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On the Derived Category of Strongly Homotopy Associative Algebras

Master's thesis in Mathematical Sciences

Supervisor: Steffen Oppermann

December 2022

Master's thesis

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Norwegian University of Science and Technology
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Abstract

In this thesis, we study the homotopy theory of associative dg-algebras, conilpotent coassociative dg-coalgebras, and strongly homotopy associative algebras. We employ twisting morphisms to show that the cobar-bar construction defines a Quillen equivalence between conilpotent dg-coalgebras and dg-algebras. Every A_∞-algebra is a bifibrant object of the category of conilpotent dg-coalgebras, and the three associated homotopy categories are all equivalent.

Similarly, there are Quillen equivalences between comodule categories associated to conilpotent dg-coalgebras and module categories associated to dg-algebras. Every polydule of an A_∞-algebra is considered to be a bifibrant object of a comodule category, and the derived module category, homotopy category of the comodule category, and the derived polydule category are all equivalent.

Sammendrag

I denne avhandlingen studerer vi homotopiteorien til assosiative dg-algebraer, konilpotente koassosiative dg-koalgebraer og sterkt homotopi-assosiative algebraer. Vi bruker vridde morfier for å vise at kobar-bar konstruksjonen definerer en Quillen-ekvivalens mellom konilpotente dg-koalgebraer og dg-algebraer. Enhver A_∞-algebra er et bifibrant objekt i kategorien av konilpotente dg-koalgebraer, og de tre assosierte homotopikategoriene er ekvivalente.

På samme måte, er det Quillen-ekvivalenser mellom komodulkategorier assosiert til konilpotente dg-koalgebraer og modulkategorier assosiert til dg-algebraer. Enhver polydul til en A_∞-algebra kan ansees som et bifibrant objekt i en komodulkategori, og den deriverte modulkategorien, homotopikategorien til komodulkategorien og den deriverte polydulkategorien er alle ekvivalente.

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Introduction

A differential graded algebra, or simply dg-algebra, is an associative algebra where the underlying object is a cochain complex. Any dg-algebra A naturally carries homotopical information, and we get a graded algebra by considering the homology algebra H^*A. When we are working with homology algebras, there are many more morphisms than the morphisms coming from the differential graded structure. To understand homology algebras in the context of their dg-counterparts, we should restrict our attention solely to those morphisms from this structure. This leads us to the definition of a quasi-isomorphism, that is, morphisms $f : A \to B$ between dg-algebras such that $H^*f : H^*A \to H^*B$ is an isomorphism.

Localization is involved when constructing this category of homology algebras $\mathsf{HoAlg}^\bullet_{\mathbb{K}}$. We say that

$$\mathsf{HoAlg}^\bullet_{\mathbb{K}} = \mathsf{Alg}^\bullet_{\mathbb{K}}[\mathsf{Qis}^{-1}].$$

Localization works by adding morphisms, and we add new morphisms such that at least the intended class of morphisms we want to be invertible is invertible. The problem with this is that controlling how many morphisms we add is difficult, so figuring out which dg-algebras are quasi-isomorphic is not a simple process.

There is a weaker structure called strongly homotopy associative algebras, or A_∞-algebras. An A_∞-algebra is almost a dg-algebra, but the multiplication may fail to be associative. Instead, we assume that the associator is null-homotopic and an infinite hierarchy of homotopies controls this homotopy. By considering an A_∞-algebra A up to homotopy, we see that the homotopy algebra A defines a graded algebra.

It is becoming well known that quasi-isomorphisms $f : A \to B$ between A_∞-algebras admit a homotopy inverse. When we localize the category of A_∞-algebras at quasi-isomorphism, there is an equivalence to the homotopy category

$$\mathsf{HoAlg}_\infty = \mathsf{Alg}_\infty[\mathsf{Qis}^{-1}] \simeq {}^{\mathsf{Alg}_\infty}/{\sim}.$$

Using this construction, we can bypass the localization construction. Instead of adding new morphisms to invert the quasi-isomorphisms, we can identify homotopic morphisms.

What might be surprising is that there is an equivalence of categories,

$$\mathsf{HoAlg}^\bullet_{\mathbb{K}} \simeq \mathsf{HoAlg}_\infty.$$

This equivalence is given by localizing the non-full inclusion functor $i : \mathsf{Alg}^\bullet_{\mathbb{K}} \to \mathsf{Alg}_\infty$ at quasi-isomorphisms. We may say that a quasi-isomorphism $f : A \to B$ between dg-algebras admits a homotopy inverse of the corresponding A_∞-algebras. Similarly, we bypass the localization construction by considering homotopy algebras,

$$\mathsf{HoAlg}_{\mathbb{K}} \simeq {}^{\mathsf{Alg}_\infty}/\sim,$$

This result is still true if we consider quasi-isomorphisms $f : M \to N$ between A-modules. If we consider M and N as A-polydules, that is, A_∞-modules, the morphism f admits a homotopy inverse. With this in mind, there are equivalences of categories,

$$D_\infty A \simeq K_\infty A \simeq DA.$$

Here, $D_\infty A$ and $K_\infty A$ denote the derived and homotopy category of the category of A_∞-modules, respectively.

In this thesis, we investigate a proof provided by Lefèvre-Hasegawa [Lef03] on the homotopy invertibility of quasi-isomorphisms. In our approach, we will take a lot of inspiration from Loday and Vallette [LV12]. We wish to elaborate upon Lefèvre-Hasegawa's work to make this particular instance clearer and more accessible. Many of the concepts we will discuss here for associative algebras have been generalized to many different algebras. See, for instance, [Val20] for a generalization to Koszul operads.

The thesis is split into three different chapters.

Chapter 1 - The Bar and Cobar Construction

In Chapter 1, we develop the theory of dg-algebras and dg-coalgebras. We try to make the theory of coalgebras more intuitive by comparing how they differ from algebras. The augmented algebras and conilpotent coalgebras are of utmost importance in this thesis.

The essential tool developed in this chapter is the bar and cobar construction, denoted as B and Ω, respectively. Twisting morphisms play a unique role as they define a functor, represented by the bar and cobar construction. Thus, we have an adjoint pair of functors,

$$\begin{tikzcd} \mathsf{coAlg}^\bullet_{\mathbb{K},\mathsf{conil}} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Alg}^\bullet_{\mathbb{K},+} \arrow[l, shift left, "B"] \end{tikzcd}$$

Lastly, we define A_∞-algebras in terms of the bar construction. We will think of these as the algebras which make the bar construction fully faithful on the image of quasi-free conilpotent dg-coalgebras. We can thus think of an A_∞-algebra in two different ways, either as a dg-algebra with strong homotopy associativity or as a conilpotent dg-coalgebra. Both points of view will be fruitful.

Chapter 2 - Homotopy Theory of Algebras

Chapter 2 aims to explain some of the homotopy theories of dg-algebras, conilpotent dg-coalgebras, and A_∞-algebras. We start by giving an exposition on model categories, having a special interest in Whitehead's theorem, the fundamental theorem of model categories, and Quillen equivalences.

We upgrade the cobar-bar adjunction into a Quillen equivalence, identifying the homotopy category of dg-algebras and conilpotent dg-coalgebras. The category of A_∞-algebra will be equivalent to the bifibrant conilpotent dg-coalgebras. This will allow us to show the first claim,

$$\mathsf{HoAlg}_{\mathbb{K}} \simeq {}^{\mathsf{Alg}_\infty}\!/\!\sim.$$

Chapter 3 - Derived Categories of Strongly Homotopy Associative Algebras

In the final chapter, we investigate the homotopy theory of modules over dg-algebras and comodules over dg-coalgebras. We will further develop the theory of twisting morphisms to obtain Quillen equivalences,

$$\begin{tikzcd} \mathsf{coMod}^C \arrow[r, shift left, "L_\alpha"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}^A \arrow[l, shift left, "R_\alpha"] \end{tikzcd}$$

We prove the fundamental theorem of twisting morphisms, which allows us to characterize whenever a twisting morphism defines a Quillen equivalence.

A_∞-modules of A, called A-polydules are defined to be objects being the converse of R_α whenever $C = BA$. We may then see that A-polydules are the bifibrant BA-comodules. We will then define the derived category of polydules, $D_\infty A$. We will conclude the thesis by showing that,

$$D_\infty A \simeq K_\infty A \simeq DA.$$

Prerequisites

We assume the reader is familiar with homological algebra, category theory, triangulated categories, and Kan extensions. The theory of monads, simplicial sets, spectral sequences, and symmetric monoidal categories will also be applied. At the end of the thesis, four appendixes are supplied, recalling the definitions and most important results, which we will use throughout this thesis.

Chapter 1

Bar and Cobar Construction

In Stasheff [Sta63], a strongly homotopy associative algebra, or A_∞-algebra, over a field is a graded vector space together with homogenous linear maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ satisfying some homotopical relations; this will be made precise later. We will regard m_2 as a multiplication of A, but it is not a priori associative. We choose m_3 to be a homotopy of m_2's associator. In this manner, we know that the homotopy of A is an associative algebra. The maps m_n corresponds uniquely to a map $m^c : BA \to \overline{A}[1]$, which extends to a coderivation $m^c : BA \to BA$ of the bar construction of A. With this relation, we will define an A_∞-algebra to be a coalgebra on the form BA, and we will prefer to do so in this thesis.

To understand the bar construction, we will first study it on associative algebras. Given a differential graded coassociative coalgebra C and a differential graded associative algebra A, we say that a homogenous linear transformation $\alpha : C \to A$ is twisting if it satisfies the Maurer-Cartan equation;

$$\partial\alpha + \alpha \star \alpha = 0.$$

Let Tw(C, A) be the set of twisting morphisms from C to A. It defines a functor Tw : coAlg$_{\mathbb{K}}^{op} \times$ Alg$_{\mathbb{K}} \to Ab$, which is represented in both arguments. Moreover, these representations give rise to an adjoint pair of functors called the bar and cobar construction.

$$\begin{tikzcd} \mathsf{Alg}^{\bullet}_{\mathbb{K},+} \arrow[r, shift right, "B"'] \arrow[r, phantom, "\perp"] & \mathsf{Coalg}^{\bullet}_{\mathbb{K},\mathsf{conil}} \arrow[l, shift right, "\Omega"'] \end{tikzcd}$$

This chapter will follow the notions and progression presented in Loday and Vallette [LV12] to develop the theory for the bar-cobar adjunction, which will be the basis for our discussion of A_∞-algebras.

1.1 Algebras and Coalgebras

1.1.1 Algebras

This section reviews associative algebras over a field $\mathbb{K}$. We denote the category of such algebras $\mathsf{Alg}_{\mathbb{K}}$, and we will study some of its properties before dualizing these to the context of coalgebras.

Definition 1.1.1 ($\mathbb{K}$-Algebra)**.** Let $\mathbb{K}$ be a field with unit 1. A $\mathbb{K}$-algebra A, or an algebra A over $\mathbb{K}$, is a vector space with structure morphisms called multiplication and unit,

$$(\cdot_A) : A \otimes_{\mathbb{K}} A \to A$$
$$1_A : \mathbb{K} \to A,$$

satisfying the associativity and identity laws.

$$\begin{aligned} &\text{(associativity)} && (a \cdot_A b) \cdot_A c = a \cdot_A (b \cdot_A c) \\ &\text{(unitality)} && 1_A(1) \cdot_A a = a = a \cdot_A 1_A(1) \end{aligned}$$

Whenever A does not possess a unit morphism, we will call A a non-unital algebra. In this case, only the associativity law must hold.

By abuse of notation, we will confuse the unit of $\mathbb{K}$ with the unit of A. Since 1_A is a ring homomorphism, this is well-defined. However, when we use the unit as a morphism, we will stick to the 1_A notation. When there is no confusion, we will exchange the symbol $(\cdot_A)$ with words in A. In other words, variable concatenation replaces $(\cdot_A)$.

Definition 1.1.2 (Algebra homomorphisms)**.** Let A and B be algebras. Then $f : A \to B$ is an algebra homomorphism if

1. f is $\mathbb{K}$-linear
2. $f(ab) = f(a)f(b)$
3. $f \circ 1_A = 1_B$

Whenever A and B are non-unital, we must drop the condition that f preserves units.

Definition 1.1.3 (Category of algebras)**.** We let $\mathsf{Alg}_{\mathbb{K}}$ denote the category of $\mathbb{K}$-algebras. Its objects consist of every algebra A, and the morphisms are algebra homomorphisms. The sets of morphisms between A and B are denoted as $\mathsf{Alg}_{\mathbb{K}}(A, B)$.

Let $\widehat{\mathsf{Alg}}_{\mathbb{K}}$ denote the category of non-unital algebras. Its objects consist of every non-unital algebra A, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between A and B are denoted as $\widehat{\mathsf{Alg}}_{\mathbb{K}}(A, B)$.

There is an equivalent description of algebras by considering the symmetric monoidal category $(\mathsf{Mod}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{Z})$. Observe that given any algebra A in $\mathsf{Mod}_{\mathbb{K}}$, the triple $(A, (\cdot_A), 1_A)$ is a monoid.

There is thus an isomorphism of categories, namely $\mathsf{Alg}_{\mathbb{K}}$ is the category of monoids in $\mathsf{Mod}_{\mathbb{K}}$. The algebra axioms are then equivalent to the commutative diagrams below.

$$\begin{tikzcd}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[r, "(\cdot_A)\otimes id_{\mathbb{K}}"] \arrow[d, "id_{\mathbb{K}}\otimes(\cdot_A)"] & A \otimes_{\mathbb{K}} A \arrow[d, "(\cdot_A)"] \\
A \otimes_{\mathbb{K}} A \arrow[r, "(\cdot_A)"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_{\mathbb{K}} \mathbb{K} \arrow[r, "id_A \otimes 1_A"] \arrow[dr, "\simeq"'] & A \otimes_{\mathbb{K}} A \arrow[d, "(\cdot_A)"] & \mathbb{K} \otimes_{\mathbb{K}} A \arrow[l, "1_A \otimes id_A"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}$$

In any symmetric monoidal category $\mathcal{C}$, we may reformulate these definitions by using the monoidal structure. Section 3 will introduce electronic circuits inspired by some of the proofs found in [LV12]. These conventions will give us a graphical calculus of morphisms in $\mathcal{C}$.

We supply some examples of algebras one may encounter in nature.

Example 1.1.4. Let $\mathbb{K}$ be any field. The field is trivially an algebra over itself.

Example 1.1.5. The complex numbers $\mathbb{C}$ is an algebra over $\mathbb{R}$, as it is a vector space over $\mathbb{R}$, and complex multiplication respects scalar multiplication.

Example 1.1.6. Let $\mathbb{K}$ be any field. The ring of n-dimensional matrices $M_n(\mathbb{K})$ is an algebra over $\mathbb{K}$. The multiplication is matrix multiplication, and the unit is the n-dimensional identity matrix.

Augmented algebras will be central to our discussion. An algebra A is augmented if an algebra homomorphism splits the algebra into an augmentation ideal and a unit component. We make this precise with the following definition

Definition 1.1.7 (Augmented algebras)**.** A $\mathbb{K}$-algebra A is augmented if there is an algebra homomorphism $\varepsilon_A : A \to \mathbb{K}$. We refer to the pair (A, ε_A) as the augmented algebra.

Given this algebra homomorphism, we know it has to preserve the unit. Thus the kernel Ker$\varepsilon_A \subseteq A$ is almost A, but without its unit. In the module category $Mod_{\mathbb{K}}$, the morphism ε_A is automatically a split-epimorphism, where the splitting is the unit 1_A. Thus as a module, we have $A \simeq \overline{A} \oplus \mathbb{K}$, where $\overline{A} =$ Kerε_A. $\overline{A}$ is called the augmentation ideal or the reduced algebra of A.

A morphism $f : A \to B$ of augmented algebras is an algebra homomorphism, but with the added condition that it must preserve the augmentation, i.e., $\varepsilon_B \circ f = \varepsilon_A$. The collection of all augmented algebras over $\mathbb{K}$ together with the morphisms defines the category of augmented algebras over $\mathbb{K}$, $\mathsf{Alg}_{\mathbb{K},+}$.

Given an augmented algebra A, taking kernels of ε_A gives a functor $\overline{\_} : \mathsf{Alg}_{\mathbb{K},+} \to \widehat{\mathsf{Alg}}_{\mathbb{K}}$. This functor is well-defined on morphisms of augmented algebras, as each morphism is required to preserve the splitting. This functor has a quasi-inverse, given by the free augmentation $\_^+ : \widehat{\mathsf{Alg}}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K},+}$. Given a non-unital algebra A, the free augmentation is defined as $A^+ = A \oplus \mathbb{K}$, where the multiplication is given by:

$$(a, k)(a', k') = (aa' + ak' + a'k, kk').$$

The unit is given by the element $(0, 1)$. We summarize this in the statement below.

Proposition 1.1.8. *The functors $\overline{\_}$ and $\_^+$ are quasi-inverse to each other.*

Proof. We show that the free augmentation functor is fully faithful and essentially surjective.

Let A and B be non-unital $\mathbb{K}$-algebras, and let $f, g : A \to B$ morphisms in $\widehat{\mathsf{Alg}}_{\mathbb{K}}$. Suppose that $f^+ = g^+$, then $f = \overline{f^+} = \overline{g^+} = g$. Now suppose that $h : A^+ \to B^+$, then $h = \overline{h}^+$.

Suppose that $A \in \mathsf{Alg}_{\mathbb{K},+}$. We want to show that $A \simeq \overline{A}^+$. As $\mathbb{K}$-modules, $A = \overline{A}^+$, so we propose that $id_A : A \to \overline{A}^+$ induces an isomorphism. To see that id_A is an algebra homomorphism is to see that the multiplication in A decomposes as $(a_1 + k)(a_2 + l) = (a_1a_2 + a_1l + ka_2) + kl$, where $a_1, a_2 \in \overline{A}$ and $k, l \in \mathbb{K}$. The second condition is equivalent to the existence of ε_A. id_A also preserves the augmentation as $\overline{A} \simeq \overline{\overline{A}^+}$. $\square$

There are many augmented algebras to encounter in nature. We will note some examples.

Example 1.1.9 (Group algebra). Pick any group G and any field $\mathbb{K}$. The group ring $K[G]$ is an augmented algebra where the augmentation $\varepsilon_{\mathbb{K}[G]} : \mathbb{K}[G] \to \mathbb{K}$ is given as

$$\varepsilon_{\mathbb{K}[G]}(\sum_{g \in G} k_g g) = \sum_{g \in G} k_g.$$

Among our most important example of algebras is the tensor algebra, which is also the free algebra over $\mathbb{K}$.

Example 1.1.10 (Tensor algebra). Let V be a $\mathbb{K}$-module. We define the tensor algebra $T(V)$ of V as the module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

The tensor algebra is then the algebra consisting of words in V. Given two words $v^1..v^i$ and $w^1...w^j$ in $T(V)$ we define the multiplication by the concatenation operation,

$$\begin{aligned} \nabla_{T(V)} : T(V) \otimes_{\mathbb{K}} T(V) &\to T(V), \\ (v^1...v^i) \otimes (w^1...w^j) &\mapsto v^1...v^i w^1...w^j. \end{aligned}$$

The unit is given by including $\mathbb{K}$ into $T(V)$,

$$\begin{aligned} \upsilon_{T(V)} : \mathbb{K} &\to T(V), \\ 1 &\mapsto 1. \end{aligned}$$

Observe that the tensor algebra is augmented. The projection from $T(V)$ into $\mathbb{K}$ is an algebra homomorphism, and its splitting is the inclusion $\mathbb{K} \to T(V)$. We obtain a splitting of the tensor algebra into its unit component and its augmentation ideal $T(V) \simeq \mathbb{K} \oplus \overline{T}(V)$. $\overline{T}(V)$ is called the reduced tensor algebra.

Proposition 1.1.11 (Tensor algebras are free). *The tensor algebras are the free algebras over the category of $\mathbb{K}$-modules, i.e., for any $\mathbb{K}$-module V, there is a natural isomorphism Hom$_{\mathbb{K}}(V, A) \simeq$ Alg$_{\mathbb{K}}(T(V), A)$.*

The reduced tensor algebra is the free non-unital algebra over the category of $\mathbb{K}$-modules. That is, for any $\mathbb{K}$-module V there is a natural isomorphism Hom$_{\mathbb{K}}(V, A) \simeq \widehat{\text{Alg}}_{\mathbb{K}}(\overline{T}(V), A)$.

Proof. If $f : T(V) \to A$ is an algebra homomorphism, then f must satisfy the following conditions:

- Unitality: $f(1) = 1$
- Homomorphism property: Given $v, w \in V$, then $f(vw) = f(v) \cdot_A f(w)$

By induction, we see that f is determined by where it sends the elements of V. Thus, restriction along the inclusion of V into $T(V)$ induces a bijection. □

Modules

As for rings, every algebra A has a module category.

Definition 1.1.12 (Modules). Let A be an algebra over $\mathbb{K}$. A $\mathbb{K}$-module M is said to be a left (right) A-module if there exists a structure morphism $\mu_M : A \otimes_{\mathbb{K}} M \to M$ ($\mu_M : M \otimes_{\mathbb{K}} A \to M$) called multiplication. We require that μ_M is associative and preserves the unit of A; i.e. we have the commutative diagrams in Mod$_{\mathbb{K}}$,

$$\begin{tikzcd} A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} M \arrow[r, "A\otimes\mu_M"] \arrow[d, "(\cdot_A)\otimes M"] & A \otimes_{\mathbb{K}} M \arrow[d, "\mu_M"] \\ A \otimes_{\mathbb{K}} M \arrow[r, "\mu_M"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes_{\mathbb{K}} M \arrow[r, "1_A\otimes M"] \arrow[dr, "\simeq"] & A \otimes_{\mathbb{K}} M \arrow[d, "\mu_M"] \\ & M \end{tikzcd}$$

Definition 1.1.13 (A-linear homomorphisms). Let M, N be two left A-modules. A morphism $f : M \to N$ is called A-linear if it is $\mathbb{K}$-linear and for any a in A $f(am) = af(m)$.

The category of left A-modules is denoted as Mod$_A$, where the morphisms Hom$_A(_,_)$ are A-linear. Likewise, we denote the category of right A-modules as Mod^A. There is a free functor from $\mathbb{K}$-modules to left A-modules.

Proposition 1.1.14. *Let M be a $\mathbb{K}$-module. The module $A \otimes_{\mathbb{K}} M$ is a left A-module. Moreover, it is the free left module over $\mathbb{K}$-modules, i.e. there is a natural isomorphism Hom$_{\mathbb{K}}(M, N) \simeq$ Hom$_A(A \otimes_{\mathbb{K}} M, N)$.*

Proof. We define natural transformations in each direction and then show that they are inverses.

We define morphisms ϕ and ψ as

$$\begin{aligned}\phi : \mathrm{Hom}_A(A \otimes_{\mathbb{K}} M, N) &\to \mathrm{Hom}_{\mathbb{K}}(M, N)\\ f &\mapsto f \circ (1_A \otimes M),\\ \psi : \mathrm{Hom}_{\mathbb{K}}(M, N) &\to \mathrm{Hom}_A(A \otimes_{\mathbb{K}} M, N)\\ g &\mapsto \mu_N \circ (A \otimes g).\end{aligned}$$

Pick an $f \in \mathrm{Hom}_A(A \otimes_{\mathbb{K}} M, N)$, then

$$\psi \circ \phi(f) = \mu_N \circ (A \otimes \phi(f)) = \mu_N \circ (A \otimes f(1_A \otimes M)) = f(A \otimes M) = f.$$

Pick a $g \in \mathrm{Hom}_{\mathbb{K}}(M, N)$, then

$$\phi \circ \psi(g) = \phi(\mu_N \circ (A \otimes g)) = \mu_N \circ (1_A \otimes g) = g.$$

□

Corollary 1.1.14.1. *A as a left A-module is the free left A-module over $\mathbb{K}$; i.e. for any left A-module M, $M \simeq$ Hom$_{\mathbb{K}}(\mathbb{K}, M) \simeq$ Hom$_A(A, M)$*

Categorical structure

It is convenient to understand some of the most fundamental limits and colimits to understand the category of algebras. Unfortunately, the category of algebras does not have nice kernels and cokernels; therefore, we will restrict our attention to augmented algebras.

The category of augmented algebras is pointed. Since every morphism of augmented algebras has to preserve both unit and counit, the algebra $\mathbb{K}$ is both initial and terminal.

Definition 1.1.15. Let A and B be augmented algebras. We define their direct sum $A \oplus B$ as the following limit:

$$\begin{tikzcd} A \oplus B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "\varepsilon_A"] & \mathbb{K} \end{tikzcd}$$

Notably, $A \oplus B$ is the product in Alg$_{\mathbb{K},+}$, since $\mathbb{K}$ is terminal. Calculating this limit as a kernel, it is a subobject of $A \oplus B$ in the sense of $\mathbb{K}$-modules. We have the following relation between the direct and the ordinary direct sum.

Lemma 1.1.16. *The direct sum of augmented algebras A and B is the free augmentation on the direct sum of the augmentation ideals, $A \oplus B \simeq (\overline{A} \oplus \overline{B})^+$.*

Proof. This lemma is clear from the monadicity of the forgetful functor; see Theorem A.2.10,

$$\begin{aligned} forget : \mathsf{Alg}_{\mathbb{K},+} &\to \mathsf{Mod}_{\mathbb{K}} \\ A &\mapsto \overline{A}. \end{aligned}$$

□

Observe that the injections $A \hookrightarrow A \oplus B$ and $B \hookrightarrow A \oplus B$ do not satisfy the universal property of the coproduct. Thus, the direct sum is no longer the coproduct in this category.

Definition 1.1.17. Given two augmented algebras A and B, the free product $A * B$ is defined as the following colimit:

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\upsilon_A"] \arrow[d, "\upsilon_B"] & A \arrow[d] \\ B \arrow[r] & A * B \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Notice that the free product is definitionally the coproduct. In the case of groups, the free product consists of every formal word formed from letters from each group. We extend this construction to augmented algebras, following the main idea presented by Aambø [Aam21].

Lemma 1.1.18. *Let A and B be augmented algebras. The free product is isomorphic to a quotient of the tensor algebra*

$$A * B \simeq T(\overline{A} \oplus \overline{B})/I.$$

The right-hand side is the tensor algebra over the direct sum of the underlying non-unital algebras, and I is an ideal generated by elements on the form $\langle a \otimes a' - a \cdot a', b \otimes b' - b \cdot b' \rangle$.

Proof. We have naturally injective linear morphisms

$$\begin{aligned} \iota_A : A &\hookrightarrow T(\overline{A} \oplus \overline{B})/I, \\ a &\mapsto a, \\ 1 &\mapsto 1. \end{aligned}$$

This is in fact a ring homomorphism since $\iota_A(aa') = aa' = a \otimes a' = \iota_A(a)\iota_A(a')$.

Suppose we have the following diagram.

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"'] \arrow[dr, bend right, "f"'] & T(\overline{A} \oplus \overline{B})/I \arrow[d, dashed, "h"] & B \arrow[l, hook', "\iota_B"] \arrow[dl, bend left, "g"] \\ & T & \end{tikzcd}$$

By functoriality we obtain a morphism $h = T(\overline{f} \oplus \overline{g}) : T(\overline{A} \oplus \overline{B}) \to T$. Unitality and augmentation property force this to act as the identity on the respective identities. Clearly $f = h\iota_A$ and $g = h\iota_B$.

Assume there exists another $h' : {}^{T(\overline{A}\oplus\overline{B})}/_{I} \to T$ such that $f = h'\iota_A$ and $g = h'\iota_B$. Then $h = h'$ on $A \oplus B$ part of ${}^{T(\overline{A}\oplus\overline{B})}/_{I}$. Since h' is a ring morphism, $h = h'$ on all of ${}^{T(\overline{A}\oplus\overline{B})}/_{I}$. □

The forgetful functor creates every small limit in $\mathsf{Alg}_{\mathbb{K},+}$, and the kernel is no exception to this.

Lemma 1.1.19. *Suppose that $f : A \to B$ is a morphism of augmented algebras. The kernel of f is isomorphic to* Ker$f = (\overline{Ker} f)^+$.

Proof. This lemma is clear from the monadicity of the forgetful functor. □

On the other hand, $\mathsf{Alg}_{\mathbb{K},+}$ is cocomplete as well. However, the colimits are not as simple to describe. In some cases, we can give a simple description of it. E.g., we know that the cokernel of a morphism $f : A \to B$ exists and is ${}^{\overline{B}}/_{\overline{A}}{}^+$ if A is an ideal of A. Thus A is the kernel of the cokernel morphism $g : B \to {}^{\overline{B}}/_{\overline{A}}{}^+$. Conversely, if f is the kernel morphism of g, then A is an ideal of B. In other words, we may think of an ideal as a kernel.

Given any morphism $f : A \to B$, we may consider its coimage-image factorization.

$$\begin{tikzcd}
 & A \arrow[rrr, "f"] \arrow[dr, two heads] & & & B \arrow[dr, two heads] & \\
\mathrm{Ker}f \arrow[ur, hook] \arrow[rr, "0"] & & \mathrm{coIm}f \arrow[r, hook, two heads, "\widetilde{f}"] & \mathrm{Im}f \arrow[ur, hook] \arrow[rr, "0"] & & \mathrm{coKer}f
\end{tikzcd}$$

It is clear that Imf is an ideal of B, thus coKer$f \simeq {}^{\overline{B}}/_{\overline{\mathrm{Im}f}}{}^+$. The problem is that in the category of algebras, we cannot be sure if $\widetilde{f}$ is an isomorphism, even if it is mono and epi. Thus the ordinary set-theoretic image, coImf, may not be the categorical image, Imf. We define the image as the smallest ideal of B such that coIm$f \subseteq$ Im$f \subseteq B$, and f is called regular whenever $\widetilde{f}$ is an isomorphism. In this case, the image is then the same as the set-theoretic image, and

$$\mathrm{coKer}f \simeq {}^{\overline{B}}/_{\overline{\mathrm{Im}f}}{}^+.$$

1.1.2 Coalgebras

A coalgebra is like an algebra, but we reverse every arrow. In this section, we dualize the definitions as given for algebras. For many purposes, this dualization is good, but as we will observe, some finiteness conditions are necessary. We will denote the category of coalgebras as $\mathsf{coAlg}_{\mathbb{K}}$.

Definition 1.1.20 ($\mathbb{K}$-Coalgebra)**.** Let $\mathbb{K}$ be a field. A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-module with structure morphisms called comultiplication and counit,

$$(\Delta_C) : C \to C \otimes_{\mathbb{K}} C$$
$$\varepsilon_C : C \to \mathbb{K},$$

satisfying the coassociativity and coidentity laws.

$$\text{(coassociativity)} \quad (\Delta_C \otimes id_C) \circ \Delta_C(c) = (id_C \otimes \Delta_C) \circ \Delta_C(c)$$
$$\text{(counitality)} \quad (id_C \otimes \varepsilon_C) \circ \Delta_C(c) = c = (\varepsilon_C \otimes id_C) \circ \Delta_C(c)$$

In the same way as for algebras, we say that a coalgebra is non-counital if it is without a counit.

Like algebras, coalgebras admits a single intuitive method for writing repeated application of the comultiplication. To see this, pick an element $c \in C$, we may apply the comultiplication twice on c in two different ways:

$$\Delta^2_{C,(1)}(c) = (\Delta_C \otimes C)\Delta_C(c),$$
$$\Delta^2_{C,(2)}(c) = (C \otimes \Delta_C)\Delta_C(c).$$

One should immediately note that $\Delta^2_{C,(1)}(c) = \Delta^2_{C,(2)}(c)$ is the coassociativity axiom. Hence there is a unique way to make repeated applications of Δ_C on c. We denote the n-fold repeated application of Δ_C by Δ^n_C. Since the element $\Delta^n_C(c)$ represents a finite sum in $C^{\otimes n}$, we may use Sweedlers notation [LV12],

$$\Delta^n_C(c) = \sum c_{(1)} \otimes ... \otimes c_{(n)}.$$

Definition 1.1.21 (Coalgebra homomorphism)**.** Let C and D be coalgebras. Then $f : C \to D$ is a coalgebra morphism if

1. f is $\mathbb{K}$-linear
2. $(f \otimes f) \circ \Delta_C(c) = \Delta_D(f(c))$
3. $\varepsilon_D \circ f = \varepsilon_C$

Whenever C and D are non-counital, we only require 1. and 2. for a homomorphism of non-counital coalgebras.

Definition 1.1.22 (Category of coalgebras)**.** Let $\mathsf{coAlg}_{\mathbb{K}}$ denote the category of coalgebras. Its objects consist of coalgebras C, and the morphisms are coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\mathsf{coAlg}_{\mathbb{K}}(C, D)$.

Let $\widehat{\mathsf{coAlg}}_{\mathbb{K}}$ denote the category of non-counital algebras. Its objects consist of non-counital algebras C, and the morphisms are non-counital coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\widehat{\mathsf{coAlg}}_{\mathbb{K}}(C, D)$.

At first glance, coalgebras may seem weird and unnatural, but they appear in many places in nature.

Example 1.1.23 ($\mathbb{K}$ as a coalgebra). The field $\mathbb{K}$ can be given a coalgebra structure over itself. Since $\{1\}$ is a basis for $\mathbb{K}$ we define the structure morphisms as

$$\Delta_{\mathbb{K}}(1) = 1 \otimes 1$$
$$\varepsilon(1) = 1.$$

One may check that these morphisms are indeed coassociative and counital. Thus we may regard our field as an algebra or a coalgebra over itself.

Example 1.1.24 ($\mathbb{K}[G]$ as a coalgebra). The group algebra has a natural coalgebra structure. We may take duplication of group elements as the comultiplication, i.e.

$$\Delta_{\mathbb{K}[G]}(kg) = kg \otimes g.$$

Coincidentally we have already defined the counit, and this is the augmentation $\varepsilon_{\mathbb{K}[G]}$ for the group algebra $\mathbb{K}[G]$. Recall that this was

$$\varepsilon_C(\sum k_g g) = \sum k_g.$$

One may see that these morphisms satisfy coassociativity and counitality.

Example 1.1.25 (The linear dual coalgebra). Let M be any finite-dimensional $\mathbb{K}$-module. There is a natural isomorphism $\xi : M^* \otimes_{\mathbb{K}} M^* \to (M \otimes_{\mathbb{K}} M)^*$, given on elementary tensors as

$$\xi(f \otimes g)(m \otimes n) = f(m)g(n).$$

Let A be a finite-dimensional algebra, then its linear dual A^* is a coalgebra. The linear dual of the multiplication $(\cdot_A)$ is defined as

$$(\cdot_A)^* : A^* \to (A \otimes_{\mathbb{K}} A)^*.$$

We define the comulitplication of A^* as $\xi^{-1}(\cdot_A)^*$.

The counit of A^* is the morphism 1_A^*.

Before we state our primary example, we will introduce its essential structure.

Definition 1.1.26 (Coaugmented coalgebras)**.** Let C be a coalgebra. C is coaugmented if there is a coalgebra homomorphism $\eta_C : \mathbb{K} \to C$.

Like augmented algebras, each coaugmented coalgebra splits in the category $\mathsf{Mod}_{\mathbb{K}}$. We first notice that given a coalgebra homomorphism f, the cokernel Cok$f$ is also a coalgebra. Given a coaugmentation $\eta_C : \mathbb{K} \to C$, we call Cok$\eta_C = \overline{C}$ for the coaugmentation quotient or reduced coalgebra of C. Thus, we obtain the splitting $C \simeq \overline{C} \oplus \mathbb{K}$. The reduced comultiplication, denoted $\overline{\Delta}_C$ may explicitly be given as

$$\overline{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1.$$

Example 1.1.27 (Tensor Coalgebras). Let V be a $\mathbb{K}$-module. We define the tensor coalgebra $T^c(V)$ of V as the module

$$T^c(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

Given a string $v^1...v^i$ in $T(V)$ we define the comultiplication by the deconcatenation operation,

$$\begin{aligned}\Delta_{T^c(V)} &: T^c(V) \to T^c(V) \otimes_{\mathbb{K}} T^c(V)\\ &v^1...v^i \mapsto 1 \otimes (v^1...v^i) + (\sum_{j=1}^{i-1} (v^1...v^j) \otimes (v^{j+1}...v^i)) + (v^1...v^i) \otimes 1.\end{aligned}$$

The counit is given by projecting $T^c(V)$ onto $\mathbb{K}$,

$$\begin{aligned}\varepsilon_{T^c(V)} : T^c(V) &\to \mathbb{K}\\ 1 &\mapsto 1\\ v^1...v^i &\mapsto .\end{aligned}$$

We observe that the tensor coalgebra is coaugmented, and its coaugmentation is the inclusion of $\mathbb{K}$ into $T^c(V)$. We can split $T^c(V) \simeq \mathbb{K} \oplus \overline{T}^c(V)$, where $\overline{T}^c(V)$ denotes the reduced tensor coalgebra.

Cofreeness does not come for free for the tensor coalgebra. Our problem is a mismatch in the behavior of algebras and coalgebras. The problem arises when we try to do an evaluation. Suppose that A is an algebra and that we have n elements of A, i.e., an element of $A^{\otimes n}$. On this element, we may apply the multiplication of A a maximum of n-times; there is no non-trivial empty multiplication. However, given a single element in a coalgebra C, we may use the comultiplication on this element n times, $n+1$ times, and so on ad infinitum. In the coalgebra, we may comultiply any element, possibly an infinite amount of times. This property is sometimes ill-behaved with our dualization of algebras to coalgebras.

However, the correct property was not lost when we dualized the tensor algebra to the tensor coalgebra. We did not lose the property that an element may only be comultiplied a finite number of times since $T^c(V)$ is a direct sum of $V^{\otimes n}$, i.e., any element is a finite sum of finite tensors.

This extra assumption we need for coalgebras will be called conilpotent. Let $C \simeq \mathbb{K} \oplus \overline{C}$ be a coaugmented coalgebra. We define the coradical filtration of C as a filtration $Fr_0C \subseteq Fr_1C \subseteq ... \subseteq Fr_rC \subseteq ...$ by the submodules:

$$\begin{aligned}&Fr_0C = \mathbb{K}\\ &Fr_rC = \mathbb{K} \oplus \{c \in \overline{C} \mid \forall n \geqslant r, \overline{\Delta}_C(c) = 0\}.\end{aligned}$$

Definition 1.1.28 (Conilpotent coalgebras)**.** Let C be a coaugmented coalgebra. We say that C is conilpotent if its coradical filtration is exhaustive, i.e.

$$\varinjlim_r Fr_rC \simeq C.$$

The full subcategory of conilpotent coalgebras will be denoted as $\mathsf{coAlg}_{\mathbb{K},\mathsf{conil}}$.

Proposition 1.1.29 (Conilpotent tensor coalgebra)**.** *Let V be a $\mathbb{K}$-module. The tensor coalgebra $T^c(V)$ is conilpotent.*

Proof. Let $v \in V$, then $\Delta_{T^c(V)}(v) = 1 \otimes v + v \otimes 1$ and $\overline{\Delta}_{T^c(V)}(v) = 0$. We then observe the following:

$$\begin{aligned} &Fr_0 T^c(V) = \mathbb{K}, \\ &Fr_1 T^c(V) = \mathbb{K} \oplus V, \\ &Fr_r T^c(V) = \bigoplus_{i \leqslant r} V^{\otimes i}. \end{aligned}$$

Exhaustiveness is clear from the coradical filtration. □

Proposition 1.1.30 (Cofree tensor coalgebra)**.** *The tensor coalgebra is the cofree conilpotent coalgebra over the category of $\mathbb{K}$-modules. That is, for any $\mathbb{K}$-module V and any conilpotent coalgebra C, there is a natural isomorphism Hom$_{\mathbb{K}}(\overline{C}, V) \simeq$ coAlg$_{\mathbb{K},conil}(C, T^c(V))$.*

Proof. This proposition should be evident from the description of a coalgebra homomorphism into the tensor coalgebra. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then g must satisfy the following conditions:

1. (Coaugmentation) $g(1) = 1$,
2. (Counitality) Given $c \in \overline{C}$ then $\varepsilon_{T^c(V)} \circ g(c) = 0$,
3. (Homomorphism property) Given $c \in C$ then $\Delta_{T^c(V)}(g(c)) = (g \otimes g) \circ \Delta_C(c)$.

We will construct the maps for the isomorphism explicitly. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then composing with projection gives a map $\pi \circ g : C \to V$. Note that $\pi \circ g(1) = 0$, so this is essentially a map $\pi \circ g : \overline{C} \to V$. For the other direction, let $\overline{g} : \overline{C} \to V$. We will then define g as

$$g = id_{\mathbb{K}} \oplus \sum_{i=1}^{\infty} (\otimes^i \overline{g}) \overline{\Delta}_C^{i-1}.$$

Observe that g is well-defined since the sum convergence follows from the conilpotency of C. One may check that g is a coalgebra homomorphism, which yields the result. □

Comodules

Essential to our dualization is comodules. We provide a short definition.

Definition 1.1.31 (Comodules)**.** Let C be a coalgebra. A $\mathbb{K}$-module M is said to be left (right) C-comodule if there exist a structure morphism $\omega_M : M \to C \otimes_{\mathbb{K}} M$ ($\omega_M : M \to M \otimes_{\mathbb{K}} C$) called comultiplication. We require that ω_M is coassociative with respect to the comultiplication of C and preserves the counit of C; i.e. we have the following commutative diagrams in Mod$_{\mathbb{K}}$,

$$\begin{tikzcd} C \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} M & C \otimes_{\mathbb{K}} M \arrow[l, "C\otimes\omega_M"'] \\ C \otimes_{\mathbb{K}} M \arrow[u, "\Delta_C\otimes M"] & M \arrow[l, "\omega_M"] \arrow[u, "\omega_M"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes_{\mathbb{K}} M & C \otimes_{\mathbb{K}} M \arrow[l, "\varepsilon_C\otimes M"'] \\ & M \arrow[ul, "\simeq"] \arrow[u, "\omega_M"'] \end{tikzcd}$$

Definition 1.1.32 (C-colinear homomorphism)**.** Let M, N be two left C-comodules. A morphism $g : M \to N$ is called C-colinear if it is $\mathbb{K}$-linear and for any m in M, $\omega_N(g(m)) = (id_C \otimes g)\omega_M(m)$. In Sweedlers notation, this looks like

$$\sum g(m)_{(1)} \otimes g(m)_{(2)} = \sum c_{(1)} \otimes g(m_{(2)}).$$

The category of left C-comodules is denoted as CoMod$_C$, where the morphisms Hom$_C(_,_)$ are C-colinear. We would also like to restrict our attention to those C-comodules that are conilpotent, i.e., the comodules with exhaustive coradical filtration. The coradical filtration is defined analogously, as we only care for the $\mathbb{K}$-module structure. Notice that for conilpotent coalgebras, this requirement is automatic. Likewise, we denote the category of right C-comodules as CoMod^C.

Proposition 1.1.33. *Let M be a $\mathbb{K}$-module. The module $C \otimes_{\mathbb{K}} M$ is a left C-comodule. Moreover, it is the cofree left comodule over $\mathbb{K}$-modules, i.e. there is an isomorphism Hom$_{\mathbb{K}}(N, M) \simeq$ Hom$_C(N, C \otimes_{\mathbb{K}} M)$.*

Proof. This proposition is dual to Proposition 1.1.14. We will only construct the isomorphism, as its validity is apparent.

$$\begin{aligned} \phi' : \text{Hom}_C(N, C \otimes_{\mathbb{K}} M) &\to \text{Hom}_{\mathbb{K}}(N, M) \\ f &\mapsto (\varepsilon_C \otimes M) \circ f, \\ \psi' : \text{Hom}_{\mathbb{K}}(N, M) &\to \text{Hom}_C(N, C \otimes_{\mathbb{K}} M) \\ g &\mapsto (C \otimes g) \circ \omega_N. \end{aligned}$$

□

Corollary 1.1.33.1. *C as a left C-comodule is the cofree C-comodule over $\mathbb{K}$; i.e. for any left C-comodule N, $N^* \simeq$ Hom$_{\mathbb{K}}(N, \mathbb{K}) \simeq$ Hom$_C(N, C)$.*

Categorical structure

Dual to augmented algebras, conilpotent coalgebras have colimits that are easy to calculate, while the limits are complicated. For this discussion, we will restrict our attention to coAlg$_{\mathbb{K},\text{conil}}$.

Like for augmented algebras, coAlg$_{\mathbb{K},\text{conil}}$ is a pointed category. The initial and terminal object is $\mathbb{K}$.

Definition 1.1.34. Let C and D be conilpotent coalgebras. Their direct sum $C \oplus D$ is defined as the following colimit:

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\eta_C"] \arrow[d, "\eta_D"] & C \arrow[d] \\ D \arrow[r] & C \oplus D \end{tikzcd}$$

As before, this is some abuse of notation. This direct sum will almost be the direct sum, except we have to fix the coaugmentation.

Lemma 1.1.35. *Given conilpotent coalgebras C and D, their direct sum is the free coaugmentation on the direct sum of the coaugmentation quotients, $C \oplus D \simeq (\overline{C} \oplus \overline{D})^+$.*

Proof. This lemma is clear from the comonadicity of the forgetful functor. □

Dually to before, the projection $C \oplus D \to C$ is not usually a coalgebra morphism.

Definition 1.1.36. Let C and D be two augmented algebras, the free product $C * D$ is defined as the following limit:

$$\begin{tikzcd} C * D \arrow[r] \arrow[d] & C \arrow[d, "\varepsilon_C"] \\ D \arrow[r, "\varepsilon_D"] & \mathbb{K} \end{tikzcd}$$

We proceed to describe the free product of conilpotent coalgebras. Due to it being dual to the free product of augmented algebras, this will naturally be a subobject of the tensor coalgebra.

Lemma 1.1.37. *Given to conilpotent coalgebras C and D, then $C * D \subseteq T^c(\overline{C} \oplus \overline{D})$ consists in words generated by letters in $\overline{C}$ or $\overline{D}$ on the form*

$$[\![c]\!] = \sum_{i=0}^{\infty} \Delta_C^i(c), \text{ and}$$
$$[\![d]\!] = \sum_{i=0}^{\infty} \Delta_D^i(d).$$

Proof. We define a projection $C * D \to C$ as the "identity" on the letters in C and 0 otherwise.

$$\begin{aligned} p_C : C * D &\to C \\ [\![c]\!] &\mapsto c \\ \_ &\mapsto 0 \end{aligned}$$

By definition, p_C is a coalgebra morphism as

$$p_C^{\otimes 2}(\Delta_{T^c(\overline{C}\oplus\overline{D})}[\![c]\!]) = p_C^{\otimes 2}(\sum [\![c_{(1)}]\!] \otimes [\![c_{(2)}]\!]) = \sum c_{(1)} \otimes c_{(2)}.$$

The morphisms p_C and p_D define a cone over C and D. It remains to check the universal property. Suppose there are morphisms $f : T \to C$ and $g : T \to D$.

$$\begin{tikzcd}
T \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& C * D \arrow[r, "p_C"] \arrow[d, "p_D"] & C \arrow[d, "\varepsilon_C"] \\
& D \arrow[r, "\varepsilon_D"] & \mathbb{K}
\end{tikzcd}$$

We define the morphism h as the following sum

$$h(t) = \sum_{i=1}^{\infty} [\![f(t_{(1)})]\!] \otimes [\![g(t_{(2)})]\!] \otimes \cdots \otimes [\![?(t_{(i)})]\!] + [\![g(t_{(1)})]\!] \otimes [\![f(t_{(2)})]\!] \otimes \cdots \otimes [\![?(t_{(i)})]\!],$$

where $?$ means either f or g, which is appropriate.

We have constructed this morphism to be a coalgebra morphism, and every other coalgebra morphism has to be on this form as well. Thus h is unique. □

Opposite to augmented algebras, every small colimit of conilpotent coalgebras is created by the forgetful functor.

Lemma 1.1.38. *Suppose that $f : C \to D$ is a morphism of augmented algebras. The cokernel is isomorphic to coKer$f \simeq (\overline{\text{coKer}f})^+ \simeq {}^{\overline{D}}/_{\overline{\text{Im}f}}{}^+$.*

Proof. This lemma is clear from the comonadicity of the forgetful functor. □

This time around, we will instead have a problem calculating kernels. Let $f : C \to D$ be a morphism of coalgebras. The set $\{c \in C \mid f(c) = 0\}$ is not necessarily closed under comultiplication. We require that $f^{\otimes 2}(\Delta_C(c)) = f(c_{(1)}) \otimes f(c_{(2)}) = \Delta_D(f(c)) = \Delta_D(0) = 0$, but then only one of $f(c_{(1)})$ or $f(c_{(2)})$ has to be 0.

The abovementioned construction will sometimes work. If f is a cokernel map, that is if $f : D \to {}^{\overline{D}}/_{\overline{C}}{}^+$, then $C = \{d \in D \mid f(d) = 0\}$. Whenever $f : C \to D$ is epi and regular, f will then be a cokernel map. In particular, it is enough that $f : C \to D$ is regular, as we can consider the morphism $\pi : C \to$ coImf instead of f. Since $\widetilde{f}$: coIm$f \to$ Imf is an isomorphism, Ker$f \simeq$ Kerπ, so we can use the set-theoretic description instead,

$$\text{Ker}f = \{c \in C \mid f(c) = 0\}.$$

1.1.3 Electronic Circuits

Calculations involving both algebras and coalgebras tend to become convoluted and unmanageable. Since we want to study the interplay between algebras and coalgebras, using other tools to write equations can be handy. We will develop a graphical calculus briefly mentioned in [LV12], where we take a lot of inspiration from Sobocinski's blog [Sob15]. This graphical calculus will consist of string diagrams, referred to as electronic circuits, which describe the function composition on tensors. Since we only care about the interplay of tensors, we may develop this graphical calculus in any closed symmetric monoidal category. Why do we want to introduce this abstraction? A closed symmetric monoidal category is a good category to model functions, or morphisms, which may take several variables in its argument. Moreover, in the next section, we are going to switch categories. In this manner, we can reuse the same notions and proofs.

This section will use closed symmetric monoidal categories to define electronic circuits. The definitions can be found in Appendix D. For our purposes, a closed symmetric monoidal category is a category $\mathcal{C}$ together with a bifunctor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ usually called tensor, and a unit object $Z \in \mathcal{C}$. Additionally, we have four natural isomorphisms relating the functors and the unit to what they are supposed to represent:

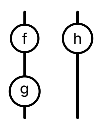
$$\begin{aligned} \text{Associator} \quad & \alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C). \\ \text{Right unit} \quad & \rho : A \otimes Z \to A. \\ \text{Left unit} \quad & \lambda : Z \otimes A \to A. \\ \text{Braiding/Symmetry} \quad & \beta : A \otimes B \to B \otimes A. \end{aligned}$$

These natural isomorphisms are supposed to satisfy some laws as well. See the appendix for the full definition.

We want to rewrite equations into string diagrams with an electronic circuit, possibly involving tensors. To illustrate with some simple examples, let $f : A \to B$, $g : B \to C$ and $h : D \to E$. We may consider the composition

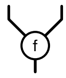
$$(g \otimes E) \circ (f \otimes h) : A \otimes D \to C \otimes E.$$

An electronic circuit is written from top to bottom and is composed of levels. The first morphisms we apply will be at the top, descending downwards with each function composition. We write each argument in the composition as a string. Thus this example above will look like the circuit below. Notice how f and h are at the same level, indicating that they are interpreted as $f \otimes h$. Thus an $\otimes$ indicates a change of string, while a $\circ$ indicates a change of level.

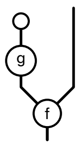


Beware that when many tensors are in use, we should remember exactly how each string is tensored. We may call adding tensors for horizontal composition and composition of morphism for vertical composition. Both have a choice in how we associate them, but both have unique choices up to isomorphism given by the associator.

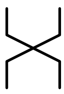
The true power of electronic circuit comes to light when we consider morphisms that, in some sense, ”creates” or ”destroys” strings. For example, a morphism of 2 variables ”destroys” a string by applying them to each other. Consider now a morphism $f : A \otimes B \to C$; we represent this morphism in an electronic circuit using a converging fork. Likewise, ”creation” of strings is seen as a diverging fork.



We may write the unit object Z without any strings in a circuit. By right and left unitality, any object A is isomorphic to $A \otimes Z \simeq A \simeq Z \otimes A$. In this manner, whenever a morphism enters or exits the unit Z, we start a new string using a source or a sink. For example, consider f as before and a morphism $g : Z \to A$, then we may write $f \circ (g \otimes B)$ as the circuit below. Again, this is only well-defined up to isomorphism by right and left unitality.

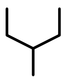


The final operation we have is braiding. When we apply the braiding morphism on the tensors, we may denote this as interchanging the strings. For example, $\beta_{A,B} : A \otimes B \to B \otimes A$ is the circuit below. Notice that by the naturality of β, we may move a braiding along the circuit. In this manner, if we have two braids, they may sometimes undo each other. In either case, we can carry a braid to either end of the circuit to ignore them during calculations.

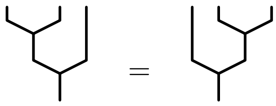


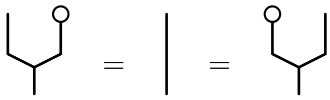
With the language of electronic circuits, we may now write down the axioms of an algebra or coalgebra electronically. The axioms state the existence of morphisms. We give the structure maps of algebras and coalgebras special notation since we will use these often.

For convenience we will let $\mathcal{C} = \mathrm{Mod}_{\mathbb{K}}$. This category is closed symmetric monoidal, with $\otimes_{\mathbb{K}}$ as the tensor. Recall that an algebra is a $\mathbb{K}$-module A together with maps $(\cdot_A) : A \otimes A \to A$ and $1_A : \mathbb{K} \to A$. We denote these morphisms electronically, as shown in the diagrams below.

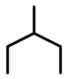
$(\cdot_A)$ =  1_A = 

We write the electronic laws for an algebra as how one would write equations. Associativity and unitality then become as follows.

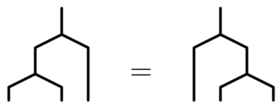
Associativity 

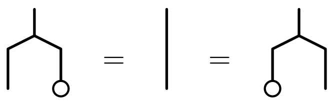
Unitality 

Dually, given a coalgebra C, we will make a similar notation. We denote the maps $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \mathbb{K}$ as the following electronic circuits.

Δ_C =  ε_C = 

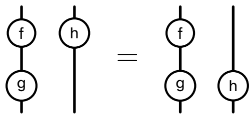
The electronic laws for C become the following diagrams.

Coassociativity 

Counitality 

This notation will be adopted for our algebras and coalgebras when convenient. The intuition for coalgebras is more accessible with electronic circuits, as we can work out a statement of algebras and then turn the diagram upside down to make it into a statement of coalgebras.

Previously we talked about braiding and how that relates to interchanging strings. In the same manner that we have a horizontal and vertical associator, we also have vertical and horizontal braiding. Horizontal braiding is the usual notion of braiding strings. On the other hand, vertical braiding refers to the function composition of tensors, which manifests in electronic circuits as sliding a morphism along a string. Whenever the given braiding of $\mathcal{C}$ is nice enough, we can get away by ignoring it whenever we move a morphism along a string. For instance, look at the category of $\mathbb{K}$-modules where we may define the braiding on elementary tensors as $\beta(a \otimes b) = b \otimes a$. In this case, the braiding is agnostic to how we move our morphisms along a string, and this means that we have the following equality of circuits.

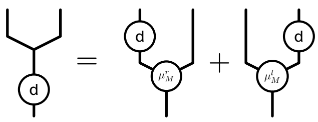


In nature, we may encounter braidings that are not as nice. In these cases, we should take a step back to figure out how to move morphisms along strings before we continue using this graphical calculation of function composition. We will meet such a braiding soon.

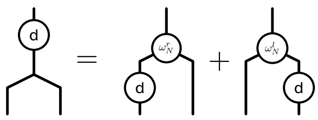
1.1.4 Derivations and DG-Algebras

This section aims to define differential graded algebras and their modules. Given an algebra A, we define a derivation as a map satisfying the Leibniz rule. In the dual case for a coalgebra, we may define a coderivation as a map satisfying the Zinbiel rule, but we will refer to these maps as derivations for brevity. Once we grasp how to make derivations, we introduce graded algebras and modules to equip these with derivations. Derivations will allow us to state the categories of differential graded algebras and cochain complexes. Throughout this section, we will also develop electronic circuits for these notions.

Definition 1.1.39 (Derivations and Coderivations)**.** Let M be an A-bimodule. A $\mathbb{K}$-linear morphism $d : A \to M$ is called a derivation if $d(ab) = d(a)b + ad(b)$, i.e. electronically,



Let N be a C-bicomodule. A $\mathbb{K}$-linear morphism $d : N \to C$ is called a coderivation if $\Delta_C \circ d = (d \otimes id_C) \circ \omega_N^r + (id_C \otimes d) \circ \omega_N^l$, i.e. electronically,



We remark that this translation between equations and electronic circuits is not at the same level of generalization. Due to this, the electronic circuit description has more advantages as it allows us to think with elements when we are only dealing with morphisms. We will use these circuits to derive results independent of the given braiding on the category.

A helpful fact about derivations is that they will always map the identity to 0. We obtain this from the Leibniz rule as one would get $d(1) = 2d(1)$, and thus $d(1) = 0$.

Proposition 1.1.40. *Let V be a $\mathbb{K}$-module and M be a $T(V)$-bimodule. A $\mathbb{K}$-linear morphism $f : V \to M$ uniquely determines a derivation $d_f : T(V) \to M$, i.e. there is an isomorphism $\mathrm{Hom}_{\mathbb{K}}(V, M) \simeq \mathrm{Der}(T(V), M)$.*

Let N be a $T^c(V)$-bicomodule. A $\mathbb{K}$-linear morphism $g : M \to V$ uniquely determines a coderivation $d_g^c : N \to T^c(V)$, i.e. there is an isomorphism $\mathrm{Hom}_{\mathbb{K}}(N, V) \simeq \mathrm{Coder}(N, T^c(V))$.

Proof. Let $a_1 \otimes ... \otimes a_n$ be an elementary tensor of $T(V)$. We define a map $d_f : T(V) \to M$ as

$$d_f(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^{n} a_1...f(a_i)...a_n$$
$$d_f(1) = 0.$$

d_f is a derivation by definition.

Restriction to V gives the natural isomorphism. Let $i : V \to T(V)$ be the inclusion, then $i^* d_f = f$. Let $d : T(V) \to M$ be a derivation, then $d_{i^*d} = d$. Suppose now that $g : M \to N$ is a morphism of $T(V)$-bimodules; then naturality follows from linearity.

In the dual case, $d_g^c : N \to T^c(V)$ is a bit tricky to define. Let $\omega_N^l : N \to N \otimes T^c(V)$ and $\omega_N^r : N \to T^c(V) \otimes N$ denote the coactions on N. Since $T^c(V)$ is conilpotent, we get the same finiteness restrictions on N. Define the reduced coactions as $\overline{\omega}_N^l = \omega_N^l - \_ \otimes 1$ and $\overline{\omega}_N^r = \omega_N^r - 1 \otimes \_$, this is well-defined by coassociativity. Observe that for any $n \in N$ there are k and $k' > 0$ such that $\overline{\omega}_N^{l^k}(n) = 0$ and $\overline{\omega}_N^{r^{k'}}(n) = 0$.

Let $n_{(k)}^{(i)}$ denote the extension of n by k coactions at position i, i.e.

$$n_{(k)}^{(i)} = \overline{\omega}_N^{r^i} \overline{\omega}_N^{l^{k-i}}(n).$$

The extension of n by k coactions is then the sum over every position i,

$$n_{(k)} = \sum_{i=0}^{k} n_{(k)}^{(i)}.$$

Observe that $n_{(0)} = n$. The grade of n is the smallest k such that $n_{(k)}$ is zero. This grading gives us the coradical filtration of N, and it is exhaustive by the finiteness restrictions given above. With this notion, every element of N has a finite grade.

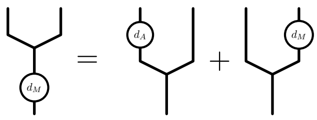
If $g : N \to V$ is a linear map, we may think of it as a map sending every element of N to an element of $T^c(V)$ of grade 1. We must extend the morphism to get a map that sends the element of grade k to grade k. Let $\pi : T^c(V) \to V$ be the linear projection and define $g_{(k)}^{(i)} = \pi \otimes ... \otimes g \otimes \pi$ as a morphism which of k tensors which is g at the i-th argument, but the projection otherwise. We define d_g^c as the sum over each coaction and coordinate,

$$d_g^c(n) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} g_{(k)}^{(i)}(n_{(k)}^{(i)}).$$

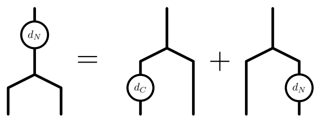
Upon closer inspection, we may observe this is the dual construction of the derivation morphism. It is well-defined as the sum is finite by the finiteness restrictions. The map is a coderivation by duality, and the natural isomorphism is post-composition with the projection map π. $\square$

Definition 1.1.41 (Differential algebra)**.** Let A be an algebra. We say that A is a differential algebra if it is equipped with a derivation $d : A \to A$. Dually, a coalgebra C is a differential coalgebra if it is equipped with a coderivation $d : C \to C$.

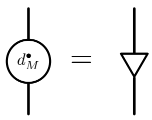
Definition 1.1.42 (A-derivation)**.** Let (A, d_A) be a differential algebra and M a left A-module. A $\mathbb{K}$-linear morphism $d_M : M \to M$ is called an A-derivation if $d_M(am) = d_A(a)m + ad_M(m)$, or electronically,



Dually, given a differential coalgebra (C, d_C) and N a left C-comodule, a $\mathbb{K}$-linear morphism $d_N : N \to N$ is a coderivation if $\omega_N \circ d_N = (d_C \otimes id_N + id_C \otimes d_N) \circ \omega_N$, or electronically,



When there is no ambiguity, we will start to adopt writing the differential in electronic circuits as a triangle,



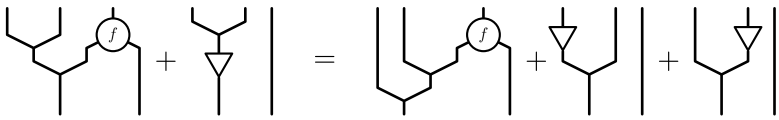
Proposition 1.1.43. *Let A be a differential algebra and M a $\mathbb{K}$-module. A $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$, i.e. there is an isomorphism Hom$_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \simeq$ Der$(A \otimes_{\mathbb{K}} M)$. Moreover, d_f is given as $((\cdot_A) \otimes id_M) \circ (id_A \otimes f) + d_A \otimes id_M$.*

Dually, if C is a differential coalgebra and N is a $\mathbb{K}$-module, then a $\mathbb{K}$-linear morphism $g : C \otimes N \to N$ uniquely determines a coderivation $d_g : C \otimes_{\mathbb{K}} N \to C \otimes_{\mathbb{K}} N$. There is an isomorphism Hom$_{\mathbb{K}}(C \otimes_{\mathbb{K}} N, N) \simeq$ Coder$(C \otimes_{\mathbb{K}} N)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_C \otimes id_N) + d_C \otimes id_N$.

Proof. We will only prove this proposition in the case of algebras. The case of coalgebras is dual.

We have to prove that the morphism $d\_$: Hom$_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \to$ Der$(A \otimes_{\mathbb{K}} M)$ is well-defined. To do this, we must check that for any morphism $f : M \to A \otimes_{\mathbb{K}} M$, the morphism d_f satisfies the Leibniz rule.

Assume that we have elements $a, b \in A$ and $m \in M$. Then $d_f(ab \otimes m) = d_f(a(b \otimes m))$. We abuse the notation to write equality between an element and a circuit. Recall that this means that we have to think of a, b, and m as generalized elements,

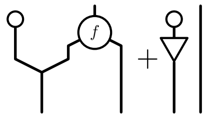
$d_f(ab \otimes m) \quad =$ 

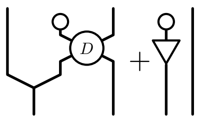
$$= d_A(a)b \otimes m + ad_f(b \otimes m).$$

Next, we show that $d\_$ has an inverse, which is given by "restriction to M," also known as

$$(1_A \otimes M)^* : \mathrm{Hom}_{\mathbb{K}}(A \otimes_{\mathbb{K}} M, N) \to \mathrm{Hom}_{\mathbb{K}}(M, N).$$

Let $f : M \to A \otimes_{\mathbb{K}} M$ be a linear map and $D : A \otimes_{\mathbb{K}} M \to A \otimes_{\mathbb{K}} M$ be a derivation, then a quick calculation verifies that $d\_$ is inverse to restriction.

$d_f \circ (1_A \otimes M) =$  $= f$

$d_{D \circ (1_A \otimes M)} \quad =$  $= D$

Notice that we use the Leibniz rule in the last equation to get the equality to D.

□

We say that a $\mathbb{K}$-module M^* admits a $\mathbb{Z}$-grading if it decomposes into either summands or factors

$$M^* = \bigoplus_{z:\mathbb{Z}} M^z \text{ or } M^* = \prod_{z:\mathbb{Z}} M^z.$$

An element of $m \in M$ is said to be homogenous if it is properly contained in a single summand, i.e., $m \in M^n$. m is then said to have degree n. We say that a morphism of graded modules $f : M^* \to N^*$ is homogenous of degree n if it preserves the grading, that is $f(M^i) \subseteq N^{n+i}$. The degree of a homogenous element m or morphism f is denoted as $|m|$ or $|f|$.

There is a distinction between the ordinary and self-enriched categories of graded modules. We are going to work with the self-enriched category, and its hom-objects are the graded module of homogenous morphisms. We denote a factor in the grading as $\mathsf{Hom}^w_{\mathbb{K}}(M^*, N^*) = \{f : M^* \to N^* \mid f \text{ is homogenous and } |f| = w\}$, so the graded hom is

$$\mathsf{Hom}^*_{\mathbb{K}} = \prod_{w\in\mathbb{Z}} \mathsf{Hom}^w_{\mathbb{K}}.$$

This category is denoted as $\mathsf{Mod}^*_{\mathbb{K}}$. In general, and whenever it makes sense, we write $\mathcal{C}^*$ as the category of $\mathbb{Z}$-graded objects from $\mathcal{C}$.

The category $\mathsf{Mod}^*_{\mathbb{K}}$ is a closed symmetric monoidal category. The tensor is given by the following formula, using the ordinary tensor of $\mathsf{Mod}_{\mathbb{K}}$,

$$M^* \otimes N^* = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p\in\mathbb{Z}} M^p \otimes_{\mathbb{K}} N^q, \text{ where } q = n - p.$$

The associator of $\mathsf{Mod}_{\mathbb{K}}$ may be lifted to this tensor. The unit is the module $\mathbb{K}$ concentrated in degree 0. Likewise, both the right and left unit transformation may be lifted from $\mathbb{K}$.

The category $\mathsf{Mod}^*_{\mathbb{K}}$ is closed, which means that the graded tensor fixed in one variable is left adjoint to the graded hom. We may obtain the graded hom as the right adjoint for the other variable by using the braiding, which we will define later. Showing closedness is done using the tensor-hom adjunction from $\mathsf{Mod}_{\mathbb{K}}$.

$$\begin{aligned}
\mathsf{Hom}^*_{\mathbb{K}}(A^* \otimes B^*, C^*) &= \prod_{w\in\mathbb{Z}} \prod_{n\in\mathbb{Z}} \mathsf{Hom}^w_{\mathbb{K}}(\bigoplus_{p\in\mathbb{Z}} A^p \otimes_{\mathbb{K}} B^{n-p}, C^n) \\
&= \prod_{w\in\mathbb{Z}} \prod_{n\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p \otimes_{\mathbb{K}} B^{n-(p+w)}, C^n) \simeq \prod_{w\in\mathbb{Z}} \prod_{n\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w),C^n})) \\
&\simeq \prod_{w\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \prod_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w),C^n})) = \prod_{w\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}^{p+w}_{\mathbb{K}}(B^*, C^*)) \\
&\simeq \prod_{w\in\mathbb{Z}} \mathsf{Hom}^w_{\mathbb{K}}(A^*, \mathsf{Hom}^*_{\mathbb{K}}(B^*, C^*)) = \mathsf{Hom}^*_{\mathbb{K}}(A^*, \mathsf{Hom}^*_{\mathbb{K}}(B^*, C^*)).
\end{aligned}$$

Following Kelly [Kel05a], we define a symmetric monoidal structure on this category. We give a braiding on homogenous elementary tensors as

$$\beta(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

It is immediate that $\beta_{A,B}$ is inverse to $\beta_{B,A}$. Observe that this category also admits a braiding where we don't introduce a sign. However, this does not work when we want to add differentials to our graded modules, so we stick with this sign. This braiding is also commonly known as the Koszul sign convention.

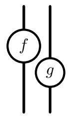
Since $\mathsf{Mod}^*_{\mathbb{K}}$ is a closed symmetric monoidal category, it admits electronic circuits. Thus the previous results we have proved by electronic circuits also apply to this category, as the proof is identical in this language. One should note that the specific implementation may differ as vertical braiding works differently. The application of two homogenous morphisms $f : A \to A'$ and $g : B \to B'$ on elements $a \in A$ and $b \in B$ on tensors is defined as

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b).$$

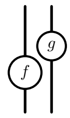
Viewing a and b as generalized elements again, we get Koszul's sign rule on morphisms. That is, given homogenous composable morphisms f, f', g, g', we get that

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g).$$

Electronically we may represent this as a 2-string circuit where a morphism on the left wants to downwards pass a morphism on its right,



$$= (-1)^{|g||f|}$$



A good way of thinking about moving components in a circuit is that whenever we move a component downwards, it has to pass over every component to the left on its current level and every component to the right of it on the level below. We introduce signs in a 2-string circuit whenever a component is moved downwards to or completely past another component on its right. If we move a component upwards completely past another component to its left, we introduce a sign. In an n-string circuit, it gets more complicated as the component may have to move past several components on both the left and right.

Unlike the other electronic equations in which we may substitute parts of an electronic circuit with other equal parts, this does not work a priori in this context because of how we defined levels. Within a 3-string circuit, the formula changes, and this is because we want to manipulate every element on a level simultaneously. If we move a left-most component downwards past many components, we may regard them as a single component on a single string. We will use this interpretation to prove an interchange of components on an n-string circuit formula.

Proposition 1.1.44. *Let $n \geqslant 1$ and suppose that we have $a_i \in A_i \to B_i$ and $b_i : B_i \to C_i$ for any $0 < i \leqslant n$. Then we get that*

$$(b_i \circ a_i) \otimes \cdots \otimes (b_n \circ a_n) = (-1)^s (b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n),$$

$$\textit{where } s = \sum_{i=1}^{n} |b_i| \Big(\sum_{1 \leqslant j < i} |a_j| \Big).$$

Proof. We prove this by induction. If $n = 1$, this is true. $s = 0$ since the sum is empty, so $b_1 \circ a_1 = (-1)^s b_1 \circ a_1$.

Assume that the conclusion holds for $n - 1$ and that we have a_i and b_i as in the hypothesis. Let $s' = \sum_{i=1}^{n-1} |b_i| (\sum_{1 \leqslant j < i} |a_j|)$, then

$$s = s' + |b_n| \Big(\sum_{i=1}^{n-1} |a_i| \Big).$$

The conclusion follows from this calculation.

$$\begin{aligned}(b_1 \circ a_1) \otimes \cdots \otimes (b_n \circ a_n) &= (-1)^{s'}((b_1 \otimes \cdots \otimes b_{n-1}) \circ (a_1 \otimes \cdots \otimes a_{n-1})) \otimes (b_n \circ a_n) \\ &= (-1)^{s'+|b_n|(\sum_{i=1}^{n-1}|a_i|)}(b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n).\end{aligned}$$

□

A final remark on this braiding is that it affects any scenario where we compose functions, and they move past each other. Since function composition factors through this tensor, moving functions around is a braiding. An important example of this is the pre-composition functor. If f and g are homogenous and composable, then

$$f^*(g) = (-1)^{|f||g|} g \circ f.$$

The graphical calculus we have developed will be the same for any symmetric monoidal category where the braiding is similar. What this means will soon be evident when we add extra structure to the objects of $\mathsf{Mod}^*_{\mathbb{K}}$.

A graded $\mathbb{K}$-module $M^\bullet$ is called a cochain complex if it comes equipped with a differential $d_M : M^\bullet \to M^\bullet$. By a differential, we mean a homogenous morphism of degree 1 such that $d_M^2 = 0$. Be cautious of bad notation, as d_M^2 might mean $d_M^2 = d_M \circ d_M$ and $d_M^2 : M^2 \to M^3$.

Given a cochain complex $M^\bullet$, we know by definition that the image of the differential lies inside the kernel of the differential. We denote this at the i'th coordinate as $B^i M \subseteq Z^i M$. $B^* M$ is the graded submodule of images, also called boundaries. $Z^* M$ is the graded submodule of kernels, also called cycles. The graded cohomology module $H^* M$ is defined as the quotient ${Z^*M}/{B^*M}$. A cochain complex is said to be exact if $H^* M \simeq 0$.

Cochain complexes are plentiful in nature.

Example 1.1.45 ($\mathbb{K}$ as a cochain complex). Let $\mathbb{K}^\bullet = (\mathbb{K}, 0)$ be the graded $\mathbb{K}$-module concentrated in degree 0 together with a 0 differential, and this is trivially a cochain complex.

Example 1.1.46 (Trivial cochain complexes). Let M^* be a graded $\mathbb{K}$-module. Let $M^\bullet = (M^*, 0)$ be the same graded module with the 0 differential, and this is also a cochain complex.

Example 1.1.47. We can create a cochain complex, as shown in the following diagram.

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id_{\mathbb{K}}} \mathbb{K} \longrightarrow 0 \longrightarrow \cdots$$

Example 1.1.48 (Cone of a chain map). Suppose that $f : A^\bullet \to B^\bullet$ is a homogenous morphism of degree 0 such that $f \circ d_A = d_B \circ f$. There is an associated cochain complex to f, which yields a short-exact sequence of cochain complexes. We define $\mathsf{cone}(f)$ at each degree by

$$\mathsf{cone}(f)^n = A^{n+1} \oplus B^n,$$
$$d^n_{\mathsf{cone}(f)} = \begin{pmatrix} d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

This complex gives us a short exact sequence,

$$B^\bullet \rightarrowtail \mathsf{cone}(f) \twoheadrightarrow A^\bullet[1].$$

Example 1.1.49 (Normalized cochain complex). Let $A : \Delta^{op} \to \mathsf{Mod}_{\mathbb{K}}$ be a simplicial $\mathbb{K}$-module. We define a collection of diagrams J^n as $J^0 = A_0$, and every other as

$$J^n = A_n \; \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{d_1} \\ \vdots \\ \xrightarrow[d_n]{} \end{array} \; A_{n-1}$$

A's normalized cochain complex is the complex given as

$$NA^{-n} = \varprojlim J^n.$$

In a complete pointed category, such as $\mathsf{Mod}_{\mathbb{K}}$, the limit is the same as the intersection of every kernel:

$$\varprojlim J^n = \bigcap_{i=1}^{n} \mathsf{Ker} d_i.$$

The differential of NA is defined to be d_0. Since we have turned the complex around, this is a morphism of degree 1. By taking the limit, we force $d_0^2 = 0$ as well.

Example 1.1.50 (Associated cochain complex). Let $A : \Delta^{op} \to \mathsf{Mod}_{\mathbb{K}}$ be a simplicial $\mathbb{K}$-module. We define a differential as

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

Let $\mathsf{C}A$ be the complex given in each degree as

$$\mathsf{C}A^{-n} = A_n.$$

d defines a differential on $\mathsf{C}A$ of degree 1.

Example 1.1.51 (Singular chain complex with $\mathbb{K}$-coefficents). Let M be a topological space. There is a simplicial set defined as $\mathsf{Sing}(M) = \mathsf{Top}(\Delta -, M) : \Delta^{op} \to \mathsf{Set}$. Here $\Delta^{[n]}$ in Top refers to the topological standard n-simplex. We get a simplicial $\mathbb{K}$-module by creating the free one, $\mathbb{K}\mathsf{Sing}(M)$. The above example defines a chain complex in $\mathsf{Mod}_{\mathbb{K}}$.

We make a distinction for some cochain complexes, which is of particular interest.

Definition 1.1.52 (Quasi-free cochain complexes). Suppose that $M^\bullet$ is a cochain complex. We say that $M^\bullet$ is quasi-free if the underlying graded module M^* is free; in other words, M^* is a tensor algebra.

Likewise, we say that $M^\bullet$ is quasi-cofree if M^* is cofree; in other words, M^* is a tensor coalgebra.

The category of cochain complexes will be denoted as $\mathsf{Mod}_{\mathbb{K}}^{\bullet}$. Note that this category is built upon $\mathsf{Mod}_{\mathbb{K}}^{*}$, and we inherit the braiding β. We want to entertain different collections of morphisms because the morphisms that respect the structure and the morphisms that make this category self-enriched are different. We will usually denote both of these categories as $\mathsf{Mod}_{\mathbb{K}}^{\bullet}$, but when we want to emphasize the structure-preserving maps, we will instead denote this as $\mathsf{Ch}(\mathbb{K})$.

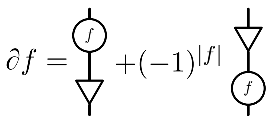
When $A^\bullet$ and $B^\bullet$ are cochain complexes the graded $\mathbb{K}$-module $\mathsf{Hom}_{\mathbb{K}}^{*}(A^\bullet, B^\bullet)$ admits a derivative. Let $f : A^\bullet \to B^\bullet$ be any homogenous morphism, then the derivative-, or boundary of f is given by

$$\partial f = (d_{B*} + d_A^*)(f) = d_B \circ f - (-1)^{|f|} f \circ d_A.$$

We see that $|\partial| = |d_{B*} + d_A^*| = 1$, and

$$\partial^2 f = (d_{B*} + d_A^*)(d_B \circ f - (-1)^{|f|} f \circ d_A) = d_B^2 f + (-1)^{|f|} d_B f d_A - (-1)^{|f|} d_B f d_A - f d_A^2 = 0.$$

Thus, $\mathsf{Hom}_{\mathbb{K}}^{\bullet}(A^\bullet, B^\bullet) = (\mathsf{Hom}_{\mathbb{K}}^{*}(A^\bullet, B^\bullet), \partial)$ is a cochain complex. We endow $Mod_{\mathbb{K}}^{\bullet}$ with these hom-objects. In an electronic circuit, we write ∂f as a sum of circuits,



Notice how this construction of $\mathsf{Hom}_{\mathbb{K}}^{\bullet}$ is the same as the (product) total complex of an anticommutative double complex. An anticommutative double complex is a graded module of cochain complexes, together with a differential between the cochain complexes. These different differentials are supposed to be anticommuting. We draw an anticommutative double complex, as shown below.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d_C^h"] & C^{-1,1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{0,1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{1,1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & \cdots \\
\cdots \arrow[r, "d_C^h"] & C^{-1,0} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{0,0} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{1,0} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & \cdots \\
\cdots \arrow[r, "d_C^h"] & C^{-1,-1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{0,-1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & C^{1,-1} \arrow[u, "d_C^v"] \arrow[r, "d_C^h"] & \cdots \\
 & \vdots \arrow[u, "d_C^v"] & \vdots \arrow[u, "d_C^v"] & \vdots \arrow[u, "d_C^v"] &
\end{tikzcd}$$

Another way of thinking of an anticommutative double complex $C^{\bullet,\bullet}$ is that it is a bigraded $\mathbb{K}$-module with a vertical and horizontal differential such that $d_C^v \circ d_C^h = -d_C^h \circ d_C^v$.

Definition 1.1.53. Let $C^{\bullet,\bullet}$ be an anticommutative double complex. We define the sum and product total complex. The differential at each $C^{p,q}$ is defined as $d_{\mathsf{Tot}C} = d^v_C + d^h_C$, and

$$\mathsf{Tot}^{\oplus}(C^{\bullet,\bullet}) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p+q=n} C^{p,q},$$
$$\mathsf{Tot}^{\prod}(C^{\bullet,\bullet}) = \prod_{n\in\mathbb{Z}} \prod_{p+q=n} C^{p,q}.$$

If $C^{\bullet,\bullet}$ is bounded, then $\mathsf{Tot}^{\oplus}(C^{\bullet,\bullet}) \simeq \mathsf{Tot}^{\prod}(C^{\bullet,\bullet})$.

If we let $\mathsf{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet} = (\prod_{p,q\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, B^q), d^*_A, d_{B*})$, then it is clear that

$$\mathsf{Hom}^\bullet_{\mathbb{K}}(A^\bullet, B^\bullet) = \mathsf{Tot}^{\prod}(\mathsf{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet}).$$

From this, we can deduce that $\mathsf{Mod}^\bullet_{\mathbb{K}}$ is a closed symmetric monoidal category. The tensor is collected from the data of $\mathsf{Hom}^\bullet_{\mathbb{K}}$. We do this by defining an anticommutative double complex $(A^\bullet \otimes_{\mathbb{K}} B^\bullet)^{\bullet,\bullet} = (\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p+q=n} A^p \otimes B^q, d_A \otimes B, A \otimes d_B)$, then the tensor is defined as

$$A^\bullet \otimes B^\bullet = \mathsf{Tot}^{\oplus}((A^\bullet \otimes B^\bullet)^{\bullet,\bullet}).$$

This tensor is left adjoint to $\mathsf{Hom}^\bullet_{\mathbb{K}}$. All the structure morphisms for a closed symmetric monoidal category are inherited from s inherited from $\mathsf{Mod}^*_{\mathbb{K}}$, and this also means that $\mathsf{Mod}^\bullet_{\mathbb{K}}$ employs the same electronic circuits as $\mathsf{Mod}^*_{\mathbb{K}}$.

The category of cochain complexes with chain maps $\mathsf{Ch}(\mathbb{K})$ is defined to have its hom-objects as $Z^0\mathsf{Hom}^\bullet_{\mathbb{K}}(A^\bullet, B^\bullet)$. By abuse of notation we may write $\mathsf{Ch}(\mathbb{K}) = Z^0\mathsf{Mod}^\bullet_{\mathbb{K}}$. Notice that this condition means that the derivative of any morphism $f : A^\bullet \to B^\bullet$ in $\mathsf{Ch}(\mathbb{K})$ is 0; i.e., that $\partial f = 0$, or $f \circ d_A = d_B \circ f$. We will call these morphisms chain maps.

The homotopy category $\mathsf{K}(\mathbb{K})$ is defined to be the quotient category of $\mathsf{Ch}(\mathbb{K})$ at null-homotopic chain maps. Observe that $\mathsf{K}(\mathbb{K}) = H^0\mathsf{Mod}^\bullet_{\mathbb{K}}$ because the chain maps $f, g : A^\bullet \to B^\bullet$ are homotopic if there is a homogenous morphism $h : A^\bullet \to B^\bullet$ of degree -1 such that $\partial h = f - g$.

A chain map $f : A^\bullet \to B^\bullet$ induces homogenous morphisms of degree 0.

$$B^*f : B^*A \to B^*B$$
$$Z^*f : Z^*A \to Z^*B$$
$$H^*f : H^*A \to H^*B$$

We say that f is a quasi-isomorphism if H^*f is an isomorphism, which is equivalent to saying that $\mathsf{cone}(f)$ is exact.

A cochain complex $N^\bullet$ is said to be contractible if id_N is null-homotopic. Then it follows for any other cochain complex $M^\bullet$ that $H^0\mathsf{Hom}^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet) \simeq 0$.

The shift functor $\_[n]$: $\text{Mod}^\bullet_{\mathbb{K}} \to \text{Mod}^\bullet_{\mathbb{K}}$ is defined on cochains $M^\bullet$ as

$$(M^\bullet, d_M)[n] = (M^\bullet[n], (-1)^n d_M).$$

With this definition, shifting is naturally isomorphic to tensoring. That is if $\mathbb{K}[n]$ denotes the field concentrated in dimension $-n$, then

$$\mathbb{K}[n] \otimes_{\mathbb{K}} M^\bullet \simeq M^\bullet[n] \simeq M^\bullet \otimes_{\mathbb{K}} \mathbb{K}[n].$$

One may see how the differential gets its sign by writing out the total tensor product. We usually call $\_[1]$ shifting, desuspension or looping; and $\_[-1]$ for inverse-shifting, suspension or delooping.

We are now ready to talk about algebras in $\text{Mod}^\bullet_{\mathbb{K}}$.

Definition 1.1.54 (Differential graded algebra)**.** $(A^\bullet, d_A)$ is a differential graded algebra if:

- $A^\bullet$ is a differential algebra in $\text{Mod}^\bullet_{\mathbb{K}}$,
- the structure morphisms $(\cdot_A)$ and 1_A are chain maps,
- and the derivation and differential coincide.

Example 1.1.55 (The unit). $\mathbb{K} = (\mathbb{K}, 0)$ is a differential graded algebra in the trivial way. It is concentrated in degree 0, and the differential is the trivial derivation.

Example 1.1.56 (De Rham complex). Given a manifold M, the exterior algebra ΩM is a differential graded algebra. See Tu [Tu11] for a thorough explanation.

In the case of differential graded algebras, we can naively define homotopies like homotopies for cochain complexes. Given morphisms $f, g : A^\bullet \to B^\bullet$, a homotopy between f and g is a morphism $h : A^\bullet \to B^\bullet$ of degree -1 such that $\partial h = f - g$. We know that such morphisms allow us to say that these morphisms are isomorphic in homotopy on the underlying cochain complexes. However, the ring structure is no longer required to be preserved. We amend this problem by (f, g)-derivations.

Definition 1.1.57. Suppose there are morphisms $f, g : A^\bullet \to B^\bullet$. We say that $h : A \to B$ is an (f, g)-derivation if $|h| = -1$ and $h \circ (\cdot_A) = (\cdot_A) \circ (f \otimes h + g \otimes h)$.

We will say that the morphisms f and g are homotopic whenever there is an (f, g)-derivation h such that $\partial h = f - g$.

Given a differential graded, or dg-algebra $A^\bullet$, we may form the category of left $A^\bullet$-modules, Mod_A.

Definition 1.1.58. $M^\bullet$ is a left $A^\bullet$-module if

- $M^\bullet$ is a cochain complex,
- there is a chain map $\mu_M : A^\bullet \otimes_{\mathbb{K}} M^\bullet \to M^\bullet$ satisfying associativity and unitality,

- d_M is an $A^\bullet$-derivation.

The hom-objects are defined analogously. We use $\mathrm{Hom}^\bullet_{A^\bullet}$ to denote the $\mathbb{K}$-linear cochain complex.

With this definition, the categories $\mathrm{Mod}_{\mathbb{K}}$ where $\mathbb{K}$ is considered as a cochain complex, and the category $\mathrm{Mod}^\bullet_{\mathbb{K}}$ is the same category because a cochain complex already satisfies the first two bullet points by definition. Being a $\mathbb{K}^\bullet$-derivation is a trivial condition, so every map meets this.

We also have the dual definition to obtain dg-coalgebras, (f, g)-coderivations and their comodules.

Definition 1.1.59. $C^\bullet$ is a differential graded coalgebra if

- $C^\bullet$ is a differential coalgebra in $\mathrm{Mod}^\bullet_{\mathbb{K}}$,
- the structure morphisms Δ_C and ε_C are chain maps,
- the coderivation and differential coincides

Definition 1.1.60. Suppose that $f, g : C^\bullet \to D^\bullet$ are morphisms of dg-coalgebras. We say that h is an (f, g)-coderivation if $\Delta h = (f \otimes h + g \otimes h)\Delta$.

Two morphisms $f, g : C^\bullet \to D^\bullet$ are said to be homotopic if there is an (f, g)-coderivation such that $\partial h = f - g$.

Definition 1.1.61. $N^\bullet$ is a left $C^\bullet$-comodule if

- $N^\bullet$ is a cochain complex,
- there is a chain map $\omega_C : N^\bullet \to C^\bullet \otimes_{\mathbb{K}} N^\bullet$ satisfying coassociativity and counitality,
- d_N is a $C^\bullet$-coderivation.

By these definitions, we may extend proposition 1.1.43 to the category of cochain complexes.

Corollary 1.1.61.1. *Let $A^\bullet$ be a differential graded algebra and $M^\bullet$ a cochain complex. A homogenous $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$ of same degree, i.e. there is an isomorphism $\mathrm{Hom}^*_{\mathbb{K}}(M^\bullet, A^\bullet \otimes_{\mathbb{K}} M^\bullet) \simeq \mathrm{Der}^*(A^\bullet \otimes_{\mathbb{K}} M^\bullet)$. Moreover, d_f is given as $(\nabla_{A^\bullet} \otimes id_M) \circ (id_A \otimes f) + d_{A \otimes M}$.*

Dually, if $C^\bullet$ is a differential graded coalgebra and $N^\bullet$ is a cochain complex, then a homogenous $\mathbb{K}$-linear morphism $g : C^\bullet \otimes N^\bullet \to N^\bullet$ uniquely determines a coderivation $d_g : C^\bullet \otimes_{\mathbb{K}} N^\bullet \to C^\bullet \otimes_{\mathbb{K}} N^\bullet$. There is an isomorphism $\mathrm{Hom}^*_{\mathbb{K}}(C^\bullet \otimes_{\mathbb{K}} N^\bullet, N^\bullet) \simeq \mathrm{Coder}^*(C^\bullet \otimes_{\mathbb{K}} N^\bullet)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_{C^\bullet} \otimes id_N) + d_{C \otimes N}$.

Proof. The same electronic circuits as in the proof of proposition 1.1.43 suffice to prove this statement. $\square$

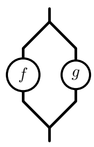
Notably, this statement carries an additional two duals. We have the same result when considering right modules, and the same proof applies in these cases.

1.2 Cobar-Bar Adjunction

1.2.1 Convolution Algebras

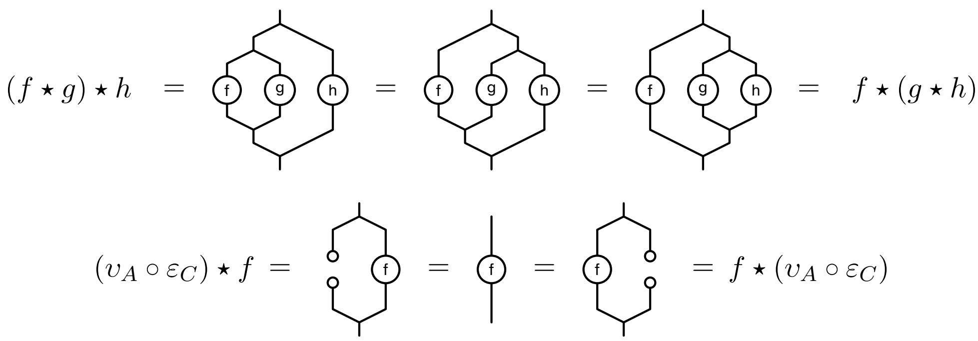
Given a coalgebra C and an algebra A, we obtain a particular product on the hom-object $\mathrm{Hom}_{\mathbb{K}}(C, A)$ by twisting the comultiplication and multiplication together. The convolution algebra forms the backbone of our proof of the cobar-bar adjunction.

Let C be a coalgebra and A an algebra, then if $f, g : C \to A$ is a $\mathbb{K}$-linear morphism we may define $f \star g = (\cdot_A)(f \otimes g)\Delta_C$. This operation is called $\star$ convolution.

$f \star g =$ 

Proposition 1.2.1 (Convolution algebra)**.** *The $\mathbb{K}$-module Hom$_{\mathbb{K}}(C, A)$ is an associative algebra when equipped with convolution $\star$: Hom$_{\mathbb{K}}(C, A) \to$ Hom$_{\mathbb{K}}(C, A)$. The unit is given by $1 \mapsto \upsilon_A \circ \varepsilon_C$.*

Proof. This proposition follows from (co)associativity and (co)unitality of (C) A.



□

This proof does not rely on braiding and lifts to any closed symmetric monoidal category.

Any algebra A may be considered a differential algebra together with the trivial derivation. That is, $(A, 0)$ is a differential algebra. For such structures, the set of A-derivations is precisely the set of A-linear morphisms. Dually, we can consider every coalgebra C as a differential coalgebra.

We may apply a trivialization of proposition 1.1.43 to A and C considered as differential (co)algebra. When we look at the module $C \otimes_{\mathbb{K}} A$, it is free over A on the right and cofree over C on the left. Consider a morphism $\alpha : C \to A$, and then there are two ways to extend α to obtain a (co)derivation. Precomposing with C's comultiplication gives us a morphism from C to the free A-module $C \otimes_{\mathbb{K}} A$,

$$(id_C \otimes \alpha) \circ \Delta_C : C \to C \otimes_{\mathbb{K}} A.$$

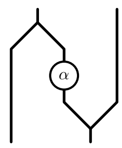
Postcomposing with the multiplication of A gives us a morphism from to the cofree C-comodule $C \otimes_{\mathbb{K}} A$ to A,

$$(\cdot_A) \circ (\alpha \otimes id_A) : C \otimes_{\mathbb{K}} A \to A.$$

When we apply proposition 1.1.43 to both morphisms, it yields the same map. Therefore it is both a derivation and a coderivation, as

$$d^r_\alpha = (id_C \otimes (\cdot_A)) \circ (id_C \otimes \alpha \otimes id_A) \circ (\Delta_C \otimes id_A)$$

$$d^r_\alpha \quad =$$

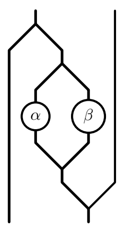


This coderivation will be very important for the rest of this thesis. In the ungraded case, we may transform it into a ring homomorphism.

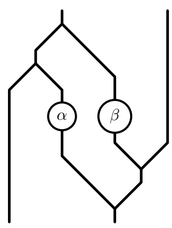
Proposition 1.2.2. $d^r_{\_}$ *: Hom*$_{\mathbb{K}}(C, A) \to$ *End*$(C \otimes_{\mathbb{K}} A)$ *is a morphism of algebras. Moreover, if* $\alpha \star \alpha = 0$*, then* $(d^r_\alpha)^2 = 0$.

Proof. The proof follows from (co)associativity and (co)unitality.

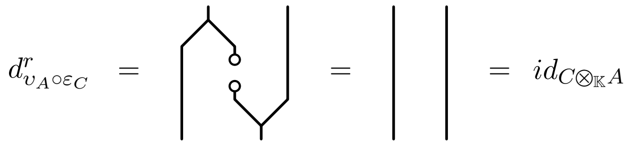
$$d^r_{\alpha\star\beta} \quad =$$



$$=$$



$$= \quad d^r_\alpha \circ d^r_\beta$$



□

This proof relies on braiding, so we will encounter problems when we try to lift this proposition to the graded case. We may observe that the above has no problem lifting, and this is because the β has no morphisms of odd degrees to the right or over itself. However, the dual will introduce some signs when lifted.

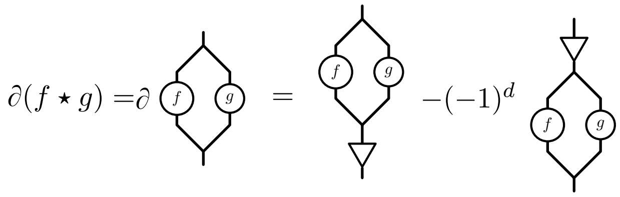
Corollary 1.2.2.1. *Suppose that C and A are differential graded (co)algebras. $d^r_{\_}$: Hom$^*_{\mathbb{K}}(C, A) \to$ End$^*(C \otimes_{\mathbb{K}} A)$ extends to a homogenous ring morphism of degree 0.*

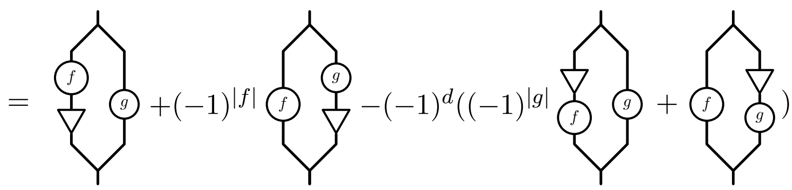
Suppose that C and A are differential graded (co)algebras. We want to expect that the differential ∂ makes (Hom$^*_{\mathbb{K}}(C, A), \star$) into a dg-algebra.

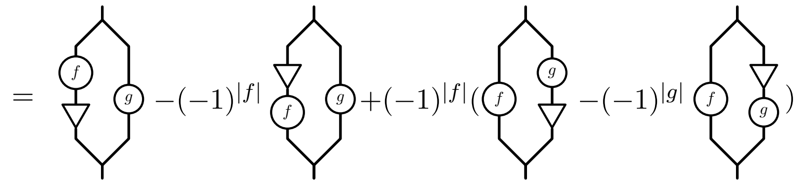
Proposition 1.2.3. *The convolution algebra (Hom$^*_{\mathbb{K}}(C, A), \star$) is a dg-algebra with differential ∂.*

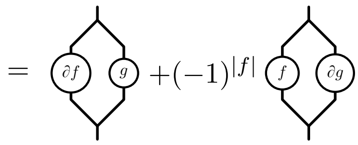
Proof. We know that (Hom$^*_{\mathbb{K}}(C, A), \star$) is a convolution algebra and that (Hom$^*_{\mathbb{K}}(C, A), \partial$) is a cochain complex. It remains to verify that the differential is compatible with the multiplication, i.e., $\partial(f \star g) = \partial f \star g + (-1)^{|f|} f \star \partial g$.

Let $f, g \in$ Hom$^*_{\mathbb{K}}(C, A)$ be two homogenous morphisms. The key property to arrive at the result is that the differential in a dg-(co)algebra is a (co)derivation. We denote the degree of $f \star g$ as $|f \star g| = |f| + |g| = d$. Then







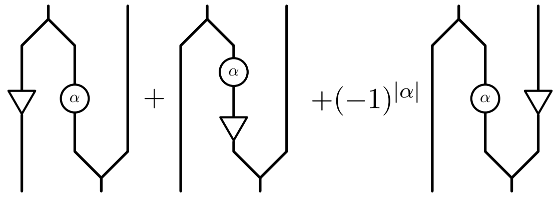


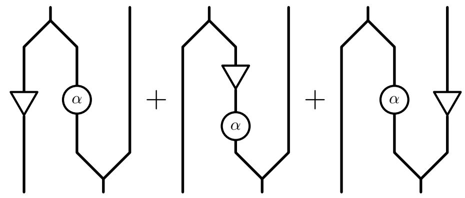
$$= \partial(f) \star g + (-1)^{|f|} f \star \partial(g)$$

□

Proposition 1.2.4. *The morphism $d^r_{\_}$: Hom$^\bullet_{\mathbb{K}}(C, A) \to$ End$^\bullet(C \otimes_{\mathbb{K}} A)$ is a chain map.*

Proof. We already know from Corollary 1.2.2.1 that $d^r_{\_}$ is a homogenous ring map. It remains to see that it commutes with the differentials. That is, $\partial d^r_\alpha = d^r_{\partial\alpha}$. We write out each summand in ∂d^r_α,

$d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha =$ 

$d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A} = (-1)^{|\alpha|}$ 

When α is of even degree, $\partial d^r_\alpha = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha - d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel, and we have

$$\partial d^r_\alpha = d_{d_A\alpha - \alpha d_C} = d_{\partial\alpha}.$$

When α is of odd degree, $\partial d^r_\alpha = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel, and we have

$$\partial d^r_\alpha = d_{d_A\alpha + \alpha d_C} = d_{\partial\alpha}.$$

□

1.2.2 Twisting Morphisms

In this section, we will define twisting morphisms from coalgebras to algebras. They are important as the bifunctor Tw(C, A) is represented in both arguments. To understand the elements of Tw, we start this section by reviewing the Maurer-Cartan equation.

Suppose that C is a coaugmented dg-coalgebra and A is an augmented dg-algebra. We say that a morphism $\alpha \in$ Hom$^*_{\mathbb{K}}(C, A)$ is twisting if it is of degree 1, is 0 on the coaugmentation of C, is 0 on the augmentation of A and satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

We say that α is an element of Tw$(C, A) \subset$ Hom$^1_{\mathbb{K}}(C, A) \subset$ Hom$^*_{\mathbb{K}}(C, A)$. Notice that these requirements means that $\mathrm{Im}\alpha|_{\overline{C}} \subseteq \overline{A}$. In light of proposition 1.2.2, every morphism between (coalgebras) algebras extends to a unique (co)derivation on the tensor product $C \otimes_{\mathbb{K}} A$. Let d^r_α denote this unique morphism. In the case of dg-coalgebras and dg-algebras, we perturb the total differential on the tensor with d^r_α, as in proposition 1.1.43. We call this derivation for the perturbated derivative,

$$d_\alpha = d_{C \otimes_{\mathbb{K}} A} + d^r_\alpha = d_C \otimes id_A + id_C \otimes d_A + d^r_\alpha.$$

Proposition 1.2.5. *Suppose that C is a dg-coalgebra and A is a dg-algebra, and $\alpha \in$ Hom$^1_{\mathbb{K}}(C, A)$. The perturbated derivation satisfies the following relation.*

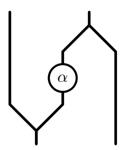
$$d_\alpha{}^2 = d^r_{\partial\alpha + \alpha\star\alpha}$$

Moreover, a morphism satisfies the Maurer-Cartan equation if and only if its associated perturbated derivative is a differential.

Proof. $d_\alpha{}^2 = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A} + d^r_\alpha{}^2$. The result is immediate by proposition 1.2.4. □

Corollary 1.2.5.1. *If $\alpha : C \to A$ is a twisting morphism, then $(C \otimes_{\mathbb{K}} A, d^\bullet_\alpha)$ is a cochain complex which is also a left C-comodule and a right A-module. We call this the right twisted tensor product, denoted as $C \otimes_\alpha A$.*

Normally $A \otimes C$ and $C \otimes A$ are isomorphic as modules. In general, it is not true that $C \otimes_\alpha A$ and $A \otimes_\alpha C$ are isomorphic since we have to choose a particular side to perform the twisting. However, if A is commutative and C is cocommutative, they are isomorphic. To illustrate, we realize the unique derivation above as a right derivative. The left derivative d^l_α is then defined analogously,

$d^l_\alpha =$ 

.

$d^l_{\_} : \mathrm{Hom}^\bullet_{\mathbb{K}}(C, A) \to \mathrm{End}^\bullet(C, A)$ does no longer define a ring morphism. Note that this still commutes with the differential. The problem lies in the ring homomorphism property. Observe that we get

$$d^l_{\alpha\star\beta} = (-1)^{|\alpha||\beta|} d^l_\beta \circ d^l_\alpha.$$

We summarize this in the next proposition.

Proposition 1.2.6. *The morphism* $d^l_{\_}$ *:* $\mathit{Hom}^\bullet_{\mathbb{K}}(C, A) \to \mathit{End}^\bullet(C, A)$ *is a skew chain map.*

Proof. This proposition is clear from the previous discussion. □

Remark 1.2.7. The functoriality of the right twisted tensor at the level of chain maps does not work. To show where it may go wrong, pick two twisting morphisms $\alpha : C \to A$ and $\beta : C' \to A'$. Given a pair of morphisms $f : C \to C'$ and $g : A \to A'$, it is unclear if $f \otimes g$ will preserve the perturbed differential, and it is not valid in general.

However, it is the case that the right twisted tensor product defines a tri-functor from the category of elements to cochain complexes,

$$\_ \otimes_{\_} \_ : \sum_{\mathrm{Coalg} \otimes \mathrm{Alg}} \mathrm{Tw} \to \mathrm{Mod}^A_C.$$

Any commutative square as below gets mapped to a morphism of its right twisted tensors. Here f is a morphism of coalgebras, and g is a morphism of algebras,

$$\begin{array}{ccccc} C & \xrightarrow{\alpha} & A & & C \otimes_\alpha A \\ \downarrow f & & \downarrow g & \rightsquigarrow & \downarrow f\otimes g \\ C' & \xrightarrow{\alpha'} & A' & & C' \otimes_{\alpha'} A' \end{array}$$

The important property to obtain this is that f and g are morphisms in their respective categories, allowing us to collapse the different compositions to the same map up to sign.

1.2.3 Bar and Cobar Construction

Eilenberg and Mac Lane first formalized the bar construction for augmented skew-commutative dg-rings [EM53]. The bar construction then served as a method to calculate the homology of Eilenberg-Mac Lane spaces. This construction was later dualized by Adams [Ada56] to obtain the cobar construction. Its first purpose was to serve as a method for constructing an injective resolution to calculate the cotor resolution [EM66]. With time, the bar-cobar construction has been subjected to many generalizations, such as a fattened tensor product on simplicially enriched, tensored, and cotensored categories [Rie14]. We will mainly follow the work of [LV12] to obtain the one-sided algebraic bar and cobar construction. The approach we will take is also slightly inspired by MacLane's canonical resolutions of comonads [Mac71].

For our purposes, the bar construction of an augmented algebra is a simplicial resolution as a cofree coalgebra structure. Given a dg-algebra, we will realize this as the total complex of its resolution. Dually, the cobar construction of a conilpotent coalgebra is a cosimplicial resolution as a free algebra structure. We will see that these constructions define an adjoint pair of functors.

An algebra A is a monoid in the monoidal category $(\mathsf{Mod}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. By proposition B.1.5, we may think of A as an augmented cosimplicial object $A : \Delta_+ \to \mathsf{Mod}_{\mathbb{K}}$. Notice that all of the cosimplicial identities follow from associativity and unitality. If A is an augmented algebra, we may instead give it the structure of an augmented simplicial set. Let $d_0^0 = \varepsilon_A$ be the augmentation. We define $d_n^n = A^{\otimes n-1} \otimes \varepsilon_A$ and set $d_n^i = A^{i-1} \otimes (\cdot_A) \otimes A^{\otimes n-i-1}$. The degeneracies are chosen to be the units, that is, the morphisms $s_n^i = A^{\otimes i} \otimes \upsilon_A \otimes A^{\otimes n-i-1}$. One may check that this structure defines an augmented simplicial object $A : \Delta_+^{op} \to \mathsf{Mod}_{\mathbb{K}}$. Observe that the cochain complex $\mathsf{C}A$ is exactly the Hochschild complex of A. We depict the simplicial object in the following diagram:

$$\begin{tikzcd}
\mathbb{K} & A \arrow[l, "\varepsilon_A"'] & A^{\otimes 2} \arrow[l, shift right, "(\cdot_A)"'] \arrow[l, shift left, "A\otimes\varepsilon_A"] & A^{\otimes 3} \arrow[l, shift right=2, "(\cdot_A)"'] \arrow[l] \arrow[l, shift left=2, "A^{\otimes 2}\otimes\varepsilon_A"] & \cdots \arrow[l, shift right=3, "(\cdot_A)"'] \arrow[l, shift right] \arrow[l, shift left] \arrow[l, shift left=3, "A^{\otimes 4}\otimes\varepsilon_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} & A \arrow[r, "s^1"] & A^{\otimes 2} \arrow[r, shift left, "s^i"] \arrow[r, shift right] & A^{\otimes 3} \arrow[r, shift left=2, "s^i"] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

The augmentation ideal $\overline{A}$ carries a natural semi-simplicial structure induced by A. As in Example 1.1.50, there is an associated cochain complex to $\overline{A}$ by restricting each of the face maps, $\overline{d}^i = d^i|_{\overline{A}} : \overline{A}^{\otimes n} \to \overline{A}^{\otimes n-1}$. The associated cochain complex is the non-unital Hochschild complex of A. We depict the semi-simplicial object as shown in the following diagram:

$$\begin{tikzcd}
\mathbb{K} & \overline{A} \arrow[l, "0"'] & \overline{A}^{\otimes 2} \arrow[l, shift right, "(\cdot_A)"'] \arrow[l, shift left, "0"] & \overline{A}^{\otimes 3} \arrow[l, shift right=2, "(\cdot_A)"'] \arrow[l] \arrow[l, shift left=2, "0"] & \cdots \arrow[l, shift right=3, "(\cdot_A)"'] \arrow[l, shift right] \arrow[l, shift left] \arrow[l, shift left=3, "0"]
\end{tikzcd}$$

As graded modules, the cochain complex $\mathsf{C}\overline{A}$ is isomorphic to $T^c(\overline{A})$. Here we think of the grading $T^c(\overline{A})$ as starting at 0 and going down to negative degrees. Consider instead the looped non-unital algebra $\overline{A}[1]$. There is a natural grading on every algebra, concentrating it in degree 0. The shift functor then changes the degree to which we concentrate the algebra. However, $\overline{A}[1]$ is no longer an associative algebra. To understand this looped multiplication, we will first consider $\mathbb{K}\{\omega\}$, where $|\omega| = -1$. We define a looped multiplication $(\cdot) : \mathbb{K}\{\omega\}^{\otimes 2} \to \mathbb{K}\{\omega\}$ as

$$\omega \cdot \omega = \omega.$$

Given an algebra A, the looped multiplication of $A[1]$ is defined as the composite

$$(\cdot_{A[1]}) = ((\cdot) \otimes (\cdot_A)) \circ (\mathbb{K}\{\omega\} \otimes \beta \otimes \overline{A}).$$

As an example, suppose that ωa_1 and ωa_2 are elements of $A[1]$, then their multiplication would look like

$$(\cdot_{A[1]})(\omega a_1 \otimes \omega a_2) = (-1)^{|a_1||\omega|}((\cdot) \otimes \cdot_A)(\omega^{\otimes 2} \otimes a_1 \otimes a_2) = (-1)^{|a_1|}\omega a_1 a_2.$$

Observe that the resulting morphism $(\cdot_{A[1]})$ is of degree 1.

Proposition 1.2.8. *Suppose that A is an augmented algebra. The differential $d_{\overline{A}[1]}$ is a coderivation for the cofree coalgebra $T^c(\overline{A}[1])$. Thus $(C\overline{A}[1], d_{\overline{A}[1]})$ is a dg-coalgebra.*

Proof. By injecting $\overline{A}[1]$ into $T^c(\overline{A}[1])$, we may think of $(\cdot_{\overline{A}[1]}) : \overline{A}[1]^{\otimes 2} \to T^c(\overline{A}[1])$ as a morphism into the tensor coalgebra. By using Proposition 1.1.40, $(\cdot_{\overline{A}[1]})$ extends uniquely into a coderivation:

$$d^c_{\overline{A}[1]} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (\cdot_{\overline{A}[1]})^{(n)}_{(i)} = d_{\overline{A}[1]}.$$

$\square$

If (A, d_A) is an augmented dg-algebra, then A is a simplicial object of $\mathrm{Mod}^{\bullet}_{\mathbb{K}}$. There is also an associated complex $\mathrm{C}A$ of A by taking the alternate sum of face maps. The complex $\mathrm{C}A$ may be seen as the total complex of the double complex represented below.

$$\begin{tikzcd}
\cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{K} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots \\
\cdots \arrow[r, "-d_A"'] & A^{-1} \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & A^0 \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & A^1 \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & \cdots \\
\cdots \arrow[r, "d_{A^{\otimes 2}}"'] & (A^{\otimes 2})^{-1} \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^0 \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^1 \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & \cdots \\
& \vdots \arrow[u] \arrow[u, shift left] \arrow[u, shift right] & \vdots x \arrow[u] \arrow[u, shift left] \arrow[u, shift right] & \vdots \arrow[u] \arrow[u, shift left] \arrow[u, shift right] &
\end{tikzcd}$$

For simplicity, we will write d_1 for the horizontal differential and d_2 for the vertical differential. $\mathrm{C}A$ is thus the total complex of the double complex above. Instead of considering the abovementioned double complex, we will consider the double complex associated with the looped algebra $\overline{A}[1]$. The following lemma states that this double complex is well-defined.

Proposition 1.2.9. *Let A be an augmented dg-algebra. The bar complex BA is the total associated cochain complex of the augmentation ideal $\overline{A}$. $(BA, d^{\bullet}_{BA})$ is the cofree conilpotent coalgebra equipped with $d^{\bullet}_{BA} = d_1 + d_2$ as coderivation.*

Proof. d_1 and d_2 are coderivations with respect to deconcatenation as comultiplication. Since the multiplication $(\cdot_A)$ is a chain map, we should have ${d^\bullet_{BA}}^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. We will show this for each element in $A^{\otimes 2}$, and the result may be extended to all of BA. Instead of decorating each a_i with an ω, we will follow Eilenberg and MacLane's notation, using brackets and bars, $\omega a_1 \otimes \omega a_2 = [a_1|a_2]$ [EM53, p. 73]. The bars in this notation are what gave this coalgebra its name.

$$
\begin{aligned}
d_1 \circ d_2[a_1|a_2] &= (-1)^{|a_1|} d_1[a_1 a_2] = (-1)^{|a_1|} d_{A[1]}[a_1 a_2] \\
&= (-1)^{|a_1|+1}[d_A(a_1 a_2)] = (-1)^{|a_1|+1}([d_A(a_1)a_2] + (-1)^{|a_1|}[a_1 d_A(a_2)]) \\
&= (-1)^{|a_1|+1}[d_A(a_1)a_2] - [a_1 d_A(a_2)]
\end{aligned}
$$

$$
\begin{aligned}
d_2 \circ d_1[a_1|a_2] &= d_2 \circ (d_{A[1]} \otimes id_{A[1]} + id_{A[1]} \otimes d_{A[1]})[a_1 \otimes a_2] \\
&= -d_2 \circ ([d_A(a_1)|a_2] + (-1)^{|a_1|+1}[a_1|d_A(a_2)]) \\
&= (-1)^{|d_A(a_1)|+1}[d_A(a_1)a_2] + (-1)^{2|a_1|+2}[a_1 d_A(a_2)] \\
&= (-1)^{|a_1|}[d_A(a_1)a_2] + [a_1 d_A(a_2)] = -d_1 \circ d_2[a_1|a_2]
\end{aligned}
$$

$\square$

Remark 1.2.10. We don't need to show that BA is a functor. This property follows from BA representing the object of Tw$(_, A)$.

On the other hand, a coalgebra C is a comonoid in Mod$_{\mathbb{K}}$. By the dual of proposition B.1.5, we may think of it as an augmented simplicial object $C : (\Delta_+)^{op} \to Mod_{\mathbb{K}}$. Dually, all of the simplicial identities follow from coassociativity and counitality. A coaugmented coalgebra C may be given an augmented cosimplicial structure in the opposite way of algebras. We then get that the coaugmentation quotient $\overline{C}$ is a semi-cosimplicial object of Mod$_{\mathbb{K}}$. Observe that $\overline{C}$ has an associated cochain complex like $\overline{A}$, but every arrow goes in the opposite direction.

$$
\begin{tikzcd}
\mathbb{K} \arrow[r, "\upsilon_C"] & C \arrow[r, shift left, "\Delta_C"] \arrow[r, shift right, swap, "A\otimes\upsilon_C"] & C^{\otimes 2} \arrow[r, shift left=2, "\Delta_C"] \arrow[r] \arrow[r, shift right=2, swap, "C^{\otimes 2}\otimes\upsilon_C"] & C^{\otimes 3} \arrow[r, shift left=3, "\Delta_C"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, swap, "C^{\otimes 4}\otimes\upsilon_C"] & \dots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{K} & C & C^{\otimes 2} \arrow[l, swap, "s_1"] & C^{\otimes 3} \arrow[l, shift right, swap, "s_i"] \arrow[l, shift left] & \dots \arrow[l, shift right=2, swap, "s_i"] \arrow[l] \arrow[l, shift left=2]
\end{tikzcd}
$$

The cobar construction is made from the suspended dg-coalgebra $C[-1]$. We may also denote suspension by tensoring with a formal generator s, such that $|s| = 1$. Then we have an isomorphism $C[-1] \simeq \mathbb{K}\{s\}\otimes C$. The cobar construction is realized as the free tensor algebra $T(\overline{C}[-1])$, where the comultiplication $\Delta_{\overline{C}[-1]}$ induces a derivation $d_{\overline{C}[-1]}$ by Proposition 1.1.40.

Remark 1.2.11. As we have chosen to define $(\cdot_{A[1]})(a_1 \otimes a_2) = (-1)^{|a_1|} a_1 a_2$, we are forced by the linear dual to define $\Delta_{C[-1]}(c) = -(-1)^{|c_{(1)}|} c_{(1)} \otimes c_{(2)}$. Here we use Sweedler's notation without sums to denote the comultiplication. Note that this really should be a sum of many different elementary tensors. Lastly, observe that this definition also agrees with Koszuls's sign rule.

The associated cochain complex CC is the total complex of the double complex below. Similarly, we want to study $C[-1]$ to obtain a similar result to the bar construction.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{-1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C \otimes \overline{C}"'] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{0} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C \otimes \overline{C}"'] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C \otimes \overline{C}"'] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & \cdots \\
\cdots \arrow[r, "d_{\overline{C}}"] & \overline{C}^{-1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \overline{C}^{0} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \overline{C}^{1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[u] \arrow[r] & \mathbb{K} \arrow[u, "0"] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots
\end{tikzcd}
$$

Proposition 1.2.12. *Let C be a coaugmented dg-coalgebra. The cobar complex ΩC is the total associated cochain complex of the suspended coaugmentation quotient $\overline{C}[-1]$. $(\Omega C, d_{\Omega C})$ is the free algebra equipped with the differential $d_{\Omega C} = d_1 + d_2$ as derivation.*

Proof. This proof is similar to the one given for the bar construction. □

Given a string of elements in the cobar $sc_1 \otimes \cdots$, we write it by using pointed brackets and bars instead,

$$sc_1 \otimes sc_2 \otimes \cdots \otimes sc_n = \langle c_1 | c_2 | \cdots | c_n \rangle.$$

The bar and cobar construction defines an adjoint pair of functors. We want to show that for any conilpotent dg-coalgebra C, the object ΩC represents a functor in the category of augmented algebras. By Yoneda's lemma, Ω does truly define a functor.

Theorem 1.2.13. *Let C be a conilpotent dg-coalgebra and A an augmented dg-algebra. The functor Tw(C, A) is represented in both arguments, i.e.*

$$\mathit{Alg}^{\bullet}_{\mathbb{K},+}(\Omega C, A) \simeq \mathit{Tw}(C, A) \simeq \mathit{coAlg}^{\bullet}_{\mathbb{K},\mathit{conil}}(C, BA).$$

Proof. We will show that ΩC represents the set of twisting morphisms in the first argument, and this shows that BA represents the second argument by using every dual proposition. Thus, C must be conilpotent to dualize the results.

Suppose that $f : \Omega C \to A$ is an augmented dg-algebra homomorphism. f is then a morphism of degree 0. By freeness, f is uniquely determined by a morphism $f\mid_{\overline{C}[-1]}: \overline{C}[-1] \to \overline{A}$ of degree 0, which corresponds to a morphism $f' : C \to A$ of degree 1 which is 0 on the augmentation and coaugmentation.

Since f is a morphism of chain complexes, it commutes with the differential, i.e.

$$\begin{aligned} & f \circ d_{\Omega C} = d_A \circ f \\ \Leftrightarrow \quad & f \circ (d_1 + d_2) = d_A \circ f \end{aligned}$$

By 1.1.11, to establish these conditions, it is enough to consider the summand where $d_1 = -d_C$ and $d_2 = \overline{\Delta}_{C[-1]}$. Then the right hand side becomes $-f' \circ d_C - (-1)^{|f|}(\cdot_A)(f' \otimes f')\Delta_C$. This is equivalent to saying that $-f' \circ d_C - f' \star f' = d_A \circ f'$. Thus f' is a twisting morphism as desired.

Since every step to establish that f' is a twisting morphism was a logical equivalence, we arrive at the desired conclusion. □

For our convenience, we will give these isomorphisms some names. Whenever $\tau : C \to A$ is a twisting morphism, we denote the induced morphism of algebras as $f_\tau : \Omega C \to A$, and the induced morphism of coalgebras as $g_\tau : C \to BA$.

Remark 1.2.14. We could have defined a twisting morphism from any coalgebra C to algebra A. In this case, we could have defined a twisting morphism as a morphism of degree 1, which satisfies the Cartan-Maurer equation. However, the cobar and bar construction on augmented algebras does not represent this definition of twisting morphisms. The subclass of twisting morphisms which also (co)restricts to twisting morphisms on its coaugmentation quotient and augmentation ideal, would be represented in this manner, which is what our definition requires.

The cobar-bar adjunction consists of a composition with the augmentation ideal (quotient) and then the (co)free tensor (co)algebra. By reversing these operations, we obtain another adjunction that is more or less the same. By abuse of language, we will call these functors for the bar and cobar construction as well, and they establish an adjoint pair between non-unital dg-algebras and reduced conilpotent dg-coalgebras. In other words, given a non-unital dg-algebra A and a reduced conilpotent dg-coalgebra C, $BA = \overline{T}^c(A[1])$ and $\Omega C = \overline{T}(C[-1])$.

$$\begin{tikzcd} \widehat{\mathsf{Alg}}^{\bullet}_{\mathbb{K}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "B"'] & \mathsf{coAlg}^{\bullet}_{\mathbb{K},\mathsf{conil},-} \arrow[l, bend right, "\Omega"'] \end{tikzcd}$$

We obtain universal elements and universal properties associated with this adjunction. Let A be an augmented dg-algebra, then the identity of the coalgebras $id_{BA} : BA \to BA$, the counit $\varepsilon_A : \Omega BA \to A$ and a twisting morphism $\pi_A : BA \to A$ are equivalent by the adjunction and representation. Dually, the identity of algebras $id_{\Omega C} : \Omega C \to \Omega C$, the unit $\eta_C : C \to B\Omega C$ and the twisting morphism $\iota_C : C \to \Omega C$ are equivalent. The morphisms π_A and ι_C are called the universal elements. We summarize their universal property in the following corollary.

Corollary 1.2.14.1. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. Any twisting morphism $\alpha : C \to A$ factors uniquely through either π_A or ι_C.*

$$\begin{tikzcd} & \Omega C \arrow[dr, dashed, "g_\alpha"] & \\ C \arrow[ur, "\iota_C"] \arrow[rr, "\alpha"] \arrow[dr, dashed, "f_\alpha"'] & & A \\ & BA \arrow[ur, "\pi_A"'] & \end{tikzcd}$$

Moreover, the morphism f_α is a morphism of dg-coalgebras, and g_α is a morphism of dg-algebras.

Definition 1.2.15 (Augmented Bar-Cobar construction)**.** Let A be an augmented dg-algebra. The (right) augmented bar construction is the right twisted tensor product $BA \otimes_{\pi_A} A$, where π_A is the universal twisting morphism.

Let C be a conilpotent dg-coalgebra. The (right) augmented cobar construction is the right twisted tensor product $C \otimes_{\iota_C} \Omega C$, where ι_C is the universal twisting morphism.

Remark 1.2.16. We could have defined the augmented bar-cobar construction as the left twisted tensor product. There is no preference for handedness. It will be specified whenever we wish to be precise about which handedness we will use. For instance, the left augmented bar construction of A.

Proposition 1.2.17. *The augmentation ideal and quotient of the augmented bar and cobar construction are acyclic, i.e., $BA\overline{\otimes}_{\pi_A}A$ ($A\overline{\otimes}_{\pi_A}BA$) and $C\overline{\otimes}_{\iota_C}\Omega C$ ($\Omega C\overline{\otimes}_{\iota_C}C$) are acyclic.*

Proof. We will postpone this proof until chapter 3; this is a part of the fundamental theorem of twisting morphisms and will not be relevant until then. $\square$

1.3 Strongly Homotopy Associative Algebras and Coalgebras

1.3.1 SHA-Algebras

We have seen from Corollary 1.2.8 that any dg-algebra A defines a dg-coalgebra $T^c(A[1])$, the bar construction, with a coderivation m^c of degree 1. Does this work in reverse? I.e., if A is a vector space such that the coalgebra $T^c(A[1])$ together with a coderivation m^c is a dg-coalgebra, is then A an algebra? The answer is no, but it leads to the definition of a strongly homotopy associative algebra.

Definition 1.3.1. An A_∞-algebra is a graded vector space A together with a differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ that is a coderivation of degree 1.

The differential m induces structure morphisms on $A[1]$. By Proposition 1.1.40, there is a natural bijection $\text{Hom}_{\mathbb{K}}(\overline{T}^c(A[1]), A[1]) \simeq \text{Coder}(\overline{T}^c(A[1]), \overline{T}^c(A[1]))$ given by the projection onto $A[1]$. Thus $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ corresponds to maps $\widetilde{m}_n : A[1]^{\otimes n} \to A[1]$ of degree 1 for any $n \geqslant 1$. We define maps $m_n : A^{\otimes n} \to A$ by the composite $s\widetilde{m}_n\omega^{\otimes n}$. Since $\omega^{\otimes n}$ is of degree $-n$, $\widetilde{m}_n$ and s is of degree 1, we get that m_n is of degree $2-n$.

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "m_n"] \arrow[d, "\omega^{\otimes n}"', "\simeq"] & A \\ A[1]^{\otimes n} \arrow[r, "\widetilde{m}_n"] & A[1] \arrow[u, "s", "\simeq"'] \end{tikzcd}$$

Remark 1.3.2. The choice of isomorphisms here is not canonical. Different choices may lead to different signs in the following formulas. We will follow the sign convention of Loday and Vallette [LV12]. This will give us the same signs as in Lefèvre-Hasegawa [Lef03], as his signs always come in a pair to cancel each other out.

Proposition 1.3.3. *An A_∞-algebra is equivalent to a graded vector space A together with homogenous morphisms $m_n : A^{\otimes n} \to A$ of degree $2-n$. Moreover, the morphism must satisfy the following relations for any $n \geqslant 1$:*

$$(\textit{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

Remark 1.3.4. We make a more convenient notation for (rel_n), called partial composition $\circ_i$,

$$m_{p+1+r} \circ_{p+1} m_q = m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}).$$

With this noation we may rewrite each (rel_n) as

$$(\text{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Before starting with the proof, we will need a lemma for checking whether a coderivation $m : T^c(A) \to T^c(A)$ is a differential.

Lemma 1.3.5. *Let $m : T^c(A) \to T^c(A)$ be a coderivation, and denote $m_n = m|_{A^{\otimes n}}$. m is a differential if and only if the following relations are satisfied,*

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Proof. By Proposition 1.1.40 we may write $m = \sum_{n=0}^{\infty}\sum_{i=0}^{n} m_{(n)}^{(i)}$. By using partial composition, we rewrite its n'th component as,

$$m_n = \sum_{q=1}^{n}\sum_{p=1}^{n} id^{\otimes(n-q)} \circ_p m_q = \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q.$$

For m^2, we denote its n'th component as m_n^2. Let $\pi : T^c(A) \to A$ denote the projection onto A. Observe the following:

$$\begin{aligned} m_n^2 &= m \circ m_n = m \circ \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q = \sum_{p+q+r=n} m \circ_{p+1} m_q, \\ \pi m_n^2 &= \pi \sum_{p+q+r=n} m \circ_{p+1} m_q = \sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q. \end{aligned}$$

By Proposition 1.1.43, every coderivation is uniquely determined by π, we get that $m^2 = 0$ if and only if

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

$\square$

Proof of Proposition 1.3.3. Let (A, m) be an A_∞-algebra. We denote the n'th component of m as $\widetilde{m}_n$. The n'th components thus define maps $m_n : A^{\otimes n} \to A$ as $m_n = s\widetilde{m}_n\omega^{\otimes n}$.

By the above lemma, we know that the n'th component of m^2 is,

$$\begin{aligned} &\sum_{p+q+r=n} \widetilde{m}_{p+1+r} \circ_{p+1} \widetilde{m}_q \\ &\quad = \sum_{p+q+r=n} \omega m_{p+1+r} s^{\otimes(p+1+r)} \circ_{p+1} \omega m_q s^{\otimes q} = \sum_{p+q+r=n} (-1)^{pq+r} \omega m_{p+1+r} \circ_{p+1} m_q s^{\otimes n}. \end{aligned}$$

The last equation is given by applying Proposition 1.1.44 twice. In other words, we want to find a parity $p = p_1 + p_2$, which determines the sign above. To get p_1 we start with moving the s on the left,

$$s^{\otimes p+1+r} \circ (id^{\otimes p} \otimes \omega m_q s^{\otimes q} \otimes id^{\otimes r}) = (-1)^{p_1}(s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

By Proposition 1.1.44,

$$p_1 = \sum_{i=1}^{n}\sum_{1\leqslant j<i} (\text{if } j = p+1 \text{ then } 1 \text{ otherwise } 0) = r.$$

In the next step, we separate the s on the right,

$$(id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) \circ s^{\otimes n} = (-1)^{p_2}(s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

We calculate p_2 to be,

$$p_2 = (2-q) \sum_{1 \leqslant j < p+1} 1 = 2p - qp.$$

Thus the parity of p is $p = 2p - qp + r = pq + r$ modulo 2.

Since suspension and loop are isomorphisms, we get that $m^2 = 0$ if and only if (rel_n) are 0 for every $n \geqslant 1$, i.e.

$$\sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

□

Given an A_∞ algebra A, we may either think of it as a differential tensor coalgebra $\overline{T}^c(A[1])$ with differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$, or as a graded vector space with morphisms $m_n : A^{\otimes n} \to A$ satisfying (rel_n). We will calculate (rel_n) for $n = 1, 2, 3$:

$$\begin{aligned}
&(\text{rel}_1) \qquad m_1 \circ m_1 = 0 \\
&(\text{rel}_2) \qquad m_1 \circ m_2 - m_2 \circ_1 m_1 - m_2 \circ_2 m_1 = 0 \\
&(\text{rel}_3) \qquad m_1 \circ m_3 - m_2 \circ_1 m_2 + m_2 \circ_2 m_2 + m_3 \circ_1 m_1 + m_3 \circ_2 m_1 + m_3 \circ_3 m_1 = 0
\end{aligned}$$

We see that (rel_1) states that m_1 should be a differential. Thus we may think of (A, m_1) as a cochain complex. Furthermore, (rel_2) says that $m_2 : (A^{\otimes 2}, m_1 \otimes id_A + id_A \otimes m_1) \to (A, m_1)$ is a morphism of chain complexes. Lastly, (rel_3) gives us a homotopy for the associator of m_2, namely m_3. Thus we may regard (A, m_1, m_2) as an algebra that is associative up to the homotopy m_3. Regarding A as a cochain complex, instead, we obtain our final equivalent definition of an A_∞-algebra.

Proposition 1.3.6. *Suppose that (A, d) is a cochain complex and that there exist morphisms $m_n : A^{\otimes n} \to A$ of degree $2 - n$ for any $n \geqslant 2$. A is an A_∞-algebra if and only it satisfies the following relations:*

$$(\textit{rel'}_n) \qquad \partial(m_n) = -\sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q$$

We define the homotopy of an A_∞-algebra to be the homology of the cochain complex (A, m_1). Since $\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2$, we get that m_2 is associative in homology. Thus for any A_∞-algebra A, the homotopy HA is an associative algebra. The operadic homology of A is defined as the homology of $(T^c(A[1]), m)$, which is the non-unital augmented Hochschild homology of A.

Example 1.3.7. Suppose that V is a cochain complex with differential d. Then V is an A_∞-algebra with trivial multiplication. In other words $m^1 = d$ and $m^i = 0$ for any $i > 1$.

Example 1.3.8. Suppose that A is a dg-algebra. Then A is an A_∞-algebra where $m^1 = d$, $m^2 = (\cdot)$ and $m^i = 0$ for any $i > 2$.

Next, we want to understand the category of A_∞-algebras. A morphism between A_∞-algebras is called an ∞-morphism. We define such an ∞-morphism $f : A \rightsquigarrow B$ between A_∞-algebras as associated dg-coalgebra homomorphism $Bf : (\overline{T}^c(A[1]), m^A) \to (\overline{T}^c(B[1]), m^B)$. Here Bf is purely formal, and we will make sense of this soon.

Proposition 1.3.9. *Let A, B be two A_∞-algebras. A collection of morphisms $f_n : A^{\otimes n} \to B$ of degree $1-n$ for any $n \geqslant 1$ defines an ∞-morphism $f : A \rightsquigarrow B$ if and only if f_1 is a morphism of chain complexes and for any $n \geqslant 2$ the following relations are satisfied:*

$$(\mathit{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k\geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k}),$$

where e is

$$e = \sum_{l=1}^{k} (1-i_l) \sum_{1\leqslant m<l} i_m.$$

Proof. Establishing the shape of this equation is immediate by the universal property of cofree coalgebras. We obtain the parity e by factoring the s to the right.

$$(f_{i_1} \otimes \cdots \otimes f_{i_k}) \circ s^{\otimes n} = (-1)^e (f_{i_1} s^{\otimes i_1} \otimes \cdots \otimes f_{i_k} s^{\otimes i_k}).$$

By Proposition 1.1.44, we arrive at the conclusion,

$$e = \sum_{l=1}^{k} |f_{i_l}| \sum_{1\leqslant m<l} |s^{\otimes i_m}| = \sum_{l=1}^{k} (1-i_l) \sum_{1\leqslant m<l} i_m$$

$\square$

Since the composition of two dg-coalgebra homomorphisms is again a dg-coalgebra homomorphism, we get that the composition of two ∞-morphisms is again an ∞-morphism. More explicitly if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ are two ∞-morphisms, then their composition is defined as

$$(fg)_n = \sum_r \sum_{i_1+\ldots+i_r=n} (-1)^e g_r (f_{i_1} \otimes \ldots \otimes f_{i_r}).$$

Here e denotes the same parity as above.

Definition 1.3.10. An ∞-morphism $f : A \rightsquigarrow B$ is called strict if $f_n = 0$ for any $n \geqslant 2$.

Definition 1.3.11. Alg_∞ denotes the category of A_∞-algebras, and the morphisms in this category are the ∞-morphisms.

Observe that we may extend the bar construction to B : $\mathsf{Alg}_\infty \to \mathsf{CoAlg}^\bullet_{\mathbb{K},\mathsf{conil}}$ to a fully faithful functor. This construction may be done explicitly by using Proposition 1.1.40. The subcategory of the essential image is the full subcategory of every quasi-cofree dg-coalgebra. Notice that the bar construction on the category of dg-algebras is a non-full injection into the category of A_∞-algebras. This inclusion gives us a recontextualization of a dg-algebra as an A_∞-algebra.

A quasi-isomorphism between A_∞-algebras is called an ∞-quasi-isomorphism. Given an ∞-morphism $f : A \rightsquigarrow B$, we say that it is an ∞-quasi-isomorphism if f_1 is a quasi-isomorphism. If we wanted to be more stringent with this definition, we would define an ∞-quasi-isomorphism to be an ∞-morphism which is a quasi-isomorphism of dg-coalgebras. We will later see that these definitions are equivalent.

A homotopy between two A_∞-algebras is a homotopy between the dg-coalgebras they define. We may trace this definition back along the quasi-inverse of the bar construction to get a new definition in terms of many morphisms.

Definition 1.3.12. Let $f, g : A \rightsquigarrow B$ be two ∞-morphisms, we say that $f \sim g$ are homotopic if there is a collection of morphisms $h_n : A^{\otimes n} \to B$ of degree $-n$ such that the following relations are satisfied for any $n \geqslant 1$:

$$f_n - g_n = \sum (-1)^s m^B_{r+1+t} \circ (f_{i_1} \otimes ... \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes ... \otimes g_{j_t}) + \sum (-1)^{j+kl} h_i \circ_{j+1} m^A_k.$$

s is some constant depending on t, r, and k, which is calculable with Koszul's sign rule. More specific details may be found in [Lef03].

One may observe that this definition of homotopy is exactly the same as requiring that the morphisms Bf and Bg are homotopic by a (Bf, Bg)-coderivation Bh.

If we have morphisms of algebras $f, g : A \to A'$ such that they are homotopic, then the (f, g)-derivation $h : A \to A'$ defines a homotopy between f and g if we consider them as strict ∞-morphisms. The relations for when $n = 2$ describes the property of being an (f, g)-derivation whenever f_2 and g_2 are both 0, which is the case by strictness. The higher relations will be trivially satisfied in this case. Thus, we may see that the bar construction maps (f, g)-derivations to (f, g)-coderivations.

As in the same case for algebras, there is also a notion of unital A_∞-algebras and augmented A_∞-algebras. For this discussion, it is essential to observe that the field $\mathbb{K}$ is also an A_∞-algebra. This algebra will be the initial algebra like it does for ordinary algebras.

Definition 1.3.13. A strictly unital A_∞-algebra is an A_∞-algebra A together with a unit morphism $v_A : \mathbb{K} \to A$ of degree 0 such that the following are satisfied:

- $m_1 \circ v_A = 0$.
- $m_2(id_A \otimes v_A) = id_A = m_2(v_A \otimes id_A)$.
- $m_i \circ_k v_A = 0$ for any $i \geqslant 3$ and $1 \leqslant k < i$.

A strictly unital ∞-morphism $f : A \rightsquigarrow B$ between strictly unital A_∞-algebras is a morphism that preserves the unit. This means that $f_1 \upsilon_A = \upsilon_B$ and $f_i \circ_k \upsilon_A = 0$ for any $i \geqslant 2$ and $1 \leqslant k < i$. The collection of strictly unital A_∞-algebras and strictly unital ∞-morphisms form a non-full subcategory of A_∞-algbras. A strict ∞-morphism which is unital at the level of chain complexes is automatically strictly unital. Strict unital will then mean strict and strictly unital. Note that $\mathbb{K}$ is strictly unital where the unit is $id_{\mathbb{K}}$.

Definition 1.3.14. An augmented A_∞-algebra is a strictly unital A_∞-algebra A together with a strict unital morphism $\varepsilon_A : A \to \mathbb{K}$. The ∞-morphism ε_A is called the augmentation of A.

The collection of augmented A_∞-algebras and strictly unital morphism is the category of augmented A_∞-algebras, denoted as $\mathsf{Alg}_{\infty,+}$. As in the same way for algebras, there is an equivalence of categories $\mathsf{Alg}_\infty \simeq \mathsf{Alg}_{\infty,+}$. The augmentation ideal, or the reduced A_∞-algebra, is the kernel of the augmentation ε_A. It does not make sense to talk about this limit a priori, as we do not know if it exists. However, we will see in Section 2.3.3 that such morphisms have kernels. This defines a functor, $\overline{\_} : \mathsf{Alg}_{\infty,+} \to \mathsf{Alg}_\infty$, where $\mathrm{Ker}\varepsilon_A = \overline{A}$. Free augmentations give the quasi-inverse to this functor. Given an A_∞-algebra A, we may construct the A_∞-algebra $A \oplus \mathbb{K}$. The structure morphisms are given by m_i^A, but there is now a unit $\upsilon_{A\oplus\mathbb{K}}$. Thus we get that $m_1(1) = 0$, $m_2(a \otimes 1) = a$ and $m_i \circ_k 1 = 0$ in the same manner. We obtain a functor $\_^+ : \mathsf{Alg}_\infty \to \mathsf{Alg}_{\infty,+}$, where $A \oplus \mathbb{K} = A^+$.

1.3.2 A_∞-Coalgebras

Dual to A_∞-algebras, we got conilpotent A_∞-coalgebras. Here we ask ourselves if the cobar construction has some converse, i.e., if C is a graded vector space such that $T(C[-1])$ together with a derivation m is a dg-algebra, is then C a coalgebra? Again, the answer to this is no, but we obtain a definition for conilpotent A_∞-coalgebras.

Definition 1.3.15. A graded vector space C is called a conilpotent A_∞-coalgebra if it is a dg-algebra of the form $(\overline{T}(C[-1]), d)$ where d is a derivation of degree 1.

Remark 1.3.16. For the rest of this thesis, an A_∞-coalgebra should be understood as a conilpotent A_∞-coalgebra unless otherwise specified.

Corollary 1.3.16.1. *C is an A_∞-coalgebra with differential d then there is a cochain complex (C, d^1), where d^1 is of degree 1, and together with morphisms $d^n : C \to C^{\otimes n}$ such that d uniquely determines each d^i for any $i > 0$. Conversely, if the morphisms d^i satisfy (rel)$_n$, then they uniquely determine a d such that C is an A_∞-coalgebra,*

$$(\mathit{rel}_n) \text{ is } \qquad \sum_{p+q+r=n} (-1)^{pq+r} d^{p+1+q} \circ^{op}_{p+1} d^q = 0$$

A morphism of A_∞-coalgebras is defined in the same manner as for A_∞-morphisms. An ∞-morphism $f : C \rightsquigarrow D$ is then either a morphism $\widetilde{f} : (T(C[-1]), m^C) \to (T(D[-1]), m^D)$ of dg-algebras; or equivalently it is a collection of morphisms $f_n : C \to D^{\otimes n}$ of degree $1-n$ such that f_1 is a morphism of chain complexes, and for any $n \geqslant 2$ the following relations are satisfied:

$$(\mathrm{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ^{op}_{p+1} m^D_q - \sum_{\substack{k \geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m^C_k \circ^{op} (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k}),$$

where e is

$$e = \sum_{l=1}^{k} (1-i_l) \sum_{1 \leqslant m < l} i_m.$$

We denote coAlg_∞ as the category of A_∞-coalgebras. Similarly, the cobar construction extends to this category and identifies A_∞-coalgebras and a subcategory of dg-algebras. This subcategory consists of every dg-algebra that is isomorphic, as an algebra, to a free tensor algebra. Lastly, every dg-coalgebra is an A_∞-coalgebra by letting every morphism $m^i = 0$ where $i > 2$, and this gives a non-full inclusion.

Chapter 2

Homotopy Theory of Algebras

Quillen envisioned a more general approach to homotopy theory, which he dubbed homotopical algebra. The structure of a model category first enclosed a homotopy theory, and now we mainly consider closed model categories. Many of the results from classical homotopy theory were recovered in the theory of model categories. The theorem which we are most concerned about is Whitehead's theorem:

Theorem 2.0.1 (Whitehead's Theorem)**.** *Let X and Y be two CW-complexes. If $f : X \to Y$ is a weak equivalence, it is also a homotopy equivalence. I.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

If we endow a Quillen model category onto the category Top, we get that a space X is bifibrant if and only if it is a CW-complex. The natural generalization is not to ask X to be a CW-complex but a bifibrant object.

Theorem 2.0.2 (Generalized Whiteheads Theorem, [Proposition 1.2.8 Hov99, p. 11])**.** *Let $\mathcal{C}$ be a model category. Suppose that X and Y are bifibrant objects of $\mathcal{C}$ and that there is a weak equivalence $f : X \to Y$. Then f is also a homotopy equivalence, i.e., there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

The category of differential graded algebras employs such a model category, and here we let the weak equivalences be quasi-isomorphisms. On the other hand, the category of differential graded coalgebras has a model structure where the weak equivalences are the maps sent to quasi-isomorphism by the cobar construction. Moreover, the bar and cobar construction defines a Quillen equivalence between these model structures. As we will see, a dg-coalgebra will be bifibrant exactly when it is an A_∞-algebra. Thus, by Whitehead's theorem, quasi-isomorphisms lift to homotopy equivalences. In this case, the derived category of A_∞-algebras is equivalent to the homotopy category of A_∞-algebras.

We will conclude this chapter by looking at the category of algebras as a subcategory of A_∞-algebras. The derived category may then be expressed as the homotopy category of A_∞-algebras, restricted to algebras.

2.1 Model categories

As one may see in literature, many semantically different definitions of model categories exist, but they are all made to be equivalent under good conditions. The difference mainly comes down to preference. This thesis will use the definitions from Mark Hovey's book "Model Categories" [Hov99]. In this section, we will define Quillen's model category. We will then prove the fundamental results about model categories, their associated homotopy category, and Quillen functors between model categories.

Before we state the definition of a model category, we need some preliminary definitions. For this section, let $\mathcal{C}$ be a category.

Definition 2.1.1 (Retract)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is a retract of a morphism $g : C \to D$ if it fits in a commutative diagram on the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "id_B"'] & D \arrow[r] & B \end{tikzcd}$$

Definition 2.1.2 (Functorial factorization)**.** A pair of functors $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ is called a functorial factorization if for any morphism $f \in \mathrm{Mor}(\mathcal{C})$, there is a factorization $f = \beta(f) \circ \alpha(f)$. We will use the notation $f_\alpha = \alpha(f)$ and $f_\beta = \beta(f)$. The following commutative diagram depict the functorial factorization:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\ & C \arrow[ur, "f_\beta"'] & \end{tikzcd}$$

Definition 2.1.3 (Lifting properties)**.** Suppose that the morphisms $i : A \to B$ and $p : C \to D$ fit inside a commutative square. i is said to have the left lifting property with respect to p, or p has the right lifting property with respect to i if there is an $h : B \to C$ such that the two triangles commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

Remark 2.1.4. We will call the left lifting property LLP and the right lifting property RLP.

Definition 2.1.5 (Wide subcategory)**.** We call a subcategory $\mathcal{W} \subset \mathcal{C}$ wide if $\mathcal{W}$ has every object $\mathcal{C}$. In particular, $\mathcal{W}$ is a subcategory having every identity morphism.

2.1.1 Model categories

Definition 2.1.6 (Model category)**.** Let $\mathcal{C}$ be a category with all finite limits and colimits. $\mathcal{C}$ admits a model structure if there are three wide subcategories, each defining a class of morphisms:

- Ac $\subset$ Mor$(\mathcal{C})$ are called weak equivalences
- Cof $\subset$ Mor$(\mathcal{C})$ are called cofibrations
- Fib $\subset$ Mor$(\mathcal{C})$ are called fibrations

In addition, we call morphisms in Cof $\cap$ Ac for acyclic cofibrations and Fib $\cap$ Ac for acyclic fibrations. Moreover, $\mathcal{C}$ has two functorial factorizations (α, β) and (γ, δ). The following axioms should be satisfied:

MC1 The class of weak equivalences satisfy the 2-out-of-3 property, i.e. if f and g are composable morphisms such that 2 out of f, g and gf are weak equivalences, then so is the third.

MC2 The three classes Ac, Cof and Fib are retraction closed, i.e., if f is a retraction of g, and g is either a weak equivalence, cofibration or fibration, then so is f.

MC3 The class of cofibrations have the left lifting property with respect to acyclic fibrations, and fibrations have the right lifting property with respect to acyclic cofibrations.

MC4 Given any morphism f, f_α is a cofibration, f_β is an acyclic fibration, f_γ is an acyclic cofibration and f_δ is a fibration.

Remark 2.1.7. The class Ac has every isomorphism, and this is because every isomorphism is a retract of some identity morphism.

Remark 2.1.8. The type of category above was first called a closed model category by Quillen [Hin01a]. In his sense, a model category does not require finite limits or finite colimits. In our case, we will explicitly state whenever a model category is non-closed, i.e., it does not have every finite limit or colimit.

A model category $\mathcal{C}$ is now defined to be a category equipped with a particular model structure. Notice that a category may admit several model structures. For more topological examples, we refer to Dwyer–Spalinski [DS95] and Hovey [Hov99].

An interesting and a not so non-trivial property of model categories is that giving all three classes Ac, Cof, and Fib is redundant. The model structure is determined by the class of weak equivalences and either cofibrations or fibrations. Thus the classes of fibrations are determined by acyclic cofibrations, and fibrations determine cofibrations. The following two results will show this.

Lemma 2.1.9 (The retract argument)**.** *Let $\mathcal{C}$ be a category. Suppose there is a factorization $f = pi$ and that f has LLP with respect to p; then f is a retract of i. Dually, if f has RLP to i, then it is a retract of p.*

Proof. We assume that $f : A \to C$ has LLP with respect to $p : B \to C$. Then we may find a lift $r : C \to B$, which realizes f as a retract of i.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dashed, "r"] & C \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"] & B \arrow[r, "p"] & C \end{tikzcd}$$

□

Proposition 2.1.10. *Let $\mathcal{C}$ be a model category. A morphism f is a cofibration (acyclic cofibration) if and only if f has LLP with respect to acyclic fibrations (fibrations). Dually, f is a fibration (acyclic fibration) if and only if it has RLP with respect to acyclic cofibrations (cofibrations).*

Proof. Assume that f is a cofibration. By **MC3**, we know that f has LLP with respect to acyclic fibrations. Assume instead that f has LLP with respect to every acyclic fibration. By **MC4**, we factor $f = f_\alpha \circ f_\beta$, where f_α is a cofibration, and f_β is an acyclic fibration. Since we assume f to have LLP with respect to f_β, by Lemma 2.1.9, we know that f is a retract of f_α. Thus by **MC2**, we know that f is a cofibration. □

Corollary 2.1.10.1. *Let $\mathcal{C}$ be a model category. (Acyclic) Cofibrations are stable under pushouts, i.e., if f is an (acyclic) cofibration, then f' is an (acyclic) cofibration.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[d, "f'"] \\ B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & D \end{tikzcd}$$

Dually, fibrations are stable under pullbacks.

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "f'"] & E \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & D \arrow[r] & F \end{tikzcd}$$

where the left-hand square is a pushout. Then f has LLP to g if and only if f' has LLP to g by the universal property of the pushout. It follows by Proposition 2.1.10 that f' is a cofibration. $\square$

Since we assume that every model category $\mathcal{C}$ admits finite limits and colimits, we know that it has both an initial and a terminal object. We let $\varnothing$ denote the initial object, and $*$ denote the terminal object.

Definition 2.1.11 (Cofibrant, fibrant and bifibrant objects)**.** Let $\mathcal{C}$ be a model category. An object X is called cofibrant if the unique morphism $\varnothing \to X$ is a cofibration. Dually, X is called fibrant if the unique morphism $X \to *$ is fibrant. If X is both cofibrant and fibrant, we call it bifibrant.

There is no reason for every object to be either cofibrant or fibrant. However, we may see that every object is weakly equivalent to an object which is either fibrant or cofibrant. In this case, we can think of X and Y being weakly equivalent if there is a weak equivalence $f : X \to Y$. We will make precise what it means for two objects to be weakly equivalent later.

Construction 2.1.12. Let X be an object of a model category $\mathcal{C}$. The morphism $i : \varnothing \to X$ has a functorial factorization $i = i_\beta \circ i_\alpha$, where $i_\alpha : \varnothing \to QX$ is a cofibration and $i_\beta : QX \to X$ is an acyclic fibration. By definition, QX is cofibrant and weakly equivalent to X.

$Q : \mathcal{C} \to \mathcal{C}$ defines a functor called the cofibrant replacement. To see this, we first look at the slice category $\varnothing/\mathcal{C}$. The objects are morphisms $f : \varnothing \to X$ for any object X in $\mathcal{C}$, while morphisms are commutative triangles. We first observe that $\varnothing/\mathcal{C} \subset \mathcal{C}^{\to}$ is a subcategory of the arrow category. Thus (α, β) may be interpreted as functors on the slice category to the arrow category. Moreover, since every arrow $f : \varnothing \to X$ is unique, we observe that this category is equivalent to $\mathcal{C}$. Thus (α, β) may be interpreted as functors on $\mathcal{C}$ into arrows. We define Q as the composition $Q = \mathrm{cod} \circ \alpha$, where $\mathrm{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ is the codomain functor.

Dually, we get a fibrant replacement functor $R : \mathcal{C} \to \mathcal{C}$. By the functorial factorizations, we have natural transformations $q : Q \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $r : \mathrm{Id}_{\mathcal{C}} \Rightarrow R$.

We collect the following properties

Lemma 2.1.13. *The cofibrant replacement Q and fibrant replacement R preserve weak equivalences.*

Proof. Suppose there is a weak equivalence $f : X \to Y$. Then there is a commutative square

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "\sim"] & QY \arrow[d, "\sim"] \\ X \arrow[r, "f", "\sim"'] & Y \end{tikzcd}$$

where every morphism is a weak equivalence by the 2-out-of-3 property. $\square$

Lemma 2.1.14 (Ken Brown's lemma)**.** *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with weak equivalences satisfying the 2-out-of-3 property. If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose that A and B are cofibrant objects and that $f : A \to B$ is a weak equivalence. Using the universal property of the coproduct, we define the map $(f, id_B) = p : A \coprod B \to B$. p has a functorial factorization into a cofibration and acyclic fibration, $p = p_\beta \circ p_\alpha$. We recollect the maps in the following pushout diagram:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & B \arrow[d, "i_2"] \arrow[ddrr, bend left, "id_B"] & & \\
A \arrow[r, "i_1"] \arrow[ur, phantom, "\lrcorner" very near end] \arrow[ddrrr, bend right, "f"'] & A \coprod B \arrow[dr, "p_\alpha"] & & \\
& & C \arrow[dr, "p_\beta"] & \\
& & & B
\end{tikzcd}$$

By Corollary 2.1.10.1, both i_1 and i_2 are cofibrations. Since f, id_B and p_β are weak equivalences, so are $p_\alpha \circ i_1$ and $p_\alpha \circ i_2$ by **MC2**. Moreover, they are acyclic cofibrations.

Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor as described above. Then by assumption, $F(p_\alpha \circ i_1)$ and $F(p_\alpha \circ i_2)$ are weak equivalences. Since a functor sends identity to identity, we also know that $F(id_B)$ is a weak equivalence. Thus by the 2-out-of-3 property $F(p_\beta)$ is a weak equivalence, as $F(p_\beta) \circ F(p_\alpha \circ i_2) = id_{F(B)}$. Again, by 2-out-of-3 property $F(f)$ is a weak equivalence, as $F(f) = F(p_\beta) \circ F(p_\alpha \circ i_1)$. □

2.1.2 Homotopy category

At its most abstract, homotopy theory is the study of categories and functions up to weak equivalences. Here, a weak equivalence may be anything, but most commonly, it is a weak equivalence in topological homotopy or a quasi-isomorphism in homological algebra. The biggest concern when dealing with such concepts is to make a functor well-defined when these chosen weak equivalences are inverted. To this end, there is a construction to amend these problems, known as derived functors. We define a homotopical category in the sense of Riehl [Rie16].

Definition 2.1.15 (Homotopical Category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is homotopical if there is a wide subcategory constituting a class of morphisms known as weak equivalences, Ac $\subset$ Mor$\mathcal{C}$. The weak equivalences should satisfy the 2-out-of-6 property, i.e. given three composable morphisms f, g and h, if gf and hg are weak equivalences, then so are f, g, h and hgf.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

Remark 2.1.16. Notice that the 2-out-of-6 property is stronger than the 2-out-of-3 property. To see this, let either f, g, or h be the identity, and then conclude with the 2-out-of-3 property.

Remark 2.1.17. The collection of weak equivalences contains every isomorphism. To see this pick an isomorphism f and f^{-1}, then the compositions are the identity on the domain and codomain, which are assumed to be in Ac.

Given such a homotopical category $\mathcal{C}$, we want to invert every weak equivalence and create the homotopy category of $\mathcal{C}$. This construction is developed in Gabriel and Zisman [Hin01b] called the calculus of fractions. This method tries to mimic localization for commutative rings in a category-theoretic fashion. We will not give an account of the existence or construction of localizations.

Definition 2.1.18. Let $\mathcal{C}$ be a homotopical category. Its homotopy category is Ho$\mathcal{C} = \mathcal{C}[\text{Ac}^{-1}]$, together with a localization functor $L : \mathcal{C} \to \text{Ho}\mathcal{C}$. The following universal property determines the localization: If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending weak equivalences to isomorphisms, then it uniquely factors through the homotopy category up to a unique natural isomorphism η.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\ & \text{Ho}\mathcal{C} \arrow[ur, dashed, "F'"'] & \end{tikzcd}$$

Definition 2.1.19. Suppose that $\mathcal{C}$ is a homotopical category. Two objects of $\mathcal{C}$ are said to be weakly equivalent if they are isomorphic in Ho$\mathcal{C}$. I.e., X and Y are weakly equivalent if there is some zig-zag relation between the objects, consisting only of weak equivalences.

$$\begin{tikzcd} X \arrow[dr, "\sim"] & & \cdots \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & Y \arrow[dl, "\sim"'] \\ & X' & & Y' & \end{tikzcd}$$

Remark 2.1.20. A renowned problem with localizations is that even if $\mathcal{C}$ is a locally small category, localizations $\mathcal{C}[S^{-1}]$ do not need to be. Thus, without a good theory of classes or higher universes, we cannot generally ensure that localization still exists as a locally small category.

From the definition of the homotopy category, a functor F admits a lift F' from the homotopy category whenever weak equivalences are mapped to isomorphisms. Moreover, if we have a functor F between homotopical categories, which preserves weak equivalences, it then induces a functor between the homotopy categories.

Definition 2.1.21 (Homotopical functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is homotopical if it preserves weak equivalences. Moreover, there is a lift of functors, as in the following diagram, where η is a natural isomorphism.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "L_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\eta"] \\ \mathrm{Ho}\mathcal{C} \arrow[r, dashed, "F'"] & \mathrm{Ho}\mathcal{D} \end{tikzcd}$$

Derived functors becomes relevant whenever we want to make a lift of non-homotopical functors. These lifts will be the closest approximation that we can make functorial. We will see that a model category is a congenial environment to work with these concepts. Firstly the problem with localizations where the homotopy category may not exist will be amended. Secondly, we will obtain a simple description of some derived functors.

Proposition 2.1.22. *Any model category $\mathcal{C}$ is a homotopical category.*

Proof. To show that a model category is homotopical, it suffices to show that Ac satisfies the 2-out-of-6 property. Assume there are 3 composable morphisms f, g, h such that $gf, hg \in$ Ac. By the 2-out-of-3 property for Ac, it is enough to show that at least one of f, g, h, fgh is a weak equivalence to deduce that every other morphism is a weak equivalence.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

To use the model structure, we will first show that we may assume f, g to be cofibrant and g, h to be fibrant. We know by **MC4** that f, g, gf may be factored into a cofibration composed with an acyclic fibration, e.g., $f = f_\beta f_\alpha$. Since gf is a weak equivalence, so is $(gf)_\alpha$ by the 2-out-of-3 property.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\ & B' \arrow[ur, "f_\beta"'] & \end{tikzcd} \quad \begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "g_\alpha"'] & & C \\ & C' \arrow[ur, "g_\beta"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[rr, "gf"] \arrow[dr, "(gf)_\alpha"'] & & C \\ & C'' \arrow[ur, "(gf)_\beta"'] & \end{tikzcd}$$

Notice that the "cofibrant approximation" of the map from A to C either goes through C' or C''. We conjoin these by taking the pullback. Since acyclic fibrations are stable under pullbacks, we get a pullback square where every morphism is an acyclic fibration. Thus the map $A \to \widetilde{C}$ is a weak equivalence by 2-out-of-3.

$$\begin{tikzcd}
A \arrow[dr, dashed, "t"] \arrow[drr, bend left, "g_\alpha f"] \arrow[ddr, bend right, "(gf)_\alpha"'] & & \\
& \widetilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & C' \arrow[d, "g_\beta"] \\
& C'' \arrow[r, "(gf)_\beta"] & C
\end{tikzcd}$$

To replace f with f_α, we must lift the composition into our "new" C, which is $\widetilde{C}$. We do this using **MC3**, as f_α is a cofibration and the pullback square above consists entirely of acyclic fibrations.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_\alpha"] & \widetilde{C} \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed, "s"] & C
\end{tikzcd}$$

To summarize, we have the following diagram, where every squiggly arrow is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[dr, "t", "\sim"'] & B' \arrow[r, "\sim"'] \arrow[d, "s"] & B \arrow[d] \arrow[dr, "\sim"] & \\
& \widetilde{C} \arrow[r, "\sim"'] & C \arrow[r] & D
\end{tikzcd}$$

We now wish to promote the arrow $s : B' \to \widetilde{C}$ into a cofibration. We do this by factoring s and t with **MC4**. Notice that s_β, t_β and t_α are weak equivalences.

$$\begin{tikzcd}
B' \arrow[rr, "s"] \arrow[dr, "s_\alpha"'] & & \widetilde{C} & A \arrow[rr, "t"] \arrow[dr, "t_\alpha"'] & & \widetilde{C} \\
& \widetilde{C}' \arrow[ur, "s_\beta"'] & & & \widetilde{C}'' \arrow[ur, "t_\beta"'] &
\end{tikzcd}$$

To obtain our final factorization, we use RLP of s_β on t_α.

$$\begin{tikzcd}
 & B' \arrow[d, "s_\alpha"] \\
A \arrow[ur, "f_\alpha"] \arrow[r] \arrow[d, "t_\alpha"'] & \widetilde{C}' \arrow[d, "s_\beta"] \\
\widetilde{C}'' \arrow[ur, dashed, "u"] \arrow[r, "t_\beta"] & \widetilde{C}
\end{tikzcd}$$

Since the bottom square only consists of weak equivalences, u has to be a weak equivalence by the 2-out-of-3 property. In this manner, we may transform our diagram into the following diagram

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[dr, "ut_\alpha", "\sim"'] & B' \arrow[r, "\sim"'] \arrow[d, "s_\alpha"] & B \arrow[d] \arrow[dr, "\sim"'] & \\
 & \widetilde{C}' \arrow[r, "\sim"'] & C \arrow[r] & D
\end{tikzcd}$$

We now have a factorization of gf into two cofibrations, followed by an acyclic fibration, in such a manner that it is compatible with the original diagram. The dual to this claim is that we may also factor hg into two fibrations preceded by an acyclic cofibration. In other words, we may assume without loss of generality that f and g are cofibrations and that g and h are fibrations.

In this case, it is enough to show the 2-out-of-6 property to show that g is an isomorphism. Consider the diagram below with lifts i and j, and these exist since we assume gf and hg to be weak equivalences.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "gf"'] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[d, "hg"] \\
C \arrow[ur, dashed, "i"] \arrow[r, "id_C"] & C \arrow[ur, dashed, "j"] \arrow[r, "h"] & D
\end{tikzcd}$$

Since the diagram is commutative, we get that $i = j$, and that g is both split-mono and split-epi, with i as its splitting. □

Since every model category is homotopical, it also has an associated homotopy category Ho$\mathcal{C}$. Let $\mathcal{C}_c$, $\mathcal{C}_f$, and $\mathcal{C}_{cf}$ denote the full subcategories consisting of cofibrant, fibrant and bifibrant objects, respectively.

Proposition 2.1.23. *Let $\mathcal{C}$ be a model category. The following categories are equivalent:*

- *Ho$\mathcal{C}$,*
- *Ho$\mathcal{C}_c$,*
- *Ho$\mathcal{C}_f$,*
- *Ho$\mathcal{C}_{cf}$.*

Proof. We only show that Ho$\mathcal{C} \simeq$ Ho$\mathcal{C}_c$, the other arguments are similar. The inclusion $i : \mathcal{C}_c \to \mathcal{C}$ preserves weak equivalences; i is homotopical and admits a lift. Moreover, since the cofibrant replacement is homotopical, it also has a lift.

$$\begin{tikzcd} \mathcal{C}_c \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Ho}\mathcal{C}_c \arrow[r, dashed, bend right, "\mathrm{Ho}\ i"'] & \mathrm{Ho}\mathcal{C} \arrow[l, dashed, bend right, "\mathrm{Ho}\ Q"'] \end{tikzcd}$$

It is clear that Ho Q is the quasi-inverse of Ho i. ☐

We still don't see how model categories will fix the size issues. To do this, we will develop the notion of homotopy equivalence, $\sim$. This homotopy equivalence will be a congruence relation on the subcategory of bifibrant objects $\mathcal{C}_{cf}$. We solve the size issues with this, together with the fact that there is an equivalence of categories Ho$\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/\sim$.

Definition 2.1.24 (Cylinder and path objects)**.** Let $\mathcal{C}$ be a model category. Given an object X, a cylinder object $X \wedge I$ is a factorization of the codiagonal map $i : X \coprod X \to X$, such that p_0 is a cofibration and that p_1 is a weak equivalence.

$$\begin{tikzcd} X \coprod X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \\ & X \wedge I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Dually, a path object X^I is a factorization of the diagonal map $i : X \to X \prod X$, such that p_0 is a weak equivalence and that p_1 is a fibration.

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \prod X \\ & X^I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Remark 2.1.25. Even though we have written $X \wedge I$ suggestively to be a functor, it is not. There may be many choices for a cylinder object. However, by using the functorial factorization from **MC4**, we get a canonical choice of a cylinder object, as it factors every map into a cofibration and an acyclic fibration. If we let the cylinder object denote this functorial choice, we can define it as a functor.

Proposition 2.1.26. *Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Given two cylinder objects $X \wedge I$ and $X \wedge I'$, they are weakly equivalent.*

Proof. It is enough to show that there exists a weak equivalence from any cylinder object into one specified cylinder object. There is such a map for the functorial cylinder object $X \wedge I$, as the morphism p_1 is an acyclic fibration, which enables a lift that is a weak equivalence by the 2-out-of-3 property.

$$\begin{tikzcd}
X \coprod X \arrow[r, "p_0"] \arrow[d, "p_0'"] & X \wedge I \arrow[d, "p_1"] \\
X \wedge I' \arrow[r, "p_1'"] \arrow[ur, dashed] & X
\end{tikzcd}$$

□

Definition 2.1.27 (Homotopy equivalence)**.** Let $f, g : X \to Y$. A left homotopy between f and g is a morphism $H : X \wedge I \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. We say that f and g are left homotopic if a left homotopy exists, and it is denoted $f \overset{l}{\sim} g$.

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, dotted, "i_0"] \arrow[drr, bend left, "f"] & & \\
X \coprod X \arrow[r, "p_0"] & X \wedge I \arrow[r, "H"] & Y \\
X \arrow[u] \arrow[ur, dotted, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

A right homotopy between f and g is a morphism $H : X \to Y^I$ such that $i_0H = f$ and $i_1H = g$. We say that f and g are right homotopic if a right homotopy exists, and it is denoted $f \overset{r}{\sim} g$.

$$\begin{tikzcd}
 & & Y \\
X \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & Y^I \arrow[r, "p_1"] \arrow[ur, dotted, "i_0"] \arrow[dr, dotted, "i_1"] & Y \prod Y \arrow[u] \arrow[d] \\
 & & Y
\end{tikzcd}$$

f and g are said to be homotopic if they are both left and right homotopic, denoted $f \sim g$. f is a homotopy equivalence if it has a homotopy inverse $h : Y \to X$, such that $hf \sim id_X$ and $fh \sim id_Y$.

It is important to note that homotopy equivalence is not a priori an equivalence relation. With the following two propositions, we can amend this by taking both fibrant and cofibrant replacements.

Proposition 2.1.28. *Let $\mathcal{C}$ be a model category, and $f, g : X \to Y$ be morphisms. We have the following:*

1. *If $f \overset{l}{\sim} g$ and $h : Y \to Z$, then $hf \overset{l}{\sim} hg$.*
2. *If Y is fibrant, $f \overset{l}{\sim} g$ and $h : W \to X$, then $fh \overset{l}{\sim} gh$.*
3. *If X is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If X is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*

Proof. $(1.)$ Assume that $f \overset{l}{\sim} g$ and $h : Y \to Z$. Let $H : X \wedge I \to Y$ denote the left homotopy between f and g. The left homotopy between hf and hg is hH.

$(2.)$ Assume that Y is fibrant, $f \overset{l}{\sim} g$ and that $h : W \to X$. Let $H : X \wedge I \to Y$ be a left homotopy. We construct a new cylinder object for the homotopy. Factor $p_1 : X \wedge I \to X$ as $q_1 \circ q_0$ where $q_0 : X \wedge I \to X \wedge I'$ is an acyclic cofibration and $q_1 : X \wedge I' \to X$ is a fibration. By the 2-out-of-3 property, q_1 is an acyclic fibration, as p_1 and q_0 are weak equivalences. $X \wedge I'$ is a cylinder object as $q_0 \circ p_0$ is a cofibration and q_1 is a weak equivalence. Since we assume Y to be fibrant we lift the left homotopy $H : X \wedge I \to Y$ to the left homotopy $H' : X \wedge I' \to Y$ with the following diagram:

$$\begin{tikzcd}
X \wedge I \arrow[r, "H"] \arrow[d, "q_0"] & Y \arrow[d] \\
X \wedge I' \arrow[r] \arrow[ur, dashed, "H'"] & *
\end{tikzcd}$$

We let WI be a cylinder object for W, where $p'_0 : W \sqcup W \to QI$ is a cofibration. We can find an appropriate homotopy needed with LLP of q_1 against p'_0, as done in the diagram below.

$$\begin{tikzcd}
W \coprod W \arrow[r, "q_0 p_0 (h \coprod h)"] \arrow[d, "p'_0"] & X \wedge I' \arrow[d, "q_1"] \\
W \wedge I \arrow[r, "hp'_1"] \arrow[ur, dashed, "k"] & X
\end{tikzcd}$$

The morphism $H'k$ is the desired left homotopy witnessing $fh \overset{l}{\sim} gh$.

$(3.)$ Assume that X is cofibrant. First, observe that a left homotopy is reflexive and symmetric. We must show that it is also transitive. Thus, assume that $f, g, h : X \to Y$ and that $H : X \wedge I \to Y$ is a left homotopy witnessing $f \overset{l}{\sim} g$ and that $H' : X \wedge I' \to Y$ is a left homotopy witnessing $g \overset{l}{\sim} h$. We first observe that $i_0 : X \to X \wedge I$ is a weak equivalence, as $id_X = p_1 i_0$ where id_X and p_1 are weak equivalences. Since X is assumed to be cofibrant, we see that $X \coprod X$ is cofibrant by the following pushout:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "inr"] \\
X \arrow[r, "inl"] & X \coprod X
\end{tikzcd}$$

Moreover, both inl and inr are cofibrations. It follows that i_0 is a cofibration as $i_0 = p_0 \circ inr$ is a composition of two cofibrations. i_0 is thus an acyclic cofibration. We define an almost cylinder object C by the pushout of i_1 and i_0'. We define the maps t and H'' by using the universal property in the following manner:

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "p_1"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "p_1'"] & C \arrow[dr, dashed, "t"] & \\
 & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "H"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "H'"] & C \arrow[dr, dashed, "H''"] & \\
 & & Y
\end{tikzcd}$$

Observe that there is a factorization of the codiagonal map $X \coprod X \xrightarrow{s} C \xrightarrow{t} X$. However, s may not be a cofibration, so we replace C with the cylinder object $X \wedge I''$ such that we have the factorization $X \coprod X \xrightarrow{s_\alpha} X \wedge I'' \xrightarrow{ts_\beta} X$. The morphism $H''s_\beta$ is then our required homotopy for $f \overset{l}{\sim} g$.

$(4.)$ Suppose that X is cofibrant and that $H : X \wedge I \to Y$ is a left homotopy for $f \overset{l}{\sim} g$. Pick a path object for Y, such that we have the factorization $Y \xrightarrow{q_0} Y^I \xrightarrow{q_1} Y \prod Y$ where q_0 is a weak equivalence and q_1 is a fibration. Again, as X is cofibrant, we get that i_0 is an acyclic cofibration, so we have the following lift of the homotopy:

$$\begin{tikzcd}
X \arrow[r, "q_0 f"] \arrow[d, "i_0"] & Y^I \arrow[d, "q_1"] \\
X \wedge I \arrow[r, "{(fp_1, H)}"] \arrow[ur, dashed, "J"] & Y \prod Y
\end{tikzcd}$$

The right homotopy is given by injecting away from f, i.e., $H' = Ji_1$. □

Corollary 2.1.28.1. *We collect the dual results of the above proposition and thus have the following.*

1. *If $f \overset{r}{\sim} g$ and $h : W \to X$, then $fh \overset{r}{\sim} gh$.*
2. *If X is cofibrant, $f \overset{r}{\sim} g$ and $h : Y \to Z$, then $hf \overset{r}{\sim} hg$.*
3. *If Y is fibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*

4. *If Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Corollary 2.1.28.2. *Homotopy is a congruence relation on $\mathcal{C}_{cf}$. Thus the category $\mathcal{C}_{cf}/\sim$ is well-defined, exists, and inverts every homotopy equivalence.*

Lemma 2.1.29 (Weird Whitehead)**.** *Let $\mathcal{C}$ be a model category. Suppose that C is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then h induces an isomorphism:*

$$\mathcal{C}(C,X)/\overset{l}{\sim} \xrightarrow[\simeq]{h_*} \mathcal{C}(C,Y)/\overset{l}{\sim}$$

Dually, if X is fibrant and $h : C \to D$ is an acyclic cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism:

$$\mathcal{C}(D,X)/\overset{r}{\sim} \xrightarrow[\simeq]{h^*} \mathcal{C}(C,X)/\overset{r}{\sim}$$

Proof. We assume $\mathcal{C}$ to be cofibrant and $h : X \to Y$ to be an acyclic fibration. We first prove that h is surjective. Let $f : C \to Y$. By RLP of h, there is a morphism $f' : C \to X$ such that $f = hf'$.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
C \arrow[r, "f"] \arrow[ur, dashed, "f'"] & Y
\end{tikzcd}$$

To show injectivity, we assume $f, g : C \to X$ such that $hf \overset{l}{\sim} hg$, in particular, there is a left homotopy $H : C \wedge I \to Y$. Remember that since C is cofibrant, the map p_0 is a cofibration. We find a left homotopy $H : C \wedge I \to X$ witnessing $f \overset{l}{\sim} g$ by the following lift.

$$\begin{tikzcd}
C \coprod C \arrow[r, "f+g"] \arrow[d, "p_0"] & X \arrow[d, "h"] \\
C \wedge I \arrow[r, "H"] \arrow[ur, dashed, "H'"] & Y
\end{tikzcd}$$

If we instead assume that both X and Y are fibrant, then the functor $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends acyclic fibrations to isomorphisms by Corollary 2.1.28.1. Ken Brown's lemma, Lemma 2.1.14, tells us then that $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends weak equivalences between fibrant objects to isomorphisms. □

Theorem 2.1.30 (Generalized Whitehead's theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a morphism of bifibrant objects. Then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose first that f is a weak equivalence. Pick a bifibrant object A, then by Lemma 2.1.29 $f_* : \mathcal{C}(A,X)/{\sim} \to \mathcal{C}(A,Y)/{\sim}$ is an isomorphism. Letting $A = Y$, we know that there is a morphism $g : Y \to X$, such that $f_*g = fg \sim id_Y$. Furthermore, by Proposition 2.1.28, since X is bifibrant, composing on the right preserves homotopy equivalence, e.g., $fgf \sim f$. By letting $A = X$, we get that $f_*gf = fgf \sim f = f_*id_X$, thus $gf \sim id_X$.

For the opposite direction, assume that f is a homotopy equivalence. We factor f into an acyclic cofibration f_γ and a fibration f_δ, i.e. $X \xrightarrow{f_\gamma} Z \xrightarrow{f_\delta} Y$. Observe that Z is bifibrant as X and Y is, in particular, f_γ is a weak equivalence of bifibrant objects, so it is a homotopy equivalence.

It is enough to show that f_δ is a weak equivalence. Let g be the homotopy inverse of f, and $H : Y \wedge I \to Y$ is a left homotopy witnessing $fg \sim id_Y$. Since Y is bifibrant, the following square has a lift.

$$\begin{tikzcd} Y \arrow[r, "f_\gamma g"] \arrow[d, "i_0"'] & Z \arrow[d, "f_\delta"] \\ Y \wedge I \arrow[r, "H"] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Let $h = H'i_1$, and then by definition, we know that $f_\delta H' i_1 = id_Y$. Moreover, H is a left homotopy witnessing $f_\gamma g \sim h$. Let $g' : Z \to X$ be the homotopy inverse of f_γ. We have the following relations $f_\delta \sim f_\delta f_\gamma g' \sim fg'$, and $hf_\delta \sim (f_\gamma g)(fg') \sim f_\gamma g' \sim id_Z$. Let $H'' : Z \wedge I \to Z$ be a left homotopy witnessing this homotopy. Since Z is bifibrant, i_0 and i_1 are weak equivalences. By the 2-out-of-3 property, H'' and hf_δ are weak equivalences. Since $f_\delta h = id_Y$, it follows that f_δ is a retract of hf_δ and is thus a weak equivalence. □

Corollary 2.1.30.1. *The category $\mathcal{C}_{cf}/{\sim}$ satisfies the universal property of the localization of $\mathcal{C}_{cf}$ by the weak equivalences. I.e. there is a categorical equivalence Ho$\mathcal{C}_{cf} \simeq \mathcal{C}_{cf}/{\sim}$.*

Proof. By generalized Whitehead's theorem, Theorem 2.1.30 weak equivalences and homotopy equivalences coincide. The corollary follows steadily from the universal property of the localization and quotient categories. □

We collect the results from above in the following theorem.

Theorem 2.1.31 (Fundamental theorem of model categories)**.** *Let $\mathcal{C}$ be a model category and denote $L : \mathcal{C} \to$ Ho$\mathcal{C}$ the localization functor. Let X and Y be objects of $\mathcal{C}$.*

1. *There is an equivalence of categories Ho$\mathcal{C} \simeq \mathcal{C}_{cf}/{\sim}$.*
2. *There are natural isomorphisms $\mathcal{C}_{cf}/{\sim}(QRX, QRY) \simeq$ Ho$\mathcal{C}(X,Y) \simeq \mathcal{C}_{cf}/{\sim}(RQX, RQY)$. Additionally, Ho$\mathcal{C}(X,Y) \simeq \mathcal{C}_{cf}/{\sim}(QX, RY)$.*
3. *The localization L identifies left or right homotopic morphisms.*
4. *A morphism $f : X \to Y$ is a weak equivalence if and only if qf is an isomorphism.*

Proof. theorem is clear by the results above. ☐

2.1.3 Quillen adjoints

We now want to study morphisms, or certain functors, between model categories. Like in the case of homotopical functors, we want these morphisms to induce a functor between the homotopy categories. However, we also want them to respect the cofibration and fibration structure, not just weak equivalences. In this way, we will instead look toward derived functors to be able to define this extension to the homotopy category. We recall the definition of a total (left/right) derived functor. In the case of model categories, we get a simple description of some of these derived functors.

Definition 2.1.32 (Total derived functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Whenever it exists, a total left derived functor of F is a functor $\mathbb{L}F$: Ho$\mathcal{C} \to$ Ho$\mathcal{D}$ with a natural transformation $\varepsilon : \mathbb{L}F \circ L \Rightarrow L \circ F$ satisfying the universal property: If G : Ho$\mathcal{C} \to$ Ho$\mathcal{D}$ is a functor. There is a natural transformation $\alpha : G \circ L \Rightarrow L \circ F$, then it factors uniquely up to unique isomorphism through ε.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L"] \\
\mathrm{Ho}\mathcal{C} \arrow[r, dashed, "\mathbb{L}F"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & \mathrm{Ho}\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L"] \\
\mathrm{Ho}\mathcal{C} \arrow[r, dashed, bend left, "\mathbb{L}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Uparrow"] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & \mathrm{Ho}\mathcal{D}
\end{tikzcd}$$

Dually, whenever it exists, a total right derived functor of F is a functor $\mathbb{R}F$: Ho$\mathcal{C} \to$ Ho$\mathcal{D}$ with a natural transformation $\eta : L \circ F \Rightarrow \mathbb{R}F \circ L$ having the opposite universal property.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L"] \\
\mathrm{Ho}\mathcal{C} \arrow[r, dashed, "\mathbb{R}F"'] \arrow[ur, phantom, "\underset{\eta}{\Swarrow}"] & \mathrm{Ho}\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "L"] \\
\mathrm{Ho}\mathcal{C} \arrow[r, dashed, bend left, "\mathbb{R}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Downarrow"] \arrow[ur, phantom, "\overset{\eta}{\Swarrow}"] & \mathrm{Ho}\mathcal{D}
\end{tikzcd}$$

Definition 2.1.33 (Deformation)**.** A left (right) deformation on a homotopical category $\mathcal{C}$ is an endofunctor Q (R) together with a natural weak equivalence $q : Q \Rightarrow Id_{\mathcal{C}}$ $(r : Id_{\mathcal{C}} \Rightarrow R)$.

A left (right) deformation on a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is a left (right) deformation Q on $\mathcal{C}$ such that F preserves weak equivalences in the image of Q.

Remark 2.1.34 (Cofibrant and fibrant replacement). If $\mathcal{C}$ is a model category, then we have a left and a right deformation. The cofibrant replacement Q defines a left deformation, and the fibrant replacement defines a right deformation. Notice that this is only because the factorization system is functorial.

Proposition 2.1.35. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. If F has a left deformation Q, then the total left derived functor $\mathbb{L}F$ exists. Moreover, the functor FQ is homotopical, and $\mathbb{L}F$ is the unique extension of FQ.*

Proof. Since we already have a candidate for the derived functor, we must check that it has the universal property. This follows by [Proposition 6.4.11 Rie16, p. 207]. □

Remark 2.1.36. There is a somewhat weaker statement by Dwyer and Spalinski [Proposition 9.3 DS95, p. 111]. If we instead ask for functors F, which have the cofibrant replacement Q (fibrant replacement R) as a left (right) deformation, we may make this proof more explicit.

With the above proposition and remark, it makes sense to define Quillen functors as left and right Quillen functors. A left Quillen functor should be left deformable by the cofibrant replacement. Moreover, for the composition of two left Quillen functors to make sense, we also need weak equivalences between cofibrant objects to be mapped to weak equivalences between cofibrant objects. We make the following definition.

Definition 2.1.37 (Quillen adjunction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. A left Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves cofibrations and acyclic cofibrations.
2. A right Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves fibrations and acyclic fibrations.
3. Suppose that (F, U) is an adjunction where $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to U. (F, U) is called a Quillen adjunction if F is a left Quillen functor and U is a right Quillen functor.

Remark 2.1.38. By Ken Brown's lemma, Lemma 2.1.14, we see that a left Quillen functor F is left deformable to the cofibrant replacement functor Q. Thus the total left derived functor is given by $\mathbb{L}F = \text{Ho}FQ$.

We will think of a morphism of model categories as a Quillen adjunction to eliminate the choice of left or right derivedness. We can choose the direction of the arrow to be along either the left or right adjoints, and we make the convention of following the left adjoint functors. We summarize the following properties.

Lemma 2.1.39. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose there is an adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : U$. The following are equivalent:*

1. *(F, U) is a Quillen adjunction.*
2. *F is a left Quillen functor.*
3. *U is a right Quillen functor.*

Proof. This lemma follows from the naturality of the adjunction. I.e., any square in $\mathcal{C}$, with the right side from $\mathcal{D}$ is commutative if and only if any square in $\mathcal{D}$ with the left side from $\mathcal{C}$ is commutative. Now, f has LLP with respect to Ug if and only if Ff has LLP with respect to g.

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & UX \arrow[d, "Ug"] \\ B \arrow[r, "l"'] \arrow[ur, dotted, "h"] & UY \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} FA \arrow[r, "k^T"] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r, "l^T"'] \arrow[ur, dotted, "h^T"] & Y \end{tikzcd}$$

□

Remark 2.1.40. We say that h^T is the transpose of h along the unique natural isomorphism witnessing the adjunction between F and U. With this notion, $(h^T)^T = h$.

Proposition 2.1.41. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The functors $\mathbb{L}F$: Ho$\mathcal{C}$ $\to$ Ho$\mathcal{D}$ and $\mathbb{R}U$: Ho$\mathcal{D}$ $\to$ Ho$\mathcal{C}$ forms an adjoint pair.*

Proof. We must show that Ho$\mathcal{D}(\mathbb{L}FX, Y) \simeq$ Ho$\mathcal{D}(X, \mathbb{R}UY)$. By using the fundamental theorem of model categories, Theorem 2.1.31, we have the following isomorphisms: Ho$\mathcal{D}(\mathbb{L}FX, Y) \simeq {}^{\mathcal{C}(FQX, RY)}/\sim$ and Ho$\mathcal{D}(X, \mathbb{R}UY) \simeq {}^{\mathcal{D}(QX, URY)}/\sim$. In other words, if we assume X to be cofibrant and Y to be fibrant, we must show that the adjunction preserves homotopy equivalences.

We show it in one direction. Suppose that the morphisms $f, g : FA \to B$ are homotopic, witnessed by a right homotopy $H : FA \to B^I$. Since we assume U to preserve products, fibrations, and weak equivalences between fibrant objects, $U(B^I)$ is a path object for UB. Thus the transpose $H^T : A \to U(B^I)$ is the desired homotopy witnessing $f^T \sim g^T$ □

Definition 2.1.42 (Quillen equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $(F, U) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. (F, U) is called a Quillen equivalence if for any cofibrant X in $\mathcal{C}$, fibrant Y in $\mathcal{D}$ such that any morphism $f : FX \to Y$ is a weak equivalence if and only if its transpose $f^T : X \to UY$ is a weak equivalence.

Proposition 2.1.43. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *Let $\eta : Id_{\mathcal{C}} \Rightarrow UF$ denote the unit, and $\varepsilon : FU \Rightarrow Id_{\mathcal{D}}$ denote the counit. The composite $Ur_F \circ \eta : Id_{\mathcal{C}_c} \Rightarrow URF|_{\mathcal{C}_c}$, and $\varepsilon \circ Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ are natural weak equivalences.*
3. *The derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories.*

Proof. Firstly observe that $2. \implies 3.$ by definition. Secondly, observe that equivalences both preserves and reflect isomorphisms. From this, we get $3. \implies 1.$. We now show $1. \implies 2.$. Pick X in $\mathcal{C}$ such that X is cofibrant. Since (F, U) is assumed to be a Quillen adjunction, FX is still cofibrant. The fibrant replacement $r_{FX} : FX \to RFX$ gives us a weak equivalence. Furthermore, since (F, U) is assumed to be a Quillen equivalence, its transpose $r_{FX}^T : X \to URFX$ is a weak equivalence. Unwinding the definition of the transpose, we get that $r_{FX}^T = Ur_{FX} \circ \eta_X$.

□

We have the following refinement.

Corollary 2.1.43.1. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *F reflects weak equivalences between cofibrant objects, and $\varepsilon \circ Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ is a natural weak equivalence.*
3. *U reflects weak equivalences between fibrant objects, and $Ur_F \circ \eta : Id_{\mathcal{C}_c} \Rightarrow URF|\mathcal{C}_c$ is a natural weak equivalence.*

Proof. We start by showing $1. \implies 2.$ and $3.$. We already know that the derived unit and counit are isomorphisms in homotopy, so we only need to show that F (U) reflects weak equivalences between cofibrant (fibrant) objects. Suppose that $Ff : FX \to FY$ is a weak equivalence between cofibrant objects. Since F preserves weak equivalences between cofibrant objects, we get that FQf is a weak equivalence; that $\mathbb{L}Ff$ is an isomorphism. By assumption, $\mathbb{L}F$ is an equivalence of categories, so f is a weak equivalence as needed.

We will show $2. \implies 1.$; the case $3. \implies 1.$ is dual. We assume that the counit map is an isomorphism in homotopy. By assumption, the derived unit $\mathbb{L}\eta$ is split-mono on the image of $\mathbb{L}F$. Moreover, the derived counit $\mathbb{R}\varepsilon$ is assumed to be an isomorphism. In particular, the derived unit $\mathbb{L}F\mathbb{L}\eta$ is an isomorphism. Unpacking this, we have a morphism, which we call $\eta'_X : FQX \to FQURFQX$, which is a weak equivalence. Since F and Q reflect weak equivalences, we get that $\eta_X : X \to URFQX$ is a weak equivalence. □

2.2 Model structures on Algebraic Categories

To understand ∞-quasi-isomorphism of strongly homotopy associative algebras, we will study different homotopy theories of various categories. Munkholm [Mun78] successfully showed that the derived category of augmented algebras is equivalent to the derived category of augmented algebras equipped with ∞-morphisms. To be more precise, he showed that certain subcategories of augmented algebras had this property. Lefevre-Hasagawas Ph.D. thesis [Lef03] builds upon this identification, but with the help of further development within the field. We will follow the approach of Lefevre-Hasegawa, by comparing the model structure for algebras and coalgebras,

2.2.1 DG-Algebras as a Model Category

Bousfield and Guggenheim [BG76] proved that the category of commutative dg-algebras had a model structure whenever the base field was a field of characteristic 0. In a joint project, Jardine's paper from 1997 [Jar97] shows that this construction may be extended to dg-algebras over any commutative ring. On the other hand, Munkholm expanded on the ideas from Bousfield and Guggenheim to get an identification of derived categories. Also, Hinich's paper from 1997 [Hin97] details another method to obtain the model category we want. We will follow the approach of Hinich, as it will be helpful later on. Notice that where Hinich uses the theory of algebraic operads to show that the category of algebras is a model category, we will give a more explicit formulation.

Let $\mathbb{K}$ be a field, and $\mathcal{C}$ be a category such that there is an adjunction $F : \mathsf{Ch}(\mathbb{K}) \rightleftharpoons \mathcal{C} : \#$, where F is left adjoint to #. Furthermore, suppose that $\mathcal{C}$ satisfies the 2 conditions:

(H0) $\mathcal{C}$ admits finite limits and every small colimit, and the functor # commutes with filtered colimits;

(H1) For M as the complex below, concentrated in 0 and 1,

$$... \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow ...$$

we have that for any $d \in \mathbb{Z}$ and for any $A \in \mathcal{C}$, the injection $A \to A \coprod F(M[d])$ induces a quasi-isomorphism $A^\# \to (A \coprod F(M[d]))^\#$.

With this adjunction in mind, we define weak equivalences, fibrations, and cofibrations as follows: Let $f \in \mathcal{C}$ be a morphism

- $f \in$ Ac if $f^\#$ is a quasi-isomorphism.
- $f \in$ Fib if $f^\#$ is surjective on each component.
- $f \in$ Cof if f has LLP to acyclic fibrations.

Theorem 2.2.1. *The category $\mathcal{C}$ equipped with the weak equivalences, fibrations, and cofibrations as defined above is a model category.*

Before we show this theorem, we need to understand the cofibrations better. Let $A \in \mathcal{C}$, $M \in \mathsf{Ch}(\mathbb{K})$ and $\alpha : M \to A^\#$ a morphism in $\mathsf{Ch}(\mathbb{K})$. We define a functor

$$h_{A,\alpha}(B) = \{(f,t) \mid f \in \mathcal{C}(A,B), t \in \mathsf{Hom}_{\mathbb{K}}^{-1}(M, B^\#) \text{ s.t. } \partial t = f^\# \circ \alpha\}.$$

Note that t is not a chain map. It is a homogenous morphism of degree -1. The differential then promotes this morphism to a chain map, and t is thus a homotopy for the composite $f^\# \circ \alpha$.

This functor is represented by an object of $\mathcal{C}$. We define this representing object $A\langle M, \alpha\rangle$ as the pushout:

$$\begin{tikzcd} F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \\ F(\mathsf{cone}(\alpha)) \arrow[r, "e"] & A\langle M, \alpha\rangle \end{tikzcd}$$

Let $i : M[1] \to \mathrm{cone}(\alpha)$ be a homogenous morphism which is the injection when considered as graded modules. Notice that we have a pair of morphisms $(a, e^T i) \in h_{A,\alpha}(A\langle M, \alpha\rangle)$.

Proposition 2.2.2. *The functor $h_{A,\alpha}$ is represented by $A\langle M, \alpha\rangle$, i.e. $h_{A,\alpha} \simeq \mathcal{C}(A\langle M, \alpha\rangle, \_)$ is a natural isomorphism. Moreover, the pair $(a, e^T i)$ is the universal element of the functor $h_{A,\alpha}$, i.e., the natural isomorphism is induced by this element under Yoneda's lemma.*

Proof. Let $(f, t) \in h_{A,\alpha}(B)$ for some $B \in \mathcal{C}$. The condition that $\partial t = f^\# \alpha$ is equivalent to say that $f^\#$ extends to a morphism $f' : \mathrm{cone}(\alpha) \to B^\#$ along t, i.e. there is a vector of morphisms $f' = \begin{pmatrix} f^\# & t \end{pmatrix}$. This construction concludes the isomorphism part, as an element (f, t) is equivalent to the diagram below, where $\widetilde{f}$ is uniquely determined.

$$\begin{tikzcd}
F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \arrow[ddr, bend left, "f"] & \\
F(\mathrm{cone}(\alpha)) \arrow[r, "e"] \arrow[drr, bend right, "f'^T"] & A\langle M, \alpha\rangle \arrow[dr, dashed, "\widetilde{f}"] & \\
 & & B
\end{tikzcd}$$

We use the adjunction to observe that the element $(a, e^T i)$ is universal to obtain naturality. □

We are now in a position to find some crucial cofibrations. We collect these morphisms into the "standard" cofibrations.

Definition 2.2.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose that f factors as a transfinite composition of morphisms on the form $A_i \to A_i\langle M_i, \alpha_i\rangle$, i.e. f factors into the diagram below, where $A_{i+1} = A_i\langle M_i, \alpha_i\rangle$.

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow B$$

- If every such M_i is a complex consisting of free $\mathbb{K}$-modules and has a 0-differential, we call f a standard cofibration.
- If every such M_i is a contractible complex and $\alpha = 0$, we call f a standard acyclic cofibration.

Proposition 2.2.4. *Every standard cofibration is a cofibration, and every standard acyclic cofibration is an acyclic cofibration.*

Remark 2.2.5. In some sense, we will see that these morphisms generate every (acyclic) cofibration.

Proof. Observe that every standard cofibration may be made iteratively from the chain complexes $\mathbb{K}[n]$, and likewise, every standard acyclic cofibration may be made iteratively from M as in $(H1)$.

We first prove that if $M \simeq \mathbb{K}[n]$, and $\alpha : M \to A^{\#}$ is any map, then the map $A \to A\langle M, \alpha\rangle$ is a cofibration; this amounts to show that it has LLP to every acyclic fibration. Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "h"] \\ A\langle M, \alpha\rangle \arrow[r, "g"] & C \end{tikzcd}$$

By the universal property of $h_{A,\alpha}$, Proposition 2.2.2, it suffices to find a pair $(f, t') \in h_{A,\alpha}$ which makes the lower triangle commute. That is, $t' : M \to B^{\#}$ is homogenous of degree -1, such that $\partial t' = f^{\#}\alpha$, and post composing h with the morphism determined by (f, t') is g. By the existence of g, there exists a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha$. Since h is an acyclic fibration, $h^{\#}$ is a surjective quasi-isomorphism. We assumed $M \simeq \mathbb{K}[n]$, so we can consider the morphism t as an element of $(C^{\#})^{n-1}$. By surjectivity of $h^{\#}$ there is an element u of $(B^{\#})^{n-1}$ such that $h^{\#}(u) = t$. Moreover, the difference $h^{\#}(\partial u - f^{\#}\alpha) = 0$, so $\partial u - f^{\#}\alpha$ factors through the kernel Ker$h^{\#}$, which is assumed to be acyclic. This element is furthermore a cycle, so by acyclicity, there is another element u' such that $\partial u' = \partial u - f^{\#}\alpha$. We may now see that $(f, u - u')$ is our desired factorization.

Secondly, we see that it is enough to prove that if M is as in (H1) and $\alpha = 0$, then the map $A \to A\langle M, \alpha\rangle$ is an acyclic cofibration. By (H1), we know that the map is already a weak equivalence, so we show that it has LLP to every acyclic fibration.

Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "h"] \\ A\langle M, \alpha\rangle \arrow[r, "g"] & C \end{tikzcd}$$

We will again use 2.2.2, so it suffices to find a t' such that $\partial t' = f^{\#}\alpha = 0$. By the existence of g, there is a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha = 0$. Since $h^{\#}$ is surjective t admits a linear homogenous lift $u : M \to B^{\#}$ such that $t = h^{\#}u$. We see that the map ∂u factors through the kernel of $h^{\#}$ as $h^{\#}\partial u = \partial h^{\#}u = \partial t = 0$. As $\partial u = 0$ is a cycle of Ker$h^{\#}$, there is a u' such that $\partial u' = \partial u$. The result follows by picking $t' = u - u'$.

$\square$

Given the above proposition, we would like to make some more convenient notation. If $M \simeq \mathbb{K}[n]$ and $\alpha : M \to Z^n(A^\#)$, s.t. $\alpha(1) = a$, we write $A\langle M, \alpha\rangle$ as $A\langle T; dT = a\rangle$ instead. Hinich calls this "adding a variable to kill a cycle." If M is the contractible complex as below and $\alpha = 0$, we write $A\langle T, S; dT = S\rangle$ for $A\langle M; dT = 0\rangle$. This construction can be thought of as "adding a variable and a cycle to kill itself."

$$\dots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \dots$$

proof of Theorem 2.2.1. **MC1** and **MC2** are satisfied. By definition, we also have the first part of **MC3**. We start by checking **MC4**.

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Given any $b \in B^\#$, let $C_b = A\langle T_b, S_b; dT_b = S_b\rangle$. We define $g_b : C_b \to B$ by the conditions that it acts on A as f, $g_b^\#(T_b) = b$ and $g_b^\#(S_b) = db$. By adding a "variable to kill a cycle" for every $b \in B$, we obtain an object C, such that the injection $A \to C$ is an acyclic standard cofibration, and the map $g : C \to B$ is a fibration. This is the desired factorization $f = f_\delta \circ f_\gamma$, where f_γ is the injection and $f_\delta = g$.

To obtain the other factorization, we want to make a standard cofibration. We already know that the map $A \to C$ is a standard cofibration, so let $C_0 = C$. From here on, we will make each C_i inductively, such that $\varinjlim C_i$ has the factorization property we desire. Notice that from C_0, there is a morphism $g_0 : C_0 \to B$, which is surjective and surjective on every kernel. This morphism may fail to be a quasi-isomorphism, so it is not an acyclic fibration.

To construct C_1 we assign to every pair of elements (c, b), such that $c \in ZC_0^\#$ and $g_0^\#(c) = db$, a variable to kill a cycle. If (c, b) is such a pair, then we add a variable T such that $dT = c$ and $g_1^\#(T) = b$. C_1 is then the complex where each cycle c has been killed by adding a variable T. Now, if we suppose that we have constructed C_i, then C_{i+1} is constructed similarly by adding a variable to kill each cycle which is a boundary in the image.

When adding a variable, we have also updated the morphism g_i by letting $g_{i+1}^\#(T) = b$. Thus in each step, we have also made a new morphism g_{i+1}. If g denotes the morphism at the colimit, it is clear that it is still a fibration and has also become a quasi-isomorphism. We can see this as every cycle which have failed to be in the homology of B has been killed.

It remains to check the last part of **MC3**. Suppose that $f : A \to B$ is an acyclic cofibration. By **MC4**, we know that it factors as $f = f_\delta \circ f_\gamma$, where f_δ is an acyclic fibration, and f_γ is a standard acyclic fibration. We thus obtain that f is a retract of f_γ by the commutative diagram below.

$$\begin{tikzcd} A \arrow[r, "f_\gamma"] \arrow[d, "f"'] & C \arrow[d, "f_\delta"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

$\square$

The following corollary will concretize what it means that the standard cofibrations generate every cofibration. This corollary is an emphasis on the last diagram in the previous proof.

Corollary 2.2.5.1. *Any (acyclic) cofibration is a retract of a standard (acyclic) cofibration.*

We may immediately apply this theorem to some familiar examples.

Corollary 2.2.5.2. *Let A be a dg-algebra over the field $\mathbb{K}$. The category Mod$_A$ of left modules is a model category.*

sketch of proof. We establish the adjunction by letting $FM = A \otimes_{\mathbb{K}} \_$. H0 is satisfied as this category is bicomplete, and we can think of filtered colimits as unions of sets. Moreover, since Mod$_A$ is an Abelian category, the forgetful functor # commutes with coproducts, or direct sums, which makes H1 trivially satisfied. $\square$

Corollary 2.2.5.3. *The categories Alg$^\bullet_{\mathbb{K}}$ (Alg$^\bullet_{\mathbb{K},+}$) are model categories.*

Proof. We establish the adjunction by letting $F = T(M)$, the tensor algebra of a cochain complex. For the same reasons as above, H0 is trivially satisfied.

Given a cochain complex $N^\bullet$, we may consider the free dg-algebra $T(N^\bullet)$. In this case, the coproduct $A * T(N^\bullet)$ has an easier description. We define a complex

$$A[N^\bullet] = A \oplus (A \otimes N^\bullet \otimes A) \oplus (A \otimes N^\bullet \otimes A \otimes N^\bullet \otimes A) \oplus \cdots.$$

The differential on $A[N^\bullet]$ is the differential induced by the tensor product. We define a multiplication on $A[N^\bullet]$ by the following formula

$$(a_1 \otimes \cdots \otimes a_i) \cdot (a'_1 \otimes \cdots \otimes a'_j) = a_1 \otimes \cdots \otimes a_i a'_1 \otimes \cdots a'_j.$$

Let $i : A \to A[N^\bullet]$ denote the inclusion, and $\iota : T(N^\bullet) \to A[N^\bullet]$ is defined by interspersing the $N^\bullet$ tensors with 1s. I.e. $\iota(n_1 \otimes \cdots \otimes n_j) = 1 \otimes n_1 \otimes 1 \otimes \cdots \otimes 1 \otimes n_j \otimes 1$.

To define a map $f : A[N^\bullet] \to T$ it is enough by the ring homomorphism property to define a map $g : A \to T$ and a map $h : T(N^\bullet) \to T$. This choice of g and h is unique for any f, establishing the universal property. I.e. $A[N^\bullet] \simeq A * T(N^\bullet)$.

To see that the map $i^\# : A^\# \to A[M^\bullet]^\#$ is a quasi-isomorphism, it is enough to see that contractible complexes are stable under tensoring. Given any contractible complex $C^\bullet$, there is a homotopy $h : C^\bullet \to C^\bullet$ such that $\partial h = id_C$. Observe that $id_N \otimes h : N^\bullet \otimes C^\bullet \to N^\bullet \otimes C^\bullet$ is a homotopy witnessing $id_{N^\bullet \otimes C^\bullet} \sim 0$. Since M is acyclic, we know that the homology of the inclusion is $H^* i = id_{H^* A}$, which shows H1. $\square$

We summarize the last result:

The category of augmented dg-algebras $\mathsf{Alg}^{\bullet}_{\mathbb{K},+}$ is a model category. Let $f : X \to Y$ be a homomorphism of augmented algebras.

- $f \in$ Ac if $f^{\#}$ is a quasi-isomorphism.
- $f \in$ Fib if $f^{\#}$ is an epimorphism (surjective onto every component).
- $f \in$ Cof if f has LLP with respect to to every acyclic fibration.

The category of augmented dg-algebras has a zero object, and this is the stalk of $\mathbb{K}$. We see that every object is fibrant, as the forgetful functor preserves the augmentation map and, by definition, is a split-epimorphism.

Remark 2.2.6. In the process of showing that $\mathsf{Alg}_{\mathbb{K},+}$ is a model category, we have not cared about functorial factorization. One may see that we get this from the constructions used to prove **MC4**. This is a technical detail which we do not need to care too much about.

2.2.2 A Model Structure on DG-Coalgebras

We now want to equip the category of dg-coalgebras with a suitable model structure. This model structure should be suitable in the sense that conilpotent dg-coalgebras will have the same homotopy theory as dg-algebras. The bar-cobar construction will be crucial in this construction, as it is a Quillen adjunction. To this end, we will follow the setup as presented by Lefevre-Hasegawa [Lef03]. His method modifies Hinich's paper [Hin01c].

Let $f : C \to D$ be a morphism of coalgebras, the category of dg-coalgebras will be equipped with the three following classes of morphisms:

- $f \in$ Ac if Ωf is a quasi-isomorphism.
- $f \in$ Fib if f has RLP with respect to every acyclic cofibration.
- $f \in$ Cof if $f^{\#}$ is a monomorphism (injective in every component).

To see that these classes of morphisms do indeed define a model structure, we will get a better description of a subclass of weak equivalences. We can only check if a morphism is a weak equivalence by calculating homologies since f is a weak equivalence if and only if $H^*\mathrm{cone}(\Omega f) \simeq 0$. Using spectral sequences to calculate these homologies is not crucial, but it gives us a method to handle the problems we will face.

Definition 2.2.7. A filtered chain map $f : M \to N$ of filtered complexes M and N is a graded quasi-isomorphism if $\mathrm{gr}f : \mathrm{gr}M \to \mathrm{gr}N$ is a quasi-isomorphism of the associated graded complexes.

Lemma 2.2.8. *Let $f : C \to C'$ be a graded quasi-isomorphism between conilpotent dg-coalgebras, then $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism.*

Proof. We do this by considering a spectral sequence. Endow C with a grading (as a vector space) induced by the coradical filtration, i.e., $c \in C$ has degree $|c| = n$ if n is the smallest number such that $\overline{\Delta}^n c = 0$. We define a filtration on ΩC by

$$F_p \Omega C = \{\langle c_1 | \cdots | c_n \rangle \mid |c_1| + ... + |c_n| \leqslant p\}$$

Since C is a dg-coalgebra, the coradical filtration respects the differential. In other words, $F_p \Omega C$ is still a cochain complex, a subcomplex of ΩC. This filtration is bounded below and exhaustive. Thus by the classical convergence theorem of spectral sequences, Theorem C.3.1, the spectral sequence converges to the homology $E\Omega C \Rightarrow H^* \Omega C$.

By definition, the 0'th page is

$$E^0_{p,q} \Omega C = {}^{(F_p \Omega C)_{p+q}} \big/ {}_{(F_{p-1} \Omega C)_{p+q}}.$$

Furthermore, notice that on this page we have the following isomorphism $E^0_{p,q} \Omega C \simeq (\Omega \text{gr} C)^{(p)}_{p+q}$, where $(\Omega \text{gr} C)^{(p)} = \{\langle c_1 | \cdots | c_n \rangle \mid |c_1| + ... + |c_n| = p\}$.

Evaluating f at the 0'th page would look like $E^0 \Omega f \simeq \Omega \text{gr} f$. By the comparison theorem, Theorem C.2.13, it is enough to check that $\Omega \text{gr} f$ is a quasi-isomorphism to see that Ωf is a quasi-isomorphism. We show that $\Omega \text{gr} f$ is a quasi-isomorphism by inspecting every cochain complex $E^0_{p,\bullet} \Omega C$.

Define a filtration G_k on $E^0_{p,\bullet} \Omega C$ as

$$G_k = \{\langle c_1 | \cdots | c_n \rangle \mid n \geqslant -k\}.$$

We see that $G_0 = E^0_{p,\bullet} \Omega C$ by definition and $G_{-p-1} \simeq 0$ on the coaugmentation quotient $\overline{C}$. The classical convergence theorem of spectral sequences defines a spectral sequence such that $EG \Rightarrow H^* E^0_{p,\bullet} \Omega C$.

To see that $\Omega \text{gr} f$ is a quasi-isomorphism, we will show that $E^0 G f$ is a quasi-isomorphism for any p. Notice that $E^0_{l,\bullet} G \subseteq (\text{gr} C[-1])^{\otimes l}$ where the total grading is p. Since f is a graded quasi-isomorphism, it follows by the Künneth-formula [Theorem 3.6.3 Wei94, p. 88] that $E^0 G f$ is a quasi-isomorphism. $\square$

This proof will serve as a template for how we approach many of the proofs we encounter. With the lemma, to show that f is a weak equivalence, it suffices to show that f is a graded quasi-isomorphism. However, to show that f is a graded quasi-isomorphism, we first need a good filtering, and once we have a filtering, we look at its spectral sequence. The mapping lemma says that it is enough to verify that a morphism becomes a quasi-isomorphism on any page to see that it is a quasi-isomorphism. We proceed then to calculate a page where we can assert that f becomes a quasi-isomorphism. If there still are problems with calculations, we look at complexes within a page on a spectral sequence and define new filtrations on these complexes to calculate the next page. We will informally call this technique for an iterated spectral sequence argument.

For completeness, we include the following statement.

Lemma 2.2.9. *Let $f : A \to A'$ be a quasi-isomorphism between dg-algebras, then $Bf : BA \to BA'$ is a graded quasi-isomorphism.*

Proof. Notice that the homology of BA may be calculated from the double complex used to define BA. In fact, at the 0'th page of the canonical spectral sequence, we have $E^0_{p,\bullet} f \simeq f^{\otimes p}$. It follows that f is a quasi-isomorphism on the 0'th page from the Künneth formula, [Theorem 3.6.3 Wei94, p. 88]. □

Let A (C) be a filtered dg-algebra (coalgebra). Given an element $a \in A$ ($c \in C$) we say that its filtered degree f-deg(a) (f-deg(c)) is the smallest number such that $a \in F_{\text{f-deg}(a)} A$ ($c \in F_{\text{f-deg}(c)} C$) but not $a \in F_{\text{f-deg}(a)-1} A$ ($c \in F_{\text{f-deg}(c)-1} C$). There is then an associated filtration on the bar (cobar) construction of this complex, defined as

$$F_p BA = \{[a_1 \mid \cdots \mid a_n] \mid \sum \text{f-deg}(a_i) \leqslant p\}$$
$$(F_p \Omega C = \{\langle c_1 \mid \cdots \mid c_n \rangle \mid \sum \text{f-deg}(c_i) \leqslant p\}).$$

We will call this the induced filtration on the bar or cobar construction.

Proposition 2.2.10. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$ is a quasi-isomorphism. The unit $\eta_C : C \to B\Omega C$ is a graded quasi-isomorphism. Moreover, $\Omega\eta_C$ is a quasi-isomorphism.*

The following proof is due to [Lef03], but with corrections given by [Kel05b]. Some minor modifications are given to the proof as it resembles a previous proof, using the method of iterated spectral sequences.

Proof. We start by showing that the counit is a quasi-isomorphism. Define the following filtration for A.

$$F_0 A = \mathbb{K}$$
$$F_1 A = A$$
$$F_p A = F_1 A$$

We see that this filtration endows A with the structure of a filtered dg-algebra. For ΩBA, we will use the induced filtration from the coradical filtration of BA.

The counit acts on ΩBA as tensor-wise projection, followed by multiplication in A. This morphism respects the filtration, so it is a filtered morphism. Notice that both filtrations are bounded below and exhaustive, so the classical convergence theorem of spectral sequences applies.

Let $E_r \Omega BA$ and $E_r A$ be the spectral sequences given by these filtrations. We have that $E^p_1 \Omega BA \simeq \text{gr}_p \Omega BA$ and $E^p_1 A \simeq \text{gr}_p A$. For $p = 1$, both complexes are isomorphic to the same complex, $\overline{A}$.

Moreover, $E_1^1\varepsilon_A = id_{\overline{A}}$. Whenever $p \neq 1$, we get that $E_1^p A \simeq 0$, so it remains to show that $E_1^p \Omega BA \simeq \mathrm{gr}_p \Omega BA$ is acyclic for any $p \geqslant 2$.

Three actions generate the differential of ΩBA: the differential on A, the multiplication on A, and the comultiplication on BA. With the induced filtration on ΩBA, we see that the multiplication on A is the only action that maps $F_p \Omega BA \to F_{p-1}\Omega BA$. Thus this action is 0 in the associated graded and the spectral sequence.

There is a homotopy of the identity given as $r : \mathrm{gr}_i \Omega BA \to \mathrm{gr}_i \Omega BA$, which is 0 except if there is an element on the form $\langle [a] \mid [\cdots] \mid [\cdots] \rangle$. In this case, r is

$$r\langle [a] \mid [\cdots] \mid \cdots \rangle = (-1)^{|a|+1} \langle [a \mid [\cdots] \mid \cdots \rangle$$

We will show that this is a homotopy by induction on i.

Let $i = 2$. Then there are two cases we must handle, either an element is on the form $\langle [a_1] \mid [a_2] \rangle$ or $\langle [a_1 \mid a_2] \rangle$. We consider the latter case first. If we apply r to this element, we are returned 0.

$$(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)\langle [a_1 \mid a_2] \rangle = r(-1)^{|a_1|+1} \langle [a_1] \mid [a_2] \rangle = \langle [a_1 \mid a_2] \rangle$$

Then we treat the former case

$$\begin{aligned}
&(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)\langle [a_1] \mid [a_2] \rangle \\
&\quad = r\langle [d_A a_1] \mid [a_2] \rangle + (-1)^{|a_1|} r \langle [a_1] \mid [d_A a_2] \rangle + d_{\Omega BA} (-1)^{|a_1|+1} \langle [a_1 \mid a_2] \rangle \\
&\quad\quad = (-1)^{|a_1|} \langle [d_A a_1 \mid a_2] \rangle - \langle [a_1 \mid d_A a_2] \rangle + \langle [a_1] \mid [a_2] \rangle \\
&\quad\quad\quad + (-1)^{|a|+1} \langle [d_A a_1 \mid a_2] \rangle + \langle [a_1 \mid d_A a_2] \rangle = \langle [a_1] \mid [a_2] \rangle.
\end{aligned}$$

This homotopy makes $id_{\mathrm{gr}_2 \Omega BA}$ null-homotopic.

To extend this argument by induction, we will observe that the terms where the differential is applied will have opposite signs, such that they cancel. The result follows for any i since the tensors far enough out to the right are not affected by r.

If C is a dg-coalgebra, we use the same technique as in Lemma 2.2.8. Consider the filtration on $B\Omega C$ given as

$$F_p B\Omega C = \{ [\langle sc_{1,1} \mid \cdots \mid sc_{1,n_1} \rangle \mid \cdots \mid \langle sc_{m,1} \mid \cdots \mid sc_{m,n_m} \rangle] \mid |c_{1,1}| + \cdots + |c_{m,n_m}| \leqslant p \}.$$

This filtration is bounded below and exhaustive, so the classical convergence theorem says that the associated spectral sequence converges. We denote this sequence as EF, and then $EF \implies H^* B\Omega C$. Let EC be the spectral sequence associated to C. Since C is conilpotent, $EC \implies H^* C$. The unit $\eta_C : C \to B\Omega C$ is now a map acting on EC^0 as the identity, sending each element in $EC^0_{p,q}$ to itself in $EF^0_{p,q}$.

On each row $EF^0_{p,\bullet}$, we make another filtration called G.

$$G_k EF^0_{p,\bullet} = \{ [\langle \ldots \rangle_1 \mid \ldots \mid \langle \ldots \rangle_n] \mid n \geqslant -k \}$$

Similarly, as in Lemma 2.2.8, this filtration is bounded below and exhaustive, so we may again apply the classical convergence theorem to obtain a spectral sequence E_pG such that $E_pG \implies H^*EF^0_{p,\bullet} \simeq EF^1_{p,\bullet}$. Since the unit acts as the identity on EC^0, it descends to a morphism $\mathrm{gr}_p C \to E_pG^0_{k,\bullet}$ which is the identity when $k = -1$ and 0 otherwise. Notice that this morphism does not hit every string of length $\geqslant 2$. However, by employing r as above, we may show that these summands are acyclic. The unit is thus an isomorphism in homology.

□

Lemma 2.2.11. *Let $f : C \to D$ be a morphism of dg-coalgebras, then:*

- *if f is a cofibration, then Ωf is a standard cofibration.*
- *if f is a weak equivalence, then Ωf is as well.*

Almost dually, let $f : A \to B$ be a morphism of dg-algebras, then:

- *if f is a fibration, then Bf is a fibration.*
- *if f is a weak equivalence, then Bf is as well.*

Proof. First, suppose that $f : C \to D$ is a cofibration. We define a filtration on D as the sum of the image of f and the coradical filtration on D: $D_i = Imf + Fr_iD$. f being a cofibration ensures us that $D_0 \simeq C$. Since D is conilpotent, we know that $D \simeq \varinjlim D_i$, and since Ω commutes with colimits there is a sequence of algebras $\Omega C \to \Omega D_1 \to ... \to \Omega D$. It is enough to show that each morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The quotient coalgebra ${}^{D_{i+1}}/_{D_i}$ only has a trivial comultiplication. Thus every element is primitive, and this means that as a cochain complex, D_{i+1} is constructed from D_i by attaching possibly very many copies of $\mathbb{K}$. We treat the case when there is only one such $\mathbb{K}$, here $D_{i+1} \simeq D_i \oplus \mathbb{K}\{x\}$ where $dx = y$ for some $y \in D_i$, which is exactly the condition for the morphism $\Omega D_i \to \Omega D_{i+1}$ to be a standard cofibration.

If f is a weak equivalence, then Ωf is a quasi-isomorphism.

By Lemma 2.1.39, or adjointness, more specifically, the property that B preserves fibrations is a consequence of Ω preserving cofibrations.

It remains to show that if $f : A \to B$ is a quasi-isomorphism, then Bf is a weak equivalence. Now, Bf is a weak equivalence if and only if ΩBf is a quasi-isomorphism. By Proposition 2.2.10, the counit $A \to \Omega BA$ is a quasi-isomorphism, so Bf is a weak equivalence by 2-out-of-3 property.

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ \Omega BA \arrow[u, "\varepsilon_A"] \arrow[r, "\Omega Bf"] & \Omega BB \arrow[u, "\varepsilon_B"'] \end{tikzcd}$$

□

We will need one more technical lemma.

Lemma 2.2.12 (Key lemma)**.** *Let A be a dg-algebra, D a dg-coalgebra, and $p : A \to \Omega D$ a fibration of algebras. The projection morphism $BA *_{B\Omega D} D \to BA$ is an acyclic cofibration.*

$$\begin{tikzcd} BA *_{B\Omega D} D \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "\eta_D"] \\ BA \arrow[r, two heads, "Bp"] & B\Omega D \end{tikzcd}$$

This proof has a slightly troubled past. In [Lef03], Lefevre-Hasegawa made a proof which was a straightforward modification of Hinich's proof [Hin01c, Key Lemma]. However, this translation does not behave as well as one would like. Keller points out that this method may sometimes work but fails in its full generality [Kel06]. The proof presented here is a modification of Vallette's proof of "A technical lemma" [Val20, Appendix B].

Proof. π being a cofibration is immediate by Corollary 2.1.10.1.

To see that π is a weak equivalence, We show that it is a graded quasi-isomorphism by Lemma 2.2.8. Since we assume p to be a fibration onto a quasi-free algebra, we may realize the algebra A as the following extension.

$$\begin{tikzcd} \cdots \arrow[r, hook] & \mathrm{cone}(d') \arrow[r, two heads, "p"] & \Omega D[1] \arrow[dll, dashed, "d'[1]"] \\ \mathrm{Ker}(p)[1] \arrow[r, hook] & \mathrm{cone}(d')[1] \arrow[r, two heads] & \cdots \end{tikzcd}$$

Between each of the extensions, there is a connecting morphism d', which comes from the differential of cone(d'). As graded modules, $A \simeq$ cone$(d') \simeq$ Ker$(p) \oplus \Omega D$. We denote $K =$ Ker(p), so that the differential of A is then the differential coming from

$$\begin{aligned} d_K &: K \to K, \\ d_{\Omega D} &: \Omega D \to \Omega D \text{ and} \\ d' &: \Omega D \to K. \end{aligned}$$

In the category $\mathsf{Alg}^\bullet_{\mathbb{K},+}$, $\oplus$ is the product. Since $B : \mathsf{Alg}^\bullet_{\mathbb{K},+} \to \mathsf{coAlg}^\bullet_{\mathbb{K},conil}$ is right adjoint, it necessarily preserves products. Thus

$$\begin{aligned} &BA \simeq B(K \oplus \Omega D) \simeq BK * B\Omega D \text{ and} \\ &BA *_{B\Omega D} D \simeq BK * D. \end{aligned}$$

Using this identification of the underlying graded modules, we may identify the morphism π with $id_{BK} * \eta_D$. If the differential of BA was not perturbed by d', then we could have appealed to the morphism π being a graded quasi-isomorphism to conclude that it is a quasi-isomorphism. Instead, we will employ some smart filtrations onto BA and $BA *_{B\Omega D} D$.

Since BK is quasi-free, by the comonadic presentation of D, we can obtain an identification of graded modules, $BK * D \subseteq T^c(\overline{K}[1] \oplus \overline{D})$. Likewise, since both BK and $B\Omega D$ are quasi-free, we realize the product as $BK * B\Omega D \simeq T^c(\overline{K}[1] \oplus (\overline{\Omega}D)[1])$.

With this description, we define filtrations as

$$F_n(BA *_{B\Omega D} D) \subseteq F_n(T^c(\overline{K}[1] \oplus \overline{D})) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus Fr_{n_i}\overline{D}) \text{ and}$$
$$F_n(BA) = F_n(T^c(\overline{K}[1] \oplus (\overline{\Omega}D)[1])) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus \widetilde{Fr}_{n_i}(\overline{\Omega}D)[1]).$$

Here Fr and $\widetilde{Fr}$ refer to the coradical and induced coradical filtration. This filtration is made to be agnostic towards K. In other words, morphisms into K are a priori filtered. Thus the part of the differential coming from d_K and d' are filtered. Likewise, the coradical filtration preserves the part of the differential coming from $d_{\Omega D}$. The differential coming from the multiplication of K and ΩD is of -1 filtered degree. $\eta_{\overline{D}}$ preserves this filtration as it acts like the identity.

The associated graded component reduces to the associated graded of D and $B\Omega D$. If we lower the degree of a n_i by 1, this component lands in the lower degree of the filtration. By cocontinuity of the tensor, we may move the associated graded into each variable. The sum handles every other component.

$$\mathrm{gr}_n(BA *_{B\Omega D} D) \simeq BK * \mathrm{gr}_n D$$
$$\mathrm{gr}_n(BA) \simeq BK * B\Omega \mathrm{gr}_n D$$

In the same manner, the morphism π then acts on each element as $id_{BK} * \mathrm{gr}(\eta_D)$.

These filtrations are bounded below. Since D and $B\Omega D$ are both conilpotent dg-coalgebras, the filtrations are also exhaustive. By the classical convergence theorem of filtered spectral sequences, we obtain spectral sequences $E(BA *_{B\Omega D} D) \implies \mathrm{H}^*(BA *_{B\Omega D} D)$ and $E(BA) \implies \mathrm{H}^*(BA)$. We want to show that the morphism of spectral sequences $id_{BK} *_{B\Omega D} \mathrm{gr}\eta_D : E(BA *_{B\Omega D} D) \to E(BA)$ eventually becomes a quasi-isomorphism, and this will happen on the first page.

To obtain this on the first page, we will define another spectral sequence $\widetilde{E}$ such that $\widetilde{E} \implies E_1$.

We start by defining new filtrations,

$$\widetilde{F}_n(BK * \mathrm{gr}D) \subseteq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k+k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus \mathrm{gr}_{n_i}\overline{D}) \text{ and}$$

$$\widetilde{F}_n(BK * B\Omega\mathrm{gr}D) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus (\bigoplus_{t=1}^{\infty} \sum_{\substack{m_1+\cdots+m_t+t \\ \leqslant n_i}} \bigotimes_{j=1}^{t} \mathrm{gr}_{m_j}\overline{D}[-1])[1]).$$

Again, these filtrations are agnostic towards K, so both parts of the differential that comes from d_K and d' are filtered. The part of the differential which comes from d_D naturally goes from $\mathrm{gr}_{n_i}\overline{D}$ to itself. The differential coming from the multiplication has already been dealt with, so these filtrations respect our differential. The morphism $id_{BK} * \mathrm{gr}(\eta_D)$ also preserves this filtration, as it acts like the identity on elements. In other words, the first filtered object is naturally a subobject of the second filtered object by identifying the elements d with $[\langle d\rangle]$.

At the 0'th page of $\widetilde{E}$, we want to show that the part of the differential coming from d' acts like 0. This is the same to say that $\mathrm{Im}d' \mid_{F_n} \subseteq F_{n-1}$. We calculate the 0'th page of the double spectral sequence as below.

$$\widetilde{E}_0^{-n}(BK * \mathrm{gr}D)[-n] \subseteq \mathrm{gr}_n(BK * \mathrm{gr}D) \simeq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k+k \\ =n}} \bigotimes_{j=1}^{k} (\overline{K}[1] \oplus \mathrm{gr}_{n_i}\overline{D})$$

$$\begin{aligned}\widetilde{E}_0^{-n}(BK * B\Omega\mathrm{gr}D)[-n] &= \mathrm{gr}_n(BK * B\Omega\mathrm{gr}D) \\ &\simeq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ =n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus (\bigoplus_{t=1}^{\infty} \sum_{\substack{m_1+\cdots+m_t+t \\ =n_i}} \bigotimes_{j=1}^{t} \mathrm{gr}_{m_j}\overline{D}[-1])[1])\end{aligned}$$

We now pick an element $([k_1]+d_1)\otimes\cdots\otimes([k_k]+d_k) \in \mathrm{gr}_n(BK * \mathrm{gr}D)$. Then $|d_1|+\cdots+|d_k|+k = n$. The differential from d' is the alternate sum of d' at each tensor argument. We illustrate what happens at the i'th argument.

$$\begin{aligned}&\widetilde{d}'(([k_1]+d_1)\otimes\cdots\otimes([k_i]+d_i)\otimes\cdots\otimes([k_k]+d_k)) \\ =&([k_1]+d_1)\otimes\cdots\otimes([k_i]+d'(d_i))\otimes\cdots\otimes([k_k]+d_k)\end{aligned}$$

Since $|[k]+d'(d_i)| = 0$, the total degree of this element goes down at least 1 if $d_i \neq 0$. If $d_i = 0$, then $d'(d_i) = 0$ anyway. In this manner, this morphism does not survive at the $\widetilde{E}_0$ page. Likewise, given an element on the form $[k_1 + \langle d_{1,1} \mid \cdots \mid d_{1,t_1}\rangle \mid \cdots \mid k_k + \langle d_{k,1} \mid \cdots \mid d_{k,t_k}\rangle]$, then $|d'(\langle d_{i,1} \mid \cdots \mid d_{i,t_i}\rangle)| = 0$. So the phenomenon occurs at the other spectral sequence as well.

In this way $\mathrm{gr}(id_{BK} * \mathrm{gr}\eta_D)$, is in fact a quasi-isomorphism between the sequences $\widetilde{E}(BK * \mathrm{gr}D) \to \widetilde{E}(BK * B\Omega\mathrm{gr}D)$ just as Lemma 2.2.10. By the classical convergence theorem, this assembles into a quasi-isomorphism on the E_1 page of the previous spectral sequences, showing that π is a graded quasi-isomorphism. $\square$

Theorem 2.2.13. *The category* $\mathsf{coAlg}^{\bullet}_{\mathbb{K},conil}$ *is a model category with the classes Ac, Fib and Cof as defined above.*

Proof. The axioms **MC1** and **MC2** are immediate. Also, fibrations having RLP with respect to acyclic cofibrations is by definition.

We show **MC4** first. Let $f : C \to D$ be a morphism of coalgebras. There is a factorization $\Omega f = pi$ of morphisms between algebras, where i is a cofibration, p is a fibration, and at least one of i and p are quasi-isomorphisms. Applying the bar construction, we get a factorization $B\Omega f = BiBp$, where Bp is a fibration, and at least one of Bi and Bp are weak equivalences.

$$\begin{tikzcd}
\Omega C \arrow[rr, "\Omega f"] \arrow[dr, "i"'] & & \Omega D & \rightsquigarrow & B\Omega C \arrow[rr, "B\Omega F"] \arrow[dr, "Bi"'] & & B\Omega D \\
& A \arrow[ur, "p"'] & & & & BA \arrow[ur, "Bp"'] &
\end{tikzcd}$$

We construct a pullback with Bp and η_D. By Lemma 2.2.12, the morphism π is an acyclic cofibration. We collect our morphisms in a big diagram. The dashed arrow exists since the rightmost square is a pullback.

$$\begin{tikzcd}
& BA *_{B\Omega D} D \arrow[dr, "q"] \arrow[dd] \arrow[dr, phantom, "\ulcorner", near start] & \\
C \arrow[ur, dashed, "j"] \arrow[rr, "f"] \arrow[dd, "\eta_C"] & & D \arrow[dd, "\eta_D"] \\
& BA \arrow[dr, "Bp"] & \\
B\Omega C \arrow[ur, "Bi"] \arrow[rr, "B\Omega f"] & & B\Omega D
\end{tikzcd}$$

First, notice that q is a fibration since fibrations are stable under pullbacks. j is a cofibration, or a monomorphism, as the composition $Bi \circ \eta_C$ is a monomorphism. Thus it remains to see that if Bi (Bp) is a weak equivalence, then j (q) is as well. We know this from the 2-out-of-3 property, as η is a natural weak equivalence, π is a weak equivalence, and Bi (Bp) is a weak equivalence.

We now show **MC3**. Suppose there are morphisms as in the square below, where i is a cofibration, and t is an acyclic cofibration.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "i"] & C \arrow[d, "t"] \\ F \arrow[r] & D \end{tikzcd}$$

We can factor t as $t = qj$ by **MC4**. Notice that t is a retract of q, i.e., there is a commutative diagram below.

$$\begin{tikzcd} C \arrow[rr, equal] \arrow[d, "j"] & & C \arrow[d, "t"] \\ BA *_{B\Omega A} D \arrow[urr] \arrow[rr, "q"] & & D \end{tikzcd}$$

To find a lift to C, we may find a lift to $BA *_{B\Omega D} D$. Since p is an acyclic fibration by construction and Ωi is a cofibration by Lemma 2.2.11, there is a lift $h : \Omega E \to A$ of algebras. We obtain our desired lift from the bar-cobar adjunction and the universal property of the pullback.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "i"] & BA *_{B\Omega D} D \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & BA \arrow[d, "Bp"] & & \Omega E \arrow[r] \arrow[d, "\Omega i"] & A \arrow[d, "p"] \\ F \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted, "h^T"] & D \arrow[r, "\eta_D"] & B\Omega D \arrow[r, phantom, "\leftrightsquigarrow"] & & \Omega F \arrow[r] \arrow[ur, dotted, "h"] & \Omega D \end{tikzcd}$$

□

We restate the corollary of the adjunction.

Corollary 2.2.13.1. *The bar-cobar construction* $\Omega : coAlg^{\bullet}_{\mathbb{K},conil} \rightleftharpoons$ *Alg*$^{\bullet}_{\mathbb{K},+}$ $: B$ *as a Quillen equivalence.*

Proof. We first observe that (B, Ω) is a Quillen adjunction by Lemma 2.2.11. Moreover, since the unit and counit are weak equivalences by Proposition 2.2.10, it follows by either Proposition 2.1.43 or its Corollary 2.1.43.1 that (B, Ω) is a Quillen equivalence. □

2.2.3 Homotopy theory of A_∞-algebras

This section aims to finalize the discussion of the homotopy theory of A_∞-algebras. We will look at the homotopy invertibility of every strongly homotopy associative quasi-isomorphism and its relation to ordinary associative algebras. This discussion will end with mentioning different results, which gives a more explicit description of fibrations, cofibrations, and homotopy equivalences. This section follows Lefevre-Hasegawa [Lef03]. Before we get to the main theorem, we start by discussing a non-closed model structure on the category of Alg$_\infty$.

Let $f : A \rightsquigarrow B$ be a morphism between A_∞-algebras, the category of A_∞-algebras will be equipped with the three following classes of morphisms:

- $f \in$ Ac if f is an ∞-quasi-isomorphism, i.e. f_1 is a quasi-isomorphism.
- $f \in$ Fib if f_1 is an epimorphism.
- $f \in$ Cof if f_1 is a monomorphism.

This category does not make a model category in the sense of a closed model category, as we lack many finite limits. It does, however, come quite close to being such a category.

Theorem 2.2.14. *The category Alg$_\infty$ equipped with the three classes as defined above satisfies:*

*a The axioms **MC1** through **MC4**.*
b Given a diagram as below, where p is a fibration, then its limit exists.

$$\begin{tikzcd} & A \arrow[d, "p"] \\ B \arrow[r] & C \end{tikzcd}$$

Before we are ready to prove this theorem, we will need some preliminary results. We will only prove the first lemma.

Lemma 2.2.15. *let A be an A_∞-algebra, and K an acyclic complex considered as an A_∞-algebra. If $g : (A, m_1^A) \to (K, m_1^K)$ is a cochain map, then it extends to an ∞-morphism $f : A \rightsquigarrow K$.*

Proof. We construct each f_i inductively. The case $i = 1$ is degenerate as we have assumed $f_1 = g$.

Assume that we have already constructed f_1 through f_n. We observe that the sum below is a cycle of $\mathrm{Hom}_{\mathbb{K}}^*(A, K)$.

$$\sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k\geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

Thus since K is acyclic, $\mathrm{Hom}_{\mathbb{K}}^*(A, K)$ is acyclic, and there exists some morphism f_{n+1} such that $\partial(fn + 1)$ is the sum above, and this says that this extension does satisfy (rel_{n+1}). □

Lemma 2.2.16 ([Lemma 1.3.3.3 Lef03, p. 44])**.** *Let $j : A \rightsquigarrow D$ be a cofibration of A_∞-algberas, and then there is an isomorphism $k : D \rightsquigarrow D'$ such that the composition $k \circ j : A \rightsquigarrow D'$ is a strict morphism of A_∞-algebras.*

Dually, if $j : A \rightsquigarrow D$ is a fibration, then there is an isomorphism $l : A' \rightsquigarrow A$ such that the composition $j \circ l : A' \rightsquigarrow D$ is a strict morphism of A_∞-algebras.

We will need the following lemma.

Proof of Theorem 2.2.14. We start by showing (b). Suppose we have a diagram of A_∞-algebras, such that g_1 is an epimorphism.

$$\begin{tikzcd} & A \arrow[d, two heads, "g"] \\ A' \arrow[r, "f"] & A'' \end{tikzcd}$$

First, notice that as dg-coalgebras, this pullback exists and defines a new dg-coalgebra $BA *_{BA''} BA'$.

Since g_1 is an epimorphism, $A[1]$ as a graded vector space splits into $A''[1] \oplus K$, where $K = \text{Ker} g_1$. The pullback is then naturally identified with $BA \prod_{BA''} BA' \simeq \overline{T}^c(K) \prod \overline{T}^c(A'[1])$ as graded vector spaces. Since the cofree coalgebra is right adjoint to forget, it commutes with products, and we get $\overline{T}^c(A'[1]) \prod \overline{T}^c(K) \simeq \overline{T}^c(A'[1] \oplus K)$. Thus the pullback is isomorphic to a cofree coalgebra as a graded coalgebra, i.e., an A_∞-algebra.

We now prove (a). **MC1** and **MC2** are immediate, so we will not prove them.

We start by proving **MC3**. Suppose that there is a square of A_∞-algebras as below, where j is a cofibration, and q is a fibration.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

By Lemma 2.2.16, we may assume that both j and q are strict morphisms. We can assume that q is an ∞-quasi-isomorphism since the proof will be analogous if j is an ∞-quasi-isomorphism instead.

Our goal is to construct a lifting in this diagram inductively. Having a lift means finding an ∞-morphism $a : C \rightsquigarrow B$, such that the following hold for any $n \geqslant 1$:

- a satisfy (rel_n).
- $a_n \circ j_1 = f_n$.
- $q_1 \circ a_n = g_n$.

We start by showing there is such an a_1. Consider the diagram below of chain complexes over $\mathbb{K}$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, tail, "j_1"] & B \arrow[d, two heads, "q_1"] \\ C \arrow[r, "g_1"] \arrow[ur, dashed, "a_1"] & D \end{tikzcd}$$

The lift exists since the category Ch($\mathbb{K}$) is a model category, Corollary 2.2.5.2. Here j_1 is a cofibration, while q_1 is an acyclic fibration, so the lift a_1 exists.

We now wish to extend this. Suppose that we have been able to create morphisms a_1 up to a_n, all satisfying the above points. A naive solution to make a_{n+1} is $b = f_{n+1}r^{\otimes n+1} + sg_{n+1} - sq_1 f_{n+1}r^{\otimes n+1}$, where $r : C \to A$ is a splitting of j_1 and $s : D \to B$ is a splitting of q_1. Notice that this morphism satisfies the two last points by definition. We will augment b to get an a_{n+1} which also satisfies (rel_{n+1}).

For our own convenience, let $-c(f_1, ..., f_n)$ denote the right hand side of (rel_{n+1}) formula. Since both j and q are strict ∞-morphisms we get the following identites:

$$(\partial b + c(a_1, ..., a_n)) \circ j_1 = \partial(b \circ j_1) + c(a_1 \circ j_1, ..., a_n \circ j_1) = \partial f_{n+1} + c(f_1, ..., f_n) = 0$$
$$q_1 \circ (\partial b + c(a_1, ..., a_n)) = \partial(q_1 \circ b) + c(q_1 \circ a_1, ..., q_1 \circ a_n) = \partial g_{n+1} + c(g_1, ..., g_n) = 0$$

We thus obtain that the cycle $\partial b + c(a_1, ..., a_n)$ factors through the cokernel of j and the kernel of q. Let us say that it factors like the diagram below:

$$C \xrightarrow{p} \text{Cok} j_1 \xrightarrow{c'} \text{Ker} q_1 \xrightarrow{i} D$$

Now, c' is a morphism between two A_∞-algebras. Since q is assumed to be an ∞-quasi-isomorphism, it follows that Kerq_1 is acyclic. Since c' is a cycle in $\text{Hom}^*_{\mathbb{K}}(\text{Cok} j_1, \text{Ker} q_1)$, it necessarily has to be in the image of the differential. Let h be a morphism such that $\partial h = c'$, and define $a_{n+1} = b - i \circ h \circ p$. One may check that this morphism satisfies all three properties.

We will now show **MC4**. Since the two properties have similar proofs, we will only show one direction. Let $f : A \rightsquigarrow B$ be an ∞-morphism, an $C = \text{cone}(id_{B[-1]})$, where the complex C is considered as an A_∞-algebra. Let $j : A \rightsquigarrow A \prod C$ be the morphism induced by id_A and $0 : A \to C$. The canonical projection $q_1 : A \oplus C \to B$ gives a lift of the following diagram.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j_1"] & B \arrow[d] \\ A \oplus C \arrow[r] \arrow[ur, "q_1"] & 0 \end{tikzcd}$$

Since we have a morphism of chain complexes lodged between an acyclic cofibration and a fibration, we use the same technique as above to construct an ∞-morphism $q : A \prod C \to B$. q is a fibration by construction. The morphism f may be factored as $f = q \circ j$, where j is an acyclic cofibration, and q is a fibration. □

This model structure can characterize the fibrant and cofibrant conilpotent dg-coalgebras.

Proposition 2.2.17. *Let C be a conilpotent dg-coalgebra. Then C is cofibrant, and C is fibrant if and only if there is a cochain complex V, such that $C \simeq T^c(V)$ as complexes.*

Proof. To see that C is cofibrant is the same as to verify that the map $\mathbb{K} \to C$ is a monomorphism, but this is clear.

We start by assuming that C is fibrant. Then there is a lift in the square below, making the unit split-mono.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "\eta_C"] & C \arrow[d, "\varepsilon_C"] \\ B\Omega C \arrow[r, "\varepsilon_{B\Omega C}"] \arrow[ur, dashed, "r"] & \mathbb{K} \end{tikzcd}$$

Define the morphism $p_1^C : C \to Fr_1 C$ as $p_1^C = Fr_1 r \circ p_1 \circ \eta_C$, where $p_1 : B\Omega C \to Fr_1 B\Omega C$ is the canonical projection on the filtration induced by the coradical filtration on C. The morphism r makes p_1 into a universal arrow in the category of conilpotent coalgebras, so $C \simeq T^c(\overline{Fr_1 C})$.

Assuming that C is isomorphic to $T^c(V)$ as coalgebras for some cochain complex V. Note that, by definition, C is an A_∞-algebra. We have a commutative square of A_∞-algebras. Since every A_∞-algebra is bifibrant, we know that this diagram has a lift, exhibiting C as a retract of $B\Omega C$.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] & C \arrow[d] \\ B\Omega C \arrow[r] \arrow[ur, dashed] & \mathbb{K} \end{tikzcd}$$

We know that ΩC is fibrant since the map $\Omega C \to \mathbb{K}$ is epi. By Lemma 2.2.11, we know that the bar construction preserves fibrations, so $B\Omega C$ is fibrant. Thus C is fibrant as well. □

The model structure of A_∞-algebras is compatible with the model structure of conilpotent dg-coalgebras in the following sense. If $f : A \rightsquigarrow A'$ is an ∞-morphism, we denote its dg-coalgebra counterpart as $Bf : BA \to BA'$. Remember that the bar construction is extended as an equivalence of categories on its image. We use this to realize Alg_∞ as a subcategory of $\mathsf{coAlg}_{\mathbb{K}}$ to obtain two different model structures on this category. The following proposition tells us that these structures do not differ.

Lemma 2.2.18. *Let A and A' be A_∞-algebras. Suppose that $f : A \rightsquigarrow A'$ is an ∞-morphism and $Bf : BA \to BA'$ is a graded quasi-isomorphism, then f is an ∞-quasi-isomorphism.*

Proof. Given $Bf : BA \to BA'$, we may reconstruct $f_i = s \circ \pi_{B[1]} Bf \circ (\omega \circ \iota_A)^{\otimes i}$.

We know that the unit η_{BA} is a graded quasi-isomorphism from Proposition 2.2.10. The inverse of the bar construction restricts this morphism to the first filtered degree, together with some shift; $B^{-1}\eta_{BA} : A \to \Omega BA$, which is is again a quasi-isomorphism by assumption. □

Proposition 2.2.19. *Let $f : A \rightsquigarrow A'$ be an ∞-morphism. Then we have the following:*

- *f is an ∞-quasi-isomorphism if and only if Bf is a weak equivalence.*
- *f_1 is a monomorphism if and only if Bf is a cofibration.*
- *f_1 is an epimorphism if and only if Bf is a fibration.*

Proof. Suppose that $f : A \to A'$ is an ∞-quasi-isomorphism. The Künneth theorem shows that $Bf : BA \to BA'$ is a graded quasi-isomorphism.

Suppose that $Bf : BA \to BA'$ is a weak equivalence. Then $B\Omega Bf : B\Omega BA \to B\Omega BA'$ is a graded quasi-isomorphism. By Proposition 2.2.10, we know that η_{BA} and $\eta_{BA'}$ are both graded quasi-isomorphism. By Lemma 2.2.18, we get that the ∞-morphisms ΩBf, $B^{-1}\eta_{BA}$ and $B^{-1}\eta_{BA'}$ are ∞-quasi-isomorphisms. By the 2-out-of-3 property, we get that f has to be as well.

The cofibrations of coAlg$^{\bullet}_{\mathbb{K},conil}$ are monomorphisms. Since B is an equivalence of categories, it must preserve and reflect monomorphisms.

Suppose that Bf is a fibration. Then it has RLP to acyclic cofibrations Bg. By the previous points, we know that g_1 is a quasi-isomorphism and a monomorphism; in particular, f has RLP to g.

Suppose that f_1 is an epimorphism and that there exists morphism fitting inside a commutative diagram as below.

$$\begin{tikzcd} BC \arrow[r, "Bh"] \arrow[d, tail, "Bg"] & BA \arrow[d, two heads, "Bf"] \\ BD \arrow[r, "Bi"] & BB \end{tikzcd}$$

Assume that Bg is an acyclic cofibration. We want to show that Bf has RLP to Bg, then Bf has to be a fibration. Notice that BA and BA' are fibrant, so the terminal morphism is a fibration. We find the lifting by considering the following diagram.

$$
\begin{tikzcd}
BC \arrow[rr, "Bh"] \arrow[dd, "Bg"] \arrow[dr, "B\eta_{BA}"] & & BA \arrow[dd, "Bf"] \\
& B\Omega BC \arrow[ur, dashed] \arrow[dd, tail, "B\Omega Bg"] & \\
BD \arrow[rr, "Bi"] \arrow[dr, "B\eta_{BD}"] & & BB \\
& B\Omega BD \arrow[uur, dashed] \arrow[ur, dashed] &
\end{tikzcd}
$$

□

2.3 The Homotopy Category of $\mathbf{Alg}_\infty$

We now have many different notions of homotopy, coming from either homological algebra or the model categorical structure. In the case for A_∞-algebras, these notions will luckily coincide.

Proposition 2.3.1 ([Proposition 1.3.4.1 Lef03, p. 49])**.** *Let C and D be two conilpotent dg-coalgebras, where $f, g : C \to D$ are two morphisms. Then:*

- *If $f \sim g$ by an (f, g)-coderivation h, then they are left homotopic.*
- *If D is fibrant, then $f \sim g$ by an (f, g)-coderivation if and only if f and g are left homotopic.*

Sketch of proof. We construct a cylinder object for C. Consider the cochain complex below, called I,

$$\cdots \longrightarrow \mathbb{K}\{e\} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{K}\{e_1, e_2\} \longrightarrow \cdots$$

concentrated in degree -1 and 0. Its comulitplication is given as

$$\Delta(e_0) = e_0 \otimes e_0, \quad \Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e) = e \otimes e_1 + e_0 \otimes e$$

The object $C \otimes I$ is now a cylinder object of C. To define a left homotopy from f to g is the same as finding a morphism H making the diagram below commute.

$$\begin{tikzcd}
C \arrow[d] \arrow[dr, dotted, "i_0"] \arrow[drr, bend left, "f"] & & \\
C \coprod C \arrow[r, "p_0"] & C \wedge I \arrow[r, "H"] & D \\
C \arrow[u] \arrow[ur, dotted, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Since we assume that f and g are homotopic, there is then an (f, g)-coderivation $h : C \to D$. To define H, there are essentially three different components we have to consider. Let H be defined as

$$H\mid_{C\otimes e_0} = f, \quad H\mid_{C\otimes e_1} = g, \text{ and } H\mid_{C\otimes e} = h$$

We see that this morphism respect the comulitplication, as h is an (f, g)-coderivation. We see that it respects the differential since $\partial h = f - g$, and that f and g are morphism of cochain complexes. Moreover, any such morphism $H : C \otimes I \to D$ defines an (f, g)-coderivation. This concludes that null homotopic morphisms are left homotopic.

To see it the other way around if D is fibrant, and the morphisms f and g are left homotopic, we may promote this homotopy to a homotopy $H : C \otimes I \to D$. The result follows by extracting the homotopy as $h = H\mid_{C\otimes e}$. □

Remark 2.3.2. In the category Alg_∞, we are now able to say that the homotopies as defined in Section 1.3 are exactly the model categorical homotopies. This follows from the fact that bifibrant objects may promote their left homotopies to right homotopies, and right homotopies to left homotopies. By the above proposition, we know as well that left homotopies, may be promoted to ordinary homotopies.

Due to this result, we may know think of homotopies to actually belong to the model categorical structure. We will make little distinction between these notions going forward.

Theorem 2.3.3. *In the category Alg_∞ we have the following:*

- *Homotopy equivalence is an equivalence relation.*
- *A morphism is an ∞-quasi-isomorphism if and only if it is a homotopy equivalence.*
- *By abuse of notation, let $\text{Alg}_{\mathbb{K}} \subseteq \text{Alg}_\infty$ be the full subcategory consisting of dg-algebras considered as A_∞-algebras. $\text{Alg}_{\mathbb{K}}$ has an induced homotopy equivalence from Alg_∞, and the inclusion $\text{Alg}_{\mathbb{K}} \to \text{Alg}_{\mathbb{K}} \subset \text{Alg}_\infty$ induces an equivalence in homotopy $\text{Alg}[Qis^{-1}] \simeq {}^{\text{Alg}_{\mathbb{K}}}\!/\!\sim$.*

Proof. We observe the first point from Corollary 2.1.28.2, and the second point is Whitehead's theorem, Theorem 2.1.30.

To see the final point, observe that the inclusion functor is given by the bar construction B. By Corollary 2.2.13.1, we know that the bar construction induces an equivalence on the homotopy categories, i.e., HoAlg $\simeq$ HocoAlg. Moreover, we know that by Theorem 2.1.31 that HocoAlg $\simeq$ $^{\text{Alg}_\infty}/\sim$. Notice that the image of B is $\text{Alg}_{\mathbb{K}}$, so in homotopy, we get that the image $^{\text{Alg}_{\mathbb{K}}}/\sim$ is equivalent to the essential image HoAlg_∞. $\square$

Chapter 3

Derived Categories of Strongly Homotopy Associative Algebras

In this chapter, we wish to study the derived categories of A_∞-algebras. This category lies at the heart of homological algebra, so it is only natural to ask what this category looks like in the case of an A_∞-algebra. In Chapter 2, we studied the relationship between the category of dg-algebras and dg-coalgebras to understand how quasi-isomorphisms between A_∞-algebras worked. In this chapter, we will instead examine the relationship between module and comodule categories to understand how quasi-isomorphisms between A_∞-modules will work. Twisting morphisms $\alpha : C \to A$ will reappear, allowing us to study the relationship between Mod^A and coMod^C.

From twisting morphisms we obtain functors $L_\alpha : \mathsf{coMod}^C \to \mathsf{Mod}^A$ and $R_\alpha : \mathsf{Mod}^A \to \mathsf{coMod}^C$, which creates an adjoint pair of functors. This adjoint pair will become a Quillen equivalence whenever the twisting morphism α is acyclic.

We wish to reuse all the methods we have gained and acquired throughout this thesis. The first part of this chapter will mostly be reformulations and recontextualizations of previous definitions, concepts, and techniques. In this chapter, we will mainly follow Lefèvre-Hasegawa [Lef03] to obtain our desired results.

3.1 Twisting Morphisms

Twisting morphisms were introduced in Chapter 1, representing the bar and cobar construction. We now want twisting morphisms and twisting tensors to play a more significant role. To define the functors L_α and R_α, the choice of a given twisting morphism will be crucial.

3.1.1 Twisted Tensor Products

Let A be an augmented dg-algebra, C a conilpotent dg-coalgebra, and $\alpha : C \to A$ a twisting morphism. In this chapter, we choose to switch the handedness of the twisted tensor. We make this choice in the hope that it lessens confusing notation. The right (left) twisted tensor product is the complex $A \otimes_\alpha C$ ($C \otimes_\alpha A$) together with the differential $d^\bullet_\alpha = d^\bullet_{C\otimes A} + d^r_\alpha$. Since we want the right twisted tensor to be associated with a right adjoint, we redefine the right perturbation as

$$d^r_\alpha = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C).$$

If M is a right A-module and N is a left C-comodule then the tensor product $M \otimes_{\mathbb{K}} N$ exists and is a $\mathbb{K}$-module with differential $d_{M\otimes N}$. We may define a perturbation to this differential as

$$d^r_\alpha = (\mu_M \otimes id_N) \circ (id_M \otimes \alpha \otimes id_N) \circ (id_M \otimes \nu_N).$$

By using the same line of thought as in Proposition 1.2.5, there is a twisted tensor product $M \otimes_\alpha N$ with differential $d^\bullet_\alpha = d_{M\otimes N} + d^r_\alpha$.

Remark 3.1.1. Adjointness forces us to define the differential of the left twisted tensor product as $d^\bullet_\alpha = d_{N\otimes M} - d^l_\alpha$. The necessity of this sign will be evident in the proof of Proposition 3.1.4.

Definition 3.1.2. Suppose that $M \in \mathsf{Mod}^A$ ($M \in \mathsf{Mod}_A$) and $N \in \mathsf{coMod}_C$ ($N \in \mathsf{coMod}^C$), then the left (right) twisted tensor product is the $\mathbb{K}$-module $M \otimes_\alpha N$ ($N \otimes_\alpha M$).

We see now that right-handedness and left-handedness for the twisted tensor product are distinct, as we only have an action or coaction from one of the chosen sides. Trying to force the other-handedness on the twisted tensors would be ill-defined.

Definition 3.1.3. Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra, such that there is a twisting morphism $\alpha : C \to A$. Given a linear map $f : N \to M$ between a right C-comodule N and a right A-module M we say that it is an α right twisted linear morphism if it satisfies

$$\partial f + f \star \alpha = 0.$$

If the handedness is unambiguous, we call it a twisted linear morphism.

This definition essentially describes a functor $\mathsf{Tw}^r_\alpha : \mathsf{coMod}^C \times \mathsf{Mod}^A \to \mathsf{Mod}_{\mathbb{K}}$, which is the collection of right twisted linear homomorphisms between a comodule and module.

Suppose that $\alpha : C \to A$ is a twisting morphism. Define the functor $L_\alpha = \_ \otimes_\alpha A : \mathsf{coMod}^C \to \mathsf{Mod}^A$ as an arbitrary right twisted tensor product with A. This functor hits Mod^A by using the free right A-module structure on A. Likewise, we define a functor $R_\alpha = \_ \otimes_\alpha C : \mathsf{Mod}^A \to \mathsf{coMod}^C$ as an arbitrary left twisted tensor product with C. This functor also hits right C-comodules by using the cofree right C-comodule structure on C.

Proposition 3.1.4. *Suppose that $\alpha : C \to A$ is a twisting morphism. The functor L_α and R_α form an adjoint pair of categories.*

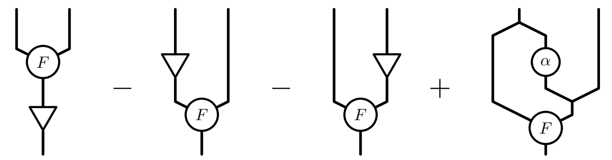
$$\begin{tikzcd} \textit{coMod}^C \arrow[r, bend left, "L_\alpha"] \arrow[r, phantom, "\perp"] & \textit{Mod}^A \arrow[l, bend left, "R_\alpha"] \end{tikzcd}$$

Proof. This proof boils down to showing coMod$^C(N, R_\alpha(M)) \simeq$ Tw$^r_\alpha(N, M) \simeq$ Mod$^A(L_\alpha(N), M)$, which is a routine calculation, much like the proof for Theorem 1.2.13.

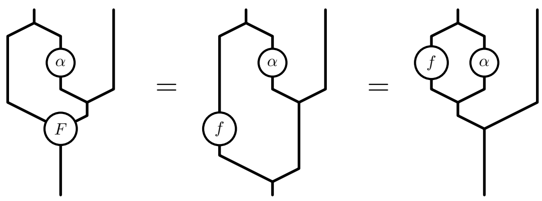
By Corollary 1.1.61.1, we have an isomorphism between $\mathbb{K}$-linear chain maps and A-linear chain maps,

$$f : N \to M \mapsto F = \mu_M(f \otimes A) : L_\alpha N \to M, \text{ and}$$
$$F : L_\alpha N \to M \mapsto f = (N \otimes 1_A)^* F : N \to M.$$

Consider first that we have an A-linear morphism $F : L_\alpha M \to M$. Then $\partial F = d_M f - f d_{L_\alpha N} = 0$. We write this electronically as

$\partial F =$  $= 0$

The first three circuits will together make up the ordinary differential of Mod$^A(N \otimes_{\mathbb{K}} A, M)$, so we will only need to consider the final circuit. Replacing F with $\mu_M(f \otimes id_A)$ we get

 $= \mu_M((f \star \alpha) \otimes A)$

By sending in the identity of A in the rightmost string at each summand, we get the condition that f is α right twisted,

$$\partial f + f \star \alpha = 0.$$

This is because $d_A(1_A) = 0$, $\mu_M(m \cdot 1_?) = m$, and $f(m) = F(m, 1_A)$.

By the abovementioned isomorphism, we are now able to deduce that any α right twisted morphism $f : N \to M$ defines an A-linear morphism $F = \mu_M(f \otimes A) : L_\alpha N \to M$.

Notice that by turning every circuit upside-down, we get equations establishing the natural isomorphism

$$Tw^r_\alpha(N, M) \simeq \mathsf{coMod}^C(N, R_\alpha M).$$

□

Let A be a dg-algebra, and M a right A-module. Recall that by the cobar-bar adjunction, Theorem 1.2.13, there exists a universal twisting morphism $\pi_A : BA \to A$. We define the bar construction of M as $B_A M = R_{\pi_A} M = M \otimes_{\pi_A} BA$. Likewise, given a conilpotent dg-coalgebra C and N a right C-comodule we define the cobar construction as $\Omega_C N = L_{\iota_C} N = N \otimes_{\iota_C} \Omega C$. In these cases we obtain adjunctions $\Omega_{BA} \dashv B_A$ and $\Omega_C \dashv B_{\Omega C}$.

Let A and B be two algebras, and $f : A \to B$ is an algebra morphism. Then f induces a functor between the module categories by restriction: $f^* : \mathsf{Mod}^B \to \mathsf{Mod}^A$. Since A and B considered as categories are small, and the category of abelian groups is cocomplete, the left Kan extension (induction) along this functor exists. This result can, for instance, be found in Riehl [Rie14].

$$\begin{tikzcd} \mathsf{Mod}^B \arrow[r, "f^*"] & \mathsf{Mod}^A \arrow[l, bend right, "f_!"'] \end{tikzcd}$$

Dually, if C and D are two coalgebras and $g : C \to D$ is a coalgebra morphism. Then g induces a functor between the module categories by composing: $g* : \mathsf{coMod}^C \to \mathsf{coMod}^D$. Since C and D considered as categories are small, and the category of abelian groups is complete, the right Kan extension (coinduction) along this functor exists.

$$\begin{tikzcd} \mathsf{coMod}^C \arrow[r, "g*"] & \mathsf{coMod}^D \arrow[l, bend left, "g^!"] \end{tikzcd}$$

Lemma 3.1.5. *Let $\tau : C \to A$ be a twisting morphism. The adjunction (L_τ, R_τ) factors as $(f_{\tau!}, f^*_\tau) \circ (L_{\iota_C}, R_{\iota_C})$ or $(L_{\pi_A}, R_{\pi_A}) \circ (g_{\tau*}, g^!_\tau)$.*

Proof. This follows from Corollary 1.2.14.1, that is $\tau = f_\tau \circ \iota_C = \pi_A \circ g_\tau$. □

Definition 3.1.6. A twisting morphism $f : C \to A$ is called acyclic if the counit of the adjunction $L_\alpha \dashv R_\alpha$ is a pointwise quasi-isomorphism.

Lemma 3.1.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The universal twisting morphisms π_A and ι_C are acyclic.*

Proof. We start with π_A. Recall that π_A is constructed as the twisting morphism corresponding to id_{BA}. This morphism is then the projection onto the first dimension of BA, that is:

$$\pi_A sa = a$$
$$\pi_A(sa \otimes ...) = 0$$

We say that π_A is acyclic if the counit $\varepsilon : L_{\pi_A} R_{\pi_A} \Rightarrow Id_{\mathsf{Mod}^A}$ at each object M is a quasi-isomorphism.

For each M in Mod^A, $L_{\pi_A} R_{\pi_A} M = M \otimes_{\pi_A} BA \otimes_{\pi_A} A$. We may split the differential into two summands, d_v and d_h. d_v is the ordinary differential on the tensor product, while $d_h = (-d^l_{\pi_A} \otimes A) + M \otimes d_2 \otimes A + d^r_{\pi_A}$. Since $(d_v + d_h)^2 = 0$ and $d_v^2 = 0$ we can observe that $d_v d_h = -d_h d_v$ and $d_h^2 = 0$. We may see this as d_v changes the homological degree while d_h does not, so if the two first equations are true, the last two must be true. We obtain an anticommutative double complex.

$$\begin{tikzcd} \cdots \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow[loop above, "d_v"] \arrow[r, "d_h"] & \cdots \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow[loop above, "d_v"] \arrow[r, "d_h"] & M \otimes A \arrow[loop above, "d_v"] \arrow[r, "0"] & 0 \end{tikzcd}$$

The total complex of this anticommutative double complex is $L_{\pi_A} R_{\pi_A} M$. Moreover, the counit induces an augmentation to this complex resolution of M, denoted as cone(ε_M).

$$\begin{tikzcd} \cdots \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow[loop above, "d_{M\otimes BA_{(i)}\otimes A}"] \arrow[r, "d_h"] & \cdots \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow[loop above, "d_{M\otimes BA_{(1)}\otimes A}"] \arrow[r, "d_h"] & M \otimes A \arrow[loop above, "d_{M\otimes A}"] \arrow[r, "\varepsilon_M"] & M \arrow[loop above, "d_M"] \arrow[r, "0"] & 0 \end{tikzcd}$$

To see that this is a resolution, we define a morphism $h : \text{cone}(\varepsilon_M) \to \text{cone}(\varepsilon_M)$ of degree -1. It works by the following formula:

$$h(m \otimes (sa_1 \otimes ... \otimes sa_n) \otimes a) = m \otimes (sa_1 \otimes ... \otimes sa_n \otimes sa) \otimes 1$$

It is clear that $id_{\text{cone}(\varepsilon_M)} = d_h h - h d_h$ and $d_v h = h d_v$. Thus to see that the cone is acyclic we let $c \in \text{cone}(\varepsilon_M)$ be a cycle, that is $(d_v + d_h)(c) = 0$. Our goal is to show that $h(c)$ is a preimage of c along $d_v + d_h$.

$$(d_v + d_h) \circ h(c) = d_v \circ h(c) + d_h \circ h(c) = h \circ d_v(c) + c + h \circ d_h(c) = h \circ (d_v + d_h)(c) + c = c$$

Next up, we show that ι_C is acyclic. Equipping C with its coradical filtration induces a filtration $F_p \Omega C$. We will freely use $|\_|$ to denote the filtered degree of every element.

$$Fr_p C = \{c \mid |c| \leqslant p\}$$
$$f_p \Omega C = \{\langle c_1 \mid \cdots \mid c_n \rangle \mid |c_1| + \cdots + |c_n| \leqslant p\}$$

Let $M \in \mathrm{Mod}^{\Omega C}$, we equip this module with a trivial filtration,

$$F_p M = M.$$

M's associated graded is then quite trivial, $\mathrm{gr}_0 M \simeq M$ and every other $\simeq 0$.

All of these three filtrations together induces a filtration on $L_{\iota_C} R_{\iota_C} M$,

$$F_p L_{\iota_C} R_{\iota_C} M = \{m \otimes c \otimes \langle c_1 \mid \cdots \mid c_n\rangle \mid |m| + |c| + |c_1| + \cdots + |c_n| \leqslant p\}.$$

We calculate the associated graded of this module.

$$\begin{aligned}
\mathrm{gr}_0 L_{\iota_C} R_{\iota_C} M &\simeq M \\
\mathrm{gr}_p L_{\iota_C} R_{\iota_C} M &\simeq \bigoplus_{i_1+i_2=p} M \otimes \mathrm{gr}_{i_1} C \otimes \mathrm{gr}_{i_2} \Omega C
\end{aligned}$$

The graded counit $\mathrm{gr}_p \varepsilon : \mathrm{gr}_p L_{\iota_C} R_{\iota_C} M \to \mathrm{gr}_p M$ becomes the identity on M when $p = 0$. To see that $\mathrm{gr}\varepsilon$ is a quasi-isomorphism, it is enough to show that $\mathrm{gr}_p L_{\iota_C} R_{\iota_C} M$ is acyclic for every $p \geqslant 1$.

Consider the graded differential component $\mathrm{gr}_p d^l_{\iota_C}$ when it acts as a morphism $\mathrm{gr}_{i_1} C \otimes \mathrm{gr}_{i_2} \Omega C \to \mathrm{gr}_{i_1+i_2} \Omega C$, which can be considered a morphism

$$\rho : \bigoplus_{i_1+i_2=p} \mathrm{gr}_{i_1} C[-1] \otimes \mathrm{gr}_{i_2} \Omega C \to \mathrm{gr}_p \Omega C,$$

which is an isomorphism by reversing the operation.

$$\begin{aligned}
\rho(sc \otimes \langle \cdots \rangle) &= \langle c \mid \cdots \rangle, \\
\rho^{-1}(\langle c \mid \cdots \rangle) &= sc \otimes \langle \cdots \rangle.
\end{aligned}$$

Since ρ is an isomorphism, $\mathrm{cone}(\rho)$ is then acyclic. By construction, we have that $\mathrm{cone}(\rho) \simeq \mathrm{gr}_p L_{\iota_C} R_{\iota_C} M$.

□

3.1.2 Model Structure on Module Categories

Let A be an augmented dg-algebra. By Corollary 2.2.5.2, we have a model structure on Mod^A defined as follows:

- $f \in \mathrm{Ac}$ is a weak equivalence if f is a quasi-isomorphism,
- $f \in \mathrm{Fib}$ is a fibration if $f^\#$ is an epimorphism,
- $f \in \mathrm{Cof}$ is a cofibration if it has LLP to acyclic fibrations.

Every object in this category is fibrant as the morphism $0 : M \to 0$ is always an epimorphism.

3.1.3 Model Structure on Comodule Categories

Unless stated otherwise, in this section, we fix A to be an augmented dg-algebra, C as a conilpotent dg-coalgebra, and $\tau : C \to A$ as an acyclic twisting morphism. We endow $\mathsf{coMod}^C_{\mathsf{conil}}$ with three classes of morphisms:

- $f \in$ Ac is a weak equivalence if $L_\tau f$ is a quasi-isomorphism.
- $f \in$ Cof is a cofibration if $f^\#$ is a monomorphism.
- $f \in$ Fib is a fibration if it har RLP to acyclic cofibrations.

Theorem 3.1.8. *The category $\mathsf{coMod}^C_{\mathsf{conil}}$ with the three classes as above form a model category. Every object is cofibrant, and those objects, which is a direct summand of $R_\tau M$ for some $M \in \mathsf{Mod}^A$, are fibrant. The adjoint pair (L_τ, R_τ) is a Quilllen equivalence.*

We will call this model structure for the canonical model structure on $\mathsf{coMod}^C_{\mathsf{conil}}$. Under the hypothesis of this theorem, we may observe that every object of $\mathsf{coMod}^C_{\mathsf{conil}}$ is cofibrant. Since every $M \in \mathsf{Mod}^A$ is fibrant, and R_τ preserves fibrant objects, we know that $R_\tau M$ is fibrant as well. By the retract argument, every direct summand of $R_\tau M$ is fibrant. If $N \in \mathsf{coMod}^C_{\mathsf{conil}}$ is fibrant, then it is a direct summand of $R_\tau L_\tau N$, which shows that the bifibrant objects of $\mathsf{coMod}^C_{\mathsf{conil}}$ is exactly the thick image of R_τ.

To be able to prove this, we will need some lemmata. This proof is essentially the same as the case for dg-coalgebras. The main difference is to show independence of the choice of twisting morphisms τ. To this end, we must establish the relationship between graded quasi-isomorphisms and weak equivalences and a technical lemma.

Recall that given a coaugmented coalgebra C, we have a filtration called the coradical filtration, defined as $Fr_i C = \mathrm{Ker}(\bar{\Delta}_C)^i$. If N is a right C-comodule we may define the coradical filtration of N as $Fr_i N = \mathrm{Ker}(\bar{\omega}^i_N)$. This filtration is admissable, meaning it is exhaustive and $Fr_0 N = 0$.

Lemma 3.1.9. *Let C be a conilpotent dg-coalgebra, M and N be right C-comodules. Then any graded quasi-isomorphism $f : M \to N$ is a weak equivalence.*

Proof. This proof is identical to Lemma 2.2.8. □

Lemma 3.1.10. *Let M and N be two objects of Mod^A. The functor R_τ sends a quasi-isomorphism $f : M \to N$ to a weak equivalence $R_\tau f : R_\tau M \to R_\tau N$.*

The unit of the adjunction $\eta : \mathrm{Id}_{\mathsf{coMod}^C} \to R_\tau L_\tau$ is a pointwise weak equivalence.

Proof. $R_\tau f$ is a weak equivalence if $L_\tau R_\tau f$ is a quasi-isomorphism. By the naturality of the counit, we have the following commutative diagram.

$$\begin{tikzcd} M & L_\tau R_\tau M \arrow[l, "\varepsilon_M"'] \arrow[d, "L_\tau R_\tau f"] \\ N \arrow[from=u, "f"] & L_\tau R_\tau N \arrow[l, "\varepsilon_N"] \end{tikzcd}$$

From the assumption, we know that all three of f, ε_M, and ε_N are quasi-isomorphisms. It follows by the 2-out-of-3 property that $L_\tau R_\tau f$ is also a quasi-isomorphism.

To show that $\eta : \mathrm{Id}_{\mathrm{coMod}} \to L_\tau R_\tau$ is a pointwise weak equivalence, we must show that $L\eta$ is a pointwise quasi-isomorphism. Since L_τ is left adjoint to R_τ we know that η is split on the image of L_τ, i.e.

$$\varepsilon_{L_\tau} \circ L_\tau \eta = id_{L_\tau}$$

Since we know that the natural isomorphisms ε and id are pointwise quasi-isomorphisms, we get by the 2-out-of-3 property that $L\eta$ is a pointwise quasi-isomorphism as well. □

Lemma 3.1.11. *The functor L_τ preserves cofibrations and sends weak equivalences to quasi-isomorphisms.*

Proof. This proof is essentially the same as Lemma 2.2.11. □

With the above lemmata, we have now established that the adjunction (L_τ, R_τ) forms a Quillen equivalence if coMod^C is a model category.

Lemma 3.1.12 ([Lemma 2.2.2.9 Lef03, p. 74])**.** *Let M be a right A-module and N a right C-comodule. Let $p : M \to L_\tau N$ be a fibration of modules. The projection $j : R_\tau M \prod_{R_\tau L_\tau N} N \to R_\tau M$ is an acyclic cofibration of comodules.*

Proof. Let $K = \mathrm{Ker}p$. Then since R_τ is a right adjoint, it preserves kernels, so $R_\tau K \simeq \mathrm{Ker}R_\tau p$. Consider the pullback square with the horizontal kernels

$$\begin{tikzcd} R_\tau K \arrow[r, tail] \arrow[d, "\simeq"] & RM \prod_{R_\tau L_\tau N} N \arrow[r, two heads] \arrow[d, "j"] & N \arrow[d, "\eta_N"] \\ R_\tau K \arrow[r, tail] & R_\tau M \arrow[r] & R_\tau L_\tau N \end{tikzcd}$$

Since $L_\tau N$ is a quasi-free module, we get that $M \simeq K \oplus L_\tau N$ as a graded module. In other words, the short exact sequences above are split when considered as graded sequences. If we apply L_τ this sequence, then L_τ has to preserve exactness at the graded level since it is additive. Thus we obtain a morphism of exact sequences, and $L_\tau j$ is a quasi-isomorphism by 5-Lemma.

$$\begin{tikzcd}
L_\tau R_\tau K \arrow[r, tail] \arrow[d, "\simeq"] & L_\tau(RM \prod_{R_\tau L_\tau N} N) \arrow[r, two heads] \arrow[d, "L_\tau j"] & L_\tau N \arrow[d, "\eta_N"] \\
L_\tau R_\tau K \arrow[r, tail] & L_\tau R_\tau M \arrow[r] & L_\tau R_\tau L_\tau N
\end{tikzcd}$$

□

Proof of Theorem 3.1.8. With the above lemmata established, this proof is identical to the proof of Theorem 2.2.13. □

3.1.4 Triangulation of Homotopy Categories

In this section, we will show that the homotopy categories are triangulated. If we look at the category Mod^A, we will observe that the category HoMod^A is the derived category $\mathcal{D}(A)$. It is not the same for the category coMod^C. Here we want HocoMod^C to be equivalent to the derived category of a ring, so we will see that the derived category is a further localization of HocoMod^C.

Furthermore, by employing the theory of triangulated categories, we will show that the model structure on coMod^C is independent of the choice of acyclic twisting morphism. Thus, every acyclic twisting morphism induces an equivalence between derived categories, as done by Keller in [Kel94].

Mod^A is an abelian category, where we employ the maximal exact structure $\mathcal{E}'$ consisting of short exact sequences in Mod^A. In other words, these short exact sequences are those which are degree-wise short exact. However, this category also has an exact structure $\mathcal{E}$, which makes Mod^A into a Frobenius category, which we will now describe.

Let $f : M \to N$ be a chain map from M to N. Then $\mathcal{E}$ contains a conflation on the form:

$$N \rightarrowtail \text{cone}(f) \twoheadrightarrow M[1]$$

We define $\mathcal{E}$ as the smallest exact structure on Mod^A, which contains every conflation arising from a chain map f. Observe that these conflations are exactly the short exact sequences of Mod^A such that they are split when regarded as graded modules, i.e., forgetting the differential. Thus the smallest such $\mathcal{E}$ is exactly the collection of every conflation arising from a chain map f.

Recall that an object M is projective (injective) if the represented functor Mod$^A(M, _)$ (Mod$^A(_, M)$) is exact. For the category (Mod$^A, \mathcal{E}$)

Proposition 3.1.13. *Let M be an object of Mod^A. The following are equivalent:*

- *M is $\mathcal{E}$-projective*
- *M is $\mathcal{E}$-injective*
- *M is contractible*

Proof. This proposition is a well-known statement from literature. See Krause [Kra21], Happel [Hap88], or Bühler [Büh10] for an account of this result. □

To see that $(\mathsf{Mod}^A, \mathcal{E})$ has both enough projectives and injectives, we consider the following conflation:

$$M \rightarrowtail \mathsf{cone}(id_M) \twoheadrightarrow M[1]$$

The complex $\mathsf{cone}(id_M)$ is contractible for any complex M. By letting M vary, we can find inflation or deflation from the identity cone to or from any complex. This concludes that $(\mathsf{Mod}^A, \mathcal{E})$ is a Frobenius category.

Let $\overline{\mathsf{Mod}}^A$ denote the injectively stable module category. Let $I(M,N)$ denote the set of chain maps from M to N, which factors through an injective object. We define the injectively stable category as the quotient of abelian groups $\overline{\mathsf{Mod}}^A(M,N) = {}^{\mathsf{Mod}^A(M,N)}/_{I(M,N)}$.

Theorem 3.1.14. *Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category, then the injectively stable category $\overline{\mathcal{C}}$ is triangulated. The additive auto-equivalence is given by cosyzygy, and the standard triangles are the conflations' images into the quotient.*

Proof. This theorem is well-known in the literature. An account for it may also be found in Krause [Kra21], Happel [Hap88], or Bühler [Büh10]. □

We thus obtain a triangulated category $\overline{\mathsf{Mod}}^A$ associated to the Frobenius pair $(\mathsf{Mod}^A, \mathcal{E})$. This category is commonly denoted as $K(A)$, and we will do this as well. Notice that with the structure given by $\mathcal{E}$, the cosyzygy is defined by the shift functor $\_[1]$. Every standard triangle is also on the form:

$$M \xrightarrow{f} N \longrightarrow \mathsf{cone}(f) \longrightarrow M[1]$$

To define the derived category $D(A)$ of A we will consider the localization of $K(A)$ at the quasi-isomorphisms, $D(A) = K(A)[\mathsf{Qis}^{-1}]$. To see that the derived category is triangulated, we realize it as a Verdier quotient of $K(A)$.

Proposition 3.1.15. *The derived category of A is equivalent to the Verdier quotient ${}^{K(A)}/_{\mathsf{Ac}}$, where Ac denotes the image of acyclic objects in $K(A)$.*

Proof. Proof may be found in Bühler [Büh10]. □

There is another way of telling the story of the derived category $D(A)$. That is to localize it at the quasi-isomorphisms directly. We may directly see that $D(A) \simeq \mathsf{Mod}^A[\mathsf{Qis}^{-1}]$ which we know is HoMod^A by definition.

Theorem 3.1.16. *The homotopy category of Mod^A is triangulated; moreover, it is the derived category $D(A)$.*

Proof. This theorem follows from the discussion above. □

The triangulated construction for the category $\mathsf{HocoMod}^C$ closely resembles that of HoMod^A. We start by studying the Frobenius pair $(\mathsf{coMod}^C, \mathcal{E})$, where $\mathcal{E}$ is the same exact structure. Notice that this exact structure only considers the underlying category of chain complexes, so this follows from the above description.

We define the injectively stable category $\overline{\mathsf{coMod}}^C = K(C)$ in the same manner. The standard triangles and the additive auto-equivalence stay the same.

At this point, things start to differ. The definition for the homotopy category $\mathsf{HocoMod}^C$ is $\mathsf{coMod}^C[\mathsf{Ac}^{-1}]$, here Ac denotes the class of weak equivalences in coMod^C. By abuse of notation, we also let $\mathsf{Ac} \subset K(C)$ be the collection of objects which are cones of weak equivalences. This subcategory is equivalent to the preimage of acyclic objects $\mathsf{Ac} \subset K(A)$ along $L_\tau : \mathsf{coMod}^C \to \mathsf{Mod}^A$. To see this, look at the image of the triangle where the cone is in Ac. For this identification, it suffices to show that $\mathsf{Ac} \subset K(C)$ is a triangulated subcategory. In this manner, $\mathsf{HocoMod}^C$ is the category ${}^{K(C)}/_{Ac}$, which is a triangulated category.

Remark 3.1.17. We may show that $\mathsf{Ac} \subset K(C)$ is a subcategory of acyclic objects, and we get that $D(C) \simeq \mathsf{HocoMod}^C[\mathsf{Qis}^{-1}]$. This is done in Lefevre-Hasegawa as [Proposition 1.3.5.1 Lef03, p. 51] [Lemma 2.2.2.11 Lef03, p. 75]. This result follows from the fact that we have an equivalence of categories $\mathsf{coMod}^C[\mathsf{fQis}^{-1}] \simeq \mathsf{HocoMod}^C$, where fQis means the collection of graded quasi-isomorphisms. Since every graded quasi-isomorphism is a quasi-isomorphism, we get the inclusion of triangulated subcategories $\langle \mathsf{cone}(\mathsf{fQis})\rangle \subseteq \langle \mathsf{cone}(\mathsf{Qis})\rangle \subseteq K(C)$.

Let $\tau : C \to A$ and $\upsilon : C \to A'$ be two acyclic twisting morphisms. These independently defines two different model structures on coMod^C by the adjunctions (L_τ, R_τ) and (L_υ, R_υ). By Lemma 3.1.5 we have the identification $(L_\tau, R_\tau) = (f_{\tau!}, f_\tau^*)(L_{\iota_C}, R_{\iota_C}) = (f_{\tau!}L_{\iota_C}, R_{\iota_C} f_\tau^*)$, and likewise for υ. To show that τ and υ define equivalent model structures on coMod^C, it is enough that both define the same structure as ι_C. By symmetry, we may assume that $\upsilon = \iota_C$. From Lemma 3.1.7, we know that ι_C is acyclic, so this assumption is well-founded.

Since we already know that (L_τ, R_τ) and $(L_{\iota_C}, R_{\iota_C})$ are Quillen equivalences, it remains to show that $(f_{\tau!}, f_\tau^*)$ is a Quillen equivalence. We get this if f_τ^* is a right Quillen functor, and it induces a triangle equivalence between $D(A)$ and $D(\Omega C)$.

We know that f_{τ}^{*} preserves fibrations (epimorphisms) because, on morphisms, this functor acts as the identity. It only changes the ring action, so epimorphisms stay epimorphisms.

It remains to show that the functor preserves quasi-isomorphisms, and we will show this by identifying the derived categories. We follow the methods given by Keller in [Kel94].

Let A be a dg-algebra. A is then free in the enriched sense; i.e. for any right A-module M, $\mathrm{Hom}^{\bullet}_{A}(A, M) \simeq M$. Recall that P is projective if it is a direct summand of A^n for some $n \in \mathbb{N}$.

Given a right bounded complex M, we know how to construct a projective resolution $p : pM \to M$. Associated with this resolution is a triangle in $K(\mathbb{K})$ consisting of the complexes M, pM, and aM, where aM is an acyclic complex.

$$M \xrightarrow{p} pM \longrightarrow aM \longrightarrow M[1]$$

In this sense, we obtain an identification $M \simeq pM$ in $D(\mathbb{K})^{-}$. By following Keller's construction, we can weaken this identification to all of $D(\mathbb{K})$ by weakening the structure of the projective resolution. In Keller's paper, he calls these complexes of property (P). We will refer to them as homotopically projective complexes since they are built up from projective complexes in a manner respecting homotopy colimits.

Definition 3.1.18. Let P be a complex of Mod^{A}. We say that P is homotopically projective if there exists a complex P', a homotopy equivalence $P \simeq P'$ and a filtration of P'.

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots \subseteq P'$$

The filtration should satisfy these properties:

(F1) P' is the colimit of the filtration.
(F2) Each inclusion $i_n : F_n \subseteq F_{n+1}$ is split as graded modules.
(F3) The quotient ${F_{n+1}}/{F_n}$ is projective.

Remark 3.1.19. The properties (F1) and (F2) may be reformulated to require that P' should be the homotopy colimit of the filtration, see Krause [Kra21]. Thus there is a canonical triangle in $K(A)$:

$$\bigoplus F_n \xrightarrow{\Phi} \bigoplus F_n \longrightarrow P' \longrightarrow \bigoplus F_p[1]$$

Φ is the unique morphism that acts as the identity and the inclusion on each summand of $\bigoplus F_p$:

$$\Phi_n = \begin{pmatrix} id_{F_n} \\ -i_n \end{pmatrix}$$

In defining a homotopically projective complex, we have required that each quotient is strictly projective. If only this were true, these objects would be ill-behaved in the homotopy category. We can weaken this assumption to (F3'): the quotient ${}^{F_{n+1}}/_{F_n}$ is homotopy equivalent to a projective complex.

Lemma 3.1.20. *If P is the colimit of a filtration admitting (F2) and (F3'), then P is homotopically projective.*

Proof. Let $\{F_n\}$ denote the filtration on P. Showing that P is homotopically projective is the same as finding a homotopy equivalence to a complex P', such that P' is the homotopy colimit of a filtration admitting (F3).

Suppose that ${}^{F_{n+1}}/_{F_n} \simeq Q_{n+1}$, where each Q_{n+1} is projective. We wish to inductively define a filtration $\{F'_n\}$ which has (F2) and (F3) and a pointwise homotopy equivalence of filtrations $f : \{F_n\} \to \{F'_n\}$. The object P' is defined as the homotopy colimit of this new filtration.

Define $F'_0 = Q_0$, and let $f_0 : F_0 \to F'_0$ be the projection onto Q_0. By assumption f_0 is a homotopy equivalence, and we have a commutative square where the vertical arrows are homotopy equivalences. Moreover, each horizontal arrow splits as a graded arrow.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & F_0 \arrow[d, "f_0"] \\
0 \arrow[r, "0"] & Q_0
\end{tikzcd}$$

Suppose that we can construct this filtration up to F'_p. By using our known homotopy equivalences, there is an isomorphism of Ext groups:

$$\mathrm{Ext}_A({}^{F_p}/_{F_{p-1}}, F_{p-1}) \simeq \mathrm{Ext}_A(Q_p, F'_{p-1})$$

Given the triangle consisting of F_{p-1}, F_p and ${}^{F_p}/_{F_{p-1}}$ there is an associated triangle with the morphisms as follows:

$$\begin{tikzcd}
F_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & F_p \arrow[r] \arrow[d, dashed] & {}^{F_p}/_{F_{p-1}} \arrow[r] \arrow[d, "\sim"] & F_{p-1}[1] \arrow[d, "f_{p-1}[1]"] \\
F'_{p-1} \arrow[r] & F'_p \arrow[r] & Q_p \arrow[r] & F'_{p-1}
\end{tikzcd}$$

By the morphism axiom, there is a morphism $f_p : F_p \to F'_p$, which is also a homotopy equivalence by the 2-out-of-3 property.

This defines a filtration $\{F'_p\}$, with (F3) and P' as its homotopy colimit. To see that P is homotopy equivalent to P', we use the maps f_p constructed to obtain a homotopy equivalence by the morphism axiom and the 2-out-of-3 property.

$$\begin{tikzcd} \bigoplus F_p \arrow[r, "\Phi"] \arrow[d, "\oplus f_p"] & \bigoplus F_p \arrow[r] \arrow[d, "\oplus f_p"] & P \arrow[r] \arrow[d, dashed, "\sim"] & \bigoplus F_p[1] \arrow[d, "\oplus f_p[1]"] \\ \bigoplus F'_p \arrow[r, "\Phi'"] & \bigoplus F'_p \arrow[r] & P' \arrow[r] & \bigoplus F'_p[1] \end{tikzcd}$$

□

The projective complexes are the complexes generated by the free module A in the sense that they are all in the smallest thick triangulated subcategory of $K(A)$ containing A. By definition, we may see that the homotopically projective complexes are the complexes in the smallest thick triangulated subcategory of $K(A)$, which is closed under well-ordered homotopy colimits and contains $K(A)$. By devissage we may extend the fully faithful property of functors on the set $\{A\}$ to the class of homotopically projective objects.

Lemma 3.1.21 (Devissage)**.** *Let $F : \mathcal{T} \to \mathcal{U}$ be a triangulated functor between triangulated categories, which commutes with arbitrary coproducts. Suppose $S \subseteq \mathcal{T}$ is a class of objects closed under shift, and denote $\langle S \rangle$ for the smallest thick triangulated subcategory (closed under well-ordered homotopy colimits). If $F|_S$ is fully faithful, then $F|_{\langle S \rangle}$ is fully faithful as well.*

Proof. The first part follows from Yoneda's lemma, Yoneda embeddings, and the 5-lemma. More details may be found in [Kra21].

To get closed under homotopy colimits, we also need that F commutes with infinite direct sums and that the set $\{S\}$ only contains small objects. □

Lemma 3.1.22. *Suppose we have F and S as above. If $F|_S = 0$, then it is 0 on all of $\langle S \rangle$.*

Proof. The same argument as above, except we have to squeeze out zeros from exact sequences. □

The acyclic assembly lemma is the final ingredient to construct a homotopically projective resolution for our complexes.

Lemma 3.1.23 (Acyclic assembly, [Lemma 2.7.3 Wei94, p. 59])**.** *Suppose that C is a double complex of R-modules. Then $Tot^{\oplus}C$ is acyclic if either:*

- *C is a lower half-plane complex with exact rows.*
- *C is a left half-plane complex with exact columns.*

Proof. We omit the proof as the following proof is in some sense very similar. $\square$

Corollary 3.1.23.1. *Suppose that C is a double complex of R-modules such that every column is exact and that the kernels along the rows give rise to exact columns, then $Tot^{\oplus}C$ is acyclic.*

Proof. We want to realize the images along the rows as the coimage along the horizontal differential. Write $Z^n(C)$ for the n-th horizontal kernel and $B^n(C)$ for the n-th horizontal image. We have a short exact sequence of complexes:

$$Z^n(C)^* \longrightarrow C^{n,*} \longrightarrow B^n(C)^*$$

Given that $C^{n,*}$ is acyclic, we get that $Z^n(C)^*$ is acyclic if and only if $B^n(C)^*$ is acyclic.

Assuming that all of these three constructions are acyclic, we make a filtration on C. Let $F_nC^{p,*} = C$ if $p \in [-n, n-1]$, $F_nC^{n,*} = Z^nC$ and $F_nC^{p,*} = 0$ otherwise.

This filtration is bounded below and exhaustive as colimits commute with colimits.

$$Tot^{\oplus}C = Tot^{\oplus} \varinjlim F_nC \simeq \varinjlim Tot^{\oplus}F_nC$$

We should be a bit careful here as the total complex is not a coproduct, but since coproducts and cokernels are calculated pointwise, we obtain the commutativity.

We apply the classical convergence theorem to the filtration to obtain a converging spectral sequence $EF_2C \implies H^*(Tot^{\oplus}C)$, but since we assume each column to be exact in the filtration, the second page is 0, so $H^*(Tot^{\oplus}C) \simeq 0$ as desired. $\square$

Theorem 3.1.24. *Suppose that P is homotopically projective, and N is acyclic. Then $K(A)(P, N) \simeq 0$.*

Given any module M, there is a homotopically projective object pM and an acyclic object aM, giving rise to a triangle in $K(A)$.

$$pM \longrightarrow M \longrightarrow aM \longrightarrow pM[1]$$

Proof. We assume that $P \simeq A$. By a devissage argument we may extend the isomorphism to all homotopically projective P.

$$K(A)(A, N) \simeq H^0\mathrm{Hom}^\bullet_A(A, N) \simeq H^0N \simeq 0$$

We want to construct two complexes, pM and aM, by taking the total complexes. We show that aM is acyclic by using Corollary 3.1.23.1. We will construct an exact sequence of complexes satisfying the assumptions to be able to use the corollary. As described by Mac Lane [Mac95], there is an exact structure $\mathcal{E}$ on Mod^R such that the collections on conflations are the short exact sequences such that the kernel functor is exact.

$$L \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} N$$

$$Z^*L \overset{Z^*f}{\rightarrowtail} Z^*M \overset{Z^*g}{\twoheadrightarrow} Z^*N$$

Since limits commute with limits, the kernel functor preserves any limit. Thus the kernel is left exact, and its only obstruction for exactness is to preserve cokernels. We may thus characterize the conflations by inflations and deflations, which are monomorphisms and epimorphisms preserved by the kernel functor. Mac Lane calls these deflations for proper epimorphisms instead.

We want to construct $\mathcal{E}$-projectives to be on the form of homotopically projective complexes. $A[-n]$ is $\mathcal{E}$-projective by the following isomorphism,

$$Z^0\mathsf{Hom}^\bullet_A(A[-n], M) \simeq M^n.$$

Define the trivialization $\mathsf{triv}M$ of M be the underlying graded module M endowed with a trivial differential. This trivial differential is the inclusion of graded modules into chain complexes. Thus we have the following isomorphism on hom-sets:

$$Z^i\mathsf{Hom}^\bullet_A(\mathsf{triv}M, \mathsf{triv}N) \simeq \mathsf{Hom}^i_A(M, N)$$

triv is then well-defined as a functor, as every morphism between chain complexes uniquely defines a morphism between their trivializations. By using the isomorphisms from Keller [Kel94] Section 2.2. we get that:

$$\begin{aligned} Z^0\mathsf{Hom}^\bullet_A(\mathsf{cone}(id_{\mathsf{triv}A}), M) &\simeq Z^0\mathsf{Hom}^\bullet_A(\mathsf{cone}(id_{\mathsf{triv}A[-1]})[1], M) \\ &\simeq \mathsf{Hom}^*_A(\mathsf{triv}A, \mathsf{triv}M[-1])^0 \simeq \mathsf{Hom}^*_A(A, M)^{-1} \simeq M^{-1}. \end{aligned}$$

This shows that if P is homotopically projective, then P and $\mathsf{cone}(id_{\mathsf{triv}P})$ are $\mathcal{E}$-projective. To see that there are enough $\mathcal{E}$-projectives, pick an arbitrary module M. Since we know there are enough projectives, let P be a projective such that there is an epimorphism $p : P \to M$. We don't know if this morphism is a deflation, so pick another projective Q such that there is an epimorphism $q : Q \to Z^*M$. Since Z^*M has a trivial differential, we know that $d_Q q = 0$. Thus this morphism extends to $q' = \begin{bmatrix} q & 0 \end{bmatrix} : \mathsf{cone}(id_{\mathsf{triv}Q}) \to M$ such that Z^*q' is an epimorphism. The morphism $\begin{bmatrix} p & q' \end{bmatrix} : P \oplus \mathsf{cone}(id_{\mathsf{triv}Q}) \to M$ is thus a deflation. $P' = P \oplus \mathsf{cone}(id_{\mathsf{triv}Q})$ shows that we have enough projectives. Moreover every $\mathsf{cone}(id_{\mathsf{triv}Q})$ is contractible, so $P' \simeq P$ in $K(A)$.

Since we have enough $\mathcal{E}$-projective, we may construct an $\mathcal{E}$-projective resolution $P'^{*,*}$ of M in the standard way. This would be analogous to taking projective covers of the kernels; see Keller

[Kel90] for details. Such resolutions are then double complexes, and the augmented resolution below is $\mathcal{E}$-acyclic.

$$\ldots \longrightarrow P'_1 \longrightarrow P'_0 \longrightarrow\!\!\!\!\rightarrow M \xrightarrow{0} 0$$

Having an $\mathcal{E}$-acyclic resolution means that each row is exact, and taking kernels along the columns preserves the exactness of the rows.

Denote the augmentation of $P'^{*,*}$ by $m : P'^{',*} \to M$. We define the complexes $pM = Tot^{\oplus}(P'^{*,*})$ and $aM = Tot^{\oplus}(\text{cone}(m))$.

pM carries a natural filtration $F_n pM$ from the double complex structure. Let $F_n pM$ be the truncated complex:

$$\ldots \longrightarrow 0 \longrightarrow P'^{n,*} \longrightarrow \ldots \longrightarrow P'^{1,*} \longrightarrow P'^{0,*} \longrightarrow 0 \longrightarrow \ldots$$

The filtration $F_n pM$ satisfies (F1) and (F2) by construction. The quotients ${}^{F_{n+1}pM}/_{f_n pM} \simeq P'_n$ which is homotopy equivalent to a projective. By Lemma 3.1.20, pM is homotopically projective.

The complex cone(m) satisfies the conditions for Corollary 3.1.23.1, aM is acyclic, and there is a triangle in $K(A)$ as desired. □

Corollary 3.1.24.1. *Let M be an arbitrary module. If P is homotopically projective, then $K(A)(P, M) \simeq K(A)(P, pM)$. If N is acyclic, then $K(A)(M, N) \simeq (aM, N)$.*

a and p are well-defined functors that commute with infinite direct sums.

Corollary 3.1.24.2. *Let $\langle A \rangle$ denote the smallest thick triangulated subcategory of $D(A)$, which is closed under homotopy colimits and contains $\{A\}$. Then $D(A) \simeq \langle A \rangle$.*

Corollary 3.1.24.3. *Suppose that $f : A \to B$ is a dg-algebra homomorphism and a quasi-isomorphism between the dg-algebras, then $D(A) \simeq D(B)$.*

Proof. f endows B with both a left and right A-module structure. We will consider B as a left A-module and a right B module. There is then a natural hom-tensor adjunction between the differential graded enriched categories.

$$\begin{tikzcd} \text{Mod}^A \arrow[r, bend left, "f_!"] \arrow[r, phantom, "\perp"] & \text{Mod}^B \arrow[l, bend left, "f^*"] \end{tikzcd}$$

The restriction functor f^* can naturally be identified with the hom functor $\mathrm{Hom}^\bullet_A(B, \_)$, and then it is evident to realize $f_!$ as $\_ \otimes_A B$. In this way, $f_!(A) \simeq B$, so $f_! : \mathrm{Hom}^\bullet_A(A, A) \to \mathrm{Hom}^\bullet_B(B, B)$ is given by f. Since we assume f to be a quasi-isomorphism, it follows that $\mathbb{L}f_! : D(A) \to D(B)$ is fully faithful on $\{A\}$.

By devissage, the functor $\mathbb{L}f_!$ is fully faithful on all of $D(A)$ since $D(A) \simeq \langle A \rangle$. As $f_!$ hits all of $D(B)$'s generators, $\mathbb{L}f_!$ is essentially surjective as well. □

Remark 3.1.25. We have ignored smallness conditions for objects. This technique does not always work, as it depends on some unstated isomorphisms, whose existence is implied by the smallness of A and B. This detail is given more care in Keller [Kel94].

With this result, we can show that HoMod^A and $\mathrm{HoMod}^{\Omega C}$ are equivalent. Since we assumed the morphism $\tau : C \to A$ to be acyclic, we would expect the morphism $f_\tau : \Omega C \to A$ to be a quasi-isomorphism. If this is the case, we know that $D(\Omega C) \simeq D(A)$.

3.1.5 The Fundamental Theorem of Twisting Morphisms

In this section, we aim to finish what we started in Chapter 1. We will prove a characterization for the acyclic twisting morphisms.

Theorem 3.1.26 (Fundamental Theorem of Twisting Morphisms)**.** *Let $\tau : C \to A$ be a twisting morphism between augmented objects. The following are equivalent:*

1. *τ is acyclic, i.e. the natural transformation $\varepsilon : L_\tau R_\tau \implies Id_{\mathsf{Mod}^A}$ is a pointwise quasi-isomorphism.*
2. *The unit transformation $\eta : Id_{\mathsf{coMod}^C} \implies R_\tau L_\tau$ is a pointwise weak equivalence.*
3. *The counit at A is a quasi-isomorphism, i.e. $\varepsilon_A : L_\tau R_\tau A \to A$ is a quasi-isomorphism.*
4. *The unit at $\mathbb{K}$ is a weak equivalence, i.e. the algebra unit υ_A and coaugmentation υ_C assembles into a weak equivalence: $\upsilon_A \otimes \upsilon_C : \mathbb{K} \to A \otimes_\tau C$.*
5. *The morphism of algebras $f_\tau : \Omega C \to A$ is a quasi-isomorphism.*
6. *The morphism of coalgebras $g_\tau : C \to BA$ is a weak equivalence.*

Proof. Notice that 1. is equivalent to 2. since $\mathbb{L}L$ and $\mathbb{R}R$ are quasi-inverse. 3. is a special case of 1. and 4. is a special case of 2. Observe that 5. and 6. are equivalent since the cobar-bar-adjunction is a Quillen equivalence, which is Corollary 2.2.13.1.

We show 3. implies 1. Let $\mathcal{T} \subseteq D(A)$ be the full subcategory consisting of objects M where ε_M is a quasi-isomorphism. This subcategory is, by assumption, non-empty and contains A. By the 5-lemma, making triangles (and smallness of A), this subcategory contains the smallest thick triangulated subcategory closed under homotopy colimits which contains A. We know this to be all of $D(A)$ by Corollary 3.1.24.2.

To show 4. implies 5. we consider the twisting morphism ι_C. Since ι_C is acyclic, we know that the counit at A is a quasi-isomorphism.

$$L_{\iota_C} R_{\iota_C} f_\tau^* A \to f_\tau^* A$$

By assumption the unit morphism $\eta_{\mathbb{K}} : \mathbb{K} \to A \otimes_\tau C$ is a weak equivalence, so the morphism $L_{\iota_C} \eta_{\mathbb{K}} : \Omega C \to L_{\iota_C} R_\tau A = L_{\iota_C} R_{\iota_C} f_\tau^* A$ is a quasi-isomorphism. Let ε' denote the counit of $L_{\iota_C} \dashv R_{\iota_C}$, then we see that $f_\tau = \varepsilon'_A \circ L_{\iota_C} \eta_{\mathbb{K}}$, so f_τ is a quasi-isomorphism by the 2-out-of-3 property.

It remains to show that $5.$ implies $1.$ Let the counit of $f_{\tau *} \dashv f_\tau^*$ be denoted as $\tilde{\varepsilon}$. Since f_τ is a quasi-isomorphism, f_τ^* descends to an equivalence between the derived categories, which is Corollary 3.1.24.3. Thus $\tilde{\varepsilon} : f_{\tau !} f_\tau^* \implies$ Id is a pointwise quasi-isomorphism. Observe that the counit factors as

$$\varepsilon = \tilde{\varepsilon} \circ f_{\tau !} \varepsilon'_{f_\tau^*}$$

By the 2-out-of-3 property, it follows that ε is a quasi-isomorphism. □

Corollary 3.1.26.1. *There is only one canonical model structure on coMod^C defined by the acyclic twisting morphisms $\tau : C \to A$, for any algebra A. I.e., each acyclic twisting morphism defines the same model structure for coMod^C.*

Proof. Apply the fundamental theorem of twisting morphisms, Theorem 3.1.26, to the discussion of Section 3.1.4. □

3.2 Polydules

3.2.1 The Bar Construction

In Section 1.3, we saw that we could extend the domain of the bar construction to obtain an equivalence of categories. This converse led us to the definition of an A_∞-algebra and recognizing them as quasi-free dg-coalgebras. By employing the adjunction L_τ : coMod$^C \rightleftharpoons$ Mod^A : R_τ, we can do something similar for modules.

Let A be an augmented dg-algebra. The bar construction of A gives us a universal adjunction L_{π_A} : coMod$^{BA} \rightleftharpoons$ Mod^A : R_{π_A}. We will call $R_{\pi_A}(\_[1]) = \_[1] \otimes_{\pi_A} BA$ for B_A, the bar construction on Mod^A. In this manner, every A-module M gives rise to a quasi-free BA-comodule $B_A M$, but does the converse of this construction work?

Let us first look at what B_A does to an A-module M. $B_A M$ is the dg-comodule which as a graded comodule is the free comodule $M[1] \otimes BA$. The differential of $B_A M$ is given by the A-module structure of M. That is, every elementary element m' of $B_A M$ is an element of M together with a finite string of elements of A.

$$m' = [m \mid\mid a_1 \mid ... \mid a_n]$$

The differential acts on m' by using the differential of $d_{M[1]\otimes BA}$ and multiplication from the right.

$$d_{B_A M}(m') = d_{M[1]\otimes BA}(m') + (-1)^{|m|+|a|}[m \cdot a_1 \mid\mid a_2 \mid ... \mid a_n]$$

By using delooping, we see that $d_{B_A M}$ defines an A-module structure for M. We may decompose $B_A M$ as:

$$B_A M = M[1] \oplus M[1] \otimes \bar{A} \oplus M[1] \otimes \bar{A}^{\otimes 2} \oplus ...$$

Let $\pi_M : R_{\pi_A} M \to M$ be the linear map that kills anything not on the form $[m]$. We denote $(d_{B_A M})_i$ by $d_{B_A M} \circ \iota_i$, where $\iota_i : M[-1] \otimes \bar{A}^{\otimes i-1} \hookrightarrow B_A M$. Proposition 1.1.43 tells us that we may recover the structure of M from the differential $d_{B_A M}$, which is done by conjugating the components of $d_{B_A M}$ with desuspension and applying projections appropriately. We recover the maps as follows:

1. The differential of M is $d_M = s \circ \pi_{M[1]} \circ (d_{B_A M})_1 \omega$
2. The right multiplication from A is $\mu_M = s \circ \pi_{M[-1]} \circ (d_{B_A M})_2 \circ \omega^{\otimes 2}$
3. For $i \geqslant 3$ we have $0 = s \circ \pi_{M[1]} \circ (d_{B_A M})_i \circ \omega^{\otimes i}$

Now, let $\widetilde{N}$ be a quasi-free BA-comodule. That is, $\widetilde{N} = N[1] \otimes BA$ as a graded comodule. We would now like that N to carry an A-module structure. Unfortunately, this does not happen in general. However, like in the case of algebras, this defines a notion of A_∞-modules to the algebra A. If we try to recover the same structure, we obtain the following structure morphisms for N:

A differential of degree 1: $m_1 = d_N = s \circ \pi_N (d_{\widetilde{N}})_1 \circ \omega$
A 2-ary operation of degree 0: $m_2 = s \circ \pi_N (d_{\widetilde{N}})_2 \circ \omega^{\otimes 2}$
A 3-ary operation of degree -1: $m_3 = s \circ \pi_N (d_{\widetilde{N}})_3 \circ \omega^{\otimes 3}$
A 4-ary operation of degree -2: ...

Let $\widetilde{m}_i$ be the looped versions of the m_i. Then the sum $\sum \widetilde{m}_i : \widetilde{N} \to N[1]$ extends to $d_{B_A N}$ by Proposition 1.1.43, i.e.

$$d_{B_A N} = (\sum \widetilde{m}_i \otimes id_{BA})(id_N \otimes \Delta_{BA}) + N[1] \otimes d_{BA}.$$

Since $d^2_{B_A N} = 0$ we get the relations (rel_n) as defined in Section 1.3 imposed on the morphisms m_i. We summarize this in the next definition.

Definition 3.2.1 (A-polydule)**.** Let A be a dg-algebra and M be a graded $\mathbb{K}$-module. We say that M is a right A-polydule if there are morphisms

$$m_i : M \otimes A^{\otimes i-1} \to M \tag{3.1}$$

of degree $|m_i| = 2 - i$ for any $i \geqslant 1$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q^?,$$

where $m_q^?$ is meant as either m_q or m_q^A, that which is appropriate.

A left A-polydule is defined analogously. If M is an A-polydule, it has the structure of an A-module where associativity is only well-defined up to strong homotopy. m_3 is a homotopy for the associator for m_2, and m_4 is like a homotopy for the associator of m_3, and so on.

The category of A-polydules is denoted as Mod_∞^A. We have defined its objects in correspondence to the bar construction. Thus every object has been uniquely defined from a quasi-free $B(A^+)$-comodule. Likewise, we will uniquely define every morphism to come from $B(A^+)$-comodule morphisms. In this manner B_{A^+} defines a fully faithful functor $B_{A^+} : \mathsf{Mod}_\infty^A \to \mathsf{coMod}^{B(A^+)}$ which is an isomorphism on the full subcategory of quasi-free $B(A^+)$-comodules.

Definition 3.2.2 (∞-morphisms)**.** Let A be a dg-algebra, and let M and N be two right A-polydules. We say that $f : M \rightsquigarrow N$ is an ∞-morphism if there are morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N$$

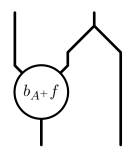
of degree $|f_i| = 1 - i$ for any $i \geqslant 1$. Furthermore, the morphism should satisfy the relations

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m_q^M = \sum_{p+q=n} m_{p+1}^N \circ_1 f_q$$

Suppose that we have the A-polydules M, N and P. If $f : M \rightsquigarrow N$ and $g : N \rightsquigarrow P$ are ∞-morphisms, then their composition is defined as

$$(gf)_n = \sum_{p+q=n} g_{p+1} \circ_1 f_q.$$

To illustrate what the bar construction does, suppose that $f : M \rightsquigarrow N$ is an ∞-morphism. The bar construction on f is then defined as



where $b_{A^+} f = \sum s \circ f_i \circ \omega^{\otimes i}$.

There is a natural inclusion on objects i : $\mathsf{Mod}^A \to \mathsf{Mod}^A_\infty$. This functor acts as the identity on each object, letting every higher $m_i = 0$:

$$\begin{aligned} i : \mathsf{Mod}^A &\to \mathsf{Mod}^A_\infty, \\ (M, d_M, \mu_M) &\mapsto (M, d_M, \mu_M, 0, 0, \cdots). \end{aligned}$$

Suppose that $f : M \to N$ is a morphism between the A-modules M and N. Then this defines an ∞-morphism $i \circ f : M \rightsquigarrow N$, such that $if_1 = f$ and $if_n = 0$ for every $n \geqslant 2$. Thus i : $\mathsf{Mod}^A \to \mathsf{Mod}^A_\infty$ is a functor.

Definition 3.2.3 (strict ∞-morphisms)**.** Let $f : M \rightsquigarrow N$ be an ∞-morphism. We say it is strict if $f_i = 0$ for every $i \geqslant 2$.

The category $\mathsf{Mod}^A_{\infty,\mathsf{strict}}$ is the non-full subcategory of Mod^A_∞ such that every ∞-morphism are strict.

3.2.2 Polydules of SHA-algebras

In the last section, we developed the notion of a polydule for augmented and ordinary algebras. We extend this notion to any A_∞-algebra.

Suppose that A is an A_∞-algebra. Recall the bar construction BA, and that this is a quasi-cofree coalgebra on the form

$$BA = \bigoplus_{i=1}^{\infty} A[1]^{\otimes i},$$

where the differential comes from the $m_i : A^{\otimes i} \to A$. To define the A-polydules, we will consider the quasi-free comodules in coMod^{BA}. This construction will be completely analogous to how it worked for ordinary dg-algebras.

Definition 3.2.4 (A-polydule)**.** Let A be an A_∞-algebra, and M a graded $\mathbb{K}$-module. We say that M is a right A-polydule if there exists morphisms

$$m_i : M \otimes A^{\otimes i-1} \to M,$$

where the degree $|m_i| = 2 - i$ for any $i \geqslant 1$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \partial(m_n) = -\sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q.$$

Definition 3.2.5 (∞-morphisms)**.** Let A be an A_∞-algebra, and let M and N be two right A-polydules. We say that $f : M \rightsquigarrow N$ is an ∞-morphism if there are morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N$$

of degree $|f_i| = 1 - i$ for any $i \geqslant 1$. Furthermore, the morphism should satisfy the relations

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m_q^M = \sum_{p+q=n} m_{p+1}^N \circ_1 f_q$$

Definition 3.2.6. Let A be an A_∞-algebra. The category Mod_∞^A has A-polydules as objects and ∞-morphisms as morphisms.

The quasi-isomorphisms in Mod_∞^A are the ∞-morphisms f such that f_1 is a quasi-isomorphism.
Remark 3.2.7. The isomorphisms of Mod_∞^A are the ∞-morphisms f where f_1 is an isomorphism.

We say that an ∞-morphism is strict if $f_i = 0$ for any $i \geqslant 2$. The category $\mathsf{Mod}_{\infty,\mathsf{strict}}^A$ is the non-full subcategory of Mod_∞^A restricted to strict ∞-morphisms.

Suppose now that A is instead a strictly unital A_∞-algebra; see Definition (1.3.13). We may define strictly unital A-polydules as an A-polydule M such that

$$m_2^M \circ (id_M \otimes \upsilon_A) = id_M$$
$$\forall i \geqslant 3 \quad m_i^M \circ (id_M \otimes ... \otimes \upsilon_A \otimes ... \otimes id_A) = 0$$

An ∞-morphism $f : M \rightsquigarrow N$ is strictly unital if

$$\forall i > 2 \quad f_i(id_M \otimes ... \otimes \upsilon_A \otimes ... \otimes id_A) = 0$$

We define the categories of strictly unital polydules with strictly unital morphisms suMod_∞^A and $\mathsf{suMod}_{\infty,\mathsf{strict}}^A$. These categories are non-full subcategories of Mod_∞^A.

Given an augmented A_∞-algebra A, see Definition 1.3.14, we obtain an equivalence of categories. Recall that the categories Alg_∞ and $\mathsf{Alg}_{\infty,+}$ were equivalent by taking the kernel of the augmentation and applying the free augmentation as its quasi-inverse. In the same manner, given a strictly unital A-polydule M, then it defines a strictly unital $\bar{A}$-polydule $\bar{M}$ by restricting the structure maps to $\bar{A}^{\otimes n}$, and this defines an equivalence of categories.

$$\begin{tikzcd} \mathsf{suMod}_\infty^A \arrow[rr, bend left, "\overline{-}"] & & \mathsf{Mod}_\infty^{\overline{A}} \arrow[ll, bend left, "-^+"] \end{tikzcd}$$

We may call its quasi-inverse for the free strict unitization. This functor takes an $\overline{A}$-polydule M and turns it into a strictly unital A-polydule by defining the structure morphism as 0 on the unit.

The reduced bar construction allows us to translate an A-polydule M to a quasi-free BA-comodule. We let $\overline{B}_A M = M[1] \otimes BA$, together with the differential coming from each m_n : $M \otimes A^{\otimes n-1} \to M$

$$d_{\overline{B}_A M} = (\sum \widetilde{m}_i \otimes id_{BA})(id_{M[1]} \otimes \Delta_{BA}) + id_{M[1]} \otimes d_{BA} = d_m + id_{M[1]} \otimes d_{BA}.$$

Likewise, we may take a quasi-free BA-comodule to obtain an A-polydule by doing the reverse bar construction, like in Proposition 1.1.43.

We will mostly restrict our attention to augmented A_∞-algebras. The reason for this is that if A is an arbitrary A_∞-algebra, then studying Mod_∞^A would be the same as studying $\mathrm{suMod}_\infty^{A^+}$. We extend the bar construction along this equivalence to a fully faithful functor $B_A : \mathrm{suMod}_\infty^A \to \mathrm{coMod}^{B\overline{A}}$. By abuse of equivalence we may write $B_{A^+} : \mathrm{Mod}_\infty^A \to \mathrm{coMod}^{BA}$.

We may also lift homotopies between quasi-free BA-comodules and A-polydules. A homotopy $B_{A^+}h : B_{A^+}M \to B_{A^+}M$ is a morphism of degree -1. Thus the collection $h_n : M \otimes A^{\otimes n-1} \to N$ has morphisms of degree $-i$. Moreover, $h : M \rightsquigarrow N$ defines a homotopy of $f, g : M \rightsquigarrow N$ if we have

$$f_n - g_n = \sum_{p+q} (-1)^p m_{p+1}^N \circ_1 h_q - \sum_{p+q+r=n} (-1)^{pq+r} h_{p+1+r} \circ_{p+1} m_q^M$$

We say that a homotopy is strictly unital if it is a strictly unital ∞-morphism.

Definition 3.2.8. Suppose there are two ∞-morphisms $f, g : M \rightsquigarrow N$ between two A-polydules, then f is homotopic to g, written as $f \sim g$, if there is a homotopy $h : M \rightsquigarrow N$ as above.

3.2.3 Universal Enveloping Algebra

Given any augmented A_∞-algebra A, there is a universal enveloping algebra UA. This algebra is universal in the sense that given any augmented algebra A' and an ∞-morphism $A' \to A$, then this factors through UA by an algebra map $A' \to UA$. By the cobar-bar adjunction, there is essentially only one way to define this algebra.

Definition 3.2.9. Let A be an A_∞-algebra. The universal enveloping algebra is the algebra defined as ΩBA.

Remark 3.2.10. In this definition, we have used the extended bar construction to A_∞-algebras and the cobar construction on dg-coalgebras.

Lemma 3.2.11. *There is an isomorphism of categories i : Mod^{UA} $\to$ suMod$_{\infty,strict}^A$ given by delooping.*

Proof. This lemma is immediate by the definition of a UA-module. To have a UA-module $M[1]$, we must have structure maps $m_i^M : M \otimes A^{\otimes i-1} \to M$ of degree $2 - i$ for any $i \geqslant 2$. Unwinding this definition and using the adjunction data establishes this isomorphism. □

We can generalize the universal enveloping algebra to the case of A_∞-algebras. This construction is very non-trivial and requires using the universal enveloping algebra relative to an operad. The necessary definitions may be found in Kriz and May [KM95].

Given an A_∞-algebra, we will denote its universal enveloping algebra UA. We have the following proposition due to Kriz and May.

Proposition 3.2.12 ([Proposition 4.10 KM95, p. 19]). *Let A be an A_∞-algebra. There is an equivalence of categories*

$$i : \textit{Mod}^{UA} \to \textit{suMod}^{A}_{\infty,\textit{strict}}.$$

With the established equivalences, we can now pull the model structure on Mod^{UA} onto $\text{suMod}^{A}_{\infty,\text{strict}}$. Recall that this is the model structure defined in Theorem 2.2.1.

3.2.4 Bipolydules

We will look at bipolydules over two A_∞-algebras A and A'. This construction will be analogous to that of A-A'-bimodules, and by considering the tensor product, it is similar to ordinary polydules.

Definition 3.2.13 (A-A'-Bipolydule). Suppose that A and A' are A_∞-algebras, and that M is a graded $\mathbb{K}$-module. M is an A-A'-bipolydule if there are morphisms

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to M,$$

such that the degree $|m_{i,j}| = 1-i-j$ for any $i, j \geqslant 0$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \sum_{\substack{n=p+q+r \\ p+1+r=s+t \\ q=u+v \\ s,t,u,v\geqslant 0}} (-1)^{pq+r} m_{s,t} \circ_{p+1} m_{u,v} = 0$$

Definition 3.2.14 (Strictly Unital A-A'-Bipolydule). Suppose that A and A' are strictly unital A_∞-algebras, and that M is an A-A'-bipolydule. We say that M is strictly unital if

$$m_{i,j}(id^{\otimes p} \otimes \upsilon_? \otimes id^{\otimes q}) = 0;$$

where $?$ is either A or A', $p \neq i$ and $(i,j) \neq (0,1)$ nor $(i,j) \neq (1,0)$. Lastly,

$$m_{1,0}(\upsilon_A \otimes id_M) = m_{0,1}(id_M \otimes \upsilon_{A'}) = id_M.$$

A morphism of bipolydules is a bit more complicated than right polydules because the left module structure induces some more signs.

Definition 3.2.15 (∞-morphisms). Let A and A' be two A_∞-algebras and let M and N be two A-A'-bipolydules. An ∞-morphism $f : M \rightsquigarrow N$ is a collection of morphisms

$$f_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to N,$$

where the degree $|f_{i,j}| = -i - j$ for any $i, , \geqslant 0$. Furthermore, the morphisms should satisfy the following relations

$$(rel_n) \qquad \sum_{\substack{n=p+q+r \\ q=s+t}} (-1)^{p(-s-t)} m_{p,q} \circ_{p+1} f_{s,t} = \sum_{n=p+q+r} (-1)^{pq+r} f_{p,r} \circ_{p+1} m_q^?,$$

where $m_q^?$ means the appropriate structure morphism.

This definition is well-defined. If $m_q^?$ is supposed to mean $m_{q_1,q_2} : A^{\otimes q_1} \otimes M \otimes B^{\otimes q_2} \to M$, then q_1 and q_2 are not uniquely determined. However, the sum will span every possibility of q_1 and q_2.

We say that an ∞-morphism is strict if $f_{0,0}$ is the only non-zero component.

The polydules assemble into categories $\mathsf{Mod}_{A,\infty}^{A'}$, $\mathsf{Mod}_{A,\infty,\mathsf{strict}}^{A'}$, $\mathsf{suMod}_{A,\infty}^{A'}$ and $\mathsf{suMod}_{A,\infty,\mathsf{strict}}^{A'}$ like in the usual sense. These definitions may seem somewhat more complicated. However, they almost reduce to the ordinary case by considering the category $\mathsf{coMod}^{BA^{op} \otimes BA'}$. We may derive a 2-sided bar-construction $B_{A^+ - A'^+} : \mathsf{Mod}_{A,\infty}^{A'} \to \mathsf{coMod}_{BA}^{BA'}$. However, we know that $\mathsf{coMod}_{BA}^{BA'} \simeq \mathsf{coMod}^{BA^{op} \otimes BA'}$. In this manner, we may argue about bipolydules with the techniques we have developed for comodules.

3.2.5 A Tensor and a Hom on Mod_∞^A

To understand the category Mod_∞^A, we would like to construct a tensor product and a hom-functor on it. In its most generality, the tensor will be a bifunctor:

$$\_ \otimes_{A'}^\infty \_ : \mathsf{Mod}_{A,\infty}^{A'} \otimes \mathsf{Mod}_{A',\infty}^{A''} \to \mathsf{Mod}_{A,\infty}^{A''}.$$

In the usual sense, given a bipolydule $M \in \mathsf{Mod}_{A,\infty}^{A'}$, it will act as a morphism

$$\_ \otimes_A^\infty M : \mathsf{Mod}_\infty^A \to \mathsf{Mod}_\infty^{A'}.$$

In particular, this functor will be a left adjoint to its corresponding hom-functor. In its most general form, the hom functor will be a bifunctor:

$$\mathsf{Hom}_{A'}^\infty : \mathsf{Mod}_{A,\infty}^{A'} \otimes \mathsf{Mod}_{A'',\infty}^{A'} \to \mathsf{Mod}_{A'',\infty}^{A}.$$

We start by describing the tensor product in the simplest case. Let A be an A_∞-algebra, and let M and N be a right and left A-polydule, respectively. We define $M \otimes_A^\infty N$ as a cochain complex

$$M \otimes_A^\infty N = M \otimes T^c(A[1]) \otimes N.$$

Its structure comes from the cotensor product of quasi-free coalgebras. Consider instead the right and left BA dg-comodules $B_{A^+}M = M[1] \otimes BA$ and $B_{A^+}N = BA \otimes N[1]$.

$$B_{A^+}M \mathbin{\square_{BA}} B_{A^+}N = \mathsf{Ker}(\omega_{B_{A^+}M}^r \otimes B_{A^+}N - B_{A^+}M \otimes \omega_{B_{A^+}N}^l)$$

Then $B_{A^+}M \,\square_{BA}\, B_{A^+}N$ is a $\mathbb{K}$ dg-module. Taking the cotensor, we restrict our attention to solely those parts of this tensor in which comultiplication from the left is the same as comultiplication from the right. An element may then be seen to be of the form

$$\begin{aligned}&[m \mid\mid a_1 \mid \cdots \mid a_n] \otimes [n]\\ +&[m \mid\mid a_1 \mid \cdots \mid a_{n-1}] \otimes [a_n \mid\mid n]\\ +&\cdots\\ +&[m \mid\mid a_1] \otimes [a_2 \mid \cdots \mid a_n \mid\mid n]\\ +&[m] \otimes [a_1 \mid \cdots \mid a_n \mid\mid n].\end{aligned}$$

There is an evident isomorphism to $M[1] \otimes BA \otimes N[1]$ by sending each of the elements above to the elements

$$[m \mid\mid a_1 \mid \cdots \mid a_n \mid\mid n].$$

Its differential is induced by the restriction of the differential on the cochain complex $B_{A^+}M \otimes B_{A^+}N$. Since $d_{B_{A^+}M \otimes B_{A^+}N}$ is well-defined on each element in $B_{A^+}M \otimes B_{A^+}N$, the restricted differential $d_{B_{A^+}M} \otimes id_{N[1]} + id_{M[1]} \otimes d_{BA} \otimes id_{N[1]} + id_{M[1]} \otimes d_{B_{A^+}N}$ on $M[1] \otimes BA \otimes N[1]$ is well-defined as well.

Definition 3.2.16 (The tensor product)**.** Let A be an A_∞-algebra, and let M and N be respectively a right and a left A-polydule. The tensor $M \otimes_A^\infty N = M \otimes BA \otimes N$ is a cochain complex with differential

$$(s \otimes id_{BA} \otimes s)(d_{B_{A^+}M} \otimes id_{N[1]} + id_{M[1]} \otimes d_{BA} \otimes id_{N[1]} + id_{M[1]} \otimes d_{B_{A^+}N})(\omega \otimes id_{BA} \otimes \omega).$$

An element of $M \otimes_A^\infty N$ may be written on the form

$$m[a_1 \mid \cdots \mid a_n]n.$$

Given A-polydules M, M', N and N' and ∞-morphisms $f : M \rightsquigarrow M'$ and $g : N \rightsquigarrow N'$, we define $f \otimes_A^\infty g$ as

$$f \otimes_A^\infty g(m[a_1 \mid \cdots \mid a_n]n) = \sum_{p+q+r=n+2} (-1)^s f_p(m, a_1, \cdots)[\cdots]g_r(\cdots, a_n, n),$$

where s is the appropriate sign derived from Koszul's sign rule. Note that as a $\mathbb{K}$-polydule, this morphism is a strict ∞-morphism. This fact will not change, even in the more general cases.

We will extend this tensor to bipolydules. Suppose that N now has the structure of an A-A'-bipolydule. The cotensor $B_{A^+}M \,\square_{BA}\, B_{A^+-A'^+}N \simeq (B_{A^+}M \,\square_{BA}\, B_{A^+}N) \otimes T^c(A'[1])$ as graded comodules. When we thus recover the structure morphisms, we may recover them at $T^c(A'[1])$. In other words, $m_{0,n} : N \otimes A'^{\otimes n-1} \to N$ induces morphisms $m_n : M \otimes_A^\infty N \otimes A'^{\otimes n-1} \to M \otimes_A^\infty N$. Thus, given a bipolydule such as N, we obtain a functor

$$\_ \otimes_A^\infty N : \mathsf{Mod}_\infty^A \to \mathsf{Mod}_\infty^{A'}.$$

We will now describe the hom functor in the simplest case. Let A be an A_∞-algebra, and let M and N be right A-polydules. We define $\mathrm{Hom}_A^\infty(M,N)$ as a cochain complex

$$\mathrm{Hom}_A^\infty(M,N) = \mathrm{Hom}_{BA}^*(B_{A^+}M, B_{A^+}N).$$

Its differential is the usual hom differential, i.e. given $f \in \mathrm{Hom}_{BA}^*(B_{A^+}M, B_{A^+}N)$ then

$$\partial f = d_{B_{A^+}N} \circ f - (-1)^{|f|} f \circ d_{B_{A^+}M}.$$

Functoriality is given by post- and pre-composition in the usual sense for dg-comodules. If we are given ∞-morphisms, we will instead consider the dg-comodule counterpart and define functoriality purely through that. Because of this, when we regard this as $\mathbb{K}$-polydule, post-, and pre-composition is a strict ∞-morphism.

To be able to get to a more complicated case, we first need a new way to encode the data of an A-polydule. The $\mathbb{K}$-module $\mathrm{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ carries a natural bimodule structure. There are actions on $\mathrm{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ on the right from the dg-endomorphism algebra $\mathrm{End}(B_{A^+}M)$, and on the left from $\mathrm{End}(B_{A^+}N)$ by composition. If we consider these dg-algebras as A_∞-algebras, then we may give $\mathrm{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ the structure of a bipolydule. The following lemma connects representations of A_∞-algebras to A-polydules.

Lemma 3.2.17 (Representation lemma, [Lemme 5.3.0.1 Lef03, p. 140]). *Let A be an A_∞-algebra, and let M be a graded $\mathbb{K}$-module. The following are equivalent:*

- *There is an ∞-morphism of A_∞-algebras $\phi : A \rightsquigarrow \mathrm{End}(M)$,*
- *M is a left A-polydule.*

Proof. We will only establish the bijection map. Proof of well-definedness may be found in [Lef03].

The bijection is given by the transpose of the tensor. Notice that as $\mathbb{K}$-linear morphisms we have the following bijections

$$\mathrm{Hom}_{\mathbb{K}}(A^{\otimes n-1}, \mathrm{End}(M)) \simeq \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n-1} \otimes M, M).$$

Thus if $\phi : A \to \mathrm{End}(M)$ is an ∞-morphism, then we may define

$$\begin{aligned} m_n : A^{\otimes n-1} \otimes M &\to M \\ (a_1 \otimes \cdots \otimes a_{n-1}) \otimes m &\mapsto \phi(a_1 \otimes \cdots \otimes a_{n-1})(m). \end{aligned}$$

On the other hand, if we have structure morphisms $m_n : A^{\otimes n-1} \otimes M \to M$, then we may define ϕ by uncurrying:

$$\begin{aligned} \phi_n : A^{\otimes n} &\to \mathrm{End}(M), \\ a_1 \otimes \cdots \otimes a_n &\mapsto (m \mapsto m_{n+1}(a_1 \otimes \cdots \otimes a_n \otimes m)). \end{aligned}$$

$\square$

Remark 3.2.18. This lemma is well-known and holds in many other aspects as well. One may, for example, recognize this in the representation theory of finite groups. A more general account of this lemma may be found as [Proposition 5.2.2. LV12, p. 139].

Corollary 3.2.18.1. *Let A and A' be two A_∞-algebras, and let M be an A-A'-bipolydule. Then there is an A_∞-morphism $\phi : A \rightsquigarrow$ End$(B_{A'^+}M)$. In particular, any End$(B_{A'^+}M)$-modules is an A-polydule.*

Proof. By Lemma 3.2.17 we obtain the ∞-morphism $\phi : A \rightsquigarrow$ End$(B_{A'^+}M)$ by transposing the structure morphisms

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to M.$$

In other words,

$$\begin{aligned} \phi_n : A^{\otimes n} &\to \mathrm{End}(B_{A'^+}M), \\ a_1 \otimes \cdots \otimes a_n &\mapsto (\\ [m \mid\mid a'_1 \mid \cdots \mid a'_l] &\mapsto d_{B_{A^+ - A'^+}M} \circ (\omega^{\otimes n} \otimes id_{M[1]} \otimes id_{A'[1]}^{\otimes l})(a_1 \otimes \cdots \otimes a_n \otimes [m \mid\mid a'_1 \mid \cdots \mid a'_l])). \end{aligned}$$

□

We are now ready to describe the hom-functor. Suppose that A and A' are A_∞-algebras, and that M is an A-A'-polydule and N a right A'-polydule. We define the A-polydule

$$\mathrm{Hom}^\infty_{A'}(M, N) = \mathrm{Hom}^*_{BA'}(B_{A'^+}M, B_{A'^+}N),$$

with structure map $\phi : A \rightsquigarrow$ End$(B_{A'^+}M)$ defined by the above corollary. In this way, we obtain a functor

$$\mathrm{Hom}^\infty_{A'}(M, \_) : \mathrm{Mod}^{A'}_\infty \to \mathrm{Mod}^A_\infty.$$

Lemma 3.2.19 (Hom-Tensor adjunction, [Lemme 4.1.1.4 Lef03, p. 115])**.** *Let A and A' be two A_∞-algebras and M an A-A'-bipolydule. There is an adjoint pair of functors*

$$\begin{tikzcd} \mathit{Mod}^A_\infty \arrow[r, bend left, "\_\otimes^\infty_A M"] \arrow[r, phantom, "\perp"] & \mathit{Mod}^{A'}_\infty \arrow[l, bend left, "\mathit{Hom}^\infty_{A'}(M{,}\_)"] \end{tikzcd}$$

Proof. We establish the natural bijection. We refer to [Lef03, Lemme 4.1.1.4] to see that it is well-defined.

Consider an ∞-morphism $f : L \otimes^\infty_A M \rightsquigarrow R$ of right A'-polydules. By consider the bar construction of A', this morphism is in correspondance with $B_{A'^+}f : L \otimes^\infty_A B_{A'^+}M \to B_{A'^+}R$. Through the ordinary tensor-hom adjunction we get a correspondance $f_i^T : L \otimes A^{\otimes i} \to$ Hom$_{BA'}(B_{A'^+}M, B_{A'^+}R)$.

□

3.2.6 Homologically Unital SHA-Algebras and Polydules

This section will define the notion of homologically unital A_∞-algebras and polydules. These notions will be weaker than strictly unitary objects, but their definition may be easier to use. As we will see, these notions almost coincide with homotopy. This section will be given without proof.

If A is an A_∞-algebra, or M is an A-polydule, we will use H^*A and H^*M to denote their homology. Note that H^*A is an associative algebra, as m_i for $i \geqslant 3$ are homotopies, witnessing associativity of H^*m_2. In the same fashion, H^*M, becomes a H^*A-module, by considering $\mathrm{H}^*m_2^M$.

Definition 3.2.20 (Homologically unital A_∞-algebra)**.** Let A be an A_∞-algebra. A morphism $v_A : \mathbb{K} \to A$ is called a homological unit, if $\mathrm{H}^*v_A : \mathbb{K} \to \mathrm{H}^*A$ is a unit in homology. We say that A equipped with a homological unit v_A is a homologically unital A_∞-algebra.

An ∞-morphism $f : A \rightsquigarrow A'$ is homologically unital if it preserves the unit in homology, i.e., $\mathrm{H}^*f : \mathrm{H}^*A \to \mathrm{H}^*A'$ is also a morphism of graded algebras.

Given two ∞-morphisms $f, f' : A \rightsquigarrow A'$, they are homotopically unital if there is a homotopy $h : A \rightsquigarrow A'$ between f and f' which is strictly unital with respect to the homological unit v_A.

We let suAlg_∞ denote the non-full subcategory of strictly unital A_∞-algebras with strictly unital ∞-morphisms, huAlg_∞ denote the non-full subcategory of homologically unital A_∞-algebras with homologically unital ∞-morphism, and uAlg_∞ denote the full subcategory of strictly unital A_∞-algebras with ∞-morphisms. Note that if A is a strictly unital A_∞-algebra, then it is also homologically unital. Thus we see that $\mathsf{suAlg}_\infty \subseteq \mathsf{huAlg}_\infty$.

To obtain a stronger relationship between homologically unital A_∞-algebras and strictly unital A_∞-algebras, we need minimal models.

Definition 3.2.21 (Minimal SHA-algebra/polydule)**.** Let A be an A_∞-algebra, and M an A-polydule. We say that A is minimal if $m_1^A = 0$, and likewise M is minimal if $m_1^M = 0$

Definition 3.2.22 (Minimal model)**.** Let A and A' be A_∞-algebras. We say that an ∞-quasi-isomorphism $f : A' \rightsquigarrow A$ is a minimal model of A.

Theorem 3.2.23 ([Corollaire 1.4.1.4 Lef03, p. 54])**.** *Let A be an A_∞-algebra. The injection from the homology H^*A into A is a minimal model of A.*

Proof. We will only construct the first component of this injection.

Since $\mathsf{Mod}^{\mathbb{K}}$ is semi-simple, A splits naturally as $A \simeq \mathrm{H}^*A \oplus K$. By definition, K is acyclic, and the inclusion $\mathrm{H}^*A \to A$ is a quasi-isomorphism. ☐

We now state the following relationship between homologically unital and strictly unital A_∞-algebras.

Theorem 3.2.24 ([Theoreme 3.2.1.1 Lef03, p. 99]). *Any minimal homologically unital A_∞-algebra is isomorphic to a minimal strictly unital A_∞-algebra.*

Corollary 3.2.24.1 (Unital strictification of A_∞-algebras, [Corollaire 3.2.1.2 Lef03, p. 99]). *Any homologically unital A_∞-algebra is homotopy equivalent to a strictly unital A_∞-algebra.*

Proof. We obtain this result by combining Theorem 3.2.23 and Theorem 3.2.24. □

Theorem 3.2.25 (Unital strictification of ∞-morphisms, [Theoreme 3.2.2.1 Lef03, p. 103]). *A homologically unital ∞-morphism of strictly unital minimal A_∞-algebras is homotopic to a strictly unital ∞-morphism.*

Theorem 3.2.26 (Unital strictification of homotopies, [Theoreme 3.2.3.1 Lef03, p. 104]). *Let A and A' be two minimal strictly unital A_∞-algebras. Let $f, g : A \rightsquigarrow A'$ be strictly unital ∞-morphisms that are homotopic, and then there is a strictly unital homotopy witnessing the homotopy $f \sim g$.*

Corollary 3.2.26.1. *Let A and A' be two A_∞-algebra, and let $f : A \rightsquigarrow A'$ be a strictly unital homotopy equivalence. Thus, there is a strictly unital homotopy equivalence $g : A' \rightsquigarrow A'$, with strictly unital homotopies witnessing that g is the homotopy inverse of f.*

With the above results, we learn that the homotopic information of strictly unital A_∞-algebras is essentially controlled by strictly unital ∞-morphism. In other words the non-full inclusion $\mathsf{suAlg}_\infty \to \mathsf{uAlg}_\infty$ induces an equivalence of categories

$$\mathsf{suAlg}_\infty/\sim \;\simeq\; \mathsf{uAlg}_\infty/\sim.$$

We also get that the unital strictification of homologically unital A_∞-algebras induces an equivalence

$$\mathsf{huAlg}_\infty/\sim \;\simeq\; \mathsf{suAlg}_\infty/\sim.$$

We also have similar results for polydules.

Definition 3.2.27. Let A be a homologically unital A_∞-algebra, and let M be an A-polydule. We say that M is homologically unital if H^*M is a unital H^*A-module.

Let M and N be two homologically unital A-polydules, and $f : M \rightsquigarrow N$ be an ∞-morphism. We say that $f : M \rightsquigarrow N$ is homologically unital if $\mathrm{H}^*f_1 : \mathrm{H}^*M \to \mathrm{H}^*N$ is a H^*A-linear morphism.

We denote the category of homologically unital A-polydules with homologically unital ∞-morphisms by huMod^A_∞. This category is a non-full subcategory of Mod^A_∞. Recall that we also have suMod^A_∞, the category of strictly unital A-polydules with strictly unital ∞-morphism. Let uMod^A_∞ denote the full subcategory of Mod^A_∞ consisting of strictly unital A-polydules. We have the same kind of results as for A_∞-algebras.

Theorem 3.2.28 (Unital strictification of A-polydules, [Theoreme 3.3.1.2 Lef03, p. 109])**.** *Let A be a strictly unital A_∞-algebra. Any minimal homologically unital A-polydule is isomorphic to a strictly unital A-polydule.*

Corollary 3.2.28.1 ([Corollaire 3.3.1.3 Lef03, p. 109])**.** *Let A be a minimal strictly unital A_∞-algebra. Any homologically unital A-polydule is homotopy equivalent to a strictly unital A-polydule.*

Theorem 3.2.29 (Unital strictification of ∞-morphisms, [Theoreme 3.3.1.4 Lef03, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M and N be minimal strictly unital A-polydules. Any ∞-morphism $f : M \rightsquigarrow N$ is homotopic to a strictly unital ∞-morphism.*

Theorem 3.2.30 (Unital strictification of homotopies, [Theoreme 3.3.1.5 Lef03, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M and N be minimal strictly unital A-polydules. Let $f, g : M \rightsquigarrow N$ be homotopic ∞-morhpisms, then there is a strictly unital homotopy between f and g.*

Proposition 3.2.31 (Minimal models, [Proposition 3.3.1.7 Lef03, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M be a strictly unital A-polydule. Then there is a minimal strictly unital A-polydule N together with a strictly unital minimal model $f : N \rightsquigarrow M$. In particular, f_1 is a quasi-isomorphism.*

Suppose that A is a minimal strictly unital A_∞-akgebra. With the above results, we are now able to deduce that the non-full inclusion $\mathsf{suMod}_\infty^A \to \mathsf{uMod}_\infty^A$ induces an equivalence

$$\mathsf{suMod}_\infty^A / \sim \; \simeq \; \mathsf{uMod}_\infty^A / \sim,$$

and the non-full inclusion $\mathsf{huMod}_\infty^A \to \mathsf{suMod}_\infty^A$ induces an equivalence

$$\mathsf{huMod}_\infty^A / \sim \; \simeq \; \mathsf{suMod}_\infty^A / \sim.$$

3.2.7 H-Unitary SHA-Algebras and Polydules

In this section, we will define notions that will help us to calculate homologies. We will define a twisting morphism between an augmented A_∞-algebra and a conilpotent dg-coalgebra. For the second part, we will define H-unitary A_∞-algebras and polydules.

Definition 3.2.32. Let A be an augmented A_∞-algebra, and let C be a conilpotent dg-coalgebra. $\tau : C \to A$ is a twisting morphism if it is of degree 1, it is 0 on the augmentation ideal and the coaugmentation quotient and

$$\sum_{i \geqslant 1} m_i \otimes (\tau^{\otimes i}) \otimes \Delta_C^i = 0.$$

Let M be an A-polydule, and N a C-comodule. Given a twisting morphism $\tau : C \to A$, we define the twisted tensor products

$$\_ \otimes_\tau C : \mathsf{Mod}_\infty^A \to \mathsf{coMod}^C,$$
$$\_ \otimes_\tau A : \mathsf{coMod}^C \to \mathsf{Mod}_\infty^A.$$

The perturbations are

$$d^r_\tau = \sum_{i=1}^{\infty}(m_i \otimes C)(M \otimes \tau^{\otimes i-1} \otimes C)(M \otimes \Delta^i_C),$$
$$d^l_\tau = \sum_{i=1}^{\infty}(N \otimes m_i)(N \otimes \tau^{\otimes i-1} \otimes A)(\nu^i_N \otimes A).$$

We define the perturbed differential of the cochain complexes $M \otimes C$ and $N \otimes A$ as

$$d^\bullet_\tau = d_{M\otimes C} + d^r_\tau, \text{ and}$$
$$d^\bullet_\tau = d_{N\otimes A} - d^l_\tau.$$

Definition 3.2.33 (Twisted tensor products)**.** Let A be an augmented A_∞-algebra, let C be a conilpotent dg-coalgebra, and let $\tau : C \to A$ be a twisting morphism. Given an A-polydule M (a C-comodule N), we define the right (left) twisted tensor product as $M \otimes_\tau C$ ($N \otimes_\tau A$) together with the perturbated differential $d^\bullet_\tau$.

Pick an augmented A_∞-algebra A. The morphism

$$\tau = i \circ s \circ \pi_1 : B\overline{A} \to A$$

is a twisting morphism. Here $\pi_1 : B\overline{A} \to \overline{A}[1]$ is the projection onto first component, and $i : \overline{A} \to A$ is the inclusion.

Lemma 3.2.34. *The morphism $\varepsilon_{B\overline{A}} \otimes_\tau \varepsilon_A : B\overline{A} \otimes_\tau A \to \mathbb{K}$ is a quasi-isomorphism.*

Proof. We have already seen this in Lemma 3.1.7. □

Twisting morphisms will be important in understanding H-unitary A_∞-algebras and polydules.

Definition 3.2.35. Let A be an A_∞-algebra. We say that A is H-unitary if the bar construction BA is acyclic.

Lemma 3.2.36. *Let A be a minimal strictly unital A_∞-algebra, and then it is H-unitary.*

Proof. The unit map $id_B A \otimes \upsilon_A[1] : BA \to BA$ is a morphism of degree -1 and is a homotopy of the identity. □

Corollary 3.2.36.1. *Any homologically unital A_∞-algebra is H-unitary.*

Proof. Pick any homologically unital A_∞-algebra A. By Corollary 3.2.24.1, there exists a strictly unital A_∞-algebra A' and an ∞-quasi-isomorphism $f : A' \rightsquigarrow A$. Applying the bar construction yields a quasi-isomorphism $Bf : BA' \to BA$. By Lemma 3.2.36, BA' is acyclic, so BA has to be acyclic. □

We have the same kind of relationships between polydules.

Definition 3.2.37. Let A be an augmented strictly unital A_∞-algebra. Any A-polydule M is H-unitary if $B_A M$ is acyclic.

Lemma 3.2.38. *Let A be a strictly unital A_∞-algebra. An A^+-polydule M is H-unitary if and only if it is homologically unital as an A-polydule.*

Proof. Suppose first that M is a homologically unital A-polydule. Then by Corollary 3.2.28.1, there is a strictly unital A-polydule M' together with an ∞-quasi-isomorphism $M' \rightsquigarrow M$. It is enough to show that $B_{A^+}M'$ is acyclic. The unit υ_A defines a homotopy of the identity

$$id_{B_{A^+}M'} \otimes \upsilon_A[1] : B_{A^+}M' \to B_{A^+}M'.$$

For the other direction, suppose that M is an H-unitary A^+-polydule. Note that we have an exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \longrightarrow \mathbb{K} \longrightarrow 0$$

Recall that $\tau = i \circ s \circ \pi_1 : BA \to A^+$. This sequence induces an exact sequence on the twisted tensors

$$0 \longrightarrow M \otimes_\tau BA \otimes_\tau A \longrightarrow M \otimes_\tau BA \otimes_\tau A^+ \longrightarrow M \otimes_\tau BA \otimes_\tau \mathbb{K} \longrightarrow 0$$

By assumption $M \otimes_\tau BA \otimes_\tau \mathbb{K} \simeq (M[1] \otimes_\tau BA)[-1] \simeq (B_{A^+}M)[-1]$ which is acyclic by assumption. Thus $M \otimes_\tau BA \otimes_\tau A$ is quasi-isomorphic to $M \otimes_\tau BA \otimes A^+$. By Lemma 3.2.34, $M \otimes_\tau BA \otimes A^+ \simeq M \otimes_\tau \mathbb{K} \simeq M$. Thus, $M \simeq M \otimes_\tau BA \otimes_\tau A$ is a strictly unital right A-polydule by freeness. $\square$

3.3 The Derived Category $D_\infty A$

3.3.1 The Derived Category of Augmented SHA-Algebras

In this section, we wish to define the derived category of strictly unital polydules of an augmented A_∞-algebra. If Qis denote the class of ∞-quasi-isomorphisms, we want the derived category to be the localization at ∞-quasi-isomorphisms, e.g.

$$\mathcal{D}_\infty A = \mathsf{suMod}_\infty^A[\mathsf{Qis}^{-1}].$$

Like in the case of algebras, we may understand the quasi-isomorphisms better. The category suMod_∞^A is not complete, but we may give it a model structure without limits in the same sense as before. Within this structure, we already know that every object is cofibrant, and the goal is to show that every object is also fibrant. With this, we can lift every ∞-quasi-isomorphism to homotopy equivalence, and we may see that the identity gives the localization from $K_\infty A \to D_\infty A$.

Within the category suMod$^A_\infty$ we define three classes of morphisms:

- $f \in Ac$ is a weak equivalence if f_1 is a quasi-isomorphism,
- $f \in Cof$ is a cofibration if f_1 is a monomorphism,
- $f \in Fib$ is a fibration if f_1 is an epimorphism,

Theorem 3.3.1. *The category suMod$^A_\infty$ is a model category without enough limits. Moreover, every object is bifibrant.*

Proof. This result is more or less identical to the proof of Theorem 2.3.3. □

Like in the case of algebras, Proposition 2.3.1, we may consider ordinary homotopies of comodules as left homotopies. In this way, we can think of the homological homotopies as model categorical homotopies. Since polydules are exactly the bifibrant comodules, we get that the homological homotopies are exactly the model categorical homotopies.

Corollary 3.3.1.1. *Homotopy equivalence defined in suMod$^A_\infty$ is an equivalence relation, and every ∞-quasi-isomorphism is a homotopy equivalence.*

Proof. This corollary follows from the above discussion, as the homological homotopies coincide with the model categorical homotopies. It is thus an equivalence relation, and Whitehead's theorem, Theorem 2.1.30, gives us a lift to an ∞-quasi-isomorphism. □

We now want this model structure on suMod$^A_\infty$ to respect the model structure on the category coMod$^{BA}_{conil}$. In other words, we want the functor B_A : suMod$^A_\infty$ $\to$ coMod$^{BA}_{conil}$ to preserve and reflect the model structure of both categories.

Lemma 3.3.2. *Let M be an object of suMod$^A_\infty$. The unit $B_A M \to R_{\iota_{BA}} L_{\iota_{BA}} B_A M$ is a quasi-isomorphism on the primitive elements.*

Proof. This proof uses the same trick as Lemma 3.1.7. Equip M, the trivial filtration, BA the coradical filtration and $\Omega BA = UA$ the induced filtration.

$$\begin{aligned} F_p M &= M, \\ Fr_p BA &= \{[a_1 \mid \cdots \mid a_n] \mid n \leqslant p\}, \\ F_p UA &= \{\langle [a_{1_1} \mid \cdots \mid a_{n_1}] \mid \cdots \mid [a_{1_k} \mid \cdots \mid a_{n_k}] \rangle \mid n_1 + \cdots + n_k \leqslant p\}. \end{aligned}$$

We see that $\mathrm{gr}_0 M[1] \simeq M[1]$ and otherwise $\simeq 0$. In the same way, $\mathrm{gr}_0\eta$ acts as the identity on $M[1]$. By the similar lemma, we know that each $\mathrm{gr}_p M[1] \otimes BA \otimes UA$ is acyclic for $p \geqslant 1$. Thus $\mathrm{gr}\eta$ is a graded quasi-isomorphism on the primitives.

□

Proposition 3.3.3. *Let M and M' be objects of suMod$_\infty^A$, together with an ∞-morphism $f : M \to M'$.*

- *f is an ∞-quasi-isomorphism if and only if $B_A f$ is a weak equivalence.*
- *f is a fibration if and only if $B_A f$ is a fibration.*
- *f is a cofibration if and only if $B_A f$ is a cofibration.*

Proof. Recall from Theorem 3.1.8 that the morphism $\iota_{BA} : BA \to UA$ is an acyclic twisting morphism. Thus the adjoint pair $(L_{\iota_{BA}}, R_{\iota_{BA}})$ defines a Quillen equivalence.

We show only the first bullet point. The last two are identical to the proof of proposition 2.2.19.

If f_1 is a quasi-isomorphism, then $B_A f$ is a graded quasi-isomorphism. So suppose that $B_A f$ is a weak equivalence instead. The unit transformation gives us a natural square.

$$\begin{tikzcd} B_A M \arrow[r] \arrow[d, "B_A f"] & R_{\iota_{BA}} L_{\iota_{BA}} B_A M \arrow[d, "R_{\iota_{BA}} L_{\iota_{BA}} B_A f"] \\ B_A M' \arrow[r] & R_{\iota_{BA}} L_{\iota_{BA}} B_A M' \end{tikzcd}$$

In this case, $R_{\iota_{BA}} = B_A i$, so this diagram is in the image of B_A. Since B_A is fully faithful, we consider this diagram in suMod$_\infty^A$ instead.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & iL_{\iota_{BA}} BM \arrow[d, "iL_{\iota_{BA}} Bf"] \\ M' \arrow[r] & iL_{\iota_{BA}} BM' \end{tikzcd}$$

Since $B_A f$ is a weak equivalence, $iL_{\iota_{BA}} B_A f$ is an ∞-quasi-isomorphism by definition. By the above lemma, the horizontal maps are ∞-quasi-isomorphisms. Thus by the 2-out-of-3 property, f is an ∞-quasi-isomorphism. □

There is a homotopy category associated with every augmented A_∞-algebra. Since homotopy equivalence $\sim$ in suMod$_\infty^A$ defines a congruence relation, we may construct the homotopy category $K_\infty A$.

Corollary 3.3.3.1. *The identity gives the localization $K_\infty A \to D_\infty A$. Moreover, $K_\infty A = D_\infty A$.*

Remark 3.3.4. The name homotopy category comes from homological algebra and has a priori nothing to do with the homotopy category Ho(suMod$^A_\infty$). However, in this particular case, these naming conventions coincide.

Lemma 3.3.5. *The composition J : Mod^{UA} $\to$ suMod$^A_{\infty,\text{strict}}$ $\to$ suMod$^A_\infty$ given by $J = \iota \circ i$, induces an equivalence of categories:*

$$DUA \simeq D_\infty A.$$

Proof. Consider the commutative square:

$$\begin{tikzcd}
\mathsf{Mod}^{UA} \arrow[r, "i"] \arrow[d, "R_{\iota_{BA}}"] & \mathsf{suMod}^A_{\infty,\mathrm{strict}} \arrow[d, "\iota"] \\
\mathsf{coMod}^{BA} & \mathsf{suMod}^A_\infty \arrow[l, "B_A"']
\end{tikzcd}$$

Since the three functors $R_{\iota_{BA}}$, i, and B_A all induce equivalences on the derived categories, then ι has to as well. ☐

To summarize, we have established an equivalence between 5 different categories:

- $D_\infty A$, derived category of A;
- $K_\infty A$, the homotopy category associated to A;
- suMod$^A_{\infty,\text{strict}}[Qis^{-1}]$, derived category of A with only strict morphisms;
- DBA, derived category of BA as a dg-coalgebra;
- DUA derived category of the universal enveloping algebra of A.

We may see that within the derived category, all of the higher homotopic data of each morphism have been collapsed by the homotopy.

The triangulated structure on $D_\infty A$ may be lifted along these equivalences, making them triangulated as well. Note that $R_{\iota_{BA}}$ is already triangulated, and there is only one way of forcing the triangulated structure on suMod$^A_\infty$. Since suMod$^A_\infty$ isn't complete, it isn't easy to obtain a description of the triangles along any ∞-morphism f. However, this problem does not appear in suMod$^A_{\infty,\text{strict}}$, so one should think of only strict morphisms instead, but in this case, we are already working in the category Mod^{UA}.

If we let A to be an ordinary associative augmented algebra, we can obtain a similar characterization. Notice first that by Lemma 3.2.11 and Proposition 2.1.43, there is a quasi-isomorphism $UA \to A$. By Corollary 3.1.24.3, we get that their derived categories have to be equivalent. In other words, the six categories below are equivalent:

- DA, the derived category of A;
- $D_\infty A$, the derived category of A considered as an A_∞-algebra;
- $K_\infty A$, the homotopy category associated to A considered as an A_∞-algebra;
- $\mathsf{suMod}^A_{\infty,\mathsf{strict}}[\mathsf{Qis}^{-1}]$, the derived category of A considered as an A_∞-algebra considering only strict morphisms;
- and DUA, the derived category of UA.

3.3.2 The Derived Category of Strictly Unital SHA-Algebras

In this section, we will generalize the construction of the derived category to any strictly unital A_∞-algebra. Consider the strictly unital A_∞-algebra A. If we look at the augmented algebra A^+, then the augmentation $\varepsilon_A : A^+ \to \mathbb{K}$ gives $\mathbb{K}$ the structure of an A^+-polydule. We construct the following functor

$$\_ \otimes^\infty_{A^+} \mathbb{K} : \mathsf{Mod}^{A^+}_\infty \to \mathsf{Mod}^\infty_{\mathbb{K}}.$$

We may observe that this functor maps strictly unital objects into strictly unital objects

$$\_ \otimes^\infty_{A^+} \mathbb{K} : \mathsf{uMod}^{A^+}_\infty \to \mathsf{uMod}^\infty_{\mathbb{K}}.$$

The derived category $D_\infty A^+$ is equivalent to $\mathsf{uMod}^{A^+}_\infty/\sim$. Since the functor above preserves ∞-quasi-isomorphisms, it induces a functor between the derived categories

$$\_ \otimes^\infty_{A^+} \mathbb{K} : D_\infty A^+ \to D_\infty \mathbb{K}.$$

Definition 3.3.6. Let A be an A_∞-algebra. We define the derived category as the kernel

$$D_\infty A = \mathsf{Ker}(\_ \otimes^\infty_{A^+} \mathbb{K} : D_\infty A^+ \to D_\infty \mathbb{K}).$$

Theorem 3.3.7. *Let A and A' be two A_∞-algebras, and let $f : A \to A'$ be an ∞-quasi-isomorphism. The restriction*

$$f^* : \mathit{Mod}^{A'}_\infty \to \mathit{Mod}^A_\infty$$

induces an equivalence on the derived categories

$$f^* : D_\infty A' \to D_\infty A.$$

Proof. We have already seen a variant of this. Consider the diagram

$$\begin{tikzcd}
D^{A'}_\infty \arrow[r, tail] \arrow[d, "f^*"] & D_\infty A'^+ \arrow[r] \arrow[d, "(f^+)^*"] & D_\infty \mathbb{K} \arrow[d, "\simeq"] \\
D_\infty A \arrow[r, tail] & D_\infty A^+ \arrow[r] & D_\infty \mathbb{K}
\end{tikzcd}$$

By Lemma 3.3.5, we have a commutative square

$$\begin{tikzcd}
DU(A'^{+}) \arrow[r, "\simeq"] \arrow[d, "U((f^{+})^{*})", "\simeq"'] & D_\infty A'^{+} \arrow[d, "(f^{+})^{*}"] \\
DU(A^{+}) \arrow[r, "\simeq"] & D_\infty A^{+}
\end{tikzcd}$$

Since $U((f^{+})^{*})$ is an equivalence by Corollary 3.1.24.3, $((f^{+})^{*})$ is an equivalence as well. By the first diagram, f^{*} has to be an equivalence by the kernel property. □

A valuable property of the ∞-tensor is that it behaves like the ordinary tensor up to homotopy.

Lemma 3.3.8. *Let A be an A_∞-algebra. Let M be a strictly unital A-polydule. In the category uMod$_\infty^A$ we have the following:*

- *There is an ∞-quasi-isomorphism $M \otimes_A^\infty A \rightsquigarrow M$,*
- *and there is an ∞-quasi-isomorphism $M \rightsquigarrow$ Hom$_A^\infty(A, M)$.*

Proof. Since the second point is the transpose of the first point, we will only prove that $M \otimes_A^\infty \rightsquigarrow M$ is an ∞-quasi-isomorphism.

We define the multiplication morphism componentwise

$$g_{i,j} : M \otimes_A^\infty A \to M,$$
$$m \otimes [a_1 \mid \cdots \mid a_j] \otimes a \otimes a'_1 \otimes \cdots \otimes a'_{i-1} \mapsto m_{1+j+1+i-1}(m, a_1, \cdots, a_j, a, a'_1, \cdots, a'_i),$$

so that $g_i = \sum_{j=1}^{\infty} g_{i,j}$.

To see that g defines an ∞-quasi-isomorphism we calculate the homology of cone(g_1).

One may observe that the morphism

$$id_M \otimes v_A[1] \otimes id_A : M \otimes (A[1])^{\otimes i} \otimes A \to M \otimes (A[1])^{\otimes i+1} \otimes A$$

induces a homotopy between $id_{\text{cone}(g_1)}$ and 0, so g_1 is indeed a quasi-isomorphism. □

We are now going to define other categories which will look very similar to the derived category in the augmented case. It is also true that these categories will be equivalent to the derived category in the strictly unital case.

Definition 3.3.9 (Compactly generated triangulated category)**.** Let A be a strictly unital A_∞-algebra. We let $\langle A \rangle$ denote the smallest thick triangulated subcategory category of $D_\infty A^{+}$ containing A which is closed under infinite sums.

Definition 3.3.10 (Homotopy category)**.** Let A be a strictly unital A_∞-algebra. Let the homotopy category be

$$K_\infty A = \text{suMod}_\infty^A / \sim,$$

where $\sim$ is a homotopy equivalence.

We are not sure if the congruence relation generated by the homotopy equivalence is strictly greater than homotopy equivalences. However, by considering the restriction map

$$r = \begin{pmatrix} id_A & \upsilon_A \end{pmatrix} : A^+ \to A,$$

we obtain a faithful functor

$$r^* : \text{suMod}_\infty^A \to \text{suMod}_\infty^{A^+},$$

which respects homotopy equivalences. This functor also induces a fully faithful functor

$$r^*/\sim : K_\infty A \to K_\infty A^+.$$

Since homotopy equivalence is a congruence relation in the latter category, it necessarily has to be that in the former category.

Theorem 3.3.11. *Let A be a strictly unital A_∞-algebra. The following categories are equivalent:*

- $D_\infty A$
- $\langle A \rangle$
- $K_\infty A$
- *suMod*$_\infty^A[Qis^{-1}]$
- *Ho*(*suMod*$_{\infty,\textit{strict}}^A$)

Proof of $D_\infty A \simeq \langle A \rangle$. To see this, we would like to have an exact sequence of triangulated categories

$$\langle A \rangle \rightarrowtail D_\infty A^+ \twoheadrightarrow D_\infty \mathbb{K}$$

By [Proposition 3.2.8 Kra21, p. 81] it suffices to show that for any A^+-polydule M, in the triangle

$$M \otimes_{A^+}^\infty A \longrightarrow M \longrightarrow M \otimes_{A^+}^\infty \mathbb{K} \longrightarrow (M \otimes_{A^+}^\infty A)[1]$$

the objects $M \otimes_{A^+}^\infty A \in \langle A \rangle$ and $M \otimes_{A^+}^\infty \mathbb{K}$ are $\langle A \rangle$-local. An object of $M \in D_\infty A^+$ is said to be $\langle A \rangle$-local if for any $L \in \langle A \rangle$

$$D_\infty A^+(L, M) = 0.$$

We start by observing that $M \otimes_{A^+} A = M \otimes BA^+ \otimes A$, so $M \otimes^{\infty}_{A^+} A$ is in fact contained in $\langle A \rangle$.

To see that $M \otimes_{A^+} \mathbb{K}$ is $\langle A \rangle$-local, we start by considering the following triangle

$$A \otimes^{\infty}_{A^+} \mathbb{K} \longrightarrow A^+ \otimes^{\infty}_{A^+} \mathbb{K} \longrightarrow \mathbb{K} \otimes^{\infty}_{A^+} \mathbb{K} \longrightarrow (A \otimes^{\infty}_{A^+} \mathbb{K})[1]$$

By assumption, A is strictly unital, so it is also homologically unital, even if considered as an A-polydule. By Lemma 3.2.38, A is H-unitary as an A^+-polydule. Notice that $A \otimes_{A^+} \mathbb{K} = A \otimes BA^+ \otimes \mathbb{K} \simeq B_{A^+} A$. Since A is H-unitary, we get that $A \otimes^{\infty}_{A^+} \mathbb{K}$ is acyclic. Moreover, by thickness, any $L \in \langle A \rangle$ has the property that

$$L \otimes^{\infty}_{A^+} \mathbb{K} \simeq 0.$$

By acyclicity of $A \otimes^{\infty}_{A^+} \mathbb{K}$, we obtain an ∞-quasi-isomorphism

$$A^+ \otimes^{\infty}_{A^+} \mathbb{K} \to \mathbb{K} \otimes^{\infty}_{A^+} \mathbb{K}.$$

If we consider the projection

$$A^+ \otimes_{A^+} \mathbb{K} \to \mathbb{K},$$

we see that this is an ∞-quasi-isomorphism, since the cone is the bar construction of A^+. BA^+ is acyclic, as A^+ is strictly unital and thus H-unitary.

By composing these morphisms in the derived category $D_\infty A^+$, we get an isomorphism

$$\mathbb{K} \to \mathbb{K} \otimes^{\infty}_{A^+} \mathbb{K}.$$

Now, pick an arbitrary morphism $f : L \to M \otimes^{\infty}_{A^+} \mathbb{K}$. We have the following commutative diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d] & M \otimes^{\infty}_{A^+} \mathbb{K} \arrow[d, "\simeq"] \\ L \otimes^{\infty}_{A^+} \mathbb{K} \arrow[r] & M \otimes^{\infty}_{A^+} \mathbb{K} \otimes^{\infty}_{A^+} \mathbb{K} \end{tikzcd}$$

As $L \otimes^{\infty}_{A^+} \mathbb{K} \simeq 0$, the morphism f factors through 0. Thus $f = 0$. □

Proof of $D_\infty A \simeq K_\infty A$. Let M be an A^+-polydule. We evaluate $M \otimes^{\infty}_{A^+} \mathbb{K} = M \otimes BA^+ = B_{A^+} M$. In other words, M is H-unitary if and only if $M \otimes^{\infty}_{A^+} \mathbb{K}$ is acyclic. By definition, $D_\infty A$ is thus made up of every H-unitary A^+-polydules. By Lemma 3.2.38, we know that $D_\infty A$ is then formed by the homologically unital A-polydules. By Corollary 3.2.28.1, every such A-polydule is ∞-quasi-isomorphic to a strictly unital A-polydule.

For the augmented A_∞-algebra A^+ we know already that $K_\infty A^+ \simeq D_\infty A^+$. Thus $K_\infty A$ is exactly the kernel in the following diagram

$$K_\infty A \rightarrowtail K_\infty A^+ \twoheadrightarrow D_\infty \mathbb{K}$$

as the inclusion sends strictly unital polydules to H-unitary polydules. □

Proof of $K_\infty A \simeq$ suMod$^A_\infty$[Qis^{-1}]. Since there is a fully faithful functor

$$K_\infty A \rightarrowtail K_\infty A^+$$

it follows that every ∞-quasi-isomorphism in suMod$^A_\infty$ is a homotopy equivalence. Thus suMod$^A_\infty$[Qis^{-1}] $\simeq$ $K_\infty A$. □

We prove the final statement first in the case of ordinary associative algebras.

Lemma 3.3.12. *Let A be a differential graded algebra. The inclusion i : Mod^A $\to$ suMod$^A_\infty$ induces an equivalence of categories*

$$DA \simeq \textit{suMod}^A_\infty[\textit{Qis}^{-1}],$$

where $\_ \otimes^\infty_A A$ gives the inverse.

Proof. Let M be an A-polydule, and then we already know that there is an ∞-quasi-isomorphism $M \otimes^\infty_A A \rightsquigarrow M$.

Let instead M be an A-module. Then we can consider it an A-polydule by letting the higher multiplication $m_i = 0$ for any $i \geqslant 3$. Thus we see that the ∞-morphism g defined as in Lemma 3.3.8 is a strict morphism. In other words, $g = g_1$ defines a morphism of algebras.

We have already seen that the component g_1 is a quasi-isomorphism, so there is a quasi-isomorphism of modules $i(M) \otimes^\infty_A A \to M$. Thus we have proved that the derived categories $D_\infty A$ and DA composing the functors are isomorphic to applying the identity functors. Thus we get an equivalence

$$DA \simeq D_\infty A.$$

□

Before the last proof, we will need some technical lemmata.

Lemma 3.3.13 ([Proposition 7.5.0.2 Lef03, p. 171])**.** *Let A be a strictly unital A_∞-algebra, then there is a dg-algebra A' and a strictly unital acyclic cofibration*

$$A \rightsquigarrow A'.$$

Lemma 3.3.14. *[Proposition 3.2.4.5 Lef03, p. 106] Let A and A' be two strictly unital A_∞-algebras. If $i : A \rightsquigarrow A'$ is a strictly unital acyclic cofibration, then there is a strictly unital acyclic fibration $p : A' \to A$, such that $p \circ i = id_A$ and $i \circ p \sim id_{A'}$.*

Lemma 3.3.15. *[Lemme 4.1.3.15 Lef03, p. 128] Let A and B be two unital differential graded algebras. Let $f, f' : A \to B$ be two morphisms of algebras, such that they are right homotopic $f \sim_r f'$. The restriction functors*

$$f^*, f'^* : \mathit{Mod}^B \to \mathit{Mod}^A \tag{3.2}$$

induces equivalent functors on the derived category

$$f^* \simeq f'^* : DB \to DA.$$

Proof of $suMod^A_\infty[Qis^{-1}] \simeq Ho(suMod^A_{\infty,strict})$. Assume first that A is a differential-graded associative algebra. We have the following chain of faithful inclusions

$$\mathsf{Mod}^A \rightarrowtail \mathsf{suMod}^A_{\infty,\mathsf{strict}} \rightarrowtail \mathsf{suMod}^A_\infty.$$

By Lemma 3.3.12, the composition is an equivalence on the derived categories and then necessarily essentially surjective and fully faithful. The last inclusion is, by definition, essentially surjective and fully faithful on the derived categories. In this manner, all three categories are equivalent.

We will now suppose that A is an A_∞-algebra. By Lemma 3.3.13, there exists a dg-algebra A' and an acyclic cofibration

$$p : A \rightsquigarrow A'.$$

By Lemma 3.3.14, there also exists an acyclic fibration $q : A' \rightsquigarrow A$, splitting p as $q \circ p = id_A$ and $p \circ q \sim id_{A'}$.

If we are using the model structures on $\mathsf{suMod}^A_{\infty,\mathsf{strict}}$ and $\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}}$ induced by the universal enveloping algebras, the morphisms p and q induces functors

$$\begin{aligned} \mathsf{Ho}(p^*) &: \mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}}) \to \mathsf{Ho}(\mathsf{suMod}^{A}_{\infty,\mathsf{strict}}) \text{ and,} \\ \mathsf{Ho}(q^*) &: \mathsf{Ho}(\mathsf{suMod}^{A}_{\infty,\mathsf{strict}}) \to \mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}}). \end{aligned}$$

If we have that

$$\begin{aligned} \mathsf{Ho}(p^*)\mathsf{Ho}(q^*) &\simeq \mathsf{Id}_{\mathsf{Ho}(\mathsf{suMod}^{A}_{\infty,\mathsf{strict}})} \text{ and} \\ \mathsf{Ho}(q^*)\mathsf{Ho}(p^*) &\simeq \mathsf{Id}_{\mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}})}, \end{aligned}$$

then we would be done. This is because $p^* : D_\infty A' \to D_\infty A$ induces an equivalence by Theorem 3.3.7. Thus we may consider the following commutative diagram

$$\begin{tikzcd}
\mathsf{Ho}(\mathsf{suMod}^{A}_{\infty,\mathsf{strict}}) \arrow[r, "\simeq"] \arrow[d, dashed] & \mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}}) \arrow[d, "\simeq"] \\
D_\infty A & D_\infty A' \arrow[l, "\simeq"']
\end{tikzcd}$$

Here the equivalence on the right-hand side is given by the case for ordinary algebras treated earlier. Finally, by previous results we know that $D_\infty A \simeq \mathsf{suMod}^{A}_{\infty}[\mathsf{Qis}^{-1}]$.

To see that we have the equivalences as claimed, we first note that the first one is automatic by the equation $q \circ p = id_A$. We must show that $p \circ q$ is isomorphic to the identity on $\mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}})$. By the earlier argument, proving this will be the same as proving that $p \circ q$ induces an equivalence on DA'. Since $p \circ q$ is homotopic to $id_{A'}$, they induce isomorphic morphisms in the category $\mathsf{HocoAlg}_{\mathbb{K}}$ by the bar construction and Proposition 2.3.1. By Corollary 2.2.13.1, there are isomorphisms of categories

$$\mathsf{HoAlg}_{\mathbb{K}} \simeq \mathsf{HocoAlg}_{\mathbb{K}}.$$

Thus $p \circ q$ is isomorphic to $id_{A'}$ in $\mathsf{HoAlg}_{\mathbb{K}}$. We replace this morphism by taking the universal enveloping algebra. Thus there is a morphism $r : U(A') \to U(A')$ which is isomorphic to $id_{U(A')}$ and $p \circ q$ in $\mathsf{HoAlg}_{\mathbb{K}}$. Since $U(A')$ is bifibrant r lifts from a weak equivalence to a homotopy equivalence by Whitehead's theorem, Theorem 2.1.30. We get by Lemma 3.3.15 that r induces the identity functor

$$r^* \simeq \mathsf{Id}_{DU(A')} : DU(A') \to D(A').$$

Moreover, $p \circ q$ has to induce the identity as well,

$$(p \circ q)^* \simeq \mathsf{Id}_{A'} : DA' \to D(A').$$

□

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Appendix A

Monads

This appendix is a short exposition on the theory of monads and comonads. The results we use may be found in Riehl [Rie16] or Mac Lane [Mac71].

A.1 Monads and Categories of Algebras

Definition A.1.1 (Monad)**.** Let $\mathcal{C}$ be a category. We say that an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with

- a multiplication $\mu : M \circ M \Rightarrow M$
- and a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow M$

is a monad, if the following diagrams commute

$$
\begin{tikzcd}
M \circ M \circ M \arrow[r, "M(\mu)"] \arrow[d, "\mu_M"] & M \circ M \arrow[d, "\mu"] \\
M \circ M \arrow[r, "\mu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \circ \mathrm{Id}_{\mathcal{C}} \arrow[r, "M(\eta)"] \arrow[dr, equal] & M \circ M \arrow[d, "\mu"] & \mathrm{Id}_{\mathcal{C}} \circ M \arrow[l, "\eta_M"'] \arrow[dl, equal] \\
& M &
\end{tikzcd}
$$

In other words, a monad is a monoid in the category of endofunctors, $(T, \mu, \eta) \in (\mathrm{End}\mathcal{C}, \circ, \mathrm{Id}_{\mathcal{C}})$.

Lemma A.1.2 (Monads from adjunctions, [Lemma 5.1.3. Rie14, p. 155])**.** *Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ and*

- *a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$*
- *and a counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$,*

there is an associated monad (T, μ, η)*. Let* $T = GF$*, together with*

- *a multiplication given by the counit* $\mu = G(\varepsilon_F) : T \circ T \Rightarrow T$
- *and the unit* $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow T$*,*

is a monad on $\mathcal{C}$*.*

Given any monad $(T : \mathcal{C} \to \mathcal{C}, \mu, \eta)$, we say that an object $M \in \mathcal{C}$ is a T-algebra if there exists a morphism $m : T(M) \to M$ such that the following diagrams commute

$$\begin{tikzcd} T \circ T(M) \arrow[r, "T(m)"] \arrow[d, "\varepsilon_M"] & T(M) \arrow[d, "m"] \\ T(M) \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, equal] & T(M) \arrow[d, "m"] \\ & M \end{tikzcd}$$

If M and N are two T-algebras, then we say that a morphism $f : M \to N$ is a T-algebra morphism if the following diagram commute

$$\begin{tikzcd} T(M) \arrow[r, "T(f)"] \arrow[d, "m"] & T(N) \arrow[d, "n"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Definition A.1.3 (Eilenberg-Moore category)**.** The Eilenberg-Moore category or the category of algebras $\mathcal{C}^T$ is the category having

- objects as M as T-algebras
- and morphisms $f : M \to N$ as T-algebra morphisms.

There is a free functor from $\mathcal{C}$ to T-algebras

$$F^T : \mathcal{C} \to \mathcal{C}^T, \quad M \mapsto (T(M), \mu_M).$$

By forgetting the T-algebra structure, we obtain a forgetful functor

$$U^T : \mathcal{C}^T \to \mathcal{C}, \quad (M, m) \mapsto M.$$

The next lemma justifies calling these functors free and forgetful.

Lemma A.1.4 (Adjunctions from monads, [Lemma 5.2.8 Rie14, p. 162])**.** *Given any monad* $(T, \mu, \eta) : \mathcal{C} \to \mathcal{C}$*, then the pair of functors* F^T *and* U^T *defines an adjunction*

$$F^T \dashv U^T : \mathcal{C} \to \mathcal{C}^T.$$

Definition A.1.5 (Free T-algebra)**.** (M, m) is a free T-algebra if there is an object $N \in \mathcal{C}$ and an isomorphism $(M, m) \simeq F^T(N)$.

In the category of algebras $\mathcal{C}^T$, we may approximate every T-algebra M by free T-algebras. This means that we may construct a canonical free resolution of any T-algebra M.

Proposition A.1.6 (Free resolutions, [Proposition 5.4.3 Rie14, p. 169])**.** *Given any T-algebra M, then*

$$((T \circ T)(M), \mu_{TM}) \underset{\mu_M}{\overset{Tm}{\rightrightarrows}} (TM, \mu_M) \xrightarrow{\ m\ } (M, m)$$

is a colimit diagram in $\mathcal{C}^T$.

It is useful to recognize when a category is a category of some algebra. Then every object is generated by every free object, which may arise from a simpler category.

Definition A.1.7 (Monadicity)**.** Suppose that there is an adjunction $F \vdash G : \mathcal{C} \to \mathcal{D}$ and that $T = GF$. We say that the adjunction, or $G : \mathcal{D} \to \mathcal{C}$, is monadic if there exists an equivalence of categories $K : \mathcal{D} \to \mathcal{C}^T$ such that there are natural isomorphisms $G \simeq U^T \circ K$ and $F^T \simeq K \circ F$.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K", "\simeq"'] \arrow[dr, shift right, "G"'] & & \mathcal{C}^T \arrow[dl, shift right, "U^T"'] \\
& \mathcal{C} \arrow[ul, shift right, "F"'] \arrow[ur, shift right, "F^T"'] &
\end{tikzcd}$$

Many of the categories which we consider are monadic.

Example A.1.8 (Ab is monadic over Set, [Corollary 5.5.3 Rie14, p. 174]). Consider the adjoint pair of functors $\mathbb{Z}\_ \dashv$ forget : Set → Ab, where we define

$$\begin{aligned} \mathbb{Z}\_ : \mathsf{Set} &\to \mathsf{Ab}, \\ M &\mapsto \mathbb{Z}M. \end{aligned}$$

The binary operation on the group is given by formal linear combinations. This adjoint pair is monadic.

Example A.1.9 (Mod^R is monadic over Mod$^{\mathbb{K}}$). The adjoint pair of functors $\_ \otimes_{\mathbb{K}} R \dashv$ forget : Mod$^{\mathbb{K}}$ → Mod^R is monadic.

Example A.1.10 (Alg$_{\mathbb{K},+}$ is monadic over Mod$^{\mathbb{K}}$). The adjoint pair $T\_ \dashv$ forget : Mod$^{\mathbb{K}}$ → Alg$_{\mathbb{K},+}$, where T is the tensor algebra, is monadic.

Definition A.1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that a functor $G : \mathcal{D} \to \mathcal{E}$ creates limits, if the composite $GF : \mathcal{C} \to \mathcal{E}$ has a limit E, then the limit cone $\lambda : \Delta_E \Rightarrow GF$ lifts to a limit cone $\widehat{\lambda} : \Delta_D \Rightarrow F$ such that G reflects the limit E to D.

One very good property about categories of algebras is that their small limits are well-behaved. These are created by limits as in $\mathcal{C}$. We have the following result:

Theorem A.1.12 ([Theorem 5.6.5 Rie14, p. 181])**.** *A monadic functor $G : \mathcal{D} \to \mathcal{C}$*

- *creates any limits which $\mathcal{C}$ has,*
- *and creates any colimits $\mathcal{C}$ has and which are preserved by the monad T and its square $T \circ T$.*

A.2 Comonads and Categories of Coalgebras

In this section, we will dualize the definitions and results from the last section. One could think of the dual themselves, but we do this for clarity.

Definition A.2.1 (Comonad)**.** Let $\mathcal{C}$ be a category. We say that an endofunctor $W : \mathcal{C} \to \mathcal{C}$ together with

- a comultiplication $\nu : W \Rightarrow W \circ W$
- and a counit $\varepsilon : W \Rightarrow \mathrm{Id}_{\mathcal{C}}$

is a comonad, if the following diagrams commute

$$\begin{tikzcd} W \circ W \circ W & W \circ W \arrow[l, "W(\nu)"'] \\ W \circ W \arrow[u, "\nu_W"] & W \arrow[l, "\nu"] \arrow[u, "\nu"'] \end{tikzcd} \qquad \begin{tikzcd} W \circ \mathrm{Id}_{\mathcal{C}} & W \circ W \arrow[l, "W(\varepsilon)"'] \arrow[r, "\varepsilon_W"] & \mathrm{Id}_{\mathcal{C}} \circ W \\ & W \arrow[ul, equal] \arrow[u, "\nu"] \arrow[ur, equal] & \end{tikzcd}$$

Lemma A.2.2 (Comonads from adjunctions)**.** *Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ with*

- *unit η : Id$_{\mathcal{C}} \Rightarrow GF$*
- *and a counit $\varepsilon : FG \Rightarrow$ Id$_{\mathcal{D}}$,*

there is an associated comonad (W, ν, ε). Let $W = FG$, together with

- *a comulitplication given by the unit $\nu = F(\eta_G) : W \Rightarrow W \circ W$*
- *and the counit $\varepsilon : W \Rightarrow$ Id$_{\mathcal{D}}$*

is a comonad on $\mathcal{D}$.

Given any comonad $(W : \mathcal{D} \to \mathcal{D}, \nu, \varepsilon)$, we say that M is a W-coalgebra if there exists a morphism $w : M \to W(M)$ such that the following diagrams commute

$$
\begin{tikzcd}
W \circ W(M) & W(M) \arrow[l, "W(w)"'] \\
W(M) \arrow[u, "\nu_M"] & M \arrow[l, "w"] \arrow[u, "w"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M & W(M) \arrow[l, "\varepsilon_M"'] \\
 & M \arrow[ul, equal] \arrow[u, "w"']
\end{tikzcd}
$$

Given two W-coalgebras M and N we say that a morphism $f : M \to N$ is a W-coalgebra morphism if the following diagram commutes

$$
\begin{tikzcd}
W(M) \arrow[r, "W(f)"] & W(N) \\
M \arrow[u, "w"] \arrow[r, "f"] & N \arrow[u, "u"]
\end{tikzcd}
$$

Definition A.2.3 (Category of coalgebras)**.** The category of coalgebras $\mathcal{C}_W$ is the category having

- objects M as W-coalgebras
- and morphisms $f : M \to N$ as W-coalgebra morphisms.

There is a cofree functor from $\mathcal{D}$ to W-coalgebras

$$
\begin{aligned}
F_W : \mathcal{D} &\to \mathcal{D}_W, \\
M &\mapsto (W(M), \nu_M).
\end{aligned}
$$

By forgetting the W-coalgebra structure, we obtain a forgetful functor

$$
\begin{aligned}
U_W : \mathcal{D}_W &\to \mathcal{D}, \\
(M, w) &\mapsto M.
\end{aligned}
$$

Lemma A.2.4 (Adjunctions from comonads)**.** *Given any comonad $(W, \nu, \varepsilon) : \mathcal{D} \to \mathcal{D}$, the the pair of functors U_W and F_W defines an adjunction*

$$U_W \dashv F_W : \mathcal{D}_W \to \mathcal{D}.$$

In the category of coalgebras $\mathcal{D}_W$, every object may be cogenerated from cofree W-coalgebras.

Definition A.2.5 (Cofree W-coalgebras)**.** (M, w) is a cofree W-coalgebra if there is an object $N \in \mathcal{D}$ and an isomorphism $(M, w) \simeq F_W(N)$.

Proposition A.2.6 (Cofree resolutions)**.** *Given any W-coalgebra M, then*

$$
(M, m) \xrightarrow{w} (W(M), \nu_M) \overset{W(w)}{\underset{\nu_M}{\rightrightarrows}} (W \circ W(M), \nu_{W(M)})
$$

is a limit diagram in $\mathcal{D}_W$.

Definition A.2.7 (Comonadicity)**.** Suppose that there is an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ such that $W = FG$. We say that the adjunction, or the $F : \mathcal{C} \to \mathcal{D}$, is comonadic if there exists an equivalence of categories $K : \mathcal{D}_W \to \mathcal{C}$ such that there are natural isomorphisms $F \circ K \simeq F_W$ and $K \circ U_W \simeq G$.

As we would expect, we have the comonadic categories.

Example A.2.8 (coMod^C is comonadic over Mod$^{\mathbb{K}}$)*.* The adjoint pair of functors forget $\dashv$ $\_ \otimes_{\mathbb{K}} C$: coMod^C $\to$ Mod$^{\mathbb{K}}$ is comonadic.

Example A.2.9 (coAlg$_{\mathbb{K},\text{conil}}$ is comonadic over Mod$^{\mathbb{K}}$)*.* The adjoint pair of functors forget $\dashv T^c\_$: coAlg$_{\mathbb{K},\text{conil}}$ $\to$ Mod$^{\mathbb{K}}$.

Theorem A.2.10. *A comonadic functor $F : \mathcal{C} \to \mathcal{D}$*

- *creates any colimits which $\mathcal{D}$ has*
- *and creates and limits $\mathcal{D}$ has and which are preserved by the comonad W and its square $W \circ W$.*

A.3 Canonical Resolutions

As described by MacLane [Mac71, p. 180]: "Monads and their duals, the comonads, play via Δ a central role in homological algebra, ...". We will here look at a method to construct resolutions associated with comonads.

Let (W, ν, ε) be a comonad over an abelian category $\mathcal{D}$, then this is a comonoid in the category of endofunctors (End$\mathcal{D}, \circ, \text{Id}_{\mathcal{D}}$). By Proposition B.1.5, there is then a strong monoidal functor, which we denote by $W^{\circ ?}$, $W^{\circ ?}$: $\Delta_+^{\text{op}} \to$ End$\mathcal{D}$. Using the standard representation of simplicial objects, we see that the face and degeneracy maps are given as

$$\begin{array}{ccccccccc} \text{Id}_{\mathcal{D}} & & W & \xrightarrow{\nu} & W^{\circ 2} & \overset{W(\nu)}{\underset{\nu_W}{\rightrightarrows}} & W^{\circ 3} & \rightrightarrows & \cdots \\ \text{Id}_{\mathcal{D}} & \xleftarrow{\varepsilon} & W & \overset{W(\varepsilon)}{\underset{\varepsilon_W}{\leftleftarrows}} & W^{\circ 2} & \leftleftarrows & W^{\circ 3} & \leftleftarrows & \cdots \end{array}$$

Let M be an object of $\mathcal{D}$. Evaluating $W^{\circ ?}$ at M gives us a functor $W^{\circ ?}(M) : \Delta_+^{\text{op}} \to \mathcal{D}$. This may be made into a cochain complex by Example 1.1.50,

$$\cdots \longrightarrow W^{\circ 3}(M) \longrightarrow W^{\circ 2}(M) \longrightarrow W(M) \xrightarrow{\varepsilon_M} M \longrightarrow 0 \longrightarrow \cdots.$$

Definition A.3.1 (Canonical W-resoultion)**.** The cochain complex, as defined above, is the canonical W-resolution at M.

This canonical resolution is more of a recipe to see how a comonad on an abelian category induces a resolution.

Example A.3.2 (Free resolution). Let R be a $\mathbb{K}$-algebra. Then there is an adjunction $\_ \otimes_{\mathbb{K}} R \dashv$ forget : $\mathsf{Mod}^{\mathbb{K}} \to \mathsf{Mod}^{R}$. The comonad $\_ \otimes_{\mathbb{K}} R : \mathsf{Mod}^{R} \to \mathsf{Mod}^{R}$ induces free R-resolutions on every right R-module M.

$$\cdots \longrightarrow M \otimes_{\mathbb{K}} R^{\otimes 3} \longrightarrow M \otimes_{\mathbb{K}} R^{\otimes 2} \longrightarrow M \otimes_{\mathbb{K}} R \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

Appendix B

Simplicial Objects

B.1 The Simplex Category

The simplex category is, in some sense, the categorification of the standard topological simplices, Δ^n. This category carries the necessary data in order to define concepts such as homology or homotopy. This section will give a brief review of this category.

Definition B.1.1 (The simplex category)**.** The simplex category Δ consists of ordered sets $[n] = \{0, ..., n\}$ for any $n \in \mathbb{N}$. A morphism $f \in \Delta([m], [n])$ is a monotone function, i.e.

$$a \leqslant b \in [m] \implies f(a) \leqslant f(b) \in [n].$$

Definition B.1.2 (The augmented simplex category)**.** Δ_+ is called the augmented simplex category, where we add an initial object $[-1] = \varnothing$.

Definition B.1.3 (The reduced simplex category)**.** Δ_inj is called the reduced simplex category. The morphisms consist only of the injective morphisms in Δ.

Inspired by the topological simplices, the simplex category has coface and codegeneracy morphisms. The coface maps are the injective morphisms $\delta_i : [n] \to [n+1]$, while the codegeneracy maps are the surjective morphisms $\sigma_i : [n] \to [n-1]$.

$$\delta_i(k) = \begin{cases} k, \text{ if } k<i \\ k+1, \text{ otherwise} \end{cases} \qquad \sigma_i(k) = \begin{cases} k, \text{ if } k\leqslant i \\ k-1, \text{ otherwise} \end{cases}$$

Proposition B.1.4 ([Lemma Mac71, p. 177])**.** *Every morphism in Δ factors into coface and codegeneracy maps.*

This result tells us that understanding how these morphisms work in tandem will be very important in understanding the simplex category. Luckily, there are five identities that characterize these maps. These are called cosimplical identities.

1. $\delta_j\delta_i = \delta_i\delta_{j-1}$, if $i < j$
2. $\sigma_j\delta_i = \delta_i\sigma_{j-1}$, if $i < j$
3. $\sigma_j\delta_i = id$, if $i = j$ or $i = j+1$
4. $\sigma_j\delta_i = \delta_{i-1}\sigma_j$, if $i > j+1$
5. $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$, if $i \leqslant j$

If we want a more visual description of the simplex category, we may think of them in this manner. An inductive tower with an increasing amount of morphisms.

$$\begin{tikzcd}
[-1] \arrow[r] & {[0]} \arrow[r, shift left, "\delta_i"] \arrow[r, shift right] & {[1]} \arrow[r, shift left=2, "\delta_i"] \arrow[r] \arrow[r, shift right=2] & {[2]} \arrow[r, shift left=3, "\delta_i"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
[-1] & {[0]} & {[1]} \arrow[l, "\sigma_0"'] & {[2]} \arrow[l, shift right, "\sigma_i"'] \arrow[l, shift left] & \cdots \arrow[l, shift right=2, "\sigma_i"'] \arrow[l] \arrow[l, shift left=2]
\end{tikzcd}$$

The augmented simplex category has a universal monoid. Let $+ : \Delta_+ \times \Delta_+ \to \Delta_+$ be the functor acting on objects and morphisms as:

$$[m] + [n] = [m+n+1]$$
$$(f+g)(k) = \begin{cases} f(k), \text{ if } k \leqslant m \\ g(k)+m, \text{ otherwise} \end{cases}$$

$(\Delta_+, +, [-1])$ becomes a monoidal category. Unitality is satisfied as $[-1] + [m] = [1+m-1] = [m] = [m] + [-1]$. Associativity follows from the associativity of addition. Since addition acts on morphisms by juxtaposition, we get that the maps $id_{[}0] : [0] \to [0]$, $\delta_0 : [-1] \to [0]$ and $\sigma_0 : [1] \to [0]$ allows us to express any morphism in Δ by summing them.

Since the object $[0]$ is terminal, it automatically becomes a monoid in $(\Delta, +, [-1])$. The unit is the unique map $\delta_0 : [-1] \to [0]$, and the multiplication is the uniqe map $\sigma_0 : [1] \to [0]$. Associativity and unitality are automatically satisfied by the uniqueness of any morphism $f : [n] \to [0]$.

Proposition B.1.5 ([Proposition 1 Mac71, p. 175])**.** *Let $(\mathcal{C}, \otimes, Z)$ be a monoidal category. If (C, η, μ) is a monoid in $\mathcal{C}$, then there is a strong monoidal functor* $: \Delta_+ \to \mathcal{C}$*, such that $F[0] \simeq C$, $F\delta_{-1} \simeq \eta$ and $F\sigma_0 \simeq \mu$.*

B.2 Simplicial Objects

To exert the properties of the simplex category on another category $\mathcal{C}$, we look at functors from Δ into $\mathcal{C}$.

Definition B.2.1 (Simplical object)**.** A simplicial object in $\mathcal{C}$ is a functor $S : \Delta^{op} \to \mathcal{C}$.

Such an object may be viewed as a collection of objects $\{S_n\}_{n\in\mathbb{N}}$ together with face maps $d^i : S_n \to S_{n-1}$ and degeneracy maps $s^i : S_n \to S_{n+1}$. Additionally, these maps must satisfy the simplicial identities, which are dual to the cosimplical identities.

Definition B.2.2 (Augmented simplical object)**.** An augmented simplicial object is then a functor $S : \Delta_+^{op} \to \mathcal{C}$.

The restricted functor $\bar{S} : \Delta^{op} \to \mathcal{C}$ is called the augmentation ideal of S.

Definition B.2.3 (Semi-simplicial object)**.** A semi-simplicial object is a functor $S : \Delta_{inj} \to \mathcal{C}$.

Observe that a semi-simplicial object may be considered as a collection of objects $\{S_n\}$ such that we only have face maps satisfying the 1st simplicial identity.

Definition B.2.4 (cosimplical object)**.** A cosimplicial object is a functor $S : \Delta \to \mathcal{C}$.

Such an object may be regarded as a collection of objects together with coface and codegeneracy maps satisfying the cosimplicial identities.

Simplicial objects are studied across many different fields of mathematics.

Example B.2.5 (Simplicial sets). A simplicial set S is a collection of sets together with face and degeneracy maps. This is a functor $S : \Delta^{op} \to$ Set. The category of simplicial sets is usually denoted as sSet or Set_Δ.

Example B.2.6 (The standard topological n-simplex). The topological n-simplex Δ^n is a topological space. Abstracting away the n we get a functor $\Delta - : \Delta \to$ Top. In this manner, the collection of standard n-simplicies is a cosimplical object of Top.

Example B.2.7 (Rings). Any ring R is, by definition, a monoid in the category of abelian groups. By the above proposition, this monoid is uniquely determined by a strong monoidal functor $R : \Delta_+ \to$ Ab. Thus any ring is a cosimplical object of Ab.

Appendix C

Spectral Sequences

Here we will summarize spectral sequences and the classical convergence theorem of filtered spectral sequences. For a thorough account, look in Weibel [Wei94].

C.1 Filtrations

Let $\mathcal{A}$ be an abelian category. Given two objects A and B, we denote an inclusion $B \to A$ by $B \subseteq A$. This section is devoted to filtration terminology.

Definition C.1.1 (Filtration)**.** A filtration on an object A is a possibly infinite collection of inclusions

$$\cdots \subseteq A_i \subseteq A_{i+1} \subseteq A_{i+2} \subseteq \cdots \subseteq A.$$

Definition C.1.2 (Bounded filtration)**.** We say that a filtration on A is bounded below if there is an integer $s \in \mathbb{Z}$ such that

$$0 = A_s \subseteq A_{s+1} \subseteq \cdots A_i \subseteq \cdots \subseteq A.$$

We say that a filtration on A is bounded above if there is an integer $n \in \mathbb{Z}$ such that

$$\cdots \subseteq A_i \subseteq \cdots \subseteq A_t = A.$$

A filtration is bounded, or finite, if it is both bounded below and above, i.e., the filtration is finite;

$$0 = A_s \subseteq \cdots \subseteq A_i \subseteq \cdots \subseteq A_n = A.$$

Definition C.1.3 (Exhaustive filtrations)**.** A filtration on A is said to be exhaustive if $\varinjlim_i A_i \simeq A$,

$$\cdots \hookrightarrow A_i \hookrightarrow A_{i+1} \hookrightarrow \cdots \hookrightarrow \varinjlim_i A_i \simeq A.$$

Definition C.1.4 (Hausdorff filtrations)**.** A filtration on A is called Hausdorff if $\varprojlim A_i \simeq 0$.

Every bounded below filtration is Hausdorff by definition.

Definition C.1.5 (Complete filtrations)**.** Let $A/A_i = \varinjlim(A_i \to A)$. A filtration on A is called complete if $\varprojlim_i A/A_i \simeq A$,

$$A \simeq \varprojlim_i A/A_i \twoheadrightarrow \cdots \twoheadrightarrow A/A_i \twoheadrightarrow A/A_{i+1} \twoheadrightarrow \cdots$$

We denote the completion of A by $\varprojlim_i A/A_i \simeq \widehat{A}$, and we denote the completion of each subobject by $\widehat{A}_i = \varprojlim_{j\leqslant i} A_i/A_j$ There is a filtration on $\widehat{A}$ given by

$$\cdots \subseteq \widehat{A}_i \subseteq \widehat{A}_{i+1} \subseteq \cdots \subseteq \widehat{A}.$$

C.2 Spectral Sequence

For this section, we will let $\mathcal{A}$ be an abelian category. To be more precise, one should assume that $\mathcal{A}$ is bicomplete, that arbitrary coproducts of epis are epi, and that arbitrary products of monos are mono. Categories such as Mod^R for a ring R have these properties.

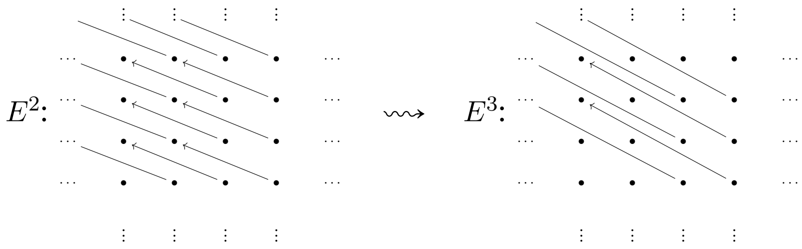
A spectral sequence is a method in which one may calculate the homology of chain complexes. For instance, there is a spectral sequence associated with each filtered chain complex. The spectral sequence will be defined in terms of pages.

Definition C.2.1 (Homology spectral sequence)**.** A homology spectral sequence E starting at page a is

- a collection of objects $E^r_{p,q}$ for any $p, q \in \mathbb{Z}$ and $r \geqslant a$,
- morhpisms $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r \circ d^r = 0$
- and isomorphisms between page $r+1$ and the homology of page r,

$$E^{r+1}_{p,q} \simeq \mathsf{Ker} d^r_{p,q} / \mathsf{Im} d^r_{p+r,q-r+1}.$$

We refer to the collection of objects $E^r_{\bullet,\bullet}$ for the r'th page of the spectral sequence E. A homology spectral sequence starting at the second page may be illustrated as



where we go from the second page to the third page by taking homology. At page r, each line along the form $(-r, r-1)$ defines a chain complex in $\mathsf{Ch}(\mathcal{A})$.

Definition C.2.2 (Cohomology spectral sequence)**.** A cohomology spectral sequence E starting at page a is

- a collection of objects $E_r^{p,q} \in \mathcal{A}$ for any $p, q \in \mathbb{Z}$ and $r \geqslant a$,
- morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$
- and isomorphisms between page $r+1$ and the homology of page r,

$$E_{r+1}^{p,q} \simeq {}^{\mathsf{Ker} d_r^{p,q}}\!/\!_{\mathsf{Im} d_r^{p-r,q+r-1}}$$

We divide a spectral sequence into diagonals. The object $E_{p,q}^r$ is said to be of degree n if $n = p+q$.

Definition C.2.3 (Bounded spectral sequence)**.** A homology spectral sequence E starting at page a is said to be bounded if there are only finitely many non-zero terms of every degree n.

Given a bounded spectral sequence E, there is a page r_0, such that for any $r \geqslant r_0$ p and q, $E_{p,q}^r \simeq E_{p,q}^{r+1}$. This stable, unchanging page will be denoted as $E^\infty = E^r$.

Definition C.2.4 (Bounded convergence)**.** A bounded homology spectral sequence is said to converge to H_* if, for each n, there is a finite filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_i H_n \subseteq \cdots \subseteq FtH_n = H_n,$$

such that $E_{p,q}^\infty \simeq {}^{F_p H_{p+q}}\!/\!_{F_{p-1} H_{p+q}}$. We write this as

$$E_{p,q}^a \Rightarrow H_{p+q}.$$

Suppose that we have a bounded homology spectral sequence E starting at page a, such that it converges $E^a \Rightarrow H$. To calculate each H_n, one would then have to solve extension problems. For instance, there is a short exact sequence

$$0 \longrightarrow F_{s+1} H_n \rightarrowtail F_{s+2} H_n \twoheadrightarrow E_{s+2,n-s-2}^\infty \longrightarrow 0.$$

In this manner, given some extra information, we could calculate the homology in terms of the ∞-page.

Definition C.2.5 (Collapse)**.** We say that a homology spectral sequence collapse at page $r \geqslant 2$ if there is at most one non-zero column or row in E^r.

Whenever a spectral sequence collapse at page r, this is automatically the ∞-page. If a spectral sequence converges $E^a \Rightarrow H$, then H_n is the unique non-zero object of degree n in E^∞.

Definition C.2.6 (∞-page)**.** Let E be a homology spectral sequence starting at page a. Define $Z^r_{p,q} = \mathrm{Ker} d^r_{p,q}$ and $B^r_{p,q} = \mathrm{Im} d^r_{p,q}$, then $E^{r+1}_{p,q} \simeq {}^{Z^r_{p,q}}/_{B^r_{p,q}}$. We define the ∞-page in terms

$$
\begin{aligned}
Z^\infty_{p,q} &= \varprojlim_{a \leqslant r} Z^r_{p,q} \text{ and} \\
B^\infty_{p,q} &= \varinjlim_{a \leqslant r} B^r_{p,q},
\end{aligned}
$$

such that

$$E^\infty_{p,q} = {}^{Z^\infty_{p,q}}/_{B^\infty_{p,q}}.$$

Definition C.2.7 (Morphism of spectral sequences)**.** A morphism of homology spectral sequences $f : E \to F$ is a collection of morphisms $f^r_{p,q} : E^r_{p,q} \to F^r_{p,q}$ such that $f^r \circ d^r_E = d^r_F \circ f^r$, and $H_* f^r \simeq f^{r+1}$.

Lemma C.2.8 (Mapping lemma, [Lemma 5.2.4 and Exercise 5.2.3 Wei94, p. 123])**.** *Let $f : E \to F$ be a morphism of spectral sequences. If $f^r : E^r \to F^r$ is an isomorphism, then $f^{r'} : E^{r'} \to F^{r'}$ is an isomorphism for any $r' \geqslant r$, and $f^\infty : E^\infty \to E^\infty$ is an isomorphism as well.*

Proof. The first statement is immediate from the functoriality of taking homology, as isomorphisms are sent to isomorphisms.

Suppose instead that for any page $r \geqslant a$, there is an isomorphism $f^r : E^r \to F^r$. Restricting this morphism to the kernels yields an isomorphism by the 5-lemma,

$$
\begin{tikzcd}
ZE^r_{p,q} \arrow[r, hook] \arrow[d, dashed, "Zf^r_{p,q}"] & E^r_{p,q} \arrow[r, "d^r_{p,q}"] \arrow[d, "\simeq"] & E^r_{p-r,q+r-1} \arrow[d, "\simeq"] \\
ZF^r_{p,q} \arrow[r, hook] & F^r_{p,q} \arrow[r, "d^r_{p,q}"] & F^r_{p-r,q+r-1}.
\end{tikzcd}
$$

Likewise, there is an isomorphism $Bf^r_{p,q} : BE^r_{p,q} \to BF^r_{p,q}$. In this manner, we obtain isomorphisms of diagrams

$$
\begin{tikzcd}
\cdots \arrow[r, hook] & ZE^r \arrow[r, hook] \arrow[d, "Zf^r"] & \cdots \arrow[r, hook] & ZE^a \arrow[d, "Zf^a"] \\
\cdots \arrow[r, hook] & ZF^r \arrow[r, hook] & \cdots \arrow[r, hook] & ZF^a
\end{tikzcd}
$$

$$\begin{tikzcd}
BE^a \arrow[r, hook] \arrow[d, "Bf^a"] & \cdots \arrow[r, hook] & BE^r \arrow[r, hook] \arrow[d, "Bf^r"] & \cdots \\
BF^a \arrow[r, hook] & \cdots \arrow[r, hook] & BF^r \arrow[r, hook] & \cdots
\end{tikzcd}$$

Thus the limits ZE^∞ and ZF^∞ and the colimits BE^∞ and BF^∞ exhibit the same universal property, respectively. By the 5-lemma, we obtain the isomorphism on the ∞-page

$$\begin{tikzcd}
BE^\infty \arrow[r, tail] \arrow[d, "\simeq"', "Bf^\infty"] & ZE^\infty \arrow[r, two heads] \arrow[d, "\simeq"', "Zf^r"] & E^\infty \arrow[d, dashed, "\simeq"', "f^\infty"] \\
BF^\infty \arrow[r, tail] & ZF^\infty \arrow[r, two heads] & F^\infty
\end{tikzcd}$$

□

Definition C.2.9 (Bounded below spectral sequences)**.** A homology spectral sequence E starting at page a is said to be bounded below if, for each degree n, there is an integer s such that if $p+q=n$, then $E^a_{p,q}=0$ for any $p<s$.

Definition C.2.10 (Regular spectral sequences)**.** A homology spectral sequence E is said to be regular if there is an r such that for any $r' \geqslant r$, we have that $d^r = 0$. In other words, $Z^\infty \simeq Z^r$.

Definition C.2.11 (Weak convergence)**.** A homology spectral sequence E weakly converges to H_* if each H_n has a filtration

$$\cdots \subseteq F_i H_n \subseteq \cdots \subseteq H_n$$

such that there are isomorphisms $E^\infty_{p,q} \simeq {}^{F_p H_{p+q}}\!/\!_{F_{p-1}H_{p+q}}$.

A problem with weak convergence, which we did not have with bounded convergence, is that the spectral sequence cannot detect the elements which may be found in either $\varprojlim F_i H_n$ or $\varinjlim F_i H_n$. This problem is amended if the filtration is exhaustive and Hausdorff; in this case, we say that the spectral sequence approaches H_*.

Definition C.2.12 (Convergence)**.** A homology spectral sequence E converges to H_* if it approaches H_*, E is regular and every H_n is complete, $H_n \simeq \widehat{H}_n$.

In this definition, we require regular because of practical reasons. One may observe that every bounded below spectral sequence which approaches $H*$ converges to $H*$. Completeness is assumed for the following theorem.

Theorem C.2.13 (Comparison Theorem, [Theorem 5.2.12 Wei94, p. 126])**.** *Let E and E' be homology spectral sequences converging to H_* and H'_*, respectively. Suppose that there is a morphism $h : H_* \to H'_*$, which is compatible with a morphism of spectral sequences $f : E \to E'$. If $f^r : E^r \to F^r$ is an isomorphism, then h is an isomorphism as well.*

Proof. There are short exact sequences and a morphism between them,

$$\begin{tikzcd} 0 \arrow[r] & {}^{F_{p-1}H_n}/_{F_sH_n} \arrow[r] \arrow[d] & {}^{F_pH_n}/_{F_sH_n} \arrow[r] \arrow[d] & E^{\infty}_{p,n-p} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & {}^{F_{p-1}H'_n}/_{F_sH'_n} \arrow[r] & {}^{F_pH'_n}/_{F_sH'_n} \arrow[r] & E'^{\infty}_{p,n-p} \arrow[r] & 0 \end{tikzcd}$$

by weak convergence. Since we assume f^{∞} to be an isomorphism, we get the isomorphism on the last component. If we fix $s \geqslant 0$, then by doing induction on $p \geqslant s$ the 5-lemma tells us that there are isomorphisms,

$$ {}^{F_pH_n}/_{F_sH_n} \simeq {}^{F_pH'n}/_{F_sH'n}. $$

Since we assume F_pH_n to be exhaustive, it follows that

$$ {}^{H_n}/_{F_sH_n} \simeq {}^{H'_n}/_{F_sH'_n}. $$

Moreover, since F_pH_n is complete, we get that $H_n \simeq H'n$ by taking the limit over s. □

C.3 Spectral Sequence of a Filtration

Associated with a filtration F on a chain complex C, there is a homology spectral sequence E starting at page 0. We define $E^0_{p,q} = {}^{F_pC_{p+q}}/_{F_{p-1}C_{p+q}}$, where the differential is induced by the associated graded. The 1-page is then the homology along each associated graded piece, $E^1_{p,q} = H_*(E^0_{p,*})$.

One may observe that the spectra sequence arising from C is the same as the spectral sequence arising from its completion $\widehat{C}$.

We describe the spectral sequence in more detail. Let $\pi_p : F_pC \to {}^{F_pC}/_{F_{p-1}C}$. We let

$$ A^r_p = \{c \in F_pC \mid d(c) \in F_{p-r}C\} $$

be the collection of cycles modulo $F_{p-r}C$. Then we define the complexes in E^0

$$ Z^r_{p,*} = \pi_p(A^r_p) \text{ and} $$
$$ B^{r+1}_{p-r,*} = \pi_{p-r}(d(A^r_p)). $$

Every page may then be described as $E^r_p = {}^{Z^r_p}/_{B^r_p}$.

The important takeaway is the following theorem.

Theorem C.3.1 (Classical convergence theorem, [Theorem 5.5.1 Wei94, p. 135])**.** *Let C be a chain complex.*

- *Suppose that the filtration on C is bounded. Then the spectral sequence E is bounded and $E^1_{p,q} \Rightarrow H_{p+q}(C)$.*
- *Suppose that the filtration on C is bounded below and exhaustive. Then the spectral sequence E is bounded below and $E^1_{p,q} \Rightarrow H_{p+q}(C)$.*

This convergence is also natural in the sense that given any morphism of chain complexes $f : C \to D$. Then the morphism in homology $H_*f : H_*C \to H_*D$ is compatible with the morphism of spectral sequences $E^1 f : EC^1 \to ED^1$.

Appendix D

Symmetric Monoidal Categories

D.1 Monoidal Categories

Here we will give a brief summary of symmetric monoidal categories. More detailed accounts may be found in Mac Lane [Mac71], Riehl [Rie14], or Kelly [Kel05a].

Definition D.1.1 (Monoidal category)**.** We say that a category $\mathcal{C}$ is a monoidal category if it comes equipped with

- a bifunctor

$$\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

- a natural isomorphism in three variables

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

- a unit object $Z \in \mathcal{C}$
- and natural isomorphisms

$$\lambda_A : Z \otimes A \to A,$$
$$\rho_A : A \otimes Z \to A.$$

Moreover, these maps should satisfy some coherence relations. The following diagrams should commute,

$$
\begin{tikzcd}[column sep=small]
 & & A \otimes (B \otimes (C \otimes D)) \arrow[dll, "\alpha_{A,B,C\otimes D}"'] \arrow[drr, "A\otimes\alpha_{B,C,D}"] & & \\
A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B\otimes C,D}"] & & & & (A \otimes B) \otimes (C \otimes D) \arrow[dl, "\alpha_{A\otimes B,C,D}"'] \\
 & (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B,C\otimes D}"] & & ((A \otimes B) \otimes C) \otimes D &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (Z \otimes B) \arrow[rr, "\alpha_{A,Z,B}"] \arrow[dr, "A\otimes\lambda_B"'] & & (A \otimes Z) \otimes B \arrow[dl, "\rho_A\otimes B"] \\
 & A \otimes B &
\end{tikzcd}
$$

The coherence diagrams allow us to think of the monoidal product $\otimes$ as an associative and unital product. If the identities give α, λ, and ρ, we say that the monoidal category is strict.

Definition D.1.2 (Lax monoidal functors)**.** Let $(\mathcal{C}, \otimes, Z)$ and $(\mathcal{D}, \boxtimes, W)$ be monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is monoidal if it comes equipped with

- a natural transformation

$$\mu_{A,B} : F(A) \boxtimes F(B) \to F(A \otimes B)$$

- and a morphism of units

$$\upsilon : W \to F(Z).$$

Furthermore, the following diagrams should commute.

$$
\begin{tikzcd}[column sep=small]
 & F(A) \boxtimes (F(B) \boxtimes F(C)) \arrow[rr, "\alpha^{\mathcal{D}}_{F(A),F(B),F(C)}"] \arrow[dl, "F(A)\boxtimes\mu_{B,C}"] & & (F(A) \boxtimes F(B)) \boxtimes F(C) \arrow[dr, "\mu_{A,B}\boxtimes F(C)"] & \\
F(A) \boxtimes (F(B \otimes C)) \arrow[dr, "\mu_{A,B\otimes C}"] & & & & F(A \otimes B) \boxtimes F(C) \arrow[dl, "\mu_{A\otimes B,C}"] \\
 & F(A \otimes (B \otimes C)) \arrow[rr, "F(\alpha^{\mathcal{C}}_{A,B,C})"] & & F((A \otimes B) \otimes C) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(A) \boxtimes W \arrow[r, "\rho^{\mathcal{D}}_{F(A)}"] \arrow[d, "F(A)\boxtimes\upsilon"] & F(A) \\
F(A) \boxtimes F(Z) \arrow[r, "\mu_{A,Z}"] & F(A \otimes Z) \arrow[u, "F(\rho^{\mathcal{C}}_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
W \boxtimes F(A) \arrow[r, "\lambda^{\mathcal{D}}_{F(A)}"] \arrow[d, "\upsilon\boxtimes F(A)"] & F(A) \\
F(Z) \boxtimes F(A) \arrow[r, "\mu_{Z,A}"] & F(Z \otimes A) \arrow[u, "F(\lambda^{\mathcal{C}}_A)"']
\end{tikzcd}
$$

The monoidal functor is said to be strong monoidal if μ is a natural isomorphism and υ is an isomorphism. If the morphisms μ and υ are given by identities, then we say that the functor is strict monoidal.

Definition D.1.3 (Monoidal natural transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be lax monoidal functors between monoidal categories. We say that a natural transformation $\theta : F \Rightarrow G$ is a monoidal natural transformation if the following diagrams commute

$$
\begin{tikzcd}
F(A) \boxtimes F(B) \arrow[r, "\mu_{A,B^F}"] \arrow[d, "\theta_A \boxtimes \theta_B"] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
G(A) \boxtimes G(B) \arrow[r, "\mu^G_{A,B}"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(Z) \arrow[dd, "\theta_Z"] \\
W \arrow[ur, "\upsilon^F"] \arrow[dr, "\upsilon^G"'] & \\
 & G(Z)
\end{tikzcd}
$$

Definition D.1.4 (Braided monoidal category)**.** Let $\mathcal{C}$ be a monoidal category. We say that the category is braided if it comes equipped with natural isomorphisms

$$\beta_{A,B} : A \otimes B \to B \otimes A,$$

which has the following commutative diagrams for any A, B and C.

$$
\begin{tikzcd}
A \otimes Z \arrow[rr, "\beta_{A,Z}"] \arrow[dr, "\rho_A"'] & & Z \otimes A \arrow[dl, "\lambda_A"] \\
 & A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\beta_{A,B\otimes C}"] \arrow[d, "\alpha^{-1}_{A,B,C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha_{C,A,B}"] \\
A \otimes (B \otimes C) \arrow[d, "A\otimes\beta_{B,C}"] & (C \otimes A) \otimes B \arrow[d, "\beta_{C,A\otimes B}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha_{A,C,B}"] & (A \otimes C) \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] \arrow[d, "\alpha_{A,B,C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha^{-1}_{B,C,A}"] \\
(A \otimes B) \otimes C \arrow[d, "\beta_{A,B}\otimes C"] & B \otimes (C \otimes A) \arrow[d, "B\otimes\beta_{C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha^{-1}_{B,A,C}"] & B \otimes (A \otimes C)
\end{tikzcd}
$$

Definition D.1.5 (Symmetric monoidal category)**.** A braided monoidal category $\mathcal{C}$ is called symmetric if the braiding β is chosen so that it has its own inverses, i.e., the following diagram commutes.

$$
\begin{tikzcd}
A \otimes B \arrow[rr, equal] \arrow[dr, "\beta_{A,B}"] & & A \otimes B \\
 & B \otimes A \arrow[ur, "\beta_{B,A}"] &
\end{tikzcd}
$$

In the case of symmetric braiding, one only has to check that either one of the braiding hexagons commutes, as the other follows from symmetry.

Definition D.1.6 (Braided lax monoidal functor)**.** We say that a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between braided categories is braided if it commutes with braiding in the sense of the following commutative diagram.

$$\begin{array}{ccc} F(A) \boxtimes F(B) & \xrightarrow{\beta^{\mathcal{D}}_{F(A),F(B)}} & F(B) \boxtimes F(A) \\ \big\downarrow{\scriptstyle \mu_{A,B}} & & \big\downarrow{\scriptstyle \mu_{B,A}} \\ F(A \otimes B) & \xrightarrow{F(\beta^{\mathcal{C}}_{A,B})} & F(B \otimes A) \end{array}$$

Definition D.1.7 (Closed symmetric monoidal category)**.** A symmetric monoidal category $(\mathcal{C}, \otimes, Z)$ is said to be closed if for any $C \in \mathcal{C}$, the functor $\_ \otimes C : \mathcal{C} \to \mathcal{C}$ has a right adjoint $[C, \_] : \mathcal{C} \to \mathcal{C}$. The object $[C, D]$ is usually called the internal hom of C.

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