North-Western European Journal of Mathematics

Hilbert points in Hilbert space-valued L^p spaces

Ole Fredrik Brevig[1] Sigrid Grepstad[2]

Received: March 4, 2022/Accepted: February 6, 2023/Online: February 20, 2023

Abstract

Let H be a Hilbert space and $(\Omega, \mathcal{F}, \mu)$ a probability space. A Hilbert point in $L^p(\Omega; H)$ is a nontrivial function φ such that $\|\varphi\|_p \leq \|\varphi + f\|_p$ whenever $\langle f, \varphi \rangle = 0$. We demonstrate that φ is a Hilbert point in $L^p(\Omega; H)$ for some $p \neq 2$ if and only if $\|\varphi(\omega)\|_H$ assumes only the two values 0 and $C > 0$. We also obtain a geometric description of when a sum of independent Rademacher variables is a Hilbert point.

Keywords: Hilbert points, vector-valued functions.

MSC: 46E40, 46B09, 60G50.

1 Introduction

Suppose that $(\Omega, \mathcal{F}, \mu)$ is a probability space and that H is a Hilbert space. For $1 \leq p \leq \infty$, consider the usual L^p spaces of H-valued Bochner integrable functions f on Ω. A *Hilbert point* in $L^p(\Omega; H)$ is a nontrivial function φ in $L^p(\Omega; H)$ which enjoys the property that

$$\langle f, \varphi \rangle = 0 \qquad \Longrightarrow \qquad \|\varphi\|_p \leq \|\varphi + f\|_p, \tag{1}$$

for every f in $L^p(\Omega; H)$. The inner product in (1) is that of the Hilbert space $L^2(\Omega; H)$, namely

$$\langle f, g \rangle = \int_\Omega \langle f(\omega), g(\omega) \rangle_H \, d\mu(\omega). \tag{2}$$

Every nontrivial function in $L^2(\Omega; H)$ is evidently a Hilbert point in $L^2(\Omega; H)$, since in this case $\langle f, \varphi \rangle = 0$ if and only if $\|\varphi + f\|_2^2 = \|\varphi\|_2^2 + \|f\|_2^2$.

If $p < 2$, then some care has to be taken when interpreting the inner product in (1). We declare here that $\langle f, \varphi \rangle = 0$ whenever f lies in the $L^p(\Omega; H)$ closure of the set of functions g in $L^\infty(\Omega; H)$ which satisfy $\langle g, \varphi \rangle = 0$. It turns out that this precaution is unnecessary, in view of our first main result.

[1]Department of Mathematics, University of Oslo, 0851 Oslo, Norway

[2]Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway

Theorem 1 – *A nontrivial function φ is a Hilbert point in $L^p(\Omega;H)$ for some $p \neq 2$ if and only if there is a constant $C > 0$ and a set E with $\mu(E) > 0$ such that*

$$\|\varphi(\omega)\|_H = \begin{cases} C, & \omega \in E, \\ 0, & \omega \notin E. \end{cases} \tag{3}$$

The functions satisfying (3) are the eigenfunctions of the nonlinear operator associated with Hölder's inequality discussed in Section 1.2 of Hedenmalm et al. (2018). Equivalently, if $p^{-1} + q^{-1} = 1$, then the nontrivial functions attaining equality in Hölder's inequality

$$\langle \varphi, \varphi \rangle \leq \|\varphi\|_p \|\varphi\|_q$$

are precisely those obeying (3). It is therefore hardly surprising that Hölder's inequality has a crucial role to play in the proof of Theorem 1.

The requirement (3) does not depend on $p \neq 2$, and so we observe the following corollary to Theorem 1: A nontrivial function φ is a Hilbert point in $L^p(\Omega;H)$ for *some* $p \neq 2$ if and only if it is a Hilbert point in $L^p(\Omega;H)$ for *every* $1 \leq p \leq \infty$.

The notion of Hilbert points was introduced in a recent paper of Brevig, Ortega-Cerdà, and Seip (2022). The focus of that paper was on the Hardy spaces of d-dimensional tori, denoted $H^p(\mathbb{T}^d)$. In contrast to the situation we have just observed, their results demonstrate that in the context of $H^p(\mathbb{T}^d)$ a nontrivial function may be a Hilbert point for only *one* exponent $p \neq 2$.

Another direct corollary of Theorem 1 (with $\Omega = \mathbb{T}^d$ and $H = \mathbb{C}$) is the following: If a Hilbert point φ in $H^p(\mathbb{T}^d)$ for some $p \neq 2$ extends to a Hilbert point in the larger space $L^p(\mathbb{T}^d)$, then φ is a constant multiple of an inner function. Theorem 1 similarly provides a new proof of Corollary 2.5 in Brevig, Ortega-Cerdà, and Seip (2022), which states that constant multiples of inner functions generate Hilbert points on $H^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$.

Information about certain Hilbert points in $H^p(\mathbb{T}^d)$ was parlayed in Section 5 of the above-mentioned paper to a new proof of the optimal constants in Khintchine's inequality for independent identically distributed Steinhaus variables for $2 < p < \infty$.

In the present paper, we will investigate independent identically distributed Rademacher variables. Let $(\omega_j)_{j\geq 1}$ be a sequence of independent random variables taking the values ± 1 with equal probability and consider the H-valued function

$$\varphi(\omega) = \sum_{j=1}^{\infty} \omega_j \mathbf{x}_j. \tag{4}$$

From the definition of the inner product (2) and a computation, it follows that if φ denotes a function of the form (4), then

$$\|\varphi\|_2^2 = \sum_{j=1}^{\infty} \|\mathbf{x}_j\|_H^2. \tag{5}$$

Our second main result is a characterization of the Hilbert points of the form (4).

Theorem 2 – *Let* $\Omega = \{-1,1\}^\infty$ *be the Cantor group and* μ *its Haar measure. The function* (4) *is a Hilbert point in* $L^p(\Omega;H)$ *for some* $p \neq 2$ *if and only if either*

(a) $(\mathbf{x}_j)_{j\geq 1}$ *is an orthogonal sequence satisfying* $\sum_{j\geq 1} \|\mathbf{x}_j\|_H^2 < \infty$.

(b) $\mathbf{x}_1 = \mathbf{x}_2$ *for a nonzero vector* $\mathbf{x}_1$ *and* $\mathbf{x}_j = 0$ *for all* $j \geq 3$.

(c) $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors satisfying* $\|\mathbf{u}\|_H = \|\mathbf{v}\|_H$ *and* $\mathbf{u} \perp \mathbf{v}$, *and*

$$\mathbf{x}_1 = \mathbf{u}, \qquad \mathbf{x}_2 = \frac{1}{2}\mathbf{u} + \frac{\sqrt{3}}{2}\mathbf{v}, \qquad \mathbf{x}_3 = \frac{1}{2}\mathbf{u} - \frac{\sqrt{3}}{2}\mathbf{v},$$

and $\mathbf{x}_j = 0$ *for all* $j \geq 4$.

We close out this introduction with a probabilistic interpretation of Theorem 2. Let us therefore think of H-valued functions f on Ω as H-valued random variables. We refer to Section 6.1 in Hytönen et al. (2017) for the definitions of some standard concepts used below. In particular, a function is called *real symmetric* if φ and $-\varphi$ have the same distribution. Functions of the form (4) are real symmetric, as are analogous sums of independent Steinhaus variables. It thus follows from Proposition 6.1.5 in Hytönen et al. (2017) that if φ is of the form (4) and f and φ are *independent*, then

$$\|f\|_p \leq \|f+\varphi\|_p \tag{6}$$

for every $1 \leq p \leq \infty$. If f is an integrable H-valued or scalar-valued random variable, then the expectation $\mathbb{E}(f) = \int_\Omega f(\omega)\,d\mu(\omega)$ exists. The *covariance* of two H-valued random variables f and g is defined by

$$\mathrm{Cov}(f,g) = \mathbb{E}(\langle f,g\rangle_H) - \langle \mathbb{E}(f), \mathbb{E}(g)\rangle_H,$$

and two H-valued random variables are said to be *uncorrelated* if $\mathrm{Cov}(f,g) = 0$. Since the real-symmetric variables φ from (4) satisfy $\mathbb{E}(\varphi) = \mathbf{0}$, we have the following reformulation of Theorem 2 which should be compared with (6): Fix $p \neq 2$. The H-valued random variables of the form (4) satisfying the requirement $\|\varphi\|_p \leq \|f+\varphi\|_p$ for every f which is uncorrelated to φ are precisely those given by Theorem 2.

Organization

This paper is comprised of two further sections. Section 2 is devoted to the proof of Theorem 1. The proof of Theorem 2 can be found in Section 3.

2 Proof of Theorem 1

We split the proof of Theorem 1 into three parts, and present these in order of increasing difficulty.

Proof (Theorem 1: Sufficiency). Assume that (3) holds, meaning that there is a constant $C > 0$ and a set E with $\mu(E) > 0$ such that

$$\|\varphi(\omega)\|_H = \begin{cases} C, & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$$

Our goal is to show that φ is a Hilbert point in $L^p(\Omega;H)$. Since φ is bounded, it is clear that φ is in $L^p(\Omega;H)$ for every $1 \leq p \leq \infty$. In particular, the orthogonal projection

$$P_\varphi f = \frac{\langle f, \varphi\rangle}{\|\varphi\|_2^2}\varphi \tag{7}$$

extends to a bounded operator on $L^p(\Omega;H)$. Using Hölder's inequality, we find that

$$\|P_\varphi\|_{L^p(\Omega;H)\to L^p(\Omega;H)} \leq \frac{\|\varphi\|_q}{\|\varphi\|_2^2}\|\varphi\|_p = 1.$$

The final equality follows from the fact that φ satisfies (3). Now if $\langle f, \varphi\rangle = 0$, then

$$\|\varphi\|_p = \|P_\varphi(\varphi + f)\|_p \leq \|\varphi + f\|_p,$$

which demonstrates that φ is a Hilbert point in $L^p(\Omega;H)$. □

The necessity part in the proof of Theorem 1 requires a separate argument for the case $p = \infty$.

Proof (Theorem 1: Necessity for $p = \infty$). Since φ is in $L^\infty(\Omega;H)$ the orthogonal projection P_φ in (7) is a bounded operator on $L^\infty(\Omega, H)$. Every f in $L^\infty(\Omega;H)$ can be orthogonally decomposed as $f = c\varphi + g$, so the assumption that φ is a Hilbert point in $L^\infty(\Omega;H)$ implies that

$$\|P_\varphi f\|_\infty = \|c\varphi\|_\infty \leq \|c\varphi + g\|_\infty = \|f\|_\infty.$$

Here we tacitly used the fact that φ is Hilbert point if and only if $c\varphi$ is a Hilbert point for every constant $c \neq 0$. This shows that $\|P_\varphi\|_{L^\infty(\Omega,H)\to L^\infty(\Omega;H)} \leq 1$. Consider

$$\psi(\omega) = \begin{cases} \frac{\varphi(\omega)}{\|\varphi(\omega)\|_H}, & \text{if } \varphi(\omega) \neq \mathbf{0}, \\ \mathbf{0}, & \text{if } \varphi(\omega) = \mathbf{0}. \end{cases}$$

Since φ is nontrivial, it is clear that $\|\psi\|_{L^\infty(\Omega,H)} = 1$. Hence $\|P_\varphi \psi\|_\infty \leq 1$, which means that

$$\frac{\|\varphi\|_1}{\|\varphi\|_2^2}\|\varphi\|_\infty \leq 1 \qquad \Longleftrightarrow \qquad \|\varphi\|_1\|\varphi\|_\infty \leq \langle \varphi, \varphi\rangle.$$

The final inequality is actually an equality, since the reverse inequality is Hölder's inequality. This is only possible if (3) holds. □

The proof we have just given can easily be adapted to work also for $2 < p < \infty$. However, we run into problems for $1 \leq p < 2$ since we cannot guarantee that the orthogonal projection P_φ in (7) is well-defined. The issue is simply that we cannot guarantee a priori that φ is in $L^2(\Omega;H)$. We circumvent this problem by using the Riesz representation theorem for $L^p(\Omega;H)$, which holds in our context since every Hilbert space enjoys the Radon–Nikodym property. We refer broadly to Chapter IV in Diestel and Uhl (1977). The following proof is inspired by arguments from Section 4.2 in Shapiro (1971).

Proof (Theorem 1: Necessity for $p < \infty$). Suppose that φ is a Hilbert point in $L^p(\Omega;H)$ where $1 \leq p < 2$ or $2 < p < \infty$. Our first goal is to show that we may assume without loss of generality that φ does not vanish. (This reduction is technically only needed in the proof of the case $p = 1$, but it simplifies the overall exposition also for $p > 1$.) Set $N = \varphi^{-1}(\{\mathbf{0}\})$. The behavior of f on N does not affect the inner product $\langle f, \varphi\rangle$ and may only increase the norm $\|\varphi + f\|_p$. When checking the Hilbert point condition, it is therefore sufficient to consider only f that vanish on N. Replacing Ω with $\Omega \setminus N$, we may therefore assume that φ does not vanish.

Let X be the subspace of $L^p(\Omega;H)$ formed by taking the closure of the set of bounded functions g which satisfy $\langle g, \varphi\rangle = 0$. The assumption that φ is a Hilbert point in $L^p(\Omega;H)$ means that if f is in X, then $\|\varphi\|_p \leq \|\varphi + f\|_p$. This implies that

$$\|\varphi\|_p = \operatorname{dist}(\varphi, X) = \inf_{f\in X}\|\varphi + f\|_p. \tag{8}$$

It follows from (8) and the Hahn–Banach theorem that there exists a linear functional Φ on $L^p(\Omega;H)$ such that $\Phi(\varphi) = 1$, $\Phi(f) = 0$ for every f in X and $\|\Phi\| = \|\varphi\|_p^{-1}$. In particular, the norm of Φ is attained at φ. Now we bring into play the Riesz representation theorem, which tells us that $\Phi(\cdot) = \langle \cdot, \psi\rangle$ for some unique function ψ in $L^q(\Omega;H)$ where $p^{-1} + q^{-1} = 1$. However, this means that

$$\frac{1}{\|\varphi\|_p} = \|\Phi\| = \|\psi\|_q \qquad \text{and} \qquad 1 = \Phi(\varphi) = \langle \varphi, \psi\rangle,$$

which shows that $\langle \varphi, \psi\rangle = \|\varphi\|_p\|\psi\|_q$. Since we have attained equality in Hölder's inequality and since φ does not vanish, we necessarily conclude that

$$\psi(\omega) = \frac{\|\varphi(\omega)\|_H^{p-2}}{\|\varphi\|_p^p}\varphi(\omega).$$

In view of what we have just established, it follows that if g is in $L^\infty(\Omega;H)$, then

$$\langle g,\varphi\rangle = 0 \qquad \Longrightarrow \qquad \langle g, \|\varphi\|_H^{p-2}\varphi\rangle = 0. \tag{9}$$

Let F_1 and F_2 be sets of positive measure and define

$$g = \left(\int_{F_1} \|\varphi\|_H \, d\mu\right)^{-1} \chi_{F_1}\frac{\varphi}{\|\varphi\|_H} - \left(\int_{F_2} \|\varphi\|_H \, d\mu\right)^{-1} \chi_{F_2}\frac{\varphi}{\|\varphi\|_H}.$$

Here χ_{F_1} and χ_{F_2} are the characteristic (scalar) functions of F_1 and F_2, respectively. Since φ does not vanish and since φ is in $L^1(\Omega;H)$ by Hölder's inequality, it is clear that g is in $L^\infty(\Omega;H)$ and that $\langle g,\varphi\rangle = 0$. It therefore follows from (9) that

$$\langle g, \|\varphi\|_H^{p-2}\varphi\rangle = 0 \qquad \Longrightarrow \qquad \frac{\int_{F_1} \|\varphi\|_H^{p-1}\, d\mu}{\int_{F_1} \|\varphi\|_H \, d\mu} = \frac{\int_{F_2} \|\varphi\|_H^{p-1}\, d\mu}{\int_{F_2} \|\varphi\|_H \, d\mu}. \tag{10}$$

There are now two cases. If $p > 2$, then it follows from (10) that

$$\inf_{\omega\in F_1} \|\varphi(\omega)\|_H^{p-2} \leq \frac{\int_{F_1} \|\varphi\|_H^{p-1}\, d\mu}{\int_{F_1} \|\varphi\|_H \, d\mu} = \frac{\int_{F_2} \|\varphi\|_H^{p-1}\, d\mu}{\int_{F_2} \|\varphi\|_H \, d\mu} \leq \sup_{\omega\in F_2} \|\varphi(\omega)\|_H^{p-2}. \tag{11}$$

Since F_1 and F_2 are arbitrary sets of positive measure, we deduce from this that $\|\varphi(\omega)\|_H^{p-2} = C$ for almost every ω in Ω which means that (3) holds. The same argument works for $1 \leq p < 2$, provided we first swap inf and sup in (11). □

As discussed in Brevig, Ortega-Cerdà, and Seip (2022), the term Hilbert point is meant to indicate that we are dealing with points in a Banach space around which the geometry of the Banach space behaves like a Hilbert space. In the present paper, we restrict ourselves to mentioning the following immediate corollary of Theorem 1.

Corollary 1 – *A nontrivial function φ is a Hilbert point in $L^p(\Omega;H)$ for some $p \neq 2$ if and only if φ is in $L^2(\Omega;H)$ and the orthogonal projection*

$$P_\varphi f = \frac{\langle f,\varphi\rangle}{\|\varphi\|_2^2}\varphi$$

extends to a norm 1 *operator on $L^p(\Omega;H)$.*

Proof. If φ is a Hilbert point in $L^p(\Omega;H)$, then it follows from Theorem 1 that φ is in $L^2(\Omega;H)$ and that $\|P_\varphi\|_{L^p(\Omega;H)\to L^p(\Omega;H)} \leq 1$ by Hölder's inequality. To see that actually $\|P_\varphi\|_{L^p(\Omega;H)\to L^p(\Omega;H)} = 1$ it is sufficient to set $f = \varphi$.

Conversely, suppose that φ is in $L^2(\Omega;H)$ and that the orthogonal projection P_φ extends to a norm 1 operator on $L^p(\Omega;H)$. It now follows at once that φ is a Hilbert point since if $\langle f,\varphi\rangle = 0$, then $\|\varphi\|_p = \|P_\varphi(\varphi+f)\|_p \leq \|\varphi+f\|_p$. □

3 Proof of Theorem 2

Let it be known that all norms and inner products in the present section will be with respect to some fixed Hilbert space H. This is a notational departure from what has been previously employed.

To warm up, we use Theorem 1 to check that the three cases (a), (b) and (c) of Theorem 2 indeed describe Hilbert points φ.

Proof (Theorem 2: Sufficiency). We are going to use Theorem 1.

(a) If the sequence $(\mathbf{x}_j)_{j\geq 1}$ is orthogonal and $\sum_{j\geq 1}\|\mathbf{x}_j\|^2 < \infty$, then

$$\|\varphi(\omega)\|^2 = \sum_{j=1}^{\infty}\|\mathbf{x}_j\|^2 = C$$

for every ω in Ω since $|\pm 1| = 1$. Hence φ is a Hilbert point.

(b) If $\varphi(\omega) = \omega_1\mathbf{x}_1 + \omega_2\mathbf{x}_1$, then $\varphi(\pm 1, \pm 1) = \pm 2\mathbf{x}_1$ and $\varphi(\pm 1, \mp 1) = \mathbf{0}$. Since either $\|\varphi(\omega)\| = 0$ or $\|\varphi(\omega)\| = 2\|\mathbf{x}_1\| = C$, we see that φ is a Hilbert point.

(c) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors of equal length with $\mathbf{u}\perp\mathbf{v}$ and set

$$\varphi(\omega) = \omega_1\mathbf{u} + \omega_2\left(\frac{1}{2}\mathbf{u} + \frac{\sqrt{3}}{2}\mathbf{v}\right) + \omega_3\left(\frac{1}{2}\mathbf{u} - \frac{\sqrt{3}}{2}\mathbf{v}\right).$$

There are eight choices of ω. We first compute

$$\varphi(1,1,1) = 2\mathbf{u}, \qquad \varphi(-1,1,1) = \mathbf{0}, \qquad \varphi(1,\pm 1,\mp 1) = \mathbf{u}\pm\sqrt{3}\mathbf{v}.$$

The four remaining choices of ω produce the same vectors multiplied by -1. For our purposes it is therefore sufficient to consider these four. Since $\mathbf{u}\perp\mathbf{v}$ and $\|\mathbf{u}\| = \|\mathbf{v}\|$ we get

$$\|\mathbf{u}\pm\sqrt{3}\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 3\|\mathbf{v}\|^2} = 2\|\mathbf{u}\| = C,$$

and hence φ is a Hilbert point. □

The more difficult "necessity" part of the proof requires some preparation in the form of a trio of lemmas on elementary Hilbert space geometry.

Lemma 1 – *Suppose that $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ are nonzero vectors in H such that*

$$\|\mathbf{u}_0\| = \|\mathbf{u}_0 + \mathbf{u}_1\| = \|\mathbf{u}_0 + \mathbf{u}_2\|.$$

(a) *If* $\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|=0$, *then*

$$\mathbf{u}_1=-\frac{1}{2}\mathbf{u}_0+\frac{\sqrt{3}}{2}\mathbf{v} \qquad \textit{and} \qquad \mathbf{u}_2=-\frac{1}{2}\mathbf{u}_0-\frac{\sqrt{3}}{2}\mathbf{v},$$

where $\mathbf{v}$ *is a vector which satisfies* $\mathbf{v}\perp\mathbf{u}_0$ *and* $\|\mathbf{v}\|=\|\mathbf{u}_0\|$.

(b) *If* $\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|=\|\mathbf{u}_0\|$, *then* $\mathbf{u}_1\perp\mathbf{u}_2$.

Proof. For (a) we see that $\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|=0$ implies that $\mathbf{u}_1=-(\mathbf{u}_0+\mathbf{u}_2)$. Since $\|\mathbf{u}_0+\mathbf{u}_2\|=\|\mathbf{u}_0+\mathbf{u}_1\|$ by assumption, we see that $\|\mathbf{u}_1\|=\|\mathbf{u}_0+\mathbf{u}_1\|$. Expanding, we obtain

$$\|\mathbf{u}_1\|^2=\|\mathbf{u}_0\|^2+2\langle\mathbf{u}_0,\mathbf{u}_1\rangle+\|\mathbf{u}_1\|^2,$$

which means that $\mathbf{u}_1=-\frac{1}{2}\mathbf{u}_0+\widetilde{\mathbf{u}}_1$ where $\widetilde{\mathbf{u}}_1\perp\mathbf{u}_0$. It follows that

$$\mathbf{u}_2=-(\mathbf{u}_0+\mathbf{u}_1)=-\frac{1}{2}\mathbf{u}_0-\widetilde{\mathbf{u}}_1,$$

and since $\|\mathbf{u}_1\|=\|\mathbf{u}_0\|$ we conclude that $\|\widetilde{\mathbf{u}}_1\|=\frac{\sqrt{3}}{2}\|\mathbf{u}_0\|$. This implies the stated result.

For (b) we compute

$$\begin{aligned}\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|^2&=\|\mathbf{u}_0+\mathbf{u}_1\|^2+2\langle\mathbf{u}_0+\mathbf{u}_1,\mathbf{u}_2\rangle+\|\mathbf{u}_2\|^2\\&=\|\mathbf{u}_0+\mathbf{u}_1\|^2+2\langle\mathbf{u}_1,\mathbf{u}_2\rangle+\|\mathbf{u}_0+\mathbf{u}_2\|^2-\|\mathbf{u}_0\|^2.\end{aligned}$$

Since $\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|=\|\mathbf{u}_0+\mathbf{u}_1\|=\|\mathbf{u}_0+\mathbf{u}_2\|=\|\mathbf{u}_0\|$, we see that $\langle\mathbf{u}_1,\mathbf{u}_2\rangle=0$. □

Lemma 2 – *Suppose that* $\mathbf{u}_0,\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ *are nonzero vectors in* H *such that*

$$\|\mathbf{u}_0\|=\|\mathbf{u}_0+\mathbf{u}_1\|=\|\mathbf{u}_0+\mathbf{u}_2\|=\|\mathbf{u}_0+\mathbf{u}_3\|$$

and that

$$\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|, \quad \|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_3\|, \quad \|\mathbf{u}_0+\mathbf{u}_2+\mathbf{u}_3\| \quad \textit{and} \quad \|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3\|$$

are (independently) equal either to 0 *or to* $\|\mathbf{u}_0\|$. *Then*

$$\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2\|=\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_3\|=\|\mathbf{u}_0+\mathbf{u}_2+\mathbf{u}_3\|=\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3\|=\|\mathbf{u}_0\|.$$

Proof. We begin by demonstrating that

$$\|\mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3\|=0 \tag{12}$$

is impossible. If (12) holds, then $\mathbf{u}_1=-(\mathbf{u}_0+\mathbf{u}_2+\mathbf{u}_3)$. Hence $\|\mathbf{u}_1\|=\|\mathbf{u}_0+\mathbf{u}_2+\mathbf{u}_3\|$ is equal to either 0 or $\|\mathbf{u}_0\|$. By our assumption that $\mathbf{u}_1$ is nonzero it follows that $\|\mathbf{u}_1\|=\|\mathbf{u}_0\|$. Combining this with the assumption that $\|\mathbf{u}_0+\mathbf{u}_1\|=\|\mathbf{u}_0\|$ as in the proof of Lemma 1 (a), we conclude that $\mathbf{u}_1 = -\frac{1}{2}\mathbf{u}_0 + \widetilde{\mathbf{u}}_1$ where $\widetilde{\mathbf{u}}_1 \perp \mathbf{u}_0$. By symmetry, the same holds also for $\mathbf{u}_2$ and $\mathbf{u}_3$. By orthogonality, we find that

$$\|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3\| \geq \frac{1}{2}\|\mathbf{u}_0\|,$$

contradicting the assumption that $\mathbf{u}_0$ is nonzero in view of (12). Hence

$$\|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3\| = \|\mathbf{u}_0\|. \tag{13}$$

By symmetry, it remains to show that

$$\|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2\| = 0 \tag{14}$$

is impossible. If (14) holds, then $\mathbf{u}_1 = -(\mathbf{u}_0 + \mathbf{u}_2)$. As in the proof of Lemma 1 (a) we get that $\mathbf{u}_1 = -\frac{1}{2}\mathbf{u}_0 + \widetilde{\mathbf{u}}_1$ where $\widetilde{\mathbf{u}}_1 \perp \mathbf{u}_0$ and $\|\widetilde{\mathbf{u}}_1\| = \frac{\sqrt{3}}{2}\|\mathbf{u}_0\|$. By (14) we find that $\mathbf{u}_2 = -\frac{1}{2}\mathbf{u}_0 - \widetilde{\mathbf{u}}_1$. By (13), we know that $\|\mathbf{u}_3\| = \|\mathbf{u}_0\|$, which means that $\mathbf{u}_3 = -\frac{1}{2}\mathbf{u}_0 + \widetilde{\mathbf{u}}_3$ where $\widetilde{\mathbf{u}}_3 \perp \mathbf{u}_0$ and $\|\widetilde{\mathbf{u}}_3\| = \frac{\sqrt{3}}{2}\|\mathbf{u}_0\|$. We now look at the final two expressions

$$\|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_3\| = \|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\| \qquad \text{and} \qquad \|\mathbf{u}_0 + \mathbf{u}_2 + \mathbf{u}_3\| = \|-\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\|,$$

which are by assumption either equal to 0 or to $\|\mathbf{u}_0\|$. There are two possible cases.

1. If (at least) one is equal to 0, say $\|-\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\| = 0$, then $\widetilde{\mathbf{u}}_3 = \widetilde{\mathbf{u}}_1$ and hence

$$\|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\| = 2\|\widetilde{\mathbf{u}}_1\| = \sqrt{3}\|\mathbf{u}_0\|.$$

   This contradicts the assumption that $\|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\|$ is equal to 0 or $\|\mathbf{u}_0\|$, since $\mathbf{u}_0$ is nonzero by assumption. A similar contradiction is reached if we start by assuming that $\|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\| = 0$.

2. If both are equal to $\|\mathbf{u}_0\|$, then

$$0 = \|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\|^2 - \|-\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\|^2 = 4\langle \widetilde{\mathbf{u}}_1, \widetilde{\mathbf{u}}_3 \rangle.$$

   This shows that $\|\widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_3\|^2 = \frac{3}{2}\|\mathbf{u}_0\|^2$ which cannot be equal to $\|\mathbf{u}_0\|^2$ since $\mathbf{u}_0$ is nonzero by assumption.

We conclude that (14) is false and hence $\|\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2\| = \|\mathbf{u}_0\|$. □

For a sequence of vectors $(\mathbf{u}_j)_{j\geq 1}$ in H and a subset J of $\mathbb{N}$, let

$$\mathbf{u}(J) = \sum_{j \in J} \mathbf{u}_j,$$

and let $|J|$ denote the cardinality of J. We are now ready to establish the third and final geometric lemma.

Lemma 3 – *Let $\mathbf{u}_0$ be a nonzero vector in H, and for an index set J with $|J| \geq 3$ suppose that $(\mathbf{u}_j)_{j\in J}$ is a sequence of nonzero vectors in H with*

$$\|\mathbf{u}_0 + \mathbf{u}_j\| = \|\mathbf{u}_0\|$$

for all j in J. If $\|\mathbf{u}_0 + \mathbf{u}(K)\|$ is (independently) equal to either 0 or $\|\mathbf{u}_0\|$ for every finite subset K of J, then $\|\mathbf{u}_0 + \mathbf{u}(K)\| = \|\mathbf{u}_0\|$ for every finite subset K of J.

Proof. The proof is based on Lemma 2 and induction. If $|K| = 0$ or $|K| = 1$, then there is nothing to prove. If $|K| = 2$ or $|K| = 3$, then the result follows directly from Lemma 2. Suppose therefore that $|K| \geq 4$. Without loss of generality, we may assume that $\{1,2,3\} \subset K$ and write

$$\mathbf{u}(K) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}(\widetilde{K}),$$

where $\widetilde{K} = K \setminus \{1,2,3\}$. By the induction hypothesis,

$$\|\mathbf{u}_0 + \mathbf{u}(\widetilde{K})\| = \|\mathbf{u}_0 + \mathbf{u}(\widetilde{K}) + \mathbf{u}_1\| = \|\mathbf{u}_0 + \mathbf{u}(\widetilde{K}) + \mathbf{u}_2\| = \|\mathbf{u}_0 + \mathbf{u}(\widetilde{K}) + \mathbf{u}_3\| = \|\mathbf{u}_0\|.$$

By Lemma 2 (with $\mathbf{u}_0$ replaced by $\mathbf{u}_0 + \mathbf{u}(\widetilde{K})$), we find that $\|\mathbf{u}_0 + \mathbf{u}(K)\| = \|\mathbf{u}_0\|$. □

Proof (Theorem 2: Necessity). Suppose that φ is a Hilbert point in $L^p(\Omega;H)$ of the form

$$\varphi(\omega) = \sum_{j=1}^{\infty} \omega_j \mathbf{x}_j.$$

Since φ is in $L^p(\Omega;H)$ by assumption, we can combine (5) with Khintchine's inequality (see e.g. Corollary 3.2.24 in Hytönen et al. (2016)) for $1 \leq p < 2$ or with Hölder's inequality for $2 < p \leq \infty$, to conclude that

$$\sum_{j=1}^{\infty} \|\mathbf{x}_j\|^2 < \infty.$$

It now follows from Theorem 3.2 in Kahane (1985) that the series $\varphi(\omega)$ converges in H for almost every ω in Ω. We may replace $\mathbf{x}_j$ by $-\mathbf{x}_j$ without affecting whether φ is a Hilbert point, which allows us to assume without loss of generality that

$$\varphi(\mathbf{1}) = \sum_{j=1}^{\infty} \mathbf{x}_j$$

converges in H. If φ is not identically equal to $\mathbf{0}$, we may also assume that $\varphi(\mathbf{1}) \neq \mathbf{0}$. Let us for simplicity set $\mathbf{u}_0 = \varphi(\mathbf{1})$.

If we start from $\omega = \mathbf{1}$ and change the sign of ω_j for a finite set of indices, then the series for $\varphi(\omega)$ will remain convergent in H. Specifically, changing the sign of one ω_j we obtain the vectors

$$\varphi(\omega) = \mathbf{u}_0 - 2\mathbf{x}_j \tag{15}$$

for $j = 1, 2, 3, \ldots$. Since φ is a Hilbert point by assumption, we know from Theorem 1 that the norms of each of the vectors in (15) are equal to either 0 or to $\|\mathbf{u}_0\|$. Let J denote the index set of those vectors in $(\mathbf{x}_j)_{j\geq 1}$ which are nonzero and which satisfy $\|\mathbf{u}_0 - 2\mathbf{x}_j\| = \|\mathbf{u}_0\|$. There are four cases to be considered.

1. If $|J| = 0$, then any nonzero vector $\mathbf{x}_j$ satisfies $\|\mathbf{u}_0 - 2\mathbf{x}_j\| = 0$, and thus $\mathbf{x}_j = \frac{1}{2}\mathbf{u}_0$. Since $\sum_{j\geq 1}\mathbf{x}_j = \mathbf{u}_0$, it follows that $\mathbf{x}_j = 0$ for all but two indices, and we are in case (b) of Theorem 2.

2. If $|J| = 1$, then we can without loss of generality assume that $J = \{1\}$. For all $j \geq 2$ we therefore have either $\mathbf{x}_j = 0$ or $\|\mathbf{u}_0 - 2\mathbf{x}_j\| = 0$. In the latter case it follows that $\mathbf{x}_j = \frac{1}{2}\mathbf{u}_0$. Hence

$$\mathbf{u}_0 - \mathbf{x}_1 = \sum_{j=2}^{\infty}\mathbf{x}_j = c\mathbf{u}_0,$$

   where $c = k/2$ for a nonnegative integer k. This is only possible if $\mathbf{x}_j = 0$ for all but finitely many j. We get $\mathbf{x}_1 = (1-c)\mathbf{u}_0$, and since $\|\mathbf{u}_0 - 2\mathbf{x}_1\| = \|\mathbf{u}_0\|$ and $\mathbf{x}_1 \neq 0$, it follows that $c = 0$. This means that we are in case (a) of Theorem 2 with only one nonzero vector in the sequence.

3. In the case $|J| = 2$, we use Lemma 1. We assume without loss of generality that $J = \{1, 2\}$, and consider two subcases.

   First, if

$$\|\mathbf{u}_0 - 2(\mathbf{x}_1 + \mathbf{x}_2)\| = \|\mathbf{u}_0\|,$$

   then by Lemma 1 (b) we conclude that $\mathbf{x}_1 \perp \mathbf{x}_2$. Suppose for the purposes of contradiction that there is some $\mathbf{x}_k$ with $k \geq 3$ such that $\|\mathbf{u}_0 - 2\mathbf{x}_k\| = 0$ and $\mathbf{x}_k = \frac{1}{2}\mathbf{u}_0$. Choosing $\omega_j = -1$ for $j = 1, k$ and $\omega_j = 1$ for every other j, we get

$$\varphi(\omega) = \mathbf{u}_0 - 2(\mathbf{x}_1 + \mathbf{x}_k) = -2\mathbf{x}_1.$$

   Since φ is a Hilbert point and $\mathbf{x}_1 \neq \mathbf{0}$, we must have $\|\mathbf{x}_1\| = \frac{1}{2}\|\mathbf{u}_0\|$. The same argument shows that $\|\mathbf{x}_2\| = \frac{1}{2}\|\mathbf{u}_0\|$. Choosing $\omega_j = -1$ for $j = 1, 2, k$ and $\omega_j = 1$ for every other j, the assumption that φ is a Hilbert point implies that $\|\mathbf{x}_1 + \mathbf{x}_2\|$ is equal to either 0 or $\frac{1}{2}\|\mathbf{u}_0\|$. Since $\mathbf{x}_1 \perp \mathbf{x}_2$, we get that

$$\|\mathbf{x}_1 + \mathbf{x}_2\| = \frac{\sqrt{2}}{2}\|\mathbf{u}_0\|,$$

   which is a contradiction. Hence $\mathbf{x}_j = \mathbf{0}$ for every $j \geq 3$, and we are in case (a) of Theorem 2 with only two nonzero vectors in the sequence.

Second, if

$$\|\mathbf{u}_0 - 2(\mathbf{x}_1 + \mathbf{x}_2)\| = 0,$$

then Lemma 1 (a) shows that

$$\mathbf{x}_1 = \frac{1}{4}\mathbf{u}_0 + \frac{\sqrt{3}}{4}\mathbf{v} \qquad \text{and} \qquad \mathbf{x}_2 = \frac{1}{4}\mathbf{u}_0 - \frac{\sqrt{3}}{4}\mathbf{v}$$

where $\mathbf{v} \perp \mathbf{u}_0$ and $\|\mathbf{v}\| = \|\mathbf{u}_0\|$. For $j \geq 3$ we have either $\mathbf{x}_j = 0$ or $\|\mathbf{u}_0 - 2\mathbf{x}_j\| = 0$ and $\mathbf{x}_j = \frac{1}{2}\mathbf{u}_0$. Since

$$\mathbf{u}_0 - \mathbf{x}_1 - \mathbf{x}_2 = \sum_{j=2}^{\infty} \mathbf{x}_j = \frac{1}{2}\mathbf{u}_0,$$

we see that there is a single index $j \geq 3$ such that $\mathbf{x}_j = \frac{1}{2}\mathbf{u}_0$ and $\mathbf{x}_j = \mathbf{0}$ for every other $j \geq 3$, and we are in case (c) of Theorem 2.

4. In the case $|J| \geq 3$ we rely on Lemma 3, which says that

$$\left\|\mathbf{u}_0 - 2\sum_{j\in K} \mathbf{x}_j\right\| = \|\mathbf{u}_0\|,$$

for any finite subset K of J. Using Lemma 1 (b) iteratively, we find that $(\mathbf{x}_j)_{j\in J}$ is an orthogonal sequence. It remains to show that there are no other nonzero vectors in $(\mathbf{x}_j)_{j\geq 1}$. Arguing as in the first subcase of the previous case, we conclude that $\mathbf{x}_j = \mathbf{0}$ for every j which is not in J. Thus we are in case (a) of Theorem 2 with $|J|$ nonzero vectors. □

Acknowledgments

Sigrid Grepstad is supported by Grant 275113 of the Research Council of Norway.

References

Brevig, O. F., J. Ortega-Cerdà, and K. Seip (2022). “Hilbert points in Hardy spaces”. *Algebra i Analiz* **34** (3), pp. 131–158 (cit. on pp. 18, 22).

Diestel, J. and J. J. Uhl Jr. (1977). *Vector measures*. Mathematical Surveys, No. 15. With a foreword by B. J. Pettis. American Mathematical Society, Providence, R.I., pp. xiii+322 (cit. on p. 21).

Hedenmalm, H. et al. (2018). "Sharpening Hölder's inequality". *J. Funct. Anal.* **275** (5), pp. 1280–1319 (cit. on p. 18).

Hytönen, T. et al. (2016). *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*. **63**. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, pp. xvi+614 (cit. on p. 26).

Hytönen, T. et al. (2017). *Analysis in Banach spaces. Vol. II*. **67**. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Probabilistic methods and operator theory. Springer, Cham, pp. xxi+616 (cit. on p. 19).

Kahane, J.-P. (1985). *Some random series of functions*. Second. **5**. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xiv+305 (cit. on p. 26).

Shapiro, H. S. (1971). *Topics in approximation theory*. Lecture Notes in Mathematics, Vol. 187. With appendices by Jan Boman and Torbjörn Hedberg. Springer-Verlag, Berlin-New York, pp. viii+275 (cit. on p. 21).

Contents