A NOTE ON THE MEAN VALUES OF THE DERIVATIVES OF ζ'/ζ

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Abstract. Assuming the Riemann hypothesis, we obtain a formula for the mean value of the k-derivative of ζ'/ζ, depending on the pair correlation of zeros of the Riemann zeta-function. This formula allows us to obtain new equivalences to Montgomery's pair correlation conjecture. This extends a result of Goldston, Gonek, and Montgomery where the mean value of ζ'/ζ was considered.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. The Riemann hypothesis (RH) states that the non-trivial zeros ρ of $\zeta(s)$ have the form $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. We will assume RH throughout this paper.

1.1. **Montgomery's pair correlation conjecture.** In 1973, Montgomery [17] defined the pair correlation function

$$N(\beta, T) := \sum_{\substack{0<\gamma,\gamma'\leqslant T \\ 0<\gamma-\gamma'\leqslant \frac{2\pi\beta}{\log T}}} 1,$$

where the double sum runs over the ordinates γ, γ' of two sets of non-trivial zeros of $\zeta(s)$, counted with multiplicity. Since there are $\sim T\log T/(2\pi)$ non-trivial zeros of $\zeta(s)$ with ordinates in the interval $(0, T]$ as $T \to \infty$, the function $N(\beta, T)$ counts the number of pairs of zeros within β times the average spacing between zeros. The pair correlation conjecture of Montgomery asserts that

$$N(\beta, T) \sim \frac{T\log T}{2\pi}\int_0^{\beta}\left\{1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right\}\mathrm{d}u, \quad \text{as } T\to\infty \ \text{ for any fixed } \beta > 0. \tag{1.1}$$

Assuming RH, there are several known equivalences[1] to this conjecture. Define the function

$$F(\alpha, T) := \frac{2\pi}{T\log T}\sum_{0<\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma'),$$

introduced by Montgomery [17], where $\alpha \in \mathbb{R}$, $T \geqslant 2$, and $w(u) = 4/(4+u^2)$. Using this function, Goldston [12] showed that the pair correlation conjecture (1.1) is equivalent to

$$\int_b^{b+\ell} F(\alpha, T)\,\mathrm{d}\alpha \sim \ell, \quad \text{as } T\to\infty \ \text{ for any fixed } b\geqslant 1 \text{ and } \ell > 0. \tag{1.2}$$

Another equivalence for the pair correlation conjecture is related to the second moment of ζ'/ζ. In fact, Goldston, Gonek, and Montgomery [14, Theorem 3] established that the pair correlation conjecture is equivalent

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[1] For an equivalence of the pair correlation conjecture related to the asymptotic formula for an integral of Selberg connected with primes in short intervals, see [10, 12, 14].

to the asymptotic

$$I(a,T) := \int_1^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2 \mathrm{d}t \sim \left(\frac{1-e^{-2a}}{4a^2}\right) T\log^2 T, \quad \text{as } T\to\infty \text{ for any fixed } a>0. \tag{1.3}$$

Since Montgomery's pair correlation conjecture remains a difficult open problem, the efforts have thus been concentrated in obtaining upper and lower bounds for the functions $N(\beta,T)$, $\int_b^{b+\ell} F(\alpha,T)\,\mathrm{d}\alpha$, and $I(a,T)$ in place of asymptotic formulae (see for instance [3, 4, 9, 11, 13, 14]).

1.2. **Mean values of the k-derivative of ζ'/ζ.** The main goal in this paper is to extend the technique developed by Goldston, Gonek, and Montgomery in [14] to get new equivalences of the pair correlation conjecture, related to the mean values of the derivatives of ζ'/ζ. Let $k\geqslant 0$ be an integer. For $a>0$ and $T\geqslant 2$, define the second moment of the k-derivative of ζ'/ζ as

$$I_k(a,T) = \int_1^T \left|\left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2 \mathrm{d}t.$$

With this notation, we have $I_0(a,T)=I(a,T)$.

Theorem 1. *Assume RH and let $k\geqslant 0$ be an integer. The following statements are equivalent:*

(I) $\displaystyle\int_b^{b+\ell} F(\alpha,T)\,\mathrm{d}\alpha \sim \ell,$ *as $T\to\infty$ for any fixed $b\geqslant 1$* and $\ell>0$*;*

(II) $\displaystyle I_k(a,T) \sim \left(\frac{(2k+1)!}{(2a)^{2k+2}} - \sum_{m=1}^{2k+1} \frac{m\,(2k)!}{(2k+1-m)!}\frac{e^{-2a}}{(2a)^{m+1}}\right) T(\log T)^{2k+2},$ *as $T\to\infty$ for any fixed $a>0$.*

Note that Theorem 1 gives new equivalences for the pair correlation conjecture. When $k=0$, it recovers the equivalence for the asymptotic formula (1.3). Moreover, our result shows the dependence of the asymptotic formulae for $I_k(a,T)$ for all values of $k\geqslant 0$.

Corollary 2. *Assume RH. Then, the asymptotic formula* (II) *holds for some $k\geqslant 0$ if and only if it holds for all $k\geqslant 0$.*

One can estimate the right order of magnitude for $I_k(a,T)$, as $T\to\infty$, for a fixed $a>0$. In fact, using Proposition 5 and the uniform estimate (see, for instance [13])

$$\int_1^\beta F(\alpha,T)\,\mathrm{d}\alpha \ll \beta, \tag{1.4}$$

it follows that for fixed $k\geqslant 0$ and $a>0$, we have $I_k(a,T)\asymp_{k,a} T(\log T)^{2k+2}$.

On the other hand, Farmer proved a relation between $I(a,T)$ and a certain discrete mean value of ζ'/ζ. For $k\geqslant 0$ an integer, define

$$D_k(a,T) = \sum_{0<\gamma\leqslant T} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(\frac{1}{2}+\frac{a}{\log T}+i\gamma\right).$$

Then, Farmer [7, Lemma 3b] established that, for a fixed $a>0$,

$$D_0(a,T) = \frac{1}{2\pi}\, I_0\!\left(\frac{a}{2},T\right) + O\big(T^{\varepsilon}\big), \quad \text{for } T\geqslant 2 \text{ and } \varepsilon>0 \text{ sufficiently small.} \tag{1.5}$$

In particular, using (1.3) we obtain that the pair correlation conjecture is equivalent to

$$D_0(a,T) \sim \left(\frac{1-e^{-a}}{2\pi a^2}\right) T\log^2 T, \quad \text{as } T\to\infty \text{ for any fixed } a>0.$$

Extending (1.5) for $D_k(a,T)$ and using Theorem 1 we arrive at the following corollary.

Corollary 3. *Assume RH and let $k \geqslant 0$ be an integer. The following statements are equivalent:*

(I) $\displaystyle\int_b^{b+\ell} F(\alpha,T)\,\mathrm{d}\alpha \sim \ell,$ *as $T \to \infty$ for any fixed $b \geqslant 1$ and $\ell > 0$;*

(II) $\displaystyle D_k(a,T) \sim \frac{1}{2\pi}\left(\frac{(2k+1)!}{a^{2k+2}} - \sum_{m=1}^{2k+1} \frac{m\,(2k)!}{(2k+1-m)!}\frac{e^{-a}}{a^{m+1}}\right)T(\log T)^{2k+2},$ *as $T \to \infty$ for any fixed $a > 0$.*

1.3. **Related results.** We would like to point out that for some objects related to the Riemann zeta-function, there are results where one relates the asymptotic formula of their second moments to suitably weighted integrals of $F(\alpha,T)$. For instance, Goldston [11, Theorem 1] showed, under RH, that

$$\int_0^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2}\log\log T + \frac{T}{2\pi^2}\left[\int_1^\infty \frac{F(\alpha,T)}{\alpha^2}\mathrm{d}\alpha + \gamma_0 - \sum_{m=2}^\infty \sum_p \left(\frac{1}{m} - \frac{1}{m^2}\right)\frac{1}{p^m}\right] + o(T), \quad \text{as } T \to \infty,$$

where $\pi S(t)$ denote the argument of the Riemann zeta-function at the point $\frac{1}{2}+it$, and γ_0 is Euler's constant. Recently, this has been extended to the iterates of the function $S(t)$ (see [6, Theorem 1]). Note that, assuming (1.2), by integration by parts and (1.4) we get

$$\int_0^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2}\log\log T + \frac{T}{2\pi^2}\left[1 + \gamma_0 - \sum_{m=2}^\infty \sum_p \left(\frac{1}{m} - \frac{1}{m^2}\right)\frac{1}{p^m}\right] + o(T), \quad \text{as } T \to \infty.$$

We refer the reader to Farmer [7, 8] for other results related to pair correlation and certain asymptotic formulae.

2. The representation formula for $I_k(a,T)$

In this section, we establish a representation formula for the second moment of the k-derivative of ζ'/ζ, related to the function $F(\alpha,T)$. It can be seen as an extension of [14, Theorem 1]. The Poisson kernel plays an important role in our formula. For $b > 0$, let $h_b : \mathbb{R} \to \mathbb{R}$ be the Poisson kernel defined as

$$h_b(x) = \frac{b}{b^2 + x^2}, \tag{2.1}$$

and let $\ell_b : \mathbb{R} \to \mathbb{R}$ be an auxiliary function[2] defined as

$$\ell_b(x) = \frac{b^2 - x^2}{(b^2+x^2)^2}. \tag{2.2}$$

The following technical lemma about the derivatives of h_b and ℓ_b will be useful for us.

Lemma 4. *Let $k \geqslant 0$ be an even integer. Then, for all $x \in \mathbb{R}$ we have*

$$\left|(h_b)^{(k)}(x)\right| \ll_k \frac{1}{b^{k-1}(b^2+x^2)}, \quad \textit{and} \quad \left|(\ell_b)^{(k)}(x)\right| \ll_k \frac{1}{b^k(b^2+x^2)}.$$

Proof. Let us prove the first estimate for $b = 1$. For any $k \geqslant 0$, it is easy to see by induction that

$$(h_1)^{(k)}(x) = \frac{P(x)}{(1+x^2)^{2^k}},$$

where P is a polynomial of degree at most $2^{k+1} - k - 2$. In particular, when $k = 2m$ with $m \in \mathbb{Z}$ we have

$$\left|(h_1)^{(2m)}(x)\right| \ll_m \frac{1}{(1+x^2)^{m+1}}.$$

[2] The function ℓ_b has previously been used to bound the real part of the derivative of ζ'/ζ (see [5, Theorem 3]).

In the general case, since $h_b(x) = h_1(x/b)/b$, it follows that

$$\left|(h_b)^{(2m)}(x)\right| = \frac{1}{b^{2m+1}}\left|(h_1)^{(2m)}\left(\frac{x}{b}\right)\right| \ll_m \frac{b}{(b^2+x^2)^{m+1}} \leqslant \frac{1}{b^{2m-1}(b^2+x^2)}.$$

We conclude the first estimate. The proof of the second estimate is similar. □

Proposition 5. *Assume RH and let $k \geqslant 1$ be a fixed integer. Then, for $0 < a \ll 1$ and $T \geqslant 3$ we have*

$$I_k(a,T) = \frac{(-1)^k}{2^{2k}\pi^{2k}}(\log T)^{2k+1}\sum_{0<\gamma,\gamma'\leqslant T}\big(h_{a/\pi}\big)^{(2k)}\left((\gamma-\gamma')\frac{\log T}{2\pi}\right)w(\gamma-\gamma') + O\left(\frac{T(\log T)^{2k+1}}{a^{2k-1}} + \frac{(\log T)^{2k+4}}{a^{2k+2}}\right),$$

where $h_{a/\pi}$ is defined in (2.1) *and $w(u) = 4/(4+u^2)$. In particular, for a fixed $a > 0$,*

$$I_k(a,T) = \left(\int_0^1 \alpha^{2k+1}e^{-2a\alpha}\,\mathrm{d}\alpha + \int_1^\infty \alpha^{2k}e^{-2a\alpha}F(\alpha,T)\,\mathrm{d}\alpha + o(1)\right)T(\log T)^{2k+2}, \quad \text{as } T\to\infty. \tag{2.3}$$

Proof. We start obtaining a bound for $\left(\zeta'/\zeta\right)^{(k)}$. Let $s = \sigma + it$, with $\frac{1}{2} < \sigma \leqslant \frac{3}{2}$ and $t \geqslant 2$. From the partial fraction decomposition for ζ'/ζ [18, Eq. 2.12.7]

$$\frac{\zeta'}{\zeta}(s) = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{2.4}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ and $B = -\mathrm{Re}\sum_\rho \rho^{-1}$. Taking k derivatives in (2.4) and using the estimate[3]

$$\left(\frac{\Gamma'}{\Gamma}\right)^{(k)}(w) = O\left(\frac{1}{|w|^k}\right), \quad \text{for} \quad \mathrm{Re}\, w \geqslant \sigma_0 > 0,$$

it follows that

$$\left(\frac{\zeta'}{\zeta}\right)^{(k)}(s) = (-1)^k\,k!\sum_\rho \frac{1}{(s-\rho)^{k+1}} + O\left(\frac{1}{|t|^k}\right). \tag{2.5}$$

Since $\sum_{|\gamma-t|\leqslant 1} 1 = O(\log t)$, we have

$$\left|\sum_{\gamma>t+1}\frac{1}{(s-\rho)^{k+1}}\right| \leqslant \sum_{n\geqslant 1}\left\{\sum_{t+n<\gamma\leqslant t+n+1}\frac{1}{|t-\gamma|^{k+1}}\right\} \leqslant \sum_{n\geqslant 1}\left\{\sum_{t+n<\gamma\leqslant t+n+1}\frac{1}{n^{k+1}}\right\} \ll \sum_{n\geqslant 1}\frac{\log(t+n)}{n^{k+1}} \ll \log t.$$

Similarly, we can prove the same estimate when the sum runs over $\gamma < t-1$. Therefore, in (2.5) we obtain,[4] for $\frac{1}{2} < \sigma \leqslant \frac{3}{2}$ and $t \geqslant 2$,

$$\left|\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right| \ll \frac{\log t}{(\sigma-\frac{1}{2})^{k+1}}. \tag{2.6}$$

Now, let us prove Proposition 5. Using the elementary identity $|w|^2 = 2(\mathrm{Re}\,\{w\})^2 - \mathrm{Re}\,\{w^2\}$ for all $w \in \mathbb{C}$, we write

$$\int_1^T\left|\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right|^2\mathrm{d}t = 2\int_1^T\left(\mathrm{Re}\,\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right)^2\mathrm{d}t - \mathrm{Re}\int_1^T\left(\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right)^2\mathrm{d}t. \tag{2.7}$$

We estimate the second integral on the right-hand side of (2.7) by pulling the contour to the right, up to the line $\mathrm{Re}\,s = \frac{3}{2}$ (see [14, p. 111]). In fact, to estimate the vertical edge at $\mathrm{Re}\,s = \frac{3}{2}$ we use the representation

[3] It can be proved as the proof of Stirling's formula, but starting after taking k derivatives in [1, Eq. (34) in p. 202].

[4] The estimate (2.6) also holds when $k = 0$.

as a Dirichlet series of $(\zeta'/\zeta)^{(k)}(s)$, and for the upper horizontal edge we use the estimate (2.6). Therefore, in (2.7) we get

$$\int_1^T \left|\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right|^2 \mathrm{d}t = 2\int_1^T \left(\mathrm{Re}\,\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right)^2 \mathrm{d}t + O\left(\frac{\log^2 T}{(\sigma-\frac{1}{2})^{2k+1}}\right). \tag{2.8}$$

On the other hand, note that

$$\begin{aligned}
&(-1)^k\,k!\ \mathrm{Re}\left\{\sum_\rho \frac{1}{(s-\rho)^{k+1}}\right\}\\
&\quad = \sum_\gamma \mathrm{Re}\left\{(-i)^k\frac{d^k}{dx^k}\left(\frac{1}{(\sigma-\frac{1}{2})+ix}\right)\right\}\Bigg|_{x=t-\gamma}\\
&\quad = \sum_\gamma \mathrm{Re}\left\{(-i)^k\frac{d^k}{dx^k}\left(\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+x^2}\right)+(-i)^{k+1}\frac{d^k}{dx^k}\left(\frac{x}{(\sigma-\frac{1}{2})^2+x^2}\right)\right\}\Bigg|_{x=t-\gamma}\\
&\quad = \sum_\gamma \left\{\mathrm{Re}\left\{(-i)^k\right\}\frac{d^k}{dx^k}\left(\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+x^2}\right)+\mathrm{Re}\left\{(-i)^{k+1}\right\}\frac{d^{k-1}}{dx^{k-1}}\left(\frac{(\sigma-\frac{1}{2})^2-x^2}{((\sigma-\frac{1}{2})^2+x^2)^2}\right)\right\}\Bigg|_{x=t-\gamma}.
\end{aligned}$$

Therefore, taking the real part of (2.5) we arrive at

$$\mathrm{Re}\,\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it) + O\left(\frac{1}{|t|^k}\right) = \sum_\gamma f_{k,\sigma}(t-\gamma), \tag{2.9}$$

where $f_{k,\sigma}(x) = \mathrm{Re}\left\{(-i)^k\right\}(h_{\sigma-1/2})^{(k)}(x) + \mathrm{Re}\left\{(-i)^{k+1}\right\}(\ell_{\sigma-1/2})^{(k-1)}(x)$, and the functions $h_{\sigma-1/2}$ and $\ell_{\sigma-1/2}$ are defined in (2.1) and (2.2) respectively. Using the Fourier transforms[5]

$$\widehat{h_b}\,(y) = \pi e^{-2\pi b|y|} \quad \text{and} \quad \widehat{\ell_b}\,(y) = 2\pi^2|y|e^{-2\pi b|y|},$$

the Fourier transform of $f_{k,\sigma}$ is given by

$$\widehat{f_{k,\sigma}}(y) = \left(\left(\mathrm{Re}\left\{i^k\right\}\right)^2 y^k + \left(\mathrm{Re}\left\{i^{k+1}\right\}\right)^2 y^{k-1}|y|\right)(-1)^k 2^k \pi^{k+1} e^{-2\pi(\sigma-1/2)|y|}. \tag{2.10}$$

Now, we square (2.9), integrate from 1 to T, and use (2.6) to get

$$\int_1^T \left(\mathrm{Re}\,\left(\frac{\zeta'}{\zeta}\right)^{(k)}(\sigma+it)\right)^2 \mathrm{d}t + O\left(\frac{\log^2 T}{(\sigma-\frac{1}{2})^{k+1}}\right) = \int_1^T\left(\sum_\gamma f_{k,\sigma}(t-\gamma)\right)^2 \mathrm{d}t. \tag{2.11}$$

We proceed to analyze the right-hand side of (2.11). From Lemma 4, it follows that[6]

$$|f_{k,\sigma}(x)| \ll \frac{h_{\sigma-1/2}(x)}{(\sigma-1/2)^k},$$

and using Montgomery's argument [17] we can restrict the inner sum over the zeros of $\zeta(s)$ such that $0<\gamma\leqslant T$ and extend the integral to all $t\in\mathbb{R}$, with a final error at most $\ll (\sigma-1/2)^{-2k}\log^3 T + (\sigma-1/2)^{-2k-2}\log^2 T$ (see [14, p. 113]). Therefore, from (2.10) and using the fact that $f_{k,\sigma}$ is even,

$$\int_{-\infty}^{\infty}\left(\sum_{0<\gamma\leqslant T} f_{k,\sigma}(t-\gamma)\right)^2 \mathrm{d}t = \sum_{0<\gamma,\gamma'\leqslant T}\left(f_{k,\sigma} * f_{k,\sigma}\right)(\gamma-\gamma') = \sum_{0<\gamma,\gamma'\leqslant T}\left(\widehat{f_{k,\sigma}}\right)^2(\gamma-\gamma')$$

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[5] For a function $f\in L^1(\mathbb{R})$, we define its Fourier transform as $\widehat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i y x}\, f(x)\,\mathrm{d}x$, and the convolution of f and g is defined as $(f*g)(y) = \int_{-\infty}^{\infty} f(x)\,g(y-x)\,\mathrm{d}x$.

[6] We highlight that depending on the parity of k, only one of the terms of $f_{k,\sigma}$ appears.

$$= \pi(-1)^k \sum_{0<\gamma,\gamma'\leqslant T} \big(h_{2\sigma-1}\big)^{(2k)}(\gamma-\gamma').$$

We want to add the weight $w(\gamma-\gamma')$ to the last sum. In fact, Lemma 4 gives the bound

$$\Bigg| \sum_{0<\gamma,\gamma'\leqslant T} \big(h_{2\sigma-1}\big)^{(2k)}(\gamma-\gamma')\big(1-w(\gamma-\gamma')\big)\Bigg| \ll \frac{1}{(2\sigma-1)^{2k-1}} \sum_{0<\gamma,\gamma'\leqslant T} \frac{4}{4+(\gamma-\gamma')^2}$$
$$\ll \frac{T\log T\, F(0,T)}{(2\sigma-1)^{2k-1}} \ll \frac{T\log^2 T}{(2\sigma-1)^{2k-1}},$$

where in the last estimate we have used (2.14). Thus,

$$\int_1^T \Bigg(\sum_\gamma f_{k,\sigma}(t-\gamma)\Bigg)^2 \mathrm{d}t = \pi(-1)^k \sum_{0<\gamma,\gamma'\leqslant T} \big(h_{2\sigma-1}\big)^{(2k)}(\gamma-\gamma')\, w(\gamma-\gamma')$$
$$+ O\bigg(\frac{T\log^2 T}{(2\sigma-1)^{2k-1}} + \frac{\log^3 T}{(2\sigma-1)^{2k}} + \frac{\log^2 T}{(2\sigma-1)^{2k+2}}\bigg).$$

Now, considering that $\sigma = \frac{1}{2}+\frac{a}{\log T}$ for $0<a\ll 1$ and using the fact that $h_{2\sigma-1}(x) = h_{a/\pi}(x\log T/2\pi)\log T/2\pi$ for $x\in\mathbb{R}$, we obtain in (2.11)

$$\int_1^T \Bigg(\mathrm{Re}\,\bigg(\frac{\zeta'}{\zeta}\bigg)^{(k)}(\sigma+it)\Bigg)^2 \mathrm{d}t = \frac{(-1)^k}{2^{2k+1}\pi^{2k}}(\log T)^{2k+1} \sum_{0<\gamma,\gamma'\leqslant T} \big(h_{a/\pi}\big)^{(2k)}\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg) w(\gamma-\gamma')$$
$$+ O\bigg(\frac{T(\log T)^{2k+1}}{a^{2k-1}} + \frac{(\log T)^{2k+4}}{a^{2k+2}}\bigg).$$

Inserting it in (2.8) we conclude that

$$\int_1^T \Bigg|\bigg(\frac{\zeta'}{\zeta}\bigg)^{(k)}\bigg(\frac{1}{2}+\frac{a}{\log T}+it\bigg)\Bigg|^2 \mathrm{d}t = \frac{(-1)^k}{2^{2k}\pi^{2k}}(\log T)^{2k+1} \sum_{0<\gamma,\gamma'\leqslant T} \big(h_{a/\pi}\big)^{(2k)}\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg) w(\gamma-\gamma')$$
$$+ O\bigg(\frac{T(\log T)^{2k+1}}{a^{2k-1}} + \frac{(\log T)^{2k+4}}{a^{2k+2}}\bigg). \tag{2.12}$$

From Fourier inversion, it is known that for any function $R\in L^1(\mathbb{R})$ such that $\widehat{R}\in L^1(\mathbb{R})$ we have the formula (see [17, Eq. (3)])

$$\sum_{0<\gamma,\gamma'\leqslant T} R\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg) w(\gamma-\gamma') = \frac{T\log T}{2\pi}\int_{-\infty}^{\infty} \widehat{R}(\alpha)\,F(\alpha,T)\,\mathrm{d}\alpha.$$

Applying this formula to the function $(h_{a/\pi})^{(2k)}$ and using the fact that $\widehat{(h_b)^{(2k)}}(y) = (-1)^k 2^{2k}\pi^{2k+1}y^{2k}e^{-2\pi b|y|}$, we get in (2.12) that, for a fixed $a>0$,

$$\int_1^T \Bigg|\bigg(\frac{\zeta'}{\zeta}\bigg)^{(k)}\bigg(\frac{1}{2}+\frac{a}{\log T}+it\bigg)\Bigg|^2 \mathrm{d}t = \frac{T(\log T)^{2k+2}}{2}\int_{-\infty}^{\infty} \alpha^{2k}e^{-2a|\alpha|}\,F(\alpha,T)\,\mathrm{d}\alpha + O\big(T(\log T)^{2k+1}\big). \tag{2.13}$$

Refining the original work of Montgomery [17], Goldston and Montgomery [15, Lemma 8] proved that, under RH,

$$F(\alpha,T) = \big(T^{-2|\alpha|}\log T + |\alpha|\big)(1+o(1)), \quad \text{as } T\to\infty, \tag{2.14}$$

uniformly for $0\leqslant|\alpha|\leqslant 1$. Using (2.14) and the fact that $F(\alpha,T)=F(-\alpha,T)$ for all $\alpha\in\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \alpha^{2k}e^{-2a|\alpha|}\,F(\alpha,T)\,\mathrm{d}\alpha = 2\int_0^1 \alpha^{2k+1}e^{-2a\alpha}\,\mathrm{d}\alpha + 2\int_1^{\infty} \alpha^{2k}e^{-2a\alpha}F(\alpha,T)\,\mathrm{d}\alpha + o(1).$$

Inserting this in (2.13) we arrive at (2.3). □

3. A Tauberian lemma and the Proof of Theorem 1

3.1. **A Tauberian lemma.** The following lemma can be seen as a generalization[7] of [14, Lemma 2], where the case $G \equiv 1$ was considered. The proof uses Karamata's method and some examples of these Tauberian lemmas are given in [18, Section 7.12].

Lemma 6. *Let $f(\alpha, T) \geqslant 0$ be a function such that the function $\alpha \mapsto f(\alpha, T)$ is continuous for each $T \geqslant 2$ fixed, and for $\beta > 0$ and $T \geqslant 2$,*

$$\int_0^\beta f(\alpha, T)\, \mathrm{d}\alpha \ll \beta + 1. \tag{3.1}$$

Let G be a polynomial such that $G(\alpha) > 0$ for $\alpha \in [0, \infty)$. The following statements are equivalent:

(A) $\displaystyle\int_0^\infty f(\alpha, T)\, G(\alpha)\, e^{-b\alpha}\, \mathrm{d}\alpha \sim \int_0^\infty G(\alpha)\, e^{-b\alpha}\, \mathrm{d}\alpha,$ *as $T \to \infty$ for any fixed $b > 0$.*

(B) $\displaystyle\frac{1}{d - c}\int_c^d f(\alpha, T)\, \mathrm{d}\alpha \sim 1,$ *as $T \to \infty$ for any fixed $0 \leqslant c < d$.*

Proof. Let us start assuming (A). Let $0 \leqslant c < d$ be fixed, and define the function $h : [0, 1] \to \mathbb{R}$ by

$$h(u) = \begin{cases} 0, & \text{if } \ 0 \leqslant u < e^{-d} \\ \dfrac{1}{u\, G(-\log u)}, & \text{if } \ e^{-d} \leqslant u \leqslant e^{-c} \\ 0, & \text{if } \ e^{-c} < u \leqslant 1. \end{cases}$$

By the Weierstrass approximation theorem, for any $\varepsilon > 0$ sufficiently small we can construct a polynomial $P(u) = \sum_{n=0}^N a_n u^n$ (depending on ε) such that

$$h(u) \leqslant P(u) \ \text{ for all } u \in [0, 1], \quad \text{and} \quad \int_0^1 (P(u) - h(u))^2\, \mathrm{d}u = O(\varepsilon). \tag{3.2}$$

Defining the function $Q(\alpha) = e^{-\alpha} P\big(e^{-\alpha}\big)$, it follows that[8]

$$\frac{\chi_{[c,d]}(\alpha)}{G(\alpha)} \leqslant Q(\alpha)$$

for all $\alpha \geqslant 0$. Recalling that $G(\alpha) > 0$ we have

$$\begin{aligned} \int_c^d f(\alpha, T)\, \mathrm{d}\alpha \leqslant \int_0^\infty f(\alpha, T)\, G(\alpha)\, Q(\alpha)\, \mathrm{d}\alpha &= \int_0^\infty f(\alpha, T)\, G(\alpha) \sum_{n=0}^N a_n e^{-(n+1)\alpha}\, \mathrm{d}\alpha \\ &= \sum_{n=0}^N a_n \int_0^\infty f(\alpha, T)\, G(\alpha)\, e^{-(n+1)\alpha}\, \mathrm{d}\alpha. \end{aligned}$$

Taking lim sup as $T \to \infty$ and using (A) we arrive at

$$\limsup_{T \to \infty} \int_c^d f(\alpha, T)\, \mathrm{d}\alpha \leqslant \sum_{n=0}^N a_n \int_0^\infty G(\alpha)\, e^{-(n+1)\alpha}\, \mathrm{d}\alpha \tag{3.3}$$

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[7] See [2] for another extension of [14, Lemma 2] depending of certain measures.

[8] Here $\chi_{[c,d]}(\alpha)$ denotes the characteristic function of the interval $[c, d]$.

By a change of variables, the definition of h, the Cauchy-Schwarz inequality and (3.2), one can see that

$$\begin{aligned}\sum_{n=0}^{N} a_n \int_0^\infty G(\alpha)\, e^{-(n+1)\alpha}\, \mathrm{d}\alpha &= \int_0^1 G(-\log u)\, P(u)\, \mathrm{d}u \\ &= \int_0^1 G(-\log u)\, h(u)\, \mathrm{d}u + \int_0^1 G(-\log u)\, (P(u) - h(u))\, \mathrm{d}u \\ &= \int_{e^{-d}}^{e^{-c}} \frac{1}{u}\, \mathrm{d}u + O\left(\left(\int_0^1 G^2(-\log u)\, \mathrm{d}u\right)^{1/2} \left(\int_0^1 (P(u) - h(u))^2\, \mathrm{d}u\right)^{1/2}\right) \\ &= d - c + O(\varepsilon^{1/2}).\end{aligned}$$

Letting $\varepsilon \to 0$ and combining this with (3.3), we conclude that

$$\limsup_{T\to\infty} \int_c^d f(\alpha, T)\, \mathrm{d}\alpha \leqslant d - c.$$

Similarly, we can proceed to prove that

$$d - c \leqslant \liminf_{T\to\infty} \int_c^d f(\alpha, T)\, \mathrm{d}\alpha.$$

Therefore we obtain (B). Let us prove that (B) implies (A). Using integration by parts and (3.1), we see that

$$\int_0^\infty f(\alpha, T)\, G(\alpha)\, e^{-b\alpha}\, \mathrm{d}\alpha = -\int_0^\infty \left(\int_0^\alpha f(\beta, T)\, \mathrm{d}\beta\right) \big(G(\alpha)\, e^{-b\alpha}\big)'\, \mathrm{d}\alpha. \tag{3.4}$$

Finally, using (B), the dominated convergence theorem, and integration by parts one more time, we conclude. □

3.2. **Proof of Theorem 1.** Since the case $k = 0$ was considered in the work of Goldston, Gonek and Montgomery (see [14, Theorem 3]), assume $k \geqslant 1$. Using the identity[9]

$$\int_0^1 \alpha^{2k+1} e^{-2a\alpha}\, \mathrm{d}\alpha + \int_1^\infty \alpha^{2k} e^{-2a\alpha}\, \mathrm{d}\alpha = \frac{(2k+1)!}{(2a)^{2k+2}} - \sum_{m=1}^{2k+1} \frac{m\,(2k)!}{(2k+1-m)!}\, \frac{e^{-2a}}{(2a)^{m+1}}, \quad \text{for any } a > 0,$$

and (2.3) we have that (II) is equivalent to

$$\int_1^\infty \alpha^{2k} e^{-2a\alpha}\, F(\alpha, T)\, \mathrm{d}\alpha \sim \int_1^\infty \alpha^{2k} e^{-2a\alpha}\, \mathrm{d}\alpha.$$

A translation gives that (II) is equivalent to

$$\int_0^\infty (\alpha+1)^{2k} e^{-2a\alpha}\, F(\alpha+1, T)\, \mathrm{d}\alpha \sim \int_0^\infty (\alpha+1)^{2k} e^{-2a\alpha}\, \mathrm{d}\alpha.$$

Using Lemma 6 with the function $f(\alpha, T) = F(\alpha+1, T)$, $G(\alpha) = (\alpha+1)^{2k}$, and $b = 2a$ we conclude the proof. We remark that the additional constraint (3.1) follows from (1.4).

4. Proof of Corollary 3

Assume RH. From [18, p. 340], for each $n \in \mathbb{N}$ there is $T_n \in (n, n+1)$ such that for $-1 \leqslant \sigma \leqslant 2$,

$$\left|\frac{\zeta'}{\zeta}(\sigma + iT_n)\right| \ll (\log T_n)^2. \tag{4.1}$$

[9] See [16, Eq. 3.351-1 and 3.351-2].

Now, let $k \geqslant 1$ be an integer, $0 < a \ll 1$ and $T \geqslant 4$, $T \notin \mathbb{N}$. Choose $n \in \mathbb{N}$ such that $T, T_n \in (n, n+1)$ and T_n satisfies (4.1). Note that $\log T_n \asymp \log T$. Using integration by parts k times and the bound (2.6), we have

$$\int_1^{T_n} \left| \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2} + \frac{a}{\log T} + it\right)\right|^2 \mathrm{d}t = \int_1^{T_n} \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2} + \frac{a}{\log T} + it\right) \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2} + \frac{a}{\log T} - it\right) \mathrm{d}t$$
$$= \frac{1}{i}\int_{\frac{1}{2}-\frac{a}{\log T}+i}^{\frac{1}{2}-\frac{a}{\log T}+iT_n} \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(s + \frac{2a}{\log T}\right) \left(\frac{\zeta'}{\zeta}\right)^{(k)}(1-s)\, \mathrm{d}s$$
$$= \frac{1}{i}\int_{\frac{1}{2}-\frac{a}{\log T}+i}^{\frac{1}{2}-\frac{a}{\log T}+iT_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(s + \frac{2a}{\log T}\right) \frac{\zeta'}{\zeta}(1-s)\, \mathrm{d}s + O\left(\frac{(\log T)^{2k+3}}{a^{2k+1}}\right).$$

We use the residue theorem on the rectangle with vertices $\frac{1}{2} - \frac{a}{\log T} + i, 2+i, 2+iT_n$ and $\frac{1}{2} - \frac{a}{\log T} + iT_n$ (since RH holds, the function $(\zeta'/\zeta)^{(2k)}\left(s + \frac{2a}{\log T}\right)$ is analytic in this rectangle) and the bounds (2.6) and (4.1) to deduce that

$$\int_1^{T_n} \left| \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2} + \frac{a}{\log T} + it\right)\right|^2 \mathrm{d}t = 2\pi \sum_{0<\gamma<T_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(\rho + \frac{2a}{\log T}\right)$$
$$+ \frac{1}{i}\int_{2+i}^{2+iT_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(s + \frac{2a}{\log T}\right) \frac{\zeta'}{\zeta}(1-s)\, \mathrm{d}s + O\left(\frac{(\log T)^{2k+4}}{a^{2k+1}}\right).$$

It is known that $\zeta(s)$ satisfies the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2} - \frac{s}{2})}{\Gamma(\frac{s}{2})}.$$

Then, we write

$$\frac{1}{i}\int_{2+i}^{2+iT_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(s + \frac{2a}{\log T}\right) \frac{\zeta'}{\zeta}(1-s)\, \mathrm{d}s = \frac{1}{i}\int_{2+i}^{2+iT_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(s + \frac{2a}{\log T}\right) \left(\frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s)\right)\mathrm{d}s. \qquad (4.2)$$

Using the estimate

$$\frac{\chi'}{\chi}(\sigma + it) = -\log\left|\frac{t}{2\pi}\right| + O\left(\frac{1}{|t|}\right), \text{ for } |t| \geqslant 1 \text{ and } |\sigma| \ll 1,$$

and the representation as a Dirichlet series of $(\zeta'/\zeta)^{(2k)}(s)$ in the right hand-side of (4.2), we integrate term by term the right-hand side of (4.2) to obtain $O(\log T)$. Therefore, we arrive at

$$\int_1^{T_n} \left| \left(\frac{\zeta'}{\zeta}\right)^{(k)}\left(\frac{1}{2} + \frac{a}{\log T} + it\right)\right|^2 \mathrm{d}t = 2\pi \sum_{0<\gamma<T_n} \left(\frac{\zeta'}{\zeta}\right)^{(2k)}\left(\rho + \frac{2a}{\log T}\right) + O\left(\frac{(\log T)^{2k+4}}{a^{2k+1}}\right).$$

We can replace T_n by T using (2.6) and $\sum_{|t-\gamma|\leqslant 1} 1 = O(\log t)$ with an error at most $\ll (\log T)^{2k+4}/a^{2k+2}$. Therefore, we conclude for $0 < a \ll 1$ and T sufficiently large, that

$$I_k(a,T) = 2\pi D_k(2a,T) + O\left(\frac{(\log T)^{2k+4}}{a^{2k+2}}\right).$$

Finally, we use Theorem 1 to conclude.

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