Learning mathematics in a context of electrical engineering

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Abstract. This paper reports from an early phase of a project where first-year students on a programme in electronic systems learn mathematics in close contact with their engineering specialisation. Using concepts from the Anthropological Theory of the Didactic (ATD), the connection between mathematics and electrical engineering will be analysed based on concrete examples. On the basis of interviews with teachers in both fields, challenges and opportunities with teaching mathematics in an engineering context are described. The analysis reveals a complex interplay between mathematics and engineering, and the teachers emphasise division of labour as a crucial issue.

Keywords: mathematics for engineers, electrical engineering, ATD, praxeology

1. Introduction

Mathematics has always been regarded as an important subject for engineering students, and many different approaches to the teaching of mathematics for engineers can be identified. The traditional approach is to teach mathematics as part of a package of general courses, often over the first two years, assuming that this will provide the students with the necessary background to make use of the mathematics in engineering courses later (Winkelman, 2009). A critique towards this approach is that it may lead to mathematics being taught with a focus only on mathematical concepts and understanding and not on applications (Loch & Lamborn, 2016). Another critique, of a more general nature, can be connected to the challenges of transferring knowledge from one context to another (e.g. Evans, 2000). Acknowledging that knowledge is context dependent, one might argue that mathematics for engineering should be learnt within the engineering context where it is going to be used. And indeed, at many universities mathematics is taught in courses specially designed for particular engineering programmes (Alpers, 2008; Enelund et al., 2011; Klingbeil & Bourne, 2014). This model gives good opportunities for including programme specific problems in the mathematics teaching, and it is assumed that this will increase the perceived relevance of mathematics. However, this solution also raises some issues. Providing specialised mathematics courses for each study programme will be expensive if the university offers a large number of study programmes, and it may cause complications for students who wish to switch from one study programme to another. Another argument used in favour of general mathematics courses is that one of the strengths of mathematics is exactly the fact that it *is* general and that one of the competencies that students should acquire by studying mathematics is to adapt to new and unknown situations. There are, however, strong arguments for creating better connections between mathematics and the engineering subjects since many students find it challenging to apply mathematics they are supposed to have learned when they need it later in the engineering courses (Carvalho & Oliveira, 2018; Harris et al., 2015). There seems to be no obvious solution to these issues and therefore it is of interest to try out different models and study these models in practice.

In this paper I will report from an early phase of a project at the Norwegian University of Science and Technology, NTNU, where the aim is to redesign mathematics courses for engineering programmes. The project is given the acronym MARTA, and its full title would translate to English as *Mathematics as a Thinking Tool.* MARTA is so far restricted to one study programme, Electronic Systems Design and Innovation (ELSYS), but will later also include other engineering programmes. MARTA is part of a process aiming at redesigning all the technology programmes at NTNU, a process referred to as Technology Studies for the Future (Fremtidens teknologistudier, 2022). This paper is based on experiences from the first semester of the project MARTA, where a basic course in mathematics is taught in close connection with the course Electronic System Design and Analysis (ESDA) to first-year students. Using the Anthropological Theory of the Didactics (ATD), (e.g., Bosch & Gascón, 2014; Chevallard, 2006), I study the discourses that develop to see how the praxeologies in mathematics and engineering influence and interact with each other. I will inquire into the challenges and opportunities that arise at the interface between mathematics and electronics, as seen from the viewpoint of the teachers in the two subjects.

2. Background and context of the study

Engineering education has from early on experienced a tension between theory and practice, between academic and professional aims. Edström (2018) describes engineering education in the United States before 1920 as highly practical. After that time a change took place, influenced by European-educated engineers with a more mathematically oriented background. Edström writes that the development was slow, with some exceptions in "newer fields, such as chemical and electrical engineering, which grew from science disciplines" (2018, p. 40). The development got a boost after the Second World War. This is reflected in a report from a committee appointed to review the state of the education at Massachusetts Institute of Technology. In this report there are several warnings against a development of engineering education towards becoming too far separated from practice and also a critique against routine learning:

> [M]any students seem to be able to graduate from the Institute on the basis of routine learning, and … though fully equipped with knowledge of standard procedures …, they lack the critical judgement, the creative imagination, the competence in handling unique situations. (Lewis, 1949, pp. 28-29)

Further, it is emphasised in the report that it is important to "explore vigorously every means for confronting the student with basic data in genuine problem situations", and a belief is expressed that it is possible to find problems that "are simple enough to be used in the early years and complex enough to be challenging" and that "abstract concepts are best taught through their applications" (Lewis, 1949, p. 29). Edström (2018) remarks that many of the issues in the Lewis report are still valid today. The more specific question of what kind of mathematics should be taught to engineers also has a long history (Alpers, 2020, p. 5). First, this question addressed only the actual content of mathematics for engineers but later also issues about the connection between mathematics and engineering and who should be teaching mathematics to engineers were included (Ahmad et al., 2001; Bajpaj, 1985; Cardella, 2008).

Several recent studies show that the tension between usefulness and scholarliness, and the challenges with applying theory to practical engineering problems, still persists (Carvalho & Oliveira, 2018; Harris et al., 2015; Loch & Lamborn, 2016). The Conceive, Design, Implement, Operate (CDIO) Initiative, launched in 2000, addresses this issue. It is described as "an innovative educational framework for producing the next generation of engineers" (www.cdio.org). Further details are given below.

The CDIO approach has three overall goals: To educate students who are able to

1. Master a deeper working knowledge of technical fundamentals
2. Lead in the creation and operation of new products, processes, and systems
3. Understand the importance and strategic impact of research and technical development on society
   (Crawley et al., 2014, p. 13)

Crawley et al. (2014) emphasise that it is not memorisation of facts and definitions, nor the simple application of a principle that is important, but *conceptual understanding*, seen as ideas that have lasting value. In addition, the CDIO approach values *contextual learning.* This means, among other things, that new concepts should be presented in situations familiar to students and in situations they recognise as important to their current and future lives (Crawley et al., 2014, pp. 32-33). The CDIO approach involves combining ideas of learning in context and maintaining deep, or conceptual, understanding (Marton & Säljö, 1976). These ideas are in line with those presented by Scanlan in 1985 in a talk about mathematics in engineering education. In his talk Scanlan concluded by stating that mathematics should be an essential part of the students' formation and "not a set of 'tools' to be acquired before proceeding to the 'important' part of the course" (Scanlan, 1985, p. 449).

The project MARTA that I am reporting from, has as its main aim to create a closer connection between mathematics and engineering programmes, while maintaining conceptual understanding in both fields. An overarching goal for the project is to develop mathematics as a 'tool for thinking'. The programme Electronic Systems Design and Innovation (ELSYS) has been chosen as a pilot for MARTA. Other programmes will follow. ELSYS is one of 17 five-year Master of Technology programmes at NTNU, admitting approximately 1700 new students in total each year, approximately 100 in ELSYS. All these programmes traditionally contain four mathematics courses distributed over the first three semesters, with almost identical content for all programmes. MARTA represents a break with the traditional model. In MARTA, the idea is to make adaptations by shifting the emphasis on various topics as well as changing the sequencing of the topics, in order to better suit the needs of the engineering programmes. It is expected that this approach will make the students better see the relevance of mathematics for their engineering specialisation. The approach is in line with the idea of contextual learning from CDIO.

This paper is based on experiences from the first semester of the five-year programme, which is also the first semester of the project. Based on these experiences, my aim is to get a better understanding of the interplay between mathematics and topics from electrical engineering, which may be of value when developing the project further.

3. Theory and Methodology

Concepts from ATD will be used in the analysis. A central notion in ATD is the notion of *praxeology*, “the basic unit into which one can analyse human action at large” (Chevallard, 2006, p. 23). A praxeology is composed of two blocks, the praxis block, P, and the logos block L. P is seen as consisting of two parts, *types of tasks* (T) and a set of *techniques* (τ) to carry out the tasks. L also consists of two parts, a *technology* (θ), or justification for the techniques used to carry out the tasks, and the *theory* (Θ), which provides the basis and support for the technological discourse (Bosch & Gascón, 2014, p. 68). I will write $P = [T, \tau]$, $L = [\theta, \Theta]$, and $\mathscr{P} = [P/L] = [T, \tau, \theta, \Theta]$ for the whole praxeology. This is often referred to as the 4T-model.

A social situation is called a didactic situation

> whenever one of its actors (Y) does something to help a person (x) or a group of persons (X) learn something (indicated by a heart ♥). A *didactic system* $S(X; Y; ♥)$ is then formed. The thing that is to be learned is called a *didactic stake* ♥ and is made up of questions or praxeological components. (Bosch & Gascón, 2014, p. 71)

In my case X can be seen as made up of students at the ELSYS programme. Y is made up of two components, Y_M and Y_E, where Y_M consists of teachers and learning resources involved in the teaching and learning of mathematics to X, and Y_E consists of the corresponding components in the Electronic System Design and Analysis (ESDA) course.

The driving force in a praxeology is the desire for X to find answers (A) to questions (Q). The questions depend on the praxeology they emerge within. In the process of finding the answers, a didactical milieu, M, is developed, consisting of material and immaterial tools that X gathers, with the help of Y, in the process of inquiring into the question Q. This situation is represented with *the reduced Herbartian schema* $S(X; Y; Q) \hookrightarrow A$ (Chevallard, 2020). The milieu is seen as consisting of several components: existing answers (A_i) offered by other persons or institutions, works (W_j) of different kinds that can be accessed, and new questions

(Q_k) that may arise during the work: $M = \{A_1, A_2, \ldots, A_m, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p\}$ (Chevallard, 2020, p. 44).

Since there are separate courses in mathematics and electronic systems, there will also be separate didactic stakes, $\heartsuit_M \neq \heartsuit_E$. Hence, there are two didactic systems, $S(X;\, Y_M;\, \heartsuit_M)$ and $S(X;\, Y_E;\, \heartsuit_E)$, and two praxeologies, one for mathematics, $\mathscr{P}_M = [T_M, \tau_M, \theta_M, \Theta_M]$, and another for electronic systems, $\mathscr{P}_E = [T_E, \tau_E, \theta_E, \Theta_E]$. Learning in context should have as a consequence that the didactic stakes in the two praxeologies should overlap ($\heartsuit_M \cap \heartsuit_E \neq \emptyset$), and therefore I find it of interest to study the interplay between $\mathscr{P}_M$ and $\mathscr{P}_E$, within the didactic system $S(X;\, Y_E;\, \heartsuit_E)$. I focus on the system $S(X;\, Y_E;\, \heartsuit_E)$ since I consider $\mathscr{P}_E$ to be the central praxeology in ELSYS, with $\mathscr{P}_M$ playing a role as a "supporting praxeology" for $\mathscr{P}_E$. On the basis of selected questions from $\mathscr{P}_E$, I will identify elements of the milieu used to answer these questions. In particular, I will be looking for similarities and differences regarding technologies (θ) and techniques (τ) that are applied to solve a given task, coming from $\mathscr{P}_E$. The aim of this investigation is to answer the following question: In which ways can techniques and technologies from mathematics and electronic systems in combination contribute to finding answers to questions arising in $S(X;\, Y_E;\, \heartsuit_E)$?

4. Previous relevant research

One issue regarding mathematics in engineering education is to find the right balance between theory and practice. Flegg et al. (2012, p. 718) argue that "[w]ithout the explicit connection between theory and practice, the mathematical content of engineering programs may not be seen by students as relevant". They also claim that in cases where mathematics departments teach the mathematical content to the engineering students, the engineering departments may have little idea of what mathematical content the students are exposed to. Loch and Lamborn (2016) observed that first-year mathematics is often seen as irrelevant and distracting by engineering students, who are more interested in applied engineering subjects. This lack of relevance was attributed partly to mathematics being taught in a 'mathematical' way, "with a focus on mathematical concepts and understanding rather than applications" (Loch & Lamborn, 2016, p. 30). Loch and Lamborn report from a project where higher year engineering students were asked to create multimedia artefacts meant to show the relevance of mathematics. The project resulted in two animated videos showing how mathematics was used to plan and construct a building and a car. In interviews with first-year students after they had seen the videos, some students said that the videos did demonstrate the relevance of mathematics, and that “there is probably a reason we’re being taught what we’re being taught” (Loch & Lamborn, 2016, p. 38). However, students also reported that they found the videos overwhelming because of the amount of mathematics that was shown. Regarding the purpose of mathematics for engineers, Cardella (2008) claims that mathematics should be more than learning some specific topics. It is about learning a way of working and thinking that is of value for the work as an engineer. Faulkner et al. (2019) use the term “mathematical maturity” to cover what many teachers in engineering subjects hope that students learn from their mathematics coursework.

Booth (2004) discusses various approaches to learning mathematics by presenting a table of different strategies, with corresponding intentions and goals. These approaches constitute a hierarchy where the lowest level is made up of the strategy “Just learning” with the intention “To learn the content” and the goal “To know the content for use when needed”. The highest level is made up of the strategy “Studied reflection” with the intention “To be able to take different perspectives on problems” and “To relate content to the world outside of mathematics”. The goal is here formulated as “To be able to use mathematics to solve problems” and “To understand how mathematics applies to other situations” (Booth, 2004, p. 15). Scanlan, a professor of electrical engineering, warned against seeing mathematics for engineers just as a set of tools, but rather as an essential part of the students’ formation (Scanlan, 1985, p. 449). It could be argued that in order to be able to use mathematics in a meaningful way, e.g. in engineering, it is necessary to learn mathematics to the level of *studied reflection* (Booth, 2004). This could also be related to *mathematical maturity* (Faulkner et al., 2019).

Booth also argues that mathematics should not be taught by engineers but that “mathematicians and engineers could unite some of their courses so that the students experienced a team of teachers leading their learning of mathematics in the world of engineering they intend to enter” (Booth, 2004, p. 21). This is in line with the ideas of contextual teaching from CDIO (Crawley et al., 2014), and also with the ideas behind MARTA.

Gueudet and Quéré (2018) report that a gap can be observed between mathematics taught in mathematics courses and the way mathematics is used to solve problems in engineering courses. An important explanation that they give for this gap is that the mathematics courses do not make enough connections. As examples of relevant connections, the authors list links between mathematics and the real world, between different mathematical contents and between different representations (Gueudet & Quéré, 2018). Connections are also seen as important by Wolf and Biehler (2016) who present 10 examples of what they denote as *authentic problems* in mathematics for mechanical engineering. To secure connection, one of the basic principles that is presented, is that the problem should be authentic in the sense that it should not just be a dressed-up mathematical problem with unrealistic numbers (Wolf & Biehler, 2016).

Authentic problems are also discussed by Schmidt and Winsløw (2021), using the theory of didactic transposition (part of ATD). They create a model for what they call *Authentic Problems from Engineering*, defined as "a problem which comes from current research and innovation in some specific institution of scholarly engineering" (Schmidt & Winsløw, 2021, p. 266). In their paper, they present a model for task design, where tasks in the mathematics course are created, based on the problems from engineering. As an example, they present an assignment based on the problem to compute and control the magnetic field induced by a so-called Halbach magnet (Schmidt & Winsløw, 2021, p. 272).

Recently several researchers have shown how ATD can be a useful tool for investigating mathematics for engineering students, (e.g. González-Martín, 2021; González-Martín & Hernandes-Gomes, 2017, 2018, 2019; Peters et al., 2017). The main focus of González-Martín and Hernandes-Gomes is to compare presentations in Calculus textbooks with presentations in textbooks for professional engineering courses, to identify connections between the fields. Most of the examples presented by these authors are from mechanical engineering, but also a course in electricity and magnetism is studied (González-Martín, 2021). The results, in particular in mechanical engineering, indicate a lack of connection between the praxeologies. A similar analysis on the topic of Fourier series in mathematics and signal theory has been made by Rønning (2021). Also here, there are differences but it seems that signal theory makes more explicit use of results from mathematics than what may seem to be the case in mechanical engineering.

Summing up, it seems that there are two main challenges that are reported on. One is that students do not see the relevance of mathematics for their engineering profession. This in turn may reduce the motivation for mathematics, and perhaps also for the study as a whole, and may lead to drop-out (Faulkner et al., 2019). The second challenge is the lack of connection between mathematics and engineering subjects (e.g., Flegg et al., 2012; Gueudet & Quéré, 2018; Loch & Lamborn, 2016). Recently, some approaches to create connections have been presented (e.g., Schmidt & Winsløw, 2021; Wolf & Biehler, 2016). There seems to be agreement that it is important to develop problem solving abilities. This can be expressed as making mathematics a tool for thinking. And for this to happen, deep knowledge is required (e.g. Booth, 2004; Cardella, 2008; Crawley et al., 2014; Scanlan, 1985), as well as good problems.

5. Analysis of data

The question raised in this paper is the following: In which ways can techniques and technologies from $\mathscr{P}_M$ and $\mathscr{P}_E$ contribute to finding answers to questions arising in the didactic system $S(X; Y_E; \heartsuit_E)$? As data for the study, I used teaching material (problem sheets, lecture notes, textbooks, video lectures) from the ESDA course. With the video lectures (Lundheim, 2019) as the main source, supported by a textbook that was recommended for the students (Nilsson & Riedel, 2011), I performed an open coding of utterances as representing a *technique* or a *technology*. In each case, I also coded according to whether I saw the utterance as arising from $\mathscr{P}_M$ or from $\mathscr{P}_E$. To further strengthen my analysis, I conducted a joint interview with both the mathematics and the ESDA teacher after the end of the semester. The purpose of the interview was to get further insight into issues arising from studying the teaching material, as well as getting insight into the teachers' experiences from the first semester of the project. The interview was audio recorded and partly transcribed. From the teaching material I selected as my main example a situation with modelling an electric circuit (see Figure 1). This example provides the main question for the reduced Herbartian schema $S(X; Y; Q) \hookrightarrow A$ (see Section 5.1). In the interview, I inquired into the techniques and technologies behind the main example and I asked both teachers to formulate their ideas about learning and teaching in context, and to explicate their view on how the two subjects could mutually support each other. I intend to show some possibilities for making connections between mathematics and electrical engineering, and to show the interplay between the praxeologies $\mathscr{P}_M$ and $\mathscr{P}_E$ in making this connection. The analysis will show that knowledge from both praxeologies is needed to solve the given problem.

5.1. Example: An electric circuit

The electric circuit I will use as an example is illustrated in Figure 1. This, and similar circuits, are used frequently in the early phase of the ESDA course and can therefore be seen as an important basic example for the students at ELSYS. The circuit consists of two resistors, with resistance R_1 and R_2, a capacitor with capacitance C and an inductor with inductance L.

The problem is to determine the voltages v_1 and v_2 at the points A and B shown in Figure 1, given the input voltage $v(t)$.

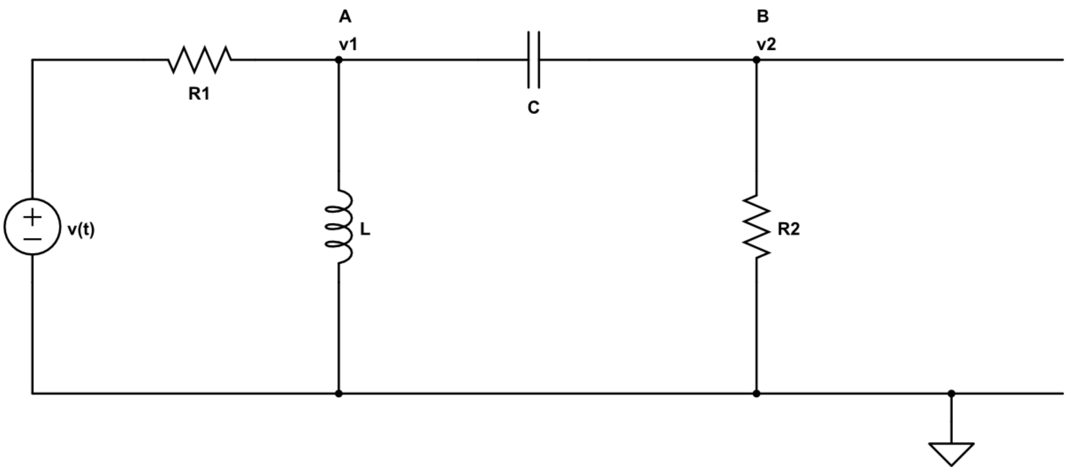

Figure 1 A circuit with a given input voltage and two unknown voltages

This is a problem from $\mathscr{P}_E$ where the question Q is to find the voltages v_1 and v_2. The expression for these voltages will be the answer A. I will discuss the reduced Herbartian schema $S(X; Y; Q) \hookrightarrow A$ for this problem by identifying elements of the didactical milieu coming both from $\mathscr{P}_E$ and from $\mathscr{P}_M$. My data for this discussion come from video lectures by Lundheim (2019) and a textbook on electric circuits (Nilsson & Riedel, 2011). Both these resources are central in the ESDA course, and hence in the didactic system $S(X; Y_E; ♥_E)$.

In the video lectures, two equations, (1) and (2), are presented, based on the currents at the points A and B, with voltages v_1 and v_2 respectively.

$$(1) \quad \frac{v_1 - v}{R_1} + \frac{1}{L}\int v_1(t)dt + C\frac{d}{dt}(v_1 - v_2) = 0$$

$$(2) \quad C\frac{d}{dt}(v_2 - v_1) + \frac{v_2}{R_2} = 0$$

Equation (1) models the current out of the node at the point A (v_1). The left-hand side of (1) contains three terms, one for the resistor, one for the inductor and one for the capacitor. I will look at how each of these terms are justified in Lundheim's (2019) presentation. For each justification, I will indicate, either by θ_E or by θ_M which praxeology I interpret the justification to be based on. The first term is justified by saying that "this is just regular circuit analysis", i.e. Ohm's law is used: The current through the resistor is proportional to the voltage over the resistor (θ_E). The two other terms are more interesting. For the second term, it is said that "the current through an inductor is proportional to the integral of the voltage over the inductor" (θ_E), and for the third term that "the current through a capacitor is proportional to the derivative of the voltage over the capacitor" (θ_E). Furthermore, the principle used is what is known as Kirchhoff's law of currents, stating that the sum of the currents out of the node at v_1 is zero (θ_E). Equation (2) is obtained in a similar way by analysing the current going out of the node at the point B (v_2). Now Lundheim observes that equation (2) is a first order differential equation whereas equation (1) contains terms including both an integral and a derivative (an integro-differential equation). To transform this to a "pure differential equation" he takes the derivative with respect to time on both sides of (1) to obtain (τ_M)

$$(1') \quad \frac{d}{dt}\frac{1}{R_1}(v_1 - v) + \frac{1}{L}v_1(t) + C\frac{d^2}{dt^2}(v_1 - v_2) = 0$$

Lundheim now observes that a system of differential equations, (1') and (2), has been obtained and that in principle this system can be solved (within $\mathscr{P}_M$). He says that he finds this to be complicated, and therefore he will look for an alternative way to find the answer A to the question Q. This "alternative way" is based on the assumption that the input signal (v) is sinusoidal. This is a reasonable assumption in $\mathscr{P}_E$, but in $\mathscr{P}_M$ it would probably be seen as a (very) special case.

The following reasoning is presented. For a given trigonometric signal $x(t) = A\cos(\omega t + \varphi)$, define its complex form $X(t) = Ae^{j(\omega t+\varphi)} = Ae^{j\varphi}e^{j\omega t}$. Then $x(t) = \operatorname{Re} X(t)$. The complex number $Ae^{j\varphi}$ is called the *phasor* or the *complex amplitude* of the signal[1]. An important point made is that $\frac{d}{dt}X(t) = j\omega X(t)$ and $\int X(t)dt = \frac{1}{j\omega}X(t)$. Although this technique is purely mathematical, it would rarely be seen as a technique for differentiating and integrating in $\mathscr{P}_M$, since it would apply only to a very limited choice of functions. These functions, however, play a very important role in $\mathscr{P}_E$ and therefore it makes sense to introduce this technique.

Applying this technique to the system of equations (1) and (2) and replacing the voltages v with their complex form V, the following system of algebraic equations is obtained.

$$(3) \quad \frac{V_1 - V}{R_1} + \frac{1}{Li\omega}V_1 + Ci\omega(V_1 - V_2) = 0$$

$$(4) \quad Ci\omega(V_2 - V_1) + \frac{V_2}{R_2} = 0$$

[1] j is used for the imaginary unit, in accordance with the tradition in $\mathscr{P}_E$.

Solving this system for V_1 and V_2 the unknown voltages v_1 and v_2 are obtained by taking the real part. The given task T belongs to $\mathscr{P}_E$ but the techniques and technologies belong to $\mathscr{P}_M$ (properties of complex numbers). However, the techniques, although purely mathematical, would not have been given such a prominent role in a mathematical praxeology. This shows that the choice of technique may depend on the praxeology: A technique (τ_M) from $\mathscr{P}_M$ is considered more important because it is used in $\mathscr{P}_E$ compared to if it had been used in $\mathscr{P}_M$.

I now return to the modelling process resulting in the system of the integro-differential equation (1) and the differential equation (2), to take a closer look at the justifications for setting up these equations. Of particular interest are the terms $\frac{1}{L}\int v_1(t)dt$ and $C\frac{d}{dt}(v_1 - v_2)$ in (1). For the term with the integral (the inductor), the principle used is that the current through an inductor is proportional to the integral of the voltage. For the term with the derivative (the capacitor), it is claimed that the current through a capacitor is proportional to the derivative of the voltage. These are technologies (θ_E) from $\mathscr{P}_E$ leading to the application of techniques (τ_M) from $\mathscr{P}_M$.

Concerning the capacitor, Nilsson and Riedel (2011) write:

> [A]pplying a voltage to the terminals of the capacitor … can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the **displacement current**. At the terminals, the displacement current is indistinguishable from the conduction current. The current is proportional to the rate at which the voltage across the capacitor varies with time. (p. 204)

This technology (θ_E) gives the relation $I = C\frac{dv}{dt}$. For the inductor, Nilsson and Riedel just state that the following relation holds, $v = L\frac{dI}{dt}$ (Eq. 6.1, p. 198). Then they state: "Note from Eq. 6.1 that the voltage across the terminals of an inductor is proportional to the time rate of change of the current in the inductor" (p. 198). Hence, they give a mathematical interpretation of a relation between electrotechnical quantities, without justifying why this particular relation, $v = L\frac{dI}{dt}$, holds. Accepting this, again by mathematical techniques (τ_M), one gets $I = \frac{1}{L}\int v(t)dt$.

For the circuit in Figure 1 I expressed the question Q as determining the voltages v_1 and v_2 at the points A and B, given an input voltage v. The answer A in $S(X; Y; Q) \hookrightarrow A$ contains the values of the unknown voltages. In search of this answer a didactical milieu M was generated,

$M = \{A_1, A_2, \ldots, A_m, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p\}$, consisting of partial answers, A_i, *works* (results), W_j, and new questions Q_k (sub questions), used to find the answer A to the original question Q. Some of these components are formulated within $\mathscr{P}_M$ and some within $\mathscr{P}_E$. Table 1 shows the didactical milieu associated with the electric circuit in Figure 1.

Table 1. The didactical milieu for the electric circuit in Figure 1

	$\mathscr{P}_M$	$\mathscr{P}_E$
Main question, Q		• Determine the voltages v_1 and v_2 given an input voltage v.
Sub-questions, Q_j	• How to solve a system of differential equations • How to solve a system of algebraic equations	• Model the current flow at the nodes v_1 and v_2
Works, W_k	• Transforming equation (1) to equation (1') • Solving a system of algebraic equations • Properties of complex numbers	• Kirchoff's current law • Behaviour of current over resistors, capacitors and inductors • Properties of the phasor
Partial answers, A_i	• The equations (1), (1') and (2) • The equations (3) and (4) • Solution of the system of equations (3) and (4)	
Main answer, A		• The values v_1 and v_2

Table 1 shows the interplay between the praxeologies $\mathscr{P}_M$ and $\mathscr{P}_E$ for the given problem. Although both Q and A belong to $\mathscr{P}_E$ the didactical milieu also includes questions and answers of a purely mathematical character, and the process of finding the values v_1 and v_2 draws on works from $\mathscr{P}_M$. However, works from $\mathscr{P}_M$ are not sufficient. In order to model the current flow, justifications from $\mathscr{P}_E$ are needed to formulate the system of equations (1) and (2). I find the behaviour of current over capacitors and inductors to be of particular interest. Why can this be modelled with derivatives and integrals as shown in equations (1) and (2)? I will return to this question in Section 5.2. I have previously pointed out that a key word pertaining to the challenge of teaching mathematics for engineers is *connections* (e.g. Gueudet & Quéré, 2018). The analysis resulting in Table 1 shows how the didactical milieu involved in solving the problem with the circuit consists of elements from both praxeologies $\mathscr{P}_M$ and

$\mathscr{P}_E$ and that both praxeologies are essential in the path leading to the solution of the problem. In the next section I will look into how the project MARTA creates opportunities for connections, as well as going deeper into some of the justifications given in the analysis of the circuit.

5.2. Opportunities for connections

The example described in Section 5.1 comes from $\mathscr{P}_E$, but the analysis shows that elements from both $\mathscr{P}_E$ and $\mathscr{P}_M$ are used to solve the problem (Table 1). Therefore, it is necessary that the students have some knowledge from mathematics in order to make sense of what is going on. This is in itself nothing special, so to see what extra can be gained by teaching mathematics and electronics in close connection, I interviewed the teachers Marc, who was teaching the mathematics course, and Eric, who was teaching the ESDA course to the same students in their first semester. When asked about the main differences in the current approach compared to a traditional approach, Marc emphasises that in addition to changing the sequencing of topics, he tries to include circuits into mathematics as often as possible. He continues: "But I don't know the electronics and it is difficult to find circuits that give good mathematical problems. Then I have to ask Eric or look in a textbook". Here Eric comments that a crucial point is *division of labour*. "I think that mathematics must live on its own premises, and that the learning goals in mathematics must be mathematical. We cannot make plans that presuppose that the mathematicians know a lot of electronics. The most important is continuous communication."

Marc gives an example of a circuit which is modelled by a non-linear differential equation. The mathematical purpose of this example was to motivate the introduction of numerical methods for solving differential equations, and Marc felt that the students thought it was fun. The problem was given as solving the differential equation using Euler's explicit and implicit methods, and it was just claimed that the differential equation would model the given circuit. The purpose of this example was purely mathematical, namely to introduce Euler's methods. This could have been done without connection to the electrical circuit, but the circuit worked as a link between the praxeologies, perhaps contributing to the students seeing increased relevance.

Below is a dialogue following another of Marc's examples.

Eric The mathematicians have a habit of setting all values of the components equal to one, because then it gets much tidier. With this, the physics disappear.

Marc I defend this based on the principle of division of labour. The mathematical principles are easier to comprehend if you leave out the physical constants.

Eric Then you are left with the structure of the problem. I think this is the kind of division of labour we should have. We can "dress the problem up" later.

Marc I think it is a good pedagogical trick to clean away the mess when you learn something for the first time.

This dialogue shows a fundamental difference between the praxeologies. In engineering one is concerned with units and with physical constants that are important for understanding the physical principles. In mathematics, however, one is more concerned with the structure, and this structure may come better to the fore if for example (non-zero) constants are set equal to one. Based on the principle of division of labour, both teachers find that this difference is not problematic, but on the contrary, that it can be an asset.

One issue that Eric finds particularly important is that the close connection to mathematics gives the possibility to justify principles from electrical engineering better. As an example, he mentions *the principle of superposition*. This is explained in the following way by Nilsson and Riedel:

> A linear system obeys the principle of **superposition,** which states that whenever a system is excited, or driven, by more than one independent source of energy, the total response is the sum of the individual responses. (Nilsson & Riedel, 2011, p. 144).

The strategy chosen in the book by Nilsson and Riedel is to deactivate all sources of energy but one, and study the system that is then created (τ_E). Solving for the currents in each of the circuits with just one source of energy, it is claimed, with reference to the principle of superposition (θ_E), that the complete solution is obtained by adding the currents. Eric finds this argument unsatisfactory, and he is happy that he can use mathematical arguments to justify the principle. Eric says: "I was always told that, 'this is how it *is*'. Now we can argue that this is actually how it *has* to be". The mathematical justification of the superposition principle is based on linear algebra. Each of the circuits with only one source of energy can be modelled with a system of linear equations $Ax_i = b_i$, $i = 1, \ldots, n,$ where n is the number of energy sources. The complete circuit can then be modelled by $Ax = b,$ where $b = b_1 + \ldots + b_n$.

Since A is a linear operator, the complete solution is given by $x = x_1 + \ldots + x_n$ (θ_M). This is an example that a technology from mathematics is used to justify a technique in electrical engineering.

In Section 5.1 it would appear both from Lundheim (2019) and from Nilsson and Riedel (2011) that the justification for the modelling of the circuit shown in Figure 1 was somewhat unsatisfactory. I therefore asked Eric in the interview how he would justify the modelling of equation (1). Regarding the capacitor, Eric says:

> Current is the derivative of charge with respect to time. How much charge passes through a crosscut per unit of time. The number C indicates how much charge a capacitor can hold. Q=CV, so I = dQ/dt = C dV/dt.

In the justification he bases his argument on the definition of current, as the rate of change of charge with respect to time (θ_E). And the charge that a capacitor can hold is proportional to the voltage, where the proportionality constant C is a characteristic of the capacitor. This is in line with the argument given in Nilsson and Riedel (2011) that "[t]he current is proportional to the rate at which the voltage across the capacitor varies with time" (p. 204).

I observed in Section 5.1 that the argument in Nilsson and Riedel (2011) for the behaviour of the inductor was rather vague. Below is the explanation provided by Eric in the interview.

> For the inductor it is more tricky. You cannot use the concept of charge. You need flux, which is physically much heavier. So here I often use some analogies, e.g. analogy with mass. Imagine you will push a car. It is heavy in the beginning but as the car starts to roll, you need less and less force and finally the car rolls by itself. Mass as inertia. An inductor functions as inertia for the current. In the beginning high voltage is needed to get the current going, but as the current starts to flow, the voltage goes down. So an inductor exerts inertia towards changes in current. If you want a quick change in the current you need high voltage. When the current evens out, the voltage goes down. When the current is zero, the inductor works as a short circuit.

The justification he gives is in the form of an analogy, thinking of the inductor as an element that resists change, like mass at rest. The crucial formulation here is "[i]f you want a quick change in the current you need high voltage". This means that to get a large value of $\frac{dI}{dt}$, v needs to be large, motivating the relation $v = L\frac{dI}{dt}$. This example shows that Eric draws on yet another praxeology for his justification, by comparing with pushing a car. This he does because the justification within $\mathscr{P}_E$ (using flux) would not be accessible for the students at this point. Then, using a mathematical technique (τ_M), the relation can be written as $I = \frac{1}{L}\int v(t)dt$, as in equation (1).

Although recognising the value of the interplay between the two praxeologies, the teachers argue that they also, to some extent, should be kept apart. This is expressed using the expression *division of labour*. It is the role of $\mathscr{P}_M$ to work with the *structure* of a problem, and the role of $\mathscr{P}_E$ to see the problem, and its solution, in an engineering context.

6. Discussion

In the literature, there are some particular challenges that are frequently mentioned: lack of relevance of mathematics for engineers, lack of connections between mathematics and engineering, and challenges with applying mathematics to engineering problems (e.g., Carvalho & Oliviera, 2018; Flegg et al., 2012; Gueudet & Quéré, 2018; Harris et al., 2015; Loch & Lamborn, 2016). There is also criticism against mathematics being taught too "mathematically" (Loch & Lamborn, 2016). However, there is evidence to support that there is a need for a deep knowledge of mathematics, to avoid mathematics becoming just a set of tools (e.g., Booth, 2004; Cardella, 2008; Crawley et al., 2014; Scanlan, 1985).

An intention with the project MARTA is to teach mathematics and engineering in close connection, with much of the mathematics contextualised through problems and examples from engineering, in line with the ideas of the CDIO approach (Crawley et al., 2014). An overarching goal is to develop mathematics as a way of thinking (Cardella, 2008; Faulkner et al., 2019) and obtaining deep learning, both in mathematics and in the engineering subject (Crawley et al., 2014; Marton & Säljö, 1976; Scanlan; 1985).

With the above principles as a background, I performed a praxeological analysis of an example from $\mathscr{P}_E$ in order to investigate how techniques and technologies from $\mathscr{P}_E$ and $\mathscr{P}_M$ in combination contribute to finding answers to questions arising in the didactic system $S(X; Y_E;$ ♥$_E)$. The analysis shows that applications of mathematics in electrical engineering involve a complex interplay between the praxeologies to establish a functional didactical milieu. Techniques and technologies from two praxeologies are intertwined and although both the problem and the answer lie within $\mathscr{P}_E$, it is necessary to use elements from $\mathscr{P}_M$ to get to the answer. This interplay between the praxeologies I see as evidence that deep knowledge in both fields is necessary. Not only techniques, but also technologies (justifications) from $\mathscr{P}_M$

are necessary, so using mathematics just as "a set of 'tools'" (Scanlan, 1985, p. 449) will not suffice. A certain degree of "mathematical maturity" (Faulkner et al., 2019) is needed to master the interplay between the praxeologies.

I also identified some issues that are seen as important from the viewpoint both of the mathematics teacher and of the electronic systems teacher. Their main message is that of division of labour. They recognise that they enter the work with the students with different competencies. They work closely together but the mathematics teacher says that "I don't know the electronics" and he admits that he finds it difficult to find examples from electrical engineering that give good mathematical problems. The electronic systems teacher says that "I think that mathematics must live on its own premises", and he recognises the value of mathematics for example to see the structure behind a method. It will be of great interest in the further work with the project to get information from the students, both in surveys reflecting their perceptions of the collaboration between the fields, and in direct observation of students working on problems. Another issue is to see the effect of including other study programmes into the project. It will not be sustainable to have specially designed mathematics courses for each study programme, so an important line of inquiry will be to study the interplay between the praxeology $\mathscr{P}_M$ and a given praxeology $\mathscr{P}_Z$, where Z represents an engineering field, for various choices of Z.

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