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Gabor frames for rational functions

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Abstract We study the frame properties of the Gabor systems

$$\mathfrak{G}(g;\alpha,\beta) := \{e^{2\pi i\beta m x} g(x-\alpha n)\}_{m,n\in\mathbb{Z}}.$$

In particular, we prove that for Herglotz windows g such systems always form a frame for $L^2(\mathbb{R})$ if $\alpha,\beta>0, \alpha\beta\le 1$. For general rational windows $g\in L^2(\mathbb{R})$ we prove that $\mathfrak{G}(g;\alpha,\beta)$ is a frame for $L^2(\mathbb{R})$ if $0<\alpha,\beta,\alpha\beta<1, \alpha\beta\notin\mathbb{Q}$ and $\hat{g}(\xi)\ne 0, \xi>0$, thus confirming Daubechies conjecture for this class of functions. We also discuss some related questions, in particular sampling in shift-invariant subspaces of $L^2(\mathbb{R})$.

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1 Introduction and main results

We investigate the Gabor systems generated by the linear combinations of the Cauchy kernels, i.e. by the windows of the form

$$g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i w_k}. \tag{1.1}$$

We describe a new wide class of such functions for which the corresponding Gabor systems possess the frame property for all rectangular lattices of density at least one.

For the general rational windows of the form (1.1) we discover that the frame property depends on rationality of the product $\alpha\beta$ and also give the precise estimate of the lattice density $(\alpha\beta)^{-1}$ which grantees the frame property of the corresponding Gabor system. Finally we study sampling in shift-invariant spaces, generated by the windows of the form (1.1) as well as some related matters.

One of the central motives of this article is hinted by the Daubechies conjecture [3, p. 981] which assumes that Gabor system is a frame for all $\alpha\beta < 1$ whenever g is a positive function with positive Fourier transform. This conjecture has been disproved by Janssen [8], yet in all known examples of functions which generates a Gabor system for all $\alpha\beta < 1$ we encounter some kind of positivity.

1.1 Gabor systems

Given a function $g \in L^2(\mathbb{R})$ we denote by $\pi_{x,y} g$ its time frequency shifts

$$\pi_{x,y} g(t) = e^{2\pi i y t} g(t - x), \; x, y \in \mathbb{R}. \tag{1.2}$$

For $\alpha, \; \beta > 0$ consider the Gabor system

$$\mathfrak{G}(g; \alpha, \beta) = \{\pi_{\alpha m, \beta n} g; \; m, n \in \mathbb{Z}\}. \tag{1.3}$$

We say that $\mathfrak{G}(g; \alpha, \beta)$ is a *frame* in $L^2(\mathbb{R})$ if the frame inequality

$$A\|f\|^2 \le \sum_{m,n} \left|\langle f, \pi_{\alpha m, \beta n} g\rangle\right|^2 \le B\|f\|^2, \; f \in L^2(\mathbb{R}) \tag{1.4}$$

holds for some $A > 0$ and $B < \infty$.

Gabor systems are widely used in signal analysis and quantum mechanics because of time-frequency localization of their elements $\pi_{\alpha m,\beta n}g$. For sufficiently dense lattices $\alpha\mathbb{Z}\times\beta\mathbb{Z}$ the supports of $\pi_{\alpha m,\beta n}g$ "cover" the whole time-frequency plane and the frame inequality (1.4) provides stable reconstruction of a signal f from the inner products $\langle f,\pi_{\alpha m,\beta n}g\rangle$. On the other hand $\mathfrak{G}(g;\alpha,\beta)$ never forms a frame if $\alpha\beta>1$ (see e.g. [10]). We refer the reader to [3,10,15], for the detailed history, as well as setting and discussion of the problem.

1.2 Frame set

The fundamental problem of the Gabor analysis is to describe the *frame set* of the window g:

$$\mathcal{F}(g)=\{(\alpha,\ \beta);\alpha,\ \beta>0 \text{ and } \mathfrak{G}(g;\alpha,\beta) \text{ is a frame in } L^2(\mathbb{R})\}.$$

If $\alpha\beta=1$ the complete characterization of the frame set can be given in terms of the Zak tarnsform $\mathcal{Z}g$ of the window g (see e.g. [10, Ch.8]) but for $\alpha\beta<1$ the frame set $\mathcal{F}(g)$ may be very complicated even for elementary functions g, see e.g. [2,4]. Even the simpler question: for which g does $\mathcal{F}(g)$ contain the whole set $\Pi:=\{(\alpha,\beta);\alpha,\beta>0,\ \alpha\beta<1\}$ is also very difficult.

The answer has been obtained for the Gaussian e^{-x^2} [17,24,25], truncated $\chi_{(0,\infty)}(x)e^{-x}$ and symmetric $e^{-|x|}$ exponential functions [6,7], the hyperbolic secant $(e^x+e^{-x})^{-1}$ [9]. Despite numerous efforts very little progress has been done until 2011. A breakthrough was achieved in [13] and later in [12] where the authors considered the class of totally positive functions of finite type and, by using another approach, Gaussian totally positive functions of finite type. These results can be viewed as a contribution to the original conjecture of Daubechies which relates the frame property to the positivity of function and its Fourier transform. Our results are to large extend motivated by [13] since the Fourier transforms of totally positive functions of finite type have the form $g(t)=P(t)^{-1}$, where P is a polynomial with simple zeroes located on the imaginary axis, such functions of course admit representation (1.1).

1.3 Herglotz functions

We suggest another approach based on interpolation by entire functions and dynamical systems. This approach allows us to describe the frame set for Herglotz functions, to study the frames with irrational densities as well as some related problems.

By Herglotz function we mean a function of the form (1.1) with $a_k>0$. Such functions appear naturally in the spectral theory of the Jacobi matrices and the Schrödinger equations. This class is in a sense opposite to the class of totally positive functions: while the coefficients a_k in the representation (1.1) of the totally positive functions have interlacing signs and also satisfy a number of additional relations, they are just positive for Herglotz functions. We will consider Herglotz functions with poles in the upper half-plane. In this case we encountered another kind of positivity related to the Gabor frame property.

1.4 Main results

Theorem 1.1 *Let g be a Herglotz function*

$$g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i w_k}, \quad a_k > 0, w_k > 0. \tag{1.5}$$

Then

$$\mathcal{F}(g) = \{(\alpha, \ \beta); \alpha\beta \leq 1\}. \tag{1.6}$$

For the general function of the form (1.1) relation (1.6) does not hold generally speaking. Amazingly we *almost always* have the frame property if $\alpha\beta \notin \mathbb{Q}$:

Theorem 1.2 *Let g be of the form (1.1) and be such that $m_0(\xi) = \sum_{k=1}^{N} a_k e^{2\pi\xi w_k} \neq 0$, $\xi > 0$ and $\Re w_k \neq \Re w_l$ for $k \neq l$. Then the Gabor system $\mathfrak{G}(g; \alpha, \beta)$ is a frame in $L^2(\mathbb{R})$ for any $(\alpha, \beta) \in \Pi$ such that $\alpha\beta \notin \mathbb{Q}$.*

Remark 1.3 Note that we only need to assume that $m_0(\xi) \neq 0$ for $\xi > 0$ and not all $\xi \in \mathbb{R}$. This is crucial, since we want to consider functions with $m_0(0) = \sum_{k=1}^{N} a_k = 0$ which are exactly rational functions g such that $g(x) = O(\frac{1}{x^2})$, $x \to \infty$.

Observe that, if $\Re w_k < 0$ for all k, we have $m_0(\xi) = \hat{g}(\xi)$, $\xi > 0$. Here and in what follows we normalize the Fourier transform as

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i t\xi} dt.$$

The next result is an important special case of Theorem 1.2.

Theorem 1.4 *Let g be a function of the form (1.1), $\Re w_k < 0$, $k = 1, 2, \ldots N$, $\Re w_k \neq \Re w_j$ for $j \neq k$ and also $\hat{g}(\xi) \neq 0$ for $\xi > 0$. Then $\mathfrak{G}(g; \alpha, \beta)$ is a frame in $L^2(\mathbb{R})$ for all $(\alpha, \beta) \in \Pi$, such that $\alpha\beta \notin \mathbb{Q}$.*

So, for the given class of functions Daubechies conjecture holds literally. Later we will see that the assumption $\alpha\beta \notin \mathbb{Q}$ cannot be dropped generally speaking.

Remark 1.5 (Rational densities with large denominators.) One can check that all arguments in the proof of Theorem 1.4 remain true if $\alpha = \frac{p}{q}$ is a rational number with sufficiently big denominator. In particular, this means that under the assumptions of Theorem 1.2 there exists at most countable set of exceptional rational α-s, i.e. such that $\mathfrak{G}(g; \alpha, 1)$ is not a frame, with only one (possible) accumulating point 1.

Sometimes we get into the situation when there exists at most finite set of exceptional α's, see Sect. 5.1.

1.5 Near the critical hyperbola

Another interesting question is related to the frame property of $\mathfrak{G}(g; \alpha, \beta)$ when the point (α, β) approaches the critical hyperbola $\alpha\beta = 1$.

Let g be of the form (1.1). Consider the function

$$\mathcal{Z}(z, \xi) = \sum_{k=1}^{N} \frac{a_k e^{2\pi \xi w_k}}{1 - z e^{2\pi w_k/\alpha}}.$$

This function can be viewed as an analog of the Zak transform for our setting.

Theorem 1.6 *Let* $\Re w_k > 0$, $k = 1, \ldots, N$ *and also* $\Re \mathcal{Z}(e^{2\pi i t}, \xi) > 0$ *for all* $t \in \mathbb{R}$, $\xi \in \mathbb{R}$. *Then there exists* $\alpha_0 < 1$ *such that* $\mathfrak{G}(g; \alpha, 1)$ *is a frame in* $L^2(\mathbb{R})$ *for all* $\alpha \in (\alpha_0, 1]$.

By the renormalization we have

$$\mathfrak{G}(g(\cdot/\beta); \alpha, \beta) \text{ is a frame in } L^2(\mathbb{R})$$

if $\alpha\beta \in (\alpha_0, 1]$.

Together with Theorem 1.6 this will lead us to Theorem 1.7.

Theorem 1.7 *Let g be of the form* (1.1), $\Re w_k < 0$, $k = 1, \ldots, N$, $\Re w_k \neq \Re w_j$ *for* $j \neq k$, *and* $\hat{g}(\xi) \neq 0$, $\xi > 0$. *Let also*

$$\Re \mathcal{Z}(e^{2\pi i t}, \xi) > 0, \quad t \in \mathbb{R}, \xi \in \mathbb{R}.$$

Then $\mathfrak{G}(g; \alpha, 1)$ *is a frame in* $L^2(\mathbb{R})$ *for all* $\alpha \in (0, 1]$, *except perhaps a finite number of rational exceptional values.*

We also highlight the following corollary of Theorem 1.6.

Corollary 1.8 *Let g be of the form (1.1), $w_k < 0$, $a_k \in \mathbb{R}$. If $\hat{g}$ is a positive, decreasing, convex function on the positive semiaxis $\mathbb{R}_+$, then $\mathfrak{G}(g; \alpha, \beta)$ is a frame for any pair (α, β) sufficiently close to the critical hyperbola $\alpha\beta = 1$, (i.e. for $\alpha > \alpha(\beta)$).*

1.6 Large densities

The condition $\alpha\beta \notin \mathbb{Q}$ in Theorem 1.2 cannot be omitted generally speaking. For example the system $\mathfrak{G}(g; \alpha, \beta)$ fails to be a frame if $g(t) = -g(-t)$ and $g(t) = O(t^{-2})$, $t \to \infty$ and $\alpha\beta = \frac{n-1}{n}$, $n = 2, 3, \ldots$ see [19]. For these cases the density of the non-frame lattice is at most 2.

It is known that for arbitrary window function g in the Wiener class the system $\mathfrak{G}(g; \alpha, \beta)$ becomes a frame if (α, β) belongs to some vicinity of $(0, 0)$ in the time-frequency plane. This vicinity depends on g, of course. A lot of effort has been spent in order to determine the size of this vicinity, see e.g. [1,3,23].

For the windows of the form (1.1) this size depends on the number of summands. This gives a partial answer to the question, formulated in [3].

Theorem 1.9 *Let g be of the form (1.1), $\Re w_k \neq \Re w_l$, $k \neq l$. Then*

$$\left\{(\alpha, \beta) : \alpha\beta \leq \frac{1}{N}\right\} \subset \mathcal{F}_g.$$

This therorem is almost precise: we will see that there exists a window g of the form (1.1) and $\alpha, \beta > 0$ with $\alpha\beta = \frac{1}{N-1}$ such that the corresponding Gabor system $\mathfrak{G}(g; \alpha, \beta)$ does not constitute a frame in $L^2(\mathbb{R})$ (see Proposition 5.3).

In particular there exist non-frame rational Gabor systems with lattices of arbitrary large density.

1.7 Concluding remarks

Infinite number of poles

We are able to generalize Theorem 1.1 to a class of Herglotz functions with infinite number of poles.

Theorem 1.10 *Let g be of the form*

$$g(x) = \sum_{k=1}^{\infty} \frac{a_k}{x - i w_k},$$

$w_k > 0$, $w_{k+1} - w_k \geq 1$ *and also* $0 < a_k < 2^{-2^{2^{w_k}}}$. *Then*

$$\mathcal{F}(g) = \{(\alpha, \beta) : \alpha, \beta > 0, \alpha\beta \leq 1\}.$$

Two kernels

Using our approach we can completely describe the frame set $\mathcal{F}_g$ for all functions $g(x) = \frac{a_1}{x - iw_1} + \frac{a_2}{x - iw_2}$, $a_1, a_2, w_1, w_2 \in \mathbb{C}$. In particular, for $w_1, w_2 \in \mathbb{R}$ we have $\Pi \subset \mathcal{F}_g$. The detailed proofs will appear elsewhere.

Completeness

In contrast to the frame property the completeness of rational Gabor systems does not require $\alpha\beta \notin \mathbb{Q}$. It can be obtained from Example 4.1. in [21] that for $\alpha\beta > 1$ the Gabor system $\mathfrak{G}(f; \alpha, \beta)$ is incomplete in $L^2(\mathbb{R})$ for any function $f \in L^2(\mathbb{R})$. This is the only restriction in case of windows of form (1.1).

Theorem 1.11 *Let the function* $g(x) = \sum \frac{a_k}{x - iw_k}$ *be such that* $\Re w_k \neq \Re w_l$, $k \neq l$. *Then the system* $\mathfrak{G}(g; \alpha, \beta)$ *is complete in* $L^2(\mathbb{R})$ *for all* $\alpha, \beta > 0$, $\alpha\beta \leq 1$.

For the windows of the form $g(t) = e^{-\gamma t^2} R(t)$ where $R(t)$ is a rational function similar statement has been proved in [11].

Multiple poles The right hand-side of (1.1) represent rational function in $L^2(\mathbb{R})$ with simple poles. We restrict ourselves to this case to avoid non-essential technical complications only.

1.8 The structure of the paper

The article is organized as follows. In Sect. 2 we give necessary and sufficient conditions for a rational Gabor system to be a frame. This characterization will be used throughout the whole article. In Sect. 3 we prove Theorem 1.1 and highlight connections to dynamical systems. In Sect. 4 we prove Theorems 1.2 and 1.4. Finally, in Sect. 5 we prove Theorem 1.6, Theorem 1.7, Theorem 1.9, and construct counterexamples. In Sect. 6 we discuss connections with the theory of shift-invariant subspaces.

Throughout this paper, $U(z) \lesssim V(x)$ (equivalently $V(z) \gtrsim U(z)$) means that there exists a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set. We write $U(z) \asymp V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2 Frame Criterion

In this section we give necessary and sufficient conditions for an arbitrary rational function $g \in L^2(\mathbb{R})$ to generate a frame for given α, β. This is the key step in the proofs of Theorems 1.1–1.4.

Let

$$g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i w_k}, \ a_k, w_k \in \mathbb{C}, \ a_k \neq 0, \ \Re w_k \neq 0. \tag{2.1}$$

2.1 Multipliers m_s and the main criterion

Given $\alpha, \beta > 0$, $\alpha\beta \leq 1$, we study the frame property in $L^2(\mathbb{R})$ of the Gabor system

$$\mathfrak{G}(g; \alpha, \beta) = \{g_{m,n}(t)\}_{m,n\in\mathbb{Z}}, \ g_{m,n}(t) = e^{2\pi i\beta t n} g(t - \alpha m). \tag{2.2}$$

It is immediate that the system $\mathfrak{G}(g; \alpha, \beta)$ is a frame if and only if the system $\mathfrak{G}(g_\beta; \alpha\beta, 1)$ is a frame, $g_\beta(t) = g(t/\beta)$. Since g_β is also a rational function it suffices to consider only the case $\beta = 1$, $\alpha \in (0, 1]$ which we will assume from now on.

We need the following notation. For $k = 1, \ldots, N$ and $s = 0, \ldots, N-1$ let

$$A_{k,s} = (-1)^s \sum_{j_1 < j_2, \ldots, < j_s, j_l \neq k} e^{\frac{2\pi}{\alpha}(w_{j_1} + \cdots + w_{j_s})}, \tag{2.3}$$

the sum is taken over pairwise different j_l's such that $j_l \neq k$. Put

$$m_s(\xi) = \sum_{k=1}^{N} a_k A_{k,s} e^{2\pi \xi w_k}. \tag{2.4}$$

Theorem 2.1 *The following statements are equivalent:*

(i) $\mathfrak{G}(g; \alpha, 1)$ *is a frame in* $L^2(\mathbb{R})$,

(ii)

$$\int_0^{\frac{1}{\alpha}} \sum_{n\in\mathbb{Z}} \left| \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \asymp ||G||^2_{L^2(\mathbb{R})}, \ G \in L^2(\mathbb{R}). \tag{2.5}$$

2.2 Proof of Theorem 2.1: Step 1

In order to establish the frame property of (2.2) we need to prove

$$\sum_{m,n} |\langle g_{m,n}, f\rangle|^2 \asymp \|f\|^2, \quad f \in L^2(\mathbb{R}). \tag{2.6}$$

Since all functions $m_s(\xi)$ are bounded the upper estimate is always true. We have

$$\left(\sum_{m,n} |\langle g_{m,n}, f\rangle|^2\right)^{\frac{1}{2}} = \sup\left\{\left|\sum_{m,n} c_{m,n}\langle g_{m,n}, f\rangle\right|; \sum_{m,n} |c_{m,n}|^2 \le 1\right\}. \tag{2.7}$$

2.3 Step 2

Given $\mathbf{c} = \{c_{m,n}\} \in l^2(\mathbb{Z}\times\mathbb{Z})$, we fix n and consider

$$S_n = \sum_m c_{m,n}\langle g_{m,n}, f\rangle = \int_{-\infty}^{\infty} \overline{f(t)} e^{2\pi i n t} \sum_{k=1}^{N} a_k \sum_m \frac{c_{m,n}}{t-(\alpha m + i w_k)} dt. \tag{2.8}$$

Denote

$$p_j(t) = 1 - e^{i\frac{2\pi}{\alpha}(t - i w_j)}; \;\; P(t) = \prod_{j=1}^{N} p_j(t). \tag{2.9}$$

The coefficients of the trigonometric polynomial $P(t)p_j(t)^{-1}$ are represents by (2.3):

$$P_k(t) = P(t)p_k(t)^{-1} = \sum_{s=0}^{N-1} A_{k,s} e^{i\frac{2\pi}{\alpha}st}. \tag{2.10}$$

Consider the entire function

$$h_n(t) = \left(1 - e^{i\frac{2\pi}{\alpha}t}\right)\sum_m \frac{c_{m,n}}{t-\alpha m}. \tag{2.11}$$

We have

$$\sum_m \frac{c_{m,n}}{t-(\alpha m + i w_k)} = \frac{P_k(t)}{P(t)} h_n(t - i w_k), \tag{2.12}$$

respectively

$$S_n = \int_{-\infty}^{\infty} \frac{\overline{f(t)}}{P(t)} e^{2\pi i n t} \sum_{k=1}^{N} a_k P_k(t) h_n(t - i w_k) dt. \tag{2.13}$$

By combining the classical sampling and the Paley–Wiener theorems we have

$$h_n(t) = \int_0^{\frac{1}{\alpha}} e^{2\pi i t \xi} \check{h}_n(\xi) d\xi, \tag{2.14}$$

with

$$\check{h}_n \in L^2(0, 1/\alpha);\ \ \|\check{h}_n\|_{L^2(0,1/\alpha)} \asymp \|\{c_{m,n}\}_m\|_{l^2(\mathbb{Z})}. \tag{2.15}$$

Let $g(t) = f(t)/\overline{P(t)}$. Since $|P(t)| \asymp 1$, $t \in \mathbb{R}$, we have $\|f\| \asymp \|g\|$. Now

$$S_n = \int_{-\infty}^{\infty} \overline{g(t)} e^{2\pi i n t} \sum_{s=0}^{N-1} e^{i \frac{2\pi}{\alpha} s t} M_s(t) dt, \tag{2.16}$$

where

$$M_s(t) = \sum_{k=1}^{N} a_k A_{k,s} h_n(t - i w_k) = \int_0^{\frac{1}{\alpha}} e^{2\pi i \xi t} \check{h}_n(\xi) \sum_{k=1}^{N} a_k A_{k,s} e^{2\pi \xi w_k} d\xi. \tag{2.17}$$

2.4 Step 3

The Parseval's identity now yields

$$S_n = \int_0^{\frac{1}{\alpha}} \left[\sum_{s=0}^{N-1} \overline{G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi)} \right] \check{h}_n(\xi) d\xi, \tag{2.18}$$

where G is the Fourier transform of g which satisfies $\|G\|_{L^2} = \|g\|_{L^2} \asymp \|f\|_{L^2}$.

Finally,

$$\sum_{m,n} |\langle g_{m,n}, f\rangle|^2 \asymp \sup \left\{ \sum_n \int_0^{1/\alpha} \left[\overline{\sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi)} \right] \check{h}_n(\xi) d\xi; \right.$$

$$\left. \sum_n \|\check{h}_n\|^2_{L^2(0,1/\alpha)} \le 1 \right\}. \tag{2.19}$$

2.5 Step 4

Observe that the sequence $\{\check{h}_n\}_{n\in\mathbb{Z}}$ runs through the whole $l^2(\mathbb{Z}, L^2(0, 1/\alpha))$ as $\{c_{m,n}\}$ runs through the whole $l^2(\mathbb{Z}\times\mathbb{Z})$ and also $\|\{\check{h}_n\}_{n\in\mathbb{Z}}\|_{l^2(\mathbb{Z},L^2(0,1/\alpha))} \asymp \|\{c_{m,n}\}\|_{l^2(\mathbb{Z}\times\mathbb{Z})}$.

Set

$$\check{h}_n(\xi) = \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) \overline{m_s(\xi)}. \tag{2.20}$$

Without loss of generality we may assume $\|f\| \asymp \|G\| \asymp 1$. Therefore $\|\{\check{h}_n\}_{n\in\mathbb{Z}}\|_{l^2(\mathbb{Z},L^2(0,1/\alpha))} \asymp \|\{c_{m,n}\}\|_{\ell^2(\mathbb{Z}\times\mathbb{Z})} \asymp 1$. Hence the system $\mathfrak{G}(g; \alpha, 1)$ is a frame if and only if

$$\int_0^{\frac{1}{\alpha}} \sum_{n\in\mathbb{Z}} \left| \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \asymp 1, \;\; G \in L^2(\mathbb{R}), \;\; \|G\| \asymp 1 \tag{2.21}$$

which is equivalent to (2.5). □

3 Frame property for Herglotz functions

In this section we prove Theorem 1.1.

3.1 Frobenius matrices

The proof of Theorem 1.1 is based on the Lemma 3.1 about Frobenius matrices which was communicated to us by Ivan Bochkov. First we recall the definition of Frobenius matrix.

Definition 1 Given a polynomial $p(z) = z^n + b_{n-1}z^{n-1} + \ldots + b_0$ with the leading coefficient 1 by the Frobenius matrix associated with p we mean the matrix

$$F(p) = \begin{pmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{3.1}$$

We refer the reader to [27] and [20] for the detailed presentation of properties of such matrices. In particular, p is the characteristic polynomial of $F(p)$. The next lemma is the key step in the proof of Theorem 1.1, we also think that it is of independent interest.

Lemma 3.1 *Let the sequence $\{\mu_k\}_{k=1}^{n+1}$ be such that $1 > \mu_1 > \mu_2 > \dots > \mu_{n+1} > 0$. Consider the set $\mathcal{P}$ of all polynomials $p(x) = (x - \lambda_1)(x - \lambda_2)\dots(x-\lambda_n)$ such that their zeroes interlace with μ_k's, i.e. $\mu_k \ge \lambda_k \ge \mu_{k+1}$. Then there exist constants $C > 0$, $0 < c < 1$ depending only on the numbers μ_k such that for any $m = 1, 2, \ldots$ and for any polynomials $p_1, \dots, p_m \in \mathcal{P}$ we have*

$$||F(p_1)F(p_2)\dots F(p_m)|| \le Cc^m. \tag{3.2}$$

We do not specify matrix norm here since all norms in finite-dimensional space are equivalent. We postpone the proof and first obtain Theorem 1.1 from Lemma 3.1.

3.2 Proof of Theorem 1.1 Step 1

As before we assume $\beta = 1$, $\alpha \le 1$ and prove the relation (2.5). We truncate the integral in (2.5) and prove the stronger estimate

$$\int_0^1 \sum_{l\in\mathbb{Z}} \left| \sum_{s=0}^{N-1} G\left(\xi + l + \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \gtrsim ||G||_2^2, \quad G \in L^2(\mathbb{R}). \tag{3.3}$$

Here as before the functions $m_s(\xi)$ are determined in (2.4).

Assume $0 < w_1 < \ldots < w_N$ and let

$$n = N - 1,\ \mu_k = e^{-2\pi w_k/\alpha}, k = 1, 2, \dots n + 1.$$

We have $1 > \mu_1 > \mu_2 > \cdots > \mu_{n+1} > 0$. Since $a_k > 0$, $w_k > 0$ we have $m_n(\xi) \neq 0$. We claim that the roots of the polynomial

$$p_\xi(z) = \frac{m_0(\xi) + m_1(\xi)z + \cdots + m_n(\xi)z^n}{m_n(\xi)}$$

satisfy the assumptions of the Lemma 3.1. This follows from the identity

$$\frac{\sum_{s=0}^{N-1} m_s(\xi)z^s}{\prod_{k=1}^{N}(1 - ze^{\frac{2\pi}{\alpha}w_k})} = \sum_{k=1}^{N} \frac{a_k e^{2\pi\xi w_k}}{1 - ze^{\frac{2\pi}{\alpha}w_k}}$$

and the positivity of a_k's and w_k's.

The estimate (3.3) is equivalent to

$$||LG||_2 \gtrsim ||G||_2, \; G \in L^2(\mathbb{R}), \tag{3.4}$$

where the operator $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by the formula

$$(LG)(\xi) = \sum_{s=0}^{n} G\left(\xi + \frac{s}{\alpha}\right) m_s(\{\xi\}).$$

Here $\{\xi\}$ denotes the fractional part of ξ.

3.3 Step 2

From the definition of m_s we have

$$m_n(\{\xi\}) = (-1)^n e^{\frac{2\pi}{\alpha}(w_1+\ldots+w_N)} \sum_{k=1}^{N} a_k e^{2\pi\{\xi\}w_k - \frac{2\pi}{\alpha}w_k}. \tag{3.5}$$

The absolute value of $m_n(\{\xi\})$ is bounded from above and from below by some positive constants. Without loss of generality we can consider the operator PG instead of LG

$$(PG)(\xi) = \frac{(LG)(\xi)}{m_n(\{\xi\})}.$$

It suffices to construct the inverse for P, i.e. solve

$$PG = h, \; h \in L^2(\mathbb{R}).$$

We rewrite this equation in the form

$$G\left(\xi+\frac{n}{\alpha}\right)=h(\xi)-\sum_{s=0}^{n-1}\frac{m_s(\{\xi\})}{m_n(\{\xi\})}G\left(\xi+\frac{s}{\alpha}\right). \tag{3.6}$$

3.4 Step 3

We transform equation (3.6) to a dynamical system. Consider the vector-functions Γ, $H\in L^2(\mathbb{R},\mathbb{C}^n)$:

$$\Gamma(\xi)=\begin{pmatrix}G(\xi+\frac{n-1}{\alpha})\\ G(\xi+\frac{n-2}{\alpha})\\ \vdots\\ G(\xi)\end{pmatrix},\qquad H(\xi)=\begin{pmatrix}h(\xi)\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

In this notation equation (3.6) can be rewritten as

$$\Gamma\left(\xi+\frac{1}{\alpha}\right)=F\left(p_{\{\xi\}}\right)\Gamma(\xi)+H(\xi),$$

where $p_\xi(z)=z^n+\frac{m_{n-1}(\xi)}{m_n(\xi)}z^{n-1}+\cdots+\frac{m_0(\xi)}{m_n(\xi)}$.

Iterating this formula we get

$$\begin{aligned}\Gamma\left(\xi+\frac{1}{\alpha}\right)=&\ F\left(p_{\{\xi\}}\right)F\left(p_{\{\xi-\frac{1}{\alpha}\}}\right)\Gamma\left(\xi-\frac{1}{\alpha}\right)\\ &+F\left(p_{\{\xi\}}\right)H\left(\xi-\frac{1}{\alpha}\right)+H(\xi).\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\Gamma\left(\xi+\frac{1}{\alpha}\right)=&\ F\left(p_{\{\xi\}}\right)F\left(p_{\{\xi-\frac{1}{\alpha}\}}\right)\dots F\left(p_{\{\xi-\frac{k}{\alpha}\}}\right)\Gamma\left(\xi-\frac{k}{\alpha}\right)\\ &+\sum_{s=0}^{k}F\left(p_{\{\xi\}}\right)F\left(p_{\{\xi-\frac{1}{\alpha}\}}\right)\dots F\left(p_{\{\xi-\frac{s-1}{\alpha}\}}\right)H\left(\xi-\frac{s}{\alpha}\right).\end{aligned}$$

By Lemma 3.1 the coefficients in front of $\Gamma(\xi-\frac{k}{\alpha})$ and $H(\xi-\frac{s}{\alpha})$ decay exponentially. By passing to the limit we obtain

$$\Gamma\left(\xi+\frac{1}{\alpha}\right)=\sum_{s=0}^{\infty}F\left(p_{\{\xi\}}\right)F\left(p_{\{\xi-\frac{1}{\alpha}\}}\right)\dots F\left(p_{\{\xi-\frac{s-1}{\alpha}\}}\right)H\left(\xi-\frac{s}{\alpha}\right).$$

Thus, we get $||\Gamma||_2 \leq C||H||_2 = C||h||_2$. Hence, $||G||_2 \leq C'||h||_2 = C'||PG||_2$. That is, $||PG||_2 \gtrsim ||G||_2$ and, subsequently, $||LG||_2 \gtrsim ||G||_2$ which is the desired estimate (3.4). □

3.5 Proof of Lemma 3.1: Preliminaries

We are going to construct a norm $||v||_*$ on $\mathbb{R}^n$ such that for all $p \in \mathcal{P}$ we have $||F(p)||_* \leq c < 1$. This implies by induction that $||F(p_1)\dots F(p_m)||_* \leq c^m$. Since all norms on the finite-dimensional space are equivalent we get the result.

We will actually construct a norm in which $F(p)^T$ is contractive uniformly for all $p \in \mathcal{P}$. If we are able to do so, then matrices $F(p)$ will be contractive in the dual norm.

The proof consists of two steps. In the first step we show that it is enough to consider only polynomials p such that λ_k's is either μ_k or μ_{k+1} for all k. Moreover, we will show that among them we can actually study only those for which all λ_k are distinct, that is polynomials $p_l(x) = \prod_{k\neq l}(x-\mu_k)$, $l = 1, \dots, n+1$. In the second step we will construct a norm in which all matrices $F(p_l)^T$ are uniformly contractive.

3.6 Step 1: Reduction to $n+1$ matrices

Let $p(x) = \prod_{k=1}^{n}(x-\lambda_k)$ be an arbitrary polynomial in $\mathcal{P}$. Assume that $\lambda_k \neq \mu_k, \mu_{k+1}$ for some k. Since $\mu_k > \lambda_k > \mu_{k+1}$ we can find positive numbers $a, b \in \mathbb{R}$ with $a+b = 1$ such that $(x-\lambda_k) = a(x-\mu_k)+b(x-\mu_{k+1})$. Denoting $q(x) = \frac{x-\mu_k}{x-\lambda_k}p(x)$ and $r(x) = \frac{x-\mu_{k+1}}{x-\lambda_k}p(x)$ we get $p(x) = aq(x) + br(x)$ and therefore $F(p)^T = aF(q)^T + bF(r)^T$. Repeating this procedure with q and r we can express $F(p)^T$ as a convex combination of matrices of the form $F(\rho)^T$ where $\rho \in \mathcal{P}$ and all its roots are from $\{\mu_1, \dots, \mu_{n+1}\}$. By the triangle inequality if $F(\rho)^T$ are contractive for all such polynomials ρ then $F(p)^T$ is contractive as well.

Now we show that it is enough to consider only the polynomials p_l. Let us consider all 2^n polynomials $p \in \mathcal{P}$ with $\lambda_k = \mu_k$ or $\lambda_k = \mu_{k+1}$ and denote by K the convex hull of the corresponding matrices $F(p)^T$. Since it is a convex hull of finitely many points it is a polytope. It is well-known that any polytope is a convex hull of its vertices which are exactly the points that can not be written as a convex combination of other points from this polytope.

Let $p(x) = \prod_{k=1}^{n}(x-\lambda_k)$ be such that $\lambda_k = \mu_k$ or $\lambda_k = \mu_{k+1}$ for all k and moreover $\lambda_l = \lambda_{l+1} = \mu_{l+1}$ for some l. There exist positive numbers a, b such that $a + b = 1$ and $(x-\mu_{l+1}) = a(x-\mu_l) + b(x-\mu_{l+2})$. Denoting $q(x) = \frac{x-\mu_l}{x-\mu_{l+1}}p(x)$ and $r(x) = \frac{x-\mu_{l+2}}{x-\mu_{l+1}}p(x)$ we get $p(x) = aq(x)+br(x)$ and therefore $F(p)^T = aF(q)^T + bF(r)^T$. On the other hand we have

$F(q)^T, F(r)^T \in K$ and both of them are not equal to $F(p)^T$. That is, we decomposed $F(p)^T$ as a convex combination of other points from K. Thus, it is not a vertex of K.

Therefore the only possible candidates for the vertices of K correspond to the polynomials with distinct roots, that is p_l's. That is, all points K are convex combinations of $F(p_l)^T$'s and so if $F(p_l)^T$'s are contractive for all l then all other matrices from K are contractive as well.

Remark 3.2 Instead of appealing to the theory of polytopes one can more carefully decompose p into the sum of two other polynomials with smaller number of repeated roots and then continue the process until there are none.

3.7 Step 2: Construction of the contractive norm for $F(p_l)^T$'s

Let us identify $\mathbb{R}^n$ with the space $\mathcal{P}_{n-1}$ of all polynomials of degree less than n in a way that $\alpha = (\alpha_{n-1}, \ \dots \alpha_0)^T \in \mathbb{R}^n$ corresponds to the polynomial $q_\alpha(z) = \alpha_{n-1}z^{n-1} + \ldots + \alpha_0 \in \mathcal{P}_{n-1}$. One can see that the action of $F(p)^T$ on the polynomial $q \in \mathcal{P}_{n-1}$ corresponds to the operation

$$q(z) \mapsto zq(z)(\text{mod } p(z)).$$

For each $l = 1, 2, \ \dots n+1$ consider the linear functional which sends the polynomial $q \in \mathcal{P}_{n-1}$ to $q(\mu_l)$. Since these are $n+1$ linear functionals on the n-dimensional vector space there is a linear dependence between them:

$$a_1 q(\mu_1) + a_2 q(\mu_2) + \ldots + a_{n+1} q(\mu_{n+1}) = 0 \tag{3.8}$$

for all polynomials $q \in \mathcal{P}_{n-1}$. Moreover, since the values of q at any n different points uniquely determine q none of a_k's vanishes. Put

$$||q||_\mu = |a_1 q(\mu_1)| + \cdots + |a_{n+1} q(\mu_{n+1})|.$$

Since q is uniquely determined by $q(\mu_1), \ldots, q(\mu_{n+1})$, this is a norm on $\mathcal{P}_{n-1}$. Note that this norm is overdetermined (n terms would have been enough to have a norm). But this overdetermination is instrumental for the proof of the contractivity. We are going to show that $||F(p_l)^T q||_\mu \leq \mu_1 ||q||_\mu$ for all l. Since $\mu_1 < 1$ this implies the result.

We fix $l \in [1, ..., n+1]$ and let $r(x) = (F(p_l)^T q)(x)$. Then

$$r(\mu_k) = \mu_k q(\mu_k) \text{ for all } k \neq l$$

and, by (3.8),

$$a_l r(\mu_l) = -\sum_{k\neq l} a_k r(\mu_k).$$

Therefore

$$\begin{aligned}
||r||_\mu &= \sum_{k\neq l} |\mu_k a_k q(\mu_k)| + |a_l r(\mu_l)| \\
&= \sum_{k\neq l} |\mu_k a_k q(\mu_k)| + |\sum_{k\neq l} a_k \mu_k q(\mu_k)| \\
&= \mu_1 \left(\sum_{k\neq l} \left| \frac{\mu_k}{\mu_1} a_k q(\mu_k) \right| + \left| \sum_{k\neq l} a_k \frac{\mu_k}{\mu_1} q(\mu_k) \right| \right).
\end{aligned} \tag{3.9}$$

For $s_k \in [0, 1]$, $x_k \in \mathbb{R}$, $k \neq l$ we have

$$\left|\sum_{k\neq l} s_k x_k\right| - \left|\sum_{k\neq l} x_k\right| \leq \left|\sum_{k\neq l} x_k(1-s_k)\right| \leq \sum_{k\neq l} |x_k|(1-s_k).$$

Hence,

$$\sum_{k\neq l} |s_k x_k| + \left|\sum_{k\neq l} s_k x_k\right| \leq \sum_{k\neq l} |x_l| + \left|\sum_{k\neq l} x_k\right|.$$

Setting $x_k = a_k q(\mu_k)$, $s_k = \frac{\mu_k}{\mu_1}$ we get

$$||r||_\mu \leq \mu_1 ||q||_\mu, \tag{3.10}$$

as required. □

4 Irrational densities

In this section we prove Theorems 1.2–1.4. The main ingridient of the proofs is careful analysis of the finite matrices with rows of the form $(0, ..., m_0(\xi), ..., m_{N-1}(\xi), 0, .., 0)$.

4.1 Preliminaries

Without loss of generality we may assume, as before, $\beta = 1$, $\alpha \in (0, 1) \setminus \mathbb{Q}$ because the rescaling $g \mapsto g_\beta(t) := g(t/\beta)$ as in Sect. 3 leads one just to rescaling of the corresponding functions $m_0(\xi)$, $\hat{g}(\xi)$.

Put

$$M(\xi) = (m_0(\xi), m_1(\xi), ..., m_{N-1}(\xi)).$$

In what follows we write $\mathbb{O}_k$ for zero row of length k; when the length is clear from the context we suppress the subscript k. So the row $(0, ..., m_0(\xi), ..., m_{N-1}(\xi), 0, ..., 0)$ can be written as $(\mathbb{O}, m_0(\xi), ..., m_{N-1}(\xi), \mathbb{O})$.

We assume $\alpha > 1/2$. The case $\alpha < 1/2$ follows since $\alpha\mathbb{Z} \times \mathbb{Z} \subset 2\alpha\mathbb{Z} \times \mathbb{Z}$.

Let $\tau = \frac{1}{\alpha} - 1$. For any fixed $\xi \in (1, \frac{1}{\alpha})$ consider the sequence

$$\{\xi, \xi - 1, \xi - 1 + \tau, \xi - 1 + 2\tau, ..., \xi - 1 + k_1\tau\},$$

here $k_1 \in \mathbb{N}$ is the first number such that $\xi - 1 + k_1\tau \in (1, 1/\alpha)$. We repeat the procedure starting from $\xi - 1 + k_1\tau$ and take the first k_2 such that $\xi - 1 + (k_1 + k_2)\tau \in (1, 1/\alpha)$, and so on. After l repetitions we obtain the sequence

$$\begin{aligned} S_{\xi,l} := \{&\xi, \xi - 1, \xi - 1 + \tau, \xi - 1 + 2\tau, ..., \xi - 1 + k_1\tau, \xi - 2 + k_1\tau, \\ &\xi - 2 + (k_1 + 1)\tau ..., \xi - 2 + (k_1 + k_2)\tau), \\ &..., \\ &\xi - l + K\tau, \xi - l - 1 + K\tau\}, \end{aligned}$$

where $K = k_1 + k_2 + \cdots + k_l$. With any such sequence $S(\xi, l)$ we associate finite $(K + l + 2) \times (K + N)$ matrix $D = D(\xi, l)$,

$$D = D(\xi, l) = \begin{pmatrix} M(\xi) & \mathbb{O} & \\ M(\xi - 1) & \mathbb{O} & \\ \mathbb{O}_1 & M(\xi - 1 + \tau) & \mathbb{O} \\ \mathbb{O}_2 & M(\xi - 1 + 2\tau) & \mathbb{O} \\ & \cdots & \\ \mathbb{O}_{k_1 - 1} & M(\xi - 1 + (k_1 - 1)\tau) & \mathbb{O} \\ \mathbb{O}_{k_1} & M(\xi - 1 + k_1\tau) & \mathbb{O} \\ \mathbb{O}_{k_1} & M(\xi - 2 + k_1\tau) & \mathbb{O} \\ \mathbb{O}_{k_1+1} & M(\xi - 2 + (k_1 + 1)\tau) & \mathbb{O} \\ & \cdots & \\ \mathbb{O}_K & M(\xi - l + K\tau) & \\ \mathbb{O}_K & M(\xi - l - 1 + K\tau) & \end{pmatrix}.$$

We put attention of the reader to the (a bit) non-traditional form of representation of this matrix: each "column" consists of strings of various length. In the next section we will see how does this matrix appear and also explain its structure in more details.

The next lemma is the key technical step in the proof of Theorem 1.2.

Lemma 4.1 *There exist* $\hat{\xi} \in (1, \frac{1}{\alpha})$, $l \in \mathbb{N}$, $l > N$, *and* $\delta > 0$ *such that for any* $\xi \in [\hat{\xi} - \delta, \hat{\xi} + \delta]$ *the rank of the matrix* $D(\xi, l)$ *is* $K + N$.

We postpone the proof of this lemma and first deduce Theorem 1.2 from Lemma 4.1.

4.2 Proof of Theorem 1.4

We have to establish relation (2.5). The $\lesssim$ part of (2.5) is straightforward since all $m_s(\xi)$ are bounded. In order to prove the opposite inequality it suffices to construct the left inverse $\mathcal{L}^{-1}$ to the operator $\mathcal{L} : L^2(\mathbb{R}) \to \ell^2(L^2(0, 1/\alpha))$ defined by the relation

$$\mathcal{L} : G \mapsto \left\{ \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) \right\}_n . \tag{4.1}$$

As before, we restrict ourselves to the case $\alpha > 1/2$ and denote $\tau = \frac{1}{\alpha} - 1$.

Given $\gamma = \{\gamma_n(\xi)\}_{n\in\mathbb{Z}} \in \Im\mathcal{L}$ we have to solve the infinite sequence of equations with respect to $\{G(\xi + n + s/\alpha)\}_n$

$$\sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) = \gamma_n(\xi), \qquad n \in \mathbb{Z}, \quad \xi \in \left(0, \frac{1}{\alpha}\right). \tag{4.2}$$

We use notations from the previous section. For $\xi \in (1, 1/\alpha)$ we will try to choose a subsystem of (4.2) which can be resolved with respect to variables

$$\left\{ G(\xi + \frac{j}{\alpha}) \right\}_{j\in\mathbb{Z}} . \tag{4.3}$$

This leads us to the matrix $D(\xi, l)$. Indeed, we have $\xi \in (1, 1/\alpha)$, $\xi - 1 \in (0, \tau)$. Two equations in (4.2) written for ξ with $n = 0$ and for $\xi - 1$ with $n = 1$ contain *the same* selection of variables $G(\xi + s/\alpha)$, $s = 0, 1, \ldots, N - 1$. The coefficients in these equations belong to the strings $M(\xi)$, $M(\xi - 1)$. We complete these strings by the corresponding number of zeroes and obtain the first two rows of the matrix $D(\xi, l)$.

The equation in (4.2) with $\xi - 1 + \tau$ instead of ξ and $n = 2$ has the form

$$\sum_{s=0}^{N-1} G\left(\xi + \frac{s+1}{\alpha}\right) m_s(\xi - 1 + \tau) = \gamma_2(\xi - 1 + \tau).$$

This equation contains the variables $\{G(\xi + s/\alpha)\}_{s=1}^{N}$, its coefficients are the elements of the string $M(\xi - 1 + \tau)$. Completing this string by one zero on the left and by the corresponding amount of zeroes on right we obtain the third row in $D(\xi, l)$.

Repeating this procedure as described in the previous section we obtain the whole matrix $D(\xi, l)$. We remark that the total number of unknowns increases by one when we add the equation related to the shift of the argument by τ and remains the same, when we add the equation related to the shift of the argument by -1. This will allow us to extract a subsystem of (4.2) which contains the same amount of equations and variables to be determined.

Moreover, for each $\xi \in (1, 1/\alpha)$ we can explicitly write the matrix of the operator applied to the sequence $\{G(\xi + j/\alpha)\}_{j\in\mathbb{Z}}$. This matrix consist of single and double strings $M(\cdot)$ shifted with respect to each other.

$$L_\xi = \begin{pmatrix} & \dots & \\ \mathbb{O}_{-k_{-1}} & M(\xi + 1 - k_{-1}\tau) & \mathbb{O} \\ \mathbb{O}_{-k_{-1}} & M(\xi - k_{-1}\tau) & \mathbb{O} \\ & \dots & \\ \mathbb{O}_{-2} & M(\xi - 2\tau) & \mathbb{O} \\ \mathbb{O}_{-1} & M(\xi - \tau) & \mathbb{O} \\ \mathbb{O}_{0} & M(\xi) & \mathbb{O} \\ \mathbb{O}_{0} & M(\xi - 1) & \mathbb{O} \\ \mathbb{O}_{+1} & M(\xi - 1 + \tau) & \mathbb{O} \\ \mathbb{O}_{+2} & M(\xi - 1 + 2\tau) & \mathbb{O} \\ & \dots & \\ \mathbb{O}_{k_1} & M(\xi - 1 + k_1\tau) & \mathbb{O} \\ \mathbb{O}_{k_1} & M(\xi - 2 + k_1\tau) & \mathbb{O} \\ & \dots & \\ \mathbb{O}_{k_1+k_2} & M(\xi - 2 + (k_1 + k_2)\tau) & \mathbb{O} \\ \mathbb{O}_{k_1+k_2} & M(\xi - 3 + (k_1 + k_2)\tau) & \mathbb{O} \\ & \dots & \end{pmatrix}, \tag{4.4}$$

here $\mathbb{O}_q$ indicates the shift of the corresponding string $M(\cdot)$ to the left or to the right depending on the sign of q.

The structure of all rows of the matrix L_ξ is similar. Therefore, we can start with $\xi - 1 + k_1\tau$ instead of ξ. Similarly, if $\xi \notin (1, \frac{1}{\alpha})$ we can first add to it $r\tau$ for some $r \in \mathbb{N}$ so that $\xi + r\tau \in (1, \frac{1}{\alpha})$ and proceed from there.

Choose l, $\hat{\xi}$ and δ as in Lemma 4.1 and, for each $t \in \mathbb{Z}$, denote by ξ_t the point of t-th return of the original point ξ into the interval $(1, 1/\alpha)$:

$$\xi_t = \xi - t + (k_1 + k_2 + \cdots + k_t)\tau,$$

so that the corresponding couple of "double rows" in (4.4) has the form

$$\begin{pmatrix} \mathbb{O}_{k_1+\cdots+k_t} & M(\xi_t) & \mathbb{O} \\ \mathbb{O}_{k_1+\cdots+k_t} & M(\xi_t - 1) & \mathbb{O} \end{pmatrix}.$$

In this notation we have $\xi = \xi_0$. Since $\alpha \notin \mathbb{Q}$ the set ξ_t is dense in $(1, 1/\alpha)$.

By shifting of the numeration we may assume that $\xi_0 \in [\hat{\xi} - \delta, \hat{\xi} + \delta]$, and also we can choose $t \in \mathbb{N}$ so that the point $\xi_{-t} = \xi + t - (k_{-1} + \cdots + k_{-t})\tau \in [\hat{\xi} - \delta, \hat{\xi} + \delta]$. Consider the submatrix of L_ξ located between the rows $(\mathbb{O}, M(\xi + t - (k_{-1} + k_{-2} + \cdots + k_{-t})\tau), \mathbb{O})$ and $(\mathbb{O}, M(\xi - l + (k_1 + k_2 + \cdots + k_l)\tau), \mathbb{O})$:

$$C = C(\xi, l, t) = \begin{pmatrix} M(\xi + t - (k_{-1} + \cdots + k_{-t})\tau) & \mathbb{O} & \\ M(\xi + t - 1 - (k_{-1} + k_{-2} + \cdots + k_{-t})\tau) & \mathbb{O} & \\ & \cdots & \\ \mathbb{O}_{k_{-1}+\cdots+k_{-t}-2} & M(\xi - 2\tau) & \mathbb{O} \\ \mathbb{O}_{k_{-1}+\cdots+k_{-t}-1} & M(\xi - \tau) & \mathbb{O} \\ \mathbb{O}_{k_{-1}+\cdots+k_{-t}} & D(\xi, l) & \end{pmatrix}.$$

The matrix $C(\xi, l, t)$ has $\nu := N + K + (k_{-1} + k_{-2} + \cdots + k_{-t})$ columns.

4.3 Rank of the matrix $C(\theta, t, l)$

We will show that $rank(C(\xi, l, t)) = \nu$. To this end we choose collection of ν rows of $C(\xi, l, t)$ which spans the whole space $\mathbb{R}^\nu$. By Lemma 4.1 there is a square non-degenerate submatrix $E(\xi, l)$ of $D(\xi, l)$ of size $(K+N) \times (K+N)$. We keep the rows which correspond to $E(\xi, l)$ and eliminate the rest of the rows $D(\xi, l)$. Further we eliminate each second row in the couples of double rows, i.e. the rows which contain the strings $M(\xi + p - 1 - (k_{-1} + \cdots + k_{-p}))$, $p = 1, ..., t$. The remaining rows form $\nu \times \nu$ matrix of the form

$$\begin{pmatrix} M(\xi + t - (k_{-1} + \ldots + k_{-t})\tau) & \mathbb{O} & \\ \mathbb{O}_1 & M(\xi + t - 1 - (k_{-1} + \ldots + k_{-t} - 1)\tau) & \mathbb{O} \\ & \cdots & \\ \mathbb{O}_{k_{-1}+\cdots+k_{-t}-1} & M(\xi - \tau) & \mathbb{O} \\ \mathbb{O}_{k_{-1}+\cdots+k_{-t}} & E(\xi, l) & \end{pmatrix}.$$

This is a block-diagonal matrix $\begin{pmatrix} X & Y \\ 0 & E(\xi, l) \end{pmatrix}$. In addition X is an upper-triangular matrix, its diagonal elements are values $m_0(\xi)$ for some point

$\xi \in (\tau, 1+\tau)$. They do not vanish and bounded away from zero by the assumption on m_0. Thus the matrix $C(\xi, l)$ indeed has the full rank.

4.4 End of the proof of Theorem 1.4

We can now find $\overrightarrow{G} = \{G(\xi + j/\alpha)\}_{j=-k_1-\ldots-k_t}^{K+N}$ which solves the system of equations (4.2) which corresponds to the selected rows of the matrix $C(\xi, l)$. The equations of this system which correspond to the rest of the rows of $C(\xi, l)$ will be met automatically since we assume $\gamma \in \Im L_\xi$. In addition we have

$$\|\overrightarrow{G}\|_2 \geq C\|\tilde{\gamma}\|, \tag{4.5}$$

where $\tilde{\gamma}$ is the section of the sequence γ corresponding the rows of $C(\xi, l)$. The constant C depends on t and the estimate from below for $|\det E(\xi, l)|$.

We observe that number t may be chosen uniformly bounded with respect to ξ. Indeed, we have an irrational motion with step τ and it is well known that it lands into any given interval in the bounded number of steps regardless of the starting point.

One can choose $\varepsilon > 0$ so that, for each $\xi \in [\xi_0 - \delta, \xi_0 + \delta]$ we have $|\det E(\xi, l)| > \varepsilon$ for the corresponding submatrix $E(\xi, l)$ of $D(\xi, l)$. Therefore the constant C in (4.5) can be chosen uniformly with respect to $\xi \in [\xi_0 - \delta, \xi_0 + \delta]$.

It remains to note that the operator L_ξ can be decomposed into the operators $C(\xi)$ with finite overlapping. So, finally we have constructed the bounded left inverse to (4.1).

Remark 4.2 The proof of Theorem 1.2 can be roughly decomposed into the following ideas: we can use nonvanishing of the function m_0 to shift the attention from the number ξ to the number $\xi - \tau \mod \frac{1}{\alpha}$. Since $\tau = \frac{1}{\alpha} - 1$ and $\frac{1}{\alpha}$ are incommensurable in this way we can come close to any given point on the interval $[0, 1/\alpha]$. Thus, it is enough to prove the corresponding bound for any single $\xi_0 \in (0, 1/\alpha)$ (and its small vicinity), which is done in Lemma 4.1.

4.5 Proof of Lemma 4.1: Step 1

First we observe the identity

$$\frac{\sum_{s=0}^{N-1} m_s(\xi) z^s}{\prod_{k=1}^{N}(1 - z e^{\frac{2\pi}{\alpha} w_k})} = \sum_{k=1}^{N} \frac{a_k e^{2\pi \xi w_k}}{1 - z e^{\frac{2\pi}{\alpha} w_k}}, \tag{4.6}$$

this follows from the definition of the functions m_s, see (2.4). Let

$$u_j = e^{\frac{2\pi}{\alpha} w_j},\ j = 1, ...N.$$

Fix $j \in \{1, ..., N\}$ and compare the residues at $z = u_j^{-1}$ on both sides in (4.6).

$$\sum_{s=0}^{N-1} m_s(\xi) u_j^{-s} = a_j u_j^{1-N} e^{2\pi\xi w_j} \prod_{l \neq j} (u_j - u_l). \tag{4.7}$$

Assume that the number $l > N$ is already found. The $(K+l+2) \times (K+N)$ matrix $D(\xi, l)$ is composed from $l+1$ “double” rows of the form

$$\begin{gathered}(\mathbb{O}_{k_1+\dots+k_s}, M(\xi - s + (k_1 + \dots + k_s)\tau), \mathbb{O}),\\ (\mathbb{O}_{k_1+\dots+k_s}, M(\xi - s - 1 + (k_1 + \dots + k_s)\tau), \mathbb{O})\end{gathered}$$

with “single” rows in between.

In order to transform it to a square matrix it suffices to eliminate $l + 2 - N$ rows. Let us eliminate the second rows in the appropriate number of double rows except the first and last ones. The remaining second rows are of the form

$$(\mathbb{O}_{k_1+\dots+k_s}, M(\xi - s - 1 + (k_1 + \dots + k_s)\tau), \mathbb{O})$$

for $s = Q_1, Q_2, ..., Q_{N-1}$ for some $0 = Q_1 < Q_2 < ... < Q_{N-1} = l$. Denote the resulting matrix by $F(\xi, l)$ and let

$$q_j = k_{Q_{j-1}} + \dots + k_{Q_j}$$

be the “distance” between the rows with numbers Q_{j-1} and Q_j.

We will choose the numbers $Q_1, ..., Q_{N-1}$, and also $\hat{\xi} \in (1, 1/\alpha)$, $\delta > 0$, so that $\det F(\xi, l) \neq 0$, $\xi \in (\hat{\xi} - \delta, \hat{\xi} + \delta)$.

It follows from (4.7) that, for each $j = 1, 2, ..., N$,

$$F(\xi, l) \begin{pmatrix} u_j^{K+N-1} \\ u_j^{K+N-2} \\ \dots \\ u_j \\ 1 \end{pmatrix} = a_j e^{2\pi \xi w_j} \prod_{l \neq j} (u_j - u_l) \begin{pmatrix} u_j^K \\ u_j^K e^{-2\pi w_j} \\ u_j^{K-1} e^{-2\pi w_j} e^{2\pi \tau w_j} \\ u_j^{K-2} e^{-2\pi w_j} e^{2\pi 2\tau w_j} \\ \cdots \\ u_j^{K-q_2} e^{-2\pi Q_2 w_j} e^{2\pi q_2 \tau w_j} \\ u_j^{K-q_2} e^{-2\pi (Q_2+1) w_j} e^{2\pi q_2 \tau w_j} \\ \cdots \\ e^{-2\pi Q_{N-1} w_j} e^{2\pi K \tau w_j} \\ e^{-2\pi (Q_{N-1}+1) w_j} e^{2\pi K \tau w_j} \end{pmatrix}$$

$$= a_j e^{2\pi \xi w_j} \prod_{l \neq j} (u_j - u_l) V_j(l).$$

For each $j = 1, 2, ..., N$, $V_j(\xi, l)$ is a column of size $K + N$. We observe that $V_j(l)$ is independent of ξ.

Put

$$W(l) := \begin{pmatrix} u_1^{K+N-1} & u_2^{K+N-1} & \dots & u_N^{K+N-1} & \\ u_1^{K+N-2} & u_2^{K+N-2} & \dots & u_N^{K+N-2} & \mathbb{O}_{N \times K} \\ \vdots & \vdots & \vdots & \vdots & \mathbb{I}_K \\ u_1 & u_2 & \dots & u_N & \\ 1 & 1 & \dots & 1 & \end{pmatrix}$$

here $\mathbb{O}_{N \times K}$ is the zero $N \times K$ matrix and $\mathbb{I}_K$ is the identity matrix of the size $K \times K$.

For any choice of $Q_1, ..., Q_{N-1}$ the determinant $d(\xi) = \det(F(\xi, l) W(l))$ is an exponential polynomial, i.e., a finite sum of the form

$$d(\xi) = \sum_j \alpha_j e^{\xi \beta_j}.$$

We are going to prove that for an appropriate choice of $Q_1, ..., Q_{N-1}$, this polynomial does not vanish identically. This would prove Lemma 4.1. We have

$$d(\xi) = e^{2\pi(w_1 + \cdots + w_N)\xi} \prod_{j=1}^{N} a_j \prod_{j \neq l} (u_j - u_l) \det(V_1, ..., V_N, F_{N+1}, ..., F_{N+K}),$$

where F_j is a j'th column of the matrix $F(\xi, l)$. Observe that $e^{2\pi(w_1+\cdots+w_N)\xi} \prod_{k=1}^{N} a_j \prod_{j\neq l}(u_j - u_l) \neq 0$, which follows from the assumption that $\Re w_j \neq \Re w_l$, $j \neq l$.

4.6 Step 2

It remains to choose $Q_1, ..., Q_{N-1}$ so that $\det(V_1, ..., V_N, F_{N+1}, ..., F_{N+K})$ is non-zero. Note that this determinant is also an exponential polynomial in ξ. Therefore, it suffices to find at least one non-zero coefficient. We assume that $\Re w_1 > \Re w_2 > ... > \Re w_N$ and we are going to choose $Q_1, ..., Q_{N-1}$ so that the term $e^{2\pi K w_1 \xi}$ participates in our polynomial with a non-zero coefficient.

We have

$$M(\xi - 1) - e^{-2\pi w_1} M(\xi) = (J_0, J_1, ..., J_{N-1}),$$

where J_s does not contain the frequency $e^{2\pi \xi w_1}$. We do the following transformations which do not change the determinant. Each remaining couple of double rows of the matrix F has the form

$$R_s = (\mathbb{O}, M(\xi - Q_s + (k_1 + \cdots + k_{Q_s})\tau), \mathbb{O}),$$
$$R_s' = (\mathbb{O}, M(\xi - Q_s - 1 + (k_1 + \cdots + k_{Q_s})\tau), \mathbb{O}).$$

We replace the row R_s' by $T_s = e^{-2\pi w_1} R_s - R_s'$ which is now free from the terms containing $e^{2\pi\xi w_1}$. Next, we rearrange the rows of $F(\xi, l)$ in such a way that the new rows T_s go after the first row of $F(\xi, l)$. This yields a rearrangment of the rows of the matrix $(V_1, ..., V_N, F_{N+1}, ..., F_{N+K})$ which after this rearrangment acquires a transparent block structure

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix},$$

where $N \times N$ matrix X and $K \times N$ matrix Z are independent on ξ, T is a $K \times K$ lower triangular matrix while the $N \times K$ matrix Y does not contain terms with $e^{2\pi\xi w_1}$.

Since T is lower-triangular the coefficient of $e^{2\pi K w_1 \xi}$ comes from the diagonal elements of the matrix T only. The diagonal elements of the matrix T are equal to $m_{N-1}(\xi + ...)$ and have non-zero coefficients in front of $e^{2\pi\xi w_1}$. Thus, it remains to prove that $\det X \neq 0$.

We have

$$X = \begin{pmatrix} u_1^K & u_2^K & \dots & u_N^K \\ 0 & u_2^K(e^{-2\pi w_2} - e^{-2\pi w_1}) & \dots & u_N^K(e^{-2\pi w_N} - e^{-2\pi w_1}) \\ 0 & u_2^{K-q_2}(e^{-2\pi w_2} - e^{-2\pi w_1})e^{2\pi w_2(q_2\tau - Q_2)} & \dots & u_N^{K-q_2}(e^{-2\pi w_N} - e^{-2\pi w_1})e^{2\pi w_N(q_2\tau - Q_2)} \\ & & \dots & \\ 0 & (e^{-2\pi w_2} - e^{-2\pi w_1})e^{2\pi w_2(K\tau - Q_{N-1})} & \dots & (e^{-2\pi w_N} - e^{-2\pi w_1})e^{2\pi w_N(K\tau - Q_{N-1})} \end{pmatrix}.$$

Note that since $\Re w_1 \neq \Re w_j$, $j > 1$ all the factors $e^{-2\pi w_j} - e^{-2\pi w_1}$ are non-zero. We have

$$\det X = u_1^K \prod_{j=2}^{N} (e^{-2\pi w_j} - u^{-2\pi w_1}) \det \begin{pmatrix} u_2^K & \dots & u_N^K \\ u_2^{K-q_2} e^{2\pi w_2(q_2\tau - Q_2)} & \dots & u_N^{K-q_2} e^{2\pi w_N(q_2\tau - Q_2)} \\ & \dots & \\ e^{2\pi w_2(K\tau - Q_{N-1})} & \dots & e^{2\pi w_N(K\tau - Q_{N-1})} \end{pmatrix}$$
$$= u_1^K \prod_{j=2}^{N} (e^{-2\pi w_j} - e^{-2\pi w_1}) \det X'.$$

Now we finally choose Q_s in such a way that $1 \ll Q_2 \ll Q_3 \ll \dots \ll Q_{N-1}$. This implies that $1 \ll q_2 \ll q_3 \ll \dots \ll q_{N-1}$. Note that since all numbers $(q_2 + \dots + q_s)\tau - Q_s$ are in $(0, \frac{1}{\alpha})$ corresponding exponents are uniformly bounded from above and from below. It remains to observe that, since $|u_2| > |u_3| > \dots > |u_N|$, when we expand $\det X'$ as a sum over all permutations the diagonal term containing $u_2^K u_3^{K-q_2} \dots$ will dominate everything else and so $\det X'$ is non-zero. The lemma is proved. □

4.7 Proof of Theorem 1.11

Modifying above arguments we can prove that the system $\mathfrak{G}(g; \alpha, \beta)$ is complete. Moreover, since the function m_0 is a priori almost everywhere non-zero and the matrix $D(\xi, l)$ almost always has full rank we do not need the assumptions about m_0 and the irrationality of $\alpha\beta$.

For $\alpha\beta > 1$ the system $\mathfrak{G}(g; \alpha, \beta)$ is incomplete in $L^2(\mathbb{R})$. This follows from the general theorem in [21]. However in our case we have a direct proof. Looking at the proof of Theorem 1.2 we can see that if $\alpha\beta > 1$ then there exists an infinite-dimensional space of function which are orthogonal to all $\mathfrak{G}(g; \alpha, \beta)$. If $\alpha\beta = 1$, then the frame operator is unitarily equivalent to the multipication by the Zak transform. In our case Zak transform is analytic and hence almost everywhere non-zero, so we have completeness in this case as well.

5 Other results

In this section we prove Theorems 1.6, 1.7, 1.9 and, in addition, we construct some counterexamples.

5.1 Near the critical hyperbola (proof of Theorems 1.6 and 1.7)

Let the condition of Theorem 1.6 be met. The beginning of the proof repeats that of Theorem 1.1. We also use some notation as in Theorem 1.1. Let

$$(LG)(\xi) = \sum_{s=0}^{N-1} G\left(\xi + \frac{s}{\alpha}\right) m_s(\{\xi\}).$$

It suffices to prove (see Sect. 3.2, Step 1) that for some α_0,

$$\|LG\|_2 \gtrsim \|G\|_2, \quad G \in L^2(\mathbb{R}),$$

for each $\alpha > \alpha_0$.

We further follow Steps 2 and 3 Sects. 2.3, 2.4 by showing that the norm of the corresponding product of Frobenius matrices decays at least as a geometric progression:

$$\|F(p_{\{\xi\}})F(p_{\{\xi-\frac{1}{\alpha}\}})\dots F(p_{\{\xi-l/\alpha\}})\| \le Cq^l, \quad \text{for some } q \in (0, 1). \tag{5.1}$$

First we note that the spectrum of $F(p_\xi)$ coincides with the zero set of the polynomial

$$p_\xi(z) = \frac{m_0(\xi) + m_1(\xi)z + \cdots + m_{N-1}(\xi)z^{N-1}}{m_{N-1}(\xi)}.$$

On the other hand, from the assumption of Theorem 1.6 we have

$$\Re\mathcal{Z}(z, \xi) = \Re\frac{m_{N-1}(\xi)p_\xi(z)}{\prod_{k=1}^{N}(1 - ze^{2\pi w_k/\alpha})} > 0, \quad |z| = 1,$$

and, by the argument principle, the number of zeroes of p_ξ (counting with multiplicities) inside the unit disk $\mathbb{D}$ is N. Actually they are located in a smaller disk $\{z : |z| < \rho\}$, where $\rho < 1$ is chosen so that $\Re\mathcal{Z}(\rho e^{2\pi it}, \xi) > 0$, $\xi \in [0, 1]$, $t \in \mathbb{R}$, and $\rho > e^{-2\pi w_k/\alpha}$, $k = 1, \dots, N$.

Therefore the spectral radius of the matrices $F(p_\xi)$ is strictly less than 1. In particular,

$$\|F^M(p_\xi)\| \le q < 1, \xi \in [0, 1], \text{ for sufficently large } M, \tag{5.2}$$

where $\|\cdot\|$ is an operator norm of matrix. This inequality is *uniform* with respect to $\xi \in [0, 1]$ (numbers q and M do not depend on ξ).

Given this M one can choose α sufficiently close to 1 so that the matrices $F(p_{\{\xi-\frac{k}{\alpha}\}})$ and $F(p_{\{\xi-\frac{j}{\alpha}\}})$ are arbitary close to each other if $|k-j| < M$, so for $M \ll l$ the product in (5.2) can be represented as a product of l/M_0 uniformly strictly contractive matrices. This completes the proof. □

Combining Remark 1.5 and Theorem 1.6 we get also Theorem 1.7.

Remark 5.1 The condition $\Re\mathcal{Z}(z, \xi) > 0$, $|z| = 1$, $\xi \geq 0$ can be reformulated as

$$\sum_{n\geq 0} m_0\left(\xi + \frac{n}{\alpha}\right)\cos(nt) > 0, \quad \xi, t \in \mathbb{R}.$$

Proof Put $z = e^{it}$. We have

$$\Re\mathcal{Z}(z, \xi) = \sum_{k=1}^{N} a_k e^{2\pi\xi w_k} \Re\frac{1}{1 - ze^{2\pi w_k \alpha}} =$$

$$\sum_{k=1}^{N} a_k e^{2\pi\xi w_k} \sum_{n=0}^{\infty} \Re(z^n e^{2\pi n w_k \alpha}) = \sum_{n\geq 0} m_0\left(\xi + \frac{n}{\alpha}\right)\cos(nt).$$

□

This form is useful if we want to check the inequality for all rescaled functions $g(\cdot/\beta)$ since rescaling of the window g corresponds to the rescaling of m_0.

Now, we are in position to prove Corollary 1.8. From the equation $m_0(\xi) = \hat{g}(\xi)$ we conclude that the Fourier series

$$\sum_{n\geq 0} m_0\left(\xi + \frac{n}{\alpha}\right)\cos(nt)$$

has positive, convex coefficients for any ξ. It is known that such the Fouries series are positive, which can be deduced by applying the Abel transform twice. □

5.2 Lattices with large densities

In this section we prove Theorem 1.9. The proof is similar to one of Theorem 1.2. By the standard rescaling we can assume $\beta = 1$. As in Sect. 4.2 it suffices to prove that the discrete operator L_θ defined by (4.4) satisfies

$$\|L_\xi P_\xi\| \gtrsim \|P_\xi\|.$$

Since $\alpha < \frac{1}{N}$, the variable ξ in (4.4) runs over the interval $(0, 1/\alpha) \supset (0, N)$. Given $\xi \in (0, 1/\alpha)$ take $\xi' = \xi + \frac{k}{\alpha} \geq N-1$, $\xi' \in (0, 1/\alpha)$. After may be omitting some rows the operator L_ξ can be reduced to the block-diagonal structure with $N \times N$ blocks of the form:

$$B(\xi', N) := \begin{pmatrix} M(\xi') \\ M(\xi'-1) \\ \dots \\ M(\xi'-N) \end{pmatrix}$$
$$= \begin{pmatrix} m_0(\xi') & m_1(\xi') & \dots. & m_{N-1}(\xi') \\ m_0(\xi'-1) & m_1(\xi'-1) & \dots. & m_{N-1}(\xi'-1) \\ \dots & & & \\ m_0(\xi'-(N-1)) & m_1(\xi'-(N-1)) & \dots. & m_{N-1}(\xi'-(N-1)) \end{pmatrix}.$$

Theorem 1.9 follows now from

Lemma 5.2 *Let $a_k \neq 0$, $k = 1, \dots, N$ and $\Re w_k \neq \Re w_l$, $k \neq l$. Then*

$$\det B(\xi, N) \neq 0.$$

Proof Put $y_k = e^{-2\pi w_k}$, $u_k = e^{2\pi w_k/\alpha}$, $A_k = a_k e^{2\pi \xi w_k}$, $k = 1, \dots, N$. Then

$$B = \begin{pmatrix} A_1 & A_2 & \dots & A_N \\ A_1 y_1 & A_2 y_2 & \dots & A_N y_N \\ & & \dots. & \\ A_1 y_1^{N-1} & A_2 y_2^{N-1} & \dots & A_N y_N^{N-1} \end{pmatrix} \begin{pmatrix} 1 & -\sum_{k\neq 1} u_k & \dots & (-1)^{N-1}\prod_{k\neq 1} u_k \\ 1 & -\sum_{k\neq 2} u_k & \dots & (-1)^{N-1}\prod_{k\neq 2} u_k \\ & & \dots & \\ 1 & -\sum_{k\neq N} u_k & .. & (-1)^{N-1}\prod_{k\neq N} u_k \end{pmatrix}$$
$$= XY.$$

The entries of the matrix Y are symmetric polynomials with respect to the corresponding subsets of variables $\{u_1, \dots, u_N\}$. We have

$$\det Y = \pm \prod_{k<l} (u_k - u_l) \neq 0.$$

On the other hand,

$$\det X = \prod_{k=1}^{N} A_k \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ y_1 & y_2 & \dots & y_N \\ & & \dots. & \\ y_1^{N-1} & y_2^{N-1} & \dots & y_N^{N-1} \end{pmatrix}.$$

It remains to note that this Vandermonde determinant does not vanish. □

Proposition 5.3 *For any $N > 1$ and any numbers $w_1, w_2, ..., w_N$, such that $\Re w_k \neq 0$, $\Re w_k \neq \Re w_l$ for $k \neq l$, there exist a non-zero $\{a_k\}_{k=1}^N$ such that $\mathfrak{G}(g; 1/(N-1), 1)$ is not a frame in $L^2(\mathbb{R})$. Here g is defined by* (1.1).

Proof We have $\alpha = \frac{1}{N-1}$. As before fix $\xi \in (0, N-1)$ and consider the corresponding operator L_ξ. Observe that there are only $N-1$ different rows in L_ξ. Therefore one can find a non-zero sequence $\{a_k\}_{k=1}^N$ such that $L_\xi \mathbb{1} = 0$, where $\mathbb{1} = (..., 1, 1, 1, ...)^T$. Since L_ξ has bounded entries and its rows have finitely many elements, by truncating the column of all 1's we can get an ℓ^2 sequence γ with arbitrary large ℓ^2-norm and uniformly bounded norm of $L_\xi \gamma$. Therefore L_ξ cannot be bounded from zero. □

Similar arguments lead us to the follwoing statement:

Proposition 5.4 *For any rational number α there exists rational window g such that Gabor system $\mathfrak{G}(g; \alpha, 1)$ is not a frame.*

5.3 Counterexamples

Assumption $m_0(\xi) \neq 0$ in Theorem 1.2 cannot be omitted, generally speaking. By analysing the proof of this theorem one can find a method for construction of *non-frame* Gabor systems different from the one used in Propositions 5.3–5.4. This leads us to Gabor non-frame systems with *irrational densities*.

Theorem 5.5 *Let $g(x) = \sum_{k=1}^{3} \frac{a_k}{x - i w_k}$. Assume that $\alpha > \frac{5}{6}$ and also that for some $0.99 < \xi_0 < 1$ we have*

$$m_0\left(\xi_0 - \frac{2}{\alpha} + 2\right) = m_1\left(\xi_0 - \frac{1}{\alpha} + 1\right) = m_2(\xi_0) = 0. \tag{5.3}$$

Then the system $\mathfrak{G}(g; \alpha, 1)$ is not a frame in $L^2(\mathbb{R})$.

Proof By Theorem 2.1 the necessery and sufficient condition for $\mathfrak{G}(g; \alpha, 1)$ to be a frame is

$$\int_0^{1/\alpha} \sum_{n \in \mathbb{Z}} \left| \sum_{s=0}^{2} G\left(\xi - n - \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \gtrsim ||G||^2_{L^2(\mathbb{R})}, \quad G \in L^2(\mathbb{R}).$$

Clearly this relation is not met for functions G concentrated in δ-vicinity of the point $\xi_0 - 2/\alpha$. Even if we bound left-hand side by

$$\int_0^{1/\alpha} \sum_{n\in\mathbb{Z}} 3 \sum_{s=0}^{2} \left| G(\xi - n - \frac{s}{\alpha}) \right|^2 |m_s(\xi)|^2 d\xi, \tag{5.4}$$

for sufficiently small δ we may obtain a quantity less than $\varepsilon \|G\|^2$. □

Remark 5.6 Similar result can be proved for the sum of arbitrary many kernels as long as α and ξ are close enough to 1.

For fixed α, w_1, w_2, w_3 conditions (5.3) are three homogenious linear equations in a_1, a_2, a_3. For this system to have a nontrivial solution corresponding 3×3 determinant has to be zero. We will construct numbers α, w_1, w_2, w_3 such that this determinant is zero and $\Re w_k$ are pairwise different. Moreover, in our construction we would have $w_2 = \frac{1}{2\pi}$, $w_3 = -\frac{1}{2\pi}$ and arbitrary α (rational or irrational) with $|\alpha - \frac{6}{7}| < \frac{1}{1000}$.

Let us begin with explicitly writing down the matrix corresponding to (5.3)

$$\begin{pmatrix} e^{2\pi w_1(\xi_0 - \frac{2}{\alpha}+2)} & e^{2\pi w_2(\xi_0 - \frac{2}{\alpha}+2)} & e^{2\pi w_3(\xi_0 - \frac{2}{\alpha}+2)} \\ e^{2\pi w_1(\xi_0 - \frac{1}{\alpha}+1)}(e^{\frac{2\pi}{\alpha}w_2} + e^{\frac{2\pi}{\alpha}w_3}) & e^{2\pi w_2(\xi_0 - \frac{1}{\alpha}+1)}(e^{\frac{2\pi}{\alpha}w_1} + e^{\frac{2\pi}{\alpha}w_3}) & e^{2\pi w_3(\xi_0 - \frac{1}{\alpha}+1)}(e^{\frac{2\pi}{\alpha}w_1} + e^{\frac{2\pi}{\alpha}w_2}) \\ e^{2\pi w_1\xi_0 + \frac{2\pi}{\alpha}w_2 + \frac{2\pi}{\alpha}w_3} & e^{2\pi w_2\xi_0 + \frac{2\pi}{\alpha}w_1 + \frac{2\pi}{\alpha}w_3} & e^{2\pi w_3\xi_0 + \frac{2\pi}{\alpha}w_1 + \frac{2\pi}{\alpha}w_2} \end{pmatrix}.$$

First we note that whether this determinant is zero or not does not depend on ξ_0 (in particular it is irrelevant if $0.99 < \xi_0 < 1$ or not). Therefore, without loss of generality we can assume that $\xi_0 = \frac{1}{\alpha} - 1$. Expanding the determinant (which we view as a function of w_1) and dividing it by $e^{\frac{2\pi}{\alpha}(w_1+w_2+w_3)}$ we get

$$\begin{aligned} F(w_1) &= e^{2\pi w_1}(e^{-2\pi w_2} - e^{-2\pi w_3}) - e^{-2\pi w_1}(e^{2\pi w_2} - e^{2\pi w_3}) \\ &\quad - e^{\varepsilon 2\pi w_1}(e^{-\varepsilon 2\pi w_2} - e^{-\varepsilon 2\pi w_3}) + e^{-\varepsilon 2\pi w_1}(e^{\varepsilon 2\pi w_2} - e^{\varepsilon 2\pi w_3}) + C, \end{aligned}$$

where $\varepsilon = \frac{1}{\alpha} - 1$ and C is a constant such that $F(w_2) = F(w_3) = 0$.

Let us first put $\alpha = \frac{6}{7}$. Then $\varepsilon = \frac{1}{6}$ and $e^{2\pi w_1} = (e^{\varepsilon 2\pi w_1})^6$. Denoting $e^{\varepsilon 2\pi w_1} = z$ and recalling that $w_2 = \frac{1}{2\pi}$, $w_3 = -\frac{1}{2\pi}$ the equation we rewrite as

$$\begin{aligned} &(e - e^{-1})(z^6 + z^{-6}) - (e^{1/6} - e^{-1/6})(z + z^{-1}) \\ &\quad - e^2 + e^{-2} + e^{1/3} - e^{-1/3} = 0. \end{aligned} \tag{5.5}$$

This equation has solutions $z_1 = e^{1/6} \approx 1.18$, $z_2 = e^{-1/6} \approx 0.85$. But one can numerically verify that it also has negative solutions $z_3 \approx -1.12$,

$z_4 \approx -0.89$. They correspond for example to $w_1 = 0.108 + 3i$ and $w_1 = -0.111 + 3i$ respectively. For any α close to $\frac{6}{7}$ we can find a close solution by the argument principle.

So, we proved the following theorem.

Theorem 5.7 *There exists a rational window g of degree 3 and irrational number $\alpha < 1$ such that $\mathfrak{G}(g; \alpha, 1)$ is not a frame.*

6 Sampling in shift-invariant spaces

The results on Gabor frames allow us to obtain new theorems on sampling in shift-invaraint spaces, generated by rational windows. We follow mainly the pattern of [10], which in turn relies on the Jannsen's version of duality theory.

First, we remind the basic definitions. Given a function g which belongs to the Wiener amalgam space $W_0 = W(\ell^1, \mathbb{C})$, i.e., g is continuous and

$$\|g\|_{W_0} = \sum_{k\in\mathbb{Z}} \max_{x\in[k,k+1]} |g(x)| < \infty. \tag{6.1}$$

For functions g of the form (1.1) this is equivalent to the equality $\sum_{k=1}^{N} a_k = 0$.

Consider now the shift-invariant space $V^2(g)$ which consists of the functions of the form

$$f(t) = \sum_{k\in\mathbb{Z}} c_k g(t-k), \quad \{c_k\} \in \ell^2(\mathbb{Z}). \tag{6.2}$$

Clearly, $V^2(g) \subset L^2(\mathbb{R})$ and

$$\|f\|_{L^2} \le \|g\|_{W_0} \|\{c_k\}\|_2. \tag{6.3}$$

We will assume the following stability of the generator g:

Proposition 6.1 (see e.g. [22], Theorem 29) *The following properties are equivalent*

(i) *There exists $C > 0$ such that*

$$\|\sum_{k\in\mathbb{Z}} c_k g(t-k)\| \ge C\|\{c_k\}\|_2; \tag{6.4}$$

(ii)

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi - k)|^2 > 0 \quad \textit{for all } \xi \in \mathbb{R}. \tag{6.5}$$

We say that a sequence $\Lambda \subset \mathbb{R}$ is *separated* if for some $\delta > 0$, $|\lambda - \mu| > \delta$, $\lambda, \mu \in \Lambda, \lambda \neq \mu$.

A separated sequence Λ is called *sampling* for $V^2(g)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|_2^2, \quad f \in V^2(g). \tag{6.6}$$

In what follows we apply these definitions to the sequences $\alpha\mathbb{Z}$, $\alpha < 1$ and spaces $V^2(g)$ generated by the rational function g.

The relation between sampling and frame properties is given by the following statement.

Proposition 6.2 *Let $g \in W_0$ satisfy the properties from Proposition 6.1. The following are equivalent:*

(i) *The family $\mathfrak{G}(g; \alpha, 1)$ is a frame for $L^2(\mathbb{R})$;*
(ii) *There exist $A, B > 0$ such that for each $x \in [0, 1]$ and f of the form (6.2)*

$$A\|\{c_k\}\|_2^2 \leq \sum_{m \in \mathbb{Z}} |f(x - \alpha m)|^2 \leq B\|\{c_k\}\|_2^2. \tag{6.7}$$

We refer the reader to (now) classical article [12] for the proof of this proposition and also to [10] for more general sequences. For the case of characteristic functions this statement has been proved also in [2].

We combine Proposition 6.2 with Theorems 1.4 and 1.2.

Lemma 6.3 *Let the function g have the form (1.1), $\Re w_k \neq \Re w_l$, $k \neq l$, and, in addition $g \in W_0$ and $m_0(\xi) \neq 0$, $\xi > 0$. Then for each $\alpha \notin \mathbb{Q}$, $\alpha \in (0, 1)$ there exist $A_\alpha, B_\alpha > 0$ such that*

$$A_\alpha\|\{c_k\}\|_2^2 \leq \sum_{k \in \mathbb{Z}} |f(\alpha k)|^2 \leq B_\alpha\|\{c_k\}\|_2^2 \tag{6.8}$$

for each function f of the form $f(t) = \sum_k c_k g(t - k)$.

Proof It suffices to prove that g admits stable sampling, for example check the inequality (6.5). We have

$$\hat{g}(\xi) = \begin{cases} \sum_{w_k < 0} a_k e^{2\pi\xi w_k}, & \xi > 0, \\ -\sum_{w_k > 0} a_k e^{2\pi\xi w_k}, & \xi < 0. \end{cases}$$

each sum on the right hand-side has the leading term as $|\xi| \to \infty$ (it corresponds to the smallest $|w_k|$). Thus, we have (6.5). □

Theorem 6.4 *Let the function g have the form* (1.1), $\Re w_k \neq \Re w_l, k \neq l$, *and, in addition, $g \in W_0$ and $m_0(\xi) \neq 0$, $\xi > 0$. Then, for each $\alpha \in (0, 1)$ the set $\alpha\mathbb{Z}$ is a sampling for $V^2(g)$ i.e. there exist $A_\alpha, B_\alpha > 0$ such that*

$$A_\alpha \|\{c_k\}\|_2^2 \leq \sum_{k\in\mathbb{Z}} |f(\alpha k)|^2 \leq B_\alpha \|\{c_k\}\|_2^2, \quad f \in V^2(g). \tag{6.9}$$

Proof The left-hand side inequality is a direct consequence of (6.8) and (6.3). The proof of the right-hand side inequality follows the classical Plancherel-Polya pattern. Each function $f \in V^2(g)$ has the form

$$f(t) = \sum_{k=1}^{N} a_k \sum_{n=-\infty}^{\infty} \frac{c_n}{t - (n + i w_k)} \tag{6.10}$$

and thus admits analytic continuation to $\mathbb{C} \setminus \cup_{k=1}^{N} (\mathbb{Z} + i w_k)$. Denote

$$q(x) = 1 - e^{2\pi i z}, \quad Q(z) = \prod_{k=1}^{N} q(z - i w_k)$$

and consider an entire function of exponential type which in addition belongs to $L^2(\mathbb{R})$

$$F(z) = Q(z) f(z). \tag{6.11}$$

Function F belongs to the Paley–Wiener space $\mathcal{PW}_a$ for an appropriate a and, hence, satisfies Plancherel-Polya inequality

$$\sum_{k\in\mathbb{Z}} |F(\alpha k)|^2 \lesssim \|F\|_2^2.$$

□

Remark 6.5 It follows from [19] that condition $\alpha \notin \mathbb{Q}$ cannot be omitted in general. On the other hand the right inequality in (6.9) holds independently of irrationality of α.

Remark 6.6 Relation (6.8) can be viewed as a statement on sampling by linear combination of values of functions in the Paley–Wiener space. Problems of such type appear in the study of eigenfunction expansions of operator pencils, see e.g. [18]. Indeed, given f of the form (6.10) we denote

$$H(z) = q(z) \sum_{n=-\infty}^{\infty} \frac{c_n}{z - n}.$$

We then have

$$H \in \mathcal{PW}_{[0,1]}, \quad \|H\|_2 \asymp \|\{c_k\}\|_{\ell^2},$$

and (6.9) can be read as

$$\sum_{j=-\infty}^{\infty} |\sum_{k=1}^{N} a_k q(\alpha j - i w_k)^{-1} H(\alpha j - i w_k)|^2 \asymp \|H\|_2^2.$$

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