Appl. Comput. Harmon. Anal. 56 (2022) 150–175

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

The affine Wigner distribution

Eirik Berge*, Stine Marie Berge, Franz Luef

Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway

ARTICLE INFO

Article history:
Received 6 February 2020
Received in revised form 18 August 2021
Accepted 23 August 2021
Available online 27 August 2021
Communicated by Bruno Torresani

Keywords:
Wigner distributions
Wavelet analysis
Quantization
Affine group

ABSTRACT

We examine the affine Wigner distribution from a quantization perspective with an emphasis on the underlying group structure. One of our main results expresses the scalogram as (affine) convolution of affine Wigner distributions. We strive to unite the literature on affine Wigner distributions and we provide the connection to the Mellin transform in a rigorous manner. Moreover, we present an affine ambiguity function and show how this can be used to illuminate properties of the affine Wigner distribution. In contrast with the usual Wigner distribution, we demonstrate that the affine Wigner distribution is never an analytic function.

Our approach naturally leads to several applications, one of which is an approximation problem for the affine Wigner distribution. We show that the deviation for a symbol to be an affine Wigner distribution can be expressed purely in terms of intrinsic operator-related properties of the symbol. Finally, we present a *positivity conjecture* regarding the non-negativity of the affine Wigner distribution.

1. Introduction

The most studied quadratic time-frequency representation is the *Wigner distribution* defined by

$$W_f(x,\omega) := \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega t}\,dt, \quad (x,\omega)\in\mathbb{R}^{2d}. \tag{1.1}$$

Originally invented by Wigner in [33] almost a century ago, the Wigner distribution is essential in quantum mechanics as it gives the expectation values for Weyl quantization of symbols [15]. In recent decades, the Wigner distribution has found many applications in time-frequency analysis [23, Chapter 4] due to its connections with the short-time Fourier transform $V_g f$ defined precisely in (2.4). One of the more surprising connections is the convolution relation

* Corresponding author.
E-mail addresses: eirik.berge@ntnu.no (E. Berge), stine.m.berge@ntnu.no (S.M. Berge), franz.luef@ntnu.no (F. Luef).

https://doi.org/10.1016/j.acha.2021.08.006

$$|V_g f(x,\omega)|^2 = W_{P(g)} * W_f(x,\omega), \tag{1.2}$$

where P is the reflection operator $P(g)(x) := g(-x)$. The function $\mathrm{SPEC}_g f := |V_g f(x,\omega)|^2$ is called the *spectrogram* of f with window g. The spectogram is an important tool for analyzing time-frequency content and has been used extensively in the engineering literature since its introduction.

Affine Wigner distribution

Parallel to the theory of time-frequency analysis is the time-scale (or wavelet) paradigm. Although there have been many attempts at finding a suitable Wigner distribution in the time-scale setting, there is no general consensus in the literature. We will motivate a particular choice of a time-scale Wigner distribution W_{Aff}^{ψ} given by

$$W_{\mathrm{Aff}}^{\psi}(x,a) := \int_{-\infty}^{\infty} \psi\left(\frac{aue^u}{e^u-1}\right)\overline{\psi\left(\frac{au}{e^u-1}\right)} e^{-2\pi i x u}\, du, \quad (x,a) \in \mathrm{Aff}. \tag{1.3}$$

The function W_{Aff}^{ψ} is called the *affine Wigner distribution* due to its relation to the *affine group* $\mathrm{Aff} := \mathbb{R} \times \mathbb{R}_+$. It was derived through a quantization procedure in [19]. The authors showed that the affine Wigner distribution satisfies $W_{\mathrm{Aff}}^{\psi} \in L_r^2(\mathrm{Aff})$ for every $\psi \in L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, a^{-1}\, da)$, where $L_r^2(\mathrm{Aff})$ denotes all measurable functions on Aff that are square integrable with respect to the measure $a^{-1}\, da\, dx$.

The affine Wigner distribution W_{Aff}^{ψ} has appeared in the literature several times throughout the years; as a particular Bertrand distribution in [29], and as a tool for studying the quantum mechanics of the Morse potential in [28]. The basic properties of the affine Wigner distribution will be developed in a rigorous manner to fill gaps in the literature. In particular, for all sufficiently nice $\psi \in L^2(\mathbb{R}_+)$ we have the *marginal properties*

$$\int_{-\infty}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\, dx = |\psi(a)|^2 \quad \text{and} \quad \int_{0}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\, \frac{da}{a} = |\mathcal{M}(\psi)(x)|^2.$$

The symbol $\mathcal{M}(\psi)(x)$ denotes the *Mellin transform* of $\psi \in L^2(\mathbb{R}_+)$ at the point $x \in \mathbb{R}$ given by

$$\mathcal{M}(\psi)(x) = \mathcal{M}_a(\psi)(x) := \int_0^{\infty} \psi(a) a^{-2\pi i x}\, \frac{da}{a}.$$

Scalogram representation and the affine ambiguity function

The first significant contribution is to develop a connection between the affine Wigner distribution and the *scalogram* defined by

$$\mathrm{SCAL}_g f(x,a) := |\mathcal{W}_g f(x,a)|^2, \quad (x,a) \in \mathrm{Aff}, \tag{1.4}$$

where $\mathcal{W}_g f$ denotes the continuous wavelet transform of f with respect to g defined precisely in (2.8). By comparing with (1.2) in the time-frequency setting, one would expect a simple convolution relation to hold. However, as the group underlying the symmetries in the time-scale case is the non-unimodular affine group, we obtain the following result.

Theorem. *Let $f, g \in L^2(\mathbb{R})$ be such that their Fourier transforms $\widehat{f}$ and $\widehat{g}$ are supported in $\mathbb{R}_+$ and satisfy $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}_+)$. Then*

$$\mathrm{SCAL}_g f(x,a) = \left(I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right) *_{\mathrm{Aff}} \Delta W_{\mathrm{Aff}}^{\widehat{f}}\right)\left(\frac{x}{a}, \frac{1}{a}\right), \quad (x,a) \in \mathrm{Aff},$$

where Δ *and* I *denote the modular function and the involution on the affine group, respectively.*

We introduce an *affine ambiguity function* A_{Aff}^{ψ} for $\psi \in L^2(\mathbb{R}_+)$ given by

$$A_{\mathrm{Aff}}^{\psi}(x,a) := \int_0^{\infty} \psi\left(r\sqrt{a}\right)\overline{\psi\left(\frac{r}{\sqrt{a}}\right)} r^{-2\pi i x}\,\frac{dr}{r}, \quad (x,a) \in \mathrm{Aff}.$$

The affine ambiguity function is intimately related to the radar ambiguity function in time-frequency analysis [23, Chapter 4.2]. We will show that the affine Wigner distribution and the affine ambiguity function are related through the Mellin transform by

$$W_{\mathrm{Aff}}^{\psi}(x,a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b\left[\left(\frac{\sqrt{b}\log(b)}{b-1}\right)^{2\pi i y} A_{\mathrm{Aff}}^{\psi}(y,b)\right](x,a). \tag{1.5}$$

The relation (1.5) is used to show that the affine Wigner distribution preserves Schwartz functions.

Analyticity and an approximation problem

It turns out that affine Wigner distributions are never analytic functions on the upper half-plane. However, the space $L_r^2(\mathrm{Aff})$ can be completely decomposed into “almost analytic” functions as the following result shows.

Proposition. *We have the orthogonal decomposition*

$$L_r^2(\mathrm{Aff}) = \bigoplus_{n=2}^{\infty} \mathcal{A}^n(\mathrm{Aff}) \oplus \mathcal{A}^{\perp,n}(\mathrm{Aff}), \tag{1.6}$$

where $\mathcal{A}^n(\mathrm{Aff})$ *and* $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ *denote the spaces of pure poly-analytic and pure anti-poly-analytic functions of order* n*, respectively.*

As an application to the theory developed we consider the approximation problem of understanding, for a given $f \in L_r^2(\mathrm{Aff})$, the quantity

$$\inf_{\psi \in L^2(\mathbb{R}_+)} \left\| f - W_{\mathrm{Aff}}^{\psi} \right\|_{L_r^2(\mathrm{Aff})}. \tag{1.7}$$

Notice that (1.7) measures how far f is from being an affine Wigner distribution. The analogous problem in time-frequency analysis has been recently studied in [5]. For each symbol $f \in L_r^2(\mathrm{Aff})$ there is a Hilbert-Schmidt operator A_f on $L^2(\mathbb{R}_+)$ that is weakly defined by the relation

$$\langle A_f \psi, \phi \rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle_{L_r^2(\mathrm{Aff})}, \quad \psi, \phi \in L^2(\mathbb{R}_+). \tag{1.8}$$

The following result shows that the quantity (1.7) is linked to how much A_f deviates from being a rank-one operator.

Theorem. *Let* $f \in L_r^2(\mathrm{Aff})$ *be real valued. Under a mild eigenvalue assumption on* A_f *we have*

$$\inf_{\psi \in L^2(\mathbb{R}_+)} \left\| f - W_{\mathrm{Aff}}^{\psi} \right\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \|A_f\|_{op}^2},$$

where $\|\cdot\|_{\mathcal{HS}}$ and $\|\cdot\|_{op}$ are the Hilbert-Schmidt norm and operator norm, respectively. Moreover, the precise number of distinct minimizers can be deduced from the spectrum of A_f.

Motivation for the affine Wigner distribution

It is not immediately obvious why a Wigner distribution W_{Aff} in the affine setting should have the form given in (1.3). In [3] the authors define Wigner distributions W_G on a general Lie group G. In the case of $G = \text{Aff}$ we indeed have that W_G reduces to W_{Aff}. The general Wigner distribution W_G is the canonical choice for a Wigner distribution on G since it naturally related with Fourier transforms on the group. For the affine group, this relation [6, Section 5.1] takes the elegant form

$$A_f = \mathcal{F}_W^{-1}\mathcal{F}_{\text{KO}}^{-1}(f), \qquad f \in L_r^2(\text{Aff}),$$

where $\mathcal{F}_W$ is the *affine Fourier-Wigner transform* and $\mathcal{F}_{\text{KO}}$ is the *affine Fourier-Kirillov transform*. Since the affine Wigner distribution determines the affine Weyl quantization completely, this motivates further investigation into the affine Wigner distribution W_{Aff}.

Further results

The affine Wigner distribution is developed further in the follow-up paper [6]. Let us mention two results in [6] that can help to additionally motivate the affine Wigner distribution:

Quantization of Coordinate Functions: In [6, Section 3.3] we extend the affine Weyl quantization to tempered distributions. This offers the possibility of rigorously determining the quantizations A_{f_x} and A_{f_a} of the coordinate functions $f_x(x,a) = x$ and $f_a(x,a) = a$. We prove in [6, Theorem 3.11] the commutation relation

$$[A_{f_x}, A_{f_a}] = \frac{1}{2\pi i}A_{f_a}.$$

This is, up to re-normalization, precisely the infinitesimal structure of the affine group. Hence the affine Weyl quantization, and thus the affine Wigner distribution, is intimately linked with the Lie group structure of the affine group.

Cohen Class Operators: In [6, Section 6.3] we develop a theory affine Cohen class operators. This is motivated by the classical Cohen class operators on phase space [23, Section 4.5]. For a reasonable function f on Aff we define the associated *affine Cohen class function* as

$$Q_f(\psi,\phi) = W_{\text{Aff}}^{\psi,\phi} *_{\text{Aff}} \check{f}, \qquad \check{f}(x,a) := f((x,a)^{-1}).$$

This is a special case of an affine Cohen class function Q_S associated to an operator S, where one considers $S = A_f$. It turns out that any bilinear form $Q : L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \to L^\infty(\text{Aff})$ is, under a mild continuity requirement, on the form $Q = Q_S$ for some bounded operator $S : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by [6, Proposition 6.11]. As such, the affine Wigner distribution is essential in developing a well-behaved Cohen class theory on the affine group.

In addition to the two topics above, we show in [6, Proposition 6.2] that the affine Wigner distribution is also related to the localization operators of Daubechies and Paul given in [13]. Finally, in [6, Section 6.2] we relate the affine Wigner distribution to covariant integral quantizations developed by Gazeau and his collaborators in [2,7,8,20–22].

Structure of the paper

In Section 2 we outline necessary definitions and briefly review the affine group as it will be central for

many of the results we develop. In Section 3 we derive basic properties of the affine Wigner distribution. We devote Section 4 to uniting the literature and pointing out how the affine Wigner distribution can be derived by emphasizing symmetry. The convolution relation between the affine Wigner distribution and the scalogram will be proved in Section 5. In Section 6 we define the affine ambiguity function and show how this allows us to extend the affine Weyl quantization (1.8) to the distributional setting. We prove the decomposition (1.6) of $L^2_r(\mathrm{Aff})$ in Section 7. In addition to the approximation problem described above, we show in Section 8 how basic questions regarding operators on $\mathbb{R}_+$ can be answered with our framework. Finally, we discuss the affine Grossmann-Royer operator and the affine positivity conjecture in Section 9.
The authors are grateful for helpful suggestions from Eirik Skrettingland and Luís Daniel Abreu.

2. Preliminaries

The notation $\mathscr{S}(\mathbb{R}^d)$ will be used for the Schwartz space of rapidly decaying smooth functions on $\mathbb{R}^d$. We write $\mathscr{S}(\mathbb{R}_+)$ for the smooth functions $\psi : \mathbb{R}_+ \to \mathbb{C}$ such that $\Psi(x) := \psi(e^x) \in \mathscr{S}(\mathbb{R})$. The corresponding dual spaces of tempered distributions are denoted by $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}_+)$, respectively. The Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ is given by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\omega}\, dx, \quad \omega \in \mathbb{R}^d.$$

We will frequently use $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, a^{-1}\, da)$ since $a^{-1}\, da$ is the Haar measure on $\mathbb{R}_+$.

2.1. The classical Wigner distribution

We begin by recalling basic definitions from time-frequency analysis and their connection with the Heisenberg group. The *cross-Wigner transform* $W(f, g)$ of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f,g)(x,\omega) := \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}e^{-2\pi i\omega t}\, dt, \quad (x,\omega) \in \mathbb{R}^{2d}.$$

Notice that the Wigner distribution W_f given in (1.1) is precisely the diagonal term $W(f, f)$. The cross-Wigner transform satisfies the orthogonality property

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}. \tag{2.1}$$

A key feature of the Wigner distribution is its connection with the Weyl calculus: For a symbol $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ the *Weyl (pseudo-differential) operator* L_σ corresponding to the symbol σ is the operator

$$L_\sigma f := \int_{\mathbb{R}^{2d}} e^{-\pi i\xi u}\hat{\sigma}(\xi, u)T_{-u}M_\xi f\, du\, d\xi. \tag{2.2}$$

The operators T_{-u} and M_ξ in (2.2) are respectively the *time-shift operator* and the *frequency-shift operator* defined by

$$T_x f(t) := f(t - x), \qquad M_\omega f(t) := e^{2\pi i\omega t}f(t), \quad x, \omega, t \in \mathbb{R}^d.$$

The association $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and the operator L_σ maps $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}'(\mathbb{R}^d)$ by [23, Lemma 14.3.1]. Moreover, the Weyl transform is a bijection between square integrable symbols $\sigma \in L^2(\mathbb{R}^{2d})$ and Hilbert-Schmidt operators $L_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by a classical result of Poole [30, Proposition V.1].

The connection between the Weyl calculus and the cross-Wigner transform is the relation

$$\langle L_\sigma f, g\rangle_{L^2(\mathbb{R}^d)} = \langle \sigma, W(g,f)\rangle_{L^2(\mathbb{R}^{2d})},$$

for $\sigma \in L^2(\mathbb{R}^{2d})$ and $f, g \in L^2(\mathbb{R}^d)$. Since the Weyl transform is a quantization procedure, one can think of the inverse transformation $L_\sigma \mapsto \sigma$ as *dequantization*. In this terminology, the Wigner distribution W_f for $f \in L^2(\mathbb{R}^d)$ is the dequantization of the rank-one operator

$$L_{W_f} g := \langle g, f\rangle f, \qquad g \in L^2(\mathbb{R}^d). \tag{2.3}$$

The reader can consult [24, Chapter 13] and [14, Chapter 4] for more details about the Weyl transform from a quantum mechanical perspective.

Central to time-frequency analysis is the *short-time Fourier transform* $V_g f$ of $f, g \in L^2(\mathbb{R}^d)$ given by

$$V_g f(x,\omega) := \langle f, M_\omega T_x g\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}\, dt. \tag{2.4}$$

We have from [23, Lemma 4.3.1] that the cross-Wigner transform and the short-time Fourier transform are related by the formula

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{P(g)} f(2x, 2\omega),$$

where $P(g)(x) := g(-x)$. The short-time Fourier transform originates from the Schrödinger representation of the Heisenberg group, see [23, Chapter 9] for details.

2.2. The affine group

The two main operators in time-scale analysis are the time-shift operator T_x and the *dilation operator* D_a given by

$$D_a f(x) := \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right), \tag{2.5}$$

for $a > 0$ and $f \in L^2(\mathbb{R})$. One defines the *affine group* as $\mathrm{Aff} := (\mathbb{R} \times \mathbb{R}_+, \cdot_{\mathrm{Aff}})$, where the group operation is given by

$$(x,a) \cdot_{\mathrm{Aff}} (y,b) := (x + ay, ab), \quad (x,a), (y,b) \in \mathrm{Aff}.$$

The motivation for the group operation stems from calculation

$$(T_x D_a)(T_y D_b) = T_x T_{ay} D_a D_b = T_{x+ay} D_{ab}.$$

We can represent the affine group Aff and its Lie algebra $\mathfrak{aff}$ in the matrix form

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \,\middle|\, a > 0,\, x \in \mathbb{R} \right\}, \quad \mathfrak{aff} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \,\middle|\, u, v \in \mathbb{R} \right\}.$$

Essential for computations is the fact that the exponential map $\exp : \mathfrak{aff} \to \mathrm{Aff}$ given by

$$\exp\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^u & \frac{v(e^u - 1)}{u} \\ 0 & 1 \end{pmatrix}$$

is a global diffeomorpism. The left Haar measure on Aff is given by $a^{-2}\,da\,dx$, while the right Haar measure is $a^{-1}\,da\,dx$. We will use the notation $L^2_r(\mathrm{Aff})$ and $L^2_l(\mathrm{Aff})$ to indicate if we are using the right or left Haar measure, respectively. The left and right Haar measures on Aff can be written in the coordinates induced by the exponential map as

$$\frac{da\,dx}{a^2} = \frac{du\,dv}{\lambda(u)}, \qquad \frac{da\,dx}{a} = \frac{du\,dv}{\lambda(-u)},$$

where the function λ is given by

$$\lambda(u) := \frac{ue^u}{e^u - 1} = \frac{ue^{\frac{u}{2}}}{2\sinh(\frac{u}{2})}. \tag{2.6}$$

A natural way the affine group can act on $L^2(\mathbb{R})$ is by translations and dilations, namely as

$$f \longmapsto T_x D_a f, \qquad f \in L^2(\mathbb{R}). \tag{2.7}$$

This is a unitary representation, although it is not irreducible. The matrix coefficients of this representation are given by

$$\mathcal{W}_g f(x,a) := \langle f, T_x D_a g\rangle_{L^2(\mathbb{R})} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(y)\overline{g\left(\frac{y-x}{a}\right)}\,dy. \tag{2.8}$$

One typically refers to the map $(x,a) \mapsto \mathcal{W}_g f(x,a)$ as the *(continuous) wavelet transform* of f with respect to g. The continuous wavelet transform is analogous to the short-time Fourier transform and incorporates the possibility of observing f at different scales through g. Moreover, the magnifying aspect coming from the change of scales can characterize local regularity through decay properties of the wavelet transform, see [12, Theorem 2.9.2].

2.3. A quantization approach to the affine Wigner distribution

We will briefly outline a procedure described in [19] to determine the affine Wigner distribution. The theory is based on Kirillov's theory of coadjoint orbits and we refer further explanations to the aforementioned paper.

The affine group Aff acts on its Lie algebra $\mathfrak{aff}$ through the *adjoint action*

$$\mathrm{Ad}_{(x,a)}(X) := \begin{pmatrix} u & av - xu \\ 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff},\ (x,a) \in \mathrm{Aff}. \tag{2.9}$$

A representation Φ of a Lie group G on a vector space V is always accompanied by a representation Φ^* of G on the dual space V^* defined by

$$\langle \Phi(g)^*\eta, v\rangle := \langle \eta, \Phi(g^{-1})v\rangle, \quad g \in G,\ v \in V,\ \eta \in V^*,$$

where the bracket denotes the natural pairing between V and V^*. In the case of the adjoint action in (2.9) we denote the accompanied representation on $\mathfrak{aff}^*$ by Ad^* and call it the *coadjoint representation* of the affine group. We can realize $\mathfrak{aff}^*$ as matrices on the form

$$\mathfrak{aff}^* \simeq \left\{ (x,y) := \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}.$$

Any point on the form $(x, 0) \in \mathfrak{aff}^*$ is a fixed point for the coadjoint representation. The upper and lower half-planes

$$\mathcal{H}_+ := \left\{(x,y) \in \mathfrak{aff}^* \,\middle|\, y > 0\right\}, \qquad \mathcal{H}_- := \left\{(x,y) \in \mathfrak{aff}^* \,\middle|\, y < 0\right\},$$

both constitute distinct orbits. For reasons of symmetry it suffices to understand the representation corresponding to $\mathcal{H}_+$. It is convenient to identify $\mathcal{H}_+ \simeq \text{Aff}$ as sets and use the notation (x, a) for a general element in $\mathcal{H}_+$. From general coadjoint orbit theory [26, Chapter 1.2] it follows that Aff is equipped with a canonical symplectic structure. In fact, this symplectic structure is simply the right Haar measure $a^{-1}\,da\,dx$ on Aff.

The main idea of Kirillov's theory is to associate irreducible representations of the Lie group to orbits of the coadjoint representation in a one-to-one manner. A realization of the representation corresponding to $\mathcal{H}_+$ is given by acting on $\psi \in L^2(\mathbb{R}_+)$ by

$$U(x,a)\psi(r) := e^{2\pi i x r}\psi(ar) = \frac{1}{\sqrt{a}} M_x D_{\frac{1}{a}}\psi(r). \tag{2.10}$$

The representation U is (up to a normalization) the representation (2.7) on the Fourier side. Define the *Stratonovich-Weyl operator* on $L^2(\mathbb{R}_+)$ by the formula

$$\Omega(x,a)\psi(r) := a \int\limits_{\mathbb{R}^2} e^{-2\pi i(xu+av)} U\left(\frac{ve^u}{\lambda(u)}, e^u\right)\psi(r)\,du\,dv,$$

where $\psi \in L^2(\mathbb{R}_+)$, $(x,a) \in \text{Aff}$, and λ is the function defined in (2.6). The following result is given in [19, Corollary 4.3].

Proposition 2.1. *There is an isometric isomorphism between $L^2_r(\text{Aff})$ and the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$. The isomorphism sends $f \in L^2_r(\text{Aff})$ to the operator A_f on $L^2(\mathbb{R}_+)$ defined by*

$$A_f\psi(r) := \int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(x,a)\Omega(x,a)\psi(r)\,\frac{da\,dx}{a}.$$

The association $f \mapsto A_f$ is called *affine Weyl quantization*, while the direction $A_f \mapsto f$ is referred to as *affine dequantization*. Moreover, we call f the *(affine) symbol* of A_f. Recall that any Hilbert-Schmidt operator A on $L^2(\mathbb{R}_+)$ has an associated integral kernel $A_K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ so that

$$A\psi(r) = \int\limits_0^{\infty} A_K(r,s)\psi(s)\,\frac{ds}{s},$$

for all $\psi \in L^2(\mathbb{R}_+)$. If $A = A_f$, then one can recover $f \in L^2_r(\text{Aff})$ from the formula

$$f(x,a) = \int\limits_{-\infty}^{\infty} A_K\left(a\lambda(u), a\lambda(-u)\right)e^{-2\pi i x u}\,du.$$

Motivated by (2.3), the affine Wigner distribution should be defined as the affine dequantization of a rank-one operator. Hence we have the following definition.

Definition 2.2. The *affine cross-Wigner transform* acts on functions $\psi, \phi \in L^2(\mathbb{R}_+)$ by

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\phi(a\lambda(-u))}e^{-2\pi i x u}\,du$$

for $(x,a) \in \mathrm{Aff}$. We refer to the diagonal $W_{\mathrm{Aff}}^{\psi} := W_{\mathrm{Aff}}^{\psi,\psi}$ as the *affine Wigner distribution* of ψ.

If $f \in L^2_r(\mathrm{Aff})$ is the symbol of the Hilbert-Schmidt operator A_f acting on $L^2(\mathbb{R}_+)$, then

$$\langle A_f \psi, \phi\rangle_{L^2(\mathbb{R}_+)} = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi}\right\rangle_{L^2_r(\mathrm{Aff})}, \quad \psi, \phi \in L^2(\mathbb{R}_+). \tag{2.11}$$

3. Basic properties

We now derive some basic properties of the affine cross-Wigner transform. The affine cross-Wigner transform is related to the isometry $\Pi : L^2(\mathbb{R}^+ \times \mathbb{R}^+, (rs)^{-1}\,dr\,ds) \to L^2_r(\mathrm{Aff})$ given by

$$\Pi(F)(u,a) := F(a\lambda(u), a\lambda(-u)).$$

Lemma 3.1. *The affine cross-Wigner transform can be factorized as*

$$W_{\mathrm{Aff}}^{\psi,\phi} = \mathcal{F}_1 \Pi(\psi \otimes \overline{\phi}), \quad \psi, \phi \in L^2(\mathbb{R}_+),$$

where $\mathcal{F}_1$ is the Fourier transform in the first component and $\psi \otimes \overline{\phi}(r,s) := \psi(r)\overline{\phi(s)}$ for $r, s \in \mathbb{R}_+$.

The factorization in Lemma 3.1 is key for understanding essential properties of the affine cross-Wigner transform. We illustrate its use by extending the orthogonality property of the classical Wigner distribution in (2.1) to the affine setting.

Proposition 3.2. *The affine Wigner distribution satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\overline{W_{\mathrm{Aff}}^{\phi}(x,a)}\,\frac{da\,dx}{a} = |\langle \psi, \phi\rangle|^2, \tag{3.1}$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$.

Proof. We use the factorization in Lemma 3.1 and obtain

$$\begin{aligned}
\left\langle W_{\mathrm{Aff}}^{\psi}, W_{\mathrm{Aff}}^{\phi}\right\rangle_{L^2_r(\mathrm{Aff})} &= \left\langle \mathcal{F}_1\Pi\left(\psi \otimes \overline{\psi}\right), \mathcal{F}_1\Pi\left(\phi \otimes \overline{\phi}\right)\right\rangle_{L^2_r(\mathrm{Aff})} \\
&= \left\langle \Pi\left(\psi \otimes \overline{\psi}\right), \Pi\left(\phi \otimes \overline{\phi}\right)\right\rangle_{L^2_r(\mathrm{Aff})} \\
&= \left\langle \psi \otimes \overline{\psi}, \phi \otimes \overline{\phi}\right\rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}^+, (rs)^{-1}\,dr\,ds)} \\
&= |\langle \psi, \phi\rangle|^2. \quad \square
\end{aligned}$$

We will refer to (3.1) as the *affine orthogonality relation* motivated by the analogous result for the classical Wigner distribution in (2.1). Through a different (but ultimately equivalent) approach to the affine Wigner distribution taken in [9] and [28], the affine orthogonality relation is already known. The usefulness of the affine orthogonality relation can be readily demonstrated with the following two corollaries.

Corollary 3.3. *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}_+)$. Then $\{W_{\mathrm{Aff}}^{\psi_n,\psi_m}\}_{n,m\in\mathbb{N}}$ is an orthonormal basis for $L_r^2(\mathrm{Aff})$. In particular, we can expand any $f \in L_r^2(\mathrm{Aff})$ as*

$$f = \sum_{n,m=0}^{\infty} \left\langle f, W_{\mathrm{Aff}}^{\mathcal{L}_n,\mathcal{L}_m} \right\rangle W_{\mathrm{Aff}}^{\mathcal{L}_n,\mathcal{L}_m},$$

where $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is given by

$$\mathcal{L}_n(x) := \frac{e^{\frac{x}{2}}}{n!\sqrt{n+1}} \frac{d^n}{dx^n} \left(e^{-x} x^{n+1}\right). \tag{3.2}$$

Proof. The orthonormality of the functions $W_{\mathrm{Aff}}^{\psi_n,\psi_m}$ clearly follows from Proposition 3.2. To see the completeness in $L_r^2(\mathrm{Aff})$ we assume that $f \in L_r^2(\mathrm{Aff})$ satisfies

$$\left\langle f, W_{\mathrm{Aff}}^{\psi_n,\psi_m} \right\rangle_{L_r^2(\mathrm{Aff})} = 0$$

for every $n, m \in \mathbb{N}$. Then equation (2.11) implies that $A_f = 0$ and hence $f \equiv 0$. □

Corollary 3.4. *We have $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$ for $\psi, \phi \in L^2(\mathbb{R}_+)$ if and only if $\psi = c \cdot \phi$ with $|c| = 1$.*

Proof. It is clear from the definition of W_{Aff} that $\psi = c\cdot\phi$ with $|c| = 1$ implies that $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$. Conversely, if we assume that $W_{\mathrm{Aff}}^{\psi} = W_{\mathrm{Aff}}^{\phi}$ then the affine orthogonality relation (3.1) shows that

$$|\langle \psi, \phi\rangle|^2_{L^2(\mathbb{R}_+)} = \|\psi\|^4_{L^2(\mathbb{R}_+)} = \|\phi\|^4_{L^2(\mathbb{R}_+)}.$$

Hence $\|\phi\|_{L^2(\mathbb{R}_+)} = \|\psi\|_{L^2(\mathbb{R}_+)}$ and $|\langle \psi, \phi\rangle|_{L^2(\mathbb{R}_+)} = \|\psi\|_{L^2(\mathbb{R}_+)}\|\phi\|_{L^2(\mathbb{R}_+)}$. This can only happen when $\psi = c \cdot \phi$ for some $|c| = 1$. □

The *marginal properties* [23, Lemma 4.3.6] for the classical Wigner distribution strengthen a quantum mechanical interpretation of the Wigner distribution. For the affine Wigner distribution, we need an analogue of the Fourier transform on the group $\mathbb{R}^+$. This is the *Mellin transform* given by

$$\mathcal{M}(\psi)(x) = \mathcal{M}_a(\psi)(x) := \int_0^\infty \psi(a) a^{-2\pi i x} \frac{da}{a},$$

for $x \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R}_+)$. There is little consensus regarding the exponent of a in the literature and we recommend checking carefully which convention is used whenever the Mellin transform is encountered. The Mellin transform is a unitary map $\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ with inverse

$$\mathcal{M}^{-1}(f)(a) = \mathcal{M}_x^{-1}(f)(a) = \int_{-\infty}^{\infty} f(x) a^{2\pi i x}\, dx, \tag{3.3}$$

for $a \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R})$. Moreover, the Mellin transform of a dilated function satisfies

$$\mathcal{M}(D_r\psi)(x) = r^{-2\pi i x - \frac{1}{2}} \mathcal{M}(\psi)(x). \tag{3.4}$$

The following marginal properties have been stated in [31] where the proofs are referred to the unpublished Ph.D. thesis of R.G. Shenoy. We provide a new proof of this remarkable fact to fill in gaps in the original sources.

Proposition 3.5. *The affine Wigner distribution satisfies for $\psi \in \mathscr{S}(\mathbb{R}_+)$ the marginal properties*

$$\int_{-\infty}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,dx = |\psi(a)|^2,$$

$$\int_{0}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,\frac{da}{a} = |\mathcal{M}(\psi)(x)|^2.$$

Proof. The first marginal property follows from Lemma 3.1 and the realization that

$$\int_{-\infty}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,dx = \mathcal{F}_1^{-1}\left(W_{\mathrm{Aff}}^{\psi}\right)(0,a).$$

The validity of the pointwise convergence in the Fourier inversion step is clear since $\psi \in \mathscr{S}(\mathbb{R}_+)$.

For the second marginal property, we utilize a change of variables in the definition of the affine Wigner distribution to get the alternative form

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \int_{0}^{\infty} u^{-2\pi i x}\psi\left(a\frac{u\log(u)}{u-1}\right)\overline{\phi\left(a\frac{\log(u)}{u-1}\right)}\,\frac{du}{u}.$$

The isometry property of the Mellin transform can then be used to obtain

$$\begin{aligned}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,\frac{da}{a} &= \int_{0}^{\infty}\int_{0}^{\infty} u^{-2\pi i x}\psi\left(a\frac{u\log(u)}{u-1}\right)\overline{\psi\left(a\frac{\log(u)}{u-1}\right)}\,\frac{da\,du}{au}\\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} u^{-2\pi i x}\mathcal{M}_a\left(\psi\left(a\frac{u\log(u)}{u-1}\right)\right)(\beta)\overline{\mathcal{M}_a\left(\psi\left(a\frac{\log(u)}{u-1}\right)\right)(\beta)}\,\frac{d\beta\,du}{u}.\end{aligned}$$

By using the dilation relation (3.4) and the inverse Mellin transform (3.3) we end up with

$$\begin{aligned}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\psi}(x,a)\,\frac{da}{a} &= \int_{0}^{\infty}\int_{-\infty}^{\infty} u^{-2\pi i x}\left(\frac{u\log(u)}{u-1}\right)^{2\pi i\beta}\mathcal{M}_a\left(\psi\right)(\beta)\overline{\left(\frac{\log(u)}{u-1}\right)^{2\pi i\beta}\mathcal{M}_a(\psi)(\beta)}\,\frac{d\beta\,du}{u}\\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} u^{-2\pi i x}u^{2\pi i\beta}|\mathcal{M}_a(\psi)(\beta)|^2\,\frac{d\beta\,du}{u}\\ &= \int_{0}^{\infty} u^{-2\pi i x}\mathcal{M}_\beta^{-1}(|\mathcal{M}_a(\psi)(\beta)|^2)(u)\,\frac{du}{u}\\ &= \mathcal{M}_u(\mathcal{M}_\beta^{-1}(|\mathcal{M}_a(\psi)(\beta)|^2)(u))(x)\\ &= |\mathcal{M}(\psi)(x)|^2.\end{aligned}$$

Interchanging the order of integration and the pointwise convergence of the Mellin transform is easily justified under the assumption that $\psi \in \mathscr{S}(\mathbb{R}_+)$. $\square$

Remark. It follows from Proposition 3.5 that

$$\int_0^\infty \int_{-\infty}^\infty W_{\mathrm{Aff}}^\psi(x,a)\,\frac{da\,dx}{a} = \int_0^\infty |\psi(a)|^2\,\frac{da}{a} = \|\psi\|^2_{L^2(\mathbb{R}_+)},$$

for all ψ in the dense subspace $\mathscr{S}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$. If $\|\psi\|_{L^2(\mathbb{R}_+)} = 1$ and W_{Aff}^ψ is everywhere non-negative, then the affine Wigner distribution would be a probability density function on the upper half-plane. We will elaborate on this in Section 9.

If $\psi \in \mathscr{S}(\mathbb{R}_+)$ has compact support and $a \in \mathbb{R}_+$ is outside the support of ψ, then Proposition 3.5 shows that

$$\int_{-\infty}^\infty W_{\mathrm{Aff}}^\psi(x,a)\,dx = 0.$$

This extreme case can be improved with the following *finite support property*.

Proposition 3.6. *Assume $\psi \in L^2(\mathbb{R}_+)$ is countinuous and supported in $[r,s] \subset \mathbb{R}_+$. Then $W_{\mathrm{Aff}}^\psi(x,a) = 0$ for all $x \in \mathbb{R}$ whenever $a \notin [r,s]$.*

Proof. The functions $\psi(a\lambda(u))$ and $\psi(a\lambda(-u))$ are both non-zero if and only if

$$\lambda(u), \lambda(-u) \in L := \left[\frac{r}{a}, \frac{s}{a}\right].$$

If $a > s$ then $L \subset (0,1)$. Hence it suffices to show that $\lambda(u)$ and $\lambda(-u)$ can not take values in $(0,1)$ simultaneously. This follows since $\lambda(u)$ is an increasing function that only takes values in $(0,1)$ whenever $u < 0$. If $a < r$ then $L \subset (1,\infty)$. In this case, the result follows from the fact that $\lambda(u) > 1$ if and only if $u > 0$. $\square$

4. Alternative descriptions

Although the affine Wigner distribution was constructed rather recently, it has appeared in the literature several times in different disguises. We outline two instances of this and see how this enriches our understanding of the more subtle properties of the affine Wigner distribution.

Consider a function $\psi \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}_+)$ that is supported on $\mathbb{R}_+$ and let $f \in L^2(\mathbb{R})$ be such that $\hat{f} = \psi$. The affine Wigner distribution W_{Aff}^ψ is related to the Bertrand $P := (P_0, 1)$ distribution described in [29] by the formula

$$W_{\mathrm{Aff}}^\psi(x,a) = \frac{1}{a} Pf\left(a, -\frac{x}{a}\right).$$

One refers to P as the *Bertrand P_0 distribution* and it is in both the *affine class* and the *hyperbolic class* described in [29]. From this we can gauge several invariance properties of the affine Wigner distribution:

- The fact that P is in the affine class gives the invariance properties

$$W_{\mathrm{Aff}}^{M_\omega \psi}(x,a) = W_{\mathrm{Aff}}^\psi(x - a\omega, a), \qquad W_{\mathrm{Aff}}^{D_r \psi}(x,a) = \frac{1}{r} W_{\mathrm{Aff}}^\psi\left(x, \frac{a}{r}\right). \tag{4.1}$$

  These invariance properties can be summarized as

$$W_{\mathrm{Aff}}^{U(x,a)\psi}(y,b) = W_{\mathrm{Aff}}^{\psi}(y - bx, ab), \tag{4.2}$$

where U is the action of the affine group on $L^2(\mathbb{R}_+)$ given in (2.10).

- The fact that P is in the hyperbolic class gives the invariance property

$$W_{\mathrm{Aff}}^{\mathcal{H}(c,f_r)\psi}(x,a) = W_{\mathrm{Aff}}^{\psi}(x + c, a), \tag{4.3}$$

where $\mathcal{H}(c, f_r)$ is the transformation

$$\mathcal{H}(c, f_r)\psi(r) := e^{-2\pi ic\ln\left(\frac{r}{f_r}\right)}\psi(r), \quad r, f_r > 0,\, c \in \mathbb{R}.$$

Notice that the *positive reference frequency* f_r only appears on the left-hand side of (4.3).

The affine Wigner distribution W_{Aff} can be derived in another way by emphasizing invariance properties as done in [9] and [28]. From this perspective, one starts with a general quadratic distribution and require invariance under a group extension of the affine group. This will produce the distribution

$$W^{\psi}(x,a) := \int_{-\infty}^{\infty} \psi(a\lambda(u))\overline{\psi(a\lambda(-u))}e^{-2\pi iux}\mu(u)\,du,$$

where $\mu(u)$ is a weight function that satisfies $\overline{\mu(u)} = \mu(-u)$. The requirement that W^{ψ} satisfies the affine orthogonality relation in (3.1) forces $\mu \equiv 1$ so that $W^{\psi} = W_{\mathrm{Aff}}^{\psi}$. Although one gets the orthogonality relation (3.1) for free with this approach, the connection with the affine Weyl quantization in (2.11) is then obscured. The affine Wigner distribution W_{Aff} is a special case of a family of distributions that are called *tomographic distributions* in [9].

Remark. There have been other attempts at defining a notion of affine Wigner distribution that do not coincide with our definition. As an example, we refer the reader to [22] and the recent successor paper [21] where an affine Wigner-like quasi-probability is defined through a semi-classical quantization approach. Although this is different from the approach in [19] that our work is based on, it has similarities in both motivation and properties.

5. Affine convolution representation of the scalogram

Recall from the introduction that the classical Wigner distribution can represent the spectogram through convolution

$$\mathrm{SPEC}_g f(x,\omega) = W_{P(g)} * W_f(x,\omega) = W_{P(\hat{g})} * W_{\hat{f}}(\omega, -x), \tag{5.1}$$

where $P(g)(x) := g(-x)$. This relation was mentioned in [27, Eq 85] where the Wigner distribution went under the name *(instantaneous) spectrum-smoothing function*. It later appeared in [11, Eq 4.5], where it was used to show that the spectogram is a Cohen class distribution. Finally, it was put on more rigorous foundations in [17, Proposition 1.99]. By attempting to use the classical Wigner distribution to represent the scalogram in (1.4) one obtains

$$\mathrm{SCAL}_g f(x,a) = \int_{\mathbb{R}^2} W_f(\tau,\xi)W_g\left(\frac{\tau - x}{a}, a\xi\right)\,d\tau\,d\xi.$$

However, this only superficially looks like convolution as it does not incorporate one of the Haar measures on Aff.

We will use the affine Wigner distribution to get a proper convolution representation of the scalogram. Before stating the precise result, we recall some generalities from the theory of locally compact groups applied to the affine group: The *affine convolution* between two functions f, g on the affine group is given whenever it is well-defined by

$$f *_{\mathrm{Aff}} g(x,a) := \int_{-\infty}^{\infty}\int_{0}^{\infty} f(y,b)g\left((y,b)^{-1} \cdot_{\mathrm{Aff}} (x,a)\right) \frac{db\,dy}{b^2}.$$

A departure from the usual Euclidean convolution is that the affine convolution is not commutative. The *modular function* Δ on any locally compact group measures the difference between the right and left Haar measure. We refer the reader to a precise definition in [18, Chapter 2.4] as we only need that the modular function on the affine group is

$$\Delta(x,a) = \frac{1}{a}, \qquad (x,a) \in \mathrm{Aff}.$$

Finally, the *(right) involution* of a function f on the affine group is given by

$$I(f)(x,a) := \Delta(x,a)\overline{f((x,a)^{-1})} = \frac{1}{a}\overline{f\left(-\frac{x}{a}, \frac{1}{a}\right)}, \qquad (x,a) \in \mathrm{Aff}.$$

The following convolution result should be compared with (5.1).

Theorem 5.1. *Let $f, g \in L^2(\mathbb{R})$ be such that their Fourier transforms $\widehat{f}$ and $\widehat{g}$ are supported in $\mathbb{R}_+$ and satisfy $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}_+)$. Then*

$$\mathrm{SCAL}_g f(x,a) = \left(I\left(W_{\mathrm{Aff}}^{\widehat{g}}\right) *_{\mathrm{Aff}} \Delta W_{\mathrm{Aff}}^{\widehat{f}}\right)\left(\frac{x}{a}, \frac{1}{a}\right), \quad (x,a) \in \mathrm{Aff}.$$

Proof. By using Parseval's identity and that the support of the Fourier transforms is in $\mathbb{R}_+$ we obtain

$$\mathrm{SCAL}_g f(x,a) = \left|\langle f, T_x D_a g\rangle_{L^2(\mathbb{R})}\right|^2 = \left|\langle \widehat{f}, \sqrt{a}\cdot U(x,a)\widehat{g}\,\rangle_{L^2(\mathbb{R}_+)}\right|^2,$$

where $U(x,a)$ is given in (2.10). The affine orthogonality relation given in Proposition 3.2 and the invariance property given in (4.2) together show that

$$\begin{aligned}
\mathrm{SCAL}_g f(x,a) &= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b)\cdot a \cdot W_{\mathrm{Aff}}^{U(x,a)\widehat{g}}(y,b)\,\frac{db\,dy}{b} \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b)\cdot a \cdot W_{\mathrm{Aff}}^{\widehat{g}}(y-bx, ab)\,\frac{db\,dy}{b} \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} W_{\mathrm{Aff}}^{\widehat{f}}(y,b)\cdot ab \cdot W_{\mathrm{Aff}}^{\widehat{g}}\left((y,b)\cdot_{\mathrm{Aff}}(-x,a)\right)\,\frac{db\,dy}{b^2}.
\end{aligned}$$

We use the involution on the affine group to write

Table 1
Summary of invariance properties.

Transformation	Function	Affine Wigner Distribution	Affine Ambiguity Function
Identity	$\psi(r)$	$W^{\psi}_{\text{Aff}}(x,a)$	$A^{\psi}_{\text{Aff}}(x,a)$
Dilation	$D_s\psi(r)$	$\frac{1}{s}W^{\psi}_{\text{Aff}}\left(x,\frac{a}{s}\right)$	$s^{-2\pi ix-1}A^{\psi}_{\text{Aff}}(x,a)$
Modulation	$M_\omega\psi(r)$	$W^{\psi}_{\text{Aff}}(x-a\omega,a)$	No simple formula
Hyperbolic	$\mathcal{H}(c,f_r)\psi(r)$	$W^{\psi}_{\text{Aff}}(x+c,a)$	$a^{-2\pi ic}A^{\psi}_{\text{Aff}}(x,a)$

$$ab\cdot W^{\widehat{g}}_{\text{Aff}}\left((y,b)\cdot_{\text{Aff}}(-x,a)\right)=I\left(W^{\widehat{g}}_{\text{Aff}}\right)\left((-x,a)^{-1}\cdot_{\text{Aff}}(y,b)^{-1}\right).$$

Combining these observations shows that

$$\begin{aligned}\text{SCAL}_g f(x,a)&=\int_{-\infty}^{\infty}\int_0^{\infty}\frac{W^{\widehat{f}}_{\text{Aff}}(y,b)}{b}\cdot I\left(W^{\widehat{g}}_{\text{Aff}}\right)\left(\left(\frac{x}{a},\frac{1}{a}\right)\cdot_{\text{Aff}}(y,b)^{-1}\right)\frac{db\,dy}{b}\\&=\left(I\left(W^{\widehat{g}}_{\text{Aff}}\right)*_{\text{Aff}}\Delta W^{\widehat{f}}_{\text{Aff}}\right)\left(\frac{x}{a},\frac{1}{a}\right).\quad\square\end{aligned}$$

6. The affine ambiguity function

The *cross-ambiguity function* in time-frequency analysis of $f,g\in L^2(\mathbb{R})$ is defined to be

$$A(f,g)(x,\omega):=\int_{-\infty}^{\infty}f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{-2\pi it\omega}\,dt,\qquad(x,\omega)\in\mathbb{R}^2.$$

The *ambiguity function* $Af:=A(f,f)$ of $f\in L^2(\mathbb{R})$ has been frequently used in radar applications [23, Chapter 4.2]. In the affine setting, we suggest the following analogue.

Definition 6.1. The *affine cross-ambiguity function* of $\psi,\phi\in L^2(\mathbb{R}_+)$ is the function $A^{\psi,\phi}_{\text{Aff}}$ on Aff defined by

$$A^{\psi,\phi}_{\text{Aff}}(x,a):=\int_0^{\infty}\psi\left(r\sqrt{a}\right)\overline{\phi\left(\frac{r}{\sqrt{a}}\right)}r^{-2\pi ix}\,\frac{dr}{r},\quad(x,a)\in\text{Aff}.$$

Similarly as before, we call the function $A^{\psi}_{\text{Aff}}:=A^{\psi,\psi}_{\text{Aff}}$ the *affine ambiguity function*.

For the summary of invariance properties see Table 1. In [31] the authors define a different notion of affine ambiguity function under the name *wide-band ambiguity function*. Notice that the definition of $A^{\psi,\phi}_{\text{Aff}}$ incorporates the Haar measure on $\mathbb{R}_+$ in a natural way. Moreover, we will show that our definition possesses properties that justifies the terminology *affine ambiguity function*.

Lemma 6.2. *For $\psi,\phi\in L^2(\mathbb{R}_+)$ we define the functions $\Psi(x):=\psi(e^x)$ and $\Phi(x):=\phi(e^x)$ for $x\in\mathbb{R}$. Then*

$$A^{\psi,\phi}_{\text{Aff}}(\omega,e^x)=A(\Psi,\Phi)(x,\omega),\qquad(x,\omega)\in\mathbb{R}^2.$$

Moreover, the affine ambiguity function satisfies

$$|A^{\psi}_{\text{Aff}}(x,a)|<A^{\psi}_{\text{Aff}}(0,1)=\|\psi\|^2_{L^2(\mathbb{R}_+)},$$

for every $(x,a)\neq(0,1)$.

The last statement in Lemma 6.2 is a direct consequence of [23, Lemma 4.2.1]. The proof of the following orthogonality result illustrates the usefulness of Lemma 6.2.

Proposition 6.3. *The affine cross-ambiguity function satisfies the orthogonality relation*

$$\left\langle A_{\mathrm{Aff}}^{\psi_1,\phi_1}, A_{\mathrm{Aff}}^{\psi_2,\phi_2} \right\rangle_{L^2_r(\mathrm{Aff})} = \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi_1, \phi_2 \rangle}_{L^2(\mathbb{R}_+)},$$

for $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}_+)$.

Proof. Let $\Psi_i(x) := \psi_i(e^x)$ and $\Phi_i(x) := \phi_i(e^x)$ for $i = 1, 2$. Then Lemma 6.2 gives that

$$\begin{aligned} \left\langle A_{\mathrm{Aff}}^{\psi_1,\phi_1}, A_{\mathrm{Aff}}^{\psi_2,\phi_2} \right\rangle_{L^2_r(\mathrm{Aff})} &= \int_{-\infty}^{\infty} \int_{0}^{\infty} A(\Psi_1, \Phi_1)(\ln(a), x) \overline{A(\Psi_2, \Phi_2)(\ln(a), x)} \, \frac{da\, dx}{a} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\Psi_1, \Phi_1)(u, x) \overline{A(\Psi_2, \Phi_2)(u, x)} \, du\, dx. \end{aligned}$$

From [23, Lemma 4.3.4] it follows that the ambiguity function is related to the classical cross-Wigner transform by

$$W(\Psi_i, \Phi_i) = \mathcal{F}\mathcal{U}A(\Psi_i, \Phi_i), \quad i = 1, 2,$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{U}$ is the rotation $\mathcal{U}F(x, \omega) := F(\omega, -x)$ for a function F on $\mathbb{R}^2$. Hence from (2.1) we obtain that

$$\left\langle A_{\mathrm{Aff}}^{\psi_1,\phi_1}, A_{\mathrm{Aff}}^{\psi_2,\phi_2} \right\rangle_{L^2_r(\mathrm{Aff})} = \langle \Psi_1, \Psi_2 \rangle_{L^2(\mathbb{R})} \overline{\langle \Phi_1, \Phi_2 \rangle}_{L^2(\mathbb{R})} = \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}_+)} \overline{\langle \phi_1, \phi_2 \rangle}_{L^2(\mathbb{R}_+)}. \quad \square$$

Corollary 6.4. *Let* $\psi \in L^2(\mathbb{R}_+)$ *be normalized and let* $U \subset \mathrm{Aff}$ *be a Borel set. Assume there exists* $\epsilon > 0$ *such that*

$$\iint_U |A_{\mathrm{Aff}}^{\psi}(x, a)|^2 \, \frac{da\, dx}{a} \geq 1 - \epsilon. \tag{6.1}$$

Then

$$\mu_r(U) \geq (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad p > 2.$$

In particular, we have $\mu_r(U) \geq \max(2(1-\epsilon)^2, 1-\epsilon)$.

Proof. Notice that the assumption (6.1) is by Lemma 6.2 equivalent to

$$\int_{U_1} \int_{\ln(U_2)} |A\Psi(u, x)|^2 \, du\, dx \geq 1 - \epsilon,$$

where $\Psi(x) := \psi(e^x)$. We can write $A\Psi(u, x) = e^{\pi i u x} V_\Psi \Psi(u, x)$, where V is the short-time Fourier transform given in (2.4). The assumption

$$\int_{U_1}\int_{\ln(U_2)} |V_\Psi \Psi(u,x)|^2\, du\, dx \geq 1-\epsilon$$

implies by Lieb's uncertainty principle [23, Theorem 3.3.3] that we have

$$\mu_r(U) = |U_1 \times \ln(U_2)| \geq (1-\epsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2}{p-2}}, \qquad p > 2.$$

The final claim follows from considering $p = 4$ and $p \to \infty$. □

We now relate the affine ambiguity function to the affine Wigner distribution. Define

$$\Theta(y,b) := \left(\frac{\sqrt{b}\log(b)}{b-1}\right)^{2\pi i y},$$

for $y \in \mathbb{R}$ and $b > 0$ with the convention that $\Theta(y,1) = 1$ for all $y \in \mathbb{R}$. If we write $b = e^u$ for $u = \log(b)$, then

$$\frac{\sqrt{b}\log(b)}{b-1} = \sqrt{\lambda(u)\lambda(-u)},$$

where λ is the function given in (2.6). Hence we can think of $\Theta(y,b)$ as arising from a symmetrization of the function λ. We leave the verification of the following result to the reader as it is straightforward.

Lemma 6.5. *For $\psi, \phi \in L^2(\mathbb{R}_+)$ we have the equality*

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\Theta(y,b) \cdot A_{\mathrm{Aff}}^{\psi,\phi}(y,b)\right](x,a),$$

where $(x,a) \in \mathrm{Aff}$ and $\mathcal{M}$ is the Mellin transform.

It is of importance to extend the affine Weyl quantization to tempered distributions. To do this, we first need the following definition.

Definition 6.6. Let $\mathscr{S}(\mathrm{Aff})$ denote the smooth functions $f : \mathrm{Aff} \to \mathbb{C}$ that satisfy

$$(x,\omega) \longmapsto f(x, e^\omega) \in \mathscr{S}(\mathbb{R}^2).$$

The space $\mathscr{S}(\mathrm{Aff})$ is called the *rapidly decaying smooth functions* on Aff. The dual space of $\mathscr{S}(\mathrm{Aff})$ will be denoted by $\mathscr{S}'(\mathrm{Aff})$ and called the *tempered distributions* on Aff.

The following result illustrates how we can use the Mellin transform and the affine ambiguity function to deduce properties of the affine Wigner distribution.

Proposition 6.7. *For $\psi, \phi \in \mathscr{S}(\mathbb{R}_+)$ the affine Wigner distribution satisfies $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathscr{S}(\mathrm{Aff})$.*

Proof. Let $\Psi(x) := \psi(e^x)$ and $\Phi(x) := \phi(e^x)$. By Lemma 6.2 and Lemma 6.5 we want to show that

$$(x,\omega) \longmapsto \mathcal{M}_y^{-1} \otimes \mathcal{M}_b \left[\Theta(y,b) \cdot A^{\Psi,\Phi}(\log(b), y)\right](x, e^\omega) \in \mathscr{S}(\mathbb{R}^2).$$

The cross-ambiguity function A is a map $A : \mathscr{S}(\mathbb{R}) \times \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}^2)$ by [23, Theorem 11.2.5]. Hence $A(y, b) := A^{\Psi,\Phi}(\log(b), y) \in \mathscr{S}(\mathrm{Aff})$. Since $\Theta(y, b)$ is a smooth function with polynomially bounded derivatives, the same goes for the product $\Theta(y, b) \cdot A(y, b)$. The claim follows since the Mellin transform is related to the Fourier transform by the formula $\mathcal{M}(\psi)(x) = \mathcal{F}(\Psi)(x)$. □

Corollary 6.8. *The affine Weyl quantization A_f of $f \in \mathscr{S}'(\mathrm{Aff})$ is well-defined as an operator*

$$A_f : \mathscr{S}(\mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}_+).$$

Example 6.9. Consider the point measure $\delta_{\mathrm{Aff}}(x, a) \in \mathscr{S}'(\mathrm{Aff})$ defined by

$$\langle \delta_{\mathrm{Aff}}(x, a), f \rangle = \overline{f(x, a)},$$

for $f \in \mathscr{S}(\mathrm{Aff})$ and $(x, a) \in \mathrm{Aff}$. We compute for $\psi, \phi \in \mathscr{S}(\mathbb{R}_+)$ that

$$\left\langle A_{\delta_{\mathrm{Aff}}(x,a)}\psi, \phi \right\rangle = \left\langle \delta_{\mathrm{Aff}}(x, a), W_{\mathrm{Aff}}^{\phi,\psi} \right\rangle = \overline{W_{\mathrm{Aff}}^{\phi,\psi}(x, a)} = W_{\mathrm{Aff}}^{\psi,\phi}(x, a).$$

Hence the operator $A_{\delta_{\mathrm{Aff}}(x,a)}$ is weakly defined through the values of the affine Wigner distribution.

7. An almost analytic decomposition

Recall that analytic and anti-analytic functions f are characterized by the equations $\partial_{\bar{z}} f(z) = 0$ and $\partial_z f(z) = 0$, respectively. The fact that $W_{\mathrm{Aff}}^{\psi,\phi} \in L_r^2(\mathrm{Aff})$ when $\psi, \phi \in L^2(\mathbb{R}_+)$ allows us to exclude (anti-)analytic functions from being affine Wigner distributions.

Proposition 7.1. *There are no analytic or anti-analytic functions in the space $L_r^2(\mathrm{Aff})$.*

Proof. The conclusion is easier to obtain by looking at the isomorphic spaces in the unit disc $\mathbb{D}$ by applying the standard linear fractional transformation. Under this transformation, the analytic functions in $L_r^2(\mathrm{Aff})$ are transformed to the analytic functions f in the unit disc satisfying the integrability condition

$$\int_{\mathbb{D}} \frac{|f(z)|^2}{1 - |z|^2}\, dz < \infty. \tag{7.1}$$

Any such analytic function will have to vanish as it approaches the boundary circle. Thus they are identically zero inside the unit disc as well by the unique continuation principle for analytic functions. The case of anti-analytic functions is similar. □

Remark. Proposition 7.1 shows a big difference between the affine Wigner distribution and both the classical Wigner distribution and the wavelet transform; the classical Wigner distribution can produce Gaussians, while one can obtain plenty of analytic functions from the wavelet transform as shown in [12, Chapter 2.5].

A function $f : \mathrm{Aff} \to \mathbb{C}$ is called *(anti-)poly-analytic of order $n \in \mathbb{N}$* if $\partial_{\bar{z}}^n f = 0$ ($\partial_z^n f = 0$). We write $f \in \mathcal{A}^n(\mathrm{Aff})$ ($f \in \mathcal{A}^{\perp,n}(\mathrm{Aff})$) to signify that f is (anti-)poly-analytic of order n, but not (anti-)poly-analytic of order $n - 1$. The following result is inspired by [32] and shows that $L_r^2(\mathrm{Aff})$ decomposes completely into poly-analytic and anti-poly-analytic functions.

Proposition 7.2. *The space $L_r^2(\mathrm{Aff})$ has the orthogonal decomposition*

$$L_r^2(\mathrm{Aff}) = \bigoplus_{n=2}^{\infty} \left(\mathcal{A}^n(\mathrm{Aff}) \oplus \mathcal{A}^{\perp,n}(\mathrm{Aff})\right).$$

Proof. Notice first that

$$L_r^2(\mathrm{Aff}) \simeq \left(L^2(\mathbb{R}_+, dx) \otimes L^2(\mathbb{R}_+, a^{-1}\, da)\right) \oplus \left(L^2(\mathbb{R}_-, dx) \otimes L^2(\mathbb{R}_+, a^{-1}\, da)\right).$$

Hence for $n \geq 2$ it suffices to show the decompositions

$$\mathcal{A}^n(\mathrm{Aff}) \simeq L^2(\mathbb{R}_+, dx) \otimes \mathrm{span}\left\{\mathcal{L}_{n-2}\right\}, \qquad \mathcal{A}^{\perp,n}(\mathrm{Aff}) \simeq L^2(\mathbb{R}_-, dx) \otimes \mathrm{span}\left\{\mathcal{L}_{n-2}\right\},$$

where $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is the orthogonal basis for $L^2(\mathbb{R}_+)$ defined in (3.2). We will only show the decomposition of $\mathcal{A}^n(\mathrm{Aff})$ since the decomposition of $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ is similar.

Consider the map $\Phi : L_r^2(\mathrm{Aff}) \to L_r^2(\mathrm{Aff})$ given by

$$\Phi f(x, a) = \mathcal{F}_1(f)\left(x, \frac{a}{2|x|}\right),$$

where $\mathcal{F}_1$ is the Fourier transform in the first component. It is straightforward to check that Φ is a unitary map. The image of $\mathcal{A}^n(\mathrm{Aff})$ under Φ consists of all functions in $L_r^2(\mathrm{Aff})$ that satisfy

$$\Phi \circ \partial_{\overline{z}}^n \circ \Phi^{-1} f = \Phi(\partial_x + i\partial_a)^n \Phi^{-1} f = 0, \tag{7.2}$$

but do not satisfy (7.2) for $n-1$. A computation shows that functions $f \in L_r^2(\mathrm{Aff})$ satisfying (7.2) are precisely those that satisfy the homogeneous equation

$$|x|^n \left(\mathrm{sign}(x) + 2\partial_a\right)^n f(x, a) = 0, \tag{7.3}$$

but do not satisfy (7.3) for $n-1$. It is well-known that the solution is precisely

$$f(x, a) = g(x)\mathcal{L}_{n-2}(a), \qquad g \in L^2(\mathbb{R}_+, dx).$$

Hence we obtain the decomposition for $\mathcal{A}^n(\mathrm{Aff})$ and the result follows. □

Remark. Notice that Proposition 7.2 does not claim that $\mathcal{A}^n(\mathrm{Aff})$ and $\mathcal{A}^{\perp,n}(\mathrm{Aff})$ are orthogonal as $\mathcal{A}^n(\mathrm{Aff}) \cap \mathcal{A}^{\perp,n}(\mathrm{Aff}) \neq \{0\}$ for all $n \geq 2$. The poly-analytic functions have appeared prominently in the work of Abreu, see e.g. [1], in the context of wavelet analysis and sampling theory.

8. Applications

8.1. An approximation problem

Let us use the notation

$$\mathfrak{W}(\mathrm{Aff}) := \left\{ W_{\mathrm{Aff}}^{\psi} \,\middle|\, \psi \in L^2(\mathbb{R}_+) \right\} \subset L_r^2(\mathrm{Aff}),$$

and call $\mathfrak{W}(\mathrm{Aff})$ the *affine Wigner space*. The affine orthogonality relation (3.1) implies that $\mathfrak{W}(\mathrm{Aff})$ is a closed subset of $L_r^2(\mathrm{Aff})$. Although we can create orthonormal bases for $L_r^2(\mathrm{Aff})$ by using the affine cross-Wigner transform as done in Corollary 3.3, the space $\mathfrak{W}(\mathrm{Aff})$ is a proper subset of $L_r^2(\mathrm{Aff})$.

It is natural to ask how far a function $f \in L_r^2(\mathrm{Aff})$ is from being in $\mathfrak{W}(\mathrm{Aff})$. Hence we are interested in the following *affine Wigner approximation problem*

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})}. \tag{8.1}$$

The analogous problem for the classical Wigner distribution has been recently investigated in [5]. Our quantization based approach will as a byproduct produce a new proof of the classical Wigner approximation problem in [5].

For $g = W^{\psi}_{\mathrm{Aff}}$ it follows from (2.11) and the affine orthogonality relation (3.1) that A_g is the rank-one operator

$$A_g \phi = \langle \phi, \psi \rangle \psi. \tag{8.2}$$

The converse is also clear, so there is a one-to-one correspondence between affine Wigner distributions and positive rank-one operators. Hence the distance (8.1) should somehow be related to how far A_f is from being a rank-one operator. In Corollary 8.2 we will see that this heuristic is correct for a large class of functions $f \in L^2_r(\mathrm{Aff})$. We use the notation

$$\lambda^+_{\max}(A_f) := \max\left\{ \max_{\lambda \in \mathrm{Spec}(A_f)} \lambda, 0 \right\}.$$

Theorem 8.1. *The affine Wigner approximation problem for a real-valued function $f \in L^2_r(\mathrm{Aff})$ has the explicit solution*

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L^2_r(\mathrm{Aff})} = \sqrt{\|f\|^2_{L^2_r(\mathrm{Aff})} - \lambda^+_{\max}(A_f)^2}. \tag{8.3}$$

A minimizing function $h \in \mathfrak{W}(\mathrm{Aff})$ to the affine Wigner approximation problem always exists. Moreover, when $\lambda^+_{\max}(A_f) > 0$ the number of minimizers is equal to the multiplicity of $\lambda^+_{\max}(A_f)$.

Proof. Notice that A_f is self-adjoint since

$$\langle A_f \psi, \phi \rangle = \left\langle A_{\overline{f}} \psi, \phi \right\rangle = \left\langle \overline{f}, W^{\phi,\psi}_{\mathrm{Aff}} \right\rangle = \overline{\left\langle f, W^{\psi,\phi}_{\mathrm{Aff}} \right\rangle} = \langle \psi, A_f \phi \rangle,$$

for $\psi, \phi \in L^2(\mathbb{R}_+)$. Thus the spectral theory for compact, self-adjoint operators implies that the spectrum $\mathrm{Spec}(A_f) = \{\lambda_k\}_{k=0}^{\infty}$ of A_f is countable with $0 \in \mathrm{Spec}(A_f)$ as the only possible accumulation point. Moreover, there is by [18, Theorem 1.52] an orthonormal basis $\{\phi_k\}_{k=0}^{\infty}$ for $L^2(\mathbb{R}_+)$ such that ϕ_k is an eigenvector for A_f corresponding to the eigenvalue λ_k. The convention is that eigenvalues are repeated according to their multiplicity.

We claim that we can write $A_f = \sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$, where the convergence is in the Hilbert-Schmidt norm. Notice that convergence of $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ to A_f is guaranteed in the operator norm from the theory of compact operators [10, Theorem 3.5]. Hence it suffices to show that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ converges in the Hilbert-Schmidt norm; this will imply together with the norm inequality $\|\cdot\|_{op} \leq \|\cdot\|_{\mathcal{HS}}$ that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ must converge to A_f in the Hilbert-Schmidt norm. Due to completeness, it suffices to show that $\sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k}$ is a Cauchy sequence. For $n, m \in \mathbb{N}$ with $n < m$ we have

$$\left\| \sum_{k=n}^{m} \lambda_k \phi_k \otimes \overline{\phi_k} \right\|^2_{\mathcal{HS}} = \sum_{k,k'=n}^{m} \lambda_k \overline{\lambda_{k'}} \left\langle \phi_k \otimes \overline{\phi_k}, \phi_{k'} \otimes \overline{\phi_{k'}} \right\rangle_{\mathcal{HS}} = \sum_{k=n}^{m} |\lambda_k|^2,$$

where $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm. The claim follows since A_f is Hilbert-Schmidt.

We can now by Proposition 2.1 write

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L^2_r(\mathrm{Aff})} = \inf_{\psi \in L^2(\mathbb{R}_+)} \left\| \sum_{k=0}^{\infty} \lambda_k \phi_k \otimes \overline{\phi_k} - \psi \otimes \overline{\psi} \right\|_{\mathcal{HS}}. \tag{8.4}$$

Assume that $\lambda_j = \lambda_{\max}^+(A_f)$. Then (8.4) is clearly minimized when $\psi = \sqrt{\lambda_j}\phi_j$. By orthogonality, we can rewrite (8.4) and obtain

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \lambda_{\max}^+(A_f)^2} = \sqrt{\|f\|_{L_r^2(\mathrm{Aff})}^2 - \lambda_{\max}^+(A_f)^2}.$$

We always have a minimizer as we can take $h = W_{\mathrm{Aff}}^{\psi}$. The statement about uniqueness of minimizers is clear from (8.4). □

Remarks.

- From the spectral theory of compact, self-adjoint operators, it also follows that the eigenspaces corresponding to non-zero eigenvalues are finite-dimensional. Hence, for a given $f \in L_r^2(\mathrm{Aff})$, there is at most a finite number of minimizers $h_1, \ldots, h_k \in \mathfrak{W}(\mathrm{Aff})$ so that

$$\inf_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \|f - h_i\|_{L_r^2(\mathrm{Aff})}, \qquad i = 1, \ldots, k.$$

- The proof of Theorem 8.1 goes through almost verbatim to show the analogous result for the classical Wigner distribution. The analogous formula to (8.3) for the classical Wigner distribution was shown in [5, Theorem 3] using a variational calculus approach. That the number of minimizers can be easily deduced from the spectrum of the quantized operator seems new even for the classical Wigner distribution.
- Assume that $f \in L_r^2(\mathrm{Aff})$ is such that A_f is a negative operator. Then $\lambda_{\max}^+(A_f) = 0$ and it is clear from (8.4) that the zero function is the unique minimizer.

Corollary 8.2. *Let $f \in L_r^2(\mathrm{Aff})$ be real-valued and assume that*

$$\lambda_{\max}^+(A_f) = \max_{\lambda \in \mathrm{Spec}(A_f)} |\lambda|.$$

Then

$$\min_{g \in \mathfrak{W}(\mathrm{Aff})} \|f - g\|_{L_r^2(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \|A_f\|_{op}^2}. \tag{8.5}$$

Proof. Since A_f is self-adjoint it follows from [18, Proposition 1.24] that

$$\|A_f\|_{op} = \max_{\lambda \in \mathrm{Spec}(A_f)} |\lambda|. \quad \square$$

Remark. Notice that under the assumptions in Corollary 8.2, the heuristic we presented regarding rank-one operators holds true: If A_f is a rank-one operator, then the Hilbert-Schmidt norm and the operator norm coincide. Hence (8.5) is zero and thus f is in the affine Wigner space $\mathfrak{W}(\mathrm{Aff})$. Conversely, the equations

$$\|A_f\|_{op}^2 = \max_{\lambda \in \mathrm{Spec}(A_f)} \lambda^2, \qquad \|A_f\|_{\mathcal{HS}}^2 = \sum_{\lambda \in \mathrm{Spec}(A_f)} \lambda^2 \tag{8.6}$$

imply that (8.5) is zero precisely when A_f is a rank-one operator.

Example 8.3. Let $f \in L_r^2(\mathrm{Aff})$ be such that A_f is a positive operator with rank $k > 0$. Then (8.6) implies that

$$\|A_f\|_{op}^2 \geq \frac{\|A_f\|_{\mathcal{HS}}^2}{k}.$$

Hence we obtain from (8.5) that

$$\min_{g\in\mathfrak{W}(\mathrm{Aff})} \|f-g\|_{L^2_r(\mathrm{Aff})} = \sqrt{\|A_f\|_{\mathcal{HS}}^2 - \|A_f\|_{op}^2} \le \sqrt{\frac{k-1}{k}}\|A_f\|_{\mathcal{HS}} = \sqrt{\frac{k-1}{k}}\|f\|_{L^2_r(\mathrm{Aff})}.$$

This has the following consequence: Let $f_1, f_2 \in L^2_r(\mathrm{Aff})$ both correspond to positive operators A_{f_1} and A_{f_2} with finite rank. If $\mathrm{rank}(A_{f_1}) \ll \mathrm{rank}(A_{f_2})$, then f_1 will be closer to the affine Wigner space than f_2, unless the energy of f_2 is significantly smaller than that of f_1.

8.2. Dilation invariant operators

An operator $A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is said to be *dilation invariant* if

$$A = D_r \circ A \circ D_r^*, \tag{8.7}$$

for all $r > 0$ where D_r is the dilation operator in (2.5). We use the affine Weyl quantization to show the following result.

Proposition 8.4. *There are no non-zero dilation invariant Hilbert-Schmidt operators on* $L^2(\mathbb{R}_+)$.

Proof. Assume by contradiction that $A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a dilation invariant Hilbert-Schmidt operator and write $A = A_f$ for $f \in L^2_r(\mathrm{Aff})$. It follows from (4.1) that

$$W_{\mathrm{Aff}}^{D_{\frac{1}{r}}\psi, D_{\frac{1}{r}}\phi}(x,a) = r \cdot W_{\mathrm{Aff}}^{\psi,\phi}(x, ra), \qquad \psi,\phi \in L^2(\mathbb{R}_+),$$

for $r > 0$ and $(x,a) \in \mathrm{Aff}$. Hence (2.11) implies that

$$\left\langle D_r A_f D_{\frac{1}{r}}\psi, \phi\right\rangle = \left\langle f, W_{\mathrm{Aff}}^{D_{\frac{1}{r}}\phi, D_{\frac{1}{r}}\psi}\right\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} r f\left(x, \frac{a}{r}\right) W_{\mathrm{Aff}}^{\psi,\phi}(x,a)\,\frac{da\,dx}{a}.$$

On the other hand, since A_f is dilation invariant we also have

$$\left\langle D_r A_f D_{\frac{1}{r}}\psi, \phi\right\rangle = \langle A_f\psi, \phi\rangle = \left\langle f, W_{\mathrm{Aff}}^{\phi,\psi}\right\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} f(x,a)\, W_{\mathrm{Aff}}^{\psi,\phi}(x,a)\,\frac{da\,dx}{a}.$$

This forces $f \in L^2_r(\mathrm{Aff})$ by Corollary 3.3 to satisfy the homogeneity relation

$$f(x,a) = r f\left(x, \frac{a}{r}\right),$$

for all $r > 0$ and almost every $(x,a) \in \mathrm{Aff}$. However, this implies that

$$\|f\|_{L^2_r(\mathrm{Aff})}^2 = \int_{-\infty}^{\infty}\int_0^{\infty} |f(x,a)|^2\,\frac{da\,dx}{a} = r^2 \int_{-\infty}^{\infty}\int_0^{\infty} \left|f\left(x,\frac{a}{r}\right)\right|^2\,\frac{da\,dx}{a} = r^2\|f\|_{L^2_r(\mathrm{Aff})}^2.$$

Hence f is not in $L^2_r(\mathrm{Aff})$ unless $f = 0$, in which case A_f is the zero operator. □

Remark. Notice that the proof of Proposition 8.4 actually shows that there can be no non-zero Hilbert-Schmidt operator A that satisfies (8.7) even for a single $r \neq 1$.

Example 8.5. Although we showed in Proposition 8.4 that there are no non-zero dilation invariant Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$, there are non-zero projections in $L^2(\mathbb{R}_+)$ that are dilation invariant. As an example, consider the orthogonal projection $P : L^2(\mathbb{R}_+) \to \mathcal{M}_{(0,\infty)}$ where $\mathcal{M}_{(0,\infty)}$ is the space of all $\psi \in L^2(\mathbb{R}_+)$ such that the Mellin transform of ψ satisfies

$$\operatorname{supp}(\mathcal{M}(\psi)) \subset \mathbb{R}_+.$$

The projection P is dilation invariant due to (3.4).

8.3. Trace class operators

Finally, we give an application to trace-class operators motivated by [14, Proposition 162].

Proposition 8.6. *Let $T : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be a trace-class operator. Then we can write $T = A_f \circ A_g$ for $f, g \in L^2_r(\mathrm{Aff})$. Moreover, the trace of T can be calculated by the formula*

$$\operatorname{Tr}(T) = \operatorname{Tr}(A_f \circ A_g) = \int_{-\infty}^{\infty} \int_0^{\infty} f(x,a) g(x,a) \, \frac{da\, dx}{a}.$$

Proof. Any trace-class operator T on $L^2(\mathbb{R}_+)$ can be written as a composition of two Hilbert-Schmidt operators $T = A \circ B$. The bijective correspondence in Proposition 2.1 shows that $A = A_f$ and $B = A_g$ for $f, g \in L^2_r(\mathrm{Aff})$. Finally, we have

$$\operatorname{Tr}(T) = \operatorname{Tr}(A_f \circ A_g) = \langle A_g, A_f^* \rangle_{\mathcal{HS}} = \langle g, \overline{f} \rangle_{L^2_r(\mathrm{Aff})} = \int_{-\infty}^{\infty} \int_0^{\infty} f(x,a) g(x,a) \, \frac{da\, dx}{a}. \quad \square$$

Remark. Notice that

$$\overline{\operatorname{Tr}(T)} = \operatorname{Tr}(T^*) = \operatorname{Tr}\left(A_g^* \circ A_f^*\right) = \operatorname{Tr}\left(A_{\overline{g}} \circ A_{\overline{f}}\right) = \int_{-\infty}^{\infty} \int_0^{\infty} \overline{f(x,a) g(x,a)} \, \frac{da\, dx}{a}.$$

In particular, the trace of T is real-valued whenever f and g are real-valued.

9. Further research

9.1. The affine Grossmann-Royer operator

A standard tool for deriving properties of the classical Wigner distribution is the *Grossmann-Royer operator* $\widehat{R}(x, \omega)$ defined by the relation

$$W(f,g)(x,\omega) = \left\langle \widehat{R}(x,\omega) f, g \right\rangle_{L^2(\mathbb{R}^d)},$$

for $f, g \in L^2(\mathbb{R}^d)$ and $(x, \omega) \in \mathbb{R}^{2d}$. An essential property of the Grossmann-Royer operator $\widehat{R}(x, \omega)$ is that

$$\left\| \widehat{R}(x,\omega) f \right\|_{L^2(\mathbb{R}^d)} = 2^d \cdot \|f\|_{L^2(\mathbb{R}^d)},$$

for all $f \in L^2(\mathbb{R}^d)$ and $(x, \omega) \in \mathbb{R}^{2d}$. This is immensely useful; to see that the classical cross-Wigner transform is bounded one simply needs to apply Cauchy-Schwarz inequality to obtain

$$\sup_{(x,\omega)\in\mathbb{R}^{2d}} |W(f,g)(x,\omega)| \leq \left\| \widehat{R}(x,\omega)f \right\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} = 2^d \cdot \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{9.1}$$

Analogously, we define the *affine Grossmann-Royer operator* $\widehat{R}_{\mathrm{Aff}}(x, a)$ by the relation

$$W_{\mathrm{Aff}}^{\psi,\phi}(x,a) = \left\langle \widehat{R}_{\mathrm{Aff}}(x,a)\psi, \phi \right\rangle_{L^2(\mathbb{R}_+)},$$

for $\psi, \phi \in \mathscr{S}(\mathbb{R}_+)$ and $(x, a) \in \mathrm{Aff}$. We restrict our attention to Schwartz functions for convenience since then $W_{\mathrm{Aff}}^{\psi,\phi} \in \mathscr{S}(\mathrm{Aff})$ and hence have well-defined point values. Notice that the affine Grossmann-Royer operator $\widehat{R}_{\mathrm{Aff}}(x, a)$ is precisely the affine Weyl quantization of the point mass $\delta_{\mathrm{Aff}}(x, a)$ given in Example 6.9. The affine Grossmann-Royer operator has the explicit form

$$\widehat{R}_{\mathrm{Aff}}(x,a)\psi(r) = \frac{e^{2\pi i x \lambda^{-1}\left(\frac{r}{a}\right)} \lambda^{-1}\left(\frac{r}{a}\right)\left(1 - e^{\lambda^{-1}\left(\frac{r}{a}\right)}\right)}{1 + \lambda^{-1}\left(\frac{r}{a}\right) - e^{\lambda^{-1}\left(\frac{r}{a}\right)}} \cdot \psi\left(r e^{-\lambda^{-1}\left(\frac{r}{a}\right)}\right),$$

for $\psi \in \mathscr{S}(\mathbb{R}_+)$, $r > 0$, and $(x, a) \in \mathrm{Aff}$ where λ is the function given in (2.6).

Trying to generalize the strategy in (9.1) runs into a problem: The affine Grossmann-Royer operator is not a bounded operator on $\mathscr{S}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}_+)}$. However, if $\psi \in \mathscr{S}(\mathbb{R}_+)$ is supported in the interval $\left[\frac{1}{k}, k\right]$ for some $k > 0$, then there is a constant $C_k > 0$ such that

$$\left\| \widehat{R}_{\mathrm{Aff}}(x,a)\psi \right\|_{L^2(\mathbb{R}_+)} \leq C_k \cdot \|\psi\|_{L^2(\mathbb{R}_+)}.$$

We call the optimal constant C_k in the inequality above the *k-support constant*. Hence if $\phi \in \mathscr{S}(\mathbb{R}_+)$ we have

$$\sup_{(x,a)\in\mathrm{Aff}} \left| W_{\mathrm{Aff}}^{\psi,\phi}(x,a) \right| \leq C_k \cdot \|\psi\|_{L^2(\mathbb{R}_+)} \|\phi\|_{L^2(\mathbb{R}_+)}.$$

A trivial adaption of [23, Lemma 4.3.7] gives the following *relative uncertainty principle*.

Proposition 9.1. *Let $\psi \in \mathscr{S}(\mathbb{R}_+)$ be supported in the interval $\left[\frac{1}{k}, k\right]$ for some $k > 0$ and let $U \subset \mathrm{Aff}$ be a Borel set. Assume there exists $\epsilon \geq 0$ such that*

$$\int_U W_{\mathrm{Aff}}^{\psi}(x,a) \, \frac{da\,dx}{a} \geq (1-\epsilon)\|\psi\|_{L^2(\mathbb{R}_+)}^2.$$

Then the right Haar measure of U satisfies $\mu_r(U) \geq (1-\epsilon)C_k^{-1}$.

Motivated by Proposition 9.1, it is of interest to investigate the k-support constant C_k both numerically and asymptotically. The affine Grossmann-Royer operator is investigated more thoroughly in the follow-up paper [6].

9.2. The affine positivity conjecture

One of the major results about the classical Wigner distribution is regarding positivity; when is W_f a non-negative function on $\mathbb{R}^{2d}$? Normalized functions $f \in L^2(\mathbb{R}^d)$ such that W_f is non-negative would generate probability density functions on $\mathbb{R}^{2d}$ that represent the time-frequency distribution of f. However, a well-known result of Hudson [23, Theorem 4.4.1] shows that this can only happen for suitably perturbed Gaussians.

Turning to the affine setting, we would like to determine the normalized functions $\psi \in L^2(\mathbb{R}_+)$ such that W_{Aff}^{ψ} is a non-negative function on the affine group. In [28] the authors showed that the affine Wigner distribution $W_{\mathrm{Aff}}^{\psi_s}$ is non-negative if ψ_s is the so called *Morse ground state*

$$\psi_s(r) := \frac{r^s e^{-\frac{r}{2}}}{\Gamma(2s)}, \qquad s \geq 0.$$

We will only consider ψ_s for $s > 0$ as $\psi_0 \notin L^2(\mathbb{R}_+)$. More generally, one can use the invariance properties (4.2) and (4.3) to show that the affine Wigner distribution W_{Aff}^{ψ} of

$$\psi(r) = Cr^{-i(x+ia)}e^{i(y+ib)r} \qquad C \in \mathbb{C},\ (x,a),(y,b) \in \mathrm{Aff}, \tag{9.2}$$

is non-negative. The functions (9.2) are the *generalized Klauder wavelets* in [16, Equation (41)] that are in $L^2(\mathbb{R}_+)$. It is of interest to determine the following *affine positivity conjecture*, which is a reformulation of an open question posed in [16]:

If W_{Aff}^{ψ} is non-negative for $\psi \in L^2(\mathbb{R}_+)$, then ψ is a generalized Klauder wavelet.

The Klauder wavelets have in [25] been shown to be the only functions that generate analytic spaces for the continuous wavelet transform. This gives a concrete connection between Klauder wavelets and Gaussians, since Gaussians are the only functions in the classical case that generate analytic spaces for the short-time Fourier transform by [4, Theorem 3.1].

References

[1] L.D. Abreu, Super-wavelets versus poly-Bergman spaces, Integral Equ. Oper. Theory 73 (2) (2012) 177–193.

[2] S.T. Ali, J.-P. Antoine, J.-P. Gazeau, Coherent States, Wavelets, and Their Generalizations, second edition, Springer-Verlag, New York, 2014.

[3] S.T. Ali, N.M. Atakishiyev, S.M. Chumakov, K.B. Wolf, The Wigner function for general Lie groups and the wavelet transform, Ann. Henri Poincaré 1 (2000) 685–714.

[4] G. Ascensi, J. Bruna, Model space results for the Gabor and wavelet transforms, IEEE Trans. Inf. Theory 55 (5) (2009) 2250–2259.

[5] J.S. Ben-Benjamin, L. Cohen, N.C. Dias, P. Loughlin, J.N. Prata, What is the Wigner function closest to a given square integrable function?, SIAM J. Math. Anal. 50 (5) (2018) 5161–5197.

[6] E. Berge, S.M. Berge, F. Luef, E. Skrettingland, Affine quantum harmonic analysis, arXiv preprint, arXiv:2102.13006, 2021.

[7] H. Bergeron, J.-P. Gazeau, Integral quantizations with two basic examples, Ann. Phys. 344 (2014) 43–68.

[8] H. Bergeron, J.-P. Gazeau, Variations à la Fourier-Weyl-Wigner on quantizations of the plane and the half-plane, Entropy 20 (10) (2018) 1–16.

[9] J. Bertrand, P. Bertrand, A class of affine Wigner functions with extended covariance properties, J. Math. Phys. 33 (7) (1992) 2515–2527.

[10] P. Busch, P. Lahti, J.-P. Pellonpää, K. Ylinen, Quantum Measurement, Springer, 2016.

[11] T.A.C.M. Classen, W.F.G. Mecklenbräuker, The Wigner distribution - a tool for time-frequency signal analysis. Part III: relations with other time-frequency signal transformations, Philips J. Res. 35 (5) (1980) 372–389.

[12] I. Daubechies, Ten Lectures on Wavelets, SIAM, 1992.

[13] I. Daubechies, T. Paul, Time-frequency localisation operators - a geometric phase space approach: II. The use of dilations, Inverse Probl. 4 (3) (1988) 661–680.

[14] M. de Gosson, The Wigner Transform, World Scientific Publishing Company, 2017.

[15] E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, Wigner Distributions in Quantum Mechanics, J. Phys. Conf. Ser, vol. 87, IOP Publishing, 2007, p. 012010.

[16] P. Flandrin, Separability, positivity, and minimum uncertainty in time–frequency energy distributions, J. Math. Phys. 39 (8) (1998) 4016–4040.

[17] G.B. Folland, Harmonic Analysis in Phase Space, Princeton University Press, 1989.

[18] G.B. Folland, A Course in Abstract Harmonic Analysis, Chapman and Hall/CRC, 2016.

[19] V. Gayral, J.M. Gracia-Bondía, J.C. Várilly, Fourier analysis on the affine group, quantization and noncompact Connes geometries, J. Noncommut. Geom. 2 (2008) 215–261.
[20] J.-P. Gazeau, C. Habonimana, Signal analysis and quantum formalism: quantizations with no Planck constant, in: Landscapes of Time-Frequency Analysis - ATFA 2019, Birkhäuser, Basel, 2020, pp. 135–155.
[21] J.-P. Gazeau, T. Koide, R. Murenzi, 2-D covariant affine integral quantization(s), Adv. Oper. Theory 5 (2020) 901–935.
[22] J.P. Gazeau, R. Murenzi, Covariant affine integral quantization(s), J. Math. Phys. 57 (5) (2016) 052102.
[23] K. Gröchenig, Foundations of Time-Frequency Analysis, Springer Science & Business Media, 2001.
[24] B.C. Hall, Quantum Theory for Mathematicians, Springer, 2013.
[25] N. Holighaus, G. Koliander, Z. Průša, L.D. Abreu, Characterization of analytic wavelet transforms and a new phaseless reconstruction algorithm, IEEE Trans. Signal Process. 67 (15) (2019) 3894–3908.
[26] A.A. Kirillov, Lectures on the Orbit Method, American Mathematical Soc., 2004.
[27] W.D. Mark, Spectral analysis of the convolution and filtering of non-stationary stochastic processes, J. Sound Vib. 11 (1) (1970) 19–63.
[28] B. Molnár, M. Benedict, J. Bertrand, Coherent states and the role of the affine group in the quantum mechanics of the Morse potential, J. Phys. A 34 (14) (2001) 3139.
[29] A. Papandreou-Suppappola, F. Hlawatsch, G.F. Boudreaux-Bartels, Quadratic time-frequency representations with scale covariance and generalized time-shift covariance: a unified framework for the affine, hyperbolic, and power classes, Digit. Signal Process. 8 (1) (1998) 3–48.
[30] J.C. Pool, Mathematical aspects of the Weyl correspondence, J. Math. Phys. 7 (1) (1966) 66–76.
[31] R.G. Shenoy, T.W. Parks, Wide-band ambiguity functions and affine Wigner distributions, Signal Process. 41 (3) (1995) 339–363.
[32] N.L. Vasilevski, On the structure of Bergman and poly-Bergman spaces, Integral Equ. Oper. Theory 33 (4) (1999) 471–488.
[33] E.P. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40 (5) (1932) 749–759.