Fourth conference of the International Network for Didactic Research in University Mathematics

INDRUM 2022

October 19-22

Hannover

Leibniz University Hannover

PROCEEDINGS

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INDRUM 2022 is an ERME topic conference

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Leibniz Universität Hannover

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TWG3: Teaching and learning of linear and abstract algebra, logic, reasoning and proof

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INTRODUCTION

In this group, there were 10 oral communications and 4 posters that were organised in two thematics: linear and abstract algebra, presented in the first parallel session; logic, reasoning, and proof presented in the second parallel session. The list of papers and posters is in an annexe of the document. The group comprised 19 participants from various countries. Each paper was allocated a 15-minute presentation, followed by 5-minute discussion with the audience. During the two first discussion sessions, a slot was devoted to the presentation of posters linked to the theme of the corresponding parallel session. For the discussion following each of both sessions, we split into three non-thematic subgroups around two papers or a paper and one or two posters. Previously to the conference, for each paper, two registered participants were invited to prepare a few slides to act as critical friends during the conference. These slides have been used during the discussion sessions in small groups. This was followed by a collective discussion.

During the third discussion session, we decided to split into two thematic subgroups: Linear and abstract algebra; Logic, reasoning, and proof, to discuss the papers and posters linked to the thematic and let emerge first ideas. Seven attended the subgroup on linear and abstract algebra, and 12 attended the subgroup on logic, reasoning, and proof. During the fourth session, we go on working in thematic subgroups to reflect collaboratively on the new ideas that have emerged during the conference. Each subgroup has prepared a report on its theme that has been presented in the closing session.

TRENDS AND PERSPECTIVES IN LINEAR AND ABSTRACT ALGEBRA

In this section, we provide a summary of presentations and discussions regarding linear and abstract algebra concluding with a brief synthesis of emerging topics/issues and questions, and further research directions.

The main topics in the papers and posters presented

There were four main topics in the papers and posters; Eigentheory (Piori and Lyse-Olsen & Fleischmann, Wawro & Thompson), vector spaces (Can, Aguilar & Trigueros), Gauss algorithm (My Hahn) and computational thinking (Turgut). Regarding eigentheory, we underline the following three main *themes*:

- Semiotic analysis of signs (including gestures) produced by students during a collective activity aimed at making sense of eigenvectors and eigenvalues.
- Students' conceptions of eigenvectors and eigenvalues
  - Representations and formal elements used in students' descriptions
  - Focus on task design for individual learning activities
- Student reasoning about eigenequations (or not) in quantum mechanics

Piroi explores eigentheory teaching and learning processes in her continuing PhD research and provided the preliminary findings. The focus of the presented paper was an investigation of students' *collective* meaning-making processes within the lens of the theory of objectification, as a sociocultural theory. The paper described an activity that was created especially to support these processes of objectification. University engineering freshmen worked collaboratively to rethink eigentheory principles and rules while working in small groups. The usage of various semiotic resources by students, as well as how they relate to one another and how they have evolved, are then discussed. Under the same topic, Lyse-Olsen & Fleischmann examine students' understanding of eigenvectors at an early stage of their linear algebra instruction. Students' various explanations of eigenvectors are examined in relation to the mathematical objects they choose to depict (algebraic, geometric, or abstract representations), as well as the *formalism* they employed. Students nevertheless demonstrated their ability to switch between many representations and descriptions and produce unique concept images, even when the modes of description that were presented to them appeared to impact their own choice of description.

Wawro & Thompson's poster focused on student reasoning in quantum mechanics regarding matrix equations as eigenequations. Wawro & Thompson particularly explored how students would be able to distinguish between eigenequations and (quantum mechanics) matrix equations, and how this connects to their justifications for eigentheory in both mathematical and quantum contexts.

Regarding the Gauss algorithm, My Hahn presented research about constructing online *cloze style* (fill-in-the-blank with drop-down menu) questions to help students improve their understanding of and ability to use mathematical language. Examples given from discussing solution processes for systems of linear equations were provided. The paper discussed the preliminary findings of the analysis as well as Steinbring's *epistemological triangle* as a potential analytical tool for (aforementioned) comprehension processes.

Regarding vector spaces, Can, Aguilar & Trigueros focused on a teaching strategy for the learning of the concept of vector space using non-standard binary operations with a diversity of sets to promote student reflection on the vector space axioms more generally. The design is based on the APOS theory using its ACE cycle as a didactic approach, and a group of engineering students solved the provided task. Using sets and binary operations that aren't typically covered in a first course in linear algebra encouraged students to think critically about the axioms that define vector spaces.

Regarding computational thinking (CT), Turgut presented an emerging framework for integrating CT into teaching and learning linear algebra. The framework refers to three teaching principles of linear algebra, theory of instrumental genesis and CT. The paper presents a vignette in terms of GeoGebra's specific tools, functions and commands to teach the system of linear equations within the lens of CT.

Discussions, emerging topics and questions

The variety of theoretical and analytical frameworks available to support and guide the various research goals and the flexible way of using these frameworks were one of the main points that were discussed in the group. Presenters referred to different theoretical/conceptual lenses, such as APOS theory, the three teaching principles of linear algebra, theory of objectification, multimodal paradigm and semiotic bundle, knowledge in pieces, symbolic forms, modes of representation/thinking and (levels of) formalism. The broad spectrum of lenses not only raised discussion about the theoretical/conceptual frameworks in themselves, but also brought us to elaborate on how they should be used, the possibility of selecting the topic as a point of departure, and the role of emerging frameworks about eigentheory in research.

The second point that the group discussed was the role of *task design* in our research. For example, where to start and which has a priority; (i) designing to overcome students' epistemological issues, (ii) designing innovative teaching-learning environments, or (iii) both in a synchronised way. The group discussed the function of guiding/orienting frameworks in task design too, like the role of Realistic Mathematics Education theory. The group also discussed the emerging role of CT (Wing, 2006) as a (possible) mediator context to create mathematical meanings of linear algebra topics (and also in other STEM fields), like exploring the system of linear equations and Gram-Schmidt diagonalisation etc.

A third point was about aspects and nature of *formalism* and *representations* in linear algebra. The role of representations, shifting between them (i.e. semiotic registers, in the sense of Duval) and how this could inform researchers to design tasks were discussed. The instructor's role was at the heart of discussion on some occasions, with a particular role to make a shared discussion at the end of each teaching episode.

Future directions for next INDRUM conferences

The following points have emerged after our thematic group discussions regarding linear and abstract algebra. The first point was about the role and purpose of theoretical/conceptual frameworks in our work. The group underlined that some frameworks provide orientation (for example in task design) and a better understanding of the observed phenomenon, but such frameworks come with some constraints. The second point was about the integration with computer science (e.g., CT, computer graphics etc.), and (possibility of) digital assessment of linear and abstract algebra in particular. The third point was the contribution of linear and abstract algebra to the professional development pre-service and in-service mathematics teachers.

TRENDS AND PERSPECTIVES IN LOGIC, REASONING AND PROOF

In this section, we first present the main issues that emerge from the papers presentations; then we present the main elements that emerged from the discussion, namely the new trends and the perspective.

The main issues in the papers and posters presented

The main issues in the papers presented in this thematic were: Multi proofs analysis as a means to foster conceptualisation (Viviane Durand-Guerrier); Comparison of various proof assistants at the interface between mathematics and computer sciences (Evmorfia Iro Bartzia, Antoine Meyer & Julien Narboux); the support of CAS (Kinga Szücs); the use of cloze test and multichoice questionnaire (My Hahn); students reasoning on visual words problems (Francesco Beccuti); the generic power of proofs in number theory (Véronique Battie); refutation beyond counter-examples (Alon Pinto and Jason Cooper); linguistic issues (Dimitri Lipper, Thomas Karavi & Angeliki Mali). During the discussion sessions, two main issues have emerged: mathematical and epistemological needs and overview of analysis criteria.

Mathematical and epistemological needs

There is a necessity to go beyond the illusion of transparency of mathematical knowledge (Artigue, 1991) for strengthening the a priori analysis in a didactic perspective. The question is “how to do this?”. Relying on the research experience of some of the participants, we list the following practices: write down our own proofs, individually and then sharing with our research team and beyond, including researchers and practitioners who specialise in the mathematical domain at stake; making a review of existing proofs in the literature as well in mathematics education as in domain-specific mathematics; analysing historical proofs; considering philosophical perspective; analysing experts’ practises; considering implicit and explicit norms among various communities. The discussion showed that the introduction of proof assistants and computer scientist methods raised new epistemological questions or calls for revisiting more classical ones. This appears as a challenge for research on proof and proving in mathematics education (Hanna, De Villiers & Reid, 2019).

Overview of proof analysis criteria

In the literature and in the participants' research experience and practice, there are various types of analysis criteria that are used in research on proof and proving, both for designing, and for a priori and a posteriori analysis, depending on the goal and the research questions. Nevertheless, we have tried to provide a non-exhaustive list as the first contribution to a collective state-of-the-art that the group considered worthwhile.

When the criteria were explicitly discussed in a paper presented in the TWG, we mention the name of the author(s) in brackets.

The first group of criteria refer to language, level of formalisation and semiotic register. A second group concerns norms and expectations, depending on the audience. A third group addresses the epistemological and pragmatic issue of the type of proof expected or provided, formal proof, operational proof, experimental proof (Dimitri Lipper, Thomas Karavi & Angeliki Mali) and Proof schemata: procedural versus conceptual, meaning versus ritual etc.

The fourth group of criteria concerns the proof structure analysis: organising and operative dimensions (Veronique Battie), data, hypothesis and introduction of objects (Viviane Durand-Guerrier), direct or indirect proof (Alon Pinto & Jason Cooper, visualisation (Francisco Beccuti). The fifth group of criteria refers to the precise analysis of a particular proof: steps in the proving process, logical validity, modes of inference, clarity, modularities, encapsulation, etc.

Perspectives for next INDRUM conferences

The first one is the relevance of revisiting epistemological questions in light of the increasing use of proof assistants in both research in mathematics and in mathematics education. The second one aims to deepen the logical dimension of analysis for the teaching and learning of proof and proving, which have already been shown to be relevant, but remain underrepresented in many research on proof and proving. The third one concerns the necessity of going on designing and implementing activities aiming at improving proof and proving skills at university, taking into consideration the specificity of the condition and constraints in university mathematics education. Another issue consists in exploring more systematically the possible contribution of proof and proving to address university mathematics students' difficulties, depending on the mathematical domain.

Finally, we consider that introducing specific work on proof and proving in University teacher training would be valuable, to initiate a change in the way proof is taught in general at university: moving from proof made in front of students, to students' proof elaboration and analysis.

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