auslander–Reiten triangles and Grothendieck groups of triangulated categories*, Algebr. Represent. Theory 25 (2022), 1379-1387.

[H3] J. Haugland, K. M. Jacobsen and S. Schroll, *The role of gentle algebras in higher homological algebra*, Forum Math. 34 (2022), no. 5, 1255-1275.

[H4] J. Haugland and M. H. Sandøy, *Higher Koszul duality and connections with n-hereditary algebras.*

[H5] R. Bennett-Tennenhaus, J. Haugland, M. H. Sandøy and A. Shah, *The category of extensions and a characterisation of n-exangulated functors.*

[H6] J. August, J. Haugland, K. M. Jacobsen, S. Kvamme, Y. Palu and H. Treffinger, *A characterisation of higher torsion classes.*

The notion of *Grothendieck groups* plays a key role in the first two papers of the thesis, while all except the second paper are within the area of *higher homological algebra*. Both higher homological algebra and Grothendieck groups are closely connected to the study of *subcategory structures*, although from different perspectives. In this introduction, we give an overview of relevant background material and highlight how the topics of this thesis revolve around increasing the understanding of subcategories.

Higher homological algebra. The research field of higher homological algebra was initiated by Iyama [12, 13] and concerns higher-dimensional generalisations of categories that are central in representation theory. Given a positive integer n, important classes of examples include n-abelian (or more generally n-exact) and $(n+2)$-angulated categories as defined in [17, Definition 3.1] and [8, Section 1], respectively. A fundamental role in these higher structures is played by distinguished sequences with $n+2$ objects. For $n = 1$, one recovers the short exact sequences and distinguished triangles of abelian and triangulated categories, and the theory obtained corresponds to classical homological algebra.

Categories exhibiting a higher homological structure primarily arise as certain subcategories, known as *n-cluster tilting subcategories*, of abelian and triangulated categories. The definition of such subcategories, which is given below, plays a crucial role in this thesis. Recall that a subcategory $\mathcal{U}$ of a category $\mathcal{C}$ is called *functorially finite* if every object in $\mathcal{C}$ admits both a left and a right $\mathcal{U}$-approximation. Moreover, in the case where $\mathcal{C}$ is abelian, the subcategory $\mathcal{U}$ is called *generating* (resp. *cogenerating*) if for each object X in $\mathcal{C}$ there exists an epimorphism $U \twoheadrightarrow X$ (resp. monomorphism $X \rightarrowtail U$) with U in $\mathcal{U}$. The subcategory $\mathcal{U}$ is called *generating-cogenerating* if it is both generating and cogenerating. In the definition below, we write $\operatorname{Ext}^i_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(X, \Sigma^i Y)$ in the case where the ambient category $\mathcal{C}$ is triangulated with suspension functor Σ.

Definition (See [14, 19, 21])**.** A functorially finite subcategory $\mathcal{U}$ of an abelian or triangulated category $\mathcal{C}$ is *n-cluster tilting* if it is generating-cogenerating (in the abelian case) and

$$\begin{aligned}\mathcal{U} &= \{X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0 \;\text{ for }\; 1 \le i \le n-1\} \\ &\{X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(X, \mathcal{U}) = 0 \;\text{ for }\; 1 \le i \le n-1\}.\end{aligned}$$

For some basic examples of n-cluster tilting subcategories, see for instance [H3, Example 2.2]. The theorem below plays a fundamental role in higher homological algebra, and demonstrates the importance of n-cluster tilting subcategories. The first part of the result is due to Jasso [17], while the second is shown by Geiss, Keller and Oppermann [8].

Theorem (See [17, Theorem 1] and [8, Theorem 1])**.** *Let $\mathcal{U}$ be an n-cluster tilting subcategory of an abelian or triangulated category $\mathcal{C}$. The following statements hold:*

(1) *If $\mathcal{C}$ is abelian, then $\mathcal{U}$ is n-abelian.*
(2) *If $\mathcal{C}$ is triangulated and $\mathcal{U}$ is closed under Σ^n, then $\mathcal{U}$ carries the structure of an $(n+2)$-angulated category.*

The minor difference in the formulation of the conclusions in (1) and (2) above is due to the fact that being n-abelian is a property of a category, while in order to state that $\mathcal{U}$ is $(n+2)$-angulated, one needs to specify the suspension functor and a class of distinguished $(n+2)$-angles, see [8, Section 1].

A converse to part (1) of the theorem above has recently been shown, establishing that any n-abelian category is equivalent to an n-cluster tilting subcategory of an ambient abelian category [6, 22]. The research field of higher homological algebra can hence be interpreted as the study of important subcategories that relate to higher homological phenomena. As demonstrated in both the third and fourth paper of this thesis as well as in work by other authors, the existence of a subcategory allowing a higher homological structure is intimately related to global homological properties of the ambient category [H3, H4, 2, 3, 15, 17, 26, 28, 29].

Herschend, Liu and Nakaoka introduced n-exangulated categories in [10], giving a higher analogue of extriangulated categories as defined by Nakaoko and Palu [23]. The notion of n-exangulated categories unifies and extends the definitions of n-exact and $(n+2)$-angulated categories. In the first and fifth paper of this thesis, we work in the general framework of n-exangulated categories.

The importance of the study of higher structures has become increasingly evident as connections to various branches of mathematics have been developed. Higher homological phenomena are strongly related to higher Auslander–Reiten theory and representation theory of finite dimensional algebras [12–14]. The research field has connections to commutative algebra, algebraic K-theory, commutative and non-commutative algebraic geometry, symplectic geometry, combinatorics and conformal field theory [1, 4, 5, 7, 9, 16, 20, 25, 30]. Within the area of algebraic geometry, higher homological algebra recently provided an important ingredient allowing the proof of the Donovan–Wemyss conjecture in the context of the minimal model program [18].

Grothendieck groups. Let $\mathcal{T}$ be an essentially small triangulated category with suspension functor denoted by Σ. We start this section by giving the definition of the Grothendieck group of $\mathcal{T}$. This notion is a triangulated analogue of the classical definition of the Grothendieck group of an algebra, or more generally of an essentially small abelian category, which plays an important role in many branches of representation theory. We moreover present a result of Thomason relating the subgroup structure of the Grothendieck group to the subcategory structure of the triangulated category [27].

As $\mathcal{T}$ is essentially small, the collection of isomorphism classes $\langle X\rangle$ of objects X in $\mathcal{T}$ forms a set. We use the notation $\mathcal{F}(\mathcal{T})$ for the free abelian group generated by such isomorphism classes.

Definition. The *Grothendieck group* of $\mathcal{T}$ is the quotient $K_0(\mathcal{T}) = \mathcal{F}(\mathcal{T})/\mathcal{R}(\mathcal{T})$, where $\mathcal{R}(\mathcal{T})$ is the subgroup of $\mathcal{F}(\mathcal{T})$ generated by the subset

$$\{\langle X\rangle - \langle Y\rangle + \langle Z\rangle \mid X \to Y \to Z \to \Sigma X \text{ is a distinguished triangle in } \mathcal{T}\}.$$

The equivalence class $\langle X\rangle + \mathcal{R}(\mathcal{T})$ represented by an object X in $\mathcal{T}$ is denoted by $[X]$.

The definition above is used directly in the second paper of this thesis, while it is used as a background for defining an analogous notion in the setup of n-exangulated categories in the first paper. This more general definition, see [H1, Definition 4.1], covers both the triangulated and the abelian case, as well as higher-dimensional generalisations.

A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *dense* if each object in $\mathcal{T}$ is a direct summand of an object in $\mathcal{S}$. It should be noted that each triangulated subcategory of $\mathcal{T}$ is dense in a uniquely determined thick subcategory. Whenever a classification of thick subcategories is known, for instance if $\mathcal{T}$ is the bounded derived category of a commutative ring [11, 24], the theorem below hence yields a full classification of all triangulated subcategories.

Theorem (See [27, Theorem 2.1]). *There is a one-to-one correspondence*

$$\{\textit{subgroups of } K_0(\mathcal{T})\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{dense subcategories of } \mathcal{T}\},$$

where $f(H)$ for $H \subseteq K_0(\mathcal{T})$ is the full subcategory

$$f(H) = \{X \in \mathcal{T} \mid [X] \in H\}$$

and $g(\mathcal{S})$ for $\mathcal{S} \subseteq \mathcal{T}$ is the subgroup

$$g(\mathcal{S}) = \langle [X] \in K_0(\mathcal{T}) \mid X \in \mathcal{S}\rangle.$$

The classification theorem above is generalised to the context of n-exangulated categories in the first paper of this thesis, see [H1, Theorem 5.1], establishing that even though n-exangulated categories constitute a significantly bigger class of categories, their subcategory structures are still intimately related to the associated Grothendieck groups.

Notes for the reader. Below is a reference list which is used for this introduction, where the papers that are part of the thesis are separated out. Note that there is also included a reference list as part of each paper. In cases where the papers are published, this is indicated on the title page. At the very end of the thesis, you can find an appendix containing a list of Norwegian translations of some central mathematical terms. Norwegian readers are strongly encouraged to look up translations for terms they are not yet familiar with.

Papers in thesis

[H1] J. Haugland, *The Grothendieck group of an n-exangulated category*, Appl. Categ. Structures **29** (2021), no. 3, 431–446.

[H2] ______, *Auslander–Reiten Triangles and Grothendieck groups of Triangulated Categories*, Algebr. Represent. Theory **25** (2022), 1379-1387.

[H3] J. Haugland, K. M. Jacobsen, and S. Schroll, *The role of gentle algebras in higher homological algebra*, Forum Math. **34** (2022), no. 5, 1255-1275.

[H4] J. Haugland and M. H. Sandøy, *Higher Koszul duality and connections with n-hereditary algebras*, arXiv:2101.12743 (2021).

[H5] R. Bennett-Tennenhaus, J. Haugland, M. H. Sandøy, and A. Shah, *The category of extensions and a characterisation of n-exangulated functors*, arXiv:2205.03097 (2022).

[H6] J. August, J. Haugland, K. M. Jacobsen, S. Kvamme, Y. Palu, and H. Treffinger, *A characterisation of higher torsion classes*, arXiv:2301.10463 (2023).

References

[1] C. Amiot, O. Iyama, and I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, Amer. J. Math. **137** (2015), no. 3, 813–857.

[2] E. Darpö and O. Iyama, *d-representation-finite self-injective algebras*, Adv. Math. **362** (2020), 106932, 50.

[3] E. Darpö and T. Kringeland, *d-Representation-finite symmetric Nakayama algebras and trivial extensions of quiver algebras*, arXiv:2103.15380 (2021).

[4] T. Dyckerhoff, G. Jasso, and Y. Lekili, *The symplectic geometry of higher Auslander algebras: symmetric products of disks*, Forum Math. Sigma **9** (2021), Paper No. e10, 49.

[5] T. Dyckerhoff, G. Jasso, and T. Walde, *Simplicial structures in higher Auslander-Reiten theory*, Adv. Math. **355** (2019), 106762, 73.
[6] R. Ebrahimi and A. Nasr-Isfahani, *Higher Auslander's Formula*, Int. Math. Res. Not. IMRN **22** (2022), 18186–18203.
[7] D. E. Evans and M. Pugh, *The Nakayama automorphism of the almost Calabi-Yau algebras associated to $SU(3)$ modular invariants*, Comm. Math. Phys. **312** (2012), no. 1, 179–222.
[8] C. Geiss, B. Keller, and S. Oppermann, *n-angulated categories*, J. Reine Angew. Math. **675** (2013), 101–120.
[9] M. Herschend, O. Iyama, H. Minamoto, and S. Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, to appear in Mem. Amer. Math. Soc.
[10] M. Herschend, Y. Liu, and H. Nakaoka, *n-exangulated categories (I): Definitions and fundamental properties*, J. Algebra **570** (2021), 531–586.
[11] M. J. Hopkins, *Global methods in homotopy theory*, Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 73–96.
[12] O. Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82.
[13] ———, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50.
[14] ———, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61.
[15] O. Iyama and S. Oppermann, *Stable categories of higher preprojective algebras*, Adv. Math. **244** (2013), 23–68.
[16] O. Iyama and M. Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math. **197** (2014), no. 3, 521–586.
[17] G. Jasso, *n-abelian and n-exact categories*, Math. Z. **283** (2016), no. 3-4, 703–759.
[18] G. Jasso, B. Keller, and F. Muro, *The Triangulated Auslander–Iyama Correspondence*, arXiv:2208.14413 (2022).
[19] P. Jørgensen, *Torsion classes and t-structures in higher homological algebra*, Int. Math. Res. Not. IMRN **13** (2016), 3880–3905.
[20] ———, *Tropical friezes and the index in higher homological algebra*, Math. Proc. Cambridge Philos. Soc. **171** (2021), no. 1, 23–49.
[21] B. Keller and I. Reiten, *representation tilted algebras are Gorenstein and stably Calabi–Yau*, Adv. Math. **211** (2007), no. 1, 123–151.
[22] S. Kvamme, *Axiomatizing subcategories of Abelian categories*, J. Pure Appl. Algebra **226** (2022), no. 4, Paper No. 106862, 27.
[23] H. Nakaoka and Y. Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. **60** (2019), no. 2, 117–193.
[24] A. Neeman, *The chromatic tower for $D(R)$*, Topology **31** (1992), no. 3, 519–532. With an appendix by Marcel Bökstedt.
[25] S. Oppermann and H. Thomas, *Higher-dimensional cluster combinatorics and representation theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1679–1737.
[26] M. H. Sandøy and L.-P. Thibault, *Classification results for n-hereditary monomial algebras*, arXiv:2101.12746 (2021).
[27] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), no. 1, 1–27.
[28] L. Vaso, *n-cluster tilting subcategories of representation-directed algebras*, J. Pure Appl. Algebra **223** (2019), no. 5, 2101–2122.
[29] ———, *n-cluster tilting subcategories for radical square zero algebras*, Journal of Pure and Applied Algebra (1) **227** (2023).
[30] N. J. Williams, *New interpretations of the higher Stasheff-Tamari orders*, Adv. Math. **407** (2022).

Paper 1

THE GROTHENDIECK GROUP OF AN n-EXANGULATED CATEGORY

Appl. Categ. Structures 29 (2021), no. 3, 431-446

Johanne Haugland

THE GROTHENDIECK GROUP OF AN n-EXANGULATED CATEGORY

JOHANNE HAUGLAND

Abstract. We define the Grothendieck group of an n-exangulated category. For n odd, we show that this group shares many properties with the Grothendieck group of an exact or a triangulated category. In particular, we classify dense complete subcategories of an n-exangulated category with an n-(co)generator in terms of subgroups of the Grothendieck group. This unifies and extends results of Thomason, Bergh–Thaule, Matsui and Zhu–Zhuang for triangulated, $(n+2)$-angulated, exact and extriangulated categories, respectively. We also introduce the notion of an n-exangulated subcategory and prove that the subcategories in our classification theorem carry this structure.

1. Introduction

The Grothendieck group of an exact category is the free abelian group generated by isomorphism classes of objects modulo the Euler relations coming from short exact sequences. Similarly, one obtains the Grothendieck group of a triangulated category by factoring out the relations corresponding to distinguished triangles. It turns out that subcategories of certain categories relate to subgroups of the associated Grothendieck groups in an interesting way. More precisely, Thomason proved that there is a one-to-one correspondence between subgroups of the Grothendieck group of a triangulated category and dense triangulated subcategories [16, Theorem 2.1]. This was generalized to $(n+2)$-angulated categories with n odd by Bergh–Thaule [3, Theorem 4.6]. Later, Matsui gave an analogous result for exact categories with a (co)generator [12, Theorem 2.7].

The notion of extriangulated categories was introduced by Nakaoka–Palu as a simultaneous generalization of exact categories and triangulated categories [13]. Many concepts and results concerning exact and triangulated categories have been unified and extended using this framework, see for instance [7] for a generalization of Auslander–Reiten theory in exact and triangulated categories to this context.

In both higher dimensional Auslander–Reiten theory and higher homological algebra, n-cluster tilting subcategories of exact and triangulated categories play a fundamental role [6, 8]. This was a starting point for developing the theory of $(n+2)$-angulated categories and n-exact categories in the sense of Geiss–Keller–Oppermann [4] and Jasso [9]. Recently, Herschend–Liu–Nakaoka defined n-exangulated categories as a higher dimensional analogue of extriangulated categories [5]. Many categories studied

2010 *Mathematics Subject Classification.* 18E10, 18E30, 18F30.

Key words and phrases. Grothendieck group, n-exangulated category, $(n+2)$-angulated category, n-exact category, n-exangulated subcategory, extriangulated subcategory.

in representation theory turn out to be n-exangulated. In particular, n-exangulated categories simultaneously generalize $(n+2)$-angulated and n-exact categories. In [5, Section 6] several explicit examples of n-exangulated categories are given. See also [11, Section 4] for a construction which yields more n-exangulated categories that are neither n-exact nor $(n+2)$-angulated.

Inspired by the classification results for triangulated, $(n+2)$-angulated and exact categories mentioned above, a natural question to ask is whether there is a similar connection between subcategories and subgroups of the Grothendieck group for n-exangulated categories. Independently of our work, Zhu–Zhuang recently gave a partial answer to this question in the case $n=1$ [17, Theorem 5.7]. In this paper we prove a more general classification result for n-exangulated categories with n odd. In our main result, Theorem 5.1, we classify dense complete subcategories of an n-exangulated category with an n-(co)generator $\mathcal{G}$ in terms of subgroups of the Grothendieck group containing the image of $\mathcal{G}$. This recovers both the result of Zhu–Zhuang for extriangulated categories and the result of Bergh–Thaule for $(n+2)$-angulated categories, as well as Thomason's and Matsui's results for triangulated and exact categories, see Corollary 5.5 for details. Our main theorem also yields new classification results for n-exact categories, as well as for n-exangulated categories which are neither $(n+2)$-angulated nor n-exact.

The paper is organized as follows. In Section 2 we recall the definition of an n-exangulated category and review some results. In Section 3 we explain terminology which is needed in our main result, such as the notion of an n-(co)generator, complete subcategories and dense subcategories. We also introduce n-exangulated subcategories and prove that the subcategories which will appear in our classification theorem carry this structure. In Section 4 we define the Grothendieck group of an n-exangulated category and discuss some basic results. In Section 5 we state and prove our main theorem and explain how this unifies and extends already known results.

2. Preliminaries on n-exangulated categories

Throughout this paper, let n be a positive integer and $\mathcal{C}$ an additive category. In this section we briefly recall the definition of an n-exangulated category and related notions, as well as some known results which will be used later. All of this is taken from [5], and we recommend to consult this paper for more detailed explanations.

Recall from [13] that an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ consists of an additive category $\mathcal{C}$, a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ and an additive realization $\mathfrak{s}$ of $\mathbb{E}$ satisfying certain axioms. The functor $\mathbb{E}$ is modelled after Ext^1. Given two objects A and C in $\mathcal{C}$, the realization $\mathfrak{s}$ associates to each element $\delta \in \mathbb{E}(C, A)$ an equivalence class $\mathfrak{s}(\delta)$ of 3-term sequences in $\mathcal{C}$ starting in A and ending in C. Exact and triangulated categories are examples of extriangulated categories, where short exact sequences and distinguished triangles play the roles of these 3-term sequences. Analogously, an n-exangulated category also consists of a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where the main difference is that we consider $(n+2)$-term sequences instead of 3-term sequences. In order to give the precise definition, we need to be able to talk about extensions and morphisms of extensions.

Definition 2.1. Let $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ be a biadditive functor. Given two objects A and C in $\mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension* or simply an *extension*. We can write such an extension δ as ${}_A\delta_C$ whenever we wish to specify the objects A and C.

Given an extension $\delta \in \mathbb{E}(C, A)$ and two morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we denote the extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \text{ and } \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$$

by $a_*\delta$ and $c^*\delta$. Notice that $\mathbb{E}(c, a)(\delta) = c^*a_*\delta = a_*c^*\delta$ in $\mathbb{E}(C', A')$ as $\mathbb{E}$ is a bifunctor.

For any pair of objects A and C, the zero element ${}_A0_C$ in $\mathbb{E}(C, A)$ is called the *split extension*.

Definition 2.2. Given extensions ${}_A\delta_C$ and ${}_B\rho_D$, a *morphism of extensions* $(a, c)\colon \delta \to \rho$ is a pair of morphisms $a \in \mathcal{C}(A, B)$ and $c \in \mathcal{C}(C, D)$ such that $a_*\delta = c^*\rho$ in $\mathbb{E}(C, B)$.

We want to associate each extension ${}_A\delta_C$ to an equivalence class of $(n + 2)$-term sequences in $\mathcal{C}$ starting in A and ending in C. Our next aim is hence to discuss some terminology which will enable us to describe the appropriate equivalence relation on the class of such $(n + 2)$-term sequences.

Definition 2.3. Let $\mathbf{C}_{\mathcal{C}}$ denote the category of complexes in $\mathcal{C}$. We define $\mathbf{C}^{n+2}_{\mathcal{C}}$ to be the full subcategory of $\mathbf{C}_{\mathcal{C}}$ consisting of complexes whose components are zero in all degrees outside of $\{0, 1, \ldots, n + 1\}$. In other words, an object in $\mathbf{C}^{n+2}_{\mathcal{C}}$ is a complex $X_\bullet = \{X_i, d_i\}$ of the form

$$X_0 \xrightarrow{d_0} X_1 \to \cdots \to X_n \xrightarrow{d_n} X_{n+1}.$$

Morphisms in $\mathbf{C}^{n+2}_{\mathcal{C}}$ are written $f_\bullet = (f_0, f_1, \ldots, f_{n+1})$, where we only indicate the terms of degree $0, 1, \ldots, n + 1$.

Our next two definitions should remind the reader about the long exact Hom-Ext-sequence associated to a short exact sequence and the long exact Hom-sequence associated to a distinguished triangle.

Definition 2.4. By the Yoneda lemma, an extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations

$$\delta_\sharp\colon \mathcal{C}(-, C) \to \mathbb{E}(-, A) \text{ and } \delta^\sharp\colon \mathcal{C}(A, -) \to \mathbb{E}(C, -).$$

For an object X in $\mathcal{C}$, the morphisms $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given by

(1) $(\delta_\sharp)_X\colon \mathcal{C}(X, C) \to \mathbb{E}(X, A)$, $f \mapsto f^*\delta$;
(2) $\delta^\sharp_X\colon \mathcal{C}(A, X) \to \mathbb{E}(C, X)$, $g \mapsto g_*\delta$.

Consider a pair $\langle X_\bullet, \delta\rangle$ with $X_\bullet$ in $\mathbf{C}^{n+2}_{\mathcal{C}}$ and $\delta \in \mathbb{E}(X_{n+1}, X_0)$. Using our natural transformations from above, we can associate to $\langle X_\bullet, \delta\rangle$ the following two sequences of functors:

(1) $\mathcal{C}(-,X_0) \xrightarrow{\mathcal{C}(-,d_0)} \cdots \xrightarrow{\mathcal{C}(-,d_n)} \mathcal{C}(-,X_{n+1}) \xrightarrow{\ \ \delta_\sharp\ \ } \mathbb{E}(-,X_0)$;

(2) $\mathcal{C}(X_{n+1},-) \xrightarrow{\mathcal{C}(d_n,-)} \cdots \xrightarrow{\mathcal{C}(d_0,-)} \mathcal{C}(X_0,-) \xrightarrow{\ \ \delta^\sharp\ \ } \mathbb{E}(X_{n+1},-)$.

We are particularly interested in pairs $\langle X_\bullet, \delta\rangle$ for which these sequences are exact.

Definition 2.5. When the two sequences of functors from above are exact, we say that the pair $\langle X_\bullet, \delta\rangle$ is an *n-exangle*. Given two n-exangles $\langle X_\bullet, \delta\rangle$ and $\langle Y_\bullet, \rho\rangle$, a *morphism of n-exangles* $f_\bullet \colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, \rho\rangle$ is a chain map $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{C}}(X_\bullet, Y_\bullet)$ for which $(f_0, f_{n+1})\colon \delta \to \rho$ is also a morphism of extensions.

In order to define our equivalence classes of $(n+2)$-term sequences, we need a notion of homotopy. Two morphisms in $\mathbf{C}^{n+2}_{\mathcal{C}}$ are said to be *homotopic* if they are homotopic as morphisms of $\mathbf{C}_{\mathcal{C}}$ in the usual way. We let the homotopy category $\mathbf{K}^{n+2}_{\mathcal{C}}$ be the quotient of $\mathbf{C}^{n+2}_{\mathcal{C}}$ by the ideal of null-homotopic morphisms.

Instead of working with $\mathbf{C}^{n+2}_{\mathcal{C}}$ and $\mathbf{K}^{n+2}_{\mathcal{C}}$, we want to fix the end-terms of our sequences.

Definition 2.6. Let A and C be objects in $\mathcal{C}$. We define $\mathbf{C}^{n+2}_{(\mathcal{C};A,C)}$ to be the subcategory of $\mathbf{C}^{n+2}_{\mathcal{C}}$ consisting of complexes $X_\bullet$ with $X_0 = A$ and $X_{n+1} = C$. Morphisms in $\mathbf{C}^{n+2}_{(\mathcal{C};A,C)}$ are given by chain maps $f_\bullet$ for which $f_0 = 1_A$ and $f_{n+1} = 1_C$.

Whenever the category $\mathcal{C}$ is clear from the context, we abbreviately denote $\mathbf{C}^{n+2}_{(\mathcal{C};A,C)}$ by $\mathbf{C}^{n+2}_{(A,C)}$. Notice that $\mathbf{C}^{n+2}_{(A,C)}$ is no longer an additive category. However, we can still take the quotient of $\mathbf{C}^{n+2}_{(A,C)}$ by the same homotopy relation as in $\mathbf{C}^{n+2}_{\mathcal{C}}$. This yields $\mathbf{K}^{n+2}_{(A,C)}$, which is a subcategory of $\mathbf{K}^{n+2}_{\mathcal{C}}$.

We are now ready to describe an equivalence relation on the class of $(n+2)$-term sequences starting in A and ending in C.

Definition 2.7. A morphism $f_\bullet \in \mathbf{C}^{n+2}_{(A,C)}(X_\bullet, Y_\bullet)$ is called a *homotopy equivalence* if it induces an isomorphism in $\mathbf{K}^{n+2}_{(A,C)}$. Two objects $X_\bullet$ and $Y_\bullet$ in $\mathbf{C}^{n+2}_{(A,C)}$ are called *homotopy equivalent* if there is some homotopy equivalence between them. We denote the homotopy equivalence class of $X_\bullet$ by $[X_\bullet]$.

It should be noted that homotopy equivalence classes taken in $\mathbf{C}^{n+2}_{(A,C)}$ and in $\mathbf{C}^{n+2}_{\mathcal{C}}$ may be different. We will only use the notation $[X_\bullet]$ for equivalence classes taken in $\mathbf{C}^{n+2}_{(A,C)}$.

We are now ready to explain our desired connection between extensions ${}_A\delta_C$ and equivalence classes $[X_\bullet]$ in $\mathbf{C}^{n+2}_{(A,C)}$.

Definition 2.8. Let $\mathfrak{s}$ be a correspondence which associates a homotopy equivalence class $\mathfrak{s}(\delta) = [X_\bullet]$ in $\mathbf{C}^{n+2}_{(A,C)}$ to each extension $\delta \in \mathbb{E}(C,A)$. We call $\mathfrak{s}$ a *realization of* $\mathbb{E}$ if it satisfies the following condition for any $\mathfrak{s}(\delta) = [X_\bullet]$ and $\mathfrak{s}(\rho) = [Y_\bullet]$:

(R0) Given any morphism of extensions $(a,c)\colon \delta \to \rho$, there exists a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{C}}(X_\bullet, Y_\bullet)$ of the form $f_\bullet = (a, f_1, \ldots, f_n, c)$. Such an $f_\bullet$ is called a *lift* of (a,c).

Whenever $\mathfrak{s}(\delta) = [X_\bullet]$, we say that $X_\bullet$ *realizes* δ. A realization $\mathfrak{s}$ is called *exact* if in addition the following conditions hold:

(R1) Given any $\mathfrak{s}(\delta) = [X_\bullet]$, the pair $\langle X_\bullet, \delta\rangle$ is an n-exangle.

(R2) Given any object A in $\mathcal{C}$, we have

$$\mathfrak{s}({}_A0_0) = [A \xrightarrow{1_A} A \to 0 \to \cdots \to 0],$$

and dually

$$\mathfrak{s}({}_00_A) = [0 \to \cdots \to 0 \to A \xrightarrow{1_A} A].$$

It is not immediately clear that the condition (R1) does not depend on our choice of representative of the class $[X_\bullet]$. For this fact, see [5, Proposition 2.16].

Based on the definition above, we can introduce some useful terminology.

Definition 2.9. Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$.

(1) An n-exangle $\langle X_\bullet, \delta\rangle$ will be called a *distinguished* n*-exangle* if $\mathfrak{s}(\delta) = [X_\bullet]$.

(2) An object $X_\bullet \in \mathbf{C}^{n+2}_{\mathcal{C}}$ will be called a *conflation* if it realizes some extension $\delta \in \mathbb{E}(X_{n+1}, X_0)$.

(3) A morphism f in $\mathcal{C}$ will be called an *inflation* if there exists some conflation $X_\bullet = \{X_i, d_i\}$ satisfying $d_0 = f$.

(4) A morphism g in $\mathcal{C}$ will be called a *deflation* if there exists some conflation $X_\bullet = \{X_i, d_i\}$ satisfying $d_n = g$.

Recall that for triangulated categories, the octahedral axiom can be replaced by a mapping cone axiom [14, 15]. This should be thought of as a background for the definition of an n-exangulated category. Before we can give the definition, we need the notion of a mapping cone in our context.

Definition 2.10. Let $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{C}}(X_\bullet, Y_\bullet)$ be a morphism with $f_0 = 1_A$ for some object $A = X_0 = Y_0$ in $\mathcal{C}$. The *mapping cone* of $f_\bullet$ is the complex $M^f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{C}}$ given by

$$X_1 \xrightarrow{d_0} X_2 \oplus Y_1 \xrightarrow{d_1} X_3 \oplus Y_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} X_{n+1} \oplus Y_n \xrightarrow{d_n} Y_{n+1},$$

where

$$d_i = \begin{cases} \begin{bmatrix} -d^X_1 \\ f_1 \end{bmatrix} & \text{if } i = 0 \\ \begin{bmatrix} -d^X_{i+1} & 0 \\ f_{i+1} & d^Y_i \end{bmatrix} & \text{if } i = 1, 2, \ldots, n-1 \\ \begin{bmatrix} f_{n+1} & d^Y_n \end{bmatrix} & \text{if } i = n. \end{cases}$$

The *mapping cocone* of a morphism $g_\bullet$ where g_{n+1} is the identity on some object, is defined dually.

Definition 2.11. An *n-exangulated category* is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ of an additive category $\mathcal{C}$, a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ and an exact realization $\mathfrak{s}$ of $\mathbb{E}$, satisfying the following axioms:

(EA1) The class of inflations in $\mathcal{C}$ is closed under composition. Dually, the class of deflations in $\mathcal{C}$ is closed under composition.

(EA2) For an extension $\delta \in \mathbb{E}(D, A)$ and a morphism $c \in \mathcal{C}(C, D)$, let $\langle X_\bullet, c^*\delta\rangle$ and $\langle Y_\bullet, \delta\rangle$ be distinguished n-exangles. Then there exists a *good lift* $f_\bullet$ of $(1_A, c)$, meaning that the mapping cone of $f_\bullet$ gives a distinguished n-exangle $\langle M^f_\bullet, (d^X_0)_*\delta\rangle$.

(EA2)$^{\mathrm{op}}$ Dual of (EA2).

The condition (EA2) is actually independent of choice of representatives of the equivalence classes $[X_\bullet]$ and $[Y_\bullet]$, see [5, Corollary 2.31]. Note that we will often not mention $\mathbb{E}$ and $\mathfrak{s}$ explicitly when we talk about an n-exangulated category $\mathcal{C}$.

Not too surprisingly, a 1-exangulated category is the same as an extriangulated category [5, Proposition 4.3]. It should also be noted that n-exact and $(n+2)$-angulated categories are n-exangulated [5, Proposition 4.34 and 4.5]. For a discussion of examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [5, Section 6.3] and [11, Section 4].

In our study of subcategories of n-exangulated categories in Section 3 and Section 5, the notion of extension-closed subcategories will be relevant.

Definition 2.12. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. A full additive subcategory $\mathcal{S} \subseteq \mathcal{C}$ which is closed under isomorphisms is called *extension-closed* if for any pair of objects A and C in $\mathcal{S}$ and any extension $\delta \in \mathbb{E}(C, A)$, there is a distinguished n-exangle $\langle X_\bullet, \delta\rangle$ with X_i in $\mathcal{S}$ for $i = 1, \dots, n$.

Extension-closed subcategories inherit structure from the ambient category in a natural way. The following result is [5, Proposition 2.35].

Proposition 2.13. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category and $\mathcal{S}$ an extension-closed subcategory of $\mathcal{C}$. Given objects A and C in $\mathcal{S}$ and an extension $\delta \in \mathbb{E}(C, A)$, let $\langle X_\bullet, \delta\rangle$ be a distinguished n-exangle with X_i in $\mathcal{S}$ for $i = 1, \dots, n$. Define $\mathfrak{t}(\delta) = [X_\bullet]$, where the equivalence class is taken in $\boldsymbol{C}^{n+2}_{(\mathcal{S};A,C)}$. The following statements hold:*

(1) The correspondence $\mathfrak{t}$ is an exact realization of the restricted functor $\mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}$, and $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}, \mathfrak{t})$ satisfies (EA2) and (EA2)$^{\mathrm{op}}$.

(2) If $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}, \mathfrak{t})$ satisfies (EA1), then it is an n-exangulated category.

We end this section by reviewing two results which will be needed throughout the rest of this paper. The following proposition should be well-known, but we include a proof as we lack an explicit reference. The conflations described in Proposition 2.14 are called *trivial*.

Proposition 2.14. *Let $\mathcal{C}$ be an n-exangulated category and A an object in $\mathcal{C}$. Then the $(n+2)$-term sequence*

$$0 \to \cdots \to 0 \to A \xrightarrow{1_A} A \to 0 \to \cdots \to 0$$

which has A in position i and $i+1$ for some $i \in \{0, 1, \ldots, n\}$ is a conflation.

Proof. By (R2), the statement is true if $i = 0$ or $i = n$. We can hence assume that both our end-terms are zero. Now, our sequence is homotopy equivalent in $\mathbf{C}^{n+2}_{(0,0)}$ to

$$0 \to 0 \to \cdots \to 0 \to 0.$$

As this sequence is a conflation, again by (R2), also the sequence we started with has to be a conflation. □

As one might expect, the coproduct of two conflations is again a conflation. For a proof of this result, see [5, Proposition 3.2].

Proposition 2.15. *Let $\mathcal{C}$ be an n-exangulated category and $X_\bullet$ and $Y_\bullet$ conflations in $\mathcal{C}$. Then also $X_\bullet \oplus Y_\bullet$ is a conflation.*

3. Subcategories and n-(co)generators

In this section we introduce the terminology which is needed in our main result, such as the notion of an n-(co)generator, complete subcategories and dense subcategories. We also define n-exangulated subcategories, and show that the subcategories which will appear in our classification theorem carry this structure.

Definition 3.1. Let $\mathcal{C}$ be an n-exangulated category. A full additive subcategory $\mathcal{G}$ of $\mathcal{C}$ is called an *n-generator* (resp. *n-cogenerator*) of $\mathcal{C}$ if for each object A in $\mathcal{C}$, there exists a conflation

$$A' \to G_1 \to \cdots \to G_n \to A$$
$$(\text{Resp. } A \to G_1 \to \cdots \to G_n \to A')$$

in $\mathcal{C}$ with G_i in $\mathcal{G}$ for $i = 1, \ldots, n$.

A 1-(co)generator is often just called a (co)generator. Our notion of a (co)generator essentially coincides with what is used in [12] and [17]. There, however, it is not assumed that the subcategory $\mathcal{G}$ is additive. Note that it would be possible to prove our results also without this extra assumption, but we have chosen this convention to simplify the statement in Proposition 4.3.

We get a trivial example of an n-(co)generator by choosing $\mathcal{G}$ to be the entire category $\mathcal{C}$. Another natural example arises if our category has enough projectives or injectives. Let us first recall what this means from [11, Definition 3.2].

Definition 3.2. Let $\mathcal{C}$ be an n-exangulated category.

(1) An object P in $\mathcal{C}$ is called *projective* if for any conflation
$$X_0 \xrightarrow{d_0} X_1 \to \cdots \to X_n \xrightarrow{d_n} X_{n+1}$$
in $\mathcal{C}$ and any morphism $f\colon P \to X_{n+1}$, there exists a morphism $g\colon P \to X_n$ such that $d_n \circ g = f$.

(2) The category $\mathcal{C}$ *has enough projectives* if for each object A in $\mathcal{C}$, there exists a conflation
$$A' \to P_1 \to \cdots \to P_n \to A$$
in $\mathcal{C}$ with P_i projective for $i = 1, \ldots, n$.

(3) We define *injective objects* and the notion of having *enough injectives* dually.

The notion of having enough projectives or injectives relates well to our definition of an n-(co)generator, as demonstrated in the example below.

Example 3.3. Let $\mathcal{C}$ be an n-exangulated category. If $\mathcal{C}$ has enough projectives, then the full subcategory $\mathcal{P} \subseteq \mathcal{C}$ of projective objects is an n-generator of $\mathcal{C}$. Dually, if $\mathcal{C}$ has enough injectives, the full subcategory $\mathcal{I} \subseteq \mathcal{C}$ of injective objects is an n-cogenerator of $\mathcal{C}$. In the case where $\mathcal{C}$ is a Frobenius n-exangulated category, as defined in [11], the subcategory $\mathcal{P} = \mathcal{I}$ is both an n-generator and an n-cogenerator of $\mathcal{C}$.

We will classify subcategories of an n-exangulated category which are dense and complete.

Definition 3.4. Let $\mathcal{C}$ be an n-exangulated category and $\mathcal{S}$ a full subcategory of $\mathcal{C}$.

(1) The subcategory $\mathcal{S}$ is *dense* in $\mathcal{C}$ if each object in $\mathcal{C}$ is a summand of an object in $\mathcal{S}$.

(2) The subcategory $\mathcal{S}$ is *complete* if given any conflation in $\mathcal{C}$ with $n+1$ of its objects in $\mathcal{S}$, also the last object has to be in $\mathcal{S}$.

Even though it is not a part of the definition, it turns out that given reasonable conditions, complete subcategories are always additive and closed under isomorphisms.

Lemma 3.5. *Let $\mathcal{C}$ be an n-exangulated category. Every complete subcategory $\mathcal{S}$ of $\mathcal{C}$ which contains 0 is additive and closed under isomorphisms.*

Proof. Let A and B be objects in $\mathcal{S}$. By taking the coproduct of two trivial conflations, we get the conflation
$$A \to A \oplus B \to B \to 0 \to \cdots \to 0.$$
As 0 is in $\mathcal{S}$, all objects in this sequence except the second one is in $\mathcal{S}$. By completeness, this means that also $A \oplus B$ is in $\mathcal{S}$, which shows additivity.

Given an isomorphism $A \xrightarrow{\cong} B$ in $\mathcal{C}$, the $(n+2)$-term sequence
$$A \xrightarrow{\cong} B \to 0 \to \cdots \to 0$$
is a conflation in $\mathcal{C}$, as it is equivalent to a trivial conflation. Consequently, if A is in $\mathcal{S}$, then also B has to be there, so $\mathcal{S}$ is closed under isomorphisms. $\square$

Notice that when a subcategory $\mathcal{S}$ of an n-exangulated category is dense, it is automatically non-empty. Whenever n is odd and $\mathcal{S}$ is both dense and complete, our subcategory necessarily contains 0. This can be seen by taking an object A in $\mathcal{S}$ and using completeness with respect to the conflation

$$A \xrightarrow{1_A} A \xrightarrow{0} A \xrightarrow{1_A} \cdots \xrightarrow{0} A \xrightarrow{1_A} A \to 0,$$

which is a sum of trivial conflations, and in which the last object is the only one not equal to A. Consequently, dense and complete subcategories are always additive and isomorphism-closed when n is odd, which will often be the case in our further work. We will show that a stronger statement is true, namely that every such subcategory is actually an *n-exangulated subcategory* of the ambient category. The key requirement of an n-exangulated subcategory is that the inclusion is an n-exangulated functor, as introduced in [1, Definition 2.31].

Definition 3.6. Let $(\mathcal{C}_1, \mathbb{E}_1, \mathfrak{s}_1)$ and $(\mathcal{C}_2, \mathbb{E}_2, \mathfrak{s}_2)$ be n-exangulated categories. An additive functor $F\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is an *n-exangulated functor* if there is a natural transformation $\eta\colon \mathbb{E}_1 \longrightarrow \mathbb{E}_2(F^{\mathrm{op}}-, F-)$ such that if $\mathfrak{s}_1(\delta) = [X_\bullet]$ for some $\delta \in \mathbb{E}_1(C, A)$, then $\mathfrak{s}_2({}_A\eta_C(\delta)) = [FX_\bullet]$.

Notice that the notation ${}_A\eta_C$ is used for the group homomorphism

$${}_A\eta_C\colon \mathbb{E}_1(C, A) \longrightarrow \mathbb{E}_2(FC, FA) = \mathbb{E}_2(F^{\mathrm{op}}C, FA)$$

given by the natural transformation η. We call η an *inclusion* if ${}_A\eta_C$ is an inclusion of abelian groups for every pair of objects A and C.

We are now ready to give the definition of an n-exangulated subcategory.

Definition 3.7. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. An *n-exangulated subcategory* of $\mathcal{C}$ is a full isomorphism-closed subcategory $\mathcal{S}$ which carries an n-exangulated structure $(\mathcal{S}, \mathbb{E}', \mathfrak{s}')$ for which the inclusion functor is n-exangulated and the associated natural transformation is an inclusion.

Our definition emphasizes that an n-exangulated subcategory *inherits* the structure of the ambient category. In particular, the biadditive functor $\mathbb{E}'$ is an additive subfunctor of the restricted functor $\mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}$ in the sense of [5, Definition 3.6]. The exact realizations $\mathfrak{s}$ and $\mathfrak{s}'$ agree, meaning that if $\mathfrak{s}'(\delta) = [X_\bullet]$ for some $\delta \in \mathbb{E}'(C, A) \subseteq \mathbb{E}(C, A)$, then $\mathfrak{s}(\delta) = [X_\bullet]$. Notice that the first equivalence class is taken in $\mathbf{C}^{n+2}_{(\mathcal{S};A,C)}$, while the second is taken in $\mathbf{C}^{n+2}_{(\mathcal{C};A,C)}$. In the case $n = 1$, the subcategories defined above should be called *extriangulated subcategories*.

For our applications in Section 5, the most important class of examples of n-exangulated subcategories will arise from extension-closed subcategories. In this case we have $\mathbb{E}' = \mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}$. We also give a basic example where $\mathbb{E}'$ is a proper subfunctor.

Example 3.8. (1) Let $\mathcal{S}$ be an extension-closed subcategory of an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and define $\mathfrak{t}$ as explained in Proposition 2.13. If the triplet $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}, \mathfrak{t})$ satisfies (EA1), then $\mathcal{S}$ is an n-exangulated subcategory of $\mathcal{C}$. Notice that the natural transformation η associated to the inclusion functor is given by ${}_A\eta_C = 1_{\mathbb{E}(C,A)}$ for objects A and C in $\mathcal{S}$.

(2) Let $\mathcal{C} = Ab$ be the category of abelian groups. This is an extriangulated category with biadditive functor $\mathbb{E} = \operatorname{Ext}^1_{\mathcal{C}}$. Let $\mathcal{S} \subseteq \mathcal{C}$ denote the subcategory of semisimple objects. Using that $\mathcal{S}$ is closed under kernels and cokernels, one can check that $\mathcal{S}$ is an abelian subcategory of $\mathcal{C}$. Consequently, one obtains that $\mathcal{S}$ is an extriangulated subcategory with biadditive functor $\mathbb{E}' = \operatorname{Ext}^1_{\mathcal{S}}$. As $\mathcal{S}$ is not extension-closed in $\mathcal{C}$, we can see that $\mathbb{E}'$ is a proper subfunctor of $\mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}$.

Let us finish this section by showing that if n is odd, every dense and complete subcategory of an n-exangulated category is an n-exangulated subcategory.

Proposition 3.9. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category with n odd and $\mathcal{S}$ a dense and complete subcategory of $\mathcal{C}$. The following statements hold:*

(1) The subcategory $\mathcal{S}$ is extension-closed.
(2) The triplet $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}^{\mathrm{op}}\times\mathcal{S}}, \mathfrak{t})$, with $\mathfrak{t}$ as defined in Proposition 2.13, is an n-exangulated subcategory of $\mathcal{C}$.

Proof. As n is odd, it follows from Lemma 3.5 that the subcategory $\mathcal{S}$ is additive and isomorphism-closed.

Let A and C be objects in $\mathcal{S}$ and consider an extension $\delta \in \mathbb{E}(C, A)$. As $\mathcal{C}$ is n-exangulated, there is a distinguished n-exangle $\langle X_\bullet, \delta\rangle$ in $\mathcal{C}$ with $X_\bullet$ given by

$$A \to X_1 \to \cdots \to X_n \to C.$$

The objects X_i are not necessarily contained in $\mathcal{S}$, but we will show that we can pick another representative of the equivalence class $[X_\bullet]$ for which this is satisfied.

For $i = 1, \ldots, n-1$, use that $\mathcal{S}$ is dense and let X_i' be an object such that $X_i \oplus X_i'$ is in $\mathcal{S}$. By adding trivial conflations involving the objects X_i and X_i' to the conflation above, we get a new conflation

$$A \to X_1 \oplus X_1' \to \cdots \to \bigoplus_{i=1}^{n-1}(X_i \oplus X_i') \to \overline{X} \to C,$$

where $\overline{X} = X_1 \oplus X_2' \oplus X_3 \oplus \cdots \oplus X_{n-1}' \oplus X_n$. Notice that each of the trivial conflations we have added are equivalent to the zero conflation, i.e. the conflation given by the $(n+2)$-term sequence where every object is zero. Hence, our new conflation represents the same equivalence class as the one we started with.

It remains to observe that every object in our new conflation except possibly $\overline{X}$ is contained in $\mathcal{S}$. As $\mathcal{S}$ is complete, this means that also $\overline{X}$ is in $\mathcal{S}$, which proves *(1)*.

For *(2)*, notice that by Proposition 2.13 and Example 3.8 it is enough to verify that (EA1) is satisfied. Let f and g be two composable inflations in $\mathcal{S}$. By the definition of $\mathfrak{t}$, inflations in $\mathcal{S}$ are also inflations in $\mathcal{C}$. As $\mathcal{C}$ satisfies (EA1), there is a conflation

$$X_0 \xrightarrow{f\circ g} X_1 \to \cdots \to X_n \to X_{n+1}$$

in $\mathcal{C}$. By assumption, we know that X_0 and X_1 are in $\mathcal{S}$, but the same is not necessarily true for the last n objects. However, we apply a similar technique as above to get a conflation where all the objects are in $\mathcal{S}$. For $i = 2, \dots, n$, let X_i' be an object such that $X_i \oplus X_i'$ is in $\mathcal{S}$. Adding trivial conflations to the conflation above yields the conflation

$$X_0 \xrightarrow{f \circ g} X_1 \to X_2 \oplus X_2' \to \cdots \to \bigoplus_{i=2}^{n} (X_i \oplus X_i') \to \overline{X},$$

where $\overline{X}$ now denotes the object $X_2 \oplus X_3' \oplus X_4 \oplus \cdots \oplus X_n' \oplus X_{n+1}$. As the first $n+1$ objects in this conflation are in $\mathcal{S}$, so is $\overline{X}$. Consequently, this is a conflation in $\mathcal{S}$, which shows that $f \circ g$ is an inflation in $\mathcal{S}$. A dual argument shows that the class of deflations in $\mathcal{S}$ is closed under composition. □

4. The Grothendieck group of an n-exangulated category

Throughout the rest of this paper, we let $\mathcal{C}$ be an essentially small category. Hence, the collection of isomorphism classes $\langle A \rangle$ of objects A in $\mathcal{C}$ forms a set, and we can consider the free abelian group $\mathcal{F}(\mathcal{C})$ generated by such isomorphism classes. We will define the Grothendieck group of an n-exangulated category $\mathcal{C}$ to be a certain quotient of this free abelian group. More precisely, we want to factor out the Euler relations coming from conflations. Given a conflation

$$X_\bullet \colon X_0 \to X_1 \to \cdots \to X_n \to X_{n+1}$$

in $\mathcal{C}$, the corresponding Euler relation is the alternating sum of isomorphism classes

$$\chi(X_\bullet) = \langle X_0 \rangle - \langle X_1 \rangle + \cdots + (-1)^{n+1} \langle X_{n+1} \rangle.$$

Definition 4.1. Let $\mathcal{C}$ be an n-exangulated category. The *Grothendieck group* of $\mathcal{C}$ is the quotient $K_0(\mathcal{C}) = \mathcal{F}(\mathcal{C}) / \mathcal{R}(\mathcal{C})$, where $\mathcal{R}(\mathcal{C})$ is the subgroup generated by the subset

$$\{\chi(X_\bullet) \mid X_\bullet \text{ is a conflation in } \mathcal{C}\} \text{ if } n \text{ is odd and}$$
$$\{\langle 0 \rangle\} \cup \{\chi(X_\bullet) \mid X_\bullet \text{ is a conflation in } \mathcal{C}\} \text{ if } n \text{ is even.}$$

We denote the equivalence class $\langle A \rangle + \mathcal{R}(\mathcal{C})$ represented by an object A in $\mathcal{C}$ by $[A]$.

It is immediate from the definition that the Grothendieck group $K_0(\mathcal{C})$ has a universal property. Namely, any homomorphism of abelian groups from $\mathcal{F}(\mathcal{C})$ satisfying the Euler relations factors uniquely through $K_0(\mathcal{C})$. More precisely, given any abelian group T and a homomorphism $t \colon \mathcal{F}(\mathcal{C}) \to T$ with $t(\mathcal{R}(\mathcal{C})) = 0$, there exists a unique homomorphism t' such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(\mathcal{C}) \arrow[r, "\pi"] \arrow[dr, "t"'] & K_0(\mathcal{C}) \arrow[d, dotted, "t'"] \\ & T, \end{tikzcd}$$

where π is the natural projection.

Let us prove some basic properties of the Grothendieck group of an n-exangulated category. These properties are well-known in the cases where our category is triangulated or exact. Note that $\langle 0\rangle$ was defined to be in $\mathcal{R}(\mathcal{C})$ whenever n is even in order for the following proposition to hold.

Proposition 4.2. *Let $\mathcal{C}$ be an n-exangulated category.*

(1) The zero element in $K_0(\mathcal{C})$ is given by $[0]$, where 0 is the zero object in $\mathcal{C}$.
(2) For objects A and B in $\mathcal{C}$, we have $[A \oplus B] = [A] + [B]$ in $K_0(\mathcal{C})$.

Proof. If n is even, the definition of $\mathcal{R}(\mathcal{C})$ immediately implies that $[0]$ is the zero element in $K_0(\mathcal{C})$.

Recall that the $(n+2)$-term sequence

$$0 \to 0 \to \cdots \to 0 \to 0$$

is a conflation in $\mathcal{C}$ by (R2). Consequently, the sum $\sum_{i=0}^{n+1}(-1)^i\langle 0\rangle$ is in $\mathcal{R}(\mathcal{C})$. If n is odd, this sum is equal to $\langle 0\rangle$, and hence $[0]$ is the zero element in $K_0(\mathcal{C})$ also in this case. This shows *(1)*.

For *(2)*, consider the sequence

$$A \to A \oplus B \to B \to 0 \to \cdots \to 0$$

with $n+2$ terms. This sequence is a conflation in $\mathcal{C}$ as it is a sum of two trivial conflations. Using *(1)*, this implies that

$$\langle A\rangle - \langle A \oplus B\rangle + \langle B\rangle \in \mathcal{R}(\mathcal{C}),$$

which yields $[A \oplus B] = [A] + [B]$ in $K_0(\mathcal{C})$. □

Notice that any element in $K_0(\mathcal{C})$ can be written as $[A] - [B]$ for some objects A and B in $\mathcal{C}$, as we can collect positive and negative terms and then use the second part of the proposition above. In the case where n is odd and our category has an n-(co)generator, we get an even nicer description.

Proposition 4.3. *Let $\mathcal{C}$ be an n-exangulated category with n odd. Let $\mathcal{G}$ be an n-(co)generator of $\mathcal{C}$. Then every element in $K_0(\mathcal{C})$ can be written as $[A] - [G]$ for some objects A in $\mathcal{C}$ and G in $\mathcal{G}$.*

Proof. Given an element in $K_0(\mathcal{C})$, we know that it can be written as $[X] - [B]$ for some objects X and B in $\mathcal{C}$. When $\mathcal{G}$ is an n-generator, there exists a conflation

$$B' \to G_1 \to \cdots \to G_n \to B$$

in $\mathcal{C}$ with G_i in $\mathcal{G}$ for $i = 1, \dots, n$. Consequently, using that n is odd, we get

$$(*) \qquad [B] = -[B'] + [G_1] - [G_2] + \cdots - [G_{n-1}] + [G_n]$$

in $K_0(\mathcal{C})$. Substituting this expression for $[B]$, the element we started with can be written as

$$\begin{aligned}[X] - [B] &= [X] + [B'] - [G_1] + [G_2] - \cdots + [G_{n-1}] - [G_n] \\ &= [X \oplus B' \oplus G_2 \oplus G_4 \oplus \cdots \oplus G_{n-1}] - [G_1 \oplus G_3 \oplus \cdots \oplus G_n],\end{aligned}$$

where we have collected positive and negative terms from the alternating sum and used Proposition 4.2. Defining A and G to be the objects in the first and second bracket respectively, we get that our element can be written as $[A] - [G]$. Note that as $\mathcal{G}$ is additive, the object G is contained in $\mathcal{G}$.

The proof in the case where $\mathcal{G}$ is an n-cogenerator is dual. □

Note that it was important in the argument above that n was assumed to be odd. If n was even, there would be no negative sign in front of the term $[B']$ in the expression $(*)$. Hence, the signs of $[X]$ and $[B']$ in our final equation would be different, and we would not reach our conclusion.

The description of elements in the Grothendieck group which is provided in Proposition 4.3 will be important in our further work. In the following, we will thus often need to assume that n is odd.

Remark 4.4. Proposition 4.3 is an n-exangulated analogue of a result from [12] concerning exact categories, which can be found in the proof of Lemma 2.8. As an $(n+2)$-angulated category has $\mathcal{G} = \{0\}$ as an n-(co)generator, Proposition 4.3 can also be thought of as a generalization of part *(3)* of [3, Proposition 2.2].

5. Classification of subcategories

Recall that $\mathcal{C}$ is assumed to be essentially small. In this section we state and prove our main result. For n odd we classify dense complete subcategories of an n-exangulated category with an n-(co)generator $\mathcal{G}$ in terms of subgroups of the Grothendieck group. The subgroups which appear in the bijection, depend on the n-(co)generator. More precisely, the subgroups have to contain

$$H_{\mathcal{G}} = \langle [G] \in K_0(\mathcal{C}) \mid G \in \mathcal{G} \rangle \leq K_0(\mathcal{C}),$$

i.e. the subgroup of $K_0(\mathcal{C})$ generated by elements represented by objects in $\mathcal{G}$. When a subgroup of $K_0(\mathcal{C})$ contains $H_{\mathcal{G}}$, we say that it contains the image of $\mathcal{G}$.

Theorem 5.1. *Let $\mathcal{C}$ be an n-exangulated category with n odd. Let $\mathcal{G}$ be an n-(co)generator of $\mathcal{C}$. There is then a one-to-one correspondence*

$$\left\{\begin{matrix}\text{subgroups of } K_0(\mathcal{C}) \\ \text{containing } H_{\mathcal{G}}\end{matrix}\right\} \overset{f}{\underset{g}{\rightleftarrows}} \left\{\begin{matrix}\text{dense complete subcategories} \\ \text{of } \mathcal{C} \text{ containing } \mathcal{G}\end{matrix}\right\},$$

where $f(H)$ is the full subcategory

$$f(H) = \{A \in \mathcal{C} \mid [A] \in H\} \subseteq \mathcal{C},$$

and $g(\mathcal{S})$ is the subgroup

$$g(\mathcal{S}) = \langle [A] \in K_0(\mathcal{C}) \mid A \in \mathcal{S} \rangle \leq K_0(\mathcal{C}).$$

Remark 5.2. The subcategories in our bijection above are n-exangulated subcategories of $\mathcal{C}$, where the n-exangulated structure is inherited from that of $\mathcal{C}$ as described in Proposition 3.9.

Proof of Theorem 5.1. We prove the theorem in the case where $\mathcal{G}$ is an n-generator. The proof when $\mathcal{G}$ is an n-cogenerator is dual.

Throughout the rest of the proof, let $\mathcal{S}$ be a dense complete subcategory of $\mathcal{C}$ containing $\mathcal{G}$ and H a subgroup of $K_0(\mathcal{C})$ containing $H_{\mathcal{G}}$. Let us first verify that the maps f and g actually end up where we claim.

Note that $g(\mathcal{S})$ is a subgroup of $K_0(\mathcal{C})$ by definition. As $\mathcal{S}$ contains $\mathcal{G}$, the subgroup $H_{\mathcal{G}}$ is contained in $g(\mathcal{S})$. Similarly, it is clear that $\mathcal{G} \subseteq f(H)$. To see that $f(H)$ is a dense subcategory, let A be an object in $\mathcal{C}$. As $\mathcal{G}$ is an n-generator, there is a conflation

$$A' \to G_1 \to \cdots \to G_n \to A$$

in $\mathcal{C}$ with G_i in $\mathcal{G}$ for $i = 1, \ldots, n$. Using that n is odd, which implies that the signs in front of $[A]$ and $[A']$ in the corresponding Euler relation agree, we get

$$[A \oplus A'] = [G_1] - [G_2] + \cdots - [G_{n-1}] + [G_n] \in H.$$

This means that $A \oplus A'$ is in $f(H)$, so the subcategory is dense in $\mathcal{C}$. To show completeness, consider a conflation

$$X_0 \to X_1 \to \cdots \to X_n \to X_{n+1}$$

in $\mathcal{C}$, where $n+1$ of the $n+2$ objects are in $f(H)$. Since

$$[X_0] - [X_1] + \cdots + (-1)^{n+1}[X_{n+1}] = 0 \in H,$$

and $n+1$ of the terms in this sum are in H, also the last term has to be there. This means that also the last object of our conflation above is in $f(H)$, so $f(H)$ is complete.

Our next step is to show that f and g are inverse bijections. The inclusion $gf(H) \subseteq H$ is immediate. For the reverse inclusion, choose an element in H. By Proposition 4.3, our element can be written as $[A]-[G]$ for some A in $\mathcal{C}$ and G in $\mathcal{G}$. As $[A] = ([A]-[G])+[G]$, and both $[A]-[G]$ and $[G]$ are in H, so is $[A]$. Hence, our element is contained in $gf(H)$, and we can conclude that $H = gf(H)$.

It remains to show that $\mathcal{S} = fg(\mathcal{S})$. Again, one of the inclusions is clear from the definitions, namely $\mathcal{S} \subseteq fg(\mathcal{S})$. For the reverse inclusion, choose an object A in $fg(\mathcal{S})$. This means that $[A]$ is in $g(\mathcal{S})$. By Lemma 5.4 below, our object A is consequently in $\mathcal{S}$, which completes our proof. $\square$

We will prove Lemma 5.4 by showing that the quotient $K_0(\mathcal{C})/g(\mathcal{S})$ is isomorphic to another group $G_{\mathcal{S}}$ consisting of equivalence classes.

Given an n-exangulated category $\mathcal{C}$ with n odd and a dense complete subcategory $\mathcal{S}$ of $\mathcal{C}$, define a relation $\sim$ on the set of isomorphism classes of objects in $\mathcal{C}$ by $\langle A \rangle \sim \langle B \rangle$ if and only if there exist objects S_A and S_B in $\mathcal{S}$ such that $A \oplus S_A \simeq B \oplus S_B$. One can check that this is an equivalence relation. Denote by $G_{\mathcal{S}}$ the quotient of the set of isomorphism classes of objects in $\mathcal{C}$ by the relation $\sim$. Elements in $G_{\mathcal{S}}$ are denoted by $\{A\}$.

Lemma 5.3. *Let $\mathcal{C}$ be an n-exangulated category with n odd and $\mathcal{S}$ a dense complete subcategory of $\mathcal{C}$. An object A in $\mathcal{C}$ is contained in $\mathcal{S}$ if and only if $\{A\} = \{0\}$ in $G_{\mathcal{S}}$.*

Proof. If A is in $\mathcal{S}$, then clearly $\{A\} = \{0\}$. Conversely, assume $\{A\} = \{0\}$. This means that there are objects S_A and S_0 in $\mathcal{S}$ such that $A \oplus S_A \simeq S_0$. Consequently, the $n+1$ last objects in the $(n+2)$-term sequence

$$A \to A \oplus S_A \to S_A \to 0 \to \cdots \to 0$$

are in $\mathcal{S}$. This is a conflation as it is the coproduct of two trivial conflations. Hence, as $\mathcal{S}$ is complete, our object A is also in $\mathcal{S}$. $\square$

Lemma 5.4. *Let $\mathcal{C}$ be an n-exangulated category with n odd. Let $\mathcal{G}$ be an n-(co)generator of $\mathcal{C}$ and $\mathcal{S}$ a dense complete subcategory of $\mathcal{C}$ which contains $\mathcal{G}$. The following statements hold:*

(1) $G_{\mathcal{S}}$ is an abelian group with binary operation $\{A\} + \{B\} := \{A \oplus B\}$ and identity element $\{0\}$.
(2) The map

$$\begin{aligned} K_0(\mathcal{C})/g(\mathcal{S}) &\xrightarrow{\simeq} G_{\mathcal{S}} \\ [A] + g(\mathcal{S}) &\longmapsto \{A\} \end{aligned}$$

is a well-defined isomorphism of groups. In particular, an object A in $\mathcal{C}$ is contained in $\mathcal{S}$ if and only if $[A]$ is in $g(\mathcal{S})$.

Proof. In order to show *(1)*, notice first that our binary operation is well-defined, commutative, associative and has $\{0\}$ as identity element. For any object A in $\mathcal{C}$, there exists an object A' such that $A \oplus A'$ is in $\mathcal{S}$, by denseness of $\mathcal{S}$. Using Lemma 5.3, this means that

$$\{A\} + \{A'\} = \{A \oplus A'\} = \{0\}.$$

Hence, the element $\{A'\}$ is the inverse of $\{A\}$, and $G_{\mathcal{S}}$ is an abelian group.

For *(2)*, let us first show that the map

$$\begin{aligned} \phi\colon K_0(\mathcal{C}) &\longrightarrow G_{\mathcal{S}} \\ [A] &\longmapsto \{A\} \end{aligned}$$

is well-defined. It suffices to show that the Euler relations are sent to zero. Consider a conflation

$$X_0 \to X_1 \to \cdots \to X_n \to X_{n+1}$$

in $\mathcal{C}$. For $i = 1, \ldots, n+1$, let X_i' be an object such that $X_i \oplus X_i'$ belongs to $\mathcal{S}$. We can get a new conflation by adding trivial conflations involving the objects X_i and X_i' to the conflation above, namely

$$\overline{X} \to \bigoplus_{i=1}^{n+1}(X_i \oplus X_i') \to \cdots \to \bigoplus_{i=n}^{n+1}(X_i \oplus X_i') \to X_{n+1} \oplus X_{n+1}',$$

where $\overline{X} = X_0 \oplus X_1' \oplus X_2 \oplus \cdots \oplus X_n' \oplus X_{n+1}$. As the $n+1$ last objects in this conflation are in $\mathcal{S}$, so is $\overline{X}$. Consequently, using Lemma 5.3, we have

$$\begin{aligned}\{0\} = \{\overline{X}\} &= \{X_0\} + \{X_1'\} + \{X_2\} + \cdots + \{X_n'\} + \{X_{n+1}\} \\ &= \{X_0\} - \{X_1\} + \{X_2\} + \cdots - \{X_n\} + \{X_{n+1}\}\end{aligned}$$

in $G_{\mathcal{S}}$, so ϕ is well-defined. It is now easy to check that ϕ is a surjective group homomorphism.

Our last step is to show that $\operatorname{Ker}(\phi) = g(\mathcal{S})$. Note that the inclusion $g(\mathcal{S}) \subseteq \operatorname{Ker}(\phi)$ follows immediately by Lemma 5.3. Using Proposition 4.3, any element in $\operatorname{Ker}(\phi)$ can be written as $[A] - [G]$ for some objects A in $\mathcal{C}$ and G in $\mathcal{G}$. This means that

$$\{0\} = \phi([A] - [G]) = \{A\} - \{G\} = \{A\},$$

where the third equality follows from Lemma 5.3 and the assumption that $\mathcal{S}$ contains $\mathcal{G}$. Consequently, again using Lemma 5.3, the object A is in $\mathcal{S}$. This yields our reverse inclusion. Combining the isomorphism $K_0(\mathcal{C})/g(\mathcal{S}) \simeq G_S$ and Lemma 5.3, we see that an object A is in $\mathcal{S}$ if and only if $[A]$ is in $g(\mathcal{S})$. $\square$

Our main theorem, Theorem 5.1, extends and unifies results by Thomason, Bergh–Thaule, Matsui and Zhu–Zhuang. We also get a classification of subcategories of n-exact categories.

Corollary 5.5. *(1)* [16, Theorem 2.1] *Let $\mathcal{C}$ be a triangulated category. Then there is a one-to-one correspondence between the dense triangulated subcategories of $\mathcal{C}$ and the subgroups of $K_0(\mathcal{C})$.*

(2) [3, Theorem 4.6] *Let $\mathcal{C}$ be an $(n+2)$-angulated category with n odd. Then there is a one-to-one correspondence between the dense complete $(n+2)$-angulated subcategories of $\mathcal{C}$ and the subgroups of $K_0(\mathcal{C})$.*

(3) [12, Theorem 2.7] *Let $\mathcal{C}$ be an exact category with a (co)generator $\mathcal{G}$. Then there is a one-to-one correspondence between the dense $\mathcal{G}$-(co)resolving subcategories of $\mathcal{C}$ and the subgroups of $K_0(\mathcal{C})$ containing the image of $\mathcal{G}$.*

(4) [17, Theorem 5.7] *Let $\mathcal{C}$ be an extriangulated category with a (co)generator $\mathcal{G}$. Then there is a one-to-one correspondence between the dense $\mathcal{G}$-(co)resolving subcategories of $\mathcal{C}$ and the subgroups of $K_0(\mathcal{C})$ containing the image of $\mathcal{G}$.*

(5) Let $\mathcal{C}$ be an n-exact category with n odd. Let $\mathcal{G}$ be an n-(co)generator of $\mathcal{C}$. Then there is a one-to-one correspondence between the dense complete subcategories of $\mathcal{C}$ containing $\mathcal{G}$ and the subgroups of $K_0(\mathcal{C})$ containing the image of $\mathcal{G}$.

Proof. Part *(5)* follows immediately from Theorem 5.1 as n-exact categories are n-exangulated.

As *(2)* implies *(1)* and *(4)* implies *(3)*, it suffices to prove *(2)* and *(4)*. To show *(4)*, notice that in the case $n = 1$, a dense subcategory containing $\mathcal{G}$ is complete if and only if it is $\mathcal{G}$-(co)resolving as defined in [17, Definition 5.3]. It is thus clear that Theorem 5.1 implies *(4)*.

In order to prove that *(2)* follows from our main result, we will use the definition of an $(n+2)$-angulated category, see [2,4].

Let $(\mathcal{C}, \Sigma)$ be an $(n+2)$-angulated category. Then $\mathcal{C}$ has enough projectives, with 0 as the only projective object. The same is true for injectives, and hence $\mathcal{G} = \{0\}$ is both an n-generator and an n-cogenerator of $\mathcal{C}$. As a subgroup necessarily contains the zero element, all subgroups of $K_0(\mathcal{C})$ will contain the image of $\mathcal{G}$.

It remains to show that every complete and dense subcategory $\mathcal{S}$ of $\mathcal{C}$ has a natural structure as an $(n+2)$-angulated subcategory, by declaring the distinguished $(n+2)$-angles in $\mathcal{C}$ with all objects in $\mathcal{S}$ to be the distinguished $(n+2)$-angles in $\mathcal{S}$. Recall that a full isomorphism-closed subcategory $\mathcal{S}$ of our $(n+2)$-angulated category $(\mathcal{C}, \Sigma)$ is an $(n+2)$-angulated subcategory if $(\mathcal{S}, \Sigma)$ itself is $(n+2)$-angulated and the inclusion is an $(n+2)$-angulated functor.

Recall from Section 3 that as n is odd, the subcategory $\mathcal{S}$ contains 0, which again implies that it is additive and isomorphism-closed. To show that $(\mathcal{S}, \Sigma)$ is $(n+2)$-angulated, the crucial parts are to check that $\mathcal{S}$ is closed under Σ and that morphisms can be completed to distinguished $(n+2)$-angles.

When we think of $\mathcal{C}$ as an n-exangulated category, the distinguished $(n+2)$-angles yield conflations when we remove the last object. To see that $\mathcal{S}$ is closed under Σ, let A be an object in $\mathcal{S}$. As

$$A \to 0 \to \cdots \to 0 \to \Sigma A \xrightarrow{1_{\Sigma A}} \Sigma A$$

is a distinguished $(n+2)$-angle in which the $n+1$ first objects are in $\mathcal{S}$, also ΣA is in $\mathcal{S}$. A dual argument shows that $\Sigma^{-1} A$ is in $\mathcal{S}$.

Let $f \colon X_0 \to X_1$ be a morphism in $\mathcal{S}$. We need to show that f can be completed to a distinguished $(n+2)$-angle in $\mathcal{S}$. As $\mathcal{C}$ is $(n+2)$-angulated, there is a distinguished $(n+2)$-angle

$$X_0 \xrightarrow{f} X_1 \to X_2 \to \cdots \to X_{n+1} \to \Sigma X_0$$

in $\mathcal{C}$. For $i = 2, \ldots, n$, use that $\mathcal{S}$ is dense and let X_i' be an object such that $X_i \oplus X_i'$ is in $\mathcal{S}$. By adding trivial $(n+2)$-angles involving the objects X_i and X_i' to the $(n+2)$-angle above, we get a new distinguished $(n+2)$-angle

$$X_0 \xrightarrow{f} X_1 \to X_2 \oplus X_2' \to \cdots \to \bigoplus_{i=2}^{n} (X_i \oplus X_i') \to \overline{X} \to \Sigma X_0,$$

where $\overline{X} = X_2 \oplus X_3' \oplus X_4 \oplus \cdots \oplus X_n' \oplus X_{n+1}$. As the $n+1$ first objects in this sequence are contained in $\mathcal{S}$, so is $\overline{X}$. Consequently, this is a distinguished $(n+2)$-angle in $\mathcal{S}$ which completes the morphism f.

The remaining axioms of an $(n+2)$-angulated category are immediately verified using the fact that $\mathcal{S}$ is full. As the distinguished $(n+2)$-angles in $\mathcal{S}$ are chosen in such a way that the inclusion functor is $(n+2)$-angulated, we can conclude that $\mathcal{S}$ is an $(n+2)$-angulated subcategory of $\mathcal{C}$. Consequently, also *(2)* follows from Theorem 5.1. $\square$

Acknowledgements. The author would like to thank her supervisor Petter Andreas Bergh for helpful discussions and comments. She would also thank Louis-Philippe Thibault for careful reading and helpful suggestions on a previous version of this paper.

References

[1] Raphael Bennett-Tennenhaus and Amit Shah, *Transport of structure in higher homological algebra*, arXiv:2003.02254v2 (2020).
[2] Petter Andreas Bergh and Marius Thaule, *The axioms for n-angulated categories*, Algebr. Geom. Topol. **13** (2013), no. 4, 2405–2428.
[3] ———, *The Grothendieck group of an n-angulated category*, J. Pure Appl. Algebra **218** (2014), no. 2, 354–366.
[4] Christof Geiss, Bernhard Keller, and Steffen Oppermann, *n-angulated categories*, J. Reine Angew. Math. **675** (2013), 101–120.
[5] Martin Herschend, Yu Liu, and Hiroyuki Nakaoka, *n-exangulated categories*, arXiv:1709.06689v3 (2017).
[6] Osamu Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61.
[7] Osamu Iyama, Nakaoka Hiroyuki, and Palu Yann, *Auslander–Reiten theory in extriangultaed categories*, arXiv:1805.03776 (2018).
[8] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614.
[9] Gustavo Jasso, *n-abelian and n-exact categories*, Math. Z. **283** (2016), no. 3-4, 703–759.
[10] Steven E. Landsburg, *K-theory and patching for categories of complexes*, Duke Math. J. **62** (1991), no. 2, 359–384.
[11] Yu Liu and Panyue Zhou, *Frobenius n-exangulated categories*, arXiv:1909.13284 (2019).
[12] Hiroki Matsui, *Classifying dense resolving and coresolving subcategories of exact categories via Grothendieck groups*, Algebr. Represent. Theory **21** (2018), no. 3, 551–563.
[13] Hiroyuki Nakaoka and Yann Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég. **60** (2019), no. 2, 117–193 (English, with English and French summaries).
[14] Amnon Neeman, *Some new axioms for triangulated categories*, J. Algebra **139** (1991), no. 1, 221–255.
[15] ———, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.
[16] Robert Wayne Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), no. 1, 1–27.
[17] Bin Zhu and Xiao Zhuang, *Grothendieck groups in extriangulated categories*, arXiv:1912.00621v4 (2019).

Department of mathematical sciences, NTNU, NO-7491 Trondheim, Norway
Email address: johanne.haugland@ntnu.no

Paper 2

AUSLANDER–REITEN TRIANGLES AND GROTHENDIECK GROUPS OF TRIANGULATED CATEGORIES

Algebr. Represent. Theory 25 (2022), 1379-1387

Johanne Haugland

AUSLANDER–REITEN TRIANGLES AND GROTHENDIECK GROUPS OF TRIANGULATED CATEGORIES

JOHANNE HAUGLAND

Abstract. We prove that if the Auslander–Reiten triangles generate the relations for the Grothendieck group of a Hom-finite Krull–Schmidt triangulated category with a (co)generator, then the category has only finitely many isomorphism classes of indecomposable objects up to translation. This gives a triangulated converse to a theorem of Butler and Auslander–Reiten on the relations for Grothendieck groups. Our approach has applications in the context of Frobenius categories.

1. Introduction

The notion of almost split sequences was introduced by Auslander and Reiten in [4], and has played a fundamental role in the representation theory of finite dimensional algebras ever since [5]. The theory of almost split sequences, later called Auslander–Reiten sequences or just AR-sequences, has also greatly influenced other areas, such as algebraic geometry and algebraic topology [2, 14].

Happel defined Auslander–Reiten triangles in triangulated categories [11]. These play a similar role in the triangulated setting as AR-sequences do for abelian or exact categories. While it is known that AR-sequences always exist in the category of finitely generated modules over a finite dimensional algebra, the situation in the triangulated case turns out to be more complicated, and the associated bounded derived category will not necessarily have AR-triangles. In fact, Happel proved that this category has AR-triangles if and only if the algebra is of finite global dimension [10, 11]. Reiten and van den Bergh showed that a Hom-finite Krull–Schmidt triangulated category has AR-triangles if and only if it admits a Serre functor [18]. More recently, Diveris, Purin and Webb proved that if a category as above is connected and has a stable component of the Auslander–Reiten quiver of Dynkin tree class, then this implies existence of AR-triangles [8].

In the abelian setting, there is a well-studied relationship between AR-sequences, representation-finiteness and relations for the Grothendieck group. From Butler [7], Auslander–Reiten [3, Proposition 2.2] and Yoshino [21, Theorem 13.7], we know that if a complete Cohen–Macaulay local ring is of

2010 *Mathematics Subject Classification.* 18E30, 18F30 (primary); 18E10, 16G70 (secondary).

Key words and phrases. Auslander–Reiten triangle, Grothendieck group, triangulated category, Frobenius category.

finite representation type, then the Auslander–Reiten sequences generate the relations for the Grothendieck group of the category of Cohen–Macaulay modules. Here we say that our ring is of finite representation type if the category of Cohen–Macaulay modules has only finitely many isomorphism classes of indecomposable objects. A converse to this theorem is given by Auslander for artin algebras [1] and by Hiramatsu in the case of a Gorenstein ring with an isolated singularity [13, Theorem 1.2], where the latter is extended by Kobayashi [15, Theorem 1.2]. Results of the type described above were recently generalized to the setup of exact categories by Enomoto [9] and to certain extriangulated categories by Padrol, Palu, Pilaud and Plamondon [16].

A natural question to ask is whether there is a similar connection between AR-triangles, representation-finiteness and the relations for the Grothendieck group in the triangulated case. Xiao and Zhu give a partial answer to this question. Namely, they show that if our triangulated category is locally finite, then the AR-triangles generate the relations for the Grothendieck group [20, Theorem 2.1]. Beligiannis generalizes and gives a converse to this result for compactly generated triangulated categories [6, Theorem 12.1].

In this paper we consider the reverse direction of Xiao and Zhu from a different point of view. We prove that if the Auslander–Reiten triangles generate the relations for the Grothendieck group of a Hom-finite Krull–Schmidt triangulated category with a (co)generator, then the category has only finitely many isomorphism classes of indecomposable objects up to translation. We conclude by an application in the context of Frobenius categories. As an example, we see that our approach recovers results of Hiramatsu and Kobayashi for Gorenstein rings.

2. Auslander–Reiten triangles and Grothendieck groups

Let R be a commutative ring. An R-linear category $\mathcal{T}$ is called *Hom-finite* provided that $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is of finite R-length for every pair of objects X, Y in $\mathcal{T}$. An additive category is called a *Krull–Schmidt category* if every object can be written as a finite direct sum of indecomposable objects having local endomorphism rings. In a Krull–Schmidt category, it is well known that every object decomposes essentially uniquely in this way.

Throughout the rest of this paper, we let $\mathcal{T}$ be an essentially small R-linear triangulated category. We also assume that $\mathcal{T}$ is a Krull–Schmidt category which is Hom-finite over R. We let $\mathrm{ind}(\mathcal{T})$ consist of the indecomposable objects of $\mathcal{T}$, while the translation functor of $\mathcal{T}$ is denoted by Σ. For simplicity, we use the notation $(A,B) = \mathrm{Hom}_{\mathcal{T}}(A,B)$ and $[A,B] = \mathrm{length}_R(\mathrm{Hom}_{\mathcal{T}}(A,B))$.

We say that $\mathcal{T}$ has finitely many isomorphism classes of indecomposable objects up to translation if there is a finite subset of $\mathrm{ind}(\mathcal{T})$ such that for any $U \in \mathrm{ind}(\mathcal{T})$, there is an integer n such that $\Sigma^n U$ is isomorphic to an object in our finite subset.

Recall from [12] that a distinguished triangle $A \to B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathcal{T}$ is an *Auslander–Reiten triangle* if the following conditions are satisfied:

(1) $A, C \in \operatorname{ind}(\mathcal{T})$;
(2) $h \neq 0$;
(3) given any morphism $t \colon W \to C$ which is not a split-epimorphism, there is a morphism $t' \colon W \to B$ such that $g \circ t' = t$.

Let $\mathcal{F}(\mathcal{T})$ denote the free abelian group generated by all isomorphism classes $[A]$ of objects A in $\mathcal{T}$, while $K_0(\mathcal{T}, 0)$ is the quotient of $\mathcal{F}(\mathcal{T})$ by the subgroup generated by the set $\{[A \oplus B] - [A] - [B] \mid A, B \in \mathcal{T}\}$. By abuse of notation, objects in $K_0(\mathcal{T}, 0)$ are also denoted by $[A]$. As $\mathcal{T}$ is a Krull–Schmidt category, the quotient $K_0(\mathcal{T}, 0)$ is isomorphic to the free abelian group generated by isomorphism classes of objects in $\operatorname{ind}(\mathcal{T})$.

Let $\operatorname{Ex}(\mathcal{T})$ be the subgroup of $K_0(\mathcal{T}, 0)$ generated by the subset

$$\left\{ [X] - [Y] + [Z] \;\middle|\; \begin{array}{l} \text{there exists a distinguished triangle} \\ X \to Y \to Z \to \Sigma X \ \text{ in } \mathcal{T} \end{array} \right\}.$$

Similarly, we let $\operatorname{AR}(\mathcal{T})$ denote the subgroup of $K_0(\mathcal{T}, 0)$ generated by

$$\left\{ [X] - [Y] + [Z] \;\middle|\; \begin{array}{l} \text{there exists an AR-triangle} \\ X \to Y \to Z \to \Sigma X \ \text{ in } \mathcal{T} \end{array} \right\}.$$

Recall from for instance [12] that the Grothendieck group of $\mathcal{T}$ is defined as $K_0(\mathcal{T}) = K_0(\mathcal{T}, 0)/\operatorname{Ex}(\mathcal{T})$.

In the proof of our main results, Theorem 2.4 and Theorem 2.5, we use the well-known fact that an equality in $K_0(\mathcal{T}, 0)$ can yield an equality in $\mathbb{Z}$. We need this in the case of $[U, -]$ and $[-, U]$ for an object U in $\mathcal{T}$, but note that the following lemma could be phrased more generally in terms of additive functors.

Lemma 2.1. *Suppose that* $a_1[X_1] + \cdots + a_r[X_r] = 0$ *in* $K_0(\mathcal{T}, 0)$ *for integers* a_i *and objects* X_i *in* $\mathcal{T}$. *Then* $a_1[U, X_1] + \cdots + a_r[U, X_r] = 0$ *and* $a_1[X_1, U] + \cdots + a_r[X_r, U] = 0$ *in* $\mathbb{Z}$ *for any object* U *in* $\mathcal{T}$.

Proof. Let $a_1[X_1] + \cdots + a_r[X_r] = 0$ in $K_0(\mathcal{T}, 0)$. If $a_i \geq 0$ for every $i = 1, 2, \ldots, r$, we use the defining relations for $K_0(\mathcal{T}, 0)$ to obtain

$$a_1[X_1] + \cdots + a_r[X_r] = [a_1 X_1 \oplus \cdots \oplus a_r X_r] = 0,$$

where $a_i X_i$ denotes the coproduct of the object X_i with itself a_i times. Consequently, the object $a_1 X_1 \oplus \cdots \oplus a_r X_r$ is zero in $\mathcal{T}$. Applying $[U, -]$ or $[-, U]$ and using additivity hence yields our desired equations.

If some of the coefficients a_i are negative, we start by moving all negative terms to the right-hand side of our equality and proceed similarly. $\square$

The lemmas below, which yield a triangulated analogue of [15, Proposition 2.8], provide an important step in the proofs of Theorem 2.4 and Theorem 2.5. Note that parts of our proof of Lemma 2.2 is much the same as the proof of [8, Lemma 2.2]. Observe also that Lemma 2.3 follows from [19, Proposition 3.1] in the case where R is an algebraically closed field, and that the argument generalizes to our context. We include complete proofs for the convenience of the reader.

Lemma 2.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ be an AR-triangle in $\mathcal{T}$. The following statements hold for an object U in $\mathcal{T}$:*

(1) The morphism $(U,B) \xrightarrow{g_*} (U,C)$ is surjective if and only if C is not a direct summand in U.
(2) The morphism $(U,A) \xrightarrow{f_*} (U,B)$ is injective if and only if $\Sigma^{-1}C$ is not a direct summand in U.
(3) The morphism $(B,U) \xrightarrow{f^*} (A,U)$ is surjective if and only if A is not a direct summand in U.
(4) The morphism $(C,U) \xrightarrow{g^*} (B,U)$ is injective if and only if ΣA is not a direct summand in U.

Proof. Note that C is a direct summand in U if and only if there exists a split epimorphism $U \to C$. By the definition of an AR-triangle, this is equivalent to g_* not being surjective, which proves *(1)*.

Our triangle yields the long-exact sequence

$$\cdots \to (U, \Sigma^{-1}B) \xrightarrow{(\Sigma^{-1}g)_*} (U, \Sigma^{-1}C) \to (U,A) \xrightarrow{f_*} (U,B) \to \cdots .$$

The morphism f_* is hence injective if and only if $(\Sigma^{-1}g)_*$ is surjective. By applying part *(1)* to the object ΣU, we see that $(\Sigma^{-1}g)_*$ is surjective if and only if C is not a direct summand in ΣU, which is equivalent to $\Sigma^{-1}C$ not being a direct summand in U. This shows *(2)*.

The statements *(3)* and *(4)* are verified dually, using that AR-triangles equivalently can be defined in terms of a factorization property for the leftmost morphism, see for instance [12]. □

Lemma 2.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ be an AR-triangle in $\mathcal{T}$. The following statements hold for an indecomposable object U in $\mathcal{T}$:*

(1) We have $[U,A] - [U,B] + [U,C] \neq 0$ if and only if $U \simeq C$ or $U \simeq \Sigma^{-1}C$.
(2) We have $[A,U] - [B,U] + [C,U] \neq 0$ if and only if $U \simeq A$ or $U \simeq \Sigma A$.

Proof. From the long exact Hom-sequence arising from our triangle, we get the exact sequence

$$0 \to K \to (U,A) \xrightarrow{f_*} (U,B) \xrightarrow{g_*} (U,C) \to L \to 0,$$

where $K = \operatorname{Ker}(f_*)$ and $L = \operatorname{Coker}(g_*)$. Splitting into short exact sequences and using our finiteness assumption, we see that the alternating sum of the lengths of the objects in the sequence vanishes. This gives the equation

$$[U,A] - [U,B] + [U,C] = \operatorname{length}_R(K) + \operatorname{length}_R(L).$$

Consequently, we have $[U,A] - [U,B] + [U,C] \neq 0$ if and only if the right-hand side of the equation is also non-zero. This means that either K or L

(or both) must be non-zero. The object K is non-zero if and only if f_* is not injective. By part Lemma 2.2 part *(2)*, this is the case if and only if $\Sigma^{-1}C$ is a direct summand in U. Similarly, the object L is non-zero if and only if g_* is not surjective. Using part *(1)* of Lemma 2.2, this is equivalent to C being a direct summand in U. As U is indecomposable, a direct summand in U is necessarily isomorphic to U, which finishes our proof of part *(1)*.

Our second statement is shown dually, using part *(3)* and *(4)* of Lemma 2.2. □

We are now ready to prove our two main results, which show that we can study representation-finiteness of our category $\mathcal{T}$ by considering the relations for the associated Grothendieck group.

Theorem 2.4. *Assume there is an object X in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(Y,X) \neq 0$ or an object X' in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(X',Y) \neq 0$ for every non-zero Y in $\mathcal{T}$. If $\mathrm{Ex}(\mathcal{T}) = \mathrm{AR}(\mathcal{T})$ in $K_0(\mathcal{T},0)$, then $\mathcal{T}$ has only finitely many isomorphism classes of indecomposable objects.*

Proof. Let X be an object with the property described above, and consider the triangle $\Sigma^{-1}X \to 0 \to X \xrightarrow{1_X} X$. As this is a distinguished triangle, we have $[\Sigma^{-1}X] + [X] \in \mathrm{Ex}(\mathcal{T})$. By the assumption $\mathrm{Ex}(\mathcal{T}) = \mathrm{AR}(\mathcal{T})$, there hence exist AR-triangles

$$A_i \to B_i \to C_i \to \Sigma A_i$$

and integers a_i for $i = 1, 2, \dots, r$ such that

$$[X] + [\Sigma^{-1}X] = \sum_{i=1}^{r} a_i([A_i] - [B_i] + [C_i])$$

in $K_0(\mathcal{T},0)$. Given an object U in $\mathcal{T}$, Lemma 2.1 now yields the equality

$$[U,X] + [U,\Sigma^{-1}X] = \sum_{i=1}^{r} a_i([U,A_i] - [U,B_i] + [U,C_i])$$

in $\mathbb{Z}$. If U is non-zero, our assumption on X implies that the left-hand side of this equation is non-zero. Hence, there must for every non-zero object U be an integer $i \in \{1,\dots,r\}$ such that $[U,A_i] - [U,B_i] + [U,C_i] \neq 0$. In particular, this is true for every $U \in \mathrm{ind}(\mathcal{T})$. By Lemma 2.3 part *(1)*, this means that any indecomposable object in $\mathcal{T}$ is isomorphic to an object in the finite set $\{C_i, \Sigma^{-1}C_i\}_{i=1}^{r}$, which yields our desired conclusion.

The proof in the dual case is similar, using Lemma 2.3 part *(2)*. □

In the theorem below, an object X in $\mathcal{T}$ is called a *generator of* $\mathcal{T}$ if

$$\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X,\Sigma^n Y) \neq 0$$

for any non-zero object Y in $\mathcal{T}$. Dually, an object X is called a *cogenerator of* $\mathcal{T}$ if $\mathrm{Hom}^*_{\mathcal{T}}(Y,X) \neq 0$ for any non-zero Y.

Theorem 2.5. *Assume that our category $\mathcal{T}$ has a generator or a cogenerator. If $\operatorname{Ex}(\mathcal{T}) = \operatorname{AR}(\mathcal{T})$ in $K_0(\mathcal{T}, 0)$, then $\mathcal{T}$ has only finitely many isomorphism classes of indecomposable objects up to translation.*

Proof. Let X be a cogenerator and consider an indecomposable object U in $\mathcal{T}$. Notice that as X is a cogenerator, there exists an integer n such that $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n U, X) \neq 0$. As in the proof of Theorem 2.4, our assumption $\operatorname{Ex}(\mathcal{T}) = \operatorname{AR}(\mathcal{T})$ implies existence of a finite family of AR-triangles

$$A_i \to B_i \to C_i \to \Sigma A_i$$

which yields an equality

$$[\Sigma^n U, X] + [\Sigma^n U, \Sigma^{-1} X] = \sum_{i=1}^{r} a_i([\Sigma^n U, A_i] - [\Sigma^n U, B_i] + [\Sigma^n U, C_i])$$

in $\mathbb{Z}$. The left-hand side of this equation is non-zero, so there is an integer $i \in \{1, \dots, r\}$ such that $[\Sigma^n U, A_i] - [\Sigma^n U, B_i] + [\Sigma^n U, C_i] \neq 0$. By applying Lemma 2.3 part *(1)*, this yields that either $\Sigma^n U \simeq C_i$ or $\Sigma^{n+1} U \simeq C_i$. Consequently, every indecomposable object in $\mathcal{T}$ can be obtained as a translation of an object in the finite set $\{C_i\}_{i=1}^r$, which yields our desired conclusion.

The proof in the case where our category $\mathcal{T}$ has a generator is dual, using Lemma 2.3 part *(2)*. $\square$

3. Application to Frobenius categories

We now move on to an application of Theorem 2.4. Throughout the rest of the paper, let $\mathcal{C}$ be an essentially small R-linear Frobenius category. Recall that a Frobenius category is an exact category with enough projectives and injectives, and in which these two classes of objects coincide. The stable category of $\mathcal{C}$, i.e. the quotient category modulo projective objects, is denoted by $\underline{\mathcal{C}}$. We assume $\mathcal{C}$ to be a Krull–Schmidt category and that the stable category $\underline{\mathcal{C}}$ is Hom-finite.

As $\mathcal{C}$ is a Frobenius category, the associated stable category is triangulated. Recall that the distinguished triangles in $\underline{\mathcal{C}}$ are isomorphic to triangles of the form $X \to Y \to Z \to \Omega^{-1} X$, where $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{C}$ and $\Omega^{-1} X$ denotes the first cosyzygy of X. Note that Ω^{-1} is a well-defined autoequivalence on the stable category. The morphism $Z \to \Omega^{-1} X$ in our distinguished triangle above is obtained from the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "1_X"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & X \arrow[r] & I(X) \arrow[r] & \Omega^{-1} X \arrow[r] & 0,
\end{tikzcd}$$

where $I(X)$ is injective and both rows are short exact sequences. For a more thorough introduction to exact categories and the stable category of a Frobenius category, see for instance [12].

Based on the correspondence between short exact sequences in a Frobenius category and distinguished triangles in its stable category, we get results also for Frobenius categories. In order to see this, we need to rephrase some of our terminology in the context of exact categories. Let us first recall that a short exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ in $\mathcal{C}$ is an *Auslander–Reiten sequence* if the following conditions are satisfied:

(1) $A, C \in \operatorname{ind}(\mathcal{C})$;
(2) the sequence does not split;
(3) given any morphism $t \colon W \to C$ which is not a split-epimorphism, there is a morphism $t' \colon W \to B$ such that $g \circ t' = t$.

Just as in the triangulated case, we let $K_0(\mathcal{C}, 0)$ denote the free abelian group generated by isomorphism classes of objects in $\mathcal{C}$ modulo the subgroup generated by the set $\{[A \oplus B] - [A] - [B] \mid A, B \in \mathcal{C}\}$. Again, we can define the subgroups $\operatorname{Ex}(\mathcal{C})$ and $\operatorname{AR}(\mathcal{C})$ of $K_0(\mathcal{C}, 0)$, but now in terms of short exact sequences instead of distinguished triangles. Namely, we let $\operatorname{Ex}(\mathcal{C})$ be the subgroup generated by the subset

$$\left\{ [X] - [Y] + [Z] \;\middle|\; \begin{array}{l} \text{there exists a short exact sequence} \\ 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{C} \end{array} \right\}$$

and $\operatorname{AR}(\mathcal{C})$ the subgroup generated by

$$\left\{ [X] - [Y] + [Z] \;\middle|\; \begin{array}{l} \text{there exists an AR-sequence} \\ 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{C} \end{array} \right\}.$$

The next lemma describes a well-known correspondence between AR-sequences in $\mathcal{C}$ and AR-triangles in $\underline{\mathcal{C}}$, see [17, Lemma 3].

Lemma 3.1. *An exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ is an AR-sequence in $\mathcal{C}$ if and only if the corresponding distinguished triangle $A \to B \to C \to \Omega^{-1}A$ in $\underline{\mathcal{C}}$ is an AR-triangle in $\underline{\mathcal{C}}$.*

We are now ready to show the following lemma regarding the subgroups $\operatorname{Ex}(\mathcal{C})$ and $\operatorname{AR}(\mathcal{C})$ of $K_0(\mathcal{C}, 0)$ and the analogous subgroups of $K_0(\underline{\mathcal{C}}, 0)$.

Lemma 3.2. *If $\operatorname{Ex}(\mathcal{C}) = \operatorname{AR}(\mathcal{C})$ in $K_0(\mathcal{C}, 0)$, then $\operatorname{Ex}(\underline{\mathcal{C}}) = \operatorname{AR}(\underline{\mathcal{C}})$ in $K_0(\underline{\mathcal{C}}, 0)$.*

Proof. Assume $\operatorname{Ex}(\mathcal{C}) = \operatorname{AR}(\mathcal{C})$ in $K_0(\mathcal{C}, 0)$ and consider a distinguished triangle in $\underline{\mathcal{C}}$. As we work with isomorphism classes of objects, we can assume that our triangle is of the form $X \to Y \to Z \to \Omega^{-1}X$, where $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{C}$. Since $\operatorname{Ex}(\mathcal{C}) = \operatorname{AR}(\mathcal{C})$, there exist AR-sequences $0 \to A_i \to B_i \to C_i \to 0$ and integers a_i for $i = 1, 2, \ldots, r$ such that

$$[X] - [Y] + [Z] = \sum_{i=1}^{r} a_i([A_i] - [B_i] + [C_i])$$

in $K_0(\mathcal{C}, 0)$, and hence also in $K_0(\underline{\mathcal{C}}, 0)$. By Lemma 3.1, the right-hand side of this equation is contained in $\operatorname{AR}(\underline{\mathcal{C}})$. Thus, we have shown that $\operatorname{Ex}(\underline{\mathcal{C}}) \subseteq \operatorname{AR}(\underline{\mathcal{C}})$. The reverse inclusion is clear. $\square$

We hence have the following corollary to Theorem 2.4.

Corollary 3.3. *Assume there is an object X in $\mathcal{C}$ such that $\mathrm{Hom}_{\underline{\mathcal{C}}}(Y, X) \neq 0$ or an object X' in $\mathcal{C}$ such that $\mathrm{Hom}_{\underline{\mathcal{C}}}(X', Y) \neq 0$ for every non-zero Y in $\underline{\mathcal{C}}$. If $\mathrm{Ex}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$ in $K_0(\mathcal{C}, 0)$, then the following statements hold:*

(1) The category $\mathcal{C}$ has only finitely many isomorphism classes of non-projective indecomposable objects.
(2) If $\mathcal{C}$ has only finitely many indecomposable projective objects up to isomorphism, then $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects.

Proof. As $\mathcal{C}$ is an essentially small R-linear Krull–Schmidt category, the same is true for the stable category $\underline{\mathcal{C}}$. As $\mathrm{Ex}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$ in $K_0(\mathcal{C}, 0)$, Lemma 3.2 yields that $\mathrm{Ex}(\underline{\mathcal{C}}) = \mathrm{AR}(\underline{\mathcal{C}})$ in $K_0(\underline{\mathcal{C}}, 0)$. The result now follows from Theorem 2.4. □

Let us consider the example where R is a complete Gorenstein local ring with an isolated singularity. Recall that the category of Cohen–Macaulay R-modules is Frobenius. As R is an isolated singularity, the associated stable category is Hom-finite, and completeness of R yields the Krull–Schmidt property. By [13, Lemma 2.1], our category has an object which satisfies the assumption in the corollary above. Since R is local, there are only finitely many isomorphism classes of indecomposable projective objects. Consequently, part *(2)* of Corollary 3.3 yields that if the AR-triangles generate the relations for the Grothendieck group of this category, then R has only finitely many isomorphism classes of indecomposable Cohen–Macaulay modules. This recovers [13, Theorem 1.2] of Hiramatsu.

Note that one could, if preferred, state Theorem 2.4 and Corollary 3.3 in terms of taking the tensor product with $\mathbb{Q}$, as in the result of Kobayashi [15, Theorem 1.2]. Hence, also Kobayashi's conclusions are recovered from our approach in the case of a complete Gorenstein ring.

Acknowledgements. The author would like to thank her supervisor Petter Andreas Bergh for helpful discussions and comments. She would also thank an anonymous referee for careful reading and suggestions which led to significant improvement of the paper.

References

[1] Maurice Auslander, *Relations for Grothendieck groups of Artin algebras*, Proc. Amer. Math. Soc. **91** (1984), no. 3, 336–340.

[2] ———, *Almost split sequences and algebraic geometry*, Representations of algebras (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 116, Cambridge Univ. Press, Cambridge, 1986, pp. 165–179.

[3] Maurice Auslander and Idun Reiten, *Grothendieck groups of algebras and orders*, J. Pure Appl. Algebra **39** (1986), no. 1-2, 1–51.

[4] ———, *Representation theory of Artin algebras. III. Almost split sequences*, Comm. Algebra **3** (1975), 239–294.

[5] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.

[6] Apostolos Beligiannis, *Auslander-Reiten triangles, Ziegler spectra and Gorenstein rings*, K-Theory **32** (2004), no. 1, 1–82.

[7] Michael C. R. Butler, *Grothendieck groups and almost split sequences*, Integral representations and applications (Oberwolfach, 1980), Lecture Notes in Math., vol. 882, Springer, Berlin-New York, 1981, pp. 357–368.

[8] Kosmas Diveris, Marju Purin, and Peter Webb, *Combinatorial restrictions on the tree class of the Auslander-Reiten quiver of a triangulated category*, Math. Z. **282** (2016), no. 1-2, 405–410.

[9] Haruhisa Enomoto, *Relations for Grothendieck groups and representation-finiteness*, J. Algebra **539** (2019), 152–176.

[10] Dieter Happel, *Auslander-Reiten triangles in derived categories of finite-dimensional algebras*, Proc. Amer. Math. Soc. **112** (1991), no. 3, 641–648.

[11] ———, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[12] ———, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[13] Naoya Hiramatsu, *Relations for Grothendieck groups of Gorenstein rings*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 559–562.

[14] Peter Jørgensen, *Calabi-Yau categories and Poincaré duality spaces*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 399–431.

[15] Toshinori Kobayashi, *Syzygies of Cohen-Macaulay modules and Grothendieck groups*, J. Algebra **490** (2017), 372–379.

[16] Arnau Padrol, Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon, *Associahedra for finite type cluster algebras and minimal relations between g-vectors*, arXiv:1906.06861 (2019).

[17] Klaus W. Roggenkamp, *Auslander-Reiten triangles in derived categories*, Forum Math. **8** (1996), no. 5, 509–533.

[18] Idun Reiten and Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[19] Peter Webb, *Bilinear forms on Grothendieck groups of triangulated categories*, Geometric and topological aspects of the representation theory of finite groups, Springer Proc. Math. Stat., vol. 242, Springer, Cham, 2018, pp. 465–480.

[20] Jie Xiao and Bin Zhu, *Relations for the Grothendieck groups of triangulated categories*, J. Algebra **257** (2002), no. 1, 37–50.

[21] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990.

Department of mathematical sciences, NTNU, NO-7491 Trondheim, Norway
E-mail address: `johanne.haugland@ntnu.no`

Paper 3

THE ROLE OF GENTLE ALGEBRAS IN HIGHER HOMOLOGICAL ALGEBRA

Forum Math. 34 (2022), no. 5, 1255-1275

Johanne Haugland Karin M. Jacobsen Sibylle Schroll

Paper 4

HIGHER KOSZUL DUALITY AND CONNECTIONS WITH n-HEREDITARY ALGEBRAS

Johanne Haugland   Mads H. Sandøy

This paper is submitted for publication and is therefore not included.

Paper 5

THE CATEGORY OF EXTENSIONS AND A CHARACTERISATION OF n-EXANGULATED FUNCTORS

Raphael Bennett-Tennenhaus   Johanne Haugland
Mads H. Sandøy   Amit Shah

This paper is submitted for publication and is therefore not included.

Paper 6

A CHARACTERISATION OF HIGHER TORSION CLASSES

Jenny August Johanne Haugland Karin M. Jacobsen
Sondre Kvamme Yann Palu Hipolito Treffinger

This paper is submitted for publication and is therefore not included.

APPENDIX: NORWEGIAN TRANSLATIONS

This appendix contains a list of Norwegian translations of terminology used in the thesis.

English	**Norwegian (bokmål)**
Auslander–Reiten triangle	Auslander-Reiten-triangel
category of extensions	utvidelseskategori
closed under n-extensions	lukket under n-utvidelser
conflation	konflasjon
deflation	deflasjon
dense	tett
extension	utvidelse
extension-closed	utvidelseslukket
extriangulated	ekstriangulert
Frobenius algebra	frobeniusalgebra
Frobenius category	frobeniuskategori
functorially finite	funktorielt endelig
gentle algebra	mild algebra
Grothendieck group	grothendieckgruppe
higher Auslander algebra	høyere auslanderalgebra
higher homological algebra	høyere homologisk algebra
higher Nakayama algebra	høyere nakayamaalgebra
higher preprojective algebra	høyere preprojektiv algebra
inflation	inflasjon
Koszul dual	koszuldual
Koszul duality	koszuldualitet
lattice	gitter
mapping cone	avbildningskjegle
n-abelian	n-abelian
n-cluster tilting	n-klynge-vippe
n-exact	n-eksakt
n-exangulated	n-eksangulert
n-hereditary	n-hereditær
n-representation finite	n-representasjonsendelig
n-representation infinite	n-representasjonsuendelig
n-torsion class	n-torsjonsklasse
$(n+2)$-angulated	$(n+2)$-angulert
realization	realisering
Serre functor	serrefunktor
tilting module	vippemodul
tilting object	vippeobjekt

ISBN 978-82-326-5599-1 (printed ver.)
ISBN 978-82-326-5310-2 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

NTNU

Norwegian University of
Science and Technology