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Perceptions of Lecturers and Engineering Students of Sophism and Paradox: The Case of Differential Equations

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Abstract: One of the important topics that many STEM (science, technology, engineering, and mathematics) students learn at the tertiary level is differential equations (DEs). Previous studies have explored students' perceptions of engaging in puzzle tasks in STEM courses; however, no study has explored lecturers' and students' perceptions toward using sophism and paradox tasks in teaching mathematics courses, including DEs. This study explores DEs lecturers' and undergraduate engineering students' perceptions of using sophism and paradox tasks in the teaching and learning of DEs. The perceptions of 17 lecturers and 134 undergraduate engineering students of sophism and paradox tasks were explored using a questionnaire and semi-structured interviews. The findings showed that more than 50% of lecturers and students perceived that sophism and paradox tasks are enjoyable and entertaining activities which improve students' mathematical understanding and problem-solving skills, and enhance thinking skills. The findings suggest that sophism and paradox tasks can be used along with routine problems in teaching DEs to provide good opportunities for students to participate more effectively in classroom discussions and motivate them to learn DEs.

Keywords: differential equations; lecturers; undergraduate engineering students; sophism; paradox; perception

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1. Introduction

One of the significant educational discussions is the need to make STEM (science, technology, engineering, and mathematics) education more attractive at the tertiary level, including for engineering students [1]. Yet, research has found that while many engineering students are capable of solving routine problems, some cannot solve real-world problems as they have not developed the requisite critical and creative thinking skills [2]. Students who are limited to textbook questions that are solved using the methods discussed in the course do not develop the problem-solving strategies needed to solve real-world problems [2,3]. One of the strategies used to overcome this challenge is active learning. Active learning is a teaching approach that "engages students in the process of learning through activities and discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group work" ([4], p. 4). Students can engage more in mathematical investigations, collaborate with each other to solve problems, and increase their performance in STEM subjects [5]. Lugosi and Uribe [5] introduced six strategies to achieve the goals of active learning, such as working in groups with discussion and feedback, raising students' interest toward curriculum content, and involving students in mathematical explorations, experiments, and projects. Problem-based learning is one of the learning models that can contribute to students' active learning [6]. Puzzle-based learning (PzBL), as a subset of problem-based learning, is identified as being one of the effective ways to develop problem-solving strategies and promote inquiry learning [4]. PzBL refers to engaging students with puzzle problems to increase students' thinking (e.g., critical, creative, and lateral thinking) and problem-solving skills [3]. Previous studies have suggested that PzBL could help students to develop their conceptual understanding and motivation to learn [3,7,8]. So, we can introduce PzBL as an active learning approach that is able to involve students in class activities and leaning through puzzle tasks.

One of the important topics that mathematics and engineering students learn at the tertiary level is differential equations (DEs). DEs are an essential tool for many scientists and engineers as they are commonly used to model real-world situations [9,10]. A few studies have explored the use of puzzle problems in university mathematical courses (e.g., [11]), and students' perceptions of such problems [7]; however, we could not find a study that has explored lecturers' and students' perceptions of using sophism and paradox tasks in the teaching and learning of DEs. This study seeks to fill that gap by exploring the perceptions of DEs lecturers and engineering students of using sophism and paradox (SoPa) tasks in the teaching and learning of DEs. In particular, lecturers' and students' perceptions about the advantages and disadvantages of using SoPa tasks in the teaching and learning of DEs are explored using a questionnaire and semi-structured interviews. The research question considered in this paper was as follows: what are DEs lecturers' and engineering students' perceptions of using sophism and paradox tasks in the teaching and learning of DEs?

2. Theoretical Background and Relevant Literature

In this section, we first describe the main tenets of problem-based learning and puzzle-based learning. Then, we discuss the attitudes toward and perceptions of mathematics and PzBL. This section ends with a presentation of the relevant literature related to the teaching and learning of DEs.

2.1. Problem-Based Learning

Problem-based learning (PBL) is a "pedagogical approach that enables students to learn while engaging actively with meaningful problems" ([12], p. 2). PBL has several characteristics, such as it being student-centered; occurring in small groups; teachers being facilitators, not dispensers of knowledge; and the problems focusing on stimulating learning and developing students' problem-solving skills [13,14]. The purpose of using PBL is to improve students' problem-solving skills and to increase their motivation toward learning. It also develops self-directed learning, critical thinking, leadership skills, effective collaboration [9,11,12], and long-term knowledge retention [15,16].

2.2. Puzzle-Based Learning

PzBL is a subset of PBL and therefore shares its main characteristics [8,17]. It also positively impacts student participation and is more entertaining for students than traditional direct instruction [8]. There are three main types of puzzle problems: *sophism*, *paradox*, and *puzzle* [7]. A sophism is "intentionally invalid reasoning that looks formally correct, but in fact contains a subtle mistake or flaw" ([7], p. 1106). A paradox refers to a "surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true" ([7], p. 1106). The third type, a puzzle, is a "non-standard, non-routine, unstructured question presented in an entertaining way" ([7], p. 1106). These three types of tasks can activate higher-order thinking (i.e., analyzing, evaluating, and creating), as students who engage in solving sophism and paradox tasks should critically analyze the given information and evaluate the reasoning provided in these tasks to verify or refute them [18], which can help students to develop a deeper conceptual understanding of mathematics [11]. Additionally, for solving puzzle problems, students sometimes need to create a new strategy to reach the correct solution, which is related to higher-order thinking [18].

Michalewicz and Michalewicz [3] described four criteria that puzzle problems should meet: *simplicity*, *generality*, be *entertaining*, and having the *eureka* factor. To meet the *simplicity* criterion, the problem should be easy to state. For *generality*, puzzle problems should explain some universal mathematical problem-solving principles and provide opportunities for students to learn how to solve future unknown problems [3], as there is a strong connection between the ability to solve puzzles and the ability to solve many real-world problems [19]. The design of puzzle problems should be rooted in students' prior knowledge and experiences and they should be used alongside routine problems when teaching to prepare students for solving real-world problems [3,8]. Furthermore, the techniques used in puzzle problems can be used in other problem-solving situations [2,8]. "The ultimate goal of puzzle-based learning is to lay a foundation to be effective problem solvers in the real world" ([19], p. 22). Puzzle problems can play an important role in attracting students to engineering and mathematics programs [19], as many engineering problems are similar to puzzle problems [20]. To meet the *entertaining* criterion, puzzle problems need to be enjoyable activities that encourage students to continue looking for a solution [3]. PzBL can enhance students' motivation and convince them that science is useful, interesting, and relevant [3]. PzBL can show students that mathematics is not scary and can be enjoyable. PzBL can also motivate students to stay in school, and continue their education to develop enough knowledge and skills to solve real-world problems [3].

Finally, regarding the *eureka* factor, the solution of a puzzle problem should not be obvious, and finding the correct direction to the problem's solution might be tedious. However, as puzzle problems are supposed to be entertaining, they should be able to retain a student's desire to solve them [8]. When students reach a correct answer after a lot of effort, that moment is called the eureka moment (Martin Gardner's Aha!) [3]. The eureka moment outweighs the frustration of the solution process and gives the problem-solver a sense of satisfaction [3]. It should be noted that a puzzle problem does not need to meet all of these criteria. For instance, not all puzzle problems meet the simplicity criterion [3]. The best puzzle problems have more than one solution: a lateral thinking as well as a conventional solution [8]. Klymchuk [7] highlighted that "interesting puzzles, paradoxes, and sophisms can engage students' emotions, creativity, and curiosity and also enhance their conceptual understanding, critical thinking skills, problem-solving strategies, and lateral thinking" (p. 1106). By creative thinking we mean thinking "flexibly enough to find novel ways to move within the constraints" ([21], p. 4). However, the novel way may have been produced earlier, but it will be new to students [21]. Furthermore, we define critical thinking as "intellectual activity which emphasizes the following skills: problem formulation, problem reformulation, evaluation, problem sensitivity" ([22], p. 654). Finally, lateral thinking (thinking outside the box) refers to solving problems with innovative approaches as opposed to using traditional and routine methods [7,8].

2.3. Students' Attitudes toward and Perceptions of Mathematics and PzBL

Attitude as a construct refers to "a cognitive, affective, and behavioral reaction the individual organizes toward himself/herself based on information, feeling, and motivation" ([23], p. 334). One's attitude develops from experiences and learning in different situations, and therefore changing one's attitude takes time and effort [24]. Perception as a closely related construct to attitude has a slightly different meaning. Perception could be defined as a process where an individual interprets and organizes a sensation based on his or her prior experiences when confronting a situation or stimuli to make a meaningful experience of the world, which could be different from reality [24].

One of the factors that impacts students' learning and their success in mathematics is their attitudes toward and perceptions of mathematics [25,26]. Several factors influence students' attitudes toward and perceptions of mathematics, such as the academic and social environment of educational institutes, the content of the courses, teaching approaches, teachers, and students' experiences with mathematics [27,28]. When students realize the importance of mathematics in their daily life, they are more engaged in learning mathematics [29]. However, previous studies indicated that some students could not perceive the association between mathematical concepts in service mathematical courses, engineering subjects, and their future careers [30,31]. Furthermore, students' attitudes toward mathematics also positively impact student participation in classrooms [26]. Teachers play an important role in shaping students' attitudes toward and perceptions of mathematics. The approaches they use to teach mathematics, how they communicate with students directly, and engage students in reverent tasks in their fields of study can impact students' attitudes and perceptions toward mathematics [28,32,33].

We only found two studies on students' attitudes toward and perceptions of PzBL in higher education. However, these studies were not related to students' perceptions of using SoPa tasks in the teaching and learning of mathematics. Additionally, previous studies have not explored mathematics lecturers' perceptions of using PzBL in the teaching and learning of mathematics. One was conducted in the context of calculus [7], and the other was conducted in introductory computer science [34]. Merrick [34] investigated both the ability of 96 students to solve puzzles and their attitudes toward PzBL. The study found that the students had positive attitudes toward PzBL, and they believed that using PzBL in teaching and learning had a positive effect on their motivation, critical thinking, and problem-solving skills. However, they did not explore students' attitudes toward engaging in each type of puzzle task separately. Klymchuk's [7] findings were in line with Merrick's [34] study in that he identified that more than 90% of the students believed solving puzzle problems improved their problem-solving skills, insight, reasoning, and general thinking skills. Furthermore, 85% of the students believed puzzle problems were enjoyable, created a pleasurable environment for students, and encouraged them to seek creative solutions for puzzle problems. However, in his study, Klymchuk [7] did not provide detailed information about students' attitudes toward and perceptions of sophism, paradox, and puzzle tasks separately, and described students' attitudes toward and perceptions of PzBL in general in a calculus course.

2.4. Teaching and Learning of Differential Equations

Students majoring in engineering and mathematics typically enroll on calculus courses in their first year of tertiary study, and then take a DEs course in the second year [10]. DEs play a vital role in engineering and mathematics. DEs are used frequently to solve real-world problems in most engineering disciplines (e.g., electrical, chemical, and mechanical engineering) [9]. There are three general approaches for solving a DE: algebraic, qualitative, and numerical [10]. Analytic methods are techniques for recovering the symbolic form of a DE solution; numerical methods are iterative techniques that provide reliable approximate solutions, usually with the help of technology [10]; and graphical methods provide "overall information about solutions by viewing solutions to differential equations geometrically and by analyzing the differential equation itself" ([10], p. 56). However, engineering and mathematics students are more inclined to use algebraic methods to solve DEs [33]. One possible reason for this is that, when taught traditionally, students can develop procedural knowledge of DEs and become capable of using algebraic techniques to solve them; however, they can fail to understand the relationship between the DE and its solution [9,10,35].

Previous studies have reported that students have three main difficulties when solving DEs: interpreting the DE meaningfully, explaining the solutions, and identifying the relationship between the DE and its solutions [9,10,36]. For instance, many students in Arslan's [6] study solved DEs without understanding their meaning; the students' mistakes were often due to poor symbol use and not recognizing the type of DE (e.g., incorrectly identifying the method for solving a first-order DE). However, other research has suggested that formulating a DE from a real-world problem is one of the most common difficulties associated with the traditional teaching of DEs [35]. Focusing on numerical and graphical approaches for solving DEs might help students to learn DEs conceptually [10]. If students could learn DEs conceptually, it is more likely that they could solve modeling tasks [37]. Furthermore, modeling with DEs helps students to develop a better understanding of how mathematics can be used to solve real-world problems, improves students' problem-solving skills, and motivates them to study mathematics [35,38]. Modeling projects increase students' ability to communicate ideas, educates them as independent learners, helps them to have a better understanding of mathematics, and prepares them for their future careers [35]. Using technology (e.g., Maple and MATLAB) when teaching DEs could also help students to improve their understanding of DEs [39].

3. Methods

In this study, a pragmatic approach [40] was taken. Both qualitative and quantitative data were collected using an explanatory sequential mixed method [40] about lecturers' and engineering students' perceptions of using SoPa tasks in the teaching and learning of DEs. This research is exploratory [41], as the previous studies related to PzBL in mathematics education have mainly focused on describing and illustrating PzBL and students' performance in PzBL (e.g., [3,8,19,42]). We collected quantitative data first to allow us to invite students with different perceptions of and performance in sophism and paradox tasks to the interviews to enrich our quantitative findings.

3.1. Data Collection

Using convenience sampling, seventeen mathematics lecturers involved in the teaching of DEs and 134 undergraduate engineering students at a public university in the east of Iran participated in the 2019–2020 academic year in this study (information about the age and gender of students in the quantitative part of the study was not recorded). This university is one of the top universities in Iran. In the Faculty of Engineering at this university, several DEs courses are offered, each with 50 to 60 students. The volunteer engineering students were from computer (N = 34 (25.4%)), metallurgical (N = 24 (17.9%)), chemical (N = 16 (11.9%)), civil (N = 15 (11.2%)), industrial (N = 14 (10.4%)), electrical (N = 12 (8.9%)), mechanical (N = 11 (8.2%)) and other (N = 8 (6%)) engineering majors. It is worth mentioning that the researchers were not part of the teaching of DEs for these students to minimize the bias that students might respond favorably to the questionnaire and interview questions to please their lecturers.

We designed a sophism and a paradox (see Appendix A) based on first-order DEs and a questionnaire to explore lecturers' and students' perceptions of SoPa tasks. These were then reviewed by three DEs lecturers (one of them was also a mathematics educator) and trialed. The refined tasks were later administered to 134 engineering students working in self-selected groups of two or three students in their DEs lectures. Students were asked to audio-record their discussions. This allowed the researchers to follow their thinking, identify how their solutions progressed, and ascertain the difficulties that students encountered. After completing the tasks, the first author detailed an optimal solution for each task, and described to students what type of tasks can be considered as a sophism and a paradox in order to help the students to identify the differences between these types of tasks. Then, the students completed the questionnaire individually. They provided their opinions separately on each item for sophism and paradox. A few weeks later, using stratified random sampling, thirteen students with different levels of performance in the tasks were invited to participate in the interviews (Table 1).

To identify who to invite, as students worked in groups, each group was rated on their performance with the tasks as follows: low (L)-, medium (M)-, or high- (H)-achieving task-solving groups. At least four students from each stratum were invited. Based on their responses to the questionnaire, individual students were then categorized as having positive (P) or negative (N) perceptions of using SoPa tasks in the teaching and learning of DEs.

Regarding the lecturers, first, the first author discussed with lecturers the characteristics of SoPa tasks with some examples (including those that the students engaged with). Then, they were invited to complete the questionnaire and participate in the semi-structured interviews. These lecturers were selected using convenience sampling (see their background information in Table 2).

Table 1. Participants' information: students.

Student Label	Gender	Performance in Solving the SoPa Tasks	Perception of SoPa Tasks
H1	Female	High	P
H2	Male	High	P
H3	Male	High	P
H4	Male	High	P
M1	Female	Medium	P
M2	Female	Medium	N
M3	Female	Medium	N
M4	Male	Medium	P
M5	Male	Medium	N
L1	Female	Low	P
L2	Female	Low	N
L3	Female	Low	P
L4	Male	Low	N

Table 2. Participants' information: lecturers.

Lecturer Code	Qualification	Years of Teaching DEs	Gender
T1	PhD in applied mathematics—optimization	20	Male
T2	PhD in applied mathematics—numerical analysis	19	Male
T3	PhD in applied mathematics—numerical analysis	19	Male
T4	PhD in statistics	15	Female
T5	PhD in applied mathematics—numerical analysis	15	Male
T6	PhD in pure mathematics—group theory	10	Male
T7	PhD in applied mathematics—differential equations	10	Female
T8	PhD in applied mathematics—numerical analysis	10	Male
T9	PhD in applied mathematics—differential equations	8	Female
T10	PhD in applied mathematics—numerical analysis	7	Female
T11	PhD in pure mathematics—algebraic graphs and combinatorics	6	Male
T12	PhD in applied mathematics—control and optimization	5	Male
T13	PhD in applied mathematics—dynamic systems and geometric theories	5	Male
T14	PhD in applied mathematics—control and optimization	3	Male
T15	PhD in pure mathematics—algebraic graphs and combinatorics	3	Female
T16	PhD in applied mathematics—optimization	2	Male
T17	PhD in applied mathematics—numerical analysis	2	Female

3.2. The Instruments

In this section, we discuss how the questionnaire and the interview guide were developed.

3.2.1. The Questionnaire

The design of the questionnaire items (Table 3) was based on the relevant literature about PzBL, e.g., [7,32], and it was structured into three themes. The first seven items, labeled as 'enjoyable and entertaining activities', were designed to explore lecturers' and students' interest in including SoPa tasks in the teaching, learning, and assessment of DEs. They also explored whether SoPa tasks may increase students' motivation, curiosity, and participation in the classroom. The next two items, labeled 'improving mathematical understanding and problem-solving skills', were designed to explore whether lecturers and students believed that the strategies learned from solving SoPa tasks could be used to solve mathematical problems. They also explored whether using SoPa tasks in the teaching of DEs could improve students' mathematical understanding. The last four items, labeled 'improving different types of thinking', were designed to investigate whether using SoPa tasks could improve students' critical, creative, and lateral thinking skills.

Table 3. The questionnaire items.

Themes	Items
Enjoyable and entertaining activities	1. The use of sophism/paradox tasks in the teaching of DEs makes the teaching entertaining and enjoyable. 2. Sophism/paradox tasks are enjoyable and entertaining activities. 3. Students can learn DEs in an entertaining way by solving sophism/paradox tasks. 4. Students' curiosity can be increased by solving sophism/paradox tasks. 5. The use of sophism/paradox tasks in teaching DEs increases students' participation in the classroom. 6. The moment of discovering the correct solution to a sophism/paradox task is very enjoyable. 7. Solving sophism/paradox tasks increases students' motivation to learn DEs.
Improving mathematical understanding and problem-solving skills	8. Engaging in solving sophism/paradox tasks improves students' problem-solving skills. 9. The use of sophism/paradox tasks in teaching DEs improves students' conceptual understanding of DEs.
Improving different types of thinking	10. To solve a sophism/paradox task, students should consider the problem from different angles. 11. Engaging in solving sophism/paradox tasks improves students' critical thinking skills. 12. Engaging in solving sophism/paradox tasks improves students' creative thinking skills. 13. Engaging in solving sophism/paradox tasks leads students to analyze other DEs problems from different angles as well.

3.2.2. Interviews

The interviews were semi-structured, one-on-one, audio recorded, and took 45–80 min. The interview guide was developed by reflecting on the literature on PzBL, e.g., [3,7,8], and the teaching and learning of DEs, e.g., [9,10], in consultation with a senior lecturer of mathematics education and piloted with two undergraduate mathematics students. At the beginning of each interview, the SoPa tasks were given to the students again, and they were asked to explain how their group solved these tasks. However, as this paper only explores lecturers' and students' perceptions of SoPa tasks, only the findings related to their perceptions are reported here. The interview questions were developed from some of the items in the questionnaire. A sample interview question was, "Do you think sophism and paradox tasks improve problem-solving skills and thinking skills? Please justify your answer".

3.3. Data Analysis

To classify students based on their responses to the SoPa tasks, we used the following procedure: if one or two SoPa tasks were solved correctly, the group members were classified as being *medium* or *high performers*, respectively. With this procedure, the other students who did not belong to these two groups were considered to be *low performers*. An example of such a performance was just checking the responses provided by Reza and Ali in Task 2.

To code responses to the questionnaire, scores for each of the 13 Likert-style items for SoPa tasks were aggregated. Strongly disagree scored 1, disagree scored 2, nor agree or disagree scored 3, agree scored 4, and strongly agree scored 5; the minimum and maximum possible scores for students' perceptions were 26 and 130, respectively, as students provided their opinions separately for SoPa tasks. Within the sample, the minimum and maximum scores were found to be 55 and 130, respectively. Students below the mean of 98.7 were categorized into Group N, and those above the mean were categorized into Group P. Using this procedure, all students from the high-achieving task-solving groups were found to be coded as P. However, students with different perceptions were found in the medium- and low-achieving groups, and so a mix was selected. Please note that if another threshold score was used, such as the median, some different codes would have been generated, but as this coding was only used to ensure that students with different perceptions of SoPa tasks were included in the interview sample, this procedure was fit for purpose.

We used Fisher's exact test to explore whether the lecturers' perceptions of SoPa tasks were significantly different from the students' perceptions for each item, and we also separately examined the difference between lecturers' and students' perceptions of SoPa tasks for each item. Fisher's exact test could be used for these purposes as it examines the association between two ordinal- or nominal-level variables [43]. Finally, the qualitative data collected through the open-ended items and the interviews were inductively coded [44] and reported following a content analysis approach [45]. The first author conducted the initial coding and then discussed and refined the emerging themes with the second author in several meetings.

3.4. Reliability and Validity

One of the reliability measures is internal consistency, referring to "how accurately the measures obtained from the research was actually quantifying what it was designed to measure" ([46], p. 195). In this study, internal consistency was estimated using Cronbach's alpha correlation coefficient. Cronbach's alpha for sophism and paradox was 0.90 and 0.88, respectively, which indicates that the questionnaire items had good internal consistency.

We also conducted confirmatory factor analysis (CFA) in AMOS to determine the construct validity of the survey items of the questionnaire using students' data. The selected themes were chosen based on the relevant literature of PzBL [2,3,7,34]. The chi-square test of CFA indicates the amount of difference between expected and observed covariance matrices. If this measure is close to 1 and not overstepping 3, this indicates a good fit [47]. The comparative fit index (CFI) indicates the model fit by calculating the difference between the data and the hypothesis model. The root mean square error of approximation (RMSEA) was used to measure the difference between the sample predicted and the sample observed. A good model fit indicates that CFI is more or equal to 0.9 [48] and a sample of RMSEA should be 0.08 or less [49]. Moreover, all factor loadings for all constructs were higher than 0.6 for sophism (Figure 1) and 0.5 for paradox (Figure 2) [50]. Table 4 shows the values of chi-square, CFI, and RMSEA for sophism and paradox, separately. The results indicated that three themes were suitable, as the values for these measures were in the acceptable range.

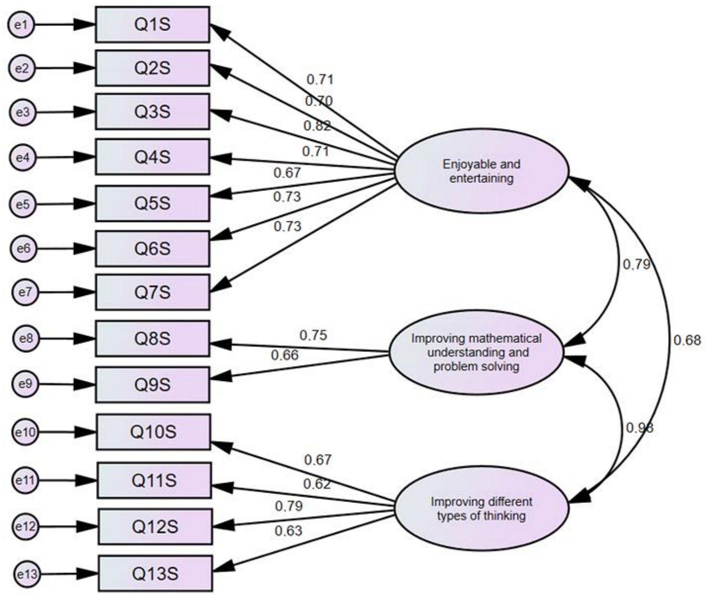

Figure 1. Standardized factor loadings for sophism.

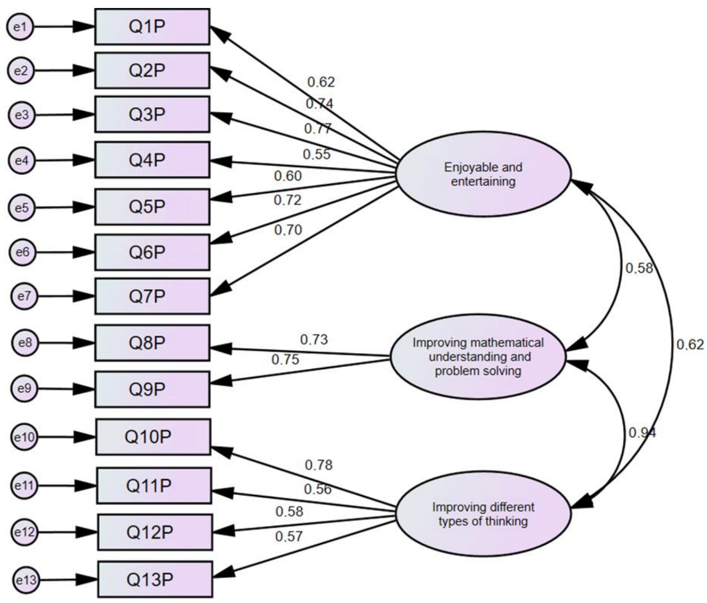

Figure 2. Standardized factor loadings for paradox.

Table 4. The outcomes of the measures for CFA.

	Chi-Square	**CFI**	**RMSEA**
Sophism	93.835	0.9	0.06
Paradox	105.200	0.9	0.07

Content validity was also used to explore the validity of the instruments. Content validity is usually explored using a literature review and with the help of experts in the field [51]. One senior lecturer of tertiary mathematics education who had the experience of teaching DEs for several years examined the content validity of the questionnaire items. His feedback was used to refine the instrument. To validate the tasks, four senior lecturers in DEs examined the validity of the tasks. Their feedback was used to improve the wording of the tasks. Then, the instruments were trialed with eleven students majoring in mathematics. We made some changes to the wording of the tasks where student responses suggested that there was a possible misinterpretation of the tasks.

To ensure our findings' *credibility* [52], we provided a *thick description* of how the study was conducted and provided several quotes when describing our qualitative findings. We also conducted *data triangulation* by using both semi-structured interviews and a questionnaire. *Multivocality* was also considered in this study as both mathematics lecturers and engineering students were invited to share their perceptions of including SoPa tasks in the teaching and learning of DEs.

4. Results

The results section presents lecturers' and students' responses to the questionnaire and the interviews. The findings regarding how the students engaged with the tasks have been published elsewhere (see [53]).

4.1. The Questionnaire Results

This section describes lecturers' and students' perceptions of SoPa tasks related to the three factors described in the questionnaire (Table 5).

4.1.1. Enjoyable and Entertaining Activities

Lecturers' and students' responses to the first two items show that more than sixty-four percent of the lecturers and more than fifty-five percent of students concurred that engaging in SoPa tasks makes the teaching and learning of DEs entertaining and enjoyable, while only a small proportion did not have such perceptions. Additionally, there was no significant difference between lecturers' and students' perceptions in these two items. Similarly, for Item 3, over fifty-five percent of lecturers and students agreed or strongly agreed that SoPa tasks increase students' curiosity, with only a minority disagreeing. However, Fisher's exact test showed that students had different perceptions of the impact of SoPa tasks on their curiosity. Sixty-seven percent of students agreed or strongly agreed that engaging with paradox tasks can increase students' curiosity, whereas this percentage was fifty-six percent for sophism tasks.

Analyzing data regarding Item 4 showed that more than eighty-two percent of lecturers and fifty percent of students perceived that SoPa tasks could increase student participation in classroom discussions. Fisher's exact test results showed a significant difference between lecturers' and students' perceptions about this item, indicating that lecturers were more positive that sophism tasks could increase students' participation in classroom discussions than students.

The results shared regarding Items 5 to 7 in Table 5 provide further evidence that over half of the lecturers and students perceived that engaging in SoPa tasks is enjoyable, that they are pleasant activities for students, and that they motivate them to learn DEs. There was no significant difference between lecturers' and students' perceptions in these three items.

4.1.2. Improving Mathematical Understanding and Problem-Solving Skills

Responses to Items 8 and 9 showed that over sixty-three percent of lecturers and students believed that engaging in SoPa tasks could help students to develop their conceptual understanding of DEs and improve their problem-solving skills. Fisher's exact test results indicated no significant difference between lecturers' and students' perceptions in these two items.

Table 5. The questionnaire results.

Themes	Items	Type	S/L *	Strongly Disagree		Disagree		Nor Agree or Disagree		Agree		Strongly Agree		*p*-Value So vs. Pa (S)	*p*-Value So vs. Pa (L)	*p*-Value Sophism (L vs. S)	*p*-Value Paradox (L vs. S)
				N	%	N	%	N	%	N	%	N	%				
Enjoyable and entertaining activities	1	Sophism	S	14	10.4	14	10.4	27	20.1	46	34.3	33	24.6	0.525	0.524	0.086	0.942
			L	0	0	1	5.9	1	5.9	12	70.6	3	17.6				
		Paradox	S	8	6	8	6	36	26.9	50	37.3	32	23.9				
			L	1	5.9	1	5.9	4	23.5	8	47.1	3	17.6				
	2	Sophism	S	11	8.2	26	19.4	22	16.4	48	35.8	27	20.1	0.093	0.950	0.391	0.114
			L	0	0	2	11.8	2	11.8	6	35.3	7	41.2				
		Paradox	S	6	4.5	13	9.7	30	22.4	58	43.3	27	20.1				
			L	1	5.9	2	11.8	2	11.8	4	23.5	8	47.1				
	3	Sophism	S	16	11.9	11	8.2	31	23.1	29	21.6	47	35.1	0.026 **	1.000	0.343	0.189
			L	1	5.9	1	5.9	1	5.9	6	35.3	8	47.1				
		Paradox	S	4	3	15	11.2	25	18.7	42	31.3	48	35.8				
			L	1	5.9	1	5.9	0	0	7	41.2	8	47.1				
	4	Sophism	S	9	6.7	21	15.7	28	20.9	33	24.6	43	32.1	0.140	0.642	0.046 **	0.737
			L	0	0	0	0	2	11.8	10	58.8	5	29.4				
		Paradox	S	4	3	13	9.7	27	20.1	49	36.6	41	30.6				
			L	0	0	0	0	3	17.6	7	41.2	7	41.2				
	5	Sophism	S	9	6.7	12	9	22	16.4	27	20.1	64	47.8	0.862	0.925	0.066	0.257
			L	0	0	0	0	5	29.4	7	41.2	4	23.5				
		Paradox	S	7	5.2	11	8.2	17	12.7	31	23.1	68	50.7				
			L	1	5.9	1	5.9	5	29.4	5	29.4	5	29.4				
	6	Sophism	S	11	8.2	12	9	36	26.9	39	29.1	36	26.9	0.795	0.733	0.641	0.374
			L	1	5.9	1	5.9	6	35.3	7	51.2	2	11.8				
		Paradox	S	8	6	8	6	37	27.6	45	33.6	36	26.9				
			L	0	0	3	17.6	4	23.5	7	41.2	3	17.6				
	7	Sophism	S	11	8.2	12	9	33	24.6	42	31.3	36	26.9	0.624	1.000	0.465	0.887
			L	0	0	1	5.9	2	11.8	7	41.2	7	41.2				
		Paradox	S	7	5.2	9	6.7	29	21.6	52	38.8	37	27.6				
			L	0	0	1	5.9	3	17.6	6	35.3	7	41.2				
Improving mathematical understanding and problem-solving skills	8	Sophism	S	10	7.5	6	4.5	28	20.9	43	32.1	47	35.1	0.168	1.000	0.579	0.335
			L	1	5.9	1	5.9	1	5.9	7	41.2	7	41.2				
		Paradox	S	2	1.5	9	6.7	27	20.1	42	31.3	54	40.3				
			L	1	5.9	1	5.9	1	5.9	7	41.2	7	41.2				
	9	Sophism	S	6	4.5	15	11.2	28	20.9	37	27.6	48	35.8	0.700	0.895	0.807	0.921
			L	0	0	1	5.9	3	17.6	4	23.5	9	52.9				
		Paradox	S	7	5.2	9	6.7	25	18.7	43	32.1	50	37.3				
			L	0	0	1	5.9	2	11.8	6	35.3	8	47.1				

Table 5. *Cont.*

Themes	Items	Type	S/L *	Strongly Disagree		Disagree		Nor Agree or Disagree		Agree		Strongly Agree		*p*-Value So vs. Pa (S)	*p*-Value So vs. Pa (L)	*p*-Value Sophism (L vs. S)	*p*-Value Paradox (L vs. S)
				N	%	N	%	N	%	N	%	N	%				
Improving different types of thinking	10	Sophism	S	6	4.5	11	8.2	28	20.9	39	29.1	50	37.3	0.757	0.706	0.669	0.060
			L	0	0	1	5.9	2	11.8	8	47.1	6	35.3				
		Paradox	S	4	3	7	5.2	32	23.9	44	32.8	47	35.1				
			L	0	0	2	11.8	0	0	9	52.9	6	35.3				
	11	Sophism	S	8	6	7	5.2	22	16.4	33	24.6	64	47.8	0.140	0.484	0.068	0.245
			L	0	0	0	0	1	5.9	1	5.9	15	88.2				
		Paradox	S	3	2.2	16	11.9	27	20.1	33	24.6	55	41				
			L	0	0	0	0	2	11.8	3	17.6	12	70.6				
	12	Sophism	S	7	5.2	9	6.7	19	14.2	49	36.6	50	37.3	0.988	0.884	0.883	0.903
			L	0	0	0	0	2	11.8	8	47.1	7	51.2				
		Paradox	S	5	3.7	9	6.7	19	14.2	51	38.1	50	37.3				
			L	0	0	1	5.9	2	11.8	9	52.9	5	29.4				
	13	Sophism	S	7	5.2	11	8.2	20	14.9	44	32.8	52	38.8	0.990	1.000	0.526	0.495
			L	0	0	0	0	4	23.5	4	23.5	9	52.9				
		Paradox	S	7	5.2	9	6.7	19	14.2	47	35.1	52	38.8				
			L	0	0	0	0	4	23.5	4	23.5	9	52.9				

* S: students, L: lecturers, So: sophism, Pa: paradox. ** The p-value is significant at 0.05 level.

4.1.3. Improving Different Types of Thinking

Lecturers' and students' responses to Item 10 and 11 showed that over eighty-two percent of the lecturers and sixty-six percent of the students agreed or strongly agreed that students need to consider different angles of the task when engaging in solving SoPa tasks and engaging with them could help students to improve their critical thinking. For Items 12 and 13, more than seventy percent of the lecturers and students believed that students' creative thinking skills could be improved by engaging in SoPa tasks and could lead students to consider different angles of DEs problems when solving them. Fisher's exact test results indicated no significant difference between lecturers' and students' perceptions in these four items.

4.2. The Interview Results

In this section, the results of the lecturers' and students' interviews about their perceptions of SoPa tasks are described.

Advantages and Disadvantages of SoPa Tasks

Lecturers' and students' responses to the interview questions about the advantages of SoPa tasks were coded into three main themes (Table 6).

Many lecturers and students believed that SoPa tasks are enjoyable and entertaining activities, improve different types of thinking (e.g., critical and lateral thinking), and help students to develop their conceptual understanding of DEs and problem-solving skills. Additionally, fourteen lecturers (82%) highlighted that routine problems could make the class boring, and that students usually memorize procedures; as a result, these procedures would be forgotten after a while:

> Many lecturers only focus on routine problems and how they can be solved. It is like you are on the road, and you just look straight ahead without paying attention to your surroundings. In my opinion, these types of tasks are like roadside which can help us to show students how fascinating it is that the concepts are related to each other . . . (T3).

Table 6. The advantages of including SoPa tasks in the teaching and learning of DEs.

Themes	Sub-Themes	S/L	So	Pa	A Sample Response
Enjoyable and entertaining activities	Entertaining and enjoyable	S	1	2	"Solving sophism and paradox tasks are enjoyable because students can come up with a correct solution themselves related to their current knowledge. Additionally, it is a nice break during a lecture" (L1).
		L	14	14	"Solving sophism motivate students, even the lazy ones . . . when students are asked to find a mistake, everyone is automatically interested in finding the invalid reasoning. It creates a competitive and enjoyable atmosphere in the lecture" (T16).
	Engaging students' minds	S	3	3	"Paradoxes and sophisms challenge students' mathematical knowledge and encourage them to improve their mathematical understanding" (M2).
		L	8	7	"Sophisms and paradoxes are very interesting problems. The nature of these problems arouses students' curiosity and engage students to find the correct solution" (T3).
	Increasing students' participation	S	0	0	
		L	2	2	"Using sophisms and paradoxes in the classroom increases the interaction between the lecturer and students" (T5).
	Increasing students' motivation to learn mathematics and solve mathematical problems	S	1	0	"Sophism break the monotony of classwork and might increase students' interest in solving problems" (M1).
		L	3	3	"Some students found DEs lectures boring. These problems can motivate students to learn DEs and participate in classroom discussions" (T8).

Table 6. *Cont.*

Themes	Sub-Themes	S/L	So	Pa	A Sample Response
Improving mathematical understanding and problem-solving skills	Improving students' mathematical understanding	S	11	7	"Sophisms and paradoxes help students to become better problem-solvers … These tasks promote deep mathematical understanding" (L3).
		L	11	11	"Sophism and paradox tasks are beneficial to use in teaching. If a student can refute a false statement, he/she has good knowledge of the topic. To do so, students need to consider different theorems simultaneously. This helps them to develop a meaningful understanding of DEs concepts" (T5).
	Increasing students' ability to solve real-world problems	S	3	2	"In the real world, sometimes engineers need to pay close attention to details, find an error in a system, or design a new model. All of these could be improved by solving sophisms" (H1).
		L	7	8	"These tasks can help students to solve real-world problems as prepare them to make decisions based on logic. They learn not to make decisions based on the appearance of the problem" (T9).
	Improving students' problem-solving skills	S	1	2	"By solving sophisms and paradoxes, students become familiar with new strategies and skills that can be used for solving mathematical problems; therefore, their problem-solving skills can be improved" (H3).
		L	10	10	"They are effective in increasing students' problem-solving skills. Students can learn DEs conceptually since they should examine the problems from different perspectives. These tasks enable students to develop new skills and strategies to solve other mathematical problems" (T13).
	Increasing the opportunities for sustainable mathematical learning	S	3	1	"To solve sophisms, students need to find relationships between different concepts. They find a solution themselves that makes the learning more sustainable for them" (M5).
		L	0	0	
	Reducing students' mathematical misunderstanding	S	2	2	"Students might identify their misunderstandings by solving sophisms and paradoxes" (L1).
		L	2	2	"Students realize their misunderstandings by solving sophism and paradox tasks because they examine the reasoning in the task several times and their accuracy would be increased" (T17).
Improving different types of thinking	Improving creativity	S	1	2	"Solving a paradox requires creativity. We need to identify relationships between different mathematical concepts to find a suitable approach" (M4).
		L	9	9	"Sophisms should be used in the classroom to cultivate thinking of engineers who play an important role in society. It could increase creativity … " (T10).
	Improving critical thinking skills	S	6	8	"To solve paradoxes and sophisms correctly, students need to critique them. They need to consider all possibilities and different aspects of the given problem" (H2).
		L	16	15	"Sophism and paradox tasks improve students' critical thinking. They need to give a reason for their judgment. I believe these tasks provide an opportunity for students to discover the relationships between mathematical concept(s)" (T1).
	Improving lateral thinking (*thinking outside the box*)	S	2	0	"Sophisms motivate students to look at the problems from different angles and use different approaches to solve them" (L1).
		L	8	8	"Sophisms and paradoxes challenge the mind, relate to various mathematical remarks, and require reasoning. Students should scrutinize the problem and look at the problem from different angles to evaluate the reasoning in the task" (T6).

Two students also suggested that SoPa tasks should be included in the teaching of mathematics at all levels. However, one noted that some students might not be interested in solving SoPa tasks:

> Depending on the characteristics of students, some are interested in solving sophism and paradox, and some are not. Those who want to master the topic are interested in solving them, and those who just focus on passing the course are not interested (H2).

Furthermore, three lecturers highlighted that using SoPa tasks will encourage students to follow the DEs with more interest and create opportunities for them to use their knowledge and skills at higher levels:

> Students need to evaluate all information and reasoning given in the task to verify or refute the reasoning in the task. In my opinion, engaging in these tasks can motivate students to follow the DEs topics with more interest. Additionally, sophism and paradox tasks are very useful for evaluating dissertations and articles. For example, sometimes we could find invalid reasoning in a published article, while the reasoning seems apparently true in the first read . . . (T16).

The negative perceptions of including SoPa tasks in the teaching and learning of DEs are categorized into four themes and presented in Table 7.

4.3. How SoPa Could Be Included in the Teaching of DEs

Fifteen lecturers (88%) and all students were unanimous in the fact that solving SoPa tasks, along with solving routine problems, helps students to consolidate their DEs knowledge and improve their conceptual understanding: "Using SoPa tasks in teaching can positively impact students' understanding. Including each DEs topic with SoPa tasks make students interested in learning the topic" (T14). Eleven lecturers (65%) suggested that SoPa tasks can be used to increase student participation in classroom discussions. Moreover, nine interviewed students (69%) highlighted that if students only solve routine problems, they learn only to apply procedures, and their opportunities for developing conceptual understanding are limited. Furthermore, it was believed that some of their misunderstandings could not be revealed by engaging in solving routine problems. A sample response was as follows:

> SoPa tasks can be used in classrooms along with routine problems. They lead to deep mathematical understanding and more attention to detail. Solving SoPa tasks helps students to develop their critical thinking, and they will learn not to accept anything without reason (M3).

Table 7. Disadvantages of SoPa tasks.

Themes	S\L	So	Pa	A Sample Response
Possibility of creating a mathematical misunderstanding or distracting students from learning mathematics	S	4	1	"If a student could not identify the wrong argument in a sophism, it could create a mathematical misunderstanding for the student" (L1).
	L	5	4	"If lecturers and students pay too much attention to sophism and paradox tasks, students may think that each task that they engage with has a trick and distract them from learning mathematics" (T8).
Lack of experience in solving SoPa tasks	S	2	2	"The teaching in our class is based on routine problems. Students do not have enough experience solving paradoxes, so there is a high possibility that students do not perform well in solving paradox tasks" (M2).
	L	6	6	"Students do not have enough experience in solving sophism and paradox tasks. Therefore, students' grades and their motivation to learn may decrease" (T15).
Time-consuming activities	S	0	1	"Finding the starting point for solving paradox tasks takes too much time" (M3).
	L	9	9	"Using these tasks is time-consuming. It can be used as long as we have the time to deal with these tasks in the classroom because it requires more discussion in the classroom" (T6).
Not appropriate for engineering students	S	1	0	"Sophisms are not appropriate for engineering students because in the problems we encounter in engineering, students can solve the problems with routine algorithms. . . . . I prefer to solve routine problems because I do not like challenging questions" (L4).
	L	0	0	

One student expressed that teaching SoPa tasks should be included not only at the tertiary level but also at primary and secondary levels:

> I believe these problems should be included from the primary level in order to help students develop their creative thinking and mathematical understanding (L2).

Sixteen lecturers (94%) believed that SoPa tasks should only be used in lectures, along with routine problems, instead of giving them to students as homework assignments. Their main reason for this was that they believed SoPa tasks could help students to think mathematically, and discussing them in lectures could avoid creating mathematical misunderstandings about the concept(s) for students. A sample response was as follows:

> It is better that first, the lecturer solves a few examples of SoPa tasks in the lecture to help students become familiar with such tasks. Then, these types of tasks can be given to students to solve in the lecture to increase students' participation. The lecturer should manage the lecture environment in a way that students feel safe to share their thoughts . . . I prefer to use these tasks in the lecture to have better control over students' thinking processes (T1).

However, of the thirteen students that were interviewed, five (H13-M45-L4) believed that SoPa tasks should be given as homework assignments. For instance, M5 said the following:

> SoPa tasks should be given to students as homework assignments in order to give students enough time to think about how they can solve them; then, students could share their solutions in tutorials.

4.4. Using SoPa Tasks in Assessments

All the lecturers except one (94%) believed that SoPa tasks could be used in assessments to evaluate students' DEs understanding: "Lecturers can assess the depth of students' knowledge and understanding of the topics by including SoPa tasks in the exams" (T1). Additionally, they believed that if this approach is used in the teaching and learning process, it should be included in the assessment; otherwise, students may not be interested in solving SoPa tasks.

However, the remaining lecturer disagreed with using SoPa tasks in the exam because the purpose of the exam is not to identify invalid reasoning: "SoPa tasks are not appropriate in exams, because exam questions should not have an educational trap ... " (T14).

Seven of the interviewed students (H24-M45-L123) suggested that if SoPa tasks were to be included in exams, they should be discussed in the classroom beforehand, and students should have had plenty of experience with solving them. All interviewed students except one believed that SoPa tasks could be given as exam questions because these questions encourage students to learn mathematics conceptually. One student disagreed with including SoPa tasks in exams:

> SoPa tasks are not suitable for assessments because solving them requires creativity and considering the problem from different angles. Only students who learned the lessons deeply are capable of solving them. Consequently, many students will fail to solve such problems and become disappointed about learning mathematics (L4).

5. Discussion and Conclusions

One of the core subjects of science and engineering is DEs [39]. DEs have applications in many disciplines, such as physics, mechanics, and electronics, to model real-world problems [9]. Active learning in mathematics involves engaging students in hands-on and collaborative activities that encourage them to explore mathematical concepts and solve problems, and prepare them to make meaningful decisions. This method could be effective in helping students to overcome their mathematics anxiety and increase their confidence and mathematical competency [5,54]. PzBL, as an active learning approach, to a great extent shares the same goals. Here, in this study, we explored mathematics lecturers' and engineering students' perceptions of using SoPa tasks to improve the teaching and learning of DEs.

This study contributes to the existing literature in that mathematics lecturers' and engineering students' perceptions of using SoPa tasks in relation to DEs have not been explored previously. In particular, no questionnaire was found about lecturers' and students' perceptions of using SoPa tasks, and therefore, for the first time, such a questionnaire was designed for this study. Furthermore, previous studies in mathematics education only used semi-structured interviews to explore students' attitudes toward and perceptions of using PzBL in general (not specifically SoPa tasks), and these were in calculus, e.g., [4]. Additionally, no study was found about lecturers' perceptions of using SoPa tasks in the teaching and learning of mathematics. The findings show that many lecturers and students perceived SoPa tasks as being entertaining and enjoyable activities that can improve mathematical understanding, problem-solving skills, and different types of thinking skills. In the following paragraphs, these findings are discussed in detail.

More than 50% of the lecturers and students who completed the SoPa questionnaire believed that SoPa tasks related to DEs were enjoyable and entertaining activities and could motivate students to learn DEs. Lecturers' and students' responses to the interview questions confirmed this finding. These findings support the idea that PzBL illustrates mathematical concepts in an entertaining way [16,55], as highlighted by Thomas et al. [5]: "Puzzles can provide additional challenges, insight, and entertainment, all of which can increase student engagement and promote independent learning" (p. 93).

The findings show that between 50% and 80% of the lecturers and students concurred that solving SoPa tasks related to DEs improves students' DEs understanding and problem-solving. Additionally, interview data corroborated this result. The students highlighted that by solving SoPa tasks, they learned new strategies that can help them to solve real-world problems that they may encounter in the future. This is consistent with previous studies about using PzBL in calculus [2,4,5] that found that puzzle problems help students to handle problems that they may come across in real life. PzBL motivates students to solve problems and learn mathematics, and encourages them to participate in classroom discussions [55]. The goal of PzBL is to "motivate students while increasing their mathematical awareness and problem-solving skills by discussing a variety of puzzles and their solution strategies" ([16], p. 23).

More than 60% of lecturers and students believed that solving SoPa tasks related to DEs helps to improve students' thinking skills. These findings are consistent with the results of lecturers' and students' responses to the interview questions. A strong relationship between thinking skills and PzBL has been reported in the literature [3]. Falkner et al. [2] mentioned that the aim of PzBL is "getting students to think about how to frame and solve unstructured problems" (p. 245). Lecturers' and students' responses showed that they believed that solving SoPa tasks is more useful for improving critical thinking and reasoning. A possible explanation for this result is that to solve SoPa tasks correctly, students should analyze the given information and evaluate the reasoning included in them, whereby these practices could improve students' critical thinking and active higher-order thinking [17].

All interviewed lecturers except one and all interviewed students were unanimous in that lecturers should discuss SoPa tasks along with routine problems in DEs lectures. Previous studies in relation to PzBL in calculus also suggested that puzzle problems can be used alongside routine problems in the teaching of mathematics [3,5]. For instance, Thomas et al. [5] highlighted that "embedding puzzle-problems in the teaching of other subjects enhances students' learning by developing their problem-solving and independent learning skills, whilst increasing their motivation to learn mathematics" (p. 122). Additionally, our findings suggest that many lecturers and students agreed that SoPa tasks could be included as an assessment tool to explore students' conceptual understanding, creativity, and critical thinking skills.

In summary, while this study worked with volunteers, and the views of lecturers and students who shied away from volunteering may be different, we believe that the findings of this study suggest that there would be advantages to using SoPa tasks in the teaching, learning, and assessment of DEs. In particular, in collaborative learning situations, such as those used in this study, SoPa tasks could help struggling and engaging students to identify the misconceptions that they might hold, support their development of conceptual understanding, and improve their ability to use their knowledge in unfamiliar situations. Including SoPa tasks in assessments would ensure that all students engage with these tasks at a deep level, though the impact of this on the length and style of assessment would need to be carefully considered. Perhaps an internally assessed collaborative activity would be the most appropriate format.

This study is not without limitations. The sample was chosen from an area that was geographically accessible to the authors. Furthermore, care should be taken when interpreting the findings because convenience sampling was used in this study. In addition, only volunteer students and lecturers participated; therefore, the findings might not represent the perceptions of all students and lecturers. Further studies are required to confirm the study findings. We encourage tertiary mathematics educators to design SoPa tasks in other mathematical domains and investigate how students perceive engaging with such tasks.

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Appendix A

In the following, the tasks are described in some detail to familiarize readers with them. The first task is a sophism as it has an argument that seems to be correct, but contains an error. The purpose of this task was to explore students' conceptual understanding of exact DEs and students' critical and analytical thinking skills. Students needed to realize that to convert a non-exact DE into an exact DE, the DE must be multiplied by an integrating factor. Furthermore, a common factor is not necessarily an integrating factor. If it is an integrating factor, its elimination impacts the exactness of the DE.

Task 2 is a paradox as it includes a claim that seems to be incorrect to students, but in fact is correct. A number of students might think that a DE only has one integrating factor. Furthermore, some students might think that because two different general solutions are found for the DE, one of the solutions must be incorrect, as by definition the general solution of a DE is unique. Students who have well understood the concepts and rules related to the exact DEs, by evaluating the responses, could realize that both general solutions are correct, and only differ in the value of the constant (i.e., c_1 and c_2).

1. Verify the following statement. Please explain your reasons.

"Factoring out a common factor and its elimination from a differential equation (DE) does not impact the exactness of a DE".
For example, a DE
$$-\frac{1}{y}\sin\frac{x}{y}\,dx + \frac{x}{y^2}\,\sin\frac{x}{y}\,dy = 0 \tag{A1}$$
is exact because $M_y = \frac{1}{y^2}\,\sin\frac{x}{y} + \frac{x}{y^3}\,\cos\frac{x}{y} = N_x$. If we factor out $\sin\frac{x}{y}$ in (A1) and then eliminate it, we have, respectively,
$$\sin\frac{x}{y}\left(-\frac{1}{y}\,dx + \frac{x}{y^2}\,dy\right) = 0$$
and
$$-\frac{1}{y}\,dx + \frac{x}{y^2}\,dy = 0. \tag{A2}$$
The DE (A2) is still exact because $M_y = \frac{1}{y^2} = N_x$. Consider now a DE
$$e^{x+y}\left(x^2+y^2\right)dx + e^{x+y}\,\left(x^2+y^3\right)dy = 0. \tag{A3}$$
Equation (A3) is not exact because
$M_y = e^{x+y}\,\left(x^2+y^2\right) + 2\,y\,e^{x+y} \neq\; e^{x+y}\,\left(x^2+y^3\right) + 2\,x\,e^{x+y} = N_x$. If we factor out e^{x+y} in (A3) and eliminate it, we have, respectively,
$$e^{x+y}\left(\left(x^2+y^2\right)dx + \left(x^2+y^3\right)dy\right) = 0$$
and
$$\left(x^2+y^2\right)dx + \left(x^2+y^3\right)dy = 0. \tag{A4}$$
This new DE (A4) is also not exact because
$$M_y = 2\,y \neq 2\,x = N_x.$$
Thus, factoring out and eliminating a common factor does not impact the exactness of a DE.

2. Reza, Ali, and Ehsan decided to study together for a DEs exam. Ehsan asked his friends how a DE

$$2ydx + xdy = 0, \quad (x, y > 0) \tag{A5}$$
can be solved with an integrating factor. Reza and Ali separately solved Equation (A5) for him. Based on their responses, Ehsan concluded that this DE has two integrating factors and both functions defined implicitly by equations $yx^2 = c_2$ and $2\,x\,\sqrt{y} = c_1$ are general solutions. Is this possible? Justify your answer.

Reza's solution:
$$\frac{N_x - M_y}{M} = -\frac{1}{2\,y} \quad\Rightarrow\quad \mu(y) = e^{-\int \frac{1}{2\,y}\,dy} = \frac{1}{\sqrt{y}}\,.$$
Now, we multiply the DE by the integrating factor, and the new DE
$$2\,\sqrt{y}\,dx + \frac{x}{\sqrt{y}}\,dy = 0 \tag{A6}$$
is exact because $M_y = \frac{1}{\sqrt{y}} = N_x$. We can solve (A6) using the standard method:
$$\int 2\,\sqrt{y}\,dx = 2\,x\sqrt{y} + Q(y).$$
Differentiation with respect to y yields
$$\frac{2\,x}{2\,\sqrt{y}} + Q'(y) = \frac{x}{\sqrt{y}} \quad\Rightarrow\quad Q'(y) = 0$$
and we set $Q(y) = 0$. Therefore, $F(x,\,y) = 2\,x\sqrt{y}$ and $2x\sqrt{y} = c_1$ is the general solution of the given DE.

Ali's solution:
$$\frac{M_y - N_x}{N} = \frac{1}{x} \;\Rightarrow\; \mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$
Multiply (A5) by the integrating factor, then the new DE
$$2yx\,dx + x^2\,dy = 0 \tag{A7}$$
is exact because $M_y = 2\,x = N_x$. We solve (A7) using the standard method:
$$\int 2yx\,dx = yx^2 + Q(y)\,.$$
Differentiate the result with respect to y :
$$x^2 + Q'(y) = x^2 \quad\Rightarrow\quad Q'(y) = 0$$
and we set $Q(y) = 0$. Therefore, $F(x,\,y) = yx^2$, and $yx^2 = c_2$ is the general solution of the given DE.

References

1. Winberg, C.; Adendorff, H.; Bozalek, V.; Conana, H.; Pallitt, N.; Wolff, K.; Olsson, T.; Roxå, T. Learning to teach STEM disciplines in higher education: A critical review of the literature. *Teach. High. Educ.* **2019**, *24*, 930–947. [CrossRef]
2. Falkner, N.; Sooriamurthi, R.; Michalewicz, Z. Teaching Puzzle-based Learning: Development of Transferable Skills. *J. Teach. Math. Comput. Sci.* **2012**, *10*, 245–268. [CrossRef]
3. Michalewicz, Z.; Michalewicz, M. *Puzzle-Based Learning*; Hybrid Publishers: Melbourne, VIC, Australia, 2008.
4. Freeman, S.; Eddy, S.; McDonough, M.; Smith, M.; Okoroafor, N.; Jordt, H.; Wenderoth, M. Active learning increases student performance in science, engineering, and mathematics. *Proc. Natl. Acad. Sci. USA* **2014**, *111*, 8410–8415. [CrossRef]
5. Lugosi, E.; Uribe, G. Active learning strategies with positive effects on students' achievements in undergraduate mathematics education. *Int. J. Math. Educ. Sci. Technol.* **2022**, *53*, 403–424. [CrossRef]
6. Lambros, A. *Problem-Based Learning in K-8 Classrooms: A Teacher's Guide to Implementation*; Corvin Press, Inc.: Thousand Oaks, CA, USA, 2002.
7. Klymchuk, S. Puzzle-based learning in engineering mathematics: Students' attitudes. *Int. J. Math. Educ. Sci. Technol.* **2017**, *48*, 1106–1119. [CrossRef]
8. Thomas, C.; Badger, M.; Ventura-Medina, E.; Sangwin, C. Puzzle-based learning of mathematics in engineering. *Eng. Educ.* **2013**, *8*, 122–134. [CrossRef]
9. Arslan, S. Traditional instruction of differential equations and conceptual learning. *Teach. Math. Its Appl. Int. J. IMA* **2010**, *29*, 94–107. [CrossRef]
10. Rasmussen, C.L. New directions in differential equations: A framework for interpreting students' understandings and difficulties. *J. Math. Behav.* **2001**, *20*, 55–87. [CrossRef]
11. Klymchuk, S.; Staples, S.G. *Paradoxes and Sophisms in Calculus*; MAA: Washington, DC, USA, 2013; Volume 45.
12. Yew, E.H.; Goh, K. Problem-based learning: An overview of its process and impact on learning. *Health Prof. Educ.* **2016**, *2*, 75–79. [CrossRef]
13. Barrows, H.S. Problem-based learning in medicine and beyond: A brief overview. *New Dir. Teach. Learn.* **1996**, *68*, 3–12. [CrossRef]
14. Hmelo-Silver, C.E. Problem-based learning: What and how do students learn? *Educ. Psychol. Rev.* **2004**, *16*, 235–266. [CrossRef]
15. Capon, N.; Kuhn, D. What's so good about problem-based learning? *Cogn. Instr.* **2004**, *22*, 61–79. [CrossRef]
16. Dochy, F.; Segers, M.; Van den Bossche, P.; Gijbels, D. Effects of problem-based learning: A meta-analysis. *Learn. Instr.* **2003**, *13*, 533–568. [CrossRef]
17. Michalewicz, Z.; Falkner, N.; Sooriamurthi, R. Puzzle-based learning: An introduction to critical thinking and problem solving. *Decis. Line* **2011**, *42*, 6–9.
18. Radmehr, F.; Vos, P. Issues and challenges for 21st century assessment in mathematics education. In *Science and Mathematics Education for 21st Century Citizens: Challenges and Ways Forwards*; Leite, L., Oldham, E., Afonso, A.S., Viseu, F., Dourado, L., Martinho, H., Eds.; Nova Science Publishers: New York, NY, USA, 2020; pp. 437–462.
19. Falkner, N.; Sooriamurthi, R.; Michalewicz, Z. Puzzle-Based Learning for Engineering and Computer Science. *IEEE Comput.* **2010**, *43*, 20–28. [CrossRef]
20. Parhami, B. A puzzle-based seminar for computer engineering freshmen. *Comput. Sci. Educ.* **2008**, *18*, 261–277. [CrossRef]
21. Ramalingam, D.; Anderson, P.; Duckworth, D.; Scoular, C.; Heard, J. *Creative Thinking: Definition and Structure*; The Australian Council for Educational Research: Camberwell, VIC, Australia, 2020.
22. Maričića, S.; Špijunović, K. Developing Critical Thinking in Elementary Mathematics Education through a Suitable Selection of Content and Overall Student Performance. *Procedia—Soc. Behav. Sci.* **2020**, *180*, 653–659. [CrossRef]
23. Aydin, I.E. Attitudes toward online communications in open and distance learning. *Turk. Online J. Distance Educ.* **2012**, *13*, 333–346.
24. Pickens, J. Attitudes and perceptions. *Organ. Behav. Health Care* **2005**, *4*, 43–76.
25. Nedaei, M.; Radmehr, F.; Drake, M. Exploring engineering undergraduate students' attitudes toward mathematical problem posing. *J. Prof. Issues Eng. Educ. Pract.* **2019**, *145*, 04019009. [CrossRef]
26. Sarouphim, K.M.; Chartouny, M. Mathematics education in Lebanon: Gender differences in attitudes and achievement. *Educ. Stud. Math.* **2017**, *94*, 55–68. [CrossRef]
27. Byers, T.; Imms, W.; Hartnell-Young, E. Comparative analysis of the impact of traditional versus innovative learning environment on student attitudes and learning outcomes. *Stud. Educ. Eval.* **2018**, *58*, 167–177. [CrossRef]
28. Ellis, J.; Kelton, M.L.; Rasmussen, C. Student perceptions of pedagogy and associated persistence in calculus. *ZDM* **2014**, *46*, 661–673. [CrossRef]
29. Attard, C. Engagement with Mathematics: What Does It Mean and What Does It Look Like? *Aust. Prim. Math. Classr.* **2012**, *17*, 9–13.
30. Flegg, J.; Mallet, D.; Lupton, M. Students' perceptions of the relevance of mathematics in engineering. *Int. J. Math. Educ. Sci. Technol.* **2012**, *43*, 717–732. [CrossRef]
31. Perdigones Borderias, A.; Gallego Vazquez, E.; Garcia Garcia, M.N.; Fernandez Alvarez, P.; Perez Martin, E.; Cerro Carrascosa, J.D. Physics and mathematics in the engineering curriculum: Correlation with applied subjects. *Int. J. Eng. Educ.* **2014**, *30*, 1509–1521.
32. Hamzeh, E. Lebanese Middle School Students' Attitudes toward Mathematics as a Subject and toward Mathematics Teachers. Unpublished. Master's Thesis, Lebanese American University, Beirut, Lebanon, 2009.

33. Klingler, K.L. Mathematic Strategies for Teaching Problem Solving: The Influence of Teaching Mathematical Problem Solving Strategies on Students' Attitudes in Middle School. Un-published. Master's Thesis, Central Florida University, Orlando, FL, USA, 2012.
34. Merrick, K.E. An empirical evaluation of puzzle-based learning as an interest approach for teaching introductory computer science. *IEEE Trans. Educ.* **2010**, *53*, 677–680. [CrossRef]
35. Czocher, J.A. How can emphasizing mathematical modeling principles benefit students in a traditionally taught differential equations course? *J. Math. Behav.* **2017**, *45*, 78–94. [CrossRef]
36. Keene, K.A. A characterization of dynamic reasoning: Reasoning with time as parameter. *J. Math. Behav.* **2007**, *26*, 230–246. [CrossRef]
37. Kwon, O.N.; Rasmussen, C.; Allen, K. Students' retention of mathematical knowledge and skills in differential equations. *Sch. Sci. Math.* **2005**, *105*, 227–239. [CrossRef]
38. Beier, J.C.; Gevertz, J.L.; Howard, K.E. Building context with tumor growth modeling projects in differential equations. *PRIMUS* **2015**, *25*, 297–325. [CrossRef]
39. Maat, S.M.; Zakaria, E. Exploring Students' Understanding of Ordinary Differential Equations Using Computer Algebraic System (CAS). *Turk. Online J. Educ. Technol.-TOJET* **2011**, *10*, 123–128.
40. Creswell, J. *Research Design: Qualitative, Quantitative, and Mixed Methods Approaches*, 4th ed.; SAGE Publication Inc.: Thousand Oaks, CA, USA, 2014.
41. Reiter, B. Theory and methodology of exploratory social science research. *Int. J. Sci. Res. Methodol.* **2017**, *5*, 129–150.
42. Badger, M.; Sangwin, C.; Ventura-Medina, E.; Thomas, C. *A Guide to Puzzle-Based Learning in STEM Subjects*. Available online: https://www.maths.ed.ac.uk/~csangwin/Publications/GuideToPuzzleBasedLearningInSTEM.pdf (accessed on 19 March 2023).
43. McDonald, J.H. *Handbook of Biological Statistics*; Sparky House Publishing: Baltimore, MD, USA, 2009; Volume 2, pp. 6–59.
44. Kennedy, B.L.; Thornberg, R. Deduction, Induction, and Abduction. In *The SAGE Handbook of Qualitative Data Collection*; Flick, U., Ed.; SAGE Publications Ltd.: Thousand Oaks, CA, USA, 2018; pp. 49–64.
45. Vaismoradi, M.; Turunen, H.; Bondas, T. Content analysis and thematic analysis: Implications for conducting a qualitative descriptive study. *Nurs. Health Sci.* **2013**, *15*, 398–405. [CrossRef] [PubMed]
46. Bolarinwa, O.A. Principles and methods of validity and reliability testing of questionnaires used in social and health science researches. *Niger. Postgrad. Med. J.* **2015**, *22*, 195–201. [CrossRef]
47. Lei, P.W.; Wu, Q. Introduction to structural equation modeling: Issues and practical considerations. *Educ. Meas. Issues Pract.* **2007**, *26*, 33–43. [CrossRef]
48. Hu, L.T.; Bentler, P.M. Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Struct. Equ. Model. A Multidiscip. J.* **1999**, *6*, 1–55. [CrossRef]
49. Browne, M.W.; Cudeck, R. Alternative ways of assessing model fit. In *Testing Structural Equation Models*; Bollen, K.A., Long, J.S., Eds.; Sage: Newbury Park, CA, USA, 1993; pp. 136–162.
50. Kilic, A.F.; Doğan, N. Comparison of confirmatory factor analysis estimation methods on mixed-format data. *Int. J. Assess. Tools Educ.* **2021**, *8*, 21–37. [CrossRef]
51. Boudreau, M.C.; Gefen, D.; Straub, D.W. Validation in information systems research: A state-of-the-art assessment. *MIS Q.* **2001**, *25*, 1–16. [CrossRef]
52. Tracy, S.J. Qualitative quality: Eight "big-tent" criteria for excellent qualitative research. *Qual. Inq.* **2010**, *16*, 837–851. [CrossRef]
53. Rezvanifard, F.; Radmehr, F.; Rogovchenko, Y. Advancing engineering students' conceptual understanding through puzzle-based learning: A case study with exact differential equations. *Math. Its Appl. Int. J. IMA* **2022**, 1–24. [CrossRef]
54. Ramirez, G.; Shaw, S.T.; Maloney, E.A. Math anxiety: Past research, promising interventions, and a new interpretation framework. *Educ. Psychol.* **2018**, *53*, 145–164. [CrossRef]
55. Parhami, B. Motivating computer engineering freshmen through mathematical and logical puzzles. *IEEE Trans. Educ.* **2009**, *52*, 360–364, Appendix: The sophism and paradox. [CrossRef]