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(Communicated by the associate editor name)

DETECTING AND DETERMINING PRESERVED MEASURES AND INTEGRALS OF BIRATIONAL MAPS

ABSTRACT. In this paper we use the method of discrete Darboux polynomials to calculate preserved measures and integrals of rational maps. The approach is based on the use of cofactors and Darboux polynomials and relies on the use of symbolic algebra tools. Given sufficient computing power, most, if not all, rational preserved integrals can be found (and even some non-rational ones). We show, in a number of examples, how it is possible to use this method to both determine and detect preserved measures and integrals of the considered rational maps, thus lending weight to a previous ansatz [13]. Many of the examples arise from the Kahan-Hirota-Kimura discretization of completely integrable systems of ordinary differential equations.

1. **Introduction.** "Most of science is a search for simple, stable properties that can answer questions which interest us." (Quote from F. Wilczek's book [27]). The above quote certainly applies to the study of ordinary differential equations (ODEs). Starting with Galileo, Kepler and Newton, and later luminaries such as Hamilton and Lagrange, the search for preserved first integrals and preserved volume forms has a long and very distinguished history. For the area of ordinary difference equations and mappings, by comparison, the corresponding search for preserved integrals and preserved volume forms is arguably still in its infancy and is mainly carried out within two subfields: discrete integrable systems [15], and numerical integration of ODEs, in particular geometric numerical integration [4]. In the discrete integrable systems subfield, two of the earliest examples of the discovery of discrete maps preserving integrals are the 2D area-preserving McMillan map, preserving a polynomial integral [25], subsequently generalized to the 2D measure-preserving QRT map, preserving a rational integral [22, 23]. In the area of numerical discretization of ODEs, one of the earliest methods used is the birational Kahan-Hirota-Kimura discretization of first order quadratic ODEs [2, 5, 7, 8], subsequently followed by other birational discretizations of ODEs of higher degree and/or higher order, such as polarization methods [12] and the methods of Hone and Quispel [6]. In a recent Letter [13], we developed a theory for detecting and calculating integrals and preserved measures of rational maps, and presented three examples illustrating its use. The algorithm we use is based on the notion of discrete Darboux polynomials and an accompanying cofactor, and it essentially only requires the solution of linear systems of equations. The details of the algorithm are presented in the next section. In this paper we shall build further on the work of Celledoni et al. [13]. After introducing notation and preliminary results, such as the Kahan map, in section 2, we shall present a total of 10 examples. The seven examples in section 3 involve determining Darboux polynomials; the three additional examples of section 4 involve parametric detection. In most cases the map we consider is obtained as the application of Kahan's method to a quadratic differential equation. Three of the examples we consider are unrelated to Kahan's method. Finally, we prove that for any quadratic Hamiltonian ODE, the modified Hamiltonian as well as the modified preserved measure of the Kahan map are both found using a particular ansatz for the choice of cofactor.

2. **Preliminaries.** In this section, we provide some preliminaries on Kahan's discretization, on measure preservation and superintegrability, on Darboux polynomials for ODEs and for discrete birational maps, and on the discrete cofactor ansatz that we will use in the remainder of the paper:

2.1. **Kahan's discretization.** Kahan [8] proposed a numerical method designed for quadratic systems of ordinary differential equations (ODEs) in $\mathbb{R}^n$ written in component form as

$$\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k + \sum_j b_{ij} x_j + c_i, \quad i = 1, \ldots, n, \tag{1}$$

where a_{ijk}, b_{ij}, c_i are arbitrary constants and all summation indices are ranging from 1 to n. The method of Kahan, also known as the Hirota–Kimura discretization [5, 7],

is a one-step method $(x_1, \ldots, x_n) \mapsto (x'_1, \ldots, x'_n)$ where

$$\frac{x'_i - x_i}{h} = \sum_{j,k} a_{ijk} \frac{x'_j x_k + x_j x'_k}{2} + \sum_j b_{ij} \frac{x_j + x'_j}{2} + c_i, \quad i = 1, \ldots, n, \qquad (2)$$

where h denotes the discrete time step. The method (2) is linearly implicit and so is its inverse, hence it defines a birational map ϕ_h.

Much of the recent interest in Kahan's method stems from its ability to preserve modified first integrals and measures of the underlying quadratic differential equation [9, 10, 11]. But even in cases where there are strong indications that Kahan's method preserves such a nearby invariant, it is not necessarily an easy task to determine its closed form.

2.2. Measure preservation and superintegrability. Consider the ODE

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \qquad (3)$$

Proposition 2.1. *(Liouville, [1]) The smooth function $M : \mathbb{R}^n \to \mathbb{R}$ is the density of an invariant of the ODE (3) iff*

$$\mathrm{div}(Mf) = 0.$$

Definition 2.2. [26] A vector field on $\mathbb{R}^n$ is superintegrable if it admits $n-1$ functionally independent constants of motion.

We now consider the birational map

$$\mathbf{x}' = \phi(\mathbf{x}),$$

with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n)$ elements of $\mathbb{R}^n$.

Definition 2.3. The map ϕ is measure preserving if there exists a smooth function Π such that

$$\Pi(\mathbf{x}')J(\mathbf{x}) = \Pi(\mathbf{x}),$$

where J is the Jacobian determinant of ϕ

$$J = \left| \frac{\partial \phi_i}{\partial x_j} \right|.$$

The preserved measure is given by $\Pi(\mathbf{x})\, d\mathbf{x}$ where $d\mathbf{x} = dx_1 \wedge \cdots \wedge dx_n$.

As mentioned in §3 of [26], for a discrete map in $\mathbb{R}^n$, the existence of $n-1$ integrals is not enough to claim (super) integrability:

Definition 2.4. [26] An n-dimensional map is superintegrable if it has $n-1$ constants of motion and it is measure-preserving.

2.3. Darboux polynomials for ordinary differential equations. For references on this topic, the reader is referred to Cheze & Combot [3] and references therein.

Consider an ODE $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$. The polynomial $\tilde{P}(\mathbf{x})$ is defined to be a Darboux polynomial of the ODE if there exists a polynomial function $\tilde{C}(\mathbf{x})$ s.t.

$$\frac{d}{dt}\tilde{P}(\mathbf{x}) = \tilde{C}(\mathbf{x})\tilde{P}(\mathbf{x}).$$

Here $\frac{d}{dt}\tilde{P}(\mathbf{x}) = f(\mathbf{x}).\nabla\tilde{P}(\mathbf{x})$, and $\tilde{C}$ is called the cofactor of $\tilde{P}$. Note that if

$$\frac{d}{dt}\tilde{P}_i(\mathbf{x}) = \tilde{C}_i(\mathbf{x})\tilde{P}_i(\mathbf{x}), \qquad i = 1, \ldots, l,$$

then

$$\frac{d}{dt}\left(\prod_i \tilde{P}_i^{\alpha_i}(\mathbf{x})\right) = \left(\sum_i \alpha_i \tilde{C}_i(\mathbf{x})\right)\left(\prod_i \tilde{P}_i^{\alpha_i}(\mathbf{x})\right),$$

so that if

$$\sum_i \alpha_i \tilde{C}_i(\mathbf{x}) = 0,$$

then

$$\prod_i \tilde{P}_i^{\alpha_i}(\mathbf{x}),$$

is an integral of the vector field f. We list some examples of cofactors and the geometric interpretation of the corresponding Darboux polynomials:

1. $\tilde{C}(\mathbf{x}) \equiv 0 \Rightarrow \dot{\tilde{P}} = 0 \Rightarrow \tilde{P}$ is a first integral.
2. $\tilde{C}(\mathbf{x}) \equiv \tilde{C} \Rightarrow \frac{d}{dt}\tilde{P} = \tilde{C}\tilde{P} \Rightarrow \tilde{P}(\mathbf{x}(t)) = \tilde{P}(\mathbf{x}(0))e^{\tilde{C}t} \Rightarrow \tilde{P}$ defines a foliation.
3. $\tilde{C}(\mathbf{x}) \equiv \mathrm{div} f(\mathbf{x}) \Rightarrow \tilde{P}$ defines the preserved measure $\frac{d\mathbf{x}}{P(\mathbf{x})}$ of the vector field f.
4. $\tilde{C}(\mathbf{x}) \equiv \tilde{C}(\mathbf{x})$, i.e. the general case $\Rightarrow \tilde{P}(\mathbf{x}) = 0$ defines a second integral, i.e. an algebraic invariant hypersurface.

2.4. **Darboux polynomials for discrete rational maps.** Consider a rational map $\mathbf{x}' = \phi(\mathbf{x})$. We define the polynomial $P(\mathbf{x})$ to be a (discrete) Darboux polynomial of the map ϕ if there exists a rational function $C(\mathbf{x})$ s.t. $P(\mathbf{x})$ satisfies the cofactor equation:

$$P(\mathbf{x}') = C(\mathbf{x})P(\mathbf{x}), \tag{4}$$

where the form of $C(\mathbf{x})$ will be prescribed below.

Note that if

$$P_i(\mathbf{x}') = C_i(\mathbf{x})P_i(\mathbf{x}), \quad i = 1, \ldots, k,$$

then

$$\left(\prod_i P_i^{a_i}(\mathbf{x}')\right) = \left(\prod_i C_i^{a_i}(\mathbf{x})\right)\left(\prod_i P_i^{a_i}(\mathbf{x})\right),$$

so that if

$$\prod_i C_i^{a_i}(\mathbf{x}) \equiv 1,$$

then

$$\prod_i P_i^{a_i}(\mathbf{x}),$$

is an integral of the map ϕ. We will see that this has the consequence that many foliations arising in this way are in factorised form.

We list some examples of discrete cofactors and the geometric interpretation of the corresponding Darboux polynomials:

1. $C(\mathbf{x}) \equiv 1 \rightarrow P' = P \rightarrow P$ is a first integral.
2. $C(\mathbf{x}) \equiv C \rightarrow P' = CP \rightarrow P(\mathbf{x}_n) = P(\mathbf{x}_0)C^n \rightarrow P$ defines a foliation.
3. $C(\mathbf{x}) \equiv J(\mathbf{x}) \rightarrow P$ defines the preserved measure $\frac{d\mathbf{x}}{P(\mathbf{x})}$ of the map ϕ.
4. $C(\mathbf{x}) \equiv C(\mathbf{x})$ i.e. the general case $\rightarrow P(\mathbf{x}) = 0$ defines a second integral, i.e. an algebraic invariant hypersurface.

1   We remark that if the map ϕ contains a timestep h, such that $\lim\limits_{h\to 0} \dfrac{\phi - 1}{h} = f(\mathbf{x})$,

2   then in the continuum limit

$$\tilde{P}(\mathbf{x}) = \lim_{h\to 0} P(\mathbf{x}),$$
$$\tilde{C}(\mathbf{x}) = \lim_{h\to 0} \frac{C(\mathbf{x}) - 1}{h},$$
$$\mathrm{div}(f(\mathbf{x})) = \lim_{h\to 0} \frac{J(\mathbf{x}) - 1}{h}, \tag{5}$$
$$\tilde{C} = \lim_{h\to 0} \frac{C - 1}{h},$$

3   the (discrete) mapping case reduces to the (continuous) ODE case.

2.5. **Jacobian factor ansatz.** The remaining question is how to prescribe the form of the discrete cofactor $C_i(\mathbf{x})$. In [13] we introduced the following ansatz: Given a rational map ϕ with Jacobian determinant

$$J(\mathbf{x}) = \frac{\prod_{i=1}^{l} K_i^{b_i}(\mathbf{x})}{\prod_{j=1}^{k} D_j^{c_j}(\mathbf{x})},$$

where the K_i and D_j are distinct factors, we try all cofactors (up to a certain polynomial degree d_1 for the numerator and d_2 for the denominator) of the form

$$C(\mathbf{x}) = \pm \frac{\prod_{i=1}^{l} K_i^{f_i}(\mathbf{x})}{\prod_{j=1}^{k} D_j^{g_j}(\mathbf{x})},$$

4   where $f_i, g_j \in \mathbb{N}_0$. This ensures that we need only check a finite number of cofactors.
5   (Note that in maps arising from Kahan's discretization, it follows from (5) that both
6   J and C are $1 + O(h)$).
7   For each such cofactor we try all possible Darboux polynomials, (again up to a
8   certain degree).
9   Given the cofactor C, the question of whether P exists, and, if so, what it is,
10 only requires solving the linear cofactor equation (4).

11 3. **Determining preserved measures and integrals.** In this section we apply
12 the aforementioned algorithm to determined preserved measures and integrals for 7
13 examples. The algorithm has been implemented in Maple[1]

14 3.1. **Example 1: finding measures and integrals of a specific 2D vector**
15 **field.** This subsection exemplifies the preservation of a modified quadratic Darboux
16 polynomial, and restates a result of [10] in terms of Darboux polynomials. We study
17 the following two-dimensional vector field

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 x_2 - 4x_2 \\ -3x_1^2 - x_2^2 + 4x_1 + 1 \end{pmatrix}. \tag{6}$$

18 The Kahan discretization ϕ_h of this vector field is given by

$$\begin{aligned} x_1' &= \frac{x_1 + h(2x_1 x_2 - 4x_2) + h^2(2x_1^2 - 2x_2^2 - 3x_1 - 2)}{D(\mathbf{x})}, \\ x_2' &= \frac{x_2 + h(-3x_1^2 - x_2^2 + 4x_1 + 1) + h^2(4x_1 x_2 - 5x_2)}{D(\mathbf{x})}, \end{aligned} \tag{7}$$

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[1]Version of 2019 was used and the codes are adapted to run on computing servers of up to 32 cores with up to 768 GBs of memory.

where the common denominator $D(\mathbf{x})$ is given by

$$D(\mathbf{x}) = 1 + h^2(3x_1^2 - x_2^2 - 8x_1 + 4).$$

The Jacobian determinant J of the Kahan map (7) is given by

$$J = C_1(\mathbf{x})C_2(\mathbf{x}),$$

where

$$C_1(\mathbf{x}) = \frac{1 + 2hx_2 + h^2(5 - 4x_1)}{D(\mathbf{x})},$$

$$C_2(\mathbf{x}) = \frac{1 + C_2^1 h + C_2^2 h^2 + C_2^3 h^3 + C_2^4 h^4}{D^2(\mathbf{x})}.$$

where

$$C_2^1 = -2hx_2,$$
$$C_2^2 = 7 - 20x_1 + 9x_1^2 + x_2^2,$$
$$C_2^3 = 26x_2 - 16x_1x_2,$$
$$C_2^4 = 28 - 28x_1 + 7x_1^2 + 3x_2^2.$$

Defining $C_3 := J$, we have used cofactors C_1, C_2 and C_3 to find the corresponding Darboux polynomials for the Kahan map (7)

$$p_{1,1} = x_1 - 2,$$
$$p_{2,1} = 1 - x_1^2 - x_2^2 + h^2(\frac{13}{3} - \frac{16}{3}x_1 + x_2^2 + \frac{7}{3}x_1^2),$$
$$p_{3,1} = 1 + h^2(3x_1^2 - x_2^2 - 8x_1 + 4),$$
$$p_{3,2} = (x_1 - 2)(1 - x_1^2 - x_2^2 + h^2(\frac{13}{3} - \frac{16}{3}x_1 + x_2^2 + \frac{7}{3}x_1^2)).$$

Here and below, $p_{i,j}$ denotes the *jth* Darboux polynomial corresponding to the cofactor C_i, i.e., $p_{i,j}$ satisfies

$$p_{i,j}(\mathbf{x}') = C_i(\mathbf{x})p_{i,j}(\mathbf{x}).$$

Note also that here $p_{3,1}(\mathbf{x}) \equiv D(\mathbf{x})$. Hence it turns out that the Kahan map (7) possesses the Darboux polynomials $p_{1,1}(\mathbf{x})$, $p_{2,1}(\mathbf{x})$ and $p_{3,1}(\mathbf{x})$, and also preserves the measure

$$\frac{d\mathbf{x}}{1 + h^2(3x_1^2 - x_2^2 - 8x_1 + 4)}.$$

Finally, the Kahan map preserves the first integral

$$\frac{(x_1 - 2)(1 - x_1^2 - x_2^2 + h^2(\frac{13}{3} - \frac{16}{3}x_1 + x_2^2 + \frac{7}{3}x_1^2))}{1 + h^2(3x_1^2 - x_2^2 - 8x_1 + 4)}.$$

Taking the continuum limit $h \to 0$, we now see in hindsight that the vector field (6) possesses two second integrals, i.e. $x_1 - 2$ and $1 - x_1^2 - x_2^2$ [2]. It also preserves the measure $dx_1 \wedge dx_2$ and the first integral $H = (x_1 - 2)(1 - x_1^2 - x_2^2)$. The fact that the affine Darboux polynomial $x_1 - 2$ is preserved by the Kahan map (7) is an example of theorem (1) of [13], which states that the Kahan discretization preserves all affine Darboux polynomials in any dimension. On the other hand, the vector field (6) preserves the integral H and the measure $dx_1 \wedge dx_2$ implying that (6) is

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[2]This is why the discrete Darboux polynomial $p_{2,1} = 1 - x_1^2 - x_2^2 + h^2(\frac{13}{3} - \frac{16}{3}x_1 + x_2^2 + \frac{7}{3}x_1^2)$ is called a *modified* Darboux polynomial
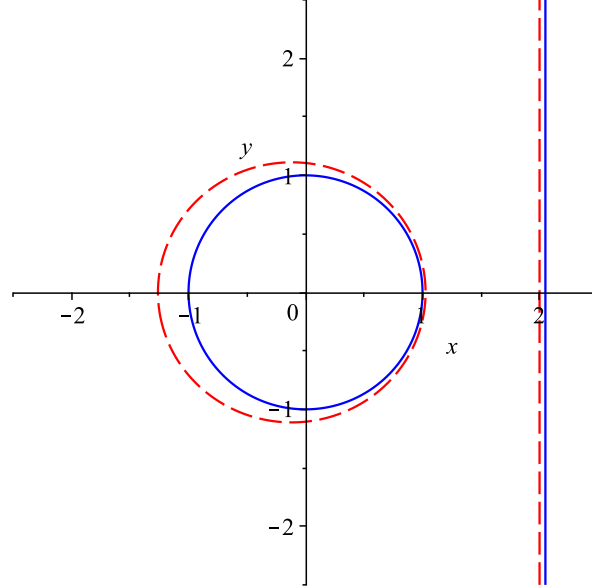
FIGURE 1. Plots of the level sets $p_{1,1} = 0$ and $p_{2,1} = 0$ of the Kahan map in Example 1 (for $h = \frac{1}{5}$), dotted red. Also shown are the corresponding second integrals of the ODE, $x_1 - 2$ and $1 - x_1^2 - x_2^2$, solid blue.

1   a Hamiltonian vector field with cubic Hamiltonian H. Therefore, the fact that the
2   Kahan method preserves a modified Hamiltonian $\tilde{H}$, and the modified densities $p_{3,1}$
3   and $p_{3,2}$ is a special case of the following theorem[3]:

4   **Theorem 3.1.** *Let H be cubic in $\mathbb{R}^n$, let K be a constant rank $2l$ antisymmetric*
5   *$n \times n$ matrix and let the vector field be given by $f = K\nabla H(\mathbf{x})$. Then:*

   (i) *$\phi_h(\mathbf{x})$ possesses the following two Darboux polynomials, both with cofactor[4]*
   $C_1(\mathbf{x}) = J$:

$$p_{1,1} = H(\mathbf{x}) \det\left(A(\mathbf{x})\right) + \frac{1}{3} h \nabla H(\mathbf{x})^t \mathrm{adj}(A(\mathbf{x})) f(\mathbf{x}),$$

$$p_{1,2} = \det\left(A(\mathbf{x})\right).$$

6   *Here $A(\mathbf{x}) = \mathbb{I} - \frac{1}{2} h f'(\mathbf{x})$, and $\mathrm{adj}(A)$ denotes the adjugate of A.*
7   (ii) *moreover, the degree of $p_{1,2}$ is at most $2l$ and the degree of $p_{1,1}$ is at most*
8   *$2l + 3$. If $n = 2l$ the degree of $p_{1,1}$ is at most $2l + 1$.*

9   *Proof.* In the proof of Proposition 4 in [10], it is shown that ϕ_h possesses the mod-
10 ified integral $\tilde{H} = \frac{p_{1,1}}{p_{1,2}}$. Proposition 5 in [10] is equivalent to the statement that
11 $p_{1,2}$ is a Darboux polynomial with cofactor J. Combining these two results, it fol-
12 lows that $p_{1,1}$ is also a Darboux polynomial with cofactor J. Part *(ii)* follows from
13 proposition 4*(i)* in [10].
14   □

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[3]Additional geometric properties of the Kahan map (7) will be given in a forthcoming preprint by Gubbiotti, Quispel and McLaren.

[4]We remind the reader that J is the Jacobian determinant of the Kahan map ϕ_h.

3.2. **Example 2: An inhomogeneous Nambu system.** We consider the following inhomogeneous Nambu system belonging to the class of systems considered in [2]

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20\,x_1x_2 + 8\,x_1x_3 + 30\,x_2{}^2 + 12\,x_2x_3 + 32\,x_1 + 48\,x_2 \\ -10\,x_1x_2 - 4\,x_1x_3 - 20\,x_2{}^2 - 8\,x_2x_3 - 16\,x_1 - 32\,x_2 \\ 16\,x_1x_2 + 10\,x_1x_3 + 32\,x_2{}^2 + 20\,x_2x_3 + 10\,x_1 + 20\,x_2 \end{pmatrix}, \quad (8)$$

which has the two integrals

$$H_1 := x_1^2 + 4x_1x_2 + 3x_2^2,$$
$$H_2 := 4x_2^2 + 5x_2x_3 + x_3^2 + 5x_2 + 8x_3,$$

and the preserved measure

$$\int dx_1 \wedge dx_2 \wedge dx_3.$$

We consider the Kahan discretization of these equations. The corresponding Jacobian determinant has the irreducible factorization

$$J = \frac{K_1(\mathbf{x})K_2(\mathbf{x})K_3(\mathbf{x})K_4(\mathbf{x})}{D(\mathbf{x})^4}, \quad (9)$$

where

$$\begin{aligned}
K_1 &= 1 + (10\,x_2 + 4\,x_3 + 16)\,h + \left(-9\,x_1{}^2 - 54\,x_1x_2 - 56\,x_2{}^2 + 20\,x_2x_3 \right. \\
&\quad \left. +4\,x_3{}^2 - 60\,x_1 - 70\,x_2 + 32\,x_3 + 64\right)h^2 + \left(-150\,x_1x_2 - 60\,x_1x_3 \right. \\
&\quad \left. -300\,x_2{}^2 - 120\,x_2x_3 - 240\,x_1 - 480\,x_2\right)h^3, \\
K_2 &= 1 + (-6\,x_1 - 12\,x_2)\,h + \left(9\,x_1{}^2 + 36\,x_1x_2 + 32\,x_2{}^2 - 8\,x_2x_3 - 4\,x_3{}^2 \right. \\
&\quad \left. -62\,x_2 - 32\,x_3 - 64\right)h^2 + \left(150\,x_1x_2 + 60\,x_1x_3 + 300\,x_2{}^2 + 120\,x_2x_3 \right. \\
&\quad \left. +240\,x_1 + 480\,x_2\right)h^3, \\
K_3 &= 1 + (6\,x_1 + 12\,x_2)\,h + \left(9\,x_1{}^2 + 36\,x_1x_2 - 28\,x_2{}^2 - 32\,x_2x_3 - 4\,x_3{}^2 \right. \\
&\quad \left. -158\,x_2 - 32\,x_3 - 64\right)h^2 + \left(150\,x_1x_2 + 60\,x_1x_3 + 300\,x_2{}^2 + 120\,x_2x_3 \right. \\
&\quad \left. +240\,x_1 + 480\,x_2\right)h^3, \\
K_4 &= 1 + h\left(-10\,x_2 - 4\,x_3 - 16\right) + \left(-9\,x_1{}^2 - 18\,x_1x_2 + 16\,x_2{}^2 + 20\,x_2x_3 \right. \\
&\quad \left. +4\,x_3{}^2 + 60\,x_1 + 170\,x_2 + 32\,x_3 + 64\right)h^2 + \left(-150\,x_1x_2 - 60\,x_1x_3 \right. \\
&\quad \left. -300\,x_2{}^2 - 120\,x_2x_3 - 240\,x_1 - 480\,x_2\right)h^3, \\
D &= 1 + \left(-9\,x_1{}^2 - 36\,x_1x_2 - 70\,x_2{}^2 - 20\,x_2x_3 - 4\,x_3{}^2 - 110\,x_2 - 32\,x_3 - 64\right)h^2 \\
&\quad + \left(90\,x_1x_2{}^2 + 36\,x_1x_2x_3 + 180\,x_2{}^3 + 72\,x_2{}^2x_3 + 294\,x_1x_2 + 60\,x_1x_3 \right. \\
&\quad \left. +588\,x_2{}^2 + 120\,x_2x_3 + 240\,x_1 + 480\,x_2\right)h^3.
\end{aligned}$$

Defining cofactors $C_i = K_i/D, i = 1 \dots 4$, we find the following Darboux polynomials $p_{i,1}$:

$$\begin{aligned}
p_{1,1} &= x_1 + x_2, \\
p_{2,1} &= 4\,x_2 + x_3 + 9, \\
p_{3,1} &= x_2 + x_3 - 1, \\
p_{4,1} &= x_1 + 3\,x_2.
\end{aligned}$$

The cofactor $C_5 = \frac{K_1 K_4}{D^2}$ gives the Darboux polynomials $p_{5,1}, p_{5,2}$:

$$p_{5,1} = (x_1 + 3\,x_2)\,(x_1 + x_2)\,,$$
$$p_{5,2} = (5\,hx_2 + 2\,hx_3 + 8\,h + 1)\,(5\,hx_2 + 2\,hx_3 + 8\,h - 1)\,,$$

and the cofactor $C_6 = \frac{K_2 K_3}{D^2}$ gives the Darboux polynomials $p_{6,1}, p_{6,2}$:

$$p_{6,1} = -\,(3\,hx_1 + 6\,hx_2 + 1)\,(3\,hx_1 + 6\,hx_2 - 1)\,,$$
$$p_{6,2} = (x_2 + x_3 - 1)(4\,x_2 + x_3 + 9).$$

From these we obtain that the preserved integrals of the Kahan map are $\frac{p_{5,1}(\mathbf{x})}{p_{5,2}(\mathbf{x})}$, $\frac{p_{6,2}(\mathbf{x})}{p_{6,1}(\mathbf{x})}$, and any combination

$$\frac{1}{p_{5,i}(\mathbf{x})p_{6,j}(\mathbf{x})}d\mathbf{x}, \qquad i,j \in \{1,2\},$$

is a preserved measure.

3.3. Example 3: Quartic Nahm system in 2D.

This subsection exemplifies the Kahan discretization of a certain class of ODEs with quartic Hamiltonians. We consider the following example whose Kahan discretization was studied in [9]

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1^2 - 12x_2^2 \\ -6x_1x_2 - 4x_2^2 \end{pmatrix}. \tag{10}$$

This ODE has a preserved integral

$$H := x_2(2x_1 + 3x_2)(x_1 - x_2)^2,$$

and a preserved measure

$$\int \frac{dx_1 \wedge dx_2}{x_2(2x_1 + 3x_2)(x_1 - x_2)}.$$

The Jacobian determinant of the Kahan discretization has the following factorization

$$J = \frac{K_1\,K_2\,K_3}{D^3},$$

where the three affine K_i are given by

$$K_1 := 1 + 3hx_1 - 8hx_2,$$
$$K_2 := 1 - 5hx_1,$$
$$K_3 := 1 + 3hx_1 + 12hx_2,$$

and the quadratic D is

$$D := 1 + hx_1 + 4hx_2 - 6h^2x_1^2 - 8h^2x_1x_2 - 36h^2x_2^2.$$

Among the cofactors $K_1^i K_2^j K_3^k / D^l$ for $i,j,k = 0,1$ and $l = 1,\ldots,3$ we consider $C_1 = \frac{K_1}{D}$, $C_2 = \frac{K_2}{D}$, and $C_3 = \frac{K_3}{D}$, satisfying $J = C_1 C_2 C_3$. The corresponding Darboux polynomials are

$$p_{1,1} = 2x_1 + 3x_2,$$
$$p_{2,1} = x_2,$$
$$p_{3,1} = x_2 - x_1,$$

leading to the preserved measure

$$\frac{dx_1 \wedge dx_2 \wedge dx_3}{p_{1,1}p_{2,1}p_{3,1}}.$$

To find the modified integral, we search for Darboux polynomials whose cofactors are of the form $C_1^i C_2^j$ for $i, j = 1, 2, \ldots$ (i.e., "super-factors" of J). Using the cofactor

$$C_4 := C_1 C_2 C_3^2,$$

we find

$$p_{4,1} = x_2(2x_1 + 3x_2)(x_1 - x_2)^2,$$
$$p_{4,2} = (1 + h(3x_1 + 2x_2))(1 - h(3x_1 + 2x_2))(1 + h(6x_2 - x_1))(1 - h(6x_2 - x_1)),$$

and $\frac{p_{4,1}}{p_{4,2}}$ is an integral of the Kahan discretization, see the corresponding example in [9].

Note that the invariant set $p_{4,2}(\mathbf{x}) = 0$, represented by a product of four lines, is shown in red in the phase plot of the Quartic Nahm system in Figure 3 of our paper [14], for timestep $h = 1/5$.

3.4. **Example 4: Lagrange top.** This subsection describes the computation of the modified Darboux polynomials of the Kahan map of one of the classical integrable tops, requiring significant computing power. Discretizations of the Lagrange top have been studied in [7] and [9], where it was shown that the Kahan map preserves a number of modified integrals. The Lagrange top reads

$$\frac{d}{dt} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} (\alpha - 1)\, m_2 m_3 + \gamma\, p_2 \\ (1 - \alpha)\, m_1 m_3 - \gamma\, p_1 \\ 0 \\ \alpha\, p_2 m_3 - p_3 m_2 \\ p_3 m_1 - \alpha\, p_1 m_3 \\ p_1 m_2 - p_2 m_1 \end{pmatrix}, \tag{11}$$

where m_i and p_i are the angular and linear momentum components and α and γ are constant parameters. The Lagrange top admits four independent integrals

$$\tilde{H}_1 = p_1^2 + p_2^2 + p_3^2,$$
$$\tilde{H}_2 = p_1 m_1 + p_2 m_2 + p_3 m_3,$$
$$\tilde{H}_3 = m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3,$$
$$\tilde{H}_4 = m_3.$$

For the Lagrange top, it suffices to treat m_3 as a free parameter by working in the variables $\bar{\mathbf{x}} = (m_1, m_2, p_1, p_2, p_3)^{\mathrm{T}}$ and look for degree-six Darboux polynomial densities in $\bar{\mathbf{x}}$. The Jacobian determinant of the Kahan map has the form

$$J = \frac{K_1 K_2}{D^3},$$

where K_1 has 377 terms and K_2 has 35 terms. Using the cofactor $C_1(\mathbf{x}) = J$, we find the following five Darboux polynomial densities

$$p_{1,1} = -256\,\gamma + 64\,h^2\gamma\left(-2\,m_3{}^2\alpha^2 + 2\,m_3{}^2\alpha + \gamma\,p_3 - m_1{}^2 - m_2{}^2 - m_3{}^2\right)$$
$$\quad + h^4 Q_{1,2}^{(4)} + h^6 Q_{1,3}^{(6)} + h^8 Q_{1,4}^{(8)},$$
$$p_{1,2} = -2048\,\gamma^3 + 256\,h^2\gamma^3\left(-2\,m_3{}^2\alpha^2 + 2\,m_3{}^2\alpha + 4\,\gamma\,p_3 - m_1{}^2 - m_2{}^2 - 2\,m_3{}^2\right)$$
$$\quad + h^4 Q_{2,2}^{(4)} + h^6 Q_{2,3}^{(6)} + h^8 Q_{2,4}^{(8)} + h^{10} Q_{2,5}^{(8)},$$
$$p_{1,3} = -2048\,m_3\,(6\,\alpha - 5) + h^2(1536\,m_3\,(1 - \alpha)(m_1^2 + m_2^2) + 6144\,\alpha\,\gamma\,m_3 p_3$$
$$\quad + 1024\,m_3^3(1 - 4\,\alpha + 10\,\alpha^2 - 8\,\alpha^3) - 512\,\gamma\,p_1 m_1 - 512\,\gamma\,p_2 m_2$$

$$- 5120\,\gamma\,m_3 p_3) + h^4 Q_{3,2}^{(5)} + h^6 Q_{3,3}^{(7)} + h^8 Q_{3,4}^{(9)} + h^{10} Q_{3,5}^{(11)} + h^{12} Q_{3,6}^{(11)},$$

$$p_{1,4} = -256\,\gamma\,m_3\,(2\,\alpha - 1) + 64\,h^2\gamma\,(-2\,\alpha^3 m_3{}^3 + 3\,\alpha^2 m_3{}^3 + 4\,\alpha\,\gamma\,m_3 p_3$$
$$- \alpha\,m_1{}^2 m_3 - \alpha\,m_2{}^2 m_3 - 3\,\alpha\,m_3{}^3 + \gamma\,p_1 m_1 + \gamma\,p_2 m_2 - 2\,\gamma\,m_3 p_3 + m_3{}^3)$$
$$+ h^4 Q_{4,2}^{(5)} + h^6 Q_{4,3}^{(7)} + h^8 Q_{4,4}^{(9)},$$

$$p_{1,5} = -65536 + h^2\big(-32768\,m_3{}^2\alpha^2 + 40960\,m_3{}^2\alpha + 16384\,(\gamma\,p_3 - m_1{}^2 - m_2{}^2$$
$$- 2\,m_3{}^2)\big) + h^4 Q_{5,2}^{(4)} + h^6 Q_{5,3}^{(6)} + h^8 Q_{5,4}^{(8)} + h^{10} Q_{5,5}^{(10)} + h^{12} Q_{5,6}^{(12)} + h^{14} Q_{5,7}^{(12)},$$

where each $Q_{j,k}^{(i)}$ is a polynomial of degree i in the variables $(m_1, m_2, m_3, p_1, p_2, p_3)$. Taking the quotients $\frac{p_{1,1}}{p_{1,5}}$, $\frac{p_{1,2}}{p_{1,5}}$, $\frac{p_{1,3}}{p_{1,5}}$ and $\frac{p_{1,4}}{p_{1,5}}$ yields four functionally independent integrals. Taking functionally dependent combinations of these, we are able to form the following integrals that are preserved by the Kahan discretization

$$H_1 = \frac{p_1^2 + p_2^2 + p_3^2 + \mathcal{O}(h^2)}{1 + \mathcal{O}(h^2)},$$

$$H_2 = \frac{p_1 m_1 + p_2 m_2 + p_3 m_3 + \mathcal{O}(h^2)}{1 + \mathcal{O}(h^2)},$$

$$H_3 = \frac{m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3 + \mathcal{O}(h^2)}{1 + \mathcal{O}(h^2)},$$

$$H_4 = m_3,$$

where the first three integrals are modified versions of the continuous integrals.

3.5. **Example 5: A Kahan map having a non-rational integral.** This example was constructed to display a non-rational integral, an exact linearization and solution, as well as a large number of affine Darboux polynomials (even more when the parameter α equals 1).

$$\frac{dx_1}{dt} = 24(x_2 - x_4)(1 - \alpha) + 9x_1^2 + 48x_1 x_3 - 40x_1 x_4 + 24x_2^2 - 48x_2 x_3$$
$$+ 48x_2 x_4 + 48x_3^2 + 24x_3 x_4 - 132x_4^2 + x_1,$$

$$\frac{dx_2}{dt} = 7(x_2 - x_4)\alpha - 2x_1^2 - 12x_1 x_3 + 12x_1 x_4 - 5x_2^2 + 12x_2 x_3 - 14x_2 x_4$$
$$- 12x_3^2 - 6x_3 x_4 + 38x_4^2 - 6x_2 + 7x_4,$$

$$\frac{dx_3}{dt} = 14(x_2 - x_4)(\alpha - 1) - 4x_1^2 - 24x_1 x_3 + 24x_1 x_4 - 14x_2^2 + 28x_2 x_3$$
$$- 28x_2 x_4 - 25x_3^2 - 12x_3 x_4 + 76x_4^2 + x_3,$$

$$\frac{dx_4}{dt} = 6(x_2 - x_4)\alpha - 2x_1^2 - 12x_1 x_3 + 12x_1 x_4 - 6x_2^2 + 12x_2 x_3 - 12x_2 x_4$$
$$- 12x_3^2 - 6x_3 x_4 + 37x_4^2 - 6x_2 + 7x_4,$$

$$(12)$$

where α is a parameter.

The Jacobian determinant of the Kahan discretization of this ODE is

$$J = \frac{K_1{}^3 K_2{}^3 K_3 K_4}{D_1{}^2 D_2{}^2 D_3{}^2 D_4{}^2},$$

$$(13)$$

where $K_1, \ldots, K_4$ are constant, and $D_1, \ldots, D_4$ are affine:

$$K_1 = 1 - \tfrac{1}{2}h,$$

$$K_2 = 1 + \tfrac{1}{2}h,$$
$$K_3 = 1 - \tfrac{1}{2}\alpha h,$$
$$K_4 = 1 + \tfrac{1}{2}\alpha h,$$
$$D_1 = 1 - \tfrac{1}{2}h - hx_1 - 4hx_4,$$
$$D_2 = 1 - \tfrac{1}{2}h - 2hx_2 + hx_3,$$
$$D_3 = 1 - \tfrac{1}{2}h - 2hx_1 - 3hx_3 - hx_4,$$
$$D_4 = 1 - \tfrac{1}{2}\alpha h - hx_2 + hx_4.$$

The following 11 affine discrete Darboux polynomials $p_{i,1}$ for $i = 1, ..., 11$ are found corresponding to the cofactors C_i

i	$p_{i,1}$	C_i
1	$1 + x_1 + 4x_4$	K_1/D_1
2	$x_1 + 4\,x_4$	K_2/D_1
3	$1 + 2x_2 - x_3$	K_1/D_2
4	$2x_2 - x_3$	K_2/D_2
5	$1 + 2x_1 + 3x_3 + x_4$	K_1/D_3
6	$2x_1 + 3x_3 + x_4$	K_2/D_3
7	$\alpha + x_2 - x_4$	K_3/D_4
8	$x_2 - x_4$	K_4/D_4
9	$x_1 - 2\,x_2 + x_3 + 4\,x_4$	$K_1K_2/(D_1D_2)$
10	$x_1 + 3x_3 - 3x_4$	$K_1K_2/(D_1D_3)$
11	$2x_1 - 2x_2 + 4x_3 + x_4$	$K_1K_2/(D_2D_3)$

(14)

It can be read off from table (14) that the three ratios $\frac{p_{1,1}}{p_{2,1}}, \frac{p_{3,1}}{p_{4,1}}$ and $\frac{p_{5,1}}{p_{6,1}}$ each have constant cofactor K_1/K_2, and that $\frac{p_{7,1}}{p_{8,1}}$ has constant cofactor K_3/K_4.

It follows that the three functions

$$H_1 = \frac{p_{1,1}p_{4,1}}{p_{2,1}p_{3,1}} = \frac{(1 + x_1 + 4x_4)\,(2x_2 - x_3)}{(x_1 + 4x_4)\,(1 + 2x_2 - x_3)}, \tag{15}$$

$$H_2 = \frac{p_{1,1}p_{6,1}}{p_{2,1}p_{5,1}} = \frac{(1 + x_1 + 4x_4)\,(2x_1 + 3x_3 + x_4)}{(x_1 + 4x_4)\,(1 + 2x_1 + 3x_3 + x_4)}, \tag{16}$$

$$H_3 = \frac{p_{1,1}p_{8,1}^a}{p_{2,1}p_{7,1}^a} = \frac{(1 + x_1 + 4x_4)\,(x_2 - x_4)^a}{(x_1 + 4x_4)\,(\alpha + x_2 - x_4)^a}, \tag{17}$$

are all integrals of the Kahan map of (12), where $a := \frac{\ln(K_1/K_2)}{\ln(K_3/K_4)}$.

We remark that in general the integral (17) is non-rational, even though the leaves $p_{1,1}/p_{2,1} = K_1/K_2$ and $p_{7,1}/p_{8,1} = K_3/K_4$ making up the integral are polynomial. [5]

We also see that

$$C_7 C_8 C_9 C_{10} C_{11} = \frac{K_1^3 K_2^3 K_3 K_4}{D_1^2 D_2^2 D_3^2 D_4^2} = J. \tag{18}$$

Hence the Kahan map of the vector field (12) has the preserved measure

$$\frac{d\mathbf{x}}{p_{7,1}p_{8,1}p_{9,1}p_{10,1}p_{11,1}}, \tag{19}$$

---

[5] Note that in the special case $\alpha = 1$, $K_3 = K_1$ and $K_4 = K_2$; the Kahan map of (12) has 3 additional affine Darboux polynomials; and the integral (17) becomes rational.

and therefore the Kahan map is super-integrable. It follows that the ODE (12) is also super-integrable, preserving the same integrals and measure as the map.

Finally, it can be read off from Table(14) and the fact that $K_1, \ldots, K_4$ are constant that

$$
\begin{aligned}
p_{1,1}(\mathbf{x}_n) &= \beta_1 p_{2,1}(\mathbf{x}_n)(K_1/K_2)^n, \\
p_{3,1}(\mathbf{x}_n) &= \beta_2 p_{4,1}(\mathbf{x}_n)(K_1/K_2)^n, \\
p_{5,1}(\mathbf{x}_n) &= \beta_3 p_{6,1}(\mathbf{x}_n)(K_1/K_2)^n, \\
p_{7,1}(\mathbf{x}_n) &= \beta_4 p_{8,1}(\mathbf{x}_n)(K_3/K_4)^n,
\end{aligned}
\tag{20}
$$

where the β_i are integration constants.

The equations (20) represent 4 linear equations in the four variables x_1, x_2, x_3, x_4. Solving them, we obtain the exact solution of the Kahan map of eq(12):

$$
\begin{aligned}
x_1(n) = \frac{1}{D}\Big(&(4((\beta_1 - \frac{7\beta_3}{4})\beta_2 + 3\beta_1\beta_3)\beta_4 r_2^n + 24((\beta_1 + \beta_3)\beta_2 + \beta_1\beta_3)\alpha \\
&+ (-4\beta_1 + 7\beta_3)\beta_2 - 12\beta_1\beta_3)r_1^{2n} + (9 - (16\beta_1 - 3\beta_2 + 5\beta_3)r_1^n)\beta_4 r_2^n \\
&+ (24(-\beta_1 - \beta_2 - \beta_3)\alpha + 16\beta_1 - 3\beta_2 + 5\beta_3)r_1^{2n} \\
&- 24 r_1^{3n}\alpha\beta_1\beta_2\beta_3 + 24\alpha - 9), \\
x_2(n) = \frac{1}{D}\Big(&(-((\beta_1 - 2\beta_3)\beta_2 + 3\beta_1\beta_3)\beta_4 r_2^n + ((-7\beta_1 - 7\beta_3)\beta_2 - 7\beta_1\beta_3)\alpha \\
&+ (\beta_1 - 2\beta_3)\beta_2 + 3\beta_1\beta_3)r_1^{2n} + 4(-\frac{1}{2} + (\beta_1 - \frac{\beta_2}{4} + \frac{\beta_3}{4})r_1^n)\beta_4 r_2^n \\
&+ ((7\beta_1 + 7\beta_2 + 7\beta_3)\alpha - 4\beta_1 + \beta_2 - \beta_3)r_1^n + 7 r_1^{3n}\alpha\beta_1\beta_2\beta_3 - 7\alpha + 2), \\
x_3(n) = \frac{1}{D}\Big(&(-2((\beta_1 - 2\beta_3)\beta_2 + \frac{7\beta_1\beta_3}{2})\beta_4 r_2^n - 14((\beta_1 + \beta_3)\beta_2 + \beta_1\beta_3)\alpha \\
&+ (2\beta_1 - 4\beta_3)\beta_2 + 7\beta_1\beta_3)r_1^{2n} + 9(-\frac{5}{9} + (\beta_1 - \frac{2\beta_2}{9} + \frac{\beta_3}{3})r_1^n)\beta_4 r_2^n + 5 \\
&+ (14(\beta_1 + \beta_2 + \beta_3)\alpha - 9\beta_1 + 2\beta_2 - 3\beta_3)r_1^n + 14\alpha(r_1^{3n}\beta_1\beta_2\beta_3 - 1)), \\
x_4(n) = \frac{1}{D}\Big(&(-((\beta_1 - 2\beta_3)\beta_2 + 3\beta_1\beta_3)\beta_4 r_2^n + ((-6\beta_1 - 6\beta_3)\beta_2 - 6\beta_1\beta_3)\alpha \\
&+ (\beta_1 - 2\beta_3)\beta_2 + 3\beta_1\beta_3)r_1^{2n} + 4(-\frac{1}{2} + (\beta_1 - \frac{\beta_2}{4} + \frac{\beta_3}{4})r_1^n)\beta_4 r_2^n \\
&+ ((6\beta_1 + 6\beta_2 + 6\beta_3)\alpha - 4\beta_1 + \beta_2 - \beta_3)r_1^n + 6 r_1^{3n}\alpha\beta_1\beta_2\beta_3 - 6\alpha + 2),
\end{aligned}
\tag{21}
$$

where

$$
r_1 = K_1/K_2, \ r_2 = K_3/K_4, \text{ and } D = (\beta_4 r_2^n - 1)(\beta_3 r_1^n - 1)(\beta_1 r_1^n - 1)(\beta_2 r_1^n - 1).
$$

The exact solution of the ODE (12) is obtained from (21) by replacing r_1^n by e^t, resp r_2^n by $e^{\alpha t}$.

3.6. **Example 6: A 4D polarization map.** This subsection exemplifies the fact that with regard to the discretization of ODEs, the application of our method is not restricted to Kahan's discretization. Here we study an application to our polarization method [12]. For a different application, cf [6].

We consider the 4-dimensional map presented in [12] (choosing $a = 2, b = 1, c = -3, d = -1, e = 1$ in their notation).

$$
x_1' = x_2,
$$

$$x_2' = \frac{2\,hx_1{}^2x_2 - 6\,hx_1{}^2x_4 - 12\,hx_1x_2x_3 - 4\,hx_1x_3x_4 - 2\,hx_2x_3{}^2 + 2\,hx_3{}^2x_4 + x_1}{D(\mathbf{x})},$$

$$x_3' = x_4,$$

$$x_4' = \frac{-4\,hx_1{}^2x_2 - 2\,hx_1{}^2x_4 - 4\,hx_1x_2x_3 + 12\,hx_1x_3x_4 + 6\,hx_2x_3{}^2 + 2\,hx_3{}^2x_4 + x_3}{D(\mathbf{x})},$$

where the quartic D is

$$\begin{aligned}
D := {}&-28\,h^2x_1{}^2x_2{}^2 + 4\,h^2x_1{}^2x_2x_4 - 40\,h^2x_1{}^2x_4{}^2 + 4\,h^2x_1x_2{}^2x_3 - 28\,h^2x_1x_2x_3x_4 \\
&- 8\,h^2x_1x_3x_4{}^2 - 40\,h^2x_2{}^2x_3{}^2 - 8\,h^2x_2x_3{}^2x_4 - 16\,h^2x_3{}^2x_4{}^2 + 1.
\end{aligned}$$

The determinant of the Jacobian of the map has the factorized form

$$J = \frac{K_1}{D^3},$$

where

$$\begin{aligned}
K_1 := {}&-128\,h^3x_1{}^3x_2{}^3 + 456\,h^3x_1{}^3x_2{}^2x_4 - 120\,h^3x_1{}^3x_2x_4{}^2 + 496\,h^3x_1{}^3x_4{}^3 \\
&+ 984\,h^3x_1{}^2x_2{}^3x_3 + 432\,h^3x_1{}^2x_2{}^2x_3x_4 + 936\,h^3x_1{}^2x_2x_3x_4{}^2 + 336\,h^3x_1{}^2x_3x_4{}^3 \\
&+ 408\,h^3x_1x_2{}^3x_3{}^2 - 864\,h^3x_1x_2{}^2x_3{}^2x_4 + 216\,h^3x_1x_2x_3{}^2x_4{}^2 - 528\,h^3x_1x_3{}^2x_4{}^3 \\
&- 488\,h^3x_2{}^3x_3{}^3 - 192\,h^3x_2{}^2x_3{}^3x_4 - 264\,h^3x_2x_3{}^3x_4{}^2 - 80\,h^3x_3{}^3x_4{}^3 \\
&- 84\,h^2x_1{}^2x_2{}^2 + 12\,h^2x_1{}^2x_2x_4 - 120\,h^2x_1{}^2x_4{}^2 + 12\,h^2x_1x_2{}^2x_3 - 84\,h^2x_1x_2x_3x_4 \\
&- 24\,h^2x_1x_3x_4{}^2 - 120\,h^2x_2{}^2x_3{}^2 - 24\,h^2x_2x_3{}^2x_4 - 48\,h^2x_3{}^2x_4{}^2 + 1.
\end{aligned}$$

Using $C_1 = J$ as cofactor, the resulting functionally independent Darboux polynomials are

$$p_{1,1} = D,$$

$$\begin{aligned}
p_{1,2} = {}&(x_1x_4 - x_2x_3)\,(4\,hx_1{}^2x_2{}^2 + 4\,hx_1{}^2x_2x_4 - 6\,hx_1{}^2x_4{}^2 + 4\,hx_1x_2{}^2x_3 - x_2x_3 \\
&- 24\,hx_1x_2x_3x_4 - 4\,hx_1x_3x_4{}^2 - 6\,hx_2{}^2x_3{}^2 - 4\,hx_2x_3{}^2x_4 + 2\,hx_3{}^2x_4{}^2 + x_1x_4),
\end{aligned}$$

$$\begin{aligned}
p_{1,3} = {}&26\,hx_1{}^3x_2{}^2x_4 + 10\,hx_1{}^3x_2x_4{}^2 + 2\,hx_1{}^3x_4{}^3 - 26\,hx_1{}^2x_2{}^3x_3 - 60\,hx_1{}^2x_2x_3x_4{}^2 \\
&- 8\,hx_1{}^2x_3x_4{}^3 - 10\,hx_1x_2{}^3x_3{}^2 + 60\,hx_1x_2{}^2x_3{}^2x_4 + 14\,hx_1x_3{}^2x_4{}^3 - 2\,hx_2{}^3x_3{}^3 \\
&+ 8\,hx_2{}^2x_3{}^3x_4 - 14\,hx_2x_3{}^3x_4{}^2 + 2\,x_1{}^2x_2{}^2 + 2\,x_1{}^2x_2x_4 + 2\,x_1x_2{}^2x_3 \\
&- 18\,x_1x_2x_3x_4 - 2\,x_1x_3x_4{}^2 - 2\,x_2x_3{}^2x_4 + x_3{}^2x_4{}^2.
\end{aligned}$$

The map thus possesses the preserved measure $\int \frac{d\mathbf{x}}{p_{1,1}}$ and the (independent) first integrals $\frac{p_{1,2}}{p_{1,1}}$ and $\frac{p_{1,3}}{p_{1,1}}$, in agreement with [12].

3.7. Example 7: sine-Gordon maps.

This subsection exemplifies the application of our method to maps arising from the theory of discrete integrable systems. We consider $(k+1)$-dimensional maps that arise as so-called $(1,k)$ reductions of the discrete sine-Gordon equation [21].

We start with the case $k = 3$, then treat the case $k = 2$, before giving a general theorem for arbitrary k.

3.7.1. *The $(1,3)$ sine-Gordon map.* The $(1,3)$ sine-Gordon map ϕ is given by

$$x_i' = x_{i+1}, \qquad i = 0, 1, 2,$$
$$x_3' = \frac{1 - \alpha x_1 x_3}{x_0(x_1 x_3 - \alpha)},$$

where α is a parameter. Using $C_1(\mathbf{x}) = J$, we find the corresponding Darboux polynomials:

$$p_{1,1} = x_3 x_2 x_1 x_0,$$
$$p_{1,2} = x_0{}^2 x_1 x_2 x_3{}^2 - \alpha\, x_0{}^2 x_2 x_3 - \alpha\, x_0 x_1{}^2 x_3 - \alpha\, x_0 x_1 x_2{}^2$$
$$\qquad - \alpha\, x_0 x_1 x_3{}^2 - \alpha\, x_0 x_2{}^2 x_3 - \alpha\, x_1{}^2 x_2 x_3 + x_1 x_2,$$
$$p_{1,3} = x_0{}^2 x_1{}^2 x_2 x_3 + x_0 x_1{}^2 x_2{}^2 x_3 + x_0 x_1 x_2{}^2 x_3{}^2 - \alpha\, x_0{}^2 x_1 x_2$$
$$\qquad - \alpha\, x_1 x_2 x_3{}^2 + x_0 x_1 + x_0 x_3 + x_3 x_2.$$

It follows that ϕ possesses the (independent) first integrals $\frac{p_{1,2}}{p_{1,1}}$ and $\frac{p_{1,3}}{p_{1,1}}$, and the preserved measure $\int \frac{d\mathbf{x}}{p_{1,1}}$. These results were found using different methods in [21].

3.7.2. *The $(1,2)$ sine-Gordon map.* The $(1,2)$ sine-Gordon map ϕ is given by

$$x_i' = x_{i+1}, \qquad i = 0, 1,$$
$$x_2' = \frac{1 - \alpha x_1 x_2}{x_0(x_1 x_2 - \alpha)}.$$

Using $C_1(\mathbf{x}) = -J$, we find the corresponding Darboux polynomials:

$$p_{1,1} = x_0 x_1 x_2,$$
$$p_{1,2} = x_0{}^2 x_1 x_2{}^2 - \alpha\, x_0{}^2 x_2 - \alpha\, x_0 x_1{}^2 - \alpha\, x_0 x_2{}^2 - \alpha\, x_1{}^2 x_2 + x_1,$$
$$p_{1,3} = x_0{}^2 x_1{}^2 x_2 + x_0 x_1{}^2 x_2{}^2 - \alpha\, x_0{}^2 x_1 - \alpha\, x_1 x_2{}^2 + x_0 + x_2.$$

It follows that ϕ possesses the (independent) first integrals $\frac{p_{1,2}}{p_{1,1}}$ and $\frac{p_{1,3}}{p_{1,1}}$, and the preserved measure $\int \frac{d\mathbf{x}}{p_{1,1}}$. Note that this is one extra first integral, that was not found using the Lax representation approach of ref [21]. Moreover, we find that there is an additional cofactor $C_2(\mathbf{x}) = J$, for which we find the corresponding Darboux polynomial

$$p_{2,1} = -x_0{}^2 x_1 x_2{}^2 + \alpha\, x_0{}^2 x_2 - \alpha\, x_0 x_1{}^2 + \alpha\, x_0 x_2{}^2 - \alpha\, x_1{}^2 x_2 + x_1.$$

Normally, a sole Darboux polynomial does not yield an integral, but, because $C_2 = -C_1$, we find that

$$H := \frac{p_{2,1}}{p_{1,1}},$$

is a so-called 2-integral [17], i.e. an integral of $\phi \circ \phi$. In this case $H(\mathbf{x}') = -H(\mathbf{x})$. This result was not found using the Lax matrix approach in [21].

3.7.3. *The $(1,k)$ sine-Gordon map.* The $(1,k)$ sine-Gordon map ϕ_k is given by

$$
\begin{aligned}
x_i' &= x_{i+1}, \qquad i = 0, \ldots, k-1, \\
x_k' &= \frac{1 - \alpha x_1 x_k}{x_0(x_1 x_k - \alpha)},
\end{aligned}
\tag{22}
$$

1 where $\lfloor \frac{k+1}{2} \rfloor$ functionally independent rational integrals for this map were found
2 using a Lax matrix approach in [21]. Denote these integrals by $H_k^n(\mathbf{x}) = \frac{N_k^n(\mathbf{x})}{D_k^n(\mathbf{x})}$,
3 $n = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor$, and define

$$\epsilon := (-1)^{k+1}. \tag{23}$$

Theorem 3.2. *For all n and k, the Darboux polynomials N_k^n and D_k^n are given by*

$$N_k^n(\mathbf{x}') = C(\mathbf{x})N_k^n(\mathbf{x}),$$
$$D_k^n(\mathbf{x}') = C(\mathbf{x})D_k^n(\mathbf{x}),$$

4 *where the cofactor $C(\mathbf{x})$ depends only on k, and is given by $C(\mathbf{x}) = \epsilon|D\phi_k(\mathbf{x})|$.*

5   The proof is given in appendix A.

6 4. **Detecting Darboux polynomials and integrals.** Given a rational map $\mathbf{x}' =$
7 $\phi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ containing k free parameters denoted by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, one
8 could ask if there exist particular choices of $\boldsymbol{\alpha}$ such that ϕ preserves additional
9 second integrals. We note that, in contrast to the *linear* cofactor equation (4), this
10 amounts to solving the *nonlinear* cofactor equation

$$p(\mathbf{x}') = C(\mathbf{x}; \boldsymbol{\alpha})p(\mathbf{x}) \tag{24}$$

11 for the Darboux polynomial indeterminants as well as the parameters, where $C(\mathbf{x}; \boldsymbol{\alpha})$
12 can be non-linear in $\boldsymbol{\alpha}$.

4.1. **Example 8: Extended McMillan map.** Consider the following rational
map $\phi(\mathbf{x})$ defined by

$$\phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 - f(x_1) \\ x_1 \end{pmatrix},$$

where

$$f(x_1) = \frac{\alpha_1 {x_1}^3 + \alpha_2 {x_1}^2 + \alpha_3 x_1 + \alpha_4}{\alpha_5 {x_1}^2 + \alpha_2 x_1 + \alpha_6},$$

13 and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_6)$ are free parameters. The integrability of a special case of this
14 map was studied in [24]. The Jacobian of the map ϕ is $J = 1$. If all parameters $\boldsymbol{\alpha}$
15 are arbitrary and using $C = J$, the equation

$$p(\phi(\mathbf{x})) = p(\mathbf{x}), \tag{25}$$

has only one solution $p_1(\mathbf{x}) = 1$. Solving the non-linear cofactor equation (24) yields
the condition $\alpha_1 = 0$. This is an integrable map known as the McMillan map [25].
Enforcing this condition, one now finds two solutions to equation (25)

$$p_1(\mathbf{x}) = 1,$$
$$p_2(\mathbf{x}) = \alpha_5 {x_1}^2{x_2}^2 + \alpha_2 \left({x_1}^2 x_2 + x_1 {x_2}^2 \right) + \alpha_3 x_1 x_2 + \alpha_6 \left({x_1}^2 + {x_2}^2 \right)$$
$$\qquad + \alpha_4 \left(x_1 + x_2 \right),$$

16 where $p_2(\mathbf{x})$ is a preserved integral of ϕ, in agreement with McMillan.

4.2. **Example 9: Two coupled Euler tops.** This subsection exemplifies an ODE that possesses a non-rational integral. This integral only seems to be preserved by the Kahan map in two special cases: one rational and one polynomial. A Kahan map that possesses a non-rational integral, that becomes rational in the continuum limit, is given in example 5.

We now consider two coupled Euler tops whose vector field is given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} a_1^2 x_2 x_3 \\ a_2^2 x_3 x_1 \\ a_3^2 x_1 x_2 + a_4^2 x_4 x_5 \\ a_5^2 x_5 x_3 \\ a_6^2 x_3 x_4 \end{pmatrix}. \tag{26}$$

This system was first presented in [20], and its integrals after discretisation were first explored in [9], where the authors present the following three independent integrals of motion

$$H_1 = a_2^2 x_1^2 - a_1^2 x_2^2, \quad H_2 = a_3^2 a_5^2 x_2^2 - a_2^2 a_5^2 x_3^2 + a_2^2 a_4^2 x_4^2, \quad H_3 = a_6^2 x_4^2 - a_5^2 x_5^2,$$

however we report here the existence of a fourth independent integral given by

$$H_4 = \frac{(a_1 x_2 + a_2 x_1)^{a_5 a_6}}{(a_5 x_5 + a_6 x_4)^{a_1 a_2}},$$

hence the system is super-integrable. To our knowledge, the integral H_4 is new. The Jacobian determinant of the Kahan map has the following factors

$$J = \frac{K_1\,K_2\,K_3\,K_4\,K_5}{D^6}.$$

The cofactors $C_i = \frac{K_i}{D}$, for $i = 1, 2, 4, 5$ admit the following linear Darboux polynomials

$$p_{1,1} = a_5 x_5 + a_6 x_4,$$
$$p_{2,1} = a_5 x_5 - a_6 x_4,$$
$$p_{4,1} = a_1 x_2 + a_2 x_1,$$
$$p_{5,1} = a_1 x_2 - a_2 x_1,$$

however, the cofactor C_3 admits no polynomial solutions, up to Darboux polynomials of degree 6. Now we look for quadratic Darboux polynomials with the cofactors $C_6 := C_1 C_2$ and $C_7 := C_4 C_5$ and get the following

$$p_{6,1} = p_{1,1}p_{2,1}, \qquad p_{6,2} = (2 - ha_5 a_6 x_3)(2 + ha_5 a_6 x_3), \tag{27}$$
$$p_{7,1} = p_{4,1}p_{5,1}, \qquad p_{7,2} = (2 - ha_1 a_2 x_3)(2 + ha_1 a_2 x_3). \tag{28}$$

We note that $p_{6,2}$ and $p_{7,2}$ also factorise. In the ODE case, if a Darboux polynomial factorises, each factor is also a Darboux polynomial. In the discrete case that need not be the case, and indeed it often is not true. (To our knowledge, this possibility was first raised in Gasull and Manosa [16]). Here for instance we have that $p_{6,2}$ and $p_{7,2}$ factorize, i.e.

$$p_{6,2} = q_{6,1}q_{6,2}, \quad p_{7,2} = q_{7,1}q_{7,2},$$

where the $q_{i,j}$ satisfy

$$q'_{6,1} = C_1 q_{6,2}, \qquad\qquad q'_{6,2} = C_2 q_{6,1}$$
$$q'_{7,1} = C_4 q_{7,2}, \qquad\qquad q'_{7,2} = C_5 q_{7,1}$$

which implies that each $q_{i,j}$ is in fact a discrete Darboux polynomial of the second iterate of the Kahan map. The Darboux polynomials from equations (27) and (28) yield two independent integrals $\frac{p_{6,1}}{p_{6,2}}$ and $\frac{p_{7,1}}{p_{7,2}}$, in agreement with [9]. We also note that no (Darboux) polynomial measures are found up to degree 6 using J as the cofactor.

We now attempt to solve the non-linear cofactor equation

$$p(\mathbf{x}') = C_3(\mathbf{x}; \boldsymbol{\alpha})p(\mathbf{x}). \tag{29}$$

For a polynomial basis of degree 2, equation (29) admits three conditions that yield non-trivial Darboux polynomials: $a_3 = 0$, $a_4 = 0$ and $a_1^2 a_2^2 = a_5^2 a_6^2$. The first two correspond to the decoupling of two of the equations and these two less interesting cases have three independent discrete integrals each. The third condition is presented in [9]. In this case the Jacobian determinant of the Kahan discretization now factors as

$$J = \frac{K_6{}^3 K_7{}^3}{D^6}.$$

Using $C_8 = \frac{K_6 K_7}{D^2}$ as the cofactor, we get the following six Darboux polynomials

$$p_{8,1} = a_2^4 x_1^2 - a_5^2 a_6^2 x_2^2,$$
$$p_{8,2} = a_2^2 x_1 x_5 - a_6^2 x_2 x_4,$$
$$p_{8,3} = a_2^2 x_1 x_4 - a_5^2 x_2 x_5,$$
$$p_{8,4} = a_2^2 a_4^2 x_4^2 - a_2^2 a_5^2 x_3^2 + a_3^2 a_5^2 x_2^2,$$
$$p_{8,5} = a_2^2 a_4^2 x_5^2 - a_2^2 a_6^2 x_3^2 + a_3^2 a_6^2 x_2^2,$$
$$p_{8,6} = 4 - a_5^2 a_6^2 x_3^2 h^2.$$

Hence, the following measures are preserved

$$\int \frac{d\mathbf{x}}{p_{8,i} p_{8,j} p_{8,k}}, \quad \text{for any} \quad i, j, k = 1, \ldots, 6,$$

and the following integrals are preserved

$$\frac{p_{8,i}}{p_{8,k}}, \quad \text{for} \quad i \neq k,$$

of which four are independent. The choice $k = 6$ yields the integrals presented in [9].

4.3. Example 10: A family of Nambu systems with rational integrals.

Here we will consider Nambu systems, of the form

$$\dot{\mathbf{x}} = c \left(\nabla H_1 \times \nabla H_2 \right), \quad \mathbf{x} \in \mathbb{R}^3, \tag{30}$$

where $c = y^{2-\alpha}$, $H_1 = \frac{x}{y}$, $H_2 = y^\alpha Q$, where α is a free parameter. Initially we take Q to be

$$Q = a_1 x^2 + a_2 y^2 + a_3 z^2 + a_4 xy + a_5 yz + a_6 zx.$$

We get the following Jacobian determinant for the Kahan map

$$J = \frac{K_1^2 K_2}{D^4}.$$

The cofactor $C_1 := K_1/D$ has the following two Darboux polynomials at degree one

$$p_{1,1} = x \text{ and } p_{1,2} = y,$$

hence the integral H_1 is preserved exactly by the Kahan method. The cofactor $C_2 := K_2/D^2$ has the following Darboux polynomial

$$p_{2,1} = Q.$$

Using $C_3 = J$, we find that the Kahan discretisation has three preserved measures corresponding to the densities

$$p_{3,1} = x^2\,Q,\ p_{3,2} = xy\,Q \text{ and } p_{3,3} = y^2 Q,$$

which yield only one independent integral, H_1.

In order to facilitate additional computations, from now on we use fixed integer coefficients $\mathbf{a} = (3, 5, 7, 11, 13, 17)$ for Q, so we can use our detection algorithm to search for any values of the parameter α such that the Kahan discretisation yields extra Darboux polynomial solutions. To do this, we solve the non-linear cofactor equation

$$p(\mathbf{x}') = |D\phi(\mathbf{x})|p(\mathbf{x})$$

for α and the Darboux polynomials p of degree 4. This gives us the following solutions for α and the corresponding additional second integral of the Kahan discretisation:

α	Integrals
-2	H_1 and H_2
-1	H_1 and H_2
0	H_1 and $\tilde{H}_{2,0}$
1	H_1 and $\tilde{H}_{2,1}$
2	H_1 and $\tilde{H}_{2,2}$

where

$$\tilde{H}_{2,0} = \frac{Q}{12 + h^2\left(1853\,xy + 3485\,xz + 938\,y^2 + 2665\,yz + 1435\,z^2\right)},$$

$$\tilde{H}_{2,1} = \frac{yQ}{1 - h^2\left(226\,x^2 + 211\,xy + 119\,xz + 64\,y^2 + 91\,yz + 49\,z^2\right)},$$

$$\tilde{H}_{2,2} = \frac{y^2 Q}{48 - h^2 A_2 - h^4\,A_4},$$

and

$$A_2 = 26616\,x^2 + 23472\,xy + 11424\,xz + 6840\,y^2 + 8736\,yz + 4704\,z^2,$$

$$A_4 = 6309873\,x^3 z - 10784832\,x^2 yz + 1918455\,x^2 z^2 + 27341015\,xy^3$$
$$+ 37337147\,xy^2 z - 7467243\,xyz^2 - 559776\,xz^3 + 14528513\,y^4$$
$$+ 37680292\,y^3 z + 19891900\,y^2 z^2 - 428064\,yz^3 - 115248\,z^4.$$

5. Concluding remarks. We have proposed an approach based on the Jacobian factor ansatz to search for the preserved measures and integrals of a birational map and applied it to a number of examples. The approach uses Darboux polynomials. We have shown that the method can be used to both determine and detect measures and integrals. Some of the examples have required the use of relatively large computer memory space and computational time. While it is possible to give theoretical bounds on the complexity of various steps in our method (e.g. the computation of the determinant, and the computation of the Darboux polynomials given the cofactor), in practice a major issue seems to be when one involves many free parameters.

In this paper Examples 1,2,3,6,7,8 could all be computed on a laptop/desktop computer, but Examples 4,9, and 10 needed more power and memory and were solved on a supercomputer.

Appendix A. Proof of Theorem 2. In [21] it was shown that $\lfloor \frac{k+1}{2} \rfloor$ functionally independent integrals of the $(1,k)$ sine-Gordon map (22) are given by the trace of the Lax matrix $L^{1,k}$:

$$Tr L^{1,k}(\mathbf{x}, \lambda) = Tr\left[\begin{pmatrix} qx_0/x_k & \lambda^{-2}/x_k \\ x_0 & q \end{pmatrix} \prod_{l=0}^{k-1} \begin{pmatrix} p & -x_{l+1} \\ -\lambda^2/x_l & px_{l+1}/x_l \end{pmatrix} \right], \quad (31)$$

where $pq = \alpha$. The individual integrals are given by the coefficients of the various powers of the spectral parameter λ in the expansion of the right-hand side of (31).

It was also shown in [21] that the sine-Gordon map ϕ_k is either measure preserving or anti measure preserving, i.e. satisfies

$$P(\mathbf{x}') = \epsilon |D\phi_k(\mathbf{x})| P(\mathbf{x}), \quad (32)$$

where $P(\mathbf{x}) := \prod_{l=0}^{k} x_l$, and ϵ is given by (23).

It is easy to see that the rhs of (31) is equal to

$$\begin{aligned}
&\mathrm{Tr}\left[\frac{1}{x_k} \begin{pmatrix} qx_0 & \lambda^{-2} \\ x_0 x_k & qx_k \end{pmatrix} \prod_{l=0}^{k-1} \frac{1}{x_l} \begin{pmatrix} px_l & -x_l x_{l+1} \\ -\lambda^2 & x_{l+1} \end{pmatrix} \right] \\
&= \mathrm{Tr}\left[\begin{pmatrix} qx_0 & \lambda^{-2} \\ x_0 x_k & qx_k \end{pmatrix} \prod_{l=0}^{k-1} \begin{pmatrix} px_l & -x_l x_{l+1} \\ -\lambda^2 & x_{l+1} \end{pmatrix} \right] / \left[\prod_{l=0}^{k} x_l \right].
\end{aligned} \quad (33)$$

We now recognize that the denominator of the integrals (33) equals the Darboux polynomial P in (32). Bearing in mind that the matrices in the trace in (33) are all polynomial, it follows using Theorem 1 that this trace is also a Darboux polynomial with the same cofactor $C = \epsilon |D\phi_k(\mathbf{x})|$, for all values of λ. $\qquad\square$

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