Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Riesz bases of exponentials for finite unions of intervals

Master's thesis in Mathematical Sciences
Supervisor: Sigrid Grepstad
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Abstract

In this thesis we study Riesz bases of exponential functions for spaces $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$. Our main focus will be on proving Kozma and Nitzan's result that given a finite union of intervals $\Omega \subset [0,1]$ we can construct a Riesz basis of exponentials $E(\Lambda)$ with integer frequencies for the space $L^2(\Omega)$. We also study certain stability and density results pertaining to Riesz bases of exponentials, such as the Paley–Wiener stability theorem, Kadec's $\frac{1}{4}$-Theorem, and Landau's necessary density conditions for Riesz bases of exponentials.

Sammendrag

I denne oppgaven studerer vi Riesz-basiser av eksponentialfunksjoner for rommet $L^2(\Omega)$, der $\Omega \subset \mathbb{R}^d$. Hovedfokuset i oppgaven er å presentere Kozma og Nitzan sitt bevis for at det finnes en Riesz-basis av eksponentialfunksjoner med heltallsfrekvenser i $L^2(\Omega)$ for enhver endelig union av intervaller $\Omega \subset [0,1]$. Vi ser også nærmere på enkelte stabilitets- og tetthetsresultater for eksponentielle Riesz-basiser, og gir detaljerte bevis for Paley–Wieners stabilitetsteorem, Kadec's $\frac{1}{4}$-Teorem og Landaus tetthetsbetingelser for eksponentielle Riesz-basiser.

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Contents

1. Introduction

In this thesis we study spanning properties of the exponential system

$$E(\Lambda) = \{e_\lambda(x) = \exp(2\pi i\langle\lambda, x\rangle) : \lambda \in \Lambda\}$$

in the space $L^2(\Omega)$, where Ω is a subset of $\mathbb{R}^d$ and $\Lambda \subset \mathbb{R}^d$ is a uniformly discrete set. This is a classical topic dating back to the work of Paley and Wiener in the 1930s [13]. In particular, we are interested in when $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$. This question may be rephrased as asking when Λ is both a set of stable sampling and interpolation in PW_Ω. The case where Ω is an interval is well-studied, and a full characterization of Riesz bases for $L^2(I)$ has been given by Hruščev, Nikol'skii and Pavlov [3]. However, in the case of disconnected sets Ω and sets in $\mathbb{R}^d$ for $d > 1$, many question have remained open.

A Riesz basis is the image of an orthogonal basis under a linear, bounded and invertible map. Thus a Riesz basis is not required to be orthogonal, but the close connection to orthogonal bases ensures that certain properties of orthogonal bases are preserved. In particular, if $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$, then every function $f \in L^2(\Omega)$ has a unique series expansion

$$f(x) = \sum_{\lambda\in\Lambda} c_\lambda e_\lambda(x).$$

Moreover, relinquishing the property of orthogonality makes Riesz bases more flexible than orthogonal bases. In particular, Riesz bases are stable under small perturbations. Thus we can think of Riesz bases of exponentials as a great alternative to orthogonal bases of exponentials for spaces that do not admit the latter.

It is well-known that there are many sets Ω where $L^2(\Omega)$ does not admit an orthogonal basis of exponentials. For suppose that it does; then it must be the case that the Fourier transform of the indicator function of Ω satisfies $\widehat{\chi_\Omega}(\lambda - \alpha) = 0$ for any pair λ, α where $\lambda \neq \alpha$. This condition is very restrictive. Suppose for instance that $\Omega = [0, 1] \cup [3/2, 2]$. Then

$$\widehat{\chi_\Omega}(x) = \frac{1 - e^{-2\pi ix} + e^{-4\pi ix} - e^{-3\pi ix}}{2\pi ix},$$

and it follows that $\widehat{\chi_\Omega}(x) = 0$ if and only if $x \in 2\mathbb{Z}$. Thus if $L^2(\Omega)$ were to have an orthogonal basis of exponentials it must be a subset of $\{e^{2\pi i(2n)x}\chi_\Omega(x)\}_{n\in\mathbb{Z}}$, which is clearly impossible. On the other hand, it is easily verified that $L^2([0, 1] \cup [3/2, 2])$ admits a Riesz basis of exponentials, with $E((1/2)\mathbb{Z} \setminus 2\mathbb{Z})$ being one possible choice.

Over the last years there have been several big breakthroughs on the topic of Riesz bases of exponentials. An important problem which remained open for decades was that of whether any finite union of intervals admits a Riesz basis of exponentials. In 2012 Kozma and Nitzan [6] were finally able to verify this. They later generalized their result to any finite union of rectangles in $\mathbb{R}^d$ [7]. Another important question which just recently saw a solution was that of whether *any* space $L^2(\Omega)$ admits a Riesz basis of exponentials. It turns out that this is not the case, as Kozma, Nitzan and Olevskii constructed a bounded set $\Omega \subset \mathbb{R}$ where no Riesz basis of exponentials for $L^2(\Omega)$ exists [8].

The main goal in this thesis is to understand the proof due to Kozma and Nitzan that any finite union of intervals Ω admits a Riesz basis of exponentials. The thesis is organized as follows. We start by introducing Riesz bases, sets of stable sampling and interpolation, and the Paley–Wiener space in section 2. We then review the Paley–Wiener stability theorem, and Kadec's $\frac{1}{4}$-theorem in section 3, as stability of Riesz bases of exponentials plays a fundamental role in the proof of Kozma and Nitzan. In section 4 we look at some important density results, originally due to Landau [9], which describe necessary conditions on Λ for $E(\Lambda)$ to serve as a Riesz basis. Finally, in section 5 we study Kozma and Nitzan's proof in detail, and observe that the main idea in the proof is to combine Riesz bases for several smaller, related spaces into one Riesz basis for $L^2(\Omega)$ in a particularly clever way.

2. Preliminaries

In this thesis we mainly work with the Hilbert space $L^2(\Omega)$ for some set $\Omega \subset \mathbb{R}^d$. Nevertheless, we begin by formulating certain definitions and results in the more general context of a Hilbert space $\mathcal{H}$. We will always assume that $\mathcal{H}$ is separable, and thus (if infinite-dimensional) isometrically isomorphic to ℓ^2. In particular we will always have a countable orthonormal basis for $\mathcal{H}$. The main sources in this section are the books of Christensen [2] and Olevskii and Ulanovskii [11].

2.1. **Frames, Riesz sequences and Riesz bases.** Given a separable Hilbert space $\mathcal{H}$ we say that a set $\{f_n\}_{n\in\mathbb{Z}}$ of elements in $\mathcal{H}$ is a frame if there exist positive constants A and B such that

$$A\|\varphi\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle \varphi, f_n\rangle|^2 \leq B\|\varphi\|^2,$$

for all $\varphi \in \mathcal{H}$. We will refer to the largest possible constant A and the smallest possible constant B as the lower and upper frame bound, respectively. The upper inequality is often referred to as Bessel's inequality. An important consequence of the lower frame bound is that frames are complete, meaning that $\overline{\text{span}}\{f_n\}_{n\in\mathbb{Z}} = \mathcal{H}$. This is equivalent to the property that only the zero-function is orthogonal to every function in $\{f_n\}_{n\in\mathbb{Z}}$. We state this consequence as a lemma and give the short proof.

Lemma 2.1. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame in a separable Hilbert space $\mathcal{H}$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is complete in $\mathcal{H}$.*

Proof. Assume that there exists a function $\varphi \in \mathcal{H}$ such that $\langle \varphi, f_n\rangle = 0$ for all $n \in \mathbb{Z}$. Then since $\{f_n\}_{n\in\mathbb{Z}}$ is a frame there exists some $A > 0$ such that

$$A\|\varphi\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle \varphi, f_n\rangle|^2 = 0$$

which implies that $\|\varphi\| = 0$, and hence $\varphi = 0$. □

We say that a frame $\{g_n\}_{n\in\mathbb{Z}}$ is a dual frame for $\{f_n\}_{n\in\mathbb{Z}}$ if

$$\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, g_n\rangle f_n,$$

for any $\varphi \in \mathcal{H}$. There always exists at least one dual frame, called the canonical dual frame, given by $\{T^{-1}f_n\}_{n\in\mathbb{Z}}$, where $T : \mathcal{H} \to \mathcal{H}$ is the frame operator

$$T\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, f_n\rangle f_n.$$

This operator is linear, bounded and invertible. The canonical dual has frame bounds $1/B$ and $1/A$ if the frame bounds of $\{f_n\}_{n\in\mathbb{Z}}$ are A and B. Observe that the canonical dual frame of $\{T^{-1}f_n\}_{n\in\mathbb{Z}}$ is $\{f_n\}_{n\in\mathbb{Z}}$. Note also that the existence of the canonical dual frame guarantees that we can express any element as a linear combination of the frame functions. However, as the dual frame need to be unique, the series expansion is in general not unique.

Given a separable Hilbert space $\mathcal{H}$ and a set $\{f_n\}_{n\in\mathbb{Z}}$ of elements in $\mathcal{H}$, we say that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence if there exist positive constants A and B such that

$$A\sum |c_n|^2 \leq \left\| \sum c_n f_n \right\|^2 \leq B\sum |c_n|^2, \tag{2.1}$$

for all finite sequences $\{c_n\}_{n\in\mathbb{Z}}$. By a finite sequence we mean that finitely many elements in the sequence are non-zero. As for frames, we call the largest possible constant A in (2.1) the lower Riesz sequence bound, and the smallest possible constant B in (2.1) the upper Riesz sequence bound for $\{f_n\}_{n\in\mathbb{Z}}$. We observe that there is a certain duality between the property of being a frame and that of being a Riesz sequence.

Theorem 2.2. *Let $\mathcal{H}$ be a separable Hilbert space which is a direct sum of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $U = \{u_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$. Assume U is the union of two disjoint sets V and W. Let P_1 and P_2 denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then P_1V is a frame in $\mathcal{H}_1$ if and only if P_2W is a Riesz sequence in $\mathcal{H}_2$.*

Proof. Assume P_2W is a Riesz sequence in $\mathcal{H}_2$. That is, there exist constants A and B such that

$$A\sum_{w\in W}|c_w|^2 \leq \left\|\sum_{w\in W} c_w w\right\|_{\mathcal{H}_2}^2 = \left\|\sum_{w\in W} c_w P_2 w\right\|_{\mathcal{H}}^2 \leq B\sum_{w\in W}|c_w|^2,$$

for all finite sequences $\{c_w\}_{w\in W}$. Let $f\in\mathcal{H}_1$. Then we may write

$$f = \sum_{u\in U} c_u u = \sum_{v\in V} c_v v + \sum_{w\in W} c_w w.$$

Note that $P_2 f = 0$, and thus it follows that

$$\left\|\sum_{v\in V} c_v P_2 v\right\|_{\mathcal{H}}^2 = \left\|\sum_{w\in W} c_w w\right\|_{\mathcal{H}_2}^2.$$

Further we see that

$$\sum_{w\in W}|c_w|^2 \leq \frac{1}{A}\left\|\sum_{w\in W} c_w w\right\|_{\mathcal{H}_2}^2 = \frac{1}{A}\left\|\sum_{v\in V} c_v P_2 v\right\|_{\mathcal{H}}^2 \leq \frac{1}{A}\sum_{v\in V}|c_v|^2 = \frac{1}{A}\sum_{v\in V}|\langle f, v\rangle|^2.$$

Now it follows that

$$\|f\|_{\mathcal{H}_1}^2 = \sum_{u\in U}|c_u|^2 = \sum_{v\in V}|c_v|^2 + \sum_{w\in W}|c_w|^2 \leq \left(1+\frac{1}{A}\right)\sum_{v\in V}|\langle f, v\rangle|^2.$$

This proves the lower frame bound, as $\langle f, v\rangle = \langle f, P_1 v\rangle$ for all $v\in V$. The upper frame bound follows from the fact that

$$\sum_{v\in V}|\langle f, v\rangle|^2 \leq \sum_{u\in U}|\langle f, u\rangle|^2 = \|f\|_{\mathcal{H}_1}^2.$$

The proof of the converse is similar, and relies on the fact that given a function $f = \sum_{w\in W} c_w w = f_1 + f_2$, we have $\langle f_1, v\rangle = -\langle f_2, v\rangle$ for all $v\in V$. A detailed proof is given in [11, Proposition 1.23]. □

Given a separable Hilbert space $\mathcal{H}$ and a set $\{f_n\}_{n\in\mathbb{Z}}$ of elements in $\mathcal{H}$ we say that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$ if $\{f_n\}_{n\in\mathbb{Z}}$ is equivalent to an orthonormal basis $\{\beta_n\}_{n\in\mathbb{Z}}$. By equivalent we mean that there exists a linear, bounded and invertible map $U:\mathcal{H}\to\mathcal{H}$ such that $U(\beta_n) = f_n$ for all $n\in\mathbb{Z}$. Given such an operator U and a Riesz basis $\{f_n\}_{n\in\mathbb{Z}}$ we call $\{(U^{-1})^*\beta_n\}_{n\in\mathbb{Z}}$ the dual Riesz basis of $\{f_n\}_{n\in\mathbb{Z}}$. By the notation $(U^{-1})^*$ we mean the adjoint of U^{-1}. Note that the dual Riesz basis is clearly also a Riesz basis in $\mathcal{H}$, as $(U^{-1})^*$ is a bounded and invertible map whenever U is. The following theorem illustrates how we can use the dual Riesz basis to find the series expansion of any function $\varphi\in\mathcal{H}$ in terms of the Riesz basis.

Theorem 2.3. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Riesz basis in a separable Hilbert space $\mathcal{H}$ and let $\{g_n\}_{n\in\mathbb{Z}}$ be its dual Riesz basis. Then for any function $\varphi \in \mathcal{H}$ we have*

$$\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, g_n \rangle f_n = \sum_{n\in\mathbb{Z}} \langle \varphi, f_n \rangle g_n.$$

Proof. Fix $\varphi \in \mathcal{H}$. Then

$$U^*\varphi = \sum_{n\in\mathbb{Z}} \langle U^*\varphi, \beta_n \rangle \beta_n = \sum_{n\in\mathbb{Z}} \langle \varphi, f_n \rangle \beta_n,$$

and thus it follows that

$$\varphi = (U^*)^{-1}U^*\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, f_n \rangle g_n.$$

Similarly,

$$U^{-1}\varphi = \sum_{n\in\mathbb{Z}} \langle U^{-1}\varphi, \beta_n \rangle \beta_n = \sum_{n\in\mathbb{Z}} \langle \varphi, g_n \rangle \beta_n,$$

and it follows that

$$\varphi = \sum_{n\in\mathbb{Z}} \langle \varphi, g_n \rangle f_n.$$

□

There are several equivalent definitions of Riesz bases. It is possible to show that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$ if and only if the map $\Psi : \mathcal{H} \to \ell^2(\mathbb{Z})$ given by $\Psi(\varphi) = \{\langle \varphi, f_n \rangle\}_{n\in\mathbb{Z}}$ is bounded and invertible. Another useful description of a Riesz basis is given below.

Theorem 2.4. *Let $\mathcal{H}$ be a separable Hilbert space and a let $\{f_n\}_{n\in\mathbb{Z}}$ be a set of elements in $\mathcal{H}$. Then $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis in $\mathcal{H}$ if and only if it is both a Riesz sequence and a frame.*

Proof. We follow Christensen [2, Lemma 3.6.5 and Theorem 3.6.6]. We start by assuming that $\{f_n\}_{n\in\mathbb{Z}}$ is both a Riesz sequence and a frame. Since $\{f_n\}_{n\in\mathbb{Z}}$ is a frame it follows that $\{f_n\}_{n\in\mathbb{Z}}$ is complete in $\mathcal{H}$. Choose $\varphi \in \operatorname{span}\{f_n\}_{n\in\mathbb{Z}}$. Then we may write

$$\varphi = \sum_{n\in\mathbb{Z}} c_n f_n,$$

where $\{c_n\}_{n\in\mathbb{Z}}$ is a finite sequence. This expression must be unique, for suppose we have two finite sequences $\{a_n\}_{n\in\mathbb{Z}}$ and $\{c_n\}_{n\in\mathbb{Z}}$ such that

$$\varphi = \sum_{n\in\mathbb{Z}} c_n f_n = \sum_{n\in\mathbb{Z}} a_n f_n.$$

Then, since $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence, there exists a constant $A > 0$ such that

$$A \sum_{n\in\mathbb{Z}} |a_n - c_n|^2 \le \left\| \sum_{n\in\mathbb{Z}} (a_n - c_n) f_n \right\|^2 = 0,$$

and it follows that $c_n = a_n$ for all $n \in \mathbb{Z}$.

Let $\{\beta_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$, and define the linear map $U : \operatorname{span}\{\beta_n\}_{n\in\mathbb{Z}} \to \mathcal{H}$ by

$$U\left(\sum_{n\in\mathbb{Z}} c_n \beta_n\right) = \sum_{n\in\mathbb{Z}} c_n f_n,$$

for all finite sequences $\{c_n\}_{n\in\mathbb{Z}}$. Let B be the upper Riesz sequence bound of $\{f_n\}_{n\in\mathbb{Z}}$. We see that

$$\|U\| = \sup_{\sum |c_n|^2=1} \left\| U\left(\sum_{n\in\mathbb{Z}} c_n\beta_n\right)\right\| = \sup_{\sum |c_n|^2=1} \left\|\sum_{n\in\mathbb{Z}} c_n f_n\right\| \leq \sqrt{B}.$$

It follows that U is bounded and $\|U\| \leq \sqrt{B}$. Since $\overline{\operatorname{span}}\{\beta_n\}_{n\in\mathbb{Z}} = \mathcal{H}$ we can extend the operator U to a bounded operator on $\mathcal{H}$.

Finally, we show that U is invertible by determining the inverse operator. Let $V : \operatorname{span}\{f_n\}_{n\in\mathbb{Z}} \to \mathcal{H}$ be the linear map given by

$$V\left(\sum_{n\in\mathbb{Z}} c_n f_n\right) = \sum_{n\in\mathbb{Z}} c_n\beta_n,$$

for any finite sequence $\{c_n\}_{n\in\mathbb{Z}}$. The operator V is bounded with $\|V\| \leq 1/\sqrt{A}$, where A is the lower Riesz sequence bound of $\{f_n\}_{n\in\mathbb{Z}}$. Since $\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{Z}} = \mathcal{H}$ we can extend the operator V to a bounded operator on $\mathcal{H}$ as well. We see that V is the inverse of U, and thus it follows that the operator U on $\mathcal{H}$ is invertible. This concludes the proof that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis.

For the converse suppose $\{f_n\}_{n\in\mathbb{Z}}$ is equivalent to the orthonormal basis $\{\beta_n\}_{n\in\mathbb{Z}}$ under the linear, bounded and invertible map U. We start by showing that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence. Let $\{c_n\}_{n\in\mathbb{Z}}$ be a finite sequence. Then

$$\left\|\sum_{n\in\mathbb{Z}} c_n f_n\right\|^2 = \left\|U\left(\sum_{n\in\mathbb{Z}} c_n\beta_n\right)\right\|^2 \leq \|U\|^2 \left\|\sum_{n\in\mathbb{Z}} c_n\beta_n\right\|^2 = \|U\|^2 \sum_{n\in\mathbb{Z}} |c_n|^2.$$

Further, we see that

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \left\|\sum_{n\in\mathbb{Z}} c_n\beta_n\right\|^2 = \left\|U^{-1}\left(\sum_{n\in\mathbb{Z}} c_n f_n\right)\right\|^2 \leq \|U^{-1}\|^2 \left\|\sum_{n\in\mathbb{Z}} c_n f_n\right\|^2.$$

It follows that $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$. Similarly we show that $\{f_n\}_{n\in\mathbb{Z}}$ is a frame. Let $\varphi \in \mathcal{H}$. Then we see that

$$\sum_{n\in\mathbb{Z}} |\langle\varphi, f_n\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle\varphi, U(\beta_n)\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle U^*\varphi, \beta_n\rangle|^2 = \|U^*\varphi\|^2 \leq \|U^*\|^2\|\varphi\|^2,$$

and that

$$\|\varphi\|^2 = \|(U^*)^{-1}U^*\varphi\|^2 \leq \|(U^*)^{-1}\|^2\|U^*\varphi\|^2 = \|(U^*)^{-1}\|^2 \sum_{n\in\mathbb{Z}} |\langle\varphi, f_n\rangle|^2.$$

It follows that $\{f_n\}_{n\in\mathbb{Z}}$ is a frame. □

2.2. **Riesz bases of exponentials and the Paley–Wiener space.** Let us now restrict our attention to the Hilbert space $L^2(\Omega)$, where Ω is some set in $\mathbb{R}^d$. Moreover, we consider Riesz bases of exponentials, meaning Riesz bases of the form

$$\{e^{2\pi i\langle\lambda_n, x\rangle}\chi_\Omega(x)\}_{n\in\mathbb{Z}},$$

where χ_Ω is the indicator function on Ω, and $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of distinct elements in $\mathbb{R}^d$. To ease notation we will use Λ as an index set and write $E(\Lambda) = \{e_\lambda\}_{\lambda\in\Lambda}$, where $e_\lambda(x) = e^{2\pi i\langle\lambda, x\rangle}\chi_\Omega(x)$. We will always assume that Λ is uniformly discrete, that is

$$\delta := \inf_{\substack{\lambda,\alpha\in\Lambda \\ \lambda\neq\alpha}} \|\lambda - \alpha\| > 0,$$

and refer to δ as the separation constant of Λ.

Observe that Riesz bases of exponentials are invariant under translation. That is, if $E(\Lambda)$ is a Riesz basis of exponentials for $L^2(\Omega)$, then

$$\{e_\lambda(x) = e^{2\pi i\langle\lambda,x\rangle}\chi_{\widetilde{\Omega}}(x)\}_{\lambda\in\Lambda}$$

is a Riesz basis for $L^2(\widetilde{\Omega})$ where $\widetilde{\Omega} = \Omega + a$ for any $a \in \mathbb{R}^d$. Further we observe that if $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$, then $E\big((1/a)\Lambda\big)$ is a Riesz basis for $L^2(a\Omega)$, for any non-zero $a \in \mathbb{R}$.

Riesz bases of exponentials for a space $L^2(\Omega)$ are closely related to sets of stable sampling and interpolation in the corresponding Paley–Wiener space PW_Ω. Given a set $\Omega \subset \mathbb{R}^d$ the Paley–Wiener space PW_Ω consists of all functions $f \in L^2(\mathbb{R}^d)$ such that f is the Fourier transform of a function $F \in L^2(\Omega)$. That is

$$f(t) = \widehat{F}(t) = \int_\Omega F(x)e^{-2\pi i\langle t,x\rangle}\,\mathrm{d}x.$$

Observe that it follows from Plancherel's theorem that PW_Ω is a Hilbert space. We say that a uniformly discrete set $\Lambda \subset \mathbb{R}^d$ is a set of stable sampling in PW_Ω if there exist constants A and B such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda}|f(\lambda)|^2 \leq B\|f\|^2,$$

for any $f \in PW_\Omega$. The right inequality is always satisfied when Ω is bounded, and this is again (as for frames) called Bessel's inequality. We call Λ a set of stable interpolation if the interpolation problem

$$\{f(\lambda)\}_{\lambda\in\Lambda} = \{c_\lambda\}_{\lambda\in\Lambda}$$

has at least one solution $f \in PW_\Omega$ for every $\{c_\lambda\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)$. We have the following duality between Riesz sequences and frames for $L^2(\Omega)$ and sets of stable interpolation and sampling for PW_Ω.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^d$ be bounded and let $\Lambda \subset \mathbb{R}^d$ be uniformly discrete. Then the following holds.*

(a) *The set $E(\Lambda)$ is a frame in $L^2(\Omega)$ if and only if Λ is a set of stable sampling in PW_Ω.*
(b) *The set $E(\Lambda)$ is a Riesz sequence in $L^2(\Omega)$ if and only if Λ is a set of stable interpolation in PW_Ω.*

Proof. Case (a) follows immediately from the fact that each $f \in PW_\Omega$ is the Fourier transform of a function $F \in L^2(\Omega)$, that is $\langle F, e_t\rangle = f(t)$. We thus have

$$\sum_{\lambda\in\lambda}|f(\lambda)|^2 = \sum_{\lambda\in\lambda}|\langle F, e_\lambda\rangle|^2,$$

and the claim follows.

To prove (b) let Ψ denote the map from $L^2(\Omega)$ to $\ell^2(\Lambda)$ given by

$$\Psi(F) = \{\langle F, e_\lambda\rangle\}_{\lambda\in\Lambda}.$$

The map is bounded by Bessel's inequality. We see that the adjoint operator is given by

$$\Psi^*(\{c_\lambda\}_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda e_\lambda.$$

It follows that Ψ is surjective if and only if Λ is a set of stable interpolation, as $\langle F, e_\lambda\rangle = f(\lambda)$ whenever $F \in L^2(\Omega)$ and $f \in PW_\Omega$ is the Fourier transform of F. Further we observe that $E(\Lambda)$ is a Riesz sequence if and only if there exists a constant $C > 0$ such that

$$\frac{1}{C}\left\|\{c_\lambda\}_{\lambda\in\Lambda}\right\| \leq \left\|\sum_{\lambda\in\Lambda} c_\lambda e_\lambda\right\| = \left\|\Psi^*(\{c_\lambda\})\right\|,$$

as the upper Riesz sequence bound follows from the fact that Λ is uniformly discrete and Ω is bounded. Thus the proof follows from the fact that a bounded linear operator Ψ between Hilbert spaces is surjective if and only if there exists some constant $C > 0$ such that

$$\|y\| \leq C\|\Psi^* y\|.$$

□

Observe that Theorem 2.5(a) holds also for unbounded Ω. It immediately follows from Theorem 2.5 that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$ if and only if Λ is both a set of stable sampling and interpolation for PW_Ω. In particular it follows that each function $f \in PW_\Omega$ can be reconstructed in a unique way from the set $\{f(\lambda)\}_{\lambda\in\Lambda}$ if Λ is a set of stable sampling and interpolation for PW_Ω. For assume $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$ with dual Riesz basis $\{h_\lambda\}_{\lambda\in\Lambda}$. Then for every $F \in L^2(\Omega)$, with Fourier transform f, we have the unique series expansion

$$F = \sum_{\lambda\in\Lambda}\langle F, e_\lambda\rangle h_\lambda = \sum_{\lambda\in\Lambda} f(\lambda) h_\lambda,$$

and taking the Fourier transform it follows that

$$f = \sum_{\lambda\in\Lambda} f(\lambda)\mathcal{F}(h_\lambda).$$

Recalling the duality between Riesz sequences and frames, we immediately get the following corollary of Theorem 2.2 as a consequence of Theorem 2.5.

Corollary 2.6. *Let $\Lambda \subset \mathbb{Z}$, and let $\Omega \subset [0,1]$. Then Λ is a set of stable sampling for PW_Ω if and only if $\Gamma = \mathbb{Z}\setminus\Lambda$ is a set of stable interpolation for $PW_{[0,1]\setminus\Omega}$.*

3. Stability

In this section we review some stability results pertaining to Riesz bases of exponentials. One of the strengths of Riesz bases of exponentials is that, unlike orthogonal bases, they are stable under small perturbations of the frequency set Λ. That is, given a Riesz basis $E(\Lambda)$ for $L^2(\Omega)$ for some bounded set $\Omega \subset \mathbb{R}^d$, there exists some $\varepsilon > 0$ such that $E(\tilde{\Lambda})$ is a Riesz basis for $L^2(\Omega)$ whenever $\tilde{\Lambda}$ is an ε-perturbation of Λ. By an ε-perturbation we mean that there exists a bijection $\Psi : \Lambda \to \tilde{\Lambda}$ such that $\|\lambda - \Psi\lambda\| < \varepsilon$ for all $\lambda \in \Lambda$. This stability property is the content of the Paley–Wiener Theorem, which we prove below. Further, we consider the orthonormal basis $E(\mathbb{Z})$ in $L^2[0,1]$ and see that $E(\Lambda)$ is a Riesz basis for $L^2[0,1]$ whenever Λ is a $(1/4)$-perturbation of $\mathbb{Z}$. This is the famous Kadec's $\frac{1}{4}$-Theorem [4].

3.1. **The Paley–Wiener stability theorem.** The Paley–Wiener stability theorem on Riesz bases of exponentials is a consequence of the more general Paley–Wiener criterion.

Theorem 3.1 (The Paley–Wiener criterion)**.** *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Riesz basis in a separable Hilbert space $\mathcal{H}$. Let A denote the lower Riesz sequence bound of $\{f_n\}_{n\in\mathbb{Z}}$. Let $\{g_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$ such that*

$$\left\|\sum_{n\in\mathbb{Z}} c_n(f_n - g_n)\right\|^2 \leq K \sum_{n\in\mathbb{Z}} |c_n|^2,$$

for all finite sequences $\{c_n\}_{n\in\mathbb{Z}}$ and any fixed constant $0 < K < A$. Then $\{g_n\}_{n\in\mathbb{Z}}$ is also a Riesz basis for $\mathcal{H}$.

Proof. We follow Young, [16, Theorem 10]. Let $T : \mathcal{H} \to \mathcal{H}$ be the linear operator defined by

$$T\left(\sum_{n\in\mathbb{Z}} c_n f_n\right) = \sum_{n\in\mathbb{Z}} c_n(f_n - g_n).$$

We see that T is well-defined, as

$$\left\|\sum_{n\in\mathbb{Z}} c_n(f_n - g_n)\right\|^2 \leq K \sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{K}{A} \left\|\sum_{n\in\mathbb{Z}} c_n f_n\right\|^2,$$

for all finite sequences $\{c_n\}_{n\in\mathbb{Z}}$, so $\sum c_n(f_n - g_n)$ converges whenever $\sum c_n f_n$ does. Moreover we see that $\|T\|^2 \leq K/A < 1$. We know that if an operator T has norm less than 1, then the operator $I - T$ is bounded and invertible. Thus $U = I - T : \mathcal{H} \to \mathcal{H}$ is bounded and invertible and $U(f_n) = f_n - (f_n - g_n) = g_n$ for all $n \in \mathbb{Z}$, so $\{g_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$. $\square$

Using the Paley–Wiener criterion, we deduce the following result, which is key in proving the Paley–Wiener theorem below.

Lemma 3.2. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Riesz basis for a separable Hilbert space $\mathcal{H}$ and let A and B be the lower and upper Riesz sequence bounds, respectively. Let c be a constant satisfying $0 < c < A^2/B$. Then every sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ which satisfies*

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n - g_n\rangle|^2 \leq c\|f\|^2,$$

for all $f \in \mathcal{H}$ is also a Riesz basis for $\mathcal{H}$.

Proof. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Riesz basis in $\mathcal{H}$. Let A and B be the lower and upper Riesz sequence bounds of $\{f_n\}_{n\in\mathbb{Z}}$, respectively. Let $\{g_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$ satisfying

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n - g_n\rangle|^2 \leq c\|f\|^2,$$

for some constant $0 < c < A^2/B$. Let $T : \mathcal{H} \to \mathcal{H}$ be the linear operator defined by

$$Tf = \sum_{n\in\mathbb{Z}} \langle f, f_n - g_n\rangle f_n.$$

Then the adjoint operator is given by

$$T^*f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle (f_n - g_n).$$

We see that

$$\|Tf\|^2 = \left\|\sum_{n\in\mathbb{Z}} \langle f, f_n - g_n\rangle f_n\right\|^2 \leq B\sum_{n\in\mathbb{Z}} |\langle f, f_n - g_n\rangle|^2 \leq Bc\|f\|^2 < A^2\|f\|^2.$$

It follows that $\|T^*\| = \|T\| < A$. Let $\{h_n\}_{n\in\mathbb{Z}}$ be the dual Riesz basis of $\{f_n\}_{n\in\mathbb{Z}}$. Now fix a function $f = \sum c_n h_n$. Then

$$\left\|\sum_{n\in\mathbb{Z}} c_n(f_n - g_n)\right\|^2 = \|T^*f\|^2 \leq \|T^*\|^2\|f\|^2 \leq A^2\left\|\sum_{n\in\mathbb{Z}} c_n h_n\right\|^2 < A\sum_{n\in\mathbb{Z}} |c_n|^2,$$

were we have used that the upper Riesz sequence bound for the dual Riesz basis $\{h_n\}_{n\in\mathbb{Z}}$ is $1/A$. It now follows from the Paley–Wiener criterion that $\{g_n\}_{n\in\mathbb{Z}}$ is a Riesz basis in $\mathcal{H}$. □

Theorem 3.3 (Paley–Wiener [13])**.** *Let $\Omega \subset \mathbb{R}$ be a bounded set and let Λ be a sequence of real numbers such that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$. Then there exists a constant η, depending on Ω and Λ, such that if a sequence $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}}$ satisfies $|\lambda_n - \gamma_n| < \eta$ for all $n \in \mathbb{Z}$ then $E(\Gamma)$ is also a Riesz basis for $L^2(\Omega)$.*

Proof. We follow the proof given in [6]. Aiming to use Lemma 3.2 we fix a function $F \in L^2(\Omega)$ and consider the sum

$$\sum_{n\in\mathbb{Z}} |\langle F(x), e^{2\pi i\lambda_n x} - e^{2\pi i\gamma_n x}\rangle|^2.$$

Let $f \in PW_\Omega$ be the Fourier transform of F. Let g be the derivative of f, and let G be the inverse Fourier transform of g. Then $G(x) = -2\pi ixF(x)$, and it follows that $\|G\|^2 \leq c\|F\|^2$, where we may choose $c = 2\pi\sup_{x\in\Omega}|x|$. Now since f is the Fourier transform of F it follows that

$$\sum_{n\in\mathbb{Z}} |\langle F(x), e^{2\pi i\lambda_n x} - e^{2\pi i\gamma_n x}\rangle|^2 = \sum_{n\in\mathbb{Z}} |f(\lambda_n) - f(\gamma_n)|^2.$$

By the mean value theorem we get

$$\sum_{n\in\mathbb{Z}} |f(\lambda_n) - f(\gamma_n)|^2 = \sum_{n\in\mathbb{Z}} |\lambda_n - \gamma_n|^2|f'(\xi_n)|^2 \leq \eta^2\sum_{n\in\mathbb{Z}} |g(\xi_n)|,$$

where $\xi_n \in (\lambda_n, \gamma_n)$ or $\xi_n \in (\gamma_n, \lambda_n)$ depending on which interval is non-empty. We have also used the assumption $|\lambda_n - \gamma_n| < \eta$. Since G is the inverse Fourier transform of g, it now follows that

$$\sum_{n\in\mathbb{Z}} |\langle F(x), e^{2\pi i\lambda_n x} - e^{2\pi i\gamma_n x}\rangle|^2 \leq \eta^2\sum_{n\in\mathbb{Z}} |\langle G(x), e^{2\pi i\xi_n x}\rangle|^2.$$

Note that for η sufficiently small the sequence $\{\xi_n\}_{n\in\mathbb{Z}}$ must be uniformly discrete since Λ is uniformly discrete. Thus by Bessel's inequality there exists a constant D such that

$$\sum_{n\in\mathbb{Z}} |\langle G(x), e^{2\pi i\xi_n x}\rangle|^2 \leq D\|G\|^2.$$

To sum up we have

$$\sum_{n\in\mathbb{Z}} |\langle F(x), e^{2\pi i\lambda_n x} - e^{2\pi i\gamma_n x}\rangle|^2 \leq \eta^2 D\|G\|^2 \leq \eta^2 cD\|F\|^2.$$

Thus for η sufficiently small it will follow from Lemma 3.2 that $E(\Gamma)$ is a Riesz basis for $L^2(\Omega)$. □

3.2. **Kadec's $\frac{1}{4}$-Theorem.** Let us now consider the special case where $\Omega = [0,1]$ and $\Lambda = \mathbb{Z}$. Since $E(\mathbb{Z})$ is an orthonormal basis in $L^2[0,1]$ the corresponding Riesz sequence bounds are $A = B = 1$. Moreover, in the proof of Theorem 3.3 we can choose $c = 2\pi$, and it is possible to show using Bessel's inequality that we can choose $D = \pi^2/2$ (see e.g. [11, Proposition 2.7]). Thus, Theorem 3.3 tells us that we may perturb $\mathbb{Z}$ by $\eta < 1/\sqrt{\pi^3} \approx 0.18$ without losing the Riesz basis property. This is not optimal, as we will see next. We start by finding the partial fraction expansions of $\cot(\pi z)$ and $\tan(\pi z)$ as we will need these in the proof of Kadec's $\frac{1}{4}$-Theorem.

Lemma 3.4. *We have the following identities:*

(a)
$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2},$$

(b)
$$\tan(\pi z) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2z}{\left(n - \frac{1}{2}\right)^2 - z^2}.$$

Proof. We use the Hadamard product

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Taking logarithms it follows that

$$\log(\pi z) - \log(\sin(\pi z)) + \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2}\right) = 0.$$

Further, taking derivatives we see that

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

This verifies (a). To prove (b) we use the Hadmard product

$$\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\left(n - \frac{1}{2}\right)^2}\right),$$

and, by taking logarithms and derivatives we arrive at

$$\tan(\pi z) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2z}{\left(n - \frac{1}{2}\right)^2 - z^2}.$$

□

Theorem 3.5 (Kadec's $\frac{1}{4}$-Theorem [4])**.** *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ such that $|\lambda_n - n| \le L$ for some $L < 1/4$ and all $n \in \mathbb{Z}$. Then $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[0,1]$.*

Note that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$ if and only if $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[0,1]$. We may thus reformulate Kadec's $\frac{1}{4}$-Theorem in the following way, which will make it easier to prove as we may then choose a convenient orthonormal basis for $L^2[-\pi,\pi]$.

Theorem 3.6. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ such that $|\lambda_n - n| \le L$ for some $L < 1/4$ and all $n \in \mathbb{Z}$. Then $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$.*

Proof. We follow the proof of Young [16, Theorem 14, page 42]. Let $\delta_n = \lambda_n - n$, and fix some $L < 1/4$. Aiming to use the Paley–Wiener criterion, let $\{c_n\}_{n=1}^N$ be a finite sequence such that $\sum |c_n|^2 \le 1$, and consider the sum

$$\left\| \sum_{n=1}^{N} c_n (e^{int} - e^{i\lambda_n t}) \right\| = \left\| \sum_{n=1}^{N} c_n e^{int}(1 - e^{i\delta_n t}) \right\|. \tag{3.1}$$

We note that the set $\{1, \sqrt{2}\cos(nt), \sqrt{2}\sin((n-\frac{1}{2})t)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$. We want to express $1 - e^{i\delta_n t}$ as a Fourier series with respect to this basis. We calculate:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - e^{i\delta_n t})\,\mathrm{d}t = 1 - \frac{\sin(\pi\delta_n)}{\pi\delta_n} \\
a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - e^{i\delta_n t})\sqrt{2}\cos(kt)\,\mathrm{d}t = \frac{(-1)^k\sqrt{2}\sin(\pi\delta_n)\delta_n}{\pi(k^2 - \delta_n^2)} \\
b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - e^{i\delta_n t})\sqrt{2}\sin\left(kt - \frac{t}{2}\right)\mathrm{d}t = \frac{i(-1)^k\sqrt{2}\cos(\pi\delta_n)\delta_n}{\pi((k-\frac{1}{2})^2 - \delta_n^2)}
\end{aligned}$$

Thus for each n we have

$$\begin{aligned}
1 - e^{i\delta_n t} = 1 - \frac{\sin(\pi\delta_n)}{\pi\delta_n} &+ \sum_{k=1}^{\infty} \frac{(-1)^k 2\sin(\pi\delta_n)\delta_n}{\pi(k^2 - \delta_n^2)}\cos kt \\
&+ \sum_{k=1}^{\infty} \frac{i(-1)^k 2\cos(\pi\delta_n)\delta_n}{\pi((k-\frac{1}{2})^2 - \delta_n^2)}\sin\left(kt - \frac{t}{2}\right).
\end{aligned}$$

Inserting the Fourier series for $1 - e^{i\delta_n t}$ in (3.1), and then interchanging the sums (which is justified by Fubini's theorem), we see that

$$\left\| \sum_{n=1}^{N} c_n e^{int}(1 - e^{i\delta_n t}) \right\| = \|A + B + C\|,$$

where

$$A := \sum_{n=1}^{N} c_n e^{int}\left(1 - \frac{\sin\pi\delta_n}{\pi\delta_n}\right),$$

$$B := \sum_{k=1}^{\infty}\sum_{n=1}^{N} \frac{(-1)^k 2\sin(\pi\delta_n)\delta_n}{\pi(k^2 - \delta_n^2)} c_n e^{int}\cos kt,$$

and

$$C := \sum_{k=1}^{\infty}\sum_{n=1}^{N} \frac{i(-1)^k 2\cos(\pi\delta_n)\delta_n}{\pi((k-\frac{1}{2})^2-\delta_n^2)} c_n e^{int} \sin\left(kt - \frac{t}{2}\right).$$

By further calculations we see that

$$\begin{aligned}
\|A\|^2 &= \left\langle \sum_{n=1}^{N} c_n e^{int}\left(1-\frac{\sin\pi\delta_n}{\pi\delta_n}\right), \sum_{k=1}^{N} c_m e^{ikt}\left(1-\frac{\sin\pi\delta_k}{\pi\delta_k}\right)\right\rangle \\
&= \sum_{n=1}^{N}\sum_{k=1}^{N}\left(1-\frac{\sin\pi\delta_n}{\pi\delta_n}\right)\left(1-\frac{\sin\pi\delta_k}{\pi\delta_k}\right) c_n\overline{c_k}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)t}\,\mathrm{d}t \\
&= \sum_{n=1}^{N}\left(1-\frac{\sin\pi\delta_n}{\pi\delta_n}\right)^2 |c_n|^2 \\
&\le \left(1-\frac{\sin\pi L}{\pi L}\right)^2. \qquad (3.2)
\end{aligned}$$

In the last inequality we have used that $\sum |c_n|^2 \le 1$ and the fact that

$$\left(1-\frac{\sin\pi\delta_n}{\pi\delta_n}\right) \le \left(1-\frac{\sin\pi L}{\pi L}\right).$$

Similarly we can show that

$$\left\|\sum_{n=1}^{N} \frac{(-1)^k 2\sin(\pi\delta_n)\delta_n}{\pi(k^2-\delta_n^2)} c_n e^{int}\cos kt\right\| \le \frac{2L\sin\pi L}{\pi(k^2-L^2)},$$

and

$$\left\|\sum_{n=1}^{N} \frac{i(-1)^k 2\cos(\pi\delta_n)\delta_n}{\pi((k-\frac{1}{2})^2-\delta_n^2)} c_n e^{int}\sin\left(kt-\frac{t}{2}\right)\right\| \le \frac{2L\cos\pi L}{\pi\big((k-\frac{1}{2})^2-L^2\big)}.$$

It then follows from Lemma 3.4 that

$$\|B\| \le \sum_{k=1}^{\infty}\frac{2L\sin\pi L}{\pi(k^2-L^2)} = \sin(\pi L)\left(\frac{1}{\pi L}-\cot(\pi L)\right) = \frac{\sin(\pi L)}{\pi L} - \cos(\pi L), \qquad (3.3)$$

and

$$\|C\| \le \sum_{k=1}^{\infty}\frac{2L\cos\pi L}{\pi\big((k-\frac{1}{2})^2-L^2\big)} = \cos(\pi L)\tan(\pi L) = \sin(\pi L). \qquad (3.4)$$

Now using the triangle inequality, and the estimates (3.2), (3.3) and (3.4) we see that

$$\begin{aligned}
\left\|\sum_{n=1}^{N} c_n(e^{int}-e^{i\lambda_n t})\right\| &\le \|A\|+\|B\|+\|C\| \\
&\le 1-\frac{\sin(\pi L)}{\pi L}+\frac{\sin(\pi L)}{\pi L}-\cos(\pi L)+\sin(\pi L) \\
&= 1-\cos\pi L+\sin\pi L = K.
\end{aligned}$$

We see that $K<1$ when $L<1/4$. Thus, by the Paley-Wiener criterion, the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$. □

We note that the constant $1/4$ in Kadec's $\frac{1}{4}$-Theorem is sharp. It is possible to show that $E(\Lambda)$ is not a Riesz basis for $L^2[0,1]$ when

$$\lambda_n = \begin{cases} n - \frac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n + \frac{1}{4}, & n < 0. \end{cases}$$

A proof of this is given in [16, p.122].

4. Density

When working with Riesz bases of exponentials the density of Λ plays a crucial role. We define the lower and upper uniform density of a uniformly discrete set $\Lambda \subset \mathbb{R}^d$ as

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{\inf_{x\in\mathbb{R}^d} \left|\Lambda \cap (x + B_r)\right|}{|B_r|},$$

and

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d} \left|\Lambda \cap (x + B_r)\right|}{|B_r|},$$

respectively, where B_r denotes the ball in $\mathbb{R}^d$ with radius r and center 0. If $D^-(\Lambda) = D^+(\Lambda)$, we denote this value by $D(\Lambda)$ and call it the uniform density of Λ.

In 1967 Landau [9] gave a necessary condition for when $E(\Lambda)$ can be a Riesz basis of exponentials for a space $L^2(\Omega)$ in terms of density. More precisely, Landau proved the following theorem, with the extra assumption that Ω is bounded in the case (a).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}$ be a set with finite measure, and let $\Lambda \subset \mathbb{R}$ be a uniformly discrete set. Then the following holds.*

(a) *If Λ is a set of stable interpolation for PW_Ω then $D^+(\Lambda) \leq |\Omega|$.*
(b) *If Λ is a set of stable sampling for PW_Ω then $D^-(\Lambda) \geq |\Omega|$.*

It follows from Theorem 4.1 that if $E(\Lambda)$ is a Riesz basis for a space $L^2(\Omega)$, then the upper and lower uniform densities must be equal and $D(\Lambda) = |\Omega|$. These density results tell us that the points in Λ can not be "too dense" (relative to the measure of Ω) in any large interval in order for Λ to be a set of stable interpolation. Similarly for Λ to be a set of stable sampling it is necessary that Λ is not "too sparse" in any large interval.

The converse of Theorem 4.1 is false in general. Let us illustrate this by two examples. We follow [11, Example 5.2 and Exercise 5.4].

Example 1. Let $\Omega = [-2,-1] \cup [1,2]$. We find that $\widehat{\chi_\Omega}(x) = (2\sin(\pi x)\cos(3\pi x))/(\pi x) \in PW_\Omega$. Let Λ be the set of zeroes of $\widehat{\chi_\Omega}$. Then $D^-(\Lambda) = 4 > |\Omega|$. However, as $\widehat{\chi_\Omega}(\lambda) = 0$ for all $\lambda \in \Lambda$, it follows that Λ can not be a set of stable sampling for PW_Ω. Note that we can create many similar examples by choosing a suitable set Ω, and then choosing Λ to be the set of zeroes of $\widehat{\chi_\Omega}$.

Example 2. Let $\Omega = [0,1/2] \cup [1,3/2] \cup [2,5/2]$. Then $D(\mathbb{Z}) = 1 < |\Omega| = 3/2$, yet we can show that $\mathbb{Z}$ is not a set of stable interpolation for PW_Ω. Aiming for a contradiction, assume that it is. Then $E(\mathbb{Z})$ is a Riesz sequence in $L^2(\Omega)$. Moreover, we see that

$$\left\|\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right\|^2_{L^2(\Omega)} = 3\left\|\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right\|^2_{L^2[0,\frac{1}{2}]},$$

and so it follows that $E(\mathbb{Z})$ is a Riesz sequence for $L^2[0,1/2]$ as well. This is clearly a contradiction, as $D(\mathbb{Z}) > 1/2$.

A special case where also sufficient density conditions can be established is that where Ω is an interval. We have the following result due to Kahane [5] and Beurling [1].

Theorem 4.2. *Let I be a finite interval in $\mathbb{R}$, and let Λ be a uniformly discrete set. Then the following holds.*

(a) *If $D^+(\Lambda) < |I|$, then Λ is a set of stable interpolation for PW_I.*

(b) *If $D^-(\Lambda) > |I|$, then Λ is a set of stable sampling for PW_I.*

In the critical cases $D^+(\Lambda) = |I|$ or $D^-(\Lambda) = |I|$, density alone is not enough to determine if Λ is a set of stable interpolation or sampling, and a finer analysis is needed. A full characterization of sets of stable sampling and interpolation for PW_I is given by Ortega-Cerdà and Seip [12] and Seip [15].

In this section we will prove Theorem 4.1(a) following Nitzan and Olevskii's proof of 2012 [10]. This means that we will show $D^+(\Lambda) \leq |\Omega|$ under the weaker condition that $E(\Lambda)$ is uniformly minimal. We will therefore first see that $E(\Lambda)$ is uniformly minimal whenever Λ is a set of stable interpolation for PW_Ω. We then show that minimality implies the density condition $D^+(\Lambda) \leq |\Omega|$. Finally, we invoke duality between sampling and interpolation to prove Theorem 4.1(b).

4.1. **Minimality of $E(\Lambda)$ when Λ is a set of stable interpolation.** We say that a system of vectors $\{h_k\}_{k\in\mathbb{Z}}$ is uniformly minimal in a Hilbert space $\mathcal{H}$ if there exists some $\delta > 0$ such that the distance $d(h_k, \operatorname{span}\{h_n\}_{n\in\mathbb{Z}\setminus\{k\}}) > \delta$ for all $k \in \mathbb{Z}$. Let us now see that if Λ is a set of stable interpolation for PW_Ω, then $E(\Lambda)$ is uniformly minimal in $L^2(\Omega)$. To do this we need the following.

Theorem 4.3. *A set $\{h_k\}_{k\in\mathbb{Z}}$ of functions is uniformly minimal in a Hilbert space $\mathcal{H}$ if and only if there exists a set of functions $\{g_k\}_{k\in\mathbb{Z}} \in \mathcal{H}$ with uniformly bounded norms such that $\{g_k\}_{k\in\mathbb{Z}}$ and $\{h_k\}_{k\in\mathbb{Z}}$ are bi-orthogonal.*

Proof. We follow the proof in [2, Lemma 3.3.1]. Assume first that $\{h_k\}_{k\in\mathbb{Z}}$ is uniformly minimal. Given $j \in \mathbb{Z}$, let P_j denote the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{span}}\{h_k\}_{k\in\mathbb{Z}\setminus\{j\}}$. Since $\{h_k\}_{k\in\mathbb{Z}}$ is uniformly minimal there exists some $\delta > 0$ such that

$$\|(I - P_j)h_j\| > \delta$$

for all j. Moreover, we see that $\langle h_j, (I - P_j)h_j\rangle = \|(I - P_j)h_j\|^2$. For $j \neq l$ we observe that $\langle h_l, (I - P_j)h_j\rangle = 0$. Thus defining

$$g_k = \frac{(I - P_k)h_k}{\|(I - P_k)h_k\|^2},$$

we see that $\{g_k\}_{k\in\mathbb{Z}}$ is bi-orthogonal to $\{h_k\}_{k\in\mathbb{Z}}$. Note also that these functions are uniformly bounded, as

$$\|g_k\| = \frac{1}{\|(I - P_k)h_k\|} < \frac{1}{\delta} = C.$$

For the converse implication, suppose that $\{h_k\}_{k\in\mathbb{Z}}$ has a uniformly bounded bi-orthogonal system $\{g_k\}_{k\in\mathbb{Z}}$. Aiming for a contradiction, assume that $h_j \in \overline{\operatorname{span}}\{h_k\}_{k\in\mathbb{Z}\setminus\{j\}}$. We then have

$$h_j = \sum_{k\in\mathbb{Z}\setminus\{j\}} c_k h_k. \tag{4.1}$$

This is a contradiction, as bi-orthogonality implies $\langle h_j, g_j\rangle = 1$, but (4.1) implies $\langle h_j, g_j\rangle = \langle \sum c_k h_k, g_j\rangle = 0$. This shows minimality.

It remains to show that $\{h_k\}_{k\in\mathbb{Z}}$ is uniformly minimal. We start by defining the functions

$$\tilde{g}_k = \frac{(I - P_k)h_k}{\|(I - P_k)h_k\|^2}.$$

Note that we here use the minimality of $\{h_k\}_{k\in\mathbb{Z}}$ to ensure that we do not divide by 0. We have already observed that $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$ is bi-orthogonal to $\{h_k\}_{k\in\mathbb{Z}}$. Fix $j \in \mathbb{Z}$, and define $f_j = g_j - \tilde{g}_j$. We see that $\langle h_k, f_j\rangle = 0$ for all $k \in \mathbb{Z}$ (including $j = k$). Thus f_j is in the orthogonal complement of

$\overline{\text{span}}\{h_k\}$. Note that $\tilde{g}_j \in \overline{\text{span}}\{h_k\}_{k\in\mathbb{Z}}$. Then the orthogonal decomposition of g_j is $g_j = f_j + \tilde{g}_j$ and by orthogonality we have

$$\|g_j\|^2 = \|\tilde{g}_j\|^2 + \|f_j\|^2 \geq \|\tilde{g}_j\|^2 = \frac{1}{\|(I-P_j)h_j\|^2}.$$

Now since $\{g_k\}_{k\in\mathbb{Z}}$ is uniformly bounded, there exists a constant C such that $\|g_j\| \leq C$ for all $j \in \mathbb{Z}$, it follows that $\|(I-P_j)h_j\| \geq 1/C$, which shows uniform minimality of $\{h_k\}_{k\in\mathbb{Z}}$. □

Corollary 4.4. *If Λ is a stable interpolation set for PW_Ω, then $E(\Lambda)$ is uniformly minimal.*

Proof. We aim to show that $E(\Lambda)$ has a bounded bi-orthogonal system. Let $C_\alpha \in \ell^2(\Lambda)$ denote the sequence where the term indexed by α is 1, and all other terms are 0. Since Λ is a set of stable interpolation there exist functions $f_\alpha \in PW_\Omega$ such that

$$f_\alpha(\lambda) = \begin{cases} 1, & \lambda = \alpha, \\ 0, & \lambda \neq \alpha. \end{cases}$$

Now let $F_\alpha \in L^2(\Omega)$ be the function satisfying $f_\alpha(\lambda) = \langle F_\alpha, e_\lambda\rangle$. By construction it follows that $\{F_\lambda\}_{\lambda\in\Lambda}$ is a bounded system bi-orthogonal to $E(\Lambda)$, and thus $E(\Lambda)$ is uniformly minimal by Theorem 4.3. □

4.2. **Proof of Theorem 4.1(a).** In this section we prove the following theorem, which is somewhat more general than Theorem 4.1(a).

Theorem 4.5. *Let $h \in PW_\Omega$ and let Λ be a uniformly discrete set. If the family $\{h_\lambda\}_{\lambda\in\Lambda}$, where $h_\lambda(x) = h(x-\lambda)$, is uniformly minimal in PW_Ω then $D^+(\Lambda) \leq |\Omega|$.*

Note that if Λ is a uniformly discrete set such that $E(\Lambda)$ is uniformly minimal in $L^2(\Omega)$, then so is $\{\widehat{\chi_\Omega}(x-\lambda)\}_{\lambda\in\Lambda} = \{e^{-2\pi i\lambda x}\widehat{\chi_\Omega}(x)\}_{\lambda\in\Lambda}$ in PW_Ω. Thus it immediately follows from Corollary 4.4 that if Λ is a set of stable interpolation for PW_Ω, then the set $\{\widehat{\chi_\Omega}(x-\lambda)\}_{\lambda\in\Lambda}$ is uniformly minimal in PW_Ω. Thus Theorem 4.1(a) is a direct consequence of Theorem 4.5. The following Lemma is key in proving Theorem 4.5.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}$ be a set with finite measure, and let $V \subseteq PW_\Omega$ be a closed subspace. Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be dual frames in V. Then*

$$0 \leq \sum_{k\in\mathbb{Z}} f_k(x)\overline{g_k(x)} \leq |\Omega|,$$

for all $x \in \mathbb{R}$. Further, if $V = PW_\Omega$, then

$$\sum_{k\in\mathbb{Z}} f_k(x)\overline{g_k(x)} = |\Omega|.$$

Proof. Let F_k and G_k be the inverse Fourier transforms of f_k and g_k respectively, and let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be dual frames in V. Then $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are dual frames in the space $W = \{\mathcal{F}^{-1}(f) : f \in V\}$ which is a closed subspace of $L^2(\Omega)$. Let $P : L^2(\Omega) \to W$ be the orthogonal projection map. Since $G_k \in W$, we have $\langle e_x, G_k\rangle = \langle Pe_x, G_k\rangle$, where $e_x(t) = e^{2\pi ixt}\chi_\Omega(t)$. Thus using the fact that $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are dual frames we see that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} f_k(x)\overline{g_k(x)} &= \sum_{k\in\mathbb{Z}}\langle F_k, e_x\rangle\langle e_x, G_k\rangle \\ &= \sum_{k\in\mathbb{Z}}\big\langle\langle e_x, G_k\rangle F_k, e_x\big\rangle \\ &= \left\langle \sum_{k\in\mathbb{Z}}\langle e_x, G_k\rangle F_k, e_x\right\rangle \\ &= \left\langle \sum_{k\in\mathbb{Z}}\langle Pe_x, G_k\rangle F_k, e_x\right\rangle \\ &= \langle Pe_x, e_x\rangle \\ &= \|Pe_x\|^2 \\ &\le \|e_x\|^2 \\ &= |\Omega|.\end{aligned}$$

To obtain equality in the inequality we must have $\|Pe_x\| = \|e_x\|$, which is clearly satisfied when $V = PW_\Omega$. □

Note that Lemma 4.6 is a slight modification of Lemmas 1 and 2 in [10]. We have weakened the assumption that $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are dual Riesz bases to merely requiring that they are dual frames. This allows us to combine the two lemmas from [10] into a single statement. Note, however, that when we apply the Lemma below we will in fact be working with dual Riesz bases. We are now ready to prove Theorem 4.5.

Proof of Theorem 4.5. If $\{h_\lambda\}_{\lambda\in\Lambda}$ is uniformly minimal in PW_Ω, then by Lemma 4.3 there exists a system $\{g_\lambda\}_{\lambda\in\Lambda} \subset PW_\Omega$ bi-orthogonal to $\{h_\lambda\}_{\lambda\in\Lambda}$, where the norms of $\{g_\lambda\}_{\lambda\in\Lambda}$ are uniformly bounded. Fix $\varepsilon > 0$ and choose b large enough to ensure that

$$\int_{|x|>b} |h(x)|^2\,\mathrm{d}x < \varepsilon^2.$$

Fix a finite interval $I \subset \mathbb{R}$ of length r and let $V = \operatorname{span}\{h_\lambda : \lambda \in I\} \subset PW_\Omega$. Since Λ is uniformly discrete, V is finite-dimensional. We see that $\{h_\lambda\}_{\lambda\in\Lambda\cap I}$ is a Riesz basis for V, since all bases are Riesz bases in finite-dimensional vector spaces. Further, we note that $\{P(g_\lambda)\}_{\lambda\in\Lambda\cap I}$ is a Riesz basis for V as well, where P denotes the orthogonal projection from PW_Ω to V. Choose an element $f \in V$. We may then write $f = \sum_{\alpha\in\Lambda\cap I} c_\alpha h_\alpha$, and thus we have

$$\sum_{\lambda\in\Lambda\cap I}\langle f, Pg_\lambda\rangle h_\lambda = \sum_{\lambda\in\Lambda\cap I}\left\langle \sum_{\alpha\in\Lambda\cap I} c_\alpha h_\alpha, Pg_\lambda\right\rangle h_\lambda = \sum_{\lambda\in\Lambda\cap I} c_\lambda h_\lambda = f.$$

As this holds for any $f \in V$, the systems $\{h_\lambda\}_{\lambda\in\Lambda\cap I}$ and $\{P(g_\lambda)\}_{\lambda\in\Lambda\cap I}$ must be dual Riesz bases. From Lemma 4.6 it follows that

$$0 \le \sum_{\lambda\in\Lambda\cap I} h_\lambda(x)\overline{Pg_\lambda(x)} \le |\Omega|, \tag{4.2}$$

for all $x \in \mathbb{R}$. Let I_b be an interval with the same center as I, but with length $r + 2b$. We choose I_b in this way to ensure that $(\mathbb{R}\setminus I_b) - \lambda \subseteq \{x \in \mathbb{R} : |x| > b\}$ for all $\lambda \in \Lambda \cap I$. Then we have the following useful estimate. Note that C is a constant that might change from line to line.

$$\begin{aligned}\left|\int_{\mathbb{R}\setminus I_b} h_\lambda(x)\overline{Pg_\lambda(x)}\,\mathrm{d}x\right|^2 &\leq \left(\int_{\mathbb{R}\setminus I_b} |h_\lambda(x)|^2\,\mathrm{d}x\right)\left(\int_{\mathbb{R}\setminus I_b} |Pg_\lambda(x)|^2\,\mathrm{d}x\right)\\ &\leq C\int_{|x|>b} |h(x)|^2\,\mathrm{d}x\\ &\leq C\varepsilon^2.\end{aligned}$$

It follows that

$$\int_{I_b} h_\lambda(x)\overline{Pg_\lambda(x)}\,\mathrm{d}x = 1 - \int_{\mathbb{R}\setminus I_b} h_\lambda(x)\overline{Pg_\lambda(x)}\,\mathrm{d}x \geq 1 - C\varepsilon.$$

Here we have used the fact that $\langle h_\lambda, Pg_\lambda\rangle = 1$ since they are dual Riesz bases. By summing over $\lambda \in \Lambda \cap I$ we see that

$$\int_{I_b} \sum_{\lambda\in\Lambda\cap I} h_\lambda(x)\overline{Pg_\lambda(x)}\,\mathrm{d}x \geq |\Lambda\cap I|(1 - C\varepsilon).$$

Now integrating (4.2) and dividing by r it follows that

$$\frac{(1-C\varepsilon)|\Lambda\cap I|}{r} \leq \frac{|\Omega|(r+2b)}{r}.$$

As we can do this for any finite interval I, it follows that

$$\frac{(1-C\varepsilon)\max_{x\in\mathbb{R}}|\Lambda\cap[x,x+r]|}{r} \leq \frac{|\Omega|(r+2b)}{r}.$$

Letting $r \to \infty$, we see that

$$(1-C\varepsilon)D^+(\Lambda) \leq |\Omega|,$$

and finally letting $\varepsilon \to 0$ we get $D^+(\Lambda) \leq |\Omega|$. □

4.3. **Proof of Theorem 4.1(b).** We will now deduce Theorem 4.1(b) from Theorem 4.1(a), using the duality between sampling and interpolation.

Proof of Theorem 4.1(b). We start by assuming Ω is bounded and follow [11, Proof of Theorem 5.1 (ii), page 48]. Let Λ be a set of stable sampling for PW_Ω. By translating we may assume without loss of generality that $\Omega \subseteq [0,a]$. It follows from Theorem 3.3 that for $0 < \varepsilon < 1/a$ sufficiently small, we may choose $\Lambda_1 \subseteq \varepsilon\mathbb{Z}$ such that Λ_1 is a set of stable sampling for PW_Ω and such that Λ_1 is also a perturbation of Λ. Then $D^-(\Lambda_1) = D^-(\Lambda)$. Let $\Gamma_1 = \varepsilon\mathbb{Z}\setminus\Lambda_1$. Then by Corollary 2.6, we have that Γ_1 is a set of stable interpolation for $PW_{[0,a]\setminus\Omega}$. It follows that

$$D^+(\Gamma_1) \leq \big|[0,a]\setminus\Omega\big| = a - |\Omega| \leq \frac{1}{\varepsilon} - |\Omega|.$$

Furthermore, as $\varepsilon\mathbb{Z} = \Gamma_1 \cup \Lambda_1$ it follows that $D^+(\Gamma_1) + D^-(\Lambda_1) = D(\varepsilon\mathbb{Z}) = 1/\varepsilon$. Thus we see that $D^-(\Lambda) = D^-(\Lambda_1) \geq |\Omega|$, when Ω is bounded.

Assume now that Ω is unbounded. Then for any $\varepsilon > 0$ there exists a bounded set $\widetilde{\Omega} \subset \Omega$ such that $|\Omega\setminus\widetilde{\Omega}| < \varepsilon$. If Λ is a set of stable sampling for PW_Ω it must be a set of stable sampling for $PW_{\widetilde{\Omega}}$ as well. Furthermore, as $\widetilde{\Omega}$ is bounded it follows that $D^-(\Lambda) \geq |\widetilde{\Omega}|$. Now letting $\varepsilon \to 0$ the claim follows. □

Remark 1. Note that $D^-(\Lambda) \geq |\Omega|$ is in fact implied by a weaker condition than Λ being a set of stable sampling. One can show that if the family of translates $\{h_\lambda(x)\}_{\lambda\in\Lambda}$ for a function $h \in PW_\Omega$ is a frame in PW_Ω then one has $D^-(\Lambda) \geq |\Omega|$. A proof of this is given in [10, Corollary 1].

Remark 2. Landaus original proof is independent of the dimension d, and also the density results presented in this section are true in any dimension d.

5. Combining Riesz bases

In this section we follow [6] and prove that given a finite union of intervals $\Omega \subset [0,1]$ we can always find a set $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$. We will work with the sets

$$A_n = \left\{ x \in \left[0, \frac{1}{N}\right] : x + \frac{j}{N} \in \Omega \text{ for exactly } n \text{ values of } j \in \{0, \dots, N-1\} \right\},$$

and

$$A_{\geq n} = \bigcup_{k=n}^{N} A_k, \tag{5.1}$$

for some conveniently chosen $N \in \mathbb{N}$. We will see that we can construct a Riesz basis of exponentials for $L^2(\Omega)$ by finding Riesz bases for the corresponding spaces $L^2(A_{\geq n})$, and then combining these into a Riesz basis of exponentials for $L^2(\Omega)$. More precisely, we establish the following theorem.

Theorem 5.1. *Fix $N \in \mathbb{N}$ and let $\Omega \subseteq [0,1]$. Assume that there exist sets $\Lambda_1, \dots, \Lambda_N \subseteq N\mathbb{Z}$ such that $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\geq n})$ for each $n \in \{1, \dots, N\}$, and let*

$$\Lambda = \bigcup_{j=1}^{N} (\Lambda_j + j).$$

Then the system $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$.

Let us briefly outline how the existence of a Riesz basis of exponentials for any finite union of intervals will follow from Theorem 5.1. We begin by observing how Theorem 5.1 can be used to show that there exists a Riesz basis $E(\Lambda)$ with integer frequencies $\Lambda \subset \mathbb{Z}$ for $L^2(I)$ for any interval I. We will then prove Theorem 5.1 in section 5.2. In section 5.3 we consider the special case where $\Omega \subset [0,1]$ is a finite union of intervals with irrational endpoints linearly independent over $\mathbb{Q}$, and see that we can then choose an N such that all sets $A_{\geq n}$ are intervals. We can thus find Riesz bases of exponentials with integer frequencies for all the spaces $L^2(A_{\geq n})$ and use Theorem 5.1 to combine these into a Riesz basis for $L^2(\Omega)$.

Finally in section 5.4, we look at the general case where there are no restrictions on the endpoints. We use an inductive argument to reduce the case of L intervals, to that of $L-1$ intervals, and once again invoke Theorem 5.1 to construct a Riesz basis for any union of intervals $\Omega \subset [0,1]$. The Riesz basis will again be of the form $E(\Lambda)$, where $\Lambda \subset \mathbb{Z}$. Note that by scaling and translating these results extend to any finite union of intervals contained in an interval of length M, but then Λ will instead be a subset of $(1/M)\mathbb{Z}$.

5.1. **Riesz bases for an interval.** Given an interval I we know that we can easily find a Riesz basis $E(\Lambda)$ for $L^2(I)$ by translating and scaling the Riesz basis $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$. However, we now establish the stronger result that if $I \subset [0,1]$, we can always find a Riesz basis $E(\Lambda)$ where $\Lambda \subset \mathbb{Z}$. This result is originally due to Seip [14], but we follow the proof of [6]. We start by proving this in the special case where $|I| < 1/4$. This is an easy consequence of Kadec's $\frac{1}{4}$-Theorem. We then extend the result to larger intervals by using Theorem 5.1.

Lemma 5.2. *Let $\eta < 1/4$. Then there exists a sequence $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2[0,\eta]$. In particular, we may choose $\Lambda = \{[n/\eta]\}_{n\in\mathbb{Z}}$.*

By the notation $[x]$ we mean the integer part of a real number x. We will let $\{x\}$ denote the fractional part of x.

Proof. Let $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}}$ be a sequence satisfying $|n - \gamma_n| < \eta$ for all $n \in \mathbb{Z}$. Then by Kadec's $\frac{1}{4}$-Theorem (Theorem 3.5), the system $E(\Gamma)$ provides a Riesz basis for $L^2[0,1]$. Moreover, by scaling we have that $E((1/\eta)\mathbb{Z})$ is an orthonormal basis, and thus a Riesz basis, for $L^2[0,\eta]$. We see that the criterion $|n - \gamma_n| < \eta$ is equivalent to

$$\left|\frac{n}{\eta} - \frac{\gamma_n}{\eta}\right| < 1,$$

so all sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ satisfying $|n/\eta - \lambda_n| < 1$ provide a Riesz basis for $L^2[0,\eta]$. Now we can easily choose $\lambda_n \in \mathbb{Z}$ since there will always be at least one integer n satisfying this inequality. In particular, we see that we may choose $\lambda_n = [n/\eta]$ for all $n \in \mathbb{Z}$. □

To generalize this result to intervals of greater length we will apply Theorem 5.1. Recalling the definition of $A_{\geq n}$ in (5.1), we will see that we can find $N \in \mathbb{N}$ such that all $NA_{\geq n}$ are intervals of length less than 1/4. Thus we may find $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(NA_{\geq n})$, and then $E(N\Lambda)$ will be a Riesz basis for $L^2(A_{\geq n})$.

Let us briefly review some results about uniformly distributed sequences. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1)$ is uniformly distributed if

$$\lim_{N\to\infty} \frac{\left|\{x_n\}_{n=1}^N \bigcap [a,b)\right|}{N} = b - a,$$

for any $0 \leq a < b \leq 1$. It follows from Weyl's Criterion that the sequence of fractional parts $\big\{\{n\alpha\}\big\}_{n\in\mathbb{N}} = \big\{\{n\alpha_1\},\ldots,\{n\alpha_d\}\big\}_{n\in\mathbb{N}}$ is uniformly distributed in $[0,1]^d$ if and only if $\{\alpha_1,\ldots,\alpha_d,1\}$ are linearly independent over the rationals. As a consequence we have the following lemma.

Lemma 5.3. *Fix $\eta > 0$ and let $0 < b \leq 1$. Then there exists some $N \in \mathbb{N}$ such that $\{Nb\} \leq \eta$.*

Proof. First assume b is rational and let $b = p/q$. Then letting $N = q$, we have $\{Nb\} = \{q\} = 0$. However if $\{b\}$ is irrational the sequence of fractional parts $\big\{\{nb\}\big\}_{n\in\mathbb{Z}}$ must be dense in $[0,1]$, and thus the result follows. □

We are now equipped to prove the following result.

Theorem 5.4. *Let $I \subseteq [0,1]$ be an interval. Then there exists a sequence $\Lambda \subseteq \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(I)$.*

Proof. By translation we may assume $I = [0,b]$. Let $\eta < 1/4$ and let N be such that $\{Nb\} \leq \eta$. We know that this is possible by Lemma 5.3. Consider the sets

$$A_n = \left\{t \in \left[0, \frac{1}{N}\right] : t + \frac{j}{N} \in [0,b] \text{ for exactly } n \text{ values of } j \in \{0,\ldots,N-1\}\right\}.$$

Note that

$$\frac{[Nb]}{N} \leq b < \frac{[Nb]+1}{N}.$$

We therefore see that

$$\left|\left\{t + \frac{j}{N}\right\}_{j=0}^{N-1} \bigcap [0,b]\right| = \begin{cases} [Nb]+1, & 0 \leq t \leq \frac{\{Nb\}}{N}, \\ [Nb], & \frac{\{Nb\}}{N} < t \leq \frac{1}{N}. \end{cases}$$

It follows that

$$A_{\geq n} = \begin{cases} \left[0, \frac{1}{N}\right], & 1 \leq n \leq [Nb], \\ \\ \left[0, \frac{\{Nb\}}{N}\right], & n = [Nb]+1, \\ \\ \emptyset, & [Nb]+2 \leq n \leq N. \end{cases}$$

We have that $E(N\mathbb{Z})$ is a Riesz basis for $[0, 1/N]$. Further, since $\{Nb\} < \eta$ we may apply Lemma 5.2 to find a set $\Lambda \subseteq \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $[0, \{Nb\}]$ in the cases $\{Nb\} \neq 0$. The set $E(N\Lambda)$ is then a Riesz basis for $[0, \{Nb\}/N]$ by scaling. Thus the conditions in Theorem 5.1 are fulfilled, and it follows that there exists $\Gamma \subseteq \mathbb{Z}$ such that $E(\Gamma)$ is a Riesz basis for $L^2(I)$. In particular we may choose

$$\Gamma = \bigcup_{j=1}^{[Nb]} (N\mathbb{Z}+j) \bigcup \left(\left\{ N \left[\frac{n}{\{Nb\}} \right] \right\}_{n \in \mathbb{Z}} + [Nb] + 1 \right),$$

if $\{Nb\} \neq 0$ and

$$\Gamma = \bigcup_{j=1}^{[Nb]} (N\mathbb{Z}+j),$$

if $\{Nb\} = 0$. □

Let us see how this construction works by considering a concrete example.

Example 3. Let $I = [0, \sqrt{2}/2]$, and let us use Theorem 5.4 to construct a Riesz basis $E(\Lambda)$ for $L^2(I)$ where $\Lambda \subset \mathbb{Z}$. Following the notation in the proof above, we observe that

$$\alpha = \left\{ 3\frac{\sqrt{2}}{2} \right\} = \frac{3\sqrt{2}}{2} - 2 \approx 0.1213 < \frac{1}{8},$$

and thus choose $N = 3$ and $\eta = 1/8$. Note that we could have chosen any other pair (N, η) satisfying $\{N\frac{\sqrt{2}}{2}\} < \eta < 1/4$, and since $\sqrt{2}/2$ is irrational there are infinitely many. We see that

$$\begin{aligned} A_{\geq 1} &= \left[0, \frac{1}{3}\right] \\ A_{\geq 2} &= \left[0, \frac{1}{3}\right] \\ A_{\geq 3} &= \left[0, \frac{1}{3}\left(\frac{3\sqrt{2}}{2} - 2\right)\right]. \end{aligned}$$

We choose $\Lambda_1 = \Lambda_2 = 3\mathbb{Z}$, as frequency sets for $A_{\geq 1}$ and $A_{\geq 2}$ respectively. Further, as suggested in the proof of Theorem 5.4 we choose $\Lambda_3 = 3\Gamma$ as frequency set for $A_{\geq 3}$ where

$$\Gamma = \left\{ \left[\frac{n}{\alpha} \right] \right\}_{n \in \mathbb{Z}} \subset \mathbb{Z}.$$

Finally, according to Theorem 5.1, the set

$$\Lambda = \bigcup_{n=1}^{3} (\Lambda_j + j) = (3\mathbb{Z}+1) \bigcup (3\mathbb{Z}+2) \bigcup (3\Gamma + 3),$$

provides a Riesz basis $E(\Lambda)$ for $L^2(I)$.

5.2. **Proof of Theorem 5.1.** Our goal in this section is to prove Theorem 5.1. We start by introducing some notation, and establishing a technical lemma which will simplify the proof. Given a set $\Omega \subset \mathbb{R}$ we let $e_\lambda(x) = e^{2\pi i \lambda x}\chi_\Omega(x)$. Fix $N \in \mathbb{N}$ and assume $\Omega \subseteq [0,1]$. Similarly to A_n and $A_{\geq n}$ we define

$$B_n = \left\{ x \in \Omega : x + \frac{j}{N} \in \Omega \text{ for exactly } n \text{ values of } j \in \{0, \ldots, N-1\} \right\},$$

and

$$B_{\geq n} = \bigcup_{k=n}^{N} B_k.$$

For $f \in L^2(\Omega)$, we denote by $f_n \in L^2(\Omega)$ the restriction of f to B_n, that is

$$f_n(x) = \begin{cases} f(x), & x \in B_n, \\ 0, & x \notin B_n. \end{cases} \tag{5.2}$$

Moreover, we let

$$f_{\geq n} = \sum_{k=n}^{N} f_k. \tag{5.3}$$

Note that $B_k \cap B_l = \emptyset$ for all $k \neq l$, and

$$\bigcup_{n=1}^{N} B_n = \Omega,$$

so $f = \sum_{n=1}^{N} f_n$. We note that

$$B_n \bigcap \left[\frac{k}{N}, \frac{k+1}{N} \right] = A_n + \frac{k}{N}$$

for any $0 \leq k \leq N-1$, and thus we may think of A_n as the set that appears if we fold B_n onto the interval $[0, 1/N]$ exactly N times. We observe that

$$|\Omega| = \sum_{n=1}^{N} |B_n| = \sum_{n=1}^{N} n|A_n| = \sum_{n=1}^{N} |A_{\geq n}|.$$

Lemma 5.5. *Fix $N \in \mathbb{N}$ and let $\Omega \subseteq [0,1]$. Assume there exist sets $\Lambda_1, \ldots, \Lambda_N \subseteq N\mathbb{Z}$ such that $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\geq n})$ for all $n \in \{1, \ldots, N\}$. Let*

$$\Lambda = \bigcup_{j=1}^{N} (\Lambda_j + j).$$

Then there exists a constant $c > 0$ such that for any $f \in L^2(\Omega)$

$$c\|f_n\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2, \tag{5.4}$$

where f_n and $f_{\geq n}$ are given in (5.2) *and* (5.3), *respectively.*

Proof. Fix $f \in L^2(\Omega)$ and $n \in \{1, \dots, N\}$. Note that $f_{\geq n}\big(x + \frac{k}{N}\big) = 0$ for all $x \in \big[0, \frac{1}{N}\big] \setminus A_{\geq n}$ and all $k \in \{0, \dots N-1\}$, since $f_{\geq n}(x) = 0$ for all $x \in \Omega \setminus B_{\geq n}$. We define

$$h_j(x) = \chi_{A_{\geq n}}(x) \sum_{k=0}^{N-1} f_{\geq n}\left(x + \frac{k}{N}\right) \exp\left(-2\pi i \frac{jk}{N}\right).$$

Note that we may view h_j either as a function in $L^2(A_{\geq n})$ or as a function in $L^2(\Omega)$. Now consider the inner product $\langle f_{\geq n}, e_\lambda \rangle$ where $\lambda \in \Lambda_j + j$ for some fixed $j \in \{1, \dots n\}$. By splitting the integral over segments of length $1/N$ we get

$$\begin{aligned}
\langle f_{\geq n}, e_\lambda \rangle_{L^2(\Omega)} &= \sum_{k=0}^{N-1} \int_{\frac{k}{N}}^{\frac{k+1}{N}} \chi_{B_{\geq n}}(x) f_{\geq n}(x) \exp(-2\pi i \lambda x)\, \mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{N-1} \chi_{B_{\geq n}}\left(x + \frac{k}{N}\right) f_{\geq n}\left(x + \frac{k}{N}\right) \exp\left(-2\pi i \lambda \left(x + \frac{k}{N}\right)\right) \mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{N-1} \chi_{A_{\geq n}}(x) f_{\geq n}\left(x + \frac{k}{N}\right) \exp\left(-2\pi i \frac{jk}{N}\right) \exp(-2\pi i \lambda x)\, \mathrm{d}x \\
&= \langle h_j, e_\lambda \rangle_{L^2(\Omega)}
\end{aligned}$$

Here we have used that since $\lambda = Nm + j$ for some $m \in \mathbb{Z}$ it follows that

$$\exp\left(-2\pi i \lambda \left(x + \frac{k}{N}\right)\right) = \exp(-2\pi i \lambda x) \exp\left(-2\pi i \frac{jk}{N}\right).$$

It is clear that $A_{\geq n} \subseteq A_{\geq j}$, and thus we may view h_j as a function in $L^2(A_{\geq j})$. By assumption $E(\Lambda_j)$ is a Riesz basis for $L^2(A_{\geq j})$. Thus letting A_j be the lower frame bound for the Riesz basis $E(\Lambda_j)$ we have

$$\sum_{\lambda \in \Lambda_j + j} |\langle f_{\geq n}, e_\lambda \rangle|^2 = \sum_{\lambda \in \Lambda_j + j} |\langle h_j, e_\lambda \rangle|^2 \geq A_j \|h_j\|^2.$$

Summing over j it follows that

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 &= \sum_{j=1}^{N} \sum_{\lambda \in \Lambda_j + j} |\langle f_{\geq n}, e_\lambda \rangle|^2 \\
&\geq \sum_{j=1}^{n} \sum_{\lambda \in \Lambda_j + j} |\langle f_{\geq n}, e_\lambda \rangle|^2 \\
&= \sum_{j=1}^{n} \sum_{\lambda \in \Lambda_j + j} |\langle h_j, e_\lambda \rangle|^2 \\
&\geq A \sum_{j=1}^{n} \|h_j\|^2,
\end{aligned}$$

where $A = \min_{j \in \{1, \dots n\}} A_j$. To conclude the proof we must show that there exists a constant $C > 0$ such that $\sum_{j=1}^{n} \|h_j\|^2 \geq C \|f_n\|^2$. To do this we define the $n \times N$ matrix

$$L = \left[\exp\left(-2\pi i \frac{jk}{N}\right)\right]_{j,k},$$

where j runs from 1 to n, and k runs from 0 to $N-1$. By a simple calculation we see that

$$L\left[f_{\geq n}\left(x+\frac{k}{N}\right)\right]_k = \left[\sum_{k=0}^{N-1} f_{\geq n}\left(x+\frac{k}{N}\right)\exp\left(-2\pi i\frac{jk}{N}\right)\right]_j = [h_j(x)]_j\,,$$

for each fixed $x \in A_n$. Recall that for each $x \in A_n$ there are exactly n different k such that $x+\frac{k}{N} \in \Omega$. Let

$$\left[g_{\geq n}\left(x+\frac{k}{N}\right)\right]_k$$

be the vector of length n corresponding to

$$\left[f_{\geq n}\left(x+\frac{k}{N}\right)\right]_k$$

in the sense that we remove the $N-n$ entries where $x+\frac{k}{N} \notin \Omega$. Then the only difference between these two vectors are $N-n$ zero-entries, as $f_{\geq n}(x) = 0$ for any $x \in [0,1]\setminus\Omega$. Further, we define L_x for a fixed x to be the matrix that results from removing the columns of L that are multiplied by these $N-n$ zeroes when we look at the product $L[f_{\geq n}(x+k/N)]_k$. Clearly, we then have

$$L_x\left[g_{\geq n}\left(x+\frac{k}{N}\right)\right]_k = L\left[f_{\geq n}\left(x+\frac{k}{N}\right)\right]_k = [h_j(x)]_j\,.$$

For each $x \in A_n$ we see that L_x is an $n\times n$ matrix, where the dependency on x is restricted to the way that x determines which columns of L are to be removed. Thus there are finitely many different matrices L_x. These matrices are all invertible, and since there are finitely many of them, there exists some constant C such that $\|L_x^{-1}\|^2 < C$ for all $x \in A_n$. Thus we have

$$\left\|\left[f_{\geq n}\left(x+\frac{k}{N}\right)\right]_k\right\|^2 = \left\|L_x^{-1}[h_j(x)]_j\right\|^2 \leq \|L_x^{-1}\|^2\,\|[h_j(x)]_j\|^2 \leq C\,\|[h_j(x)]_j\|^2\,.$$

Calculating the length of these vectors, we get

$$\frac{1}{C}\sum_{k=0}^{N-1}\left|f_{\geq n}\left(x+\frac{k}{N}\right)\right|^2 \leq \sum_{j=1}^{n}|h_j(x)|^2.$$

Note that the sum on the right hand side has (at most) n terms that are non-zero and the vector we are calculating the length of has n elements. Finally, we integrate this inequality and see that

$$\begin{aligned}\sum_{j=1}^{n}\|h_j\|^2 \geq \sum_{j=1}^{n}\|h_j\chi_{A_n}\|^2 &= \int_{A_n}\sum_{j=1}^{n}|h_j(x)|^2\,\mathrm{d}x \\ &\geq \int_{A_n}\frac{1}{C}\sum_{k=0}^{N-1}\left|f_{\geq n}\left(x+\frac{k}{N}\right)\right|^2\,\mathrm{d}x \\ &= \frac{1}{C}\int_{B_n}|f_n(x)|^2\,\mathrm{d}x \\ &= \frac{1}{C}\|f_n\|^2.\end{aligned}$$

Here we have used that $f_{\geq n} = f_n$ on B_n. This concludes the proof. □

We are now ready to prove Theorem 5.1, which we will do by proving that $E(\Lambda)$ is both a Riesz sequence and a frame.

Proof of Theorem 5.1. We start by showing that $E(\Lambda)$ is a frame in $L^2(\Omega)$. Let $f \in L^2(\Omega)$ and let $\tilde{f} \in L^2[0,1]$ be an extension of f in the sense that $\tilde{f} = f$ on Ω and $\tilde{f} \equiv 0$ elsewhere. Noting that $\Lambda \subseteq \mathbb{Z}$ we see that

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 = \sum_{n \in \mathbb{Z}} \left| \langle \tilde{f}, e_n \rangle \right|^2 = \|\tilde{f}\|^2 = \|f\|^2,$$

where we have used that $E(\mathbb{Z})$ is a frame for $L^2[0,1]$. This shows the upper frame bound with constant 1. Note that the upper frame bound, with another constant, immediately follows from Bessel's inequality.

It remains to show the lower frame bound. We start by showing that for every $n \in \{1, \dots, N\}$ there exists a constant $c > 0$ such that

$$c\|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2. \tag{5.5}$$

For any $x, y \in \mathbb{C}$ we have $|x+y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$, so for a fixed $\lambda \in \Lambda$ it follows that

$$|\langle f, e_\lambda \rangle|^2 \geq \frac{1}{2} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \left| \left\langle \sum_{k=1}^{n-1} f_k, e_\lambda \right\rangle \right|^2.$$

Summing over all $\lambda \in \Lambda$ and recalling that $\Lambda \subseteq \mathbb{Z}$ we see that

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 &\geq \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \sum_{\lambda \in \Lambda} \left| \left\langle \sum_{k=1}^{n-1} f_k, e_\lambda \right\rangle \right|^2 \\
&\geq \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \sum_{l \in \mathbb{Z}} \left| \left\langle \sum_{k=1}^{n-1} f_k, e_l \right\rangle \right|^2 \\
&= \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \left\| \sum_{k=1}^{n-1} f_k \right\|^2 \\
&= \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \sum_{k=1}^{n-1} \|f_k\|^2.
\end{aligned}$$

The last equality follows from the fact that for all $l \neq m$ we have $f_l f_m \equiv 0$ since $B_l \cap B_m = \emptyset$. Equation (5.5) now follows from Lemma 5.5. To conclude, let $\{\delta_n\}_{n=1}^N$ be a sequence of positive numbers such that $\sum_{n=1}^N \delta_n = 1$ and such that

$$\delta_n > \frac{2}{c_1} \sum_{k=n+1}^{N} \delta_k$$

for all $n \in \{1, \ldots, N\}$, where c_1 is a constant satisfying equation (5.5). This essentially means that the sequence must decrease exponentially. We then have

$$\begin{aligned}
\sum_{n=1}^{N} \delta_n \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 &\geq \sum_{n=1}^{N} \delta_n \left(c_1 \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2 \right) \\
&= \sum_{n=1}^{N} \left(c_1 \delta_n - \sum_{k=n+1}^{N} \delta_k \right) \|f_n\|^2 \\
&= \sum_{n=1}^{N} \frac{c_1}{2} \left(2\delta_n - \frac{2}{c_1} \sum_{k=n+1}^{N} \delta_k \right) \|f_n\|^2 \\
&\geq \sum_{n=1}^{N} \frac{c_1}{2} \delta_n \|f_n\|^2 \\
&\geq A \sum_{n=1}^{N} \|f_n\|^2.
\end{aligned}$$

Where $A = \frac{c_1}{2} \min_{n \in \{1,\ldots,N\}} \delta_n$. This proves that $E(\Lambda)$ is a frame for $L^2(\Omega)$ with lower frame bound A.

Let us now see that $E(\Lambda)$ is a Riesz sequence in $L^2(\Omega)$. As in the frame case, the upper bound is nearly immediate. Let $\{a_\lambda\}_{\lambda \in \Lambda}$ be a sequence in $\ell^2(\Lambda)$ and expand it to $\ell^2(\mathbb{Z})$ by letting $a_n = 0$ for all $n \notin \Lambda$. Then since $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2[0,1]$ we have

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|^2 = \left\| \sum_{n \in \mathbb{Z}} a_n e_n \right\|^2 = \sum_{\lambda \in \Lambda} |a_\lambda|^2,$$

which proves the upper Riesz sequence bound with constant 1. It remains to show the lower Riesz sequence bound. As the proof is very similar to that for the frame property above, we provide only an outline. It turns out that it suffices to show that there exists a constant $c > 0$ such that

$$\int_\Omega \left| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda(x) \right|^2 \mathrm{d}x \geq c \sum_{\lambda \in \Lambda_n + n} |a_\lambda|^2 - \sum_{j=n+1}^{N} \sum_{\lambda \in \Lambda_j + j} |a_\lambda|^2, \tag{5.6}$$

for every $n \in \{1, \ldots, N\}$. To do this we use the inequality $|x + y| \geq \frac{1}{2}|x|^2 - |y|^2$ to see that

$$\left| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda(x) \right|^2 \geq \frac{1}{2} \left| \sum_{j=1}^{n} \sum_{\lambda \in \Lambda_j + j} a_\lambda e_\lambda(x) \right|^2 - \left| \sum_{j=n+1}^{N} \sum_{\lambda \in \Lambda_j + j} a_\lambda e_\lambda(x) \right|^2.$$

Integrating over $x \in \Omega$, we see that

$$\int_\Omega \left| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda(x) \right|^2 \mathrm{d}x \geq \frac{1}{2} \int_\Omega \left| \sum_{j=1}^{n} \sum_{\lambda \in \Lambda_j + j} a_\lambda e_\lambda(x) \right|^2 \mathrm{d}x - \int_\Omega \left| \sum_{j=n+1}^{N} \sum_{\lambda \in \Lambda_j + j} a_\lambda e_\lambda(x) \right|^2 \mathrm{d}x. \tag{5.7}$$

Viewing $\{a_\lambda\}_{\lambda \in \Lambda}$ as a sequence in $\ell^2(\mathbb{Z})$ by adding zeroes to the sequence, we see that

$$\int_\Omega \left| \sum_{j=n+1}^{N} \sum_{\lambda \in \Lambda_j + j} a_\lambda e_\lambda(x) \right|^2 \mathrm{d}x = \left\| \sum_{n \in \mathbb{Z}} a_n e_n \right\|^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 = \sum_{j=n+1}^{N} \sum_{\lambda \in \Lambda_j + j} |a_\lambda|^2. \tag{5.8}$$

Further, one can show by following the proof of Lemma 5.5 that there exists a constant C such that

$$\int_{\Omega}\left|\sum_{j=1}^{n}\sum_{\lambda\in\Lambda_j+j}a_\lambda e_\lambda(x)\right|^2 \mathrm{d}x \geq C\sum_{\lambda\in\Lambda_n+n}|a_\lambda|^2. \tag{5.9}$$

Inserting (5.8) and (5.9) in (5.7) we arrive at (5.6), and this concludes the proof that $E(\Lambda)$ is a Riesz sequence in $L^2(\Omega)$. □

5.3. **Finite unions of intervals with irrational endpoints linearly independent over $\mathbb{Q}$.** We now look at the special case $\Omega \subset [0,1]$ is a union of intervals where all endpoints are irrational and linearly independent over $\mathbb{Q}$. This special case illustrates how we can construct a Riesz basis explicitly using Theorem 5.1.

Theorem 5.6. *Let $\Omega = \bigcup_{l=1}^{L}[a_l,b_l] \subseteq [0,1]$ where $\{1,a_1,\dots a_L,b_1,\dots b_L\}$ are linearly independent over $\mathbb{Q}$. Then there exists $\Lambda \subseteq \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$.*

Proof. Fix $N \in \mathbb{N}$ such that all $\{Na_l\}$ and $\{Nb_l\}$ are different and such that $\{Na_l\} < 1/2$ and $\{Nb_l\} > 1/2$ for all $l \in \{1,\dots,L\}$. This is possible since the sequence $\{na_1,\dots na_L, nb_1,\dots nb_L\}_{n\in\mathbb{Z}}$ is uniformly distributed and thus dense in $[0,1]^{2L}$. Let σ and τ be permutations of $\{1,2,\dots,L\}$ such that

$$0 < \{Na_{\sigma(1)}\} < \{Na_{\sigma(2)}\} < \cdots < \{Na_{\sigma(L)}\} < \frac{1}{2},$$

and

$$\frac{1}{2} < \{Nb_{\tau(1)}\} < \{Nb_{\tau(2)}\} < \cdots < \{Nb_{\tau(L)}\} < 1.$$

Let $\Psi : [0,1/N] \to \{0,1,\dots N\}$ be the map given by

$$\Psi(t) = \left|\left\{ j : t + \frac{j}{N} \in \Omega \text{ and } j \in \{0,1\dots N\}\right\}\right|. \tag{5.10}$$

We then have $A_{\geq n} = \{t \in [0,1/N] : \Psi(t) \geq n\}$. We observe that

$$\frac{\{Na_l\} + [Na_l]}{N} = a_l \in \Omega.$$

Thus it follows that Ψ increases with 1 at $\{Na_l\}/N$ for any $1 \leq l \leq L$. Similarly Ψ decreases with 1 at all $\{Nb_l\}/N$. Further we note that these points are the only points where Ψ can increase or decrease, as $t + j/N$ needs to be an endpoint in one of the intervals $[a_l,b_l]$ for this to happen. Let $\Psi(0) = k$. Then it follows that

$$\Psi\left(\frac{\{Na_{\sigma(j)}\}}{N}\right) = k + j,$$

and

$$\Psi\left(\frac{\{Nb_{\tau(L-j)}\}}{N}\right) = k + j,$$

for any $1 \leq j \leq L$. Thus we see that

$$A_{\geq n} = \begin{cases} \left[0, \frac{1}{N}\right], & n \leq k, \\ \left[\frac{\{Na_{\sigma(j)}\}}{N}, \frac{\{Nb_{\tau(L-j+1)}\}}{N}\right), & n = k+j, \quad 1 \leq j \leq L, \\ \emptyset, & n > k+L. \end{cases}$$

In particular $A_{\geq n}$ is always an interval. Thus by Theorem 5.4 we may find Riesz bases $E(\Lambda_n)$ for each set $L^2(NA_{\geq n}) \subseteq L^2[0,1]$, where $\Lambda_n \subseteq \mathbb{Z}$. By scaling $E(N\Lambda_n)$ will then be a Riesz basis for $L^2(A_{\geq n})$. Finally, by Theorem 5.1 it follows that there exists $\Gamma \subset \mathbb{Z}$ such that $E(\Gamma)$ is a Riesz basis for $L^2(\Omega)$.

In particular, we may construct

$$\Gamma = \bigcup_{j=1}^{k} (N\mathbb{Z} + j) \bigcup_{j=k+1}^{k+L} (N\Lambda_j + j) \subseteq \mathbb{Z},$$

where we consider the first union to be empty if $k = \Psi(0) = 0$, and Λ_j is chosen as in Theorem 5.4. That is,

$$\Lambda_j = \bigcup_{j=1}^{[Mc]} (M\mathbb{Z} + j) \bigcup \left(\left\{ M \left[\frac{n}{\{Mc\}} \right] \right\}_{n\in\mathbb{Z}} + [Mc] + 1 \right),$$

if $\{Mc\} \neq 0$, and

$$\Lambda_j = \bigcup_{j=1}^{[Mc]} (M\mathbb{Z} + j),$$

if $\{Mc\} = 0$. Here $c = \{Nb_{\tau(L-j)}\} - \{Na_{\sigma(j)}\}$ and $M \in \mathbb{N}$ is chosen such that $\{Mc\} < \eta \leq 1/4$. Note that both c and M depend on j. □

5.4. **Riesz bases for finite unions of intervals.** In this subsection we prove the existence of Riesz bases of exponentials for finite unions of intervals in full generality. That is

Theorem 5.7. *Let $\Omega \subset [0,1]$ be a union of L intervals $[a_l, b_l]$. Then there exists a Riesz basis of exponentials $E(\Lambda)$ for $L^2(\Omega)$, where Λ is a subset of $\mathbb{Z}$.*

In the proof of Theorem 5.6, it was crucial that the endpoints were irrational and linearly independent over $\mathbb{Q}$. This guaranteed that we could choose $N \in \mathbb{N}$ such that the function $\Psi : [0, 1/N] \to \{0, 1, \ldots N\}$ defined by

$$\Psi(t) = \left| \left\{ j : t + \frac{j}{N} \in \Omega \text{ and } j \in \{0, 1 \ldots N\} \right\} \right|,$$

in (5.10) was non-decreasing on $[0, 1/(2N)]$ and non-increasing on $[1/(2N), 1/N]$. This, in turn, made it possible to see that all sets $A_{\geq n}$ were intervals, and thus Theorem 5.1 could be invoked to obtain a Riesz basis of exponentials for $L^2(\Omega)$ by combining Riesz bases for the spaces $L^2(A_{\geq n})$.

When we allow the endpoints of the intervals to be any real numbers, a more delicate analysis is needed. Let us introduce the following definitions and notation. We say that two sequences $\{a_l\}_{l=1}^L$ and $\{b_l\}_{l=1}^L$ interlace if there exist permutations σ and τ of $\{1, \ldots L\}$ such that

$$a_{\sigma(1)} \leq b_{\tau(1)} \leq a_{\sigma(2)} \leq b_{\tau(2)} \leq \cdots \leq a_{\sigma(L)} \leq b_{\tau(L)},$$

or

$$b_{\tau(1)} \leq a_{\sigma(1)} \leq b_{\tau(2)} \leq a_{\sigma(2)} \leq \cdots \leq b_{\tau(L)} \leq a_{\sigma(L)}.$$

For two sequences $\{a_l\}_{l=1}^L$ and $\{b_l\}_{l=1}^L$ of numbers in $[0,1]$ we define the function $\Phi : [0,1] \to [0, 2L]$ by

$$\Phi(t) = \sum_{n=1}^{L} \chi_{[0,b_l]}(t) + \sum_{n=1}^{L} \chi_{[a_l,1]}(t), \tag{5.11}$$

and let

$$C_{\geq n} = \Phi^{-1}[n, 2L], \tag{5.12}$$

for $0 \leq n \leq 2L$. We observe that Φ is a step-function, and that Φ increases with 1 in precisely the points $\{a_l\}_{l=1}^L$ and decreases with 1 in precisely the points $\{b_l\}_{l=1}^L$.

Recall from the proof of Theorem 5.6 that we have

$$A_{\geq n} = \{t \in [0, 1/N] : \Psi(t) \geq n\} = \Psi^{-1}[n, N],$$

and that Ψ increases with 1 in precisely the points $\{Na_l\}/N$ for $1 \leq l \leq L$, and decreases with 1 in precisely the points $\{Nb_l\}/N$ for $1 \leq l \leq L$. It follows that $\Psi(x) = \Phi(Nx) - (\Psi(0) - \Phi(0))$. There is thus a close connection between the sets $C_{\geq n}$ and the sets $A_{\geq n}$. Before we prove Theorem 5.7, we need two lemmas regarding interlacing.

Lemma 5.8. *Let $\{a_l\}_{l=1}^L$ and $\{b_l\}_{l=1}^L$ be two non-decreasing sequences in $[0, 1]$ that do not interlace, and assume further that we do not have both $a_1 = 0$ and $b_L = 1$. Let Φ and $C_{\geq n}$ be defined as in (5.11) and (5.12). Then for each $n \in [0, 2L]$ the set $C_{\geq n}$ is a union of at most $L - 1$ intervals (up to a set of measure 0).*

Proof. Since the function Φ increases only in the points a_l and decreases in the points b_l, each set $C_{\geq n}$ must be a union of at most L sets of the form $[a_l, b_j]$, as well as possibly $[0, a_1]$ or $[b_L, 1]$. Further, we observe that if $[b_L, 1]$ and $[a_l, b_L]$ are both part of the union, then this is one interval. The same holds for $[0, a_1]$ and some interval $[a_1, b_j]$. Thus, each $C_{\geq n}$ is a union of at most L intervals.

Further, as the sequences $\{a_l\}_{l=1}^L$ and $\{b_l\}_{l=1}^L$ do not interlace, there is some $l \in \{1, \dots, L\}$ such that $b_j \notin [a_l, a_{l+1}]$ for any $j \in \{1, \dots, L\}$. It follows that $C_{\geq n}$ cannot contain both an interval starting in a_l and an interval starting in a_{l+1}. Thus $C_{\geq n}$ consist of at most $L - 1$ intervals. $\square$

Lemma 5.9. *Let $\{a_l\}_{l=1}^L$ and $\{b_l\}_{l=1}^L$ be two non-decreasing sequences in $[0, 1]$. If $L = 2$, assume also that we do not have both $a_1 = 0$ and $b_2 = 1$. Then there are infinitely many $N \in \mathbb{N}$ such that the sequences $\{\{Na_l\}\}_{l=1}^L$ and $\{\{Nb_l\}\}_{l=1}^L$ do not interlace.*

This is a slight reformulation of Lemma 4 in [6], and we refer the reader to [6] for the proof. Note, however, that we have restricted our sequences to $[0, 1]$ instead of arbitrary real numbers and added an extra assumption in the case $L = 2$. This is to avoid the situation where too many pairs in the sequences have integer distances, which will necessarily make the statement false. Another solution to this problem is to change the inequalities in the definition of interlacing to strict inequalities. Although slightly different from Lemma 4 in [6], the stated Lemma 5.9 is sufficent for proving Theorem 5.7.

Proof of Theorem 5.7. We use induction on the number of intervals L. The base case $L = 1$ follows from Theorem 5.4. Assume $\Omega = \cup_{l=1}^L [a_l, b_l]$ and that the theorem holds for all unions of $1 \leq K < L$ intervals. A consequence of Corollary 2.6 is that the system $E(\Lambda) \subset E(\mathbb{Z})$ is a Riesz basis for $L^2(\Omega)$ for a given set $\Omega \subset [0, 1]$ if and only if $E(\mathbb{Z} \setminus \Lambda)$ is a Riesz basis for $L^2([0, 1] \setminus \Omega)$. Thus, without loss of generality, we assume that we do not have both $a_l = 0$ and $b_j = 1$ for some l and j, as this can then be reduced to the case of $L - 1$ intervals.

Fix $N \in \mathbb{N}$ sufficiently large so that the sequences $\{\{Na_l\}\}_{l=1}^L$ and $\{\{Nb_l\}\}_{l=1}^L$ do not interlace and such that for each $1 \leq l \leq L$ there exists $k_l \in \mathbb{N}$ such that $k_l/N \in [a_l, b_l]$. The last assumption is to assure that $\Psi(0) \geq \Phi(0)$. Let us now look at the sets $C_{\geq n}$ defined in (5.12) with respect to the sequences $\{\{Na_l\}\}_{l=1}^L$ and $\{\{Nb_l\}\}_{l=1}^L$, and the sets $A_{\geq n}$ with respect to Ω and the fixed constant

N. From Lemma 5.8 it follows that the sets $C_{\geq n}$ all are unions of at most $L-1$ intervals. We know that $A_{\geq n} = \Psi^{-1}[n, \Psi(0)+L]$ and that $C_{\geq n} = \Phi^{-1}[n, 2L]$. It follows that $A_{\geq n} = (1/N)C_{\geq m}$ where $m = \max\{n - (\Psi(0) - \Phi(0)), 0\}$. In particular all the sets $A_{\geq n}$ consist of at most $L-1$ intervals. Now we apply the induction hypothesis to find Riesz bases $\Lambda_m \subset \mathbb{Z}$ for each set $L^2(C_{\geq m})$, and thus by scaling $N\Lambda_m$ will be a Riesz basis for $L^2((1/N)C_{\geq m}) = L^2(A_{\geq n})$. Finally, it follows from Theorem 5.1 that we can combine the Riesz bases for the sets $L^2(A_{\geq n})$ into a Riesz basis $E(\Lambda)$ for $L^2(\Omega)$, where $\Lambda \subset \mathbb{Z}$. $\square$

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