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THE STABILITY THRESHOLD AND DIOPHANTINE APPROXIMATION

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ABSTRACT. The purpose of this paper is to use the filtration that appeared in Ru and Vojta [Amer. J. Math. 142 (2020), pp. 957-991] to extend the result of Blum-Jonsson [Adv. Math. 365 (2020), p. 57], as well as to explore some connections between the notion of the K-stability and Diophantine approximation, especially the β-constant and the Ru-Vojta's theorem.

1. INTRODUCTION

The notion of the *K-stability* of Fano varieties is an algebro-geometric stability condition originally motivated by studies of Kähler metrics. When the base field is the complex number field, it was recently established that the existence of positive scalar curvature Kähler-Einstein metric is actually equivalent to the K-stability condition, by the works of [Tia97], [Don02], [B16], and others, including the recent celebrated result [CDS15a], [CDS15c]. This equivalence had been known before as the Yau-Tian-Donaldson conjecture (for the case of Fano varieties).

The original notion of K-stability in [Tia97], [Don02] is defined in terms of the sign of the generalised Futaki invariant on all test configurations or at least on some special test configurations (see [LX14]). Recently, there has been tremendous progress in reinterpreting K-stability in terms of invariants associated to valuations rather than test configurations. More specifically, in [BHJ17], the data of a test configuration was translated into the data of a filtration and it was shown that a nontrivial special test configuration yields a divisorial valuation. In 2016, K. Fujita [Fuj16] introduced *divisorial stability*: Let X be a $\mathbb{Q}$-*fano variety*, i.e., a projective variety over the complex number field which has at worst klt singularities such that the anticanonical divisor $-K_X$ of X is ample (as $\mathbb{Q}$-divisor). The pair $(X, -K_X)$ is said to be *divisorially stable (resp. semi-)* if the value

$$\eta(D) := \mathrm{Vol}(-K_X) - \int_0^\infty \mathrm{Vol}(-K_X - tD)dt$$

satisfies $\eta(D) > 0$ (resp. $\eta(D) \geq 0$) for any nonzero divisor D on X. Fujita [Fuj16] showed that if X is K-(semi) stable, then it is divisorially (semi) stable. Later, based on the work of [LX14], K. Fujita [Fuj19] and C. Li [Li17] independently proved that

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the K-(semi) stability and divisorial (semi) stability are indeed equivalent if one goes to the birational model and modifies the constant to

$$\tilde{\eta}(E) := A_X(E)\operatorname{Vol}(-K_X) - \int_0^\infty \operatorname{Vol}(-K_X - tE)dt$$

for any prime divisors E over X, i.e. they are prime divisors on a birational model $\pi : Y \to X$, where $A_X(E) := 1 + \operatorname{ord}_E(K_{Y/X})$ is the *log discrepancy*. Namely, *For $\mathbb{Q}$-Fano X, X is K-stable (resp. semi-) if and only if $\tilde{\eta}(E) > 0$ (resp. $\tilde{\eta}(E) \geq 0$) for any prime divisors E over X.* If we denote, for a line bundle L and a Cartier divisor D,

$$\beta(L, D) := \frac{1}{\operatorname{Vol}(L)} \int_0^\infty \operatorname{Vol}(L - tD)dt, \tag{1}$$

then it states: *For $\mathbb{Q}$-Fano X, X is K-stable (resp. semi-) if and only if $\frac{A_X(E)}{\beta(-K_X,E)} > 1$ (resp. ≥ 1) for any prime divisors E over X.* In [BJ20] (see also [FO18]), Blum-Jonsson introduced the *stability threshold* $\delta(L) = \inf_E \frac{A_X(E)}{\beta(L,E)} = \lim_{m\to\infty} \delta_m(L)$, where $\delta_m(L) := \inf\{\operatorname{lct}(D) \mid D \sim_{\mathbb{Q}} L$ of m-basis type$\}$. The result is re-formulated as follows: *For $\mathbb{Q}$-Fano X, X is K-stable (resp. semi-) if and only if $\delta(-K_X) > 1$ (resp. $\delta(-K_X) \geq 1$).*

The β-constant $\beta(L, D)$ defined in (1) played an important role in Diophantine approximation (see [MR15], [RV20]). In particular, Ru-Vojta [RV20] proved the following result, which is viewed as an extension of Schmidt's subspace theorem (for notations, see [RV20]).

Theorem A ([RV20])**.** *Let X be a projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors, both defined over a number field k. Assume that $D_1, \ldots, D_q$ intersect properly on X. Let $S \subset M_k$ be a finite set of places on k. Let L be a big line sheaf on X. Then, for every $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{j=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq (1 + \epsilon) h_L(x) \tag{2}$$

holds for all $x \in X(k)$ outside of Z.

The proof of Theorem A uses the *m-basis type* divisor chosen from a filtration which is similar to, but more sophisticated than, the filtration used in the paper of Blum-Jonsson [BJ20]. This filtration used in [RV20] is multi-variable which allows us to deal with the divisors $D_1 + \cdots + D_q$, where $D_1, \ldots, D_q$ are in general position, rather than a single divisor in the case of Blum-Jonsson [BJ20]. The purpose of this paper is to use this filtration to extend the result of Blum-Jonsson [BJ20], as well as to explore some connections among these areas. In the last section, we explore the relation between the constant $\beta(L, D)$, the Seshadri constant $\epsilon(L, D)$ and the pseudo-effective constant $T(L, D)$, and use the relation to derive some corollaries of Theorem A.

2. The Okounkov body and the β-constant

We work on the field $\mathbb{C}$ although the results hold for any algebraically closed field with characteristic zero. Throughout the paper, we use X to denote a normal projective variety of dimension n.

The β-constant. Let L be a big line bundle (we also regard L as a line sheaf or a Cartier divisor) and let D be a nonzero effective Cartier divisor on X. In [RV20] (see also [MR15]), the following constant was introduced

$$\beta(L,D) = \liminf_{m\to\infty} \frac{\sum_{j\geq 1} \dim H^0(X, mL - jD)}{m \dim H^0(X, mL)}. \tag{3}$$

The β-constant appeared in Theorem A above.

The volume function. The volume of L is defined by

$$\mathrm{Vol}(L) = \limsup_{m\to\infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

Notice that $\mathrm{Vol}(kL) = k^n \mathrm{Vol}(L)$ so the volume function can be extended to $\mathbb{Q}$-divisors. Also note that the volume function depends only on the numerical class of L, so it is defined on $\mathrm{NS}(X) := \mathrm{Div}(X)/\mathrm{Num}(X)$ and extends to a continuous function on $\mathrm{NS}(X)_{\mathbb{R}}$. By using the theory of Okounkov bodies described below, one can prove (see Theorem 2.5) that the lim inf in (3) is indeed a limit when L is big, and that $\beta(L,D)$ can be expressed through the notion of volume function as in (1).

Okounkov bodies of a graded linear series of L. An Okounkov body $\Delta(L) \subset \mathbb{R}^n$ (where $n = \dim X$) is a compact convex set designed to study the asymptotic behavior of $H^0(X, mL)$, as $m \to \infty$. They have the crucial property that $\mathrm{Vol}(\Delta) = \lim_{m\to\infty} \frac{\dim H^0(X,mL)}{m^n} = \frac{\mathrm{Vol(L)}}{n!}$. More generally, one can also attach to a graded linear series of L, i.e. $V_\bullet = \bigoplus_m V_m \subset \bigoplus_m H^0(X, mL)$, a convex body $\Delta(V_\bullet) \subset \mathbb{R}^n$ such that

$$\mathrm{Vol}(\Delta(V_\bullet)) = \lim_{m\to\infty} \frac{\dim V_m}{m^n}.$$

Here is the detailed description. Let L be a big line bundle on X. Fix a system $z = (z_1, \ldots, z_n)$ of parameters centered at a regular closed point ξ of X. It gives a rank-n valuation

$$\mathrm{ord}_z : \mathcal{O}_{X,\xi} \backslash \{0\} \to \mathbb{N}^n$$

centered at ξ as follows: expand $f \in \mathcal{O}_{X,\xi}$ as a power series

$$f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$$

and set

$$\mathrm{ord}_z(f) = \min_{lex}\{\alpha \in \mathbb{N}^n \mid a_\alpha \neq 0\},$$

where the minimum is taken with respect to the lexicographic order on $\mathbb{N}^n$. This extends to holomorphic section $s \in H^0(X, L)$ with the basic property that each graded piece has

$$\dim(\{s \in W, \mathrm{ord}_z(s) \geq_{lex} \alpha\}/\{s \in W, \mathrm{ord}_z(s) >_{lex} \alpha\}) \leq 1 \tag{4}$$

for each subspace $W \subset H^0(X, L)$. Indeed, given s_1, s_2 with $\mathrm{ord}_z(s_1) = \mathrm{ord}_z(s_2) = \alpha$, we have $s_j = c_j z^\alpha +$ (high order terms), and it immediately follows that s_1, s_2 are linearly independent modulo $\{\mathrm{ord}_z > \alpha\}$ (see also Lemma 1.3 in [LM09]). Note that (4) implies in particular that $\#(\mathrm{ord}_z(W \setminus \{0\})) = \dim W$.

Let $V_\bullet = \bigoplus_m V_m \subset \bigoplus_m H^0(X, mL)$ be a nonzero graded linear series. For $m \in \mathbb{N}$, by (4), the subset $\Gamma_m = \Gamma_m(V_\bullet) := \mathrm{ord}_z(V_m \setminus \{0\})$ has cardinality $\dim V_m$. One associates to $V_\bullet$ a semigroup

$$\Gamma(V_\bullet) := \left\{(m, \alpha) \in \mathbb{N}^{n+1} \mid \alpha \in \Gamma_m\right\}.$$

Let $\Sigma = \Sigma(V_\bullet) \subset \mathbb{R}^{n+1}$ be the closed convex cone generated by $\Gamma(V_\bullet)$. The *Okounkov body* of V with respect to z is given by

$$\Delta = \Delta_z(V_\bullet) = \{\alpha \in \mathbb{R}^n | (1, \alpha) \in \Sigma\}.$$

This is a compact convex subset of $\mathbb{R}^n$.

Remark. The Okounkov body of V depends on the choice of the system of parameters z. But the properties we are concerned about are independent of z.

For $m \geq 1$, let ρ_m be the atomic positive measure (called the *Duistermaat-Heckman measure*) on Δ given by

$$\rho_m = m^{-n} \sum_{\alpha \in \Gamma_m} \delta_{m^{-1}\alpha}$$

The following result is a special case of Theorem 1.12 in [Bo14].

Theorem 2.1 ([Bo14], Theorem 1.12)**.** *Assume that $V_\bullet$ contains an ample series, i.e. $L = A + E$ (as $\mathbb{Q}$-divisor) with A being $\mathbb{Q}$-ample and E being effective such that $H^0(X, mA) \subset V_m \subset H^0(X, mL)$. Then its Okounkov body $\Delta \subset \mathbb{R}^n$ has nonempty interior, and we have $\lim_{m\to\infty} \rho_m = \rho$ in the weak topology of measures, where ρ denotes the Lebesgue measure on $\Delta \subset \mathbb{R}^n$. In particular, the limit*

$$\mathrm{Vol}(V_\bullet) := \lim_{m\to\infty} \frac{n!}{m^n} \dim V_m \in (0, \mathrm{Vol}(L)] \tag{5}$$

exists, and equals $n! \,\mathrm{Vol}(\Delta)$.

Filtrations. We apply the above results to a special graded linear series $V_\bullet$ which is associated to a filtration $\mathcal{F}$. By a *filtration* $\mathcal{F}$ on $R(X, L) := \bigoplus_m R_m$ we mean a family $\mathcal{F}^\lambda R_m \subset R_m$ of $\mathbb{C}$-vector subspaces of R_m for $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$, satisfying

- (F1) $\mathcal{F}^\lambda R_m \subset \mathcal{F}^{\lambda'} R_m$ when $\lambda \geq \lambda'$;
- (F2) $\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$ for $\lambda > 0$;
- (F3) $\mathcal{F}^0 R_m = R_m$ and $\mathcal{F}^\lambda R_m = 0$ for $\lambda \gg 0$;
- (F4) $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.

A simple example of a filtration is given by

$$\mathcal{F}^\lambda R_m := H^0(mL - \lambda D), \tag{6}$$

where D is an effective Cartier divisor on X. Here we use the following convention: for $\lambda \in \mathbb{R}^+$ and $j \in \mathbb{N}$ with $j \leq \lambda < j+1$, we set $H^0(mL - \lambda D) = H^0(mL - jD)$.

A filtration $\mathcal{F}$ on $R(X, L)$ defines a family

$$V_\bullet^t = V_\bullet^{\mathcal{F},t} = \bigoplus_m V_m^t \tag{7}$$

of graded linear series of L, indexed by t, given by $V_m^t := \mathcal{F}^{mt} R_m$ for $m \in \mathbb{N}$.

Set

$$T_m := T_m(\mathcal{F}) := \sup\left\{t \geq 0 \mid V_m^t \neq 0\right\} \tag{8}$$

with the convention $T_m = 0$ if $R_m = 0$. By (F4) above,

$$T_{m+m'} \geq \frac{m}{m+m'} T_m + \frac{m'}{m+m'} T_{m'}.$$

so Fekete's Lemma implies that the limit

$$T(\mathcal{F}) := \lim_{m\to\infty} T_m(\mathcal{F}) \in [0, +\infty] \tag{9}$$

exists, and equals $\sup_m T_m(\mathcal{F})$. Hence

$$T(\mathcal{F}) = \sup\left\{t \geq 0 | \operatorname{Vol}\left(V_\bullet^t\right) > 0\right\}, \tag{10}$$

because $V_\bullet^t$ contains an ample linear series for any $t < T(\mathcal{F})$ (see [BC11, Lemma 1.6]), and hence, by using Theorem 2.1, $\operatorname{Vol}(V_\bullet^t) = \lim_{m\to\infty} \frac{n!}{m^n} \dim V_m^t$. We say that the filtration $\mathcal{F}$ is *linearly bounded* if $T(\mathcal{F}) < \infty$.

Let $\Delta = \Delta(L) \subset \mathbb{R}^n$ be the Okounkov body of $R(X, L)$. The filtration $\mathcal{F}$ of (R, L) induces a *concave transform*

$$G = G^{\mathcal{F}} : \Delta \to \mathbb{R}_+ \tag{11}$$

which is given by $G(\alpha) = \sup\{t \in \mathbb{R}_+ | \alpha \in \Delta^t\}$, where $\Delta^t = \Delta(V_\bullet^t) \subset \mathbb{R}^n$ is the Okounkov body associated to the graded linear series $V_\bullet^t \subset R(X, L)$. Note that, for $t' > t \geq 0$, we have $\Delta^t \supset \Delta^{t'}$, and $\Delta^0 = \Delta$ and $\Delta^t = \emptyset$ for $t > T(\mathcal{F})$. It is easy to see that $\{G \geq t\} = \Delta^t$ for $0 < t < T(\mathcal{F})$. Thus G is a concave, upper semicontinuous function on Δ with values in $[0, T(\mathcal{F})]$. As noted in the proof of [[BKMS15], Lemma 2.22], the Brunn-Minkowski inequality implies:

Proposition 2.2. *The function $t \mapsto \operatorname{Vol}(V_\bullet^t)$ is non-increasing and concave on $[0, T(\mathcal{F})]$. As a consequence, it is continuous on $\mathbb{R}_+$, except possibly at $t = T(\mathcal{F})$.*

We define the limit measure $\mu = \mu^{\mathcal{F}}$ of the filtration $\mathcal{F}$ as the pushforward

$$\mu = G_* \rho.$$

Thus μ is a positive measure on $\mathbb{R}_+$ of mass $\operatorname{Vol}(\Delta) = \frac{1}{n!}\operatorname{Vol}(L)$ with support in $[0, T(\mathcal{F})]$. Theorem 2.1 thus gives Corollary 2.3.

Corollary 2.3 (Corollary 2.4 in [BJ20])**.** *The limit measure μ satisfies*

$$\mu = -\frac{1}{n!}\frac{d}{dt}\operatorname{Vol}\left(V_\bullet^t\right) = -\frac{d}{dt}\operatorname{Vol}\left(\Delta^t\right) \tag{12}$$

and is absolutely continuous with respect to Lebesgue measure, except possibly at $t = T(\mathcal{F})$, where $\mu\{T(\mathcal{F})\} = \lim_{t\to T(\mathcal{F})-}\operatorname{Vol}\left(V_\bullet^t\right)$.

Let $N_m := \dim H^0(X, mL)$, and let $M(L)$ be the set of $m \in \mathbb{N}$ for which $N_m > 0$. Given a filtration $\mathcal{F}$, consider the *jumping numbers*

$$0 \leq a_{m,1} \leq \cdots \leq a_{m,N_m} = mT_m(\mathcal{F}),$$

defined by, for $m \in M(L)$,

$$a_{m,j} = a^{\mathcal{F}}_{m,j} = \inf\{\lambda \in \mathbb{R}_+ \mid \operatorname{codim}\mathcal{F}^\lambda R_m \geq j\}$$

for $1 \leq j \leq N_m$. Note that the non-increasing step functions $t \mapsto \dim \mathcal{F}^t R_m$ satisfy the condition that $\operatorname{codim}\mathcal{F}^t R_m = j$ if and only if $t \in (a_{m,j-1}, a_{m,j}]$. In particular, we have that

$$\frac{d}{dt}\dim \mathcal{F}^t R_m = -\sum_{j=1}^{N_m} \delta_{a_{m,j}}.$$

Define a positive measure $\mu_m = \mu_m^{\mathcal{F}}$ on $\mathbb{R}_+$ by

$$\mu_m = \frac{1}{m^n}\sum_{j=1}^{N_m} \delta_{m^{-1}a_{m,j}} = -\frac{1}{m^n}\frac{d}{dt}\dim \mathcal{F}^{mt} R_m.$$

The following result is [[BC11], Theorem 1.11].

Theorem 2.4 ([BC11], Theorem 1.11). *If $\mathcal{F}$ is linearly bounded, i.e. $T(\mathcal{F}) < +\infty$, then we have*

$$\lim_{m\to+\infty} \mu_m = \mu$$

in the weak sense of measures on $\mathbb{R}_+$.

Let D be an effective Cartier divisor on X, and consider $\mathcal{F}^\lambda R_m := H^0(mL - \lambda D)$. Then,

$$\begin{aligned}
\beta_m(L,D) : &= \frac{1}{mN_m}\sum_{j\geq 1} \dim H^0(X, mL - jD) = \frac{1}{mN_m}\sum_{j\geq 1}\dim \mathcal{F}^j R_m \\
&= \frac{1}{mN_m}\sum_{j\geq 0} j\left(\dim \mathcal{F}^j R_m - \dim \mathcal{F}^{j+1}R_m\right) \\
&= \frac{1}{mN_m}\sum_j a_{j,m} = \frac{m^n}{N_m}\int_0^\infty t d\mu_m(t).
\end{aligned}$$

Then Theorem 2.4 implies that the limit in the definition of (3) exists, i.e. the limit $\lim_{m\to\infty}\beta_m(L,D)$ exists when L is big. Moreover,

$$\begin{aligned}
\beta(L,D) = \lim_{m\to\infty}\beta_m(L,D) &= \frac{n!}{\mathrm{Vol}(L)}\int_0^\infty t d\mu(t) \\
&= \frac{1}{\mathrm{Vol}(L)}\int_0^\infty \mathrm{Vol}\left(V_\bullet^t\right) dt \\
&= \frac{1}{\mathrm{Vol}(L)}\int_0^\infty \mathrm{Vol}(L - tD)dt = \frac{1}{\mathrm{Vol}(\Delta)}\int_\Delta G d\rho.
\end{aligned}$$

We thus derive the main result of this section as follows:

Theorem 2.5. *Let X be a normal projective variety of dimension n and L be a big line bundle on X. Let D be an effective Cartier divisor on X. Then*

$$\begin{aligned}
\beta(L,D) &= \lim_{m\to\infty}\frac{1}{mN_m}\sum_{j\geq 1}\dim H^0(X, mL - jD) = \frac{1}{\mathrm{Vol}(L)}\int_0^\infty \mathrm{Vol}(L - tD)dt \\
&= \frac{1}{\mathrm{Vol}(\Delta)}\int_\Delta G d\rho.
\end{aligned}$$

3. The stability threshold introduced by Blum-Jonsson

The log canonical threshold of L. Tian [Tia87] in 1987 introduced $\alpha(L)$, the *log canonical threshold of* L, as follows: Let $h = e^{-\phi}$ be a singular metric of L with $\Theta_{L,h} \geq 0$, where $\Theta_{L,h} = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log\phi$. Let $p \in X$ and define $c_p(h) = \sup\{c \mid e^{-2c\phi}$ is locally integrable at p $\}$. Define

$$\alpha(L) = \inf_{h:\Theta_{L,h}\geq 0}\inf_{p\in X} c_p(h).$$

Tian [Tia87] proved that, *for $\mathbb{Q}$-Fano X, if $\alpha(-K_X) > \frac{n}{n+1}$, then X is K-stable.*

Let D be an effective Cartier divisor on X and $[D]$ be its associated line bundle over X. Then the canonical section s_D of $[D]$ gives a singular metric on $[D]$ with $\phi := \log|s_D|$. With the singular metric $h := e^{-\phi_D}$, we denote $\mathrm{lct}_p(D) := c_p(h)$ and $\mathrm{lct}(D) := \inf_{p\in X} c_p(h)$. $\mathrm{lct}(D)$ is called the *log canonical threshold* of D. According to Demailly (see [CS08], Appendix A),

$$\alpha(L) = \inf\{\mathrm{lct}(D) \mid D \text{ is effective}, D \sim_{\mathbb{Q}} L\},$$

where $D \sim_{\mathbb{Q}} L$ means that D is an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to L.

We also have an alternative (algebraic geometry) definition for $\mathrm{lct}(D)$ (see [BJ20]): Recall that a prime divisor E over X is a prime divisor on Y, where $\pi : Y \to X$ is a proper birational morphism and Y is normal. Then E defines a valuation $\mathbb{C}(X)^* \to \mathbb{Z}$ given by order of vanishing at the generic point of E, where $\mathbb{C}(X)^*$ is the set of nontrivial rational functions on X. The definition extends to $\mathrm{ord}_E(D)$, where D is an effective $\mathbb{Q}$-divisor: Pick $m \geq 1$ such that mD is Cartier and set $\mathrm{ord}_E(D) := m^{-1}\,\mathrm{ord}_E(f)$, where f is a local equation of mD. Equivalently, $\mathrm{ord}_E(D) = m^{-1}\,\mathrm{ord}_E(s)$, where s is the canonical section of $[mD]$. We define (see [BJ20]):

$$\mathrm{lct}(D) = \min_{E \text{ over } X} \frac{A_X(E)}{\mathrm{ord}_E(D)}, \tag{13}$$

where $A_X(E) := 1 + \mathrm{ord}_E(K_{Y/X})$ is the *log discrepancy*. We say X *has at worst klt singularities* if $A_X(E) > 0$ for all prime divisors E over X.

Blum-Jonsson's stability threshold. In [BJ20] (see also [FO18]), Blum-Jonsson introduced the stability threshold $\delta(L)$ to replace $\alpha(L)$ by replacing D with only the *m-basis type* divisors. Recall that an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ on X is of *m-basis type* (with respect to the line bundle L) if there is a basis $s_1, \dots, s_{N_m}$ of $H^0(X, mL)$ such that

$$D = \frac{1}{mN_m}(\{s_1 = 0\} + \cdots + \{s_{N_m} = 0\}).$$

Define

$$\delta_m(L) := \inf\{\mathrm{lct}(D) \mid D \sim_{\mathbb{Q}} L \text{ of m-basis type}\}, \tag{14}$$

and

$$\delta(L) = \inf_E \frac{A_X(E)}{\beta(L,E)}. \tag{15}$$

The result of Blum-Jonsson [BJ20] is as follows:

Theorem 3.1 (Blum-Jonsson [BJ20])**.** *Let X be a normal complex projective variety of dimension n with at worst klt singularities, and let L be a big line bundle on X. Then*

(a) $\lim_{m\to\infty} \delta_m(L)$ *exists and is equal to* $\delta(L)$*;*
(b) $\alpha(L) \leq \delta(L) \leq (n+1)\alpha(L)$*;*
(c) *For $\mathbb{Q}$-Fano X, X is K-semistable (K-stable) iff $\delta(-K_X) \geq 1$ ($\delta(-K_X) > 1$).*

The proof of (b) depends on the relationship of the three constants described in the next section, and (c) directly follows from the recent result of Fujita [Fuj19] and C. Li [Li17]. So here we only outline the proof of (a).

Let E be a prime divisor over X. Denote by

$$\beta_m(L,E) := \frac{1}{mN_m}\sum_j a_{m,j}, \tag{16}$$

where $a_{m,j}$ are the jumping numbers of the filtration $\mathcal{F}^\lambda R_m := H^0(m\pi^*L - \lambda E)$.

Lemma 3.2.

$$\beta_m(L,E) = \max_{\mathcal{B}} \frac{1}{mN_m} \sum_{j=1}^{N_m} \mathrm{ord}_E(s_j),$$

where the maximum is over all bases $\mathcal{B} = \{s_1, \ldots, s_{N_m}\}$ of $H^0(X, mL)$.

Proof. First consider any basis $s_1, \ldots, s_{N_m}$ of $H^0(X, mL)$. We may assume $\mathrm{ord}_E(s_1) \le \mathrm{ord}_E(s_2) \le \cdots \le \mathrm{ord}_E(s_{N_m})$. Then $\mathrm{ord}_E(s_j) \le a_{m,j}$, where $a_{m,j}$ is the j-th jumping number of the filtration $\mathcal{F}^\lambda R_m := H^0(m\pi^*L - \lambda E)$. This implies that

$$\frac{1}{mN_m} \sum_{j=1}^{N_m} \mathrm{ord}_E(s_j) \le \frac{1}{mN_m} \sum_{j=1}^{N_m} a_{m,j} = \beta_m(L,E).$$

On the other hand, we can pick the basis such that $\mathrm{ord}_E(s_j) = a_{m,j}$, and then

$$\frac{1}{mN_m} \sum_{j=1}^{N_m} \mathrm{ord}_E(s_j) = \beta_m(L,E).$$

This proves the lemma. □

Proposition 3.3. *For $m \in M(L)$, we have*

$$\delta_m(L) = \inf_E \frac{A(E)}{\beta_m(L,E)},$$

where the inf runs through prime divisors E over X.

Proof. Note that

$$\delta_m(L) = \inf_{\text{D of m-basis type}} \left(\inf_E \frac{A(E)}{\mathrm{ord}_E(D)} \right),$$

where the inner infimum runs through the prime divisors over X. Switching the order of the two infimums and applying Lemma 3.2 above yield the desired equality. □

Proof of (a) in Theorem 3.1. From (16) and Theorem 2.5, we have

$$\lim_{m\to\infty} \beta_m(L,E) = \beta(L,E).$$

This, together with Proposition 3.3, gives

$$\delta(L) = \limsup_{m\to\infty} \delta_m(L) \le \inf_E \frac{A(E)}{\beta(L,E)}. \tag{17}$$

On the other hand, given $\epsilon > 0$, there exists m_0 such that $\beta_m(L,E) \le (1+\epsilon)\beta(L,E)$ for all the prime divisors E over X. Thus

$$\delta(L) = \limsup_{m\to\infty} \delta_m(L) = \limsup_{m\to\infty} \inf_E \frac{A(E)}{\beta_m(L,E)} \ge (1+\epsilon)^{-1} \inf_E \frac{A(E)}{\beta(L,E)}.$$

Letting $\epsilon \to 0$ and combining this inequality with (17) completes the proof.

An upper bound of $\delta(L)$. We now derive an upper bound for $\delta(L)$ in terms of $\mathrm{lct}(D)$, where D is an effective Cartier divisor on X, not necessarily in $|L|$. Take a basis B of the filtration $\mathcal{F}^\lambda R_m := H^0(mL - \lambda D)$. Notice that, for any $s \in W_t := H^0(X, mL - tD)$, $\mathrm{ord}_E(s) \geq t\,\mathrm{ord}_E(D)$, so, from Lemma 3.2,

$$\begin{aligned}\beta_m(L,E) &\geq \frac{1}{mN_m}\sum_{s\in B}\mathrm{ord}_E(s) \geq \frac{1}{mN_m}\left(\sum_{t=0}^{\infty} t(\dim W_t - \dim W_{t+1})\right)\mathrm{ord}_E(D)\\ &= \frac{1}{mN_m}\left(\sum_{t=1}^{\infty}\dim W_t\right)\mathrm{ord}_E(D).\end{aligned}$$

Thus we have, from (3), (13), and Proposition 3.3, we get

$$\delta(L) \leq \frac{1}{\beta(L,D)}\,\mathrm{lct}(D).$$

Thus we proved the following result.

Theorem 3.4. *Let X be a normal complex projective variety of dimension n with at worst klt singularities, and let L a big line bundle on X. Then for any effective Cartier divisor D on X, we have*

$$\delta(L) \leq \frac{1}{\beta(L,D)}\,\mathrm{lct}(D).$$

The role of the m-basis in Ru-Vojta's result [RV20]**.** Note that the concept of m-basis is also used in the proof of the main Diophantine result in [RV20]. In particular, the following result, which is a re-formulation of Schmidt's subspace theorem, involves the m-basis. To keep the notation to a minimum, we don't recall the notation here. Instead, we refer to [RV20].

Theorem 3.5 (Theorem 2.10 in [RV20])**.** *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let X be a complete variety over k, and let L be a line bundle on X. Let $\mathcal{B}$ be a finite set of the divisors D which are of m-basis type with respect to L. Then, for any $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that*

$$\sum_{v\in S}\max_{D\in\mathcal{B}}\lambda_{D,v}(x) \leq (1+\epsilon)h_L(x) \tag{18}$$

holds for all $x \in (X \setminus Z)(k)$, where $\lambda_{D,v}$ is a local height function and h is a logarithmic height function.

The main result of this section. With the more sophisticated multi-dimensional filtration in [RV20] (see also [Aut11]), we extend Theorem 3.4 by proving the following more general result (which can be viewed as a counter-part of the arithmetic general theorem of Ru-Vojta [RV20]).

Let $D_1, \ldots, D_q$ be effective Cartier divisors on X. We say that $D_1, \ldots, D_q$ lie in *general position* if for any $I \subset \{1, \ldots, q\}$, we have $\dim(\bigcap_{i\in I}\mathrm{Supp}\,D_i) = n - \#I$ if $\#I \leq n$, and $\bigcap_{i\in I}\mathrm{Supp}\,D_i = \emptyset$ if $\#I > n$. We say that $D_1, \ldots, D_q$ *intersect properly* if for any subset $I \subset \{1, \ldots, q\}$ and any $x \in \bigcap_{i\in I}\mathrm{Supp}\,D_i$, the sequence $(\phi_i)_{i\in I}$ is a regular sequence in the local ring $\mathscr{O}_{X,x}$, where ϕ_i are the local defining functions of D_i, $1 \leq i \leq q$. It is known (see [RV20]) if $D_1, \ldots, D_q$ intersect properly, then they lie in general position. The converse holds if X is Cohen-Macaulay (this is true if X is nonsingular).

Theorem 3.6. *Let X be a normal complex projective variety of dimension n with at worst klt singularities, and let L a big line bundle on X. Then*

$$\delta(L) \leq \frac{1}{\max_{1\leq i\leq q}\beta(L,D_i)}\operatorname{lct}(D),$$

for any divisor $D = D_1 + \cdots + D_q$ with $D_1, \ldots, D_q$ intersecting properly on X.

Proof. Let $\Sigma = \left\{\sigma \subseteq \{1,\ldots,q\} \mid \bigcap_{j\in\sigma}\operatorname{Supp} D_j \neq \emptyset\right\}$. Since $D_1,\ldots,D_q$ intersect properly on X, $\#\sigma \leq n$ for $\forall\sigma\in\Sigma$. Fix an integer $b > 0$. For $\sigma \in \Sigma$, let

$$\triangle_\sigma = \left\{\mathbf{a} = (a_i) \in \mathbb{N}^{\#\sigma} \mid \sum_{i\in\sigma} a_i = b\right\}.$$

For $\mathbf{a} \in \triangle_\sigma$ and $x \in \mathbb{R}_+$, one defines the ideal $\mathcal{I}_{\mathbf{a}}(x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}_{\mathbf{a}}(x) = \sum_{\mathbf{b}} \mathscr{O}_X\left(-\sum_{i\in\sigma} b_i D_i\right),$$

where the sum is taken for $\mathbf{b} \in \mathbb{N}^{\#\sigma}$ with $\sum_{i\in\sigma} a_i b_i \geq bx$. Write L as the line sheaf $\mathscr{L}$ and consider the filtration $\mathcal{F}(\sigma;\mathbf{a})_x = H^0(X, \mathscr{L}^m \otimes \mathcal{I}_{\mathbf{a}}(x))$, which are regarded as subspaces of $H^0(X,\mathscr{L}^m)$, and let

$$F(\sigma;\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^m)}\int_0^{+\infty}(\dim \mathcal{F}(\sigma;\mathbf{a})_x)\, dx.$$

The key result from Ru-Vojta (see Proposition 6.7 in [RV20]) is that

$$F(\sigma;\mathbf{a}) \geq \min_{1\leq i\leq q}\left(\frac{1}{h^0(\mathscr{L}^m)}\sum_{m\geq 1} h^0(\mathscr{L}^m(-kD_i))\right).$$

It then gives (see Remark 6.6 in [RV20]), for any basis $\mathcal{B}_{\sigma;\mathbf{a}}$ of $H^0(X,\mathscr{L}^m)$ adapted to the above filtration,

$$\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}} \mu_{\mathbf{a}}(s) \geq \min_{1\leq i\leq q}\sum_{m\geq 1} h^0(\mathscr{L}^m(-kD_i)), \tag{19}$$

where for any $s \in H^0(X,\mathscr{L}^m)$, $\mu_{\mathbf{a}}(s) = \sup\{x\in\mathbb{R}^+ : s \in \mathcal{F}(\sigma;\mathbf{a})_x\}$.

Note that there are only finitely many ordered pairs $(\sigma,\mathbf{a})$ for $\sigma\in\Sigma, \mathbf{a}\in\triangle_\sigma$. We also note that, for any prime divisor E over X and $s \in H^0(X,\mathscr{L}^m\otimes\mathcal{I}_{\mathbf{a}}(\mu_{\mathbf{a}}(s)))$,

$$\operatorname{ord}_E(s) \geq \min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i \operatorname{ord}_E(D_i),$$

where $K = K_{\sigma,\mathbf{a},s}$ is the set of minimal elements of $\{\mathbf{b}\in\mathbb{N}^{\#\sigma} : \sum_{i\in\sigma} a_i b_i \geq b\mu_{\mathbf{a}}(s)\}$ relative to the coordinatewise partial order on $\mathbb{N}^{\#\sigma}$. The set K is a finite set. Let $t_i := \frac{\operatorname{ord}_E(D_i)}{\operatorname{ord}_E(D)}$, then $\sum_{i\in\sigma} t_i = 1$. Therefore, using $\#\sigma \leq n$, $b \leq \sum_{i\in\sigma}\lfloor (b+n)t_i\rfloor \leq b+n$, and we may choose $\mathbf{a} = (a_i) \in \triangle_\sigma$ such that $t_i \geq \frac{a_i}{b+n}$ for all $i\in\sigma$. Thus, for any $s \in \mathcal{B}_{\sigma;\mathbf{a}}$,

$$\begin{aligned}\operatorname{ord}_E(s) &\geq \min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i\operatorname{ord}_E(D_i) = \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i t_i\right)\operatorname{ord}_E(D)\\ &\geq \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma}\frac{a_i b_i}{b+n}\right)\operatorname{ord}_E(D) \geq \left(\frac{b}{b+n}\right)\mu_{\mathbf{a}}(s)\operatorname{ord}_E(D).\end{aligned}$$

Hence, by Lemma 3.2 as well as using (19),

$$\beta_m(L,E) \geq \sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}} \frac{1}{mN_m}\operatorname{ord}_E(s) \geq \frac{b}{b+n}\operatorname{ord}_E(D)\left(\min_{1\leq i\leq q}\frac{\sum_{k\geq 1} h^0(\mathscr{L}^m(-kD_i))}{mh^0(\mathscr{L}^m)}\right).$$

Thus, by Proposition 3.3 and Theorem 2.5,

$$\delta_m(L) \leq \frac{A_X(E)}{\beta_m(L,E)} \leq \left(\frac{b}{b+n}\right)\frac{1}{\max_{1\leq i\leq q}\beta(L,D_i)}\frac{A_X(\operatorname{ord}_E)}{\operatorname{ord}_E(D)}.$$

By letting $b\to\infty$ and $m\to\infty$, we get

$$\delta(L) \leq \frac{1}{\max_{1\leq i\leq q}\beta(L,D_i)}\operatorname{lct}(D).$$

□

4. Three important constants

Definition 4.1. Let L be an ample line bundle over X, we define the *Seshadri constant* $\epsilon(L,D)$ by $\epsilon(L,D) = \sup\{\gamma\in\mathbb{Q} : L-\gamma D \text{ is nef }\}$. We also define the *pseudo-effective* constant as $T(L,D) = \sup\{\gamma\in\mathbb{Q} : L-\gamma D \text{ is pseudo-effective}\}$.

Theorem 4.2. *We have* $\frac{1}{n+1}T(L,D) \leq \beta(L,D) \leq T(L,D)$.

Proof. Given a filtration $\mathcal{F}$, we show that

$$\frac{1}{n+1}T(\mathcal{F}) \leq \beta(\mathcal{F}) \leq T(\mathcal{F}), \tag{20}$$

where $T(\mathcal{F})$ is given in (9) and $\beta(\mathcal{F})$ is given by

$$\beta(\mathcal{F}) := \frac{1}{\operatorname{Vol}(L)}\int_0^\infty \operatorname{Vol}\left(V_\bullet^t\right)dt.$$

The second inequality is clear since $\operatorname{Vol}(V_\bullet^t) \leq \operatorname{Vol}(L)$ and $\operatorname{Vol}(V_\bullet^t) = 0$ for $t > T(\mathcal{F})$. The first follows from the Proposition of 2.2 which states that concavity of $t\mapsto \operatorname{vol}(V_\bullet^t)^{1/n}$ thus yields $\operatorname{Vol}(V_\bullet^t) \geq \operatorname{Vol}(L)\left(1-\frac{t}{T(\mathcal{F})}\right)^n$. Therefore (20) is proved. The theorem follows from (20) by taking the filtration $\mathcal{F}^\lambda R_m := H^0(mL-\lambda D)$ and by noticing (10). □

Combining Theorem 4.2 with Theorem A gives the following result.

Theorem 4.3. *Let X be a projective variety, and $D_1,\ldots,D_q$ be effective Cartier divisors, both defined over a number field k. Assume that $D_1,\ldots,D_q$ intersect properly on X. Let $S\subset M_k$ be a finite set of places on k. Let L be an ample line sheaf on X. Then, for every $\epsilon>0$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{j=1}^q T(L,D_j)m_S(x,D_j) \leq (n+1+\epsilon)h_L(x) \tag{21}$$

holds for all $x\in X(k)$ outside of Z.

Theorem 4.3 is reminiscent of Theorem 3.3 in [MR16], since both use the pseudo-effective constant to get bounds on the quality of Diophantine approximations.

Theorem 4.3 implies, noticing it is trivial from the definition that $\epsilon(L,D) \leq T(L,D)$, the following recent result of Levin-Heier [HL20].

Theorem 4.4. *Let X be a projective variety, and $D_1, \dots, D_q$ be effective Cartier divisors, both defined over a number field k. Assume that $D_1, \dots, D_q$ intersect properly on X. Let $S \subset M_k$ be a finite set of places on k. Let L be an ample line sheaf on X. Then, for every $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{j=1}^{q} \epsilon(L, D_j) m_S(x, D_j) \leq (n + 1 + \epsilon) h_L(x) \tag{22}$$

holds for all $x \in X(k)$ outside of Z.

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