

**Toward a Unified Account of Definitions in Mathematics Education Research: A  
Systematic Literature Review**

Hermund André Torkildsen\*†, Tore A. Forbregd\*, Eivind Kaspersen, and Trygve Solstad

Department of Teacher Education, Faculty of Social and Educational Sciences, Norwegian University of Science and  
Technology (NTNU), Trondheim, Norway

**Author Note**

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\* These authors have contributed equally to this work.

† Corresponding author at: NTNU, Department of Teacher Education, NO-7491 Trondheim,  
Norway.

### **Abstract**

In this paper, we provide a conceptual framework of the central aspects of mathematical definitions discussed in the mathematics education literature. Based on a systematic literature review, we found that characterizations of definitions in the mathematics education literature can be classified into five main themes: requirements, preferred features, role and function, nature, and types of definition. Within each theme, a refined set of aspects was found by means of inductive coding. We discuss each aspect and point out areas where we believe further research is needed. The framework can be used both as a research tool and a didactical tool, as it highlights many aspects of mathematical definitions that call for attention by teachers and researchers.

*Keywords:* mathematical definition, literature review, mathematics education, teacher education, reasoning, definition roles, definition nature, definition requirements, definition features

## 1. Introduction

What is the definition of a mathematical definition? Philosophers of mathematics, such as Whitehead, Russell, Frege, and Wittgenstein, have contemplated this question without reaching clear answers (Frege et al., 1984; Weiner, 1990; Whitehead & Russell, 1912; Wittgenstein & Bosanquet, 1989). This is perhaps evidence for a view that there cannot be a definition of mathematical definition, or at least not a trivial one. Consequently, practitioners of mathematics must come to their own understanding of what mathematical definitions entail. Nonetheless, there are examples of agreed-upon rules by which mathematical definitions should abide (e.g., Edwards & Ward, 2008; Van Dormolen & Zaslavsky, 2003). Rather than being normative in nature, these rules may be better described as advice for best practices.

Students involved in mathematical activity are expected to prove mathematical statements. These can be algebraic proofs using only symbols but also representation-based proofs (Schifter, 2009) and generic examples (Balacheff, 1988; Stylianides, 2007; Stylianides & Stylianides, 2009). Definitions are essential ingredients in a mathematical proof, which means that students, just as mathematicians, need to consult and use definitions. For example, if one wishes to prove that the product of two even numbers is even, then it is necessary to use definitions of “even number” and “product.”

While mathematicians seem to have a common understanding of the nature and use of mathematical definitions (Parameswaran, 2010), research shows that students do not necessarily share this understanding (Edwards & Ward, 2004). Mathematical definitions are stipulated (formal definitions that determine the one and only meaning of a term), but some students may see them as

extracted (everyday definitions or dictionary definitions of words), that is to say, merely a description of something. In the language of Tall and Vinner (1981), their concept image may override the concept definition, which has been exemplified by Edwards and Ward (2004).

To support further investigation into students' deductive reasoning and proving, it is important to know which aspects of mathematical definitions are relevant, discussed and highlighted in the mathematics education literature. Although this literature is rich in discussions about mathematical definitions, the many aspects of definitions that are addressed in the literature have not been collected in a coherent framework. In an effort to build such a coherent framework for mathematical definitions, we conducted a systematic literature review to answer the following research question:

*How are mathematical definitions characterized in the mathematics education literature?*

## **2. Methodology**

To answer our research question, we conducted a literature review to examine the research on aspects of mathematical definitions in the field of mathematics education. We followed the guidelines by Arksey and O'Malley (2005): (1) identifying the research question, (2) identifying relevant studies, (3) study selection, (4) charting the data, and (5) collating, summarizing, and reporting the results.

Our main goal was to collect papers that addressed mathematical definitions as objects themselves (i.e., definitions being (one of) the object(s) of study) or that contained characterizations of mathematical definitions in general. A systematic literature search was therefore carried out to find

academic publications that considered mathematical definitions in mathematics education. A coding process was then conducted on the raw material.

## **2.1. Sample**

We employed a systematic, but not exhaustive, search following guidelines provided by Arksey and O'Malley (2005) and Lipsey and Wilson (2001). In the initial search, a publication was deemed relevant if it met the following inclusion criteria: the publication (1) was related to mathematical definitions, (2) had a didactical focus, and (3) was published in a peer-reviewed journal, peer-reviewed conference proceeding, or book.

The databases used to conduct the search were the Educational Resources Information Center (ERIC) and Scopus. Our search criteria within these databases were identical and as follows: both “mathematics” and “definition” in the title, abstract, or keywords/descriptor, and “education” in any field. After this main search, we used the search engines' exclusion functionalities. Details on this process are outlined in Appendix B.

This yielded a total of  $n = 1,493$  publications (ERIC  $n = 619$ , Scopus  $n = 874$ ). The search results were then exported to the ‘ris’ or ‘nbib’ file format. Scopus allows a search result to be exported with abstracts, while for the ERIC results, we retrieved the abstracts manually. To remove duplicates (both across the databases and within each) we used a Python script that programmatically determined the similarity of papers by comparing the ratio of similarity (as text strings) of the abstracts and title. After manually confirming and removing duplicates, we were left with  $n = 1,253$  papers.

The sample was then further examined in a two-step process. First, papers were assessed by their titles and abstracts, and papers not fitting with the inclusion criteria were discarded ( $n = 1,079$ ). Many of those papers were publications in fields other than mathematics education that still contained the inclusion search words.

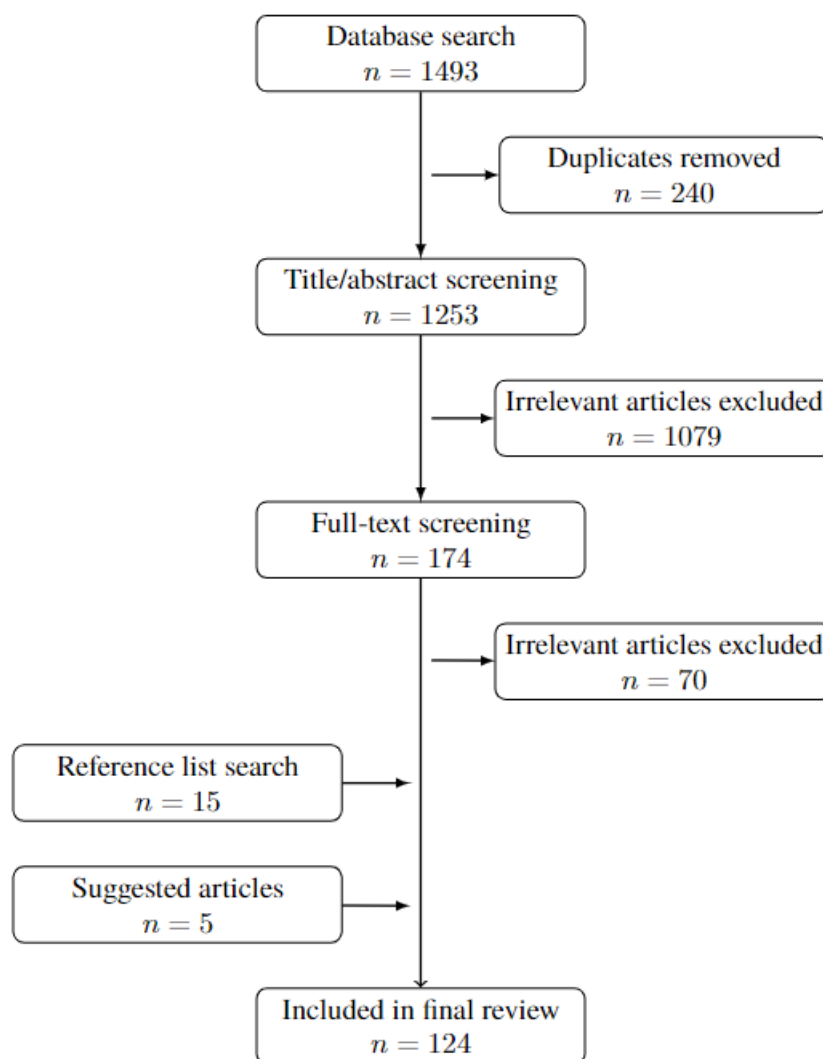
The remaining 174 candidate articles were further scrutinized by reading the papers. Upon reading the papers,  $n = 70$  publications were deemed irrelevant and excluded. The inclusion criteria in this process were as follows: (1) mathematical definitions were (one of) the main object(s) of study, or (2) general characterizations of mathematical definitions were explicitly mentioned. In this process, we first conducted a digital search for the word *definition* in each of the papers to get an overview. Many of the excluded papers were publications in the field of mathematics education that contained the word *definition* in, for example, the data materials, as examples, or as part of tasks given to informants, but did not address them as objects or mention any general characterizations. In the cases where it was unclear if statements in a paper were general characterizations of mathematical definitions, the paper was either included, or great care was taken to ascertain that the points had appeared elsewhere in the reviewed literature.

To mitigate publication bias, we examined the bibliography of the refined sample to reveal potentially relevant publications not included in the database searches. This resulted in  $n = 15$  additional publications.

After conducting the analysis and writing down our results, additional publications were suggested as potential candidates by colleagues and peers who read the paper. The steps described above were

carried out on those papers, which resulted in  $n = 5$  additional publications. The analysis procedure described in the next section was also applied to these additional papers.

This literature search strategy resulted in a final sample of  $n = 124$  publications (see Appendix B). A flow chart of the search strategy is outlined in **Figure 1**.



**Figure 1** - Systematic review search strategy

## **2.2. Analysis**

The data analysis was conducted in a multi-step process. As a first step, the first authors read the papers in the data set and highlighted quotes and paragraphs concerned with mathematical definitions. These paragraphs were then extracted from the selected papers and collected in a common list. Duplicate entries in this list were then removed.

Next, the quotes and paragraphs were coded openly in an inductive manner by the first authors individually. Paragraphs that addressed mathematical definitions as objects or described characterizations of definitions in general were extracted from the data and collected in a common list. Duplicates were then removed as in the first step described above. In this process, we looked for statements like “definitions must be,” “definitions should,” “definitions are,” “definitions are used,” and so on. Both steps had to be carried out qualitatively and manually, especially because not all statements contained the word “definition,” and because, in some cases, we had to interpret the statements to decide if they were general characterizations or not. This was first done individually, and then in a discussion group, to reach an agreement.

Subsequently, codes with common themes were merged individually by the first authors (Mayring, 2000). Obviously equivalent statements were merged first, i.e., statements that only differed in their phrasing. As a final step, a further refinement of the themes was reached by another pass of inductive coding within each theme in the previous step, both individually and in a discussion group. For example, “a definition should be minimal” and “definitions should be economical” were deemed to be sufficiently related to be merged into the theme Minimality, even though they do not necessarily mean exactly the same thing. Another example is the statement that a definition should

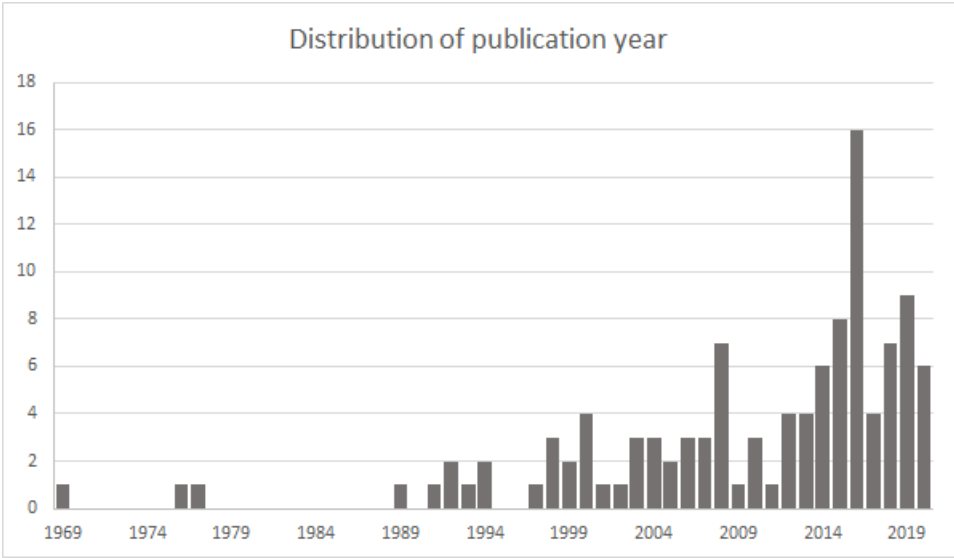


“allow to discriminate between instances and non-instances” and the criterion of “existence” of at least one instance of the defined concept. The first statement says that it should be possible to decide if an example is an element of the example space defined, while the second states that the example space cannot be empty. Although they are not saying the same thing, they both deal with the example space of the defined concept and were placed under the theme Exemplification. The differences between such statements are explained in the results section.

For every coding pass, each first author conducted individual coding, after which a discussion of the proposed categorizations was carried out, and a final agreement on categorization was reached.

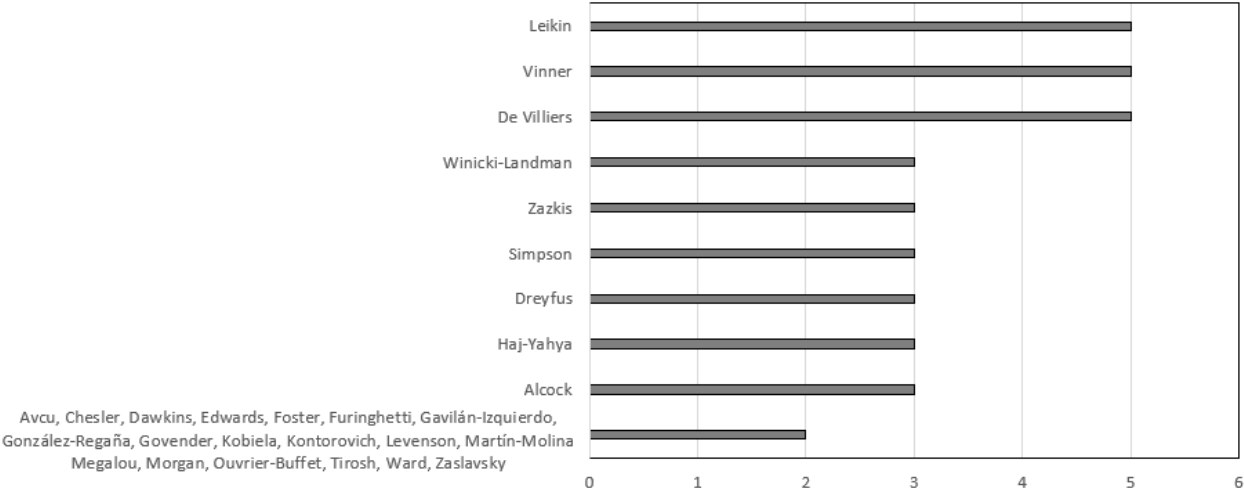
### **3. Descriptive overview**

The systematic literature search yielded 124 publications from journal articles, conference papers, book chapters, and books. The years of publication of the sample range from 1969 to 2022; **Figure 2a** shows a bar chart of the distribution of publication years. A total of 202 different authors were present in the sample, and **Figure 2b** shows authors with multiple publications. The publications in the top five most frequent journals amounted to approximately 44% of the entire sample. **Figure 2c** shows a bar chart of the most frequent journals in the sample.

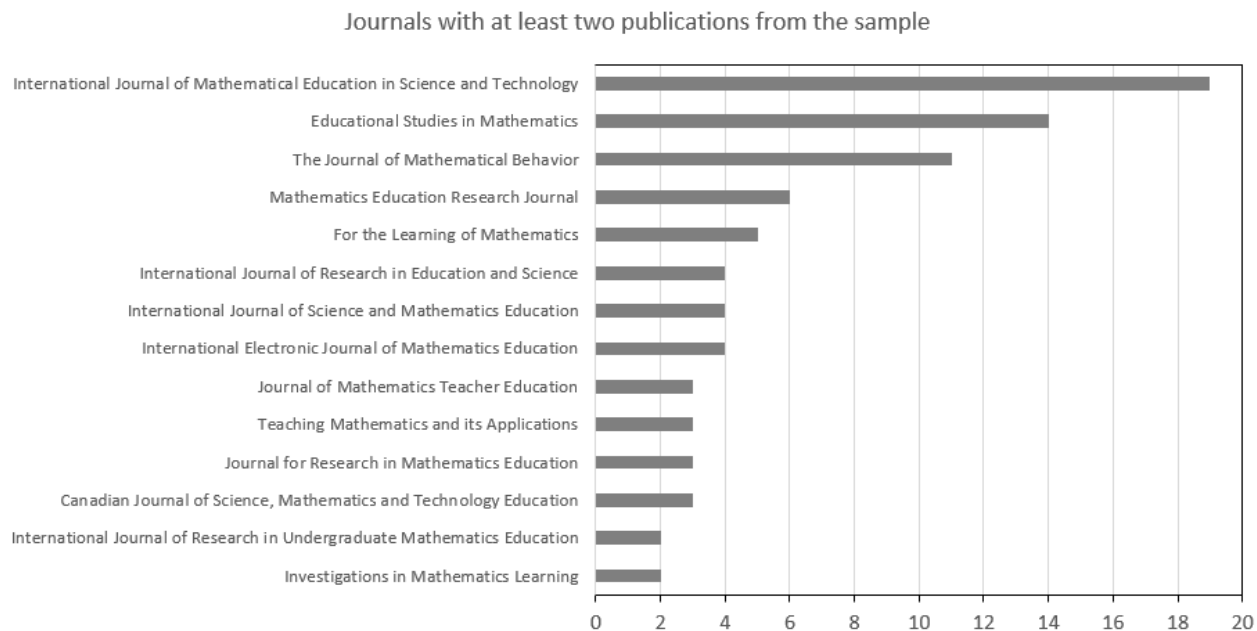


a)

Authors with at least two publications from the sample



b)

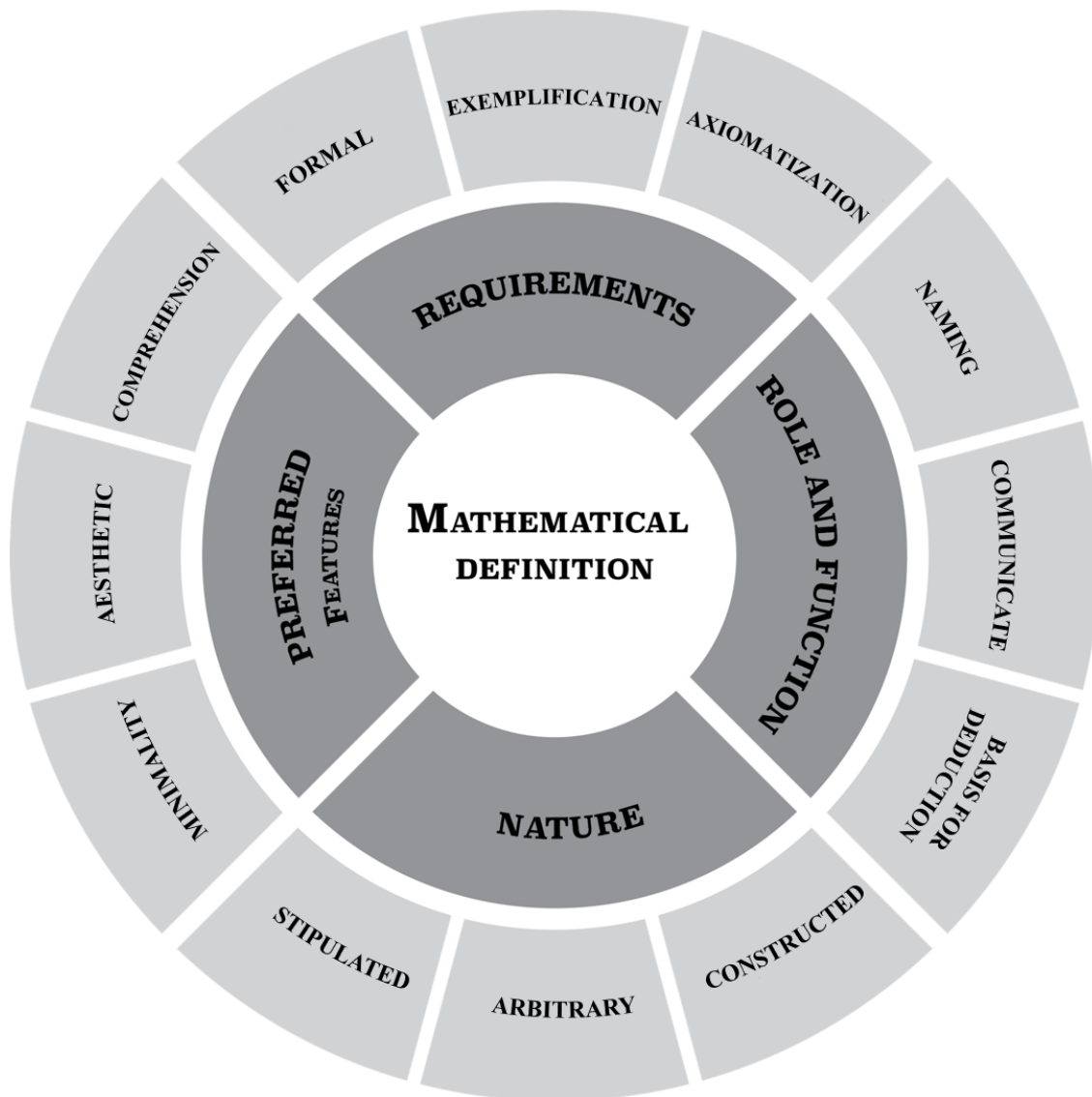


c)

*Figure 2 - Graphical visualization of selected statistics about the sample*

#### 4. Four main themes for characteristics of mathematical definitions

Our analysis revealed five emergent themes for how researchers in mathematics education characterize mathematical definitions. Four themes (Nature, Requirements, Preferred Features, and Role and Function) included characteristics of mathematical definitions, and are summarized in **Figure 3**. The fifth theme, Type, was qualitatively different from the other themes. Rather than a collection of characteristics, Type describes definitions as a two-dimensional construct. Therefore, it does not appear in Figure 3 but is described in Section 5. In the following sections, we present each theme with its corresponding categories, as well as statements and quotes from the source literature. Statements from the literature appear in italics.



*Figure 3 - The four main themes with corresponding categories*

#### **4.1. Nature**

We found an emerging theme that dealt with definitions on a meta-mathematical level, namely aspects relating to the intrinsic properties of mathematical definitions. This theme can be thought

of as properties of mathematical definitions as meta-mathematical objects, that is to say global descriptors of mathematical definitions as constructs in mathematical science.

#### **4.1.1. Stipulated**

Mathematical definitions are *stipulated* (Vinner, 1991) as opposed to extracted (lexical). Extracted definitions explain how a word is used by people, often through examples. This can be thought of as an *a posteriori* or *descriptive* definition (e.g., Foster, 2014; Foster & de Villiers, 2015; Ouvrier-Bufferet, 2010). Stipulated definitions are *formal* (e.g., Van Dormolen & Zaslavsky, 2003) and *rigorous* (e.g., Poincaré, 1969; Zazkis & Leikin, 2008) and determine the one and only meaning of a term. Terms used in a stipulated definition are placed in a new context and have only the meaning that is stated in the definition. All previous meanings of the term (if there are any) are irrelevant. The term obtains a new meaning, and previous associations of the term must be disregarded. Stipulated definitions have no truth value, as opposed to extracted definitions (Bayda & Sutliff, 2020). What this means is that extracted definitions are not always true, and they do not capture the essence of a concept completely. A chair can be defined in a dictionary as something you can sit on that usually has four legs and a rest for the back, but most people would accept a chair with three legs and no rest for the back. From the context, one would disallow the chair of a research committee. Stipulated definitions, on the other hand, are *free from connotations*, especially from everyday usage of the terms.

Foster (2014) used the term *prescriptive* to describe the stipulative property of mathematical definitions, that is, mathematical definitions prescribe the only correct meaning of a term. Others have dubbed this notion as *a priori* (De Villiers, 1994) or *constructive* (Foster & de Villiers, 2015).

An example of the relevance of this property to mathematics education is found in the work of Edwards and Ward (2004), who studied how students understand and use definitions. They found that some students, unlike mathematicians, view mathematical definitions as extracted rather than stipulated. This view profoundly impacted the students' mathematical reasoning, and the researchers concluded that even if a student can recite a definition, it does not mean that the student understands the definition. They also found that, in some cases, these students were unable to appropriately use a definition that they seemingly understood and was available to them.

#### **4.1.2. Arbitrary**

There may be many equivalent definitions of a concept. The choice of statement to use as a definition for the concept is *arbitrary*, and the other equivalent definitions become theorems that give *necessary* and *sufficient* conditions for the concept (e.g., Leikin & Zazkis, 2010; Levenson, 2012; Üreyen, 2006; Vinner, 1991; Winicki-Landman & Leikin, 2000; Zaslavsky & Shir, 2005). Of course, this does not mean that the choice of definition among a set of equivalent definitions is random. Definitions are *chosen for a purpose* (Usiskin et al., 2008), and there can be both didactical and mathematical reasons for making this choice. For example, Harel et al. (2008) suggest in their research that if students are not aware that definitions are carefully chosen for a purpose, it is unlikely that they will go beyond just memorizing them.

Two equivalent definitions may highlight different aspects of a concept, and thus they may lead to different mathematical activities (Morgan, 2005). For example, consider the following two equivalent definitions of an even number: (1) An even number is a number that can be divided into two equal groups. (2) An even number is a number for which the last digit is 0, 2, 4, 6, or 8. The

two definitions are equivalent, but they highlight different aspects of even numbers. The first definition highlights that an even number is divisible by two and that if we have an even number of marbles, we can share them fairly among two children. The second definition does not highlight these properties but provides a procedure to determine if a symbolic number is even. If a student is asked to prove that an even number multiplied by an even number is even, then the two definitions above would most likely produce different arguments (and mathematical activities).

#### 4.1.3. Constructed

Whether a mathematical concept is discovered (i.e., naming pre-existing objects) or constructed is a matter of one's philosophical persuasion. The main position of the reviewed papers, however, is that mathematical definitions are constructed. De Villiers (1998), for instance, maintained that the act of defining is a central mathematical activity, in which concepts are *constructed* or defined "into being." Morgan (2005, p. 111) states that "[definitions] are the *product of human activity*."

Likewise, Burn (as cited in Morgan, 2005) stated that

*The research mathematician may come to his results starting from special cases, which will appear as corollaries in the final version, from which he gets his ideas, which is worked with until he has a proof. Then the theorem is what has been proved. At this point he formulates his definitions so as to make the theorem and proof as neat as possible.*

Others have made similar claims (e.g., Chesler, 2012; Viirman, 2013; Fisher, 2016, Kobiela et al., 2022). In sum, the reviewed papers portrayed an image of mathematicians who construct definitions (in a shared community), and that definitions are under constant development, driven

by a need. Students are, as we have pointed out earlier, expected to be involved in the proving processes, but we observe the importance of students also being involved in the defining processes and constructing their own definitions (Borasi, 1992; De Villiers, 1998; Zandieh & Rasmussen, 2010).

## 4.2. Requirements

Several of the publications in the sample presented criteria for the definition formulation and imperative features of definitions (e.g., Van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). Naturally, this became an emerging theme. Statements were further refined into three categories, which we describe below.

### 4.2.1. Formal

This category encompasses the local criteria for a definition, namely, criteria to which a definition formulation must adhere.

According to Chesler (2012), Sánchez and García (2013), and Johnson et al. (2014), a definition must be *consistent* and *non-contradicting*. It must be consistent with the mathematical theory formed thus far. Furthermore, it must be *well-defined* (Tay & Leong, 2012); that is to say, the meaning is *unambiguous* (Burroughs & Burke, 2016; Foster & de Villiers, 2015; Morgan, 2005, 2006).

For example, if we view  $\mathbb{Q}$  as  $(\mathbb{Z} \times \mathbb{Z})/\sim$ , where  $(a, b) \sim (c, d)$  if  $ad = bc$ , and (make an attempt to) define  $f: (\mathbb{Z} \times \mathbb{Z})/\sim \rightarrow \mathbb{Z}$ , where  $f(a, b) = a + b$ , then  $f$  is not well defined, since  $f(1,2) \neq f(2,4)$ . Of course, this is transferable to fraction notation. A mathematician must often prove that



a definition is well defined, especially when dealing with equivalence classes. In general, a definition must be invariant under which representatives are chosen in the equivalence classes.

We have not found any examples of well-definedness suitable at the elementary school level in the literature. However, we can perhaps imagine a young student defining even numbers to be every other (whole) number on the number line. While the statement is true, it is not a well-defined definition, because it could describe either the odd or the even numbers depending on which number the number sequence starts from.

#### **4.2.2. Exemplification**

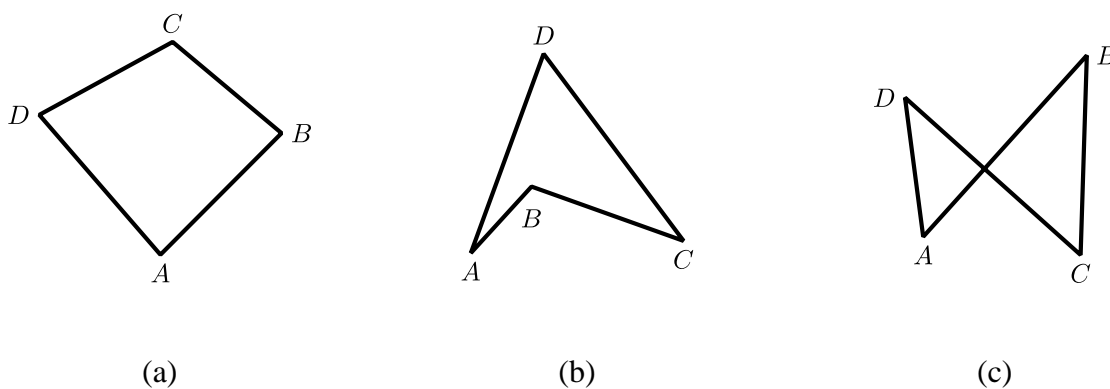
A multitude of sources contained statements pointing to the importance of the *existence* of an example/instance of a definition. That is, there must exist at least one example of the defined concept (e.g., Avcu, 2019; Edwards & Ward, 2008; Van Dormolen & Zaslavsky, 2003). Every definition must *allow discrimination between instances and non-instances* (e.g., Avcu, 2019; Edwards & Ward, 2004; Johnson et al., 2014; Morgan, 2005, 2006; Sánchez & García, 2013).

A definition is also given within a representation system and must be *invariant under change of representation* (see Sánchez & García, 2013; Zaslavsky & Shir, 2005). For example, consider the following definitions. (1) A number of unifix cubes is an even number if the unifix cubes can be divided into two equal groups. (2) An even number  $a$  is a number that can be written as a sum of two equal natural numbers (i.e., can be written as  $a = b + b$ , where  $b \in \mathbb{N}$ ). The two definitions are structurally the same; however, two different representation systems are being used.

Moreover, the example space of a mathematical definition may yield *degenerate* instances (Van Dormolen & Zaslavsky, 2003), specifically, unwanted instances of the mathematical definition that are not in accordance with our intuition of the mathematical concept. For example, Van Dormolen and Zaslavsky (2003) give as an example the following definition of quadrilaterals:

**Definition.** *A quadrilateral is a set of four points  $A, B, C, D$  of which no three are collinear, and four segments  $AB, BC, CD$  and  $DA$ .*

This definition will result in a quadrilateral of one of three shapes (see Figure 4). Van Dormolen and Zaslavsky (2003) point out that although all of the shapes fit the definition, some would consider Figure 4 (c) to be a degenerate quadrilateral. Furthermore, there are reasons to exclude Figure 4 (c) as a valid quadrilateral. For example, the theorem of interior angle sum of a quadrilateral would no longer be valid (unless we consider the intersection of the two edges, but then we would have six interior angles). Even more so, the term *interior angle* would be difficult to interpret in a meaningful manner.



**Figure 4** Three quadrilaterals that satisfy the definition

**4.2.3. Axiomatization**

The global ecosystem in which the definition formulation lives was emphasized in several sources (see Miller, 2018; Van Dormolen & Zaslavsky, 2003; Winicki-Landman & Leikin, 2000). This category may be thought of as the global criterion for a definition formulation. A definition is *part of a deductive system*, in a *hierarchical* and *noncircular* manner within itself and across existing axioms, definitions, and theorems (e.g., Aktaş, 2015; Avcu, 2019; Morgan, 2005; Sánchez & García, 2013; Wong et al., 2014). Thus, the formulations must be *consistent* within this frame of reference. A definition of a concept may employ *only previously defined concepts*; that is, the concept at hand is *a special case of a more general concept* (Van Dormolen & Zaslavsky, 2003). For example, an even number is a special case of a number, continuous function is defined by the more general concept of a function, a rectangle is a particular quadrilateral, and so on. The criterion of axiomatization ensures that new concepts are defined by means of already defined concepts, and the graph of concepts (there is a directed edge between two concepts if one is a special case of the other) gives an acyclic directed graph. Didactically, this would mean that a definition should only contain concepts that are known in the given community (Winicki-Landman & Leikin, 2000). However, according to Morgan (2005) and Aktaş (2015), an exception should be made for the case of implicit definitions.

Moreover, *equivalence* frequently appears in the reviewed literature. If there are multiple formulations of a given concept, they have to be mathematically equivalent (e.g., Avcu, 2019; Megalou, 2018; Van Dormolen & Zaslavsky, 2003; Zazkis & Leikin, 2008). For example, a square

may be defined as (1) a rhombus with right angles or (2) a quadrilateral with four mirror and four rotational symmetries. That these formulations yield the same mathematical object must be mathematically proven. Of course, this is closely related to the arbitrary nature of mathematical definitions.

### **4.3. Preferred features**

Statements that were considered to be preferred features of mathematical definitions, but not absolute mathematical requirements, were collected under this theme. This involves both mathematical and didactical features. Three categories of preferred features were found: minimality, aesthetic, and comprehension.

#### **4.3.1. Minimality**

A definition should be *economical* and *minimal* (e.g., Vinner, 1991). Minimality means that no condition in a definition can be inferred from the other conditions; that is, there is no redundancy. If we remove one condition from a minimal definition, the example space becomes properly larger. Economical can, for example, be interpreted as a definition with as few words as possible.

In the literature we reviewed, the notion of minimality appears frequently, and there has been some discussion as to whether it should be a requirement or a preferred feature (see e.g., Zaslavsky & Shir, 2005). Leikin and Winicki-Landman (2000, p. 27), for example, claim that

*...[a] mathematical definition should be minimal, we accept this requirement as essential not only from an aesthetic point of view but also from a mathematical perspective.*

On the other hand, Van Dormolen and Zaslavsky (2003, pp. 96-97), state that

*[O]ften at the time when a concept is defined there may not be sufficient knowledge to determine whether it is minimal. [...] In order to get a sound and rich mental image of the new concept, it can be advantageous to start with a description that does not satisfy the criterion of minimality.*

Borasi (1992) used the term *essentiality*, which can be expressed as being a middle-ground approach to the notion of minimality. By that she means that we must only mention necessary terms and properties so that it is possible to distinguish an instance from a non-instance. The majority of the papers, however, either view minimality as a preferred feature (e.g., Leikin & Zazkis, 2010) or recognize that it is debatable whether it is required or not (Zazkis & Leikin, 2008; Ulusoy, 2020; Avcu, 2022). In doing so, they reason that, while a minimal definition is preferred, it may not be the most accessible from a didactical perspective (Avcu, 2022; Van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005).

To mitigate the obstruction of learning due to minimal definitions, Zazkis and Leikin (2008) introduced the term *barely-not-minimal*, as definitions that *use appropriate terminology and contain necessary and sufficient conditions without being minimal* (Johnson et al., 2014, p. 288) (c.f. *essentiality*). For example, a rectangle can be defined as a parallelogram with four right angles. This definition is not minimal since it contains a theorem. It is sufficient to require a parallelogram to have a single right angle, from which it follows that all the angles are right angles. However, in certain circumstances, it is minimal enough.

### 4.3.2. Aesthetic

Arguably the most subjective of our categories, whether a definition is considered aesthetically pleasing is highly individual and context dependent. However, it is often mentioned in the literature that a definition should be *elegant* (e.g., Winicki-Landman & Leikin, 2000; Zandieh & Rasmussen, 2010; Zazkis & Leikin, 2008), *precise* (Levenson, 2012), or *clear* (Leikin & Winicki-Landman, 2000; Levenson, 2012; Miller, 2018). Minimal definitions are also considered to be more elegant.

One example, given by Vinner (1991), is the following two definitions of the absolute value of a real number  $x$ .

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad (1)$$

$$|x| = \sqrt{x^2} \quad (2)$$

While both definitions are correct and equivalent (and even minimal), one can argue that (2) is a more elegant definition than (1) since (2) uses fewer words and symbols, and in some cases (2) might be a more convenient definition for proving certain results. However, it can also be argued that definition (1) in a better way illuminates the concept of the absolute value of a real number.

This shows that what is considered more elegant in a pure mathematical context is not necessarily more elegant, or even preferred, in a didactical context. Aesthetics are therefore highly dependent on the situation.

Clarity and precision are also considered to be aspects of comprehension, as discussed in the next section.

### 4.3.3. Comprehension

This category includes statements about features of definitions that facilitate teaching and learning. A definition used in schools should be *didactically suitable* (e.g., Poincaré, 1969; Winicki-Landman & Leikin, 2000; Zazkis & Leikin, 2008). The literature highlights *intuitive* (Ouvrier-Bufferet, 2010; Zazkis & Leikin, 2008), *understandable*, *precise*, and *clear* definitions. According to Leikin and Winicki-Landman (2000), a definition must *match students' knowledge and needs*. Definitions are also considered as *starting points to understand the meaning of a concept* (Ertekin et al., 2014; Yavuzsoy-Köse et al., 2019).

Furthermore, a definition is said to be *formally operable* for an individual, if *that individual is able to use it in creating or (meaningfully) reproducing a formal argument* (Bills & Tall, 1998). Burroughs and Burke (2016) use the term *referential clarity* when a definition *has easily identifiable examples and is closely related to its natural language usage*.

Comprehension is also tied to the arbitrary nature of mathematical definitions. As pointed out in Section 4.1.2, the choice of mathematical definitions might have some impact on mathematical activity (Morgan, 2005). Chesler (2021) showed that the choice of definition of a function had some influence on informants' ability to use the definition.

## 4.4. Role and function

Another emerging theme in the literature was the role and function of mathematical definitions. Mathematical definitions play a significant role in the development and learning of mathematics.

Definitions function as vessels of mathematical concepts in the mathematical discourse, and, alongside logic, they are the foundations of mathematical deductive proofs.

#### **4.4.1. Naming**

Mathematical definitions operate as necessary tools for *naming* a collection of objects or a phenomenon and introducing new notations (e.g., Leikin & Winicki-Landman, 2001; Miller, 2018; Winicki-Landman & Leikin, 2000). That is, they are used to “replace something with something else.” Zazkis and Leikin (2008, p. 133), for example, state that defining is giving a name. Although this is mentioned in the context of logical principles a mathematical definition must follow, we interpret this to be an important purpose rather than a requirement of a mathematical definition. Moreover, according to Robinson (as cited in Edwards & Ward, 2008), stipulated definitions are an *explicit and self-conscious setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object)*.

#### **4.4.2. Communicate**

A necessary role that mathematical definitions serve is to allow practitioners to communicate mathematical ideas efficiently (Gilboa et al., 2018, p. 440). Precise definitions are paramount for developing deductive arguments and conveying them to one’s peers (Alcock & Simpson, 2016, p. 5). Poincare (as cited in Johnson et al., 2014) highlights that mathematical definitions serve the role of vehicles for the communication of concepts. Mathematical definitions allow communication through creating uniformity in the meaning of concepts (Borasi, 1992). Students must use stipulated definitions to take part in a mathematical discourse, and these definitions *allow students to communicate ideas, defend their reasoning, construct arguments, and draw logical conclusions*.



Furthermore, *using stipulated definitions correctly is one way for students to demonstrate mathematical literacy* (Bayda & Sutliff, 2020, p. 2).

#### **4.4.3. Basis for deduction**

Mathematical definitions are tools that mathematicians use to reach agreements about the nature and properties of mathematical objects (Tabach & Nachlieli, 2015). Moreover, mathematical definitions are the *basis for logical derivation*, used for deduction and structuring proofs, and they function as *the foundation stone of mathematical thinking* (Yavuzsoy-Köse et al., 2019) and lay *the foundations of proving and problem-solving* (Şen, & Güler, 2022). Definitions are constructs that extend our understanding of mathematical domains, and definitions that are fruitful in quickly enabling meaningful deductions in new domains are held in high regard (Burroughs & Burke, 2016).

Furthermore, mathematical definitions make it possible for mathematicians to *classify mathematical objects* (e.g., Zaslavsky & Shir, 2005; Alcock & Simpson, 2011), as well as to *systematize existing knowledge and produce new knowledge* (Yavuzsoy-Köse et al., 2019). To emphasize the importance of the use of mathematical definitions, Bills and Tall (1998) introduced the notion of *formally operable* definitions as a qualitative measure of whether an individual is able to employ the definition in a meaningful manner, that is to say, using it in creating or reproducing a formal argument. Among others (see e.g., Dawkins, 2014; Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008), Morgan (2006) points to a definition's *usefulness* and *applicability* as aspects to consider. A definition is stipulated with a purpose and *chosen* for a deliberate reason, for example

to make theorems, to structure proofs, and to communicate. This also taps into the nature of definitions—they are constructs of human activity and driven by a (specific) need.

One example of the usefulness of a definition and its suitability for enabling meaningful deductions in new domains is the notion of *generality* or how *generalizable* a definition is (Burroughs & Burke, 2016). A definition is generalizable if it easily extends to other closely related domains, and Usiskin and Griffin (as cited in Burroughs & Burke, 2016) give an example of the definition of interior angle in Euclidean geometry; the definition based on the intersection of two half-planes easily extends to hyperbolic geometry. Martín-Molina et al. (2018) characterized how mathematicians construct new definitions by generalizing other existing definitions, and Kobiela et al. (2022), point out how definitions play an important role in constructing new definitions.

Definitions may therefore support the development of new definitions, both equivalent and non-equivalent. For example, the definition “a number of unifix cubes is an even number if we can pair two and two cubes such that no cube is left over” does not extend to negative numbers, but the definition “an even number is a multiple of two” easily does (the example space is extended). In abstract algebra a vector space is often defined with a (rather long) list of properties. However, we can define a vector space  $V$  to be an  $R$ -module, where  $R$  is a ring, by simply requiring that  $R$  is a field (the example space becomes properly smaller, i.e., all vector spaces are modules, but not all modules are vector spaces).

We note here that Leikin & Zaskis (2007) use generality somewhat differently; that is *the ability to provide examples that are specific and precise rather than general descriptions*.

## 5. Types of definitions as a fifth main theme

A multitude of types of definitions was described in the literature. This theme was not included in the above discussion and **Figure 3** since we view this theme as independent of the other characteristics. We therefore describe this theme separately.

In some cases, there were several different names for the same type of definition. From the data analysis, we identified two independent categories (dimensions) of types of mathematical definitions. A priori, these categories are regarded as dichotomous categories. However, we do not rule out the possibility of mathematical definitions that do not exclusively belong to one category or the other.

Furthermore, it is possible to regard these dimensions as facets of the global and local aspect of mathematical definitions: procedural vs. structural (local) and inclusive vs. exclusive (global).

### 5.1. Inclusive vs. exclusive

A definition can be inclusive or exclusive (e.g., De Villiers, 1994; Usiskin et al., 2008; Winicki-Landman & Leikin, 2000, Butuner & Filiz, 2017). A definition is *exclusive (partitional)* when it describes only the concept at hand and excludes special cases as instances; that is, special cases, in the conventional hierarchical structure are regarded as unwanted degenerations. A definition is *inclusive (hierarchical)* when sets of more particular concepts are subsets of the more general concept. De Villiers (1994) gives as an example the classification of quadrilaterals: in the hierarchical classification, a square is a special case of a rectangle, while in the partitional classification the rectangle and square are considered as disjoint classes of quadrilaterals.

The term exclusive (partitional) is, to some degree, a result of a theme found in students' perception and desire to specify concepts as narrowly as possible (e.g., De Villiers, 1994; Fujita, 2012; Leikin & Winicki-Landman, 2000). The research suggests that some students want, for example, the definition of a parallelogram to exclude the rhombus as a special type of parallelogram (De Villiers, 1994). There is also the particular case of the trapezoid, for which there are two schools of thought: one in which a rectangle is a special case of trapezoid (a trapezoid has at least one pair of parallel sides) and one where it is not (a trapezoid has exactly one pair of parallel sides).

This dimension can be thought of as a global aspect of mathematical definitions. That is to say, if one regards mathematics as a stratification of mathematical concepts, the conventional way to regard mathematics is as a highly connected hierarchical system of concepts, with chains of concepts where concepts are linked by level of generality (e.g., the hierarchical structure of quadrilaterals).

To clarify this, the convention in mathematics is to strive for hierarchical mathematical definitions. De Villiers (1994) argues that this perspective may yield a much more fruitful understanding of mathematics in general, and specifically of mathematical concepts.

## **5.2. Procedural vs. structural and static vs. dynamic**

The literature under review described two forms of definition formulations. On the one hand, a mathematical concept can be given as the outcome of a prescribed process (*procedural*) (e.g., Borasi, 1992; Gilboa et al., 2018; Zaslavsky & Shir, 2005). Specifically, it describes a procedure in which the outcome will result in an instance of the mathematical concept. For instance, the algorithm for the construction of the perpendicular bisector is a procedural definition for the

perpendicular bisector. On the other hand, it is possible to define the perpendicular bisector of a segment as a locus, that is the set of points that are equidistant from the endpoints of the segment. Thus, the concept is prescribed by defining the intrinsic structure (*structural*) of the concept (e.g., Borasi, 1992; Gilboa et al., 2018; Zaslavsky & Shir, 2005).

Raychaudhuri (2008) suggests two categories, namely *static* and *dynamic* definitions. A static definition is a definition in which the object being defined is generated by the property stated in the definition (Raychaudhuri, 2008, p. 163), Raychaudhuri gives the example of the definition of the additive inverse of a number  $a$ , where  $-a$  is generated by a calculation given in the definition. A dynamic definition is a definition of an object that cannot be generated by the definition itself, that is, the process of generating the object is not explicitly stated in the definition.

## 6. Criteria for mathematical definitions

A question that inspired our investigation was what a definition of a mathematical definition could be. We do not aim to answer this question in its entirety. However, in the literature under review, we found a total of 12 different criteria or requirements for a mathematical definition. The term *criterion* was used by some of the authors of the sample, while others used phrasings such as “a definition must” have an imperative feature or required property. Here, we use criterion as a collective term that encompasses all of the aforementioned terms and phrasing. The criteria are collected in **Table 1**, and an overview of which publications in the sample the criteria can be found is provided in Appendix A.

**Table 1** - A compiled list of criteria for a mathematical definition found in the sample

Name of Criterion	Description
Hierarchy	A definition must describe any new concept as a special case of a general concept (Van Dormolen & Zaslavsky, 2003).
Equivalence	If more than one definition is given for the same concept, the equivalence of them must be proved (Van Dormolen & Zaslavsky, 2003).
Existence	It is necessary to prove that at least one instance of a newly defined concept exists (Van Dormolen & Zaslavsky, 2003).
Non-contradiction	All the properties stated in the definition should be able to coexist (Morgan, 2005).
Minimality	A definition should not contain properties that can be mathematically inferred from other parts of the definition (Vinner, 1991).
Elegance	When given a choice, definition that uses fewer words or symbols, or that looks “nicer,” should be chosen (Van Dormolen & Zaslavsky, 2003).
Axiomatization	A definition must fit into and be part of a deductive system (Van Dormolen & Zaslavsky, 2003).

Non-ambiguity	A definition must have a unique interpretation (Zaslavsky & Shir, 2005).
Degenerations	A definition may include degenerate (unexpected) instances that are non-intuitive (Van Dormolen & Zaslavsky, 2003).
Referential clarity	A definition must have easily identifiable examples, and the name of the concept must be closely related to its natural language usage (Burroughs & Burke, 2016).
Discrimination	A definition must allow instances and non-instances of the concept to be discriminated with certainty, consistency, and efficiency (Morgan, 2006).
Mathematical essence	A definition must capture and synthesize the mathematical essence of the concept (Morgan, 2006).

In this section, we have not made a distinction between which criteria were regarded as an imperative or as an optional criterion (this was done in Section 4). For instance, the criterion of minimality was in some publications considered as an imperative feature, whereas others regarded it as an optional or preferred feature (see Section 4.3.1). Appendix A, Table 2, shows that minimality was the most mentioned criterion.

The criteria in **Table 1** are also included in our analysis of emerging themes (see Section 4); however, we must point out that in our analysis, not all of the criteria were regarded as strict mathematical requirements or rules. For instance, referential clarity is difficult to interpret as a

strict criterion for a mathematical definition. In our analysis, we regard this criterion of didactical flavor as part of the function that a mathematical definition serves.

## 7. Discussion

The aim of this systematic literature review was to gather an overview of how researchers in mathematics education characterize mathematical definitions. The categorization is outlined in **Figure 3**, and it is presented in detail in Section 4. The framework offers a holistic view of mathematical definitions in the mathematics education literature and promotes a view of a mathematical definition as something more than a list of properties necessary and sufficient to characterize a mathematical concept. The framework has the potential to direct mathematics teachers, educators, and researchers' attention to the wide range of aspects of mathematical definitions.

Our investigation revealed a list of 12 criteria for mathematical definitions described in the sample (see **Table 1**). However, the criteria were of varying degrees formal or logical, and one may thus ask whether all of them are necessary or if they are independent. Moreover, our analysis did not reveal any framework or paper that collects all aspects of mathematical definitions in the mathematics education literature. We see from the overview in Appendix A, Table 2, that no publication from the sample discusses all the aspects that we found. To the best of our knowledge, this paper provides the most comprehensive framework of mathematical definitions to date.

We observe that there is a high level of agreement on the different aspects of mathematical definitions that were discussed across the literature. The one point where there was slight



controversy was whether minimality is a required property or a preferred feature of a mathematical definition (Zaslavsky & Shir, 2005). Interestingly, from Appendix A, Table 2, we see that the criterion of minimality was brought up the most in the reviewed literature. We decided to categorize minimality as a preferred feature primarily because it is not a requirement for the logical consistency of a definition—it only decreases the chance of an erroneous definition. Secondly, there are strong didactical arguments for excluding minimality as a requirement in the context of teaching and learning (e.g., Johnson et al., 2014; Van Dormolen & Zaslavsky, 2003).

From our investigation, we detected a tension between purely mathematical aspects and didactical aspects of definitions. In some cases, the necessary and preferred criteria for mathematical definitions can even be obstacles for a learner. This tension is observable for several of the aspects that are discussed in the body of the literature we have reviewed. We will give a few examples.

First, regarding the nature of mathematical definitions, Edwards and Ward (2004) present empirical evidence that some students do not treat or understand mathematical definitions as mathematicians do. They show that some students do not regard mathematical definitions as stipulated, but rather as extracted. Another point to be made about the nature of definitions is about their arbitrary and constructed nature. Some teachers and students actually hold the position that there is only one correct definition for a concept. For example, Leikin and Winicki-Landman (2001, p. 72) state that “Our reflective discussions with the teachers showed that they were not always aware that there exists a possibility to define a concept in different ways.” Furthermore, even though mathematicians might prefer a minimal definition, this is not always preferred in a didactical context, as we previously mentioned. Another point is that of aesthetics and elegance. As we have

pointed out, and in particular in the example of the absolute value of  $x$  by Vinner (1991) in Section 4.3.2, what is considered elegant in one context might not be elegant in another. To underline the point even further, although Burroughs and Burke (2016) say that the name of the term in a definition should be closely related to its natural language usage, this is clearly not the case in mathematics research. We can argue that the mathematical terms *natural*, *rational*, *complex*, *group*, *ring*, *field*, *function*, *factor*, *positive* and *negative*, *manifold*, *category*, and even *Fibonacci number*, are not. They all have their own mathematical meanings, totally independent of any natural language usage. Furthermore, the term's relation to natural language may change as the subject matter is developed. This is another example of a tension between the preferred features (and didactical aspects) of mathematical definitions and the nature of mathematical definitions in mathematical science.

It is clear that the most suitable definition at a certain level of education might not be the most preferred in a purely mathematical sense. Conversely, a mathematically sound and preferred definition, might not be the most suitable for students. These points might seem obvious, but consider the following example. It is sometimes suggested that multiplication can be introduced using an equal groups model or repeated addition (e.g., Van de Walle et al., 2019). Clearly, this restricted definition of multiplication breaks down when we want to extend the definition to rational or real numbers. We propose that it would be worthwhile to conduct more research into the observable tension among our categories of mathematical definitions.

We see mention in the body of literature under review of representation systems used in definitions. In particular, it is mentioned that a definition must be invariant under change of representation (see

Sánchez & García, 2013; Zaslavsky & Shir, 2005). While definitions are arbitrary in nature, we note that the representation system used in a definition is of an arbitrary nature as well. Several representation systems can be used to define any given mathematical concept. Vice versa, a given representation system can be used to define different mathematical concepts. This arbitrariness raises some relevant questions. For example, how can we best make a choice between possible representation systems? Which representation system is didactically more suitable in a given situation? Which representation systems do teachers use and why? More research regarding the representations used in mathematical definitions seems warranted.

## **8. Concluding remarks**

For students to be able to produce mathematical proofs, they have to know, at least to some degree, the nature and purpose of mathematical definitions. Dickerson and Pitman (2016) stress that students' inability to produce proofs often has something to do with definitions. How students come to acquire the competencies necessary for producing deductive justification and how, and to what degree, these competencies can be taught are among the key open questions raised in Harel and Weber (2020)'s review of deductive reasoning. That students are not aware that definitions serve as the basis for deductive arguments may be one such missing competence, and this points toward the structure of deductive proof and the role of definitions in this structure as areas with particular potential for the design of new didactical interventions.

Several authors stress how important it is that teachers and educators have a robust understanding of mathematical definitions. For example, Avcu (2022) mentions the importance of knowledge

about the process of defining, the nature, roles, and features of mathematical definitions, how definitions influence their pedagogical choices, and the various types of definitions.

It is interesting to note that some students who can recite definitions, and even seemingly understand and explain them, may still be unable to use them in a mathematically appropriate way in their reasoning and proving (Edwards & Ward, 2004). Based on this insight, Edwards and Ward (2004) recommend that

*The special nature of mathematical definitions should be treated as a concept in its own right, one that should be understood at some level by all college mathematics students.*

and

*[...] students also need to understand the specific role definitions play in mathematics. Thus in such courses the nature of mathematical definitions needs to be addressed more directly and more often. In particular, introduction to proof courses certainly should include a unit on the categorization and use of mathematical definitions.*

We hope that our work, and the resulting framework, can help guide teachers, educators, and researchers to emphasize important aspects of mathematical definitions in both teaching and research in mathematics education.

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## References

- Arksey, H., & O'Malley, L. (2005). Scoping studies: Towards a methodological framework. *International Journal of Social Research Methodology*, 8(1), 19-32.  
<https://doi.org/10.1080/1364557032000119616>
- Avcu, R. (2022). Pre-service middle school mathematics teachers' personal concept definitions of special quadrilaterals. *Mathematics Education Research Journal* (2022).  
<https://doi.org/10.1007/s13394-022-00412-2>
- Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 216-235). London: Hodder & Stoughton.
- Burroughs, E.A., Burke, M.J. (2016). By Definition: An Examination of the Process of Defining in Mathematics. In: Dewar, J., Hsu, Ps., Pollatsek, H. (eds) *Mathematics Education. Association for Women in Mathematics Series*, vol 7. Springer, Cham.  
[https://doi.org/10.1007/978-3-319-44950-0\\_5](https://doi.org/10.1007/978-3-319-44950-0_5)
- Dickerson, D. S., & Pitman, D. J. (2016). An examination of college mathematics majors' understandings of their own written definitions. *Journal of Mathematical Behavior*, 41, 1-9. <https://doi.org/10.1016/j.jmathb.2015.11.001>
- Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*, 111(5), 411-424. <https://doi.org/10.1080/00029890.2004.11920092>

Edwards, B. S., & Ward, M. B. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 223-232). Mathematical Association of America.  
<https://doi.org/10.5948/UPO9780883859759.018>

Forbregd, T., Kaspersen, E., Torkildsen, H. A., & Solstad, T. (2022). Compromises between required and preferred features of mathematical definitions in mathematics education. *Bringing Nordic mathematics education into the future*, 20, 41-48.

McGuinness, Brian (ed.) (1984). *Gottlob Frege: Collected Papers on Mathematics, Logic, and Philosophy*. Oxford: Blackwell.

Harel, G., & Weber, K. (2020). Deductive reasoning in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 183-190). Springer International Publishing.  
[https://doi.org/10.1007/978-3-030-15789-0\\_43](https://doi.org/10.1007/978-3-030-15789-0_43)

Johnson, H. L., Blume, G. W., Shimizu, J. K., Graysay, D., & Konnova, S. (2014). A teacher's conception of definition and use of examples when doing and teaching mathematics. *Mathematical Thinking and Learning*, 16(4), 285-311.  
<https://doi.org/10.1080/10986065.2014.953018>

Leikin, R., & Winicki-Landman, G. (2001). Defining as a vehicle for professional development of secondary school mathematics teachers. *Mathematics Education Research Journal*, 3, 62-73.

- Lipsey, M. W., & Wilson, D. B. (2001). *Practical meta-analysis* (Vol. 49). Sage Publications, Inc.
- Mayring, P. (2000). Qualitative content analysis. *Forum Qualitative Sozialforschung / Forum: Qualitative Social Research*, 1(2). <https://doi.org/10.17169/fqs-1.2.1089>
- Parameswaran, R. (2010). Expert mathematics' approach to understanding definitions. *Mathematics Educator*, 20(1), 43-51.
- Sánchez, V., & García, M. (2013). Sociomathematical and mathematical norms related to definition in pre-service primary teachers' discourse. *Educational Studies in Mathematics*, 85(2), 305-320. <https://doi.org/10.1007/s10649-013-9516-0>
- Schifter, D. (2009). Representation-based proof in the elementary grades. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and Learning Proofs across the grades* (pp. 71-86). New York: Routledge.
- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38(3), 289-321. <http://www.jstor.org/stable/30034869>
- Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, 40(3), 314-352. <https://www.jstor.org/stable/40539339>
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.

Van Dormolen, J., & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity.

*The Journal of Mathematical Behavior*, 22(1), 91-106.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2019). *Elementary and middle school*

*mathematics: Teaching developmentally*. Pearson.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D.

Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Springer Netherlands.

[https://doi.org/10.1007/0-306-47203-1\\_5](https://doi.org/10.1007/0-306-47203-1_5)

Weiner, J. (1990). Platonism, Fregean and UnFregean. In *Frege in Perspective* (pp. 176-224).

Cornell University Press. <http://www.jstor.org/stable/10.7591/j.ctvb6v64x.12>

Whitehead, A. N., & Russell, B. (1912). *Principia mathematica* (Vol. 2). University Press.

Wittgenstein, L., & Bosanquet, R. G. (1989). *Wittgenstein's lectures on the foundations of*

*mathematics, Cambridge, 1939*. University of Chicago Press.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for*

*Research in Mathematics Education*, 36(4), 317-346.

<http://www.jstor.org/stable/30035043>





Johnson et. al. (2014)				◇	◇					
Lavy & Mashiach-Eizenberg (2009)						◇	◇			
Leikin & Winicki-Landman (2000)						◇				
Leikin & Zazkis (2010)										
Morgan (2005)				◇			◇		◇	◇
Morgan (2006)							◇		◇	◇
Sánchez & García (2013)	◇	◇	◇	◇	◇		◇		◇	
Üreyen (2006)							◇			
Van Dormolen & Zaslavsky (2003)	◇	◇	◇		◇	◇	◇		◇	
Vinner (1991)							◇			
Winicki-Landman & Leikin (2000)							◇			
Zandieh & Rasmussen (2010)							◇	◇		◇
Zaslavsky & Shir (2005)			◇						◇	
Zazkis & Leikin (2008)							◇	◇		

**Abbreviations:** Hierarchy (HI), Equivalence (EQ), Existence (EX), Non-contradiction (NC), Minimality (MIN), Elegance (ELG), Axiomatization (AX), Non-ambiguity (NAM), Degenerations (DEG), Referential clarity (RCL), Discrimination (DC) and Mathematical essence (ME).

## Appendix B. Search strategy details

The initial searches in Scopus and ERIC were as follows:

<p><b>Scopus:</b></p> <p>(ABS(definition* AND mathematic*) OR TITLE(definition* AND mathematic*) OR KEY(definition* AND mathematic*)) AND ALL(education) AND LIMIT-TO(LANGUAGE,English)</p>	<p><i>N</i> = 2215</p>
<p><b>ERIC:</b></p> <p>(abstract:(definition* AND mathematic*) OR title:(definition* AND mathematic*) OR descriptor:(definition* AND mathematic*)) AND education</p>	<p><i>N</i> = 1361</p>

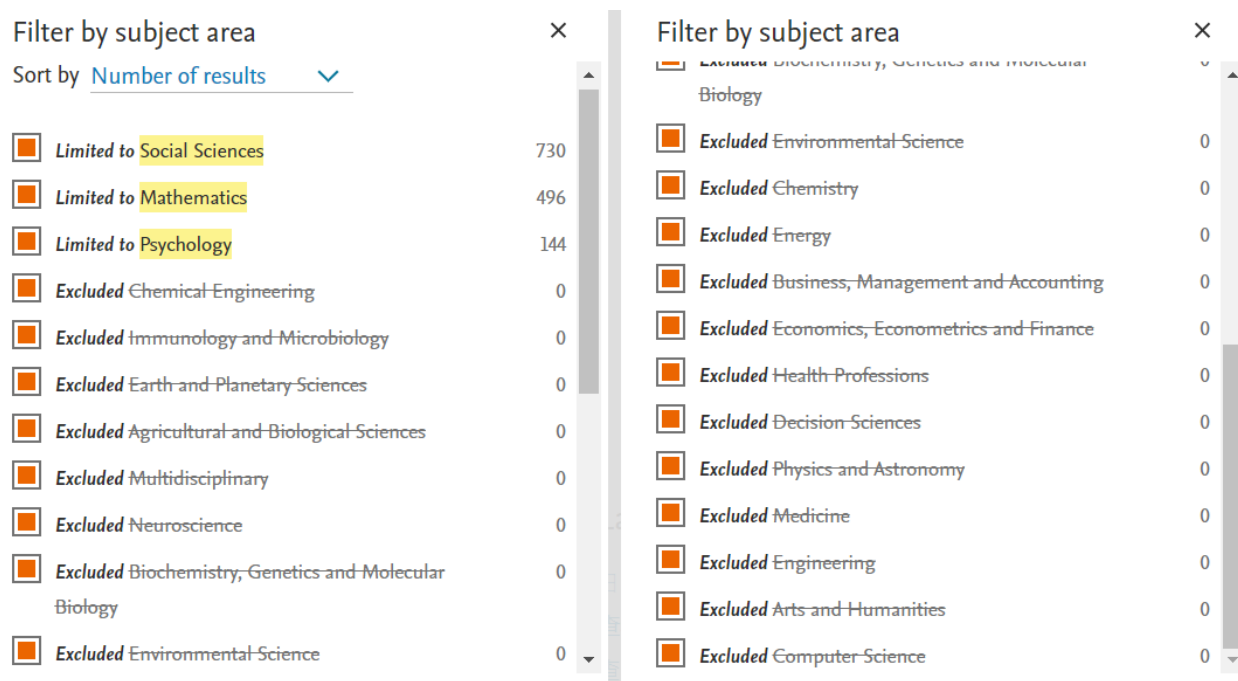
The two search engines have very different inclusion/exclusion functionalities and levels of sophistication.

### ERIC exclusion strategy:

We limited findings to peer-reviewed documents, resulting in  $N = 619$ .

### Scopus exclusion strategy:

Note that documents not written in English were excluded in the initial search. In subject area, we limited the search to social sciences, mathematics, and psychology. Further, we excluded the following, resulting in  $N = 874$ , as shown in the screenshot:



The final and last search was conducted on 30 September 2022 on both engines.

## Appendix C. Publications included in the review.

1. Aktaş, M. C. (2015). Turkish high school students' definitions for parallelograms: Appropriate or inappropriate? *International Journal of Mathematical Education in Science and Technology*, 47(4), 583-596. <https://doi.org/10.1080/0020739x.2015.1124931>
2. Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, 22(2), 28-34.
3. Alcock, L., Simpson, A. (2011). Classification and concept consistency. *Canadian Journal of Science, Mathematics and Technology Education*, 11(2), 91-106.
4. Alcock, L., & Simpson, A. (2016). Interactions between defining, explaining and classifying: The case of increasing and decreasing sequences. *Educational Studies in Mathematics*, 94(1), 5-19. <https://doi.org/10.1007/s10649-016-9709-4>
5. Altıparmak, K., & Gürcan, G. (2021). Examination of 4th grade students' definitions for square, rectangle and triangle geometric shapes. *Education Quarterly Reviews*, 4(3), 304-321.
6. Altun, S. D. G., & Konyalioglu, A. C. (2019). The influence of mistake-handling activities on mathematics education: An example of definitions. *European Journal of Educational Research*, 8(2), 467-476.
7. Arnal-Palacián, M., Claros-Mellado, J., & Sánchez-Compañía, M. T. (2020). Infinite limit of sequences and its phenomenology. *International Electronic Journal of Mathematics Education*, 15(3), em0593. <https://doi.org/10.29333/iejme/8279>
8. Asghari, A. H. (2008). Experiencing equivalence but organizing order. *Educational Studies in Mathematics*, 71(3), 219-234. <https://doi.org/10.1007/s10649-008-9173-x>
9. Avcu, R. (2019). A comparison of mathematical features of Turkish and American textbook definitions regarding special quadrilaterals. *International Journal of Mathematical Education in Science and Technology*, 50(4), 577-602. <https://doi.org/10.1080/0020739x.2018.1529338>
10. Avcu, R. (2022). Pre-service middle school mathematics teachers' personal concept definitions of special quadrilaterals. *Mathematics Education Research Journal* (2022). [https://doi.org/10.1007/s13394-022-00412-](https://doi.org/10.1007/s13394-022-00412-2)

11. Ayalon, M., & Wilkie, K. J. (2019). Exploring secondary students' conceptualization of functions in three curriculum contexts. *The Journal of Mathematical Behavior*, 56, 100718. <https://doi.org/10.1016/j.jmathb.2019.100718>
12. Bayda, N. I., & Sutliff, G. (2020). Comparing extracted and stipulated definitions in algebra 1 textbooks and Khan Academy. *International Electronic Journal of Mathematics Education*, 15(2), em0579. <https://doi.org/10.29333/iejme/7601>
13. Bell, G., & Woo, J. H. (1998). Probing the links between language and mathematical conceptualisation. *Mathematics Education Research Journal*, 10(1), 51-74. <https://doi.org/10.1007/bf03217122>
14. Berger, L. (2018). Attending to precision: Mathematical definitions in courses for pre-service and practicing teachers. *PRIMUS*, 28(8), 772-784.
15. Berman, A., & Shvartsman, L. (2016). Definitions are important: The case of linear algebra. *European Journal of Science and Mathematics Education*, 4(1), 26-32.
16. Beynon, K. A., & Zollman, A. (2015). Lacking a formal concept of limit: Advanced non-mathematics students' personal concept definitions. *Investigations in Mathematics Learning*, 8(1), 47-62. <https://doi.org/10.1080/24727466.2015.11790347>
17. Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. *PME Conference*, 2, 104-111.
18. Bolondi, G., Ferretti, F., & Maffia, A. (2020). Monomials and polynomials: The long march towards a definition. *Teaching Mathematics and its Applications*, 39(1), 1-12.
19. Borasi, R. (1992). *Learning mathematics through inquiry*. Heinemann.
20. Breen, S., & O'Shea, A. (2019). Designing mathematical thinking tasks. *PRIMUS*, 29(1), 9-20.
21. Burroughs, E.A., Burke, M.J. (2016). By Definition: An Examination of the Process of Defining in Mathematics. In: Dewar, J., Hsu, Ps., Pollatsek, H. (eds) *Mathematics Education. Association for Women in Mathematics Series*, vol 7. Springer, Cham. [https://doi.org/10.1007/978-3-319-44950-0\\_5](https://doi.org/10.1007/978-3-319-44950-0_5)
22. Butuner, S.O., & Filiz, M. (2017). Exploring Turkish mathematics teachers' content knowledge of quadrilaterals. *International Journal of Research in Education and Science*, 3(2), 395-408. DOI:10.21890/ijres.327898

23. Chesler, J. (2012). Pre-service secondary mathematics teachers making sense of definitions of functions. *Mathematics Teacher Education and Development*, 14(1), 27-40.
24. Chesler, J. D. (2021). Minding the nouns and verbs: Definitions that respect knowledge development. *Mathematics Education Research Journal*, 33(1), 79-87.
25. Chorlay, R. (2019). A pathway to a student-worded definition of limits at the secondary-tertiary transition. *International Journal of Research in Undergraduate Mathematics Education*, 5(3), 267-314.
26. Connor, J., Moss, L., & Grover, B. (2007). Student evaluation of mathematical statements using dynamic geometry software. *International Journal of Mathematical Education in Science and Technology*, 38(1), 55-63.
27. Dahl, B. (2016). First-year non-STEM majors' use of definitions to solve calculus tasks: Benefits of using concept image over concept definition? *International Journal of Science and Mathematics Education*, 15(7), 1303-1322. <https://doi.org/10.1007/s10763-016-9751-9>
28. Dawkins, P. C. (2014). How students interpret and enact inquiry-oriented defining practices in undergraduate real analysis. *The Journal of Mathematical Behavior*, 33, 88-105. <https://doi.org/10.1016/j.jmathb.2013.10.002>
29. Dawkins, P. C. (2018). Student interpretations of axioms in planar geometry. *Investigations in Mathematics Learning*, 10(4), 227-239.
30. de Moura Fonseca, D. S. S., & de Oliveira Lino Franchi, R. H. (2016). Exploring the convergence of sequences in the embodied world using GeoGebra. *Teaching Mathematics and Its Applications*, 35(2), 88-101. <https://doi.org/10.1093/teamat/hrw012>
31. De Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, 14(1), 11-18.
32. De Villiers, M. (1998). To teach definitions in geometry or teach to define? *PME Conference*, 2, 248-255.
33. Doruk, M., & Alper, Ç. (2020). Pre-service mathematics teachers' concept definitions and examples regarding sets. *International Journal of Psychology and Educational Studies* 7(2), 21-36.

34. Dickerson, D. S., & Pitman, D. J. (2016). An examination of college mathematics majors' understandings of their own written definitions. *Journal of Mathematical Behavior*, 41, 1-9. <https://doi.org/10.1016/j.jmathb.2015.11.001>
35. Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, 53(1), 5-34. <https://doi.org/10.1023/a:1024661004375>
36. Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*, 111(5), 411-424. <https://doi.org/10.1080/00029890.2004.11920092>
37. Edwards, B. S., & Ward, M. B. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: research and teaching in undergraduate mathematics education* (pp. 223-232). Mathematical Association of America. <https://doi.org/10.5948/UPO9780883859759.018>
38. Ertekin, E., Yazici, E., & Delice, A. (2014). Investigation of primary mathematics student teachers' concept images: Cylinder and cone. *International Journal of Mathematical Education in Science and Technology*, 45(4), 566-588.
39. Fernández-León, A., Gavilán-Izquierdo, J.M., González-Regaña, A.J., Martín-Molina, V., & Toscano, R. (2021). Identifying routines in the discourse of undergraduate students when defining. *Mathematics Education Research Journal*, 33(2), 301-319.
40. Fisher, B. (2016). Student-created definitions of sequence convergence: A case study. *PRIMUS*, 26(8), 770-787.
41. Flores, A., & Park, J. (2016). Students' guided reinvention of definition of limit of a sequence with interactive technology. *Contemporary Issues in Technology and Teacher Education*, 16(2), 110-126.
42. Foster, C. (2014). Confidence trick: The interpretation of confidence intervals. *Canadian Journal of Science, Mathematics and Technology Education*, 14(1), 23-34. <https://doi.org/10.1080/14926156.2014.874615>



43. Foster, C., & de Villiers, M. (2015). The definition of the scalar product: An analysis and critique of a classroom episode. *International Journal of Mathematical Education in Science and Technology*, 47(5), 750-761. <https://doi.org/10.1080/0020739x.2015.1117148>
44. Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior*, 31(1), 60-72. <https://doi.org/10.1016/j.jmathb.2011.08.003>
45. Furinghetti, F. (2000). The history of mathematics as a coupling link between secondary and university teaching. *International Journal of Mathematical Education in Science and Technology*, 31(1), 43-51. <https://doi.org/10.1080/002073900287372>
46. Furinghetti, F., & Paola, D. (2000). Definition as a teaching object: A preliminary study. *PME Conference*, 2, 289-296.
47. Gilbertson, N. J., He, J., Satyam, V. R., Smith, J. P., & Stehr, E. M. (2016). The definitions of spatial quantities in elementary curriculum materials. *Proceedings of 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (38th, Tucson, AZ, Nov 3-6, 2016).
48. Gilboa, N., Kidron, I., & Dreyfus, T. (2018). Constructing a mathematical definition: The case of the tangent. *International Journal of Mathematical Education in Science and Technology*, 50(3), 421-446. <https://doi.org/10.1080/0020739x.2018.1516824>
49. Gokbulut, Y., & Sen, M. (2019). Primary teachers knowledge of quadrilaterals. *Higher Education Studies*, 9(2), 1-9.
50. Govender, R., & de Villiers, M. (2003). Constructive evaluation of definitions in a dynamic geometry context. *Research in Mathematical Education*, 7(1), 41-58.
51. Govender, R., & De Villiers, M. (2004). A dynamic approach to quadrilateral definitions. *Pythagoras*, 2004(59), 34-45.
52. Gutiérrez, A., & Jaime, A. (1999). Preservice primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher Education*, 2(3), 253-275. <https://doi.org/10.1023/a:1009900719800>

53. Haj-Yahya, A. (2019). Can classification criteria constitute a correct mathematical definition? Pre-service and in-service teachers' perspectives. *International Journal of Research in Education and Science*, 5(1), 88-101.
54. Haj-Yahya, A. (2021). Students' conceptions of the definitions of congruent and similar triangles *International Journal of Mathematical Education in Science and Technology*.<https://doi.org/10.1080/0020739X.2021.1902008>
55. Haj-Yahya, A., Hershkowitz, R., & Dreyfus, T. (2022). Investigating students' geometrical proofs through the lens of students' definitions. *Mathematics Education Research Journal*. <https://doi.org/10.1007/s13394-021-00406-6>
56. Harel, G., Fuller, E., & Rabin, J. M. (2008). Attention to meaning by algebra teachers. *Journal of Mathematical Behavior*, 27(2), 116-127.
57. Holguín, V. A. (2016). Stages identified in university students' behavior using mathematical definitions. *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (38th, Tucson, AZ, Nov 3-6, 2016)
58. Johnson, H. L., Blume, G. W., Shimizu, J. K., Graysay, D., & Konnova, S. (2014). A teacher's conception of definition and use of examples when doing and teaching mathematics. *Mathematical Thinking and Learning*, 16(4), 285-311. <https://doi.org/10.1080/10986065.2014.953018>
59. Kabaca, T. (2017). Understanding the hierarchical classification of quadrilaterals through the ordered relation according to diagonal properties. *International Journal of Mathematical Education in Science and Technology*, 48(8), 1240-1248. <https://doi.org/10.1080/0020739x.2017.1334969>
60. Karakuş, F. (2015). Investigation into how 8th grade students define fractals. *Kuram ve Uygulamada Eğitim Bilimleri*, 15, 825-836. <https://doi.org/10.12738/estp.2015.3.2429>
61. Kemp, A., & Vidakovic, D. (2021). Ways secondary mathematics teachers apply definitions in Taxicab geometry for a real-life situation: Midset. *Journal of Mathematical Behavior*, 62, 1-20.
62. Kobiela, M., Iacono, H., Cho, S., & Chandrasekhar, V. (2022). Examining the potential of rehearsal interjections to support the teaching of mathematical practice: The case of mathematical defining. *Journal of Mathematics Teacher Education*. <https://doi.org/10.1007/s10857-022-09545-4>

63. Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education*, 46(4), 423-454.
64. Kontorovich, I., & Zazkis, R. (2017). Mathematical conventions: Revisiting arbitrary and necessary. *For the Learning of Mathematics*, 37(1), 29-34.
65. Kontorovich, I. t. (2016). We all know that  $a^0 = 1$ , but can you explain why? *Canadian Journal of Science, Mathematics and Technology Education*, 16(3), 237-246. <https://doi.org/10.1080/14926156.2016.1189623>
66. Lavy, I., & Mashiach-Eizenberg, M. (2009). The interplay between spoken language and informal definitions of statistical concepts. *Journal of Statistics Education*, 17(1). <https://doi.org/10.1080/10691898.2009.11889502>
67. Leikin, R., & Winicki-Landman, G. (2000). On equivalent and non-equivalent definitions: Part 2. *For the Learning of Mathematics*, 20(2), 24-29.
68. Leikin, R., & Winicki-Landman, G. (2001). Defining as a vehicle for professional development of secondary school mathematics teachers. *Mathematics Education Research Journal*, 3, 62-73.
69. Leikin, R., & Zazkis, R. (2010). On the content-dependence of prospective teachers' knowledge: A case of exemplifying definitions. *International Journal of Mathematical Education in Science and Technology*, 41(4), 451-466. <https://doi.org/10.1080/00207391003605189>
70. Levenson, E. (2012). Teachers' knowledge of the nature of definitions: The case of the zero exponent. *The Journal of Mathematical Behavior*, 31(2), 209-219. <https://doi.org/10.1016/j.jmathb.2011.12.006>
71. Linchesvky, L., Vinner, S., & Karsenty, R. (1992). To be or not to be minimal? Student teachers' views about definitions in geometry. *Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education*, 2, 48-55.
72. Mamona-Downs, J.K., & Megalou, F.J. (2013). Students' understanding of limiting behavior at a point for functions from  $\mathbb{R}^2$  to  $\mathbb{R}$ . *Journal of Mathematical Behavior*, 32(1), 53-68.
73. Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics*, 34, 219-248. <https://doi.org/10.1023/A:1002985109323>
74. Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study. *International Journal of*

*Mathematical Education in Science and Technology*, 49(7), 1069-1082.  
<https://doi.org/10.1080/0020739x.2018.1426795>

75. Martín-Molina, V., González-Regaña, A. J., Toscano, R., & Gavilán-Izquierdo, J. M. (2020). Differences between how undergraduate students define geometric solids and what their lecturers expect from them through the lens of the theory of commognition. *Eurasia Journal of Mathematics, Science and Technology Education*, 16(12), em1917. <https://doi.org/10.29333/ejmste/9159>
76. Megalou, F. (2018). Students' choices of definitions for the concept of limit of functions from  $\mathbb{R}^2$  to  $\mathbb{R}$ . *International Journal of Mathematical Education in Science and Technology*, 50(1), 141-156. <https://doi.org/10.1080/0020739x.2018.1489073>
77. Miller, S. M. (2018). An analysis of the form and content of quadrilateral definitions composed by novice pre-service teachers. *The Journal of Mathematical Behavior*, 50, 142-154. <https://doi.org/10.1016/j.jmathb.2018.02.006>
78. Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27(3), 249-266.
79. Morgan, C. (2005). Word, definitions and concepts in discourses of mathematics, teaching and learning. *Language and Education*, 19(2), 102-116. <https://doi.org/10.1080/09500780508668666>
80. Morgan, C. (2006). What is a definition for in school mathematics? In M. Bosch (Ed.), *European Research in Mathematics Education IV: Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education*. Sant Feliu de Guíxols: FUNDEMI IQS - Universitat Ramon Llull.
81. Mosvold, R., & Fauskanger, J. (2013). Teachers' beliefs about mathematical knowledge for teaching definitions. *International Electronic Journal of Mathematics Education*, 8(2-3), 43-61.
82. Ndemo, Z., & Mtetwa, D. (2021). Mathematics education undergraduates' personal definitions of the notion of angle of contiguity in kinematics. *Journal of Research and Advances in Mathematics Education*, 6(2), 111-127.
83. Ouvrier-Bufferet, C. (2006). Exploring mathematical definition construction processes. *Educational Studies in Mathematics*, 63(3), 259-282.

84. Ouvrier-Buffet, C. (2010). A mathematical experience involving defining processes: In-action definitions and zero-definitions. *Educational Studies in Mathematics*, 76(2), 165-182. <https://doi.org/10.1007/s10649-010-9272-3>
85. Öztürk, T., & Güven, B. (2022). A study on pre-service mathematics teachers' criteria of proof evaluation. *LUMAT: International Journal on Math, Science and Technology Education*, 10(1), 210-241. <https://doi.org/10.31129/LUMAT.10.1.1649>
86. Özyildirim-Gümüş, F., & Şahiner, Y. (2017). Investigation on how pre-service elementary mathematics teachers write and use mathematical definitions. *International Electronic Journal of Elementary Education*, 9(3), 511-522.
87. Panaoura, A., Michael-Chrysanthou, P., Gagatsis, A., Elia, I., & Philippou, A. (2016). A structural model related to the understanding of the concept of function: Definition and problem solving. *International Journal of Science and Mathematics Education*, 15(4), 723-740. <https://doi.org/10.1007/s10763-016-9714-1>
88. Pimm, D. (1993). Just a matter of definition. *Educational Studies in Mathematics*, 25(3), 261-277.
89. Poincaré, H. (1969). Mathematical definitions and teaching. *The Mathematics Teacher*, 62(4), 295-304.
90. Randahl, M. (2016). The mathematics textbook at tertiary level as curriculum material—Exploring the teacher's decision-making process. *International Journal of Mathematical Education in Science and Technology*, 47(6), 897-916. <https://doi.org/10.1080/0020739x.2015.1133853>
91. Raychaudhuri, D. (2008). Dynamics of a definition: A framework to analyse student construction of the concept of solution to a differential equation. *International Journal of Mathematical Education in Science and Technology*, 39(2), 161-177.
92. Rowlett, P., Smith, E., Corner, A. S., O'Sullivan, D., & Waldock, J. (2019). The potential of recreational mathematics to support the development of mathematical learning. *International Journal of Mathematical Education in Science and Technology*, 50(7), 972-986. <https://doi.org/10.1080/0020739x.2019.1657596>
93. Sağlam, Y., & Dost, Ş. (2015). A qualitative research on example generation capabilities of university students. *International Journal of Science and Mathematics Education*, 14(5), 979-996. <https://doi.org/10.1007/s10763-015-9624-7>

94. Sánchez, V., & García, M. (2013). Sociomathematical and mathematical norms related to definition in pre-service primary teachers' discourse. *Educational Studies in Mathematics*, 85(2), 305-320. <https://doi.org/10.1007/s10649-013-9516-0>
95. Schüler-Meyer, A. (2020). Mathematical routines in transition: Facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and its Applications*, 39(4), 237–247.
96. Seaman, C.E., Szydlik, J.E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education* 10, 167-182. <https://doi.org/10.1007/s10857-007-9033-0>
97. Şen, C., & Güler, G. (2022). Examining the effectiveness of teaching through definitions in the definition and conceptualization of number sets by pre-service mathematics teachers. *Hacettepe Egitim Dergisi*, 37(3), 969-993.
98. Shaker, H., & Berger, M. (2016). Students' difficulties with definitions in the context of proofs in elementary set theory. *African Journal of Research in Mathematics, Science and Technology Education*, 20(1), 80-90.
99. Shield, M. (2004). Formal definitions in mathematics. *The Australian Mathematics Teacher*, 60(4), 25-28.
100. Soto-Johnson, H., Hancock, B., & Oehrtman, M. (2016). The interplay between mathematicians' conceptual and ideational mathematics about continuity of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education*, 2(3), 362-389. <https://doi.org/10.1007/s40753-016-0035-0>
101. Stavrou, S. G. (2014). Common errors and misconceptions in mathematical proving by education undergraduates. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1, 1-8.
102. Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards using definitions for developing mathematical objects: The case of function. *Educational Studies in Mathematics*, 90(2), 163-187. <https://doi.org/10.1007/s10649-015-9624-0>
103. Tay, E. G., & Leong, Y. H. (2012). Explaining definitions in secondary school mathematics:  $a^0$ ,  $a^n$ ,  $0!$  *Australian Senior Mathematics Journal*, 26(3), 28-37.
104. Tirosh, D. (1999). Finite and infinite sets: Definitions and intuitions. *International Journal of Mathematical Education in Science and Technology*, 30(3), 341-349. <https://doi.org/10.1080/002073999287879>

105. Tossavainen, T., Suomalainen, H., & Mäkäläinen, T. (2016). Student teachers' concept definitions of area and their understanding about two-dimensionality of area. *International Journal of Mathematical Education in Science and Technology*, 48(4), 520-532. <https://doi.org/10.1080/0020739x.2016.1254298>
106. Tsamir, P., Tirosh, D., Levenson, E., Barkai, R., & Tabach, M. (2015). Early-years teachers' concept images and concept definitions: Triangles, circles, and cylinders. *ZDM Mathematics Education*, 47(3), 497-509.
107. Ulusoy, F. (2021). Prospective early childhood and elementary school mathematics teachers' concept images and concept definitions of triangles. *International Journal of Science and Mathematics Education*, 19(5), 1057-1078.
108. Unlu, M., & Horzum, T. (2018). Mathematics teacher candidates' definitions of prism and pyramid. *International Journal of Research in Education and Science*, 4(2), 670-685.
109. Üreyen, M. (2006). The role of definition in students' understanding with particular reference to the concavity of a function. *International Journal of Mathematical Education in Science and Technology*, 37(5), 600-604. <https://doi.org/10.1080/00207390600595090>
110. Usiskin, Z., Willmore, E., Witonsky, D., & Griffin, J. (2008). *The classification of quadrilaterals: A study of definition*. Information Age Publishing.
111. Van Dormolen, J., & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity. *The Journal of Mathematical Behavior*, 22(1), 91-106.
112. Viirman, O. (2013). The functions of function discourse—University mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, 45(4), 512-527. <https://doi.org/10.1080/0020739x.2013.855328>
113. Vinner, S. (1976). The naive concept of definition in mathematics. *Educational Studies in Mathematics*, 7(4), 413-429.
114. Vinner, S. (1977). The concept of exponentiation at the undergraduate level and the definitional approach. *Educational Studies in Mathematics*, 8(1), 17-26.
115. Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Springer Netherlands. [https://doi.org/10.1007/0-306-47203-1\\_5](https://doi.org/10.1007/0-306-47203-1_5)

116. Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20(4), 356-366.
117. Wilhelmi, M. R., Godino, J. D., & Lacasta, E. (2007). Didactic effectiveness of mathematical definitions: The case of the absolute value. *International Electronic Journal of Mathematics Education*, 2(2), 72-90.
118. Winicki-Landman, G., & Leikin, R. (2000). On equivalent and non-equivalent definitions: Part 1. *For the Learning of Mathematics*, 20(1), 17-21.
119. Wong, C. L., Chu, H., & Yap, K. C. (2014). Developing a framework for analyzing definitions: A study of “The Feynman Lectures.” *International Journal of Science Education*, 36(15), 2481-2513.
120. Yavuzsoy-Köse, N., Y. Yilmaz, T., Yeşil, D., & Yildirim, D. (2019). Middle school students’ interpretation of definitions of the parallelogram family: Which definition for which parallelogram? *International Journal of Research in Education and Science*, 5(1), 157-175.
121. Yim, J., Chong, Y. O., Song, S. H., & Kwon, S. (2008). A case study on the local organization of two mathematically gifted seventh-grade students. *Journal for the Education of the Gifted*, 32(2), 211-229.
122. Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, 29(2), 57-75. <https://doi.org/10.1016/j.jmathb.2010.01.001>
123. Zaslavsky, O., & Shir, K. (2005). Students’ conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36(4), 317-346. <http://www.jstor.org/stable/30035043>
124. Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics*, 69(2), 131-148. <https://doi.org/10.1007/s10649-008-9131-7>