Shortcomings in the *milieu* for algebraic generalisation arising from task design and vagueness in mathematical discourse

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This paper presents how the milieu for students' engagement with an algebraic generalisation task is constrained by two factors: first, by the task design; second, by the students' unawareness of the nature of a mathematical statement, combined with the teacher's use of a generic example without the students' awareness of it.

Keywords: Milieu, adidactical situation, algebraic generalisation, mathematical statement, generic example.

Introduction

The formulation of a task, as well as its mathematical, social, psychological, and didactic contexts, are important factors for students' responses on the task (Sierpinska, 2004). This paper presents an analysis of three student teachers' collaborative engagement with a task on algebraic generalisation of a shape pattern. The task is designed by their mathematics teacher educator.[18] The author has had influence neither on the design nor on the implementation of the given task. 'Task' is here understood as an assignment given to students to which they are expected to produce a solution. The paper deals with the question of how students understand the purpose of the task they are given in a *regular* teaching situation (i.e., it is not the result of didactical engineering, Artigue & Perrin-Glorian, 1991). I show how the formulation of the task and the interaction between the teacher and the students about the task constitute a gap between the teacher's intention with the task and the students' mathematical activity.

Inspired by the writing of Whitehead (1947), Devlin (1994), and others, I view mathematics as the science of patterns. A shape pattern in school mathematics is usually instantiated by some consecutive geometric configurations in an alignment imagined as continuing until infinity. Radford (2006) provides a useful characterisation of *algebraic generalisation* of patterns when he proposes that

> generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence *S*, being aware that this commonality applies to all the terms of *S* and being able to use it to provide a direct expression of whatsoever term of *S*. (Radford, 2006, p. 5)

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[18] In the rest of the paper, "students" is used to refer to student teachers, and "teacher" is used to refer to a teacher educator.

Theoretical framework

In Brousseau's (1997) theory of didactical situations in mathematics, an *adidactical situation* is a situation in which the student takes a mathematical problem as his own and solves it on the basis of its internal logic without the teacher's guidance and without trying to interpret the teacher's intention with the problem. The *devolution* of an adidactical learning situation is the act by which the teacher encourages the student to accept the responsibility for an adidactical learning situation or for a problem, and the teacher accepts the consequences of the transfer of this responsibility (Brousseau, 1997). The student cannot engage in any adidactical situation; the teacher attempts to arrange an adidactical situation that the student can handle.

In the devolution process, which is part of the broader (didactical) situation, the teacher is faced with a system, itself built up from a pair of systems; the student and a *milieu* that lacks any didactical intentions with regard to the student (Brousseau, 1997). The milieu is a subset of the students' environment with only those features that are relevant with respect to the knowledge aimed at by the teacher in the didactical situation. The concept of milieu models the elements of the material or intellectual reality on which the students act and which may be an obstacle to their actions and reasoning (Laborde & Perrin-Glorian, 2005). That is, the milieu of a didactical situation is the part of the environment that can bring feedback to students' actions to accomplish a task.

An adidactical situation is part of the didactical situation that is the broader situation with the system of interaction of the students with the milieu arranged with the purpose of the students' appropriation of the target knowledge without the teacher's intervention (Brousseau, 1997). The teacher can act on the milieu by providing new information or new equipment, for example by asking a question or directing students' attention to certain factors in the classroom situation. When the teacher acts on the milieu, she changes the knowledge needed to solve the problem (Perrin-Glorian, Deblois & Robert, 2008). Whether the student can handle an adidactical situation depends upon two conditions: first, that the student has prior knowledge that enables him to engage with the situation; second, that the milieu created by the teacher provides the student with knowings that enable him to develop the knowledge aimed at (by the teacher).

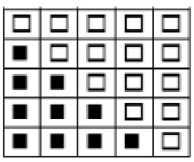
Succeeding the devolution phase, the didactical situation consists of four situations (or phases) in which the role of the teacher and the status of knowledge change (Brousseau, 1997): Situations of action, formulation, and validation are intentionally adidactical situations, whereas the situation of institutionalisation is not adidactical. In the following paragraph, these situations are briefly described (for a more elaborated explanation, see Brousseau, 1997, or Måsøval, 2011, Chapter 2).

The situation of *action* is where the students engage with the presented problem on the basis of its internal logic without the teacher's intervention. The students construct a representation of the situation which serves as a "model" that guides them in their decisions. In the situation of *formulation* the students exchange and compare observations between themselves, where the main purpose is to develop language to formulate their observations and agree on some common meanings. Here, the teacher re-enters the scene to chair the exchanges and make sure that all formulations are made "visible" in the classroom. In the situation of *validation* the students try to explain some phenomenon or verify a conjecture. Here, the teacher acts as a chair in a scientific debate and intervenes only to structure the debate and encourage the students to use more precise mathematical concepts. The situation of *institutionalisation* is where the teacher informs the students about conventional terminology and highlights definitions and theorems considered important for the contextualised knowledge (developed by the students through the preceding situations) to gain the status of cultural knowledge so that it can be used in settings other than in the original one set up by the teacher.

The adopted epistemological perspective in the paper is rooted in the theory of didactical situations: First, teaching involves the devolution to the student of an adidactical, appropriate situation; learning is the student's adaptation to this situation. Second, teaching involves the transformation of the student's responses into a piece of knowledge which can be used beyond the situation in which it is produced.

Methodology

The reported research is derived from the author's PhD project (a case study) reported in (Måsøval, 2011), where the addressed research question was: *What factors constrain students' appropriation of algebraic generality in shape patterns?* The task with which the paper deals (Task 4, presented in Figure 1) was the last of four tasks (during eight lessons of 45 minutes) on algebraic generalisation of shape patterns. Task 4 is designed for students' collaborative self-engagement. It is divided into three subtasks which are part of the milieu for situations of action, formulation, and validation, respectively: Inventing a continuation of the shape pattern represents finding a "model" that guides the students towards the intended theorem (Task 4a: action); identifying what figurate numbers that are part of the invariant structure of the pattern represents developing language to formulate the students' observations (Task 4b: formulation); and, expressing what the pattern tells them represents verifying the intended theorem (Task 4c: validation).



Thorvaldsen's museum in Copenhagen contains several floor mosaics with mathematical content. We looked at one mosaic last Friday, and here we shall look at another one. This pattern can be thought of as built up by equal, squared areas containing bright and dark mosaic tiles.

On the shape below the pattern is reproduced schematically with ■ for each of the dark squared tiles and □ for each of the bright ones.

a) If this shape were part of a sequence of shapes, what would the next one look like?

b) What kinds of figurate numbers do you find in the bright and the dark areas, and in the shape as a whole?

c) Express what the shape tells you about these numbers in terms of a mathematical statement.

Figure 1. Task 4 given to the class for work in small groups

It is relevant to notice that this categorisation of the task is based on the author's conceptualisation. At the time the data were collected, the teacher who designed the task was not acquainted with Brousseau's theory. According to the same teacher, the mathematical knowledge aimed at in Task 4 was the formulation of a mathematical statement (a theorem) represented in algebraic notation (e.g.,

$T_{n-1} + T_n = n^2$, where $T_n = 1 + 2 + 3 + \cdots + n$ denotes the *n*-th triangular number). This intention was not communicated to the students.

The data are Task 4 and a video-recorded observation of three students' collaborative engagement with Task 4 (with teacher intervention). The students are Anne, Helen, and Paul (pseudonyms), who were in their first academic year on a teacher education programme for primary and lower secondary school in Norway. The observed teacher (the one who had designed the task) is my colleague, an experienced, male teacher of mathematics. My role during data collection was to be a silent observer while video-recording the classroom interaction analysed in the paper. The teacher interacted with the students during the observed episode on the basis of the students' difficulty in understanding two of the concepts used in the task. The video-recorded episode has been transcribed and analysed through a process of open coding (using an adapted grounded theory approach, Strauss & Corbin, 1998) where concepts from the theory of didactical situations (Brousseau, 1997) have been used to make sense of what factors constrain the students' algebraic generalisation of the actual shape pattern (in this way addressing the research question).

Analysis of students' engagement with a task on algebraic generalisation

Anne, Helen, and Paul have drawn the first three elements of a shape pattern (Figure 2) which is a continuation of the element given in Task 4. They have found that for the first element of this shape pattern (the 5x5 square given in the task), the number of black components (represented by black x-es) is equal to the sum of the first four natural numbers, and the number of white components (represented by turquoise x-es) is equal to the sum of the first five natural numbers. The students have observed that this is a regularity that applies also for the next two elements of the shape pattern. That is, they have verified by inspection that for the second element (a 6x6 square), the number of black components is equal to the sum of the first five natural numbers, and the number of turquoise components is equal to the sum of the first six natural numbers, and likewise for the third element. They have, however, not identified the sums of consecutive natural numbers to be triangular numbers.

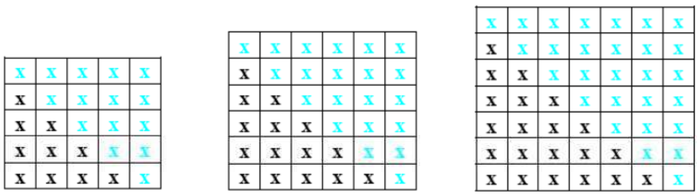


Figure 2. Continuation of the shape pattern invented by Anne, Helen, and Paul (Task 4a)

When they come to Tasks 4b and 4c, they wonder what is meant by "figurate number" and "mathematical statement", and get the teacher to help them. The teacher explains that the question about figurate numbers is about being down on the "bedrock" looking for standard numbers that it is common to have in one's "toolbox". The teacher thereafter asks what the students recognise if they look at the first element (given in the task) as a whole. The following exchange takes place:

591 Paul: Well, that it is five squared.
592 Teacher: Right.
593 Anne: Yes, it is indeed squares, and then [Pause 1-3 s]
594 Teacher: Yes, it is indeed squares, square numbers.
595 Paul: And then you have nine and sixteen as the numbers of [Pause 1-3 s] no, ten perhaps?

596 Teacher: And then there are the black and white ones? Do you recognise them?
597 Anne: Fifteen, twenty one [Pause 1-3 s] ehm

In turn 591 Paul focuses on the first element of the shape pattern (the 5x5 square). I interpret his words here to suggest that he continues to look at this element in turn 595 and refers to its number of black and white components. The numerical values he suggests at first are wrong, but he then makes a new suggestion which is right for the number of black components of the first element. The teacher does not directly respond to Paul's answer, and when the teacher asks if they recognise the black and white components (turn 596), Anne responds by giving the number of black and white components of the second element. This seems to make Paul insecure about what the teacher asks for; he wonders whether it is only the first element or it is the sequence of elements they are supposed to consider:

598 Paul: If we are supposed to see the connection, it is only *this* very shape we shall look at now? [Draws a curve with his pencil around the element given in the task] It is not the next shapes we have made [points at the succeeding elements drawn in his notebook when he says "next"]?
599 Teacher: You may well look at it as it stands there [Pause 1-3 s] uh [Pause 1-3 s] [indecipherable]
600 Paul: Not further, ok.

The teacher's response in turn 599 I interpret as confirming that it is satisfactory that the students look at the element given in the task (a 5x5 square) as a basis for finding answers to Tasks 4b and 4c. It is plausible that the teacher takes this stance as a consequence of seeing the 5x5 square as a generic example. This element of the shape pattern is an example which illustrates that the sum of the fourth and the fifth triangular numbers is equal to the fifth square number. It is generic (Rowland, 2000) in the sense that it is a representative of a class of elements which have the property that they are squares which (by the two colours) illustrate that the *n*-th square number is the sum of the $(n-1)$-th and the *n*-th triangular numbers.

These general properties are however not addressed in the classroom situation. The teacher does not express to the students that he uses the 5x5 square in the sense of a generic example, nor does he use the term "generic". What I interpret as the teacher's implicit utilisation of a generic example contributes to vagueness in the discourse: The stance taken by the teacher about the sufficiency of looking at one element of the shape pattern (genericity of the 5x5 square) is consistent with the formulation in Tasks 4b and 4c (reproduced below), a correspondence which may be expected since the task is designed by the same teacher. Application of *singular* number in the noun "the shape" indicates that the shape presented in the task is seen as generic:

> What kinds of figurate numbers do you find in the bright and the dark areas, and in *the shape* as a whole? [Task 4b, emphasis added]
> Express what *the shape* tells you about these numbers in terms of a mathematical statement. [Task 4c, emphasis added]

After having observed that the 5x5 square contains ten black components and fifteen white components, the students describe the structure of the next elements of the shape pattern. They observe that the elements develop by adding to the white components an extra row (at the top) with one more component, and that the number of black components of a successive element is the same as the number of white components of the present element. The teacher reminds the students that they have earlier written ten as a sum of the first four natural numbers, and further, tells them that numbers with this structure are referred to as triangular numbers. He refers to what I interpret as (for him) a generic example when he continues:

640 Teacher E:	So this is actually the clue here. That this element, I think I'll just tell you, that this shape represents a kind of connection between triangular numbers and square numbers.

This is succeeded by a comment by Anne that she had been insecure what was meant by the concept of "figurate numbers". After some exchanges between the teacher and her, she (re)turns attention to the concept of "mathematical statement" which so far has not been addressed explicitly by the teacher:

651 Anne:	Express what the shape tells about these numbers in terms of a mathematical statement [recitation from the task]. Are we supposed to write it as a formula or shall we formulate it?

Based on the students' conclusion on Task 4c (an explanation in natural language of the structure of the first element of the shape pattern), it is plausible that Anne in turn 651 is trying to figure out whether the teacher wants them to present the solution to Task 4c as a formula (potentially in mathematical notation) or as a formulation (potentially in natural language). The teacher responds by reinforcing attention towards the first element of the shape pattern, which I suggest he continues to use as a generic example:

652 Teacher:	Well, then you can think of that one [points at the 5x5 square presented in the task]. If you look at it as a whole, what square number is it that it [Pause 1-3 s] shows us? What position?
653 Helen:	Five or?
654 Anne:	What number in the series or?
655 Teacher:	What number in the series of square numbers, right.
656 Anne:	Well, I can imagine it is [Pause 1-3 s] the fifth then.
657 Teacher:	The fifth, right.
658 Anne:	Because that would have been good for us [smiles]
659 Teacher:	Yes. [Students laugh] Well, but here we don't have much choice, really. It is the fifth, it is twenty five, it is square number five. (Anne: uh huh). And if we think of it as composed by triangular numbers (Anne: yes) then you can think of [Pause 1-3 s] what position in the series of triangular numbers is that which these black and white [components] represent?

I interpret the teacher's utterances in turns 652, 655, 657, and 659 as an incidence of the Topaze effect (Brousseau, 1997): The answer that the students must give is determined in advance (the theorem asserting that the *n*-th square number is equal to the sum of the $(n-1)$-th and the *n*-th triangular numbers); the teacher chooses questions to which the answer can be given (turn 652). The knowledge necessary to produce these answers changes, so does its meaning. Faced with the student's continued difficulty in giving the answer, the teacher poses easier and easier questions: It is possible to answer the teacher's question in turn 659 without having to formulate the intended theorem. Hence the target knowledge has disappeared, a phenomenon referred to as the Topaze effect.

There are features in the milieu which I interpret as giving rise to the Topaze effect: First, the nature of the concept of a mathematical statement is not known to the students and is neither explained to them in plain text. Second, the teacher's use of a particular example in a generic sense, apparently without the students' consciousness about it, contributes to the students' comprehension of the particular example as representing a mathematical statement in its own right. The teacher leaves the students after turn 659, and the students collaborate to find the positions of the triangular numbers from which the fifth square number is constructed (the new task). The outcome of their engagement with Task 4c is the expression in natural language of the property of one particular shape: the fifth square number is constructed from the fourth and the fifth triangular numbers. They make no attempt to generalise this characteristic to apply to an arbitrary element of the shape pattern, neither in natural language, nor in algebraic notation.

682 Paul: Well, a person who could figure out a formula for this, he would be good [laughs].

683 Anne: No, but it is not written (Paul: no) that we shall have a formula (Paul: right). We are supposed to express it as a mathematical statement. We have done that now. It is not very good, but we have emphasised what is relevant, I think.

Recall that Anne asked the teacher if they were supposed to write the mathematical statement as a formula or just formulate it (turn 651). Paul's and Anne's utterances (turns 682 and 683) indicate that they have interpreted the teacher's response (turn 652) to Anne's question (turn 651) to mean that a mathematical statement is a formulation (in natural language) about the numbers in the shape given in the task. Further, Anne seems to conclude that a formula is different from a mathematical statement in the way she claims that they are not asked to find a formula (turn 683). It is likely that she by "formula" understands an expression in mathematical notation.

Discussion

The students remain unaware that the aim of Task 4 is to establish a theorem about a general relationship between numbers (or sequences of numbers). The generalisation process is obscured by two interrelated factors, which are interpreted as weaknesses in the milieu.

The first factor is about the design of the task: It is problematic that the task presents only one element of an imagined pattern, combined with the use of *singular* number ("the shape") when referring to the pattern. Further, there is a problem with the design of the task because the students produce appropriate solutions to the first two subtasks (with input from the teacher on the concept of figurate numbers); this, however, does not afford them with knowings that enable them to formulate the intended theorem. In the context of algebraic generalisation of shape patterns, the knowledge at stake is algebraic generalisation of arithmetic relations mapped from the elements of the pattern. For epistemological reasons, the focus in tasks on algebraic generalisation of shape patterns therefore should be on those arithmetic relations; e.g. in Task 4 (here efficiently represented in mathematical notation): that $T_4 + T_5 = 5^2$, $T_5 + T_6 = 6^2$ and so on, to subsequently encourage generalisation by algebraic thinking.

The second factor that constitutes a weakness in the milieu is about the students' unfamiliarity with the concept of mathematical statement. The students had identified the structure of the fifth, sixth and seventh elements, even if they had not been explicit about their rank (that is, they had not made the point that the fifth element is the sum of the first four natural numbers and the first five natural numbers, and likewise for the next two). They had got the teacher to come to them because they did not know what was meant by the concepts of figurate number and mathematical statement. Anne's recitation of Task 4c and her subsequent question (turn 651) indicates that the adidactical situation devolved to the students is not appropriate because it depends on knowledge they do not have (the concept of mathematical statement). The teacher, however, instead of explaining the nature of a mathematical statement, directs attention to the 5x5 square. It is relevant here that the teacher believes that the students know the concept of mathematical statement (articulated in conversation with the teacher after the lesson). It is therefore plausible that the teacher interprets Anne's question in turn 651 to signify a problem with seeing the invariant structure of the elements of the shape pattern, and not a problem with the concept of mathematical statement *per se*. For that reason, when he acts on the milieu, he tries to help them discover the structure of the elements (by utilising a generic example) so they can develop the knowledge aimed at: an equivalence relation between square numbers and the sum of two triangular numbers. But, as described above, the students' interpretation of the teacher's (generic) example as complete in itself, without attention to general properties, terminates the generalisation process. The milieu is changed, so is the knowledge needed to solve the (new) task. A gap has been created between the teacher's intention with the (original) task and the students' actions on the milieu.

Måsøval (2011) has identified that tasks on algebraic generalisation of shape patterns are of two different types, based on the mathematical object they aim at: The first type (arbitrary shape patterns) aims at a formula for the numerical value of the *n*-th member of the sequence mapped from the shape pattern; the second (conjectural shape patterns) aims at a theorem which asserts equality between two different algebraic expressions for the *n*-th member of the sequence mapped from the shape pattern. It is therefore important that those who design (or choose) tasks analyse the target the tasks aim at, whether it is a formula for the numerical value of the *n*-th element of the actual pattern (a functional relationship), or it is a general numerical statement (a theorem) decontextualised from the actual pattern. This is in order for the milieu to be designed such that the desired relationship can be explored and explicated by the students (e.g, through decomposition of elements according to the invariant structure of the pattern).

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References

Artigue, M., & Perrin-Glorian, M.-J. (1991). Didactic engineering, research and development tool: Some theoretical problems linked to this duality. *For the Learning of Mathematics, 11*(1), 13-18.

Brousseau, G. (1997). *The theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, The Netherlands: Kluwer.

Devlin, K. (1994). *Mathematics: The science of patterns.* New York, NY: Scientific American Library.

Måsøval, H. S. (2011). *Factors constraining students' appropriation of algebraic generality in shape patterns: A case study of didactical situations in mathematics at a university college*. (Unpublished doctoral dissertation). University of Agder, Kristiansand, Norway.

Perrin-Glorian, M.-J., Deblois, L., & Robert, A. (2008). Individual practising mathematics teachers: Studies of their professional growth. In K. Krainer & T. Wood (Eds.), *Participants in mathematics teacher education: Individuals, teams, communities and networks* (pp. 35-59). Rotterdam, The Netherlands: Sense.

Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting* *of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 2-21). Mérida, México: Universidad Pedagógica Nacional.

Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics*, *24*(2), 7-15.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2. ed.). Thousand Oaks, CA: Sage.

Whitehead, A. N. (1947). *Essays in science and philosophy*. New York, NY: Philosophical Library.