FROM RECURSIVE TO EXPLICIT FORMULA FOR THE *N*-TH MEMBER OF A SEQUENCE MAPPED FROM A SHAPE PATTERN

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This paper presents how two student teachers struggle to find an explicit formula for the general member of a sequence mapped from a shape pattern which they have successfully generalised in terms of a recursive formula. The paper shows how the milieu for algebraic generalisation in the observed episode is constrained in two senses: first, by the design of the task; second, by the teacher educator's intervention which presupposes prior knowledge that the student teachers do not have. It is indicated how manipulations on the geometrical configurations might enhance the milieu for algebraic generalisation of shape patterns and provide a link between recursive and explicit formulae.

Keywords: Recursive and explicit formulae, shape pattern, growth of sequence, decomposition, milieu, adidactical situation.

INTRODUCTION

There is a two-fold purpose of tasks on algebraic generalisation of shape patterns (Måsøval, 2011): One is to provide a physical or iconic context for algebraic generalisation where the aim is to promote students' algebraic thinking and justification. Here, algebra is approached through pattern generalisation. The other is to lead students to experience patterns as mathematical structures as an aim in itself. Here, algebra is a mediational means to represent invariant structures in the patterns. However, it is rather common that shape patterns are used only to produce a sequence of numbers which subsequently is generalised in terms of an algebraic formula without references to the elements of the pattern (Lannin, Barker & Townsend, 2006). Strategies of "guess-and-check" that involve superficial pattern spotting are frequently used with the consequence that students do not detect the generality of the formulae they find (Lannin et al., 2006).

THEORETICAL FRAMEWORK

In Brousseau's (1997) theory of didactical situations in mathematics, an *adidactical situation* is a situation in which the student takes a mathematical problem as his own and solves it on the basis of its internal logic without the teacher's guidance and without trying to interpret the teacher's intention with the problem. The *devolution* of an adidactical learning situation is the act by which the teacher encourages the student to accept the responsibility for an adidactical learning situation or for a problem, and the teacher accepts the consequences of the transfer of this responsibility (Brousseau, 1997). The student cannot engage in any adidactical situation; the teacher attempts to arrange an adidactical situation that the student can handle.

In the devolution process, which is part of the broader (didactical) situation, the teacher is faced with a system that consists of the student and a *milieu* "that lacks any didactical intentions with regard to the student" (Brousseau, 1997, p. 40). The milieu is a subset of the students' environment with only those features that are relevant with respect to the knowledge aimed at by the teacher in the didactical situation. The concept of milieu models the elements of the material or intellectual reality on which the students act and which may be an obstacle to their actions and reasoning (Laborde & Perrin-Glorian, 2005). That is, the milieu of a didactical situation is the part of the environment that can bring feedback to students' actions to accomplish a task.

An adidactical situation is part of the didactical situation that is the broader situation with the system of interaction of the students with the milieu arranged with the purpose of the students' appropriation of the target knowledge without the teacher's intervention (Brousseau, 1997). The teacher can act on the milieu by providing new information or new equipment, for example by asking a question or directing students' attention to certain factors in the classroom situation. When the teacher acts on the milieu, she changes the knowledge needed to solve the problem (Perrin-Glorian, Deblois & Robert, 2008). Whether the student can handle an adidactical situation depends upon two conditions: first, that the student has prior knowledge that enables him to engage with the situation; second, that the milieu created by the teacher provides the student with personal knowledge that enable him to develop the knowledge aimed at (by the teacher).

METHODOLOGY

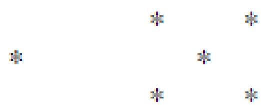
The reported research is derived from the author's PhD project (Måsøval, 2011). In the rest of the paper, "students" is used to refer to the student teachers, and "teacher" is used to refer to the teacher educator. The research question addressed in the paper is*: How do the mathematical task and the teacher's intervention constrain students' appropriation of algebraic generality in a shape pattern?* The task with which the paper deals (Figure 1) was the first of four tasks (during eight lessons) on algebraic generalisation of shape patterns. There had been a short whole-class introduction to figurate numbers (illustrated by triangular numbers) before the observed small-group lesson.

The data is a video-recorded observation of two students' collaborative engagement with Task 1 (with teacher intervention). The students are Alice and Ida, who were in their first academic year on a four-year undergraduate teacher education programme for primary and lower secondary school in Norway. The teacher is Erik, who was responsible for teaching algebra to the class of which Alice and Ida were members. The observed students worked on the same task as the rest of the class (ca. 60 students). My role during data collection was to be a non-participant observer while video-recording the interaction between the students and the teacher. The video-recorded episode has been transcribed and analysed through a process of open coding (using an adapted grounded theory approach, Strauss & Corbin, 1998) where concepts from Brousseau's (1997) theory have been used to answer the research question. For a discussion of the legitimacy of the theory of didactical situations in the analysis of the data, I refer to (Måsøval, 2011, Chapter 2.5).

An *a priori* analysis of the shape pattern in the mathematical task

Task 1 (Figure 1) has been designed by Erik for collaborative work in small groups. According to Erik, the aim of Task 1 was twofold: first, to express the regularity of the shape pattern in natural language; second, to transform the natural language expression into algebraic symbolism.

Below you see the development of the first two shapes in a pattern.



a) Draw the third and fourth shapes in this pattern. You may use the squared paper.

b) Count the number of stars in each of the shapes you have now, and put the results into a table. Explain how the number of stars increases from one shape to the next. Use this to calculate how many stars there are in the fifth shape.

c) What you have found in task b is called a recursive (or indirect) formula. Can you express it in terms of mathematical symbols?

d) Try to find a connection between the position of a shape and the number of stars in that shape. This is called an explicit formula. Can you express such a formula in terms of mathematical symbols?

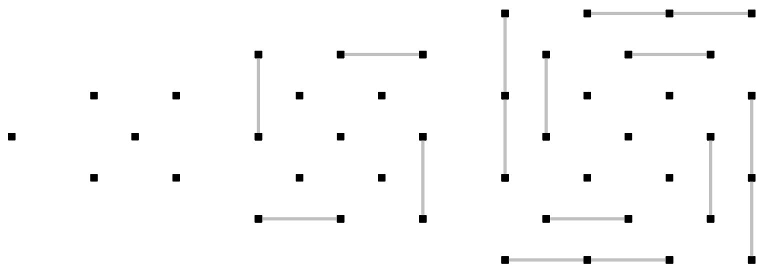
Figure 1. Task 1 given to the class for work in small groups

There are several possible ways to continue the pattern in Task 1 of which the first two geometrical configurations (elements) are given. In the *a priori* analysis I concentrate on the alternative identified by Alice and Ida (Figure 2), and relate it to the teacher's intention with the task.

Figure 2. A possible continuation of the shape pattern in Task 1

A figural approach to algebraic generality in shape patterns might involve an analysis of the invariant structure of the shape pattern by *decomposition* of its geometrical configurations according to an algorithmic rule (e.g., to isolate diagrammatically by encircling, or to paint with different colours). Below I present two possible decompositions of the actual pattern (shown in Figure 3 and Figure 4). In Figure 3 the first four elements are partitioned to illustrate that the differences of the sequence mapped from the shape pattern are multiples of four. Moreover, it is possible to see how the next element (with five dots on each side) can be made by the same rule:

Adding a line with four dots on each side of the fourth element (the same way as the other lines are placed) will complete the fifth element.



$a_1 = 1$ $a_2 = a_1 + 4 \cdot 1$ $a_3 = a_2 + 4 \cdot 2$ $a_4 = a_3 + 4 \cdot 3$

Figure 3. The first four elements illustrating that the differences are multiples of four

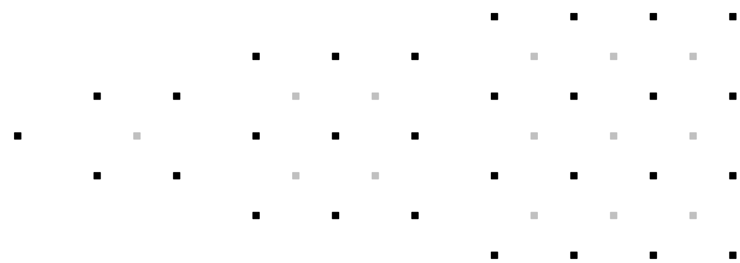
The arithmetic relations presented in Figure 3 ($a_1 = 1$, $a_2 = a_1 + 4 \cdot 1$, $a_3 = a_2 + 4 \cdot 2$, $a_4 = a_3 + 4 \cdot 3$) have references in the partitions: It is visible how each element is composed by the previous element plus four lines, each line with one dot less than the position of the current element.

Based on this, the *n*-th member of the sequence mapped from the shape pattern can be generalised by algebraic thinking (Mason, 1996) as the recursive formula $a_n = a_{n-1} + 4(n-1)$, with $a_1 = 1$. It can be noticed that the decomposition presented in Figure 3 can also serve as reference for the representation of the sequence shown in Table 1, where the *n*-th member is given in terms of an explicit formula: $b_n = 1 + \sum_{i=1}^{n-1} 4i$.

Table 1. Sequence originating from a decomposition in terms of multiples of four

Position	1	2	3	L	n
Number of dots	1	$1 + 4 \cdot 1$	$1 + 4 \cdot 1 + 4 \cdot 2$	L	$1 + \sum_{i=1}^{n-1} 4i$

A second decomposition is shown in Figure 4, where the components of the first four elements are drawn with different colours to illustrate that each element is a sum of consecutive squares. The arithmetic relations presented in Figure 4 ($c_1 = 1^2$, $c_2 = 2^2 + 1^2$, $c_3 = 3^2 + 2^2$, $c_4 = 4^2 + 3^2$) can be used to identify a relationship between the position of a member of the sequence and the rank of the squares to be added: The *n*-th member can be generalised by algebraic thinking as $c_n = n^2 + (n-1)^2$. This is an explicit formula for the *n*-th member of the sequence at stake. It is equivalent to b_n, but syntactically and semantically different (because it has been developed from a different route). The formula for the *n*-th partial sum of an arithmetic series can now be used to establish that $b_n = 1 + 4\frac{n}{2}(n-1) = 2n^2 - 2n + 1 = n^2 + (n-1)^2 = c_n$. This provides a connection between the different formulae with references to the partitions of the alternative decompositions presented.



$c_1 = 1^2$ $c_2 = 2^2 + 1^2$ $c_3 = 3^2 + 2^2$ $c_4 = 4^2 + 3^2$

Figure 4. The first four elements illustrating nested squares

Different decompositions developed by students provide opportunities to import different meanings for the algebraic symbols in formulae (corresponding to the structure of the different partitions). Moreover, it provides opportunities to engage students in meaningful manipulations of (equivalent) algebraic expressions when students show that formulae that are syntactically different can be transformed into the same expression.

ANALYSIS OF THE EPISODE

A formula for the general member of a sequence – isolated from the regularity of the shape pattern from which it is arising

The students have written the numbers mapped from the first four elements of the shape pattern down in a table where differences between successive members are written in the bottom row. Alice notices that "it is indeed the four-times table", and Ida says that "it must be sixteen and then twenty the next time." They use this to extend their table which becomes like the one shown in Table 2.

Table 2

Shape	1		2		3		4		5		6
Number of stars	1		5		13		25		41		61
		+4		+8		+12		+16		+20	

Then they reflect further on how the number of stars increases from one shape to the next (Task 1b):

28 Alice: You increase by the four-times table in a way. Each time [Pause 1-3 s] it increases by four and four each time.

29 Ida: Yes, for each new shape it increases by four.

30 Alice: Then it increases by [Pause 1-3 s] the previous [increment] plus four.

Their observations of the growth, as manifested in turns 28-30, describe properties of a pattern with quadratic growth: The first differences follow a pattern of linear growth; that is, the second differences are constant (Kalman, 1997). Alice and Ida, however, do not use the terms "linear", "quadratic", "difference", or "second difference". After they have written the first eight, and then the 19th, 42nd and 99th members of the sequence as the sum of the previous member and the difference, they arrive at the following recursive formula: $(n-1)4 + s_{n-1} = s_n$. This formula is similar to a_n presented above, except that it does not display the initial condition ($a_1 = 1$).

Alice and Ida have in the above used a numerical approach to generality. This is not surprising, given the design of the task; the focus is on counting components and putting the numbers into a table (Task 1a). Further, the students have decomposed particular numbers mapped from the shape pattern in terms of arithmetic relations, where the numbers are written as a sum of their predecessor and the difference. Algebraic thinking is then used to establish the formula $(n-1)4 + s_{n-1} = s_n$. This is based on the students' observation (of the particular members) that the difference between successive members is equal to the product of the number four and the position of the least member.

In the numerical approach employed by the students, the geometrical configurations are important only as a context to produce a sequence of numbers that subsequently is generalised in terms of a formula in algebraic symbols. However, the shape pattern in the task is a *real milieu* (Brousseau, 1997) in the sense that it can be manipulated. The students draw the next elements and count their components, but they do not decompose the geometrical configurations to analyse the invariant structure of the pattern: There is no feedback in the didactical situation that makes a structural analysis of the elements necessary.

The teacher's intention with the task (to express the regularity of the shape pattern) might have been handled by making the students create references (as suggested in the *a priori* analysis) between partitions (resulting from decomposition) of the elements on the one hand, and the symbols in the formula on the other. These two objects, partition and symbol, would be the outcomes of the situations of *action* and *formulation* (Brousseau, 1997), respectively. To express the regularity of the pattern in natural language (based on a decomposition) would correspond to an informal model of the regularity of the pattern in the situation of action, whereas the algebraic formula would correspond to a formal model of the regularity in the situation of formulation.

Alice and Ida do not make references between the iconic context and the arithmetic relations they have written down. In this way, when they generalise the arithmetic relations by algebraic thinking, it is likely that they do not experience the pattern as a mathematical structure to be an aim in itself, where algebra is a mediational means to represent the invariant structure in the pattern.

An attempt to make a connection between a recursive formula and an explicit formula does not succeed

When the students subsequently (without the teacher present) worked together to find an explicit formula for the *n*-th member of the same sequence, they use an analogue approach: In search for an explicit relationship, explained in Task 1d as "a connection between the position of a shape and the number of building blocks in that shape", they calculate the *differences* between the members of the actual sequence and their respective positions. This method I interpret as the students' erroneous application ("overgeneralisation") of the features of a recursive approach in an explicit approach to the general member of the sequence.

An explicit formula for the general member of a sequence mapped from a shape pattern can be defined as the numerical value of the n-th element expressed as a function of n. The method employed by the students is inappropriate because they establish an arithmetic relation (difference) between member and position, $f(n)-n$, instead of a functional relationship between member and position. Based on their calculation of the differences, $f(n)-n$, I infer that they have interpreted the word "connection" used in the task to mean "difference". Alice and Ida's construal is possibly influenced by their engagement with a recursive formula (Task 1c, in the same session), where they had calculated the differences between successive numbers of the sequence at stake. In search for an explicit formula (Task 1d), they produce the diagram shown in Table 3, where the commentary column (where R refers to "row") is made by me to explain how the numbers are derived. To distinguish differences from other elements of the table, differences are written in grey.

Table 3: Diagram produced by the students in search for an explicit formula

												Commentary	
	1		2		3		4		5		6	R 1	n
0		3		10		21		36		55		R 2	$f(n)-n$
	1		5		13		25		41		61	R 3	$f(n)$
		3		7		11		15		19		R 4	$(\text{R } 2)_{i+1}-(\text{R } 2)_i$
			+4		+4		+4		+4			R 5	$(\text{R } 4)_{i+1}-(\text{R } 4)_i$

The second row of Table 3 (in the commentary column symbolised by $f(n)-n$) consists of differences between members of the sequence mapped from the shape pattern (third row) and their position (first row). In constructing the numbers in the second row, the students have not focused on coordinating the referents (components and position) for the variables. If referents were added for numbers in the second row, the resulting number sentence would be: 5 [components] − 2 [position?] = 3 [components]. This is problematic since the operation of difference is not referent transforming. Further, the fourth row of Table 3 consists of first differences of $f(n)-n$, and the fifth row consists of second differences of $f(n)-n$.

When teacher Erik, who has designed the task, enters the room (on his own initiative), the students ask him if they are on the right track with respect to an explicit formula. The teacher responds by asking them what the characteristic of the recursive formula is. The students answer by describing the general nature of a recursive formula, but the teacher says that he means the particular recursive formula in Task 1. The conversation continues like this:

374	Teacher E:	It is not so easy, you know, the explicit one [laughs]. There is something, there is something about the recursive [relationship] which makes it complicated. [Pause 1-3 s] How is it if you look at the increase from one shape to the next?
375	Alice:	Ehm [Pause 1-3 s] you take the previous one and multiply. No, you take the previous increase and add four to it.
376	Teacher E:	Yes, exactly, right. You take the previous increase and add four.

When the teacher in turn 374 suggests that there is something that makes the sought explicit relationship complicated, I interpret that he attempts to explain the complexity by the *type* of growth of the sequence arising from the shape pattern. This is indicated in the same turn by the teacher's attention to the (first) differences of the sequence when he asks the students to describe "the increase from one shape to the next." He reinforces that the growth is non-constant by repeating Alice's description of the increase (turn 376). I find it plausible that his claim about the complexity of the sought explicit relationship and his attention to the fact that the first differences are non-constant, are attempts to make the students work analytically and thereby potentially deduce properties of the syntax of the explicit formula searched for. This interpretation has been approved by teacher Erik in a conversation I had with him after the observed lesson. It is consistent with a later utterance (from the transcript), where it appears that the teacher understands the numbers in the fourth row to represent the first differences of the sequence mapped from the shape pattern:

455	Teacher E:	It's just that, that [Pause 1-3 s] life would have been much easier with respect to a formula, if we had a pattern where *this* row had been a constant number [points at the fourth row in Table 3].

This interpretation by the teacher of the fourth row is possibly influenced by a comment by Alice (turn 377) about the same row of Table 3, where she claims to refer to the difference $f(6)-f(5)$ (which is equal to 20), whereas she actually refers to the difference $f(6)-6$ (which is equal to 55). Erik's interpretation of the fourth row of Table 3 as consisting of (first) differences would be in agreement with the second differences in the fifth row (which are constantly four). I interpret the teacher's intervention described in the above as an attempt to make a connection between the recursive properties of the sequence at stake (that it has *quadratic* growth) and the syntax of the desired explicit formula (that it is a polynomial of order *two*). This is however non-trivial, and there is no indication in the students' reasoning which suggests that they are able to utilise the teacher's hint: When the teacher later asks the students: "Do you have a kind of feeling which type of formulaic expressions that may emerge?" (turn 474), this is succeeded by 16 seconds of silence.

Given the school curriculum, I believe that it is most likely that Alice and Ida have no previous knowledge about different types of growth of sequences. Teacher Erik's intervention in this episode I interpret as an instance of a *metamathematical shift*

(Brousseau, 1997): It is characterised by the phenomenon that the teacher has substituted for the mathematical task (to find an explicit formula in algebraic notation) a discussion of the logic of its solution (what can be inferred about the syntax of the explicit formula from the observations about the growth of the sequence at stake). The teacher has tried to help the students improve their proficiency in establishing an explicit formula for the general member of a sequence, but the chosen method did not bring about the desired results. The unprompted utterances below indicate that the focus on a connection between recursive properties and the syntax of an explicit formula has not been helpful for the students:

611	Ida:	I'm supposed to come up with that one [explicit] [Pause 1-3 s] I'm all the time confused by the [recursive] one we figured out here.
612	Alice:	I don't see a clear distinction between recursive and explicit (Ida: no). I don't know what the different formulae are, and then I can't just shift from one to the other.

DISCUSSION

The milieu in the observed episode does not provide any feedback that requires that they analyse the pattern structurally (e.g, by decomposition) to make references between the elements of the pattern and the syntax of a formula. Quite the contrary, Task 1a focuses on counting components and putting the numbers into a table. This is in opposition to the teacher's intention, which is to express the invariant structure of the pattern. The feedback provided by the milieu with respect to an explicit formula is the concept "connection" between member and position. "Connection" is a vague (everyday) notion used instead of the mathematical concept "functional relationship" between member and position. It constitutes a weakness in the milieu because it contributes to confusion for the students in that they do not distinguish between a recursive approach (which involves *difference* between successive members) and an explicit approach (which involves member as a *function* of position).

When teacher Erik intervenes during Alice and Ida's struggle to find an explicit formula, he acts on the milieu by directing attention to a relationship between the recursive and the explicit formulae through the concept of type of growth of the sequence at stake. He thereby changes the knowledge needed to solve the problem. The analysis of the episode shows that the students cannot handle the new adidactical situation: They do not have knowledge of type of growth of sequences, neither does the milieu provide feedback that enables them to develop the knowledge necessary to utilise the teacher's intervention. However, encouraging students to connect recursive and explicit formulae is by Lannin et al. (2006) claimed to be important. Further, they recommend that tasks on generalisation of shape pattern be designed so as to promote students to remain connected to the figural representation (see also Steele, 2008). This is in line with Hewitt (1994) who warns against using contexts only to produce tables and spotting patterns in number sequences, because it does not give students insights into the structure of the original situation.

The *a priori* analysis of the actual shape pattern indicates how the process of decomposition might be used to provide feedback that could help the students remain connected to the figural representation, and hence discover and express the invariant structure of the pattern. Further, the decomposition presented in Figure 3 shows how a recursive formula and an explicit formula for the general member of the derived sequence are connected (in the way Figure 3 displays how an explicit formula is the sum of the first member and the first differences).

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