Preserving Projection Properties When Regular Two-level Designs Are Blocked

John Tyssedal [a] and Yngvild Hole Hamre [a,b]

[a] Department of Mathematical Sciences, NTNU

[b] DNB

Abstract

Regular two-level designs are useful and popular screening designs, but if they need to be run in blocks, their projection properties can dramatically deteriorate. Interactions may be fully confounded with block defining contrast(s), causing uncertainty in the identification of active factors. In this paper, we demonstrate alternative ways of blocking two-level regular designs such that their projective properties can be preserved or just weakly affected at the expense of just a small decrease in efficiency. Thereby, we can estimate effects we are normally interested in even if the design is blocked. Common regular two-level designs with 16, 32 and 64 runs are considered.

KEYWORDS: Aliasing; Doubling; D_s-efficiency; Hadamard matrices; Mirror image pair runs.

1. Introduction

Regular fractional factorial two-level designs, usually denoted as 2^{k-p} designs, are among the most popular experimental plans used for screening. They are characterized by the property that any two effects can be estimated independently of each other or are fully aliased which makes their analysis rather straight forward. Their properties and usefulness are nicely explained in several textbooks like Box et al. (2005), Wu and Hamada (2009) and Montgomery (2019).

Regular two-level designs can be constructed by allocating *k-p* factors to the principal factor columns in a full factorial with 2^{k-p} runs and the remaining *p* factor(s) to some higher order interaction column(s). This causes the full aliasing of effects. Resolution is an important concept to describe the aliasing. *By definition, a two-level design is of resolution R if no p-factor effect is aliased with any other effect containing less than R-p factors* (Box and Hunter 1961). The aliasing can be derived from the defining relation which is the set of all columns that are equal to the identity column. For instance, if factor *D* is assigned to a three-factor interaction column, say *ABC*, it creates a word in the defining relation given by *ABCD*. Any effect associated with one, two or three of these four letters will then be aliased with the effect associated with the remaining one(s). In general, the defining relation has $2^p - 1$ words and to each design a wordlength pattern can be constructed of the form $W = (A_3, A_4, \ldots, A_k)$, where A_i is the number of words of length *i*. It is assumed that no main effects are aliased with each other. The higher the resolution, the better, but designs with the same resolution can have unequal numbers of fully aliased effects. For given *k* and *p*, let d_1 and d_2 be two designs. If r is the smallest integer for which $A_r(d_1) \neq A_r(d_2)$ and $A_r(d_1) < A_r(d_2)$, d_1 is said to have less aberration than d_2. If no design has less aberration than d, it is said to have minimum aberration (Fries and Hunter 1980). This is a useful way to rank regular two-level designs.

A much cited rule for experimental work is: “Block what you can and randomize what you can’t”. Blocking is an effective way to improve the efficiency of a design when not all the experimental runs can be performed under homogenous conditions. For regular two-level designs the general rule of blocking is to assign one or a set of higher order interaction column(s), named block defining contrast(s) as block factor(s), and then associate the distinct level combinations in the column(s) with different blocks. A good blocking scheme should have as few as possible lower order interactions confounded with the block effects, and an additional wordlength pattern $W_b = (A_{2b}, A_{3b}, \ldots A_{kb})$ can be constructed, where A_{ib} is the number of *i*th order interactions confounded with the block effect(s).

In order to rank blocked regular two-level designs, a combined wordlength pattern may be constructed. Several ways of doing this have been proposed and discussed in the literature (Sitter et al. 1997; Chen and Cheng 1999; Zhang and Park 2000; Mukerje and Wu 2006; Cheng and Tsai 2009; Xu and Mee 2010; Zhao et al. 2013). Cheng and Mukerjee (2001) proposed and studied a criterion for blocking based on the alias pattern of the interactions with the purposes of maximizing the number of two-factor interactions that are neither aliased with main effects nor confounded with blocks and at the same time distributing the interactions over the alias sets as uniformly as possible.

Sun et al. (1997) used the two wordlength patterns in addition to two other criteria, the number of clear main effects and the number of clear two-factor interactions, to come up with good blocking schemes. A main effect was called clear if it was not aliased with any two-factor interaction and any block effect. Similarly, a two-factor interaction was called clear if it was not aliased with any main effect, any other two-factor interaction and any block effect. Based on these four criteria, the concept of admissibility of blocking schemes was introduced, as a way to rule out bad designs. This criterion was further explored in Mukerjee and Wu (1999).

In this paper we will mainly be concerned with using blocked regular two-level designs for screening purposes. Screening is about separating out the normally few factors, from potentially many, that can explain most of the variation in the response. Projection properties concern how good a design is when restricted to a subset of factors. It is therefore important for a screening design to have good projection properties. Box and Tyssedal (1996) defined projectivity of two-level designs as follows: *A $n \times k$ design with n runs and k factors each at two levels is said to be of projectivity P if the design contains a complete 2^P factorial in every possible subset of P out of the k factors, possibly with some points replicated.* For regular two-level designs the projectivity is always given by $P = R$-1. For more on projectivity, factor sparsity and screening, we refer to Box and Tyssedal (2001) and Tyssedal (2008).

However, recommended schemes for blocking two level regular designs may cause the projection properties of the blocked designs to deteriorate. For instance, the two-level resolution V (projectivity P = 4) design for five factors in 16 runs, denoted as the 2_V^{5-1} design, becomes a projectivity P = 1 design when blocked in two blocks using the recommended scheme.

We will in this paper present alternative ways of blocking regular designs, such that projection properties will be preserved either fully or to some extent. Reflected in our focus on screening, all the designs chosen to be blocked have good projections properties. Only blocks having an equal number of runs in each block will be considered. The designs to be blocked will have from 16 to 64 runs and will, given k and p, be minimum aberration designs taken from Wu and Hamada (2009), pages 253-257. Also, whenever we write *recommended* way of blocking, we will from now on refer to the way of blocking given there on pages 260-263. With reference to the tables presented in Sun et al. (1997), the blocked designs found in Wu and Hamada (2009) are the ones that rank best according to the number of clear main effects for 16 run designs. For designs with more than 16 runs, the number of clear two-factor interactions is used as ranking criterion.

This paper is organized as follows. In Section 2, we explain what is meant by projectivity of blocked regular two-level designs and introduce the criterion used to discriminate between different ways of blocking. A motivational example is given in Section 3. Section 4 is devoted to strategies for finding good blocking schemes, and in Section 5 we present specific ways of blocking the designs and the projectivity that is possible to obtain in each case. Several ways of blocking regular two-level designs are discussed and compared in Section 6. Concluding remarks are given in Section 7.

2. Projectivity Concepts and Evaluation Criterion

If many design factors are active, one may have to give up on estimating all higher order interactions. Evangeleras and Koukouvinos (2004) introduced the concept of generalized projectivity for two level designs as: *A* $n\times k$ *design with* n *runs and* k *factors each at two levels is said to be of generalized projectivity* P_α *if for any selection of P columns from the design all factorial effects including up to* α*- factor interactions are estimable.* In line with that, Hussain and Tyssedal (2016) defined the projectivity of blocked two-level designs as: *A blocked two-level design is of projectivity P or* P_α *if for any selection of P-columns the intercept and all factorial effects up to including P-factor interactions or* α*- factor interactions are estimable, respectively.*

Let the model for the expected response $\mathbf{Y}$ in a blocked experiment be written as

$$E\left[\mathbf{Y}\right]=\mathbf{X}_e\boldsymbol{\beta}_e+\mathbf{X}_b\boldsymbol{\beta}_b\,. \tag{1}$$

Here $\mathbf{Y}$ is a vector of n observations, $\mathbf{X}_e$ is a $n\times\left(p+1\right)$ matrix containing a column for the intercept and the effect columns (main effects and interactions under consideration), $\boldsymbol{\beta}_e=\left[\beta_0,\beta_1,\ldots,\beta_p\right]^t$ is the $\left(p+1\right)$ dimensional vector containing the intercept and the regression coefficients that are half the corresponding effects, $\mathbf{X}_b$ is a $n\times b$ matrix containing the columns for the blocks and $\boldsymbol{\beta}_b=\left[\beta_1^*,\ldots,\beta_b^*\right]^t$ is the corresponding vector of the coefficients for the b block effects. While the definition of projectivity informs what is possible to estimate, it does not tell how well the effects are estimated. A useful criterion in that context is the *D-optimality* criterion defining a *D*-optimal design as the one for which $\frac{\left|\mathbf{X}^t\mathbf{X}\right|}{n^P}$ is maximized. Here $\mathbf{X}=\left[\mathbf{X}_e,\mathbf{X}_b\right]$ is the design matrix. $\left|\mathbf{X}^t\mathbf{X}\right|$ denotes the determinant of $\mathbf{X}^t\mathbf{X}$ and is inversely proportional to the square of the volume of the confidence region of the regression coefficients and thereby directly related to estimating the effects.

When a design is blocked, it may be argued that a precise estimation of the corresponding block effect(s) is not as important as for the effects of the factors under investigation (Atkinson and Donev 1997, page 106). For $\mathbf{X} = \left[\mathbf{X}_e, \mathbf{X}_b\right]$ we have $\mathbf{X}^t\mathbf{X} = \begin{bmatrix} \mathbf{X}_e^t\mathbf{X}_e & \mathbf{X}_e^t\mathbf{X}_b \\ \mathbf{X}_b^t\mathbf{X}_e & \mathbf{X}_b^t\mathbf{X}_b \end{bmatrix}$, and the covariance matrix for the least square estimators for $\boldsymbol{\beta}_e$ is proportional to the upper left $(p+1)\times(p+1)$ submatrix of $\left[\mathbf{X}^t\mathbf{X}\right]^{-1}$ which is equal to $\left[\mathbf{X}_e^t\mathbf{X}_e - \left(\mathbf{X}_e^t\mathbf{X}_b\right)\left(\mathbf{X}_b^t\mathbf{X}_b\right)^{-1}\left(\mathbf{X}_b^t\mathbf{X}_e\right)\right]^{-1}$.

It can be shown that $\left|\mathbf{X}_e^t\mathbf{X}_e - \left(\mathbf{X}_e^t\mathbf{X}_b\right)\left(\mathbf{X}_b^t\mathbf{X}_b\right)^{-1}\left(\mathbf{X}_b^t\mathbf{X}_e\right)\right| = \dfrac{\left|\mathbf{X}^t\mathbf{X}\right|}{\left|\mathbf{X}_b^t\mathbf{X}_b\right|}$. A design that maximizes $\dfrac{\left|\mathbf{X}^t\mathbf{X}\right|}{\left|\mathbf{X}_b^t\mathbf{X}_b\right|}$ is said to be D_s-optimal (Atkinson and Donev 1997, page 106). The subscript *s* refers to the subset of *s* columns for which we are really interested in estimating the effects and equals $p+1$ for the case given above. It is then natural to define D_s-efficiency as

$$D_{s,eff} = \frac{\left[\dfrac{\left|\mathbf{X}^t\mathbf{X}\right|}{\left|\mathbf{X}_b^t\mathbf{X}_b\right|}\right]^{\frac{1}{s}}}{n}. \qquad (2)$$

where $\dfrac{1}{s}$ takes care of the increase in the determinant that occurs by increasing s. If for a given way of blocking $D_{s,eff} = 0$ for one set of *P*-factors where *s*-1 is the number of all factorial effects up to *P*-factor interactions or α- factor interactions, it is clear that also $\left|\boldsymbol{X}^t\boldsymbol{X}\right|$ has to be zero. Then the inverse of $\boldsymbol{X}^t\boldsymbol{X}$ does not exist and projectivity *P* or P_α is not obtained, respectively. Otherwise, it is natural to choose a candidate from the ones with good overall values for $D_{s,eff}$ considering both the minimum, maximum and average values over all projections onto *P* factors.

3. A Motivational Example, the 2_{IV}^{8-4} Design Arranged in Two Blocks

The 2_{IV}^{8-4} design is a resolution IV design and hence of projectivity P = 3, and all its projections onto three dimensions contain a fully replicated 2^3 design. The principal fraction of this design is given in Table 1, denoting the four principal factor columns as A, B, C and D. The generators for the four additional columns, E, F, G and H, are also given, together with two possible block defining contrasts. One is the recommended one, B_b, according to Wu and Hamada (2009). The other contrast, B_b^*, represents a possible alternative.

Table 1. The 2_{IV}^{8-4} design with two alternative ways of blocking

Run	A	B	C	D	E=ABC	F=ABD	G=ACD	H=BCD	B_b	B_b^*
1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
2	1	-1	-1	-1	1	1	1	-1	-1	1
3	-1	1	-1	-1	1	1	-1	1	-1	1
4	1	1	-1	-1	-1	-1	1	1	1	-1
5	-1	-1	1	-1	1	-1	1	1	1	1
6	1	-1	1	-1	-1	1	-1	1	-1	-1
7	-1	1	1	-1	-1	1	1	-1	-1	-1
8	1	1	1	-1	1	-1	-1	-1	1	-1
9	-1	-1	-1	1	-1	1	1	1	1	-1
10	1	-1	-1	1	1	-1	-1	1	-1	-1
11	-1	1	-1	1	1	-1	1	-1	-1	-1
12	1	1	-1	1	-1	1	-1	-1	1	1
13	-1	-1	1	1	1	1	-1	-1	1	-1
14	1	-1	1	1	-1	-1	1	-1	-1	1
15	-1	1	1	1	-1	-1	-1	1	-1	1
16	1	1	1	1	1	1	1	1	1	1

The recommended blocking scheme for arranging the design in two blocks is to let B_b = AB be the block defining contrast (any other two-factor interaction could have been chosen). The defining relation for the 2_{IV}^{8-4} design consists of 14 four letter words and one eight letter word. As a result, the four two-factor interactions AB, CE, DF and GH are fully confounded with the block defining contrast and 24 out of the 56 projections onto three dimensions are affected in the sense that one two-factor interaction cannot be estimated without being fully confounded with the block effect. Thus, when blocked the recommended way of blocking, the design becomes a P = 1 design.

Now, if we let B_b^* be the block defining contrast, we will find that for 48 projections onto three dimensions $D_{s,eff}$ = 0.917, and $D_{s,eff}$ = 1 for the eight others. Here $D_{s,eff}$ is obtained letting $\mathbf{X}_e$ be a 16×8 matrix containing a column for the intercept, the effect columns for the three main effects and their interactions and $\mathbf{X}_b = B_b^*$. Hence all the main effects and their interactions are now estimable for any three factors, and the design is of projectivity P = 3 when run in two blocks.

The notation (n,k,P) screen is used to describe a two-level projectivity P design with n runs and k factors used for screening. When run in b blocks of equal size, we will denote it a (n,k,P,b) screen. Table 2 summarizes what happens with the projectivity of some regular two-level designs when they are blocked the recommended way. As can be observed a rather severe loss in projectivity is common for many of them.

Table 2. The projectivity of some regular designs when they are blocked.

Design	Screen	Screen when blocked the recommended way
2_{IV}^{8-4}	(16, 8, 3)	(16, 8, 1, 2)
2_{V}^{5-1}	(16, 5, 4)	(16, 5, 1, 2)
2_{VI}^{6-1}	(32, 6, 5)	(32, 6, 2, 2), (32, 6, 1, 4)
2_{IV}^{7-2}	(32, 7, 3)	(32, 7, 2, 2), (32, 7, 1, 4)
2_{IV}^{8-3}	(32, 8, 3)	(32, 8, 2, 2), (32, 8, 1, 4)
2_{IV}^{9-4}	(32, 9, 3)	(32, 9, 1, 2)
2_{V}^{8-2}	(64, 8, 4)	(64, 8, 2, 2), (64, 8, 2, 4)

4. Blocking Strategies

If a design with $n = 2t$ runs is to be run in two blocks with an equal number of runs in each, this can be done in $\frac{\binom{2t}{t}}{2!}$ possible ways, and for $t \geq 4$ and a multiple of two there are

$\frac{\binom{2t}{t/2}\binom{2t-t/2}{t/2}\binom{t}{t/2}}{4!}$ possibilities for blocking into four blocks. For example, a 16 run design can be arranged in two and four blocks in 6435 and 2627625 possible ways respectively, and a 32 run design can be arranged in two blocks in 300540195 possible ways. Many of these block alternatives will have undesirable properties, and therefore some restrictions should be placed on the alternatives to investigate. We will restrict the blocking contrast(s) to be orthogonal to the main effect contrasts. The combinatorial explosion illustrated above also asks for strategies that can provide good blocking alternatives without investigating all possibilities. Several such strategies now follow.

Strategy S1. *Allocate mirror image pair runs to the same block*

A full factorial two-level design in *n* runs consists of $\frac{n}{2}$ mirror image pair runs. This is also true for some of its fractions. For example, if the factor columns in a 2^3 design are denoted A, B and C, we may add a fourth factor column D = ABC and obtain a 2_{IV}^{4-1} design. The generator of this fraction is then D = ABC and the defining relation is I = ABCD, a column of only 1's. The defining relation here consists of one word of length four, and the design consists of all level combinations for which their entry-wise product is 1. For a given level combination in the design, it is then evident that its mirror image run is also included in the design, and that this must be the case when the defining relation is a word of even length. Depending on the degree of fractioning and which fraction is chosen, the defining relation may consist of several words with a plus or a minus in front of them. The corresponding design will then consist of all level combinations that satisfy the defining relation. For a given level combination in the design, its mirror image run will also satisfy the defining relation and be included if all these words are of even length. Furthermore, if all level combinations in a design are mirror image pair runs, there can be no word of odd length in the defining relation, since a level combination that satisfies this constraint will exclude its mirror image run. For designs for which it can be used, this way of blocking assures the blocking contrast(s) to be orthogonal to main effect and odd factor interaction contrasts and reduces the number of blocking alternatives. When blocked in two blocks the reduction is from 6435 to 35 for a 16 run design and from 300540195 to 6435 for a 32 run design. Since the loss in efficiency obtained by blocking is caused by effects being confounded with the block defining contrast(s), the use of $D_{s,eff}$ to evaluate the alternatives obtained by this strategy act as a way to reduce the confounding between even factor interaction contrasts and the block defining contrast(s).

Strategy S2. *Construct blocking schemes by doubling a blocked design*

Hadamard matrices are $n \times n$ orthogonal matrices consisting of 1's and -1's. They exist for n = 1 and 2 and otherwise apparently for *n* a multiple of 4. Without loss of generality we may let the first column consist of 1's. The remaining columns will then have equally many 1's and -1's.

Several resolution IV designs can be constructed from Hadamard matrices. Let $\boldsymbol{S}_1 = 1$ and let

$$\boldsymbol{S}_n = \begin{bmatrix} \boldsymbol{S}_{n/2} & \boldsymbol{S}_{n/2} \\ \boldsymbol{S}_{n/2} & -\boldsymbol{S}_{n/2} \end{bmatrix}, \ n = 2,4,8,\ldots. \tag{3}$$

These matrices are called Sylvester type Hadamard matrices and will provide us with all saturated regular designs, with an intercept column included. For $n = 2^{2+t}$, t=1,2,3, $\ldots$the designs $\boldsymbol{D}_n = \begin{bmatrix} \boldsymbol{S}_{n/2} \\ -\boldsymbol{S}_{n/2} \end{bmatrix}$ are resolution IV designs in n runs and $n/2$ factors. Clearly, for every level combination its mirror image run will also be included. For a given regular design $\boldsymbol{D}_n$ with n runs and *k* factors, $k = 1,2,\ldots,n-1$, a regular two-level design with $2n$ runs and $2k$ ($2k+1$ if an intercept column is included in $\boldsymbol{D}_n$) factors can be constructed by doubling as

$$\boldsymbol{D}_{2n} = \begin{bmatrix} \boldsymbol{D}_n & \boldsymbol{D}_n \\ \boldsymbol{D}_n & -\boldsymbol{D}_n \end{bmatrix}. \tag{4}$$

If $\boldsymbol{D}_n$ is a resolution IV design, $\boldsymbol{D}_{2n}$ is also a resolution IV design.

Now suppose a regular design $\boldsymbol{D}_n$ with k factors has been blocked in two blocks of equal size $\boldsymbol{B}_1$ *and* $\boldsymbol{B}_2$ such that $\boldsymbol{D}_n = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}$. The design constructed by doubling is then

$$\boldsymbol{D}_{2n} = \left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_2 & \boldsymbol{B}_2 \\ \hline \boldsymbol{B}_1 & \boldsymbol{\text{-}B}_1 \\ \boldsymbol{B}_2 & \boldsymbol{\text{-}B}_2 \end{array}\right] \tag{5}$$

and has 2k factors. As it is written, it may look like it gives a way to block $\boldsymbol{D}_{2n}$ in two blocks, but two-factor interactions will be fully confounded with the block defining contrast no matter what the resolution is, and hence it will become a P = 1 design when blocked. However, both the configurations

$$\boldsymbol{D}^*_{2n} = \left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_1 & \boldsymbol{\text{-}B}_1 \\ \hline \boldsymbol{B}_2 & \boldsymbol{B}_2 \\ \boldsymbol{B}_2 & \boldsymbol{\text{-}B}_2 \end{array}\right] \tag{6}$$

and

$$\boldsymbol{D}^{**}_{2n} = \left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_2 & \boldsymbol{\text{-}B}_2 \\ \hline \boldsymbol{B}_1 & \boldsymbol{\text{-}B}_1 \\ \boldsymbol{B}_2 & \boldsymbol{B}_2 \end{array}\right] \tag{7}$$

represent valid ways of blocking a design with 2k factors in 2n runs into two blocks without having two-factor interactions fully confounded with the block effect.

The idea can be directly extended to four blocks and beyond. Suppose the regular design $\boldsymbol{D}_n$ is arranged into four blocks such that $\boldsymbol{D}_n = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_3 \\ \boldsymbol{B}_4 \end{bmatrix}$. Then a possible blocking arrangement in four blocks for a design with 2n runs and 2k factors is:

$$
\boldsymbol{D}_{2n} = \left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_4 & \boldsymbol{-B}_4 \\ \hline \boldsymbol{B}_3 & \boldsymbol{-B}_3 \\ \boldsymbol{B}_2 & \boldsymbol{B}_2 \\ \hline \boldsymbol{B}_3 & \boldsymbol{B}_3 \\ \boldsymbol{B}_4 & \boldsymbol{B}_4 \\ \hline \boldsymbol{B}_2 & \boldsymbol{-B}_2 \\ \boldsymbol{B}_1 & \boldsymbol{-B}_1 \end{array} \right] . \qquad (8)
$$

There are $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!} = 105$ ways to arrange the eight submatrices in four blocks. Nine of these, the ones obtained from creating two blocks from $[B_i \quad B_i]$, i=1,2, 3, 4 and two blocks from $[B_i \quad -B_i]$, i=1,2, 3, 4 , will have two-factor interactions fully confounded with block effects. The 96 remaining all represent ways of arranging a design in four blocks without having two-factor interactions fully confounded with block effects. This may be a very effective way to obtain good blocking alternatives. If $\boldsymbol{D}_n$ has been blocked by allocating mirror image pair runs to the same block, the blocks in $\boldsymbol{D}_{2n}$ will also consist of mirror image pair runs and the number of blocks will be a subset of all blocks obtained by using strategy S1. For in that case, arranging $\boldsymbol{D}_{2n}$ in two blocks using the blocking of $\boldsymbol{D}_n$, it follows from (2) that the minimum value of $D_{s,eff}$ for $\boldsymbol{D}_{2n}$ will be the same as for $\boldsymbol{D}_n$ considering projections onto the same dimension. Therefore, the blocks used for blocking $\boldsymbol{D}_{2n}$ should be the best ones obtained from blocking $\boldsymbol{D}_n$. The folding technique used in the blocking schemes is similar to the one suggested in Tyssedal and Samset (2010) for constructing supersaturated designs with good projection properties. For some subsets of factors this technique will reduce the partial confounding between effects and the block defining factor(s), and the average $D_{s,eff}$ for $\boldsymbol{D}_{2n}$ is expected to be higher than the one for $\boldsymbol{D}_n$. This is also observed for the cases where this method has been used.

Strategy S3*: Use of Hadamard matrices*

Given that our design columns are included in a Hadamard matrix, the additional columns are all potential columns for arranging the design in blocks. There are for instance five different Hadamard matrices for *n* = 16 giving rise to five different 16 run designs. Four of those contain the 2_{IV}^{8-4} design and three contain the 2_{V}^{5-1} design (Tyssedal and Box 2001). The use of Hadamard matrices for arranging designs in blocks is particularly useful for blocking non-regular designs, and in our case when there are words of odd length in the defining relation. The method may be very effective as will be demonstrated. However, a combinatorial explosion of the number of Hadamard matrices up to equivalence occurs for *n* = 32 (Kharagani and Tayfeh-Rezaie 2012). The number is then 13710027. For *n* > 32, the number of Hadamard matrices is to our knowledge not known. Hence checking all blocking possibilities becomes infeasible when *n* grows.

Strategy S4*: Arranging blocks in new blocks*

For some designs already blocked in two blocks, one may arrange each block in two new blocks to obtain a blocking scheme for four blocks. This may particularly be useful if the design has been obtained by doubling. For instance, the 128 run 2_{IV}^{64-57} design can be constructed from the 2_{IV}^{32-26} design by doubling and arranged in two blocks using strategy S2. If each of these blocks can be arranged in two blocks, we have a way to block the 2_{IV}^{64-57} design in four blocks.

The importance of the different strategies given here will change as the number of runs increases. If all words in the defining relation are of even length, strategy S1 may be the preferred choice for the number of runs equal to 16 and 32. For designs with more runs, S2 and S4 become more important, if they can be applied, providing us with a more feasible number of candidate-blocks to investigate. S2 may also be used when not all words in the defining relation are of even length, depending on how the design is constructed. Otherwise, in that case, S3 is our suggested strategy. Even though there is a combinatorial explosion in the number of possible Hadamard designs when n = 32, it is our experience that not that many Hadamard matrices need to be examined to obtain blocking alternatives that function well.

5. Blocking Regular Two-level Designs with 16, 32 and 64 Runs

5.1. 16 Run Designs

The 2_{IV}^{8-4} design in Table 1 is a *P* = 3 design and out of the 6435 possible ways to block it in two blocks, 6028 will preserve the projectivity while 407 will not. However, its defining relation has only words of length four and eight and thus it consists of eight mirror image pair runs, making strategy S1 suitable for blocking, leaving us with only 35 possible alternatives to check. As commented in Section 3, the recommended strategy is to use a two-factor interaction column, which will also place mirror image pair runs in the same block. Now, all two-factor interactions are aliased in strings of four, such that all the 28 two-factor interactions can be arranged in 7 sets where all two-factor interactions within one set are fully aliased. Thereby, seven of the 35 possibilities have as a result that three other two-factor interactions are fully aliased with the block effect. All the other 28 alternatives give the same minimum, maximum and average values for $D_{s,eff}$, as given in Table 5, when estimating all main effects and interactions for any three factors and thus provide us with (16,8,3,2) screens. It turns out that these are also the 28 with the highest minimum and average $D_{s,eff}$ among all the 6435 possibilities. For all of them, 48 out of the 56 projections onto three factors will have a $D_{s,eff}$ = 0.917 and eight will have a $D_{s,eff}$ =1 when divided into two blocks. The alternative block defining contrast in Table 1 is generated as $B_b^* = \frac{1}{2}\left[\text{AD+BD+CD-DE}\right]$. Hence there is only partial confounding with two-factor interactions. Using B_b^* as block generator, the 8 projections with a $D_{s,eff}$ =1 are {ABC}, {ABE}, {ACE}, {BCE}, {DFG}, {DFH}, {DGH} and {FGH}.

The 2_V^{5-1} design with defining relation I = ABCDE is a *P* = 4 screen, but it is also a $P = 5_2$ screen. Obviously, none of these properties are possible to preserve when it is blocked. The recommended way of arranging the design in two blocks is to use a two-factor interaction column as block generator. Thereby the blocked design becomes a projectivity *P*=1 design, a rather dramatic loss in projection properties. The resolution of this design is five and thus it is not built up from mirror image pairs. All the 6435 possible ways to run the 2_V^{5-1} design in two blocks were checked out. It turned out that for 60 of those the block generator is only partially confounded with two-factor interactions. For projections onto three factors, all of these block generators perform equally well, providing us with (16,5,3,2) screens. Their $D_{s,eff}$ values are given in Table 5. The design is given in Table 3 with a recommended and a suggested block generator denoted B_b and B_b^* respectively.

Table 3. The 2_V^{5-1} design with two alternative ways of blocking.

Run	A	B	C	D	E	B_b	B_b^*
1	-1	-1	-1	-1	1	1	-1
2	1	-1	-1	-1	-1	-1	-1
3	-1	1	-1	-1	-1	-1	1
4	1	1	-1	-1	1	1	-1
5	-1	-1	1	-1	-1	1	1
6	1	-1	1	-1	1	-1	1
7	-1	1	1	-1	1	-1	1
8	1	1	1	-1	-1	1	-1
9	-1	-1	-1	1	-1	1	1
10	1	-1	-1	1	1	-1	1
11	-1	1	-1	1	1	-1	-1
12	1	1	-1	1	-1	1	1
13	-1	-1	1	1	1	1	-1
14	1	-1	1	1	-1	-1	-1
15	-1	1	1	1	-1	-1	-1
16	1	1	1	1	1	1	1

The recommended block generator is B_b = AB and the suggested one is $B_b^* = \frac{1}{2}[\text{AD+AE+CE-CD}]$. When projected onto four factors, it is possible to estimate all main effects and two-factor interactions with a $D_{s,eff}$ = 0.939 for four out of five projections, using B_b^* as the block defining contrast. With B_b as block generator this is possible for only two of the five projections. For the fifth projection, the one that consists of the factors A, C, D, and E, one is only guaranteed to estimate three two-factor interactions. However, excluding one of the four two-factor interactions AD, AE, CD and CE from the model, one may estimate five of the six two-factor interactions for this projection with a $D_{s,eff}$ = 0.871. In Table 5 this design is denoted a $(16,5,4_{1+3},2)$ screen for projections onto four factors.

5.2. 32 Run Designs

In Table 4 the 2_{IV}^{16-11} design is given, with generators for the 11 factor columns F-Q.

Table 4. The 2_{IV}^{16-11} design with generators.

					ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE	ABCDE
A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
-1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	-1	1
1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
-1	-1	1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1	1
1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1
-1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	1	-1
1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1
-1	-1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	1	1	1
1	-1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	1	-1
-1	1	-1	1	-1	1	1	1	1	-1	1	-1	1	-1	1	-1
1	1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	1	1
-1	-1	1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1	-1
1	-1	1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1
-1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1	-1	-1	1
1	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	-1
-1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1
1	-1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	-1
-1	1	-1	-1	1	1	-1	-1	-1	1	1	1	-1	-1	1	-1
1	1	-1	-1	1	-1	1	1	1	-1	-1	1	-1	-1	1	1
-1	-1	1	-1	1	1	1	1	1	-1	1	1	-1	1	-1	-1
1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1
-1	1	1	-1	1	-1	-1	-1	1	-1	1	-1	1	-1	-1	1
1	1	1	-1	1	1	1	1	-1	1	-1	-1	1	-1	-1	-1
-1	-1	-1	1	1	-1	1	1	1	1	-1	1	1	-1	-1	-1
1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
-1	1	-1	1	1	1	-1	-1	1	1	-1	-1	-1	1	-1	1
1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1
-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1
-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

It has only words of even length in its defining relation and hence allow the use of strategy S1. But it can also be constructed by doubling the 2_{IV}^{8-4} design, and thereby strategy S2 can also be applied for arranging in two blocks. Using strategy S1, there are 6435 possible blocks to consider, but leaving out those that correspond to using a two-factor interaction or a four-factor interaction as block generator, there are only 6420 left. All of these make the blocked design a (32, 16, 3, 2) design, but 1380 of these have a minimum $D_{s,eff}$ = 0.88. The remaining 5040 blocks all gave the same distribution of $D_{s,eff}$, with minimum, maximum and average values given in Table 5. One of these blocking alternatives is given by B_2^* in Table A.1 in Appendix A. It can be generated as follows:

$$B_2^* = 0.25AC-0.25AD+0.25AE-0.25BC+0.5CE+0.5DE-0.25ACDE-0.25BCDE\,.$$

Based on the 28 equally good ways of arranging the 2_{IV}^{8-4} design in two blocks, it is possible to exploit strategy S2 to block the 2_{IV}^{16-11} design. Using both $\boldsymbol{D}_{2n}^{*}$ and the $\boldsymbol{D}_{2n}^{**}$, there are only 56 blocks to check, and they are all equally good according to $D_{s,eff}$. Their minimum, maximum and average values for $D_{s,eff}$ are also given in Table 5. We notice that the average $D_{s,eff}$ is only slightly smaller, despite that the number of investigated alternatives being less than 1% of those with strategy S1.

There are 2627625 possible ways to block a design with 32 runs in four blocks using strategy S1 and 2098336 of these will give us a (32, 16, 3, 4) screen. Among these, 715680 have the highest minimum $D_{s,eff}$ = 0.834 and about equal average. The one given in Table A.1 in Appendix A is one out of 50400 of these that also has a maximum $D_{s,eff}$ = 1. The two block defining contrasts are given by:

$$B_{41}^* = -0.5BD-0.5CD-0.5DE+0.5BCDE \text{ and}$$

$$B_{42}^* = -0.25AD-0.5AE+0.25BD-0.5BE-0.25CD-0.25DE+0.25ABCD+0.25ABDE+0.25ACDE-0.25BCDE$$

Due to their reasonably good projection properties and since there exists recommended blocking schemes for these designs, the 2_{VI}^{6-1}, 2_{IV}^{7-2}, 2_{IV}^{8-3} and the 2_{IV}^{9-4} designs have also been explored for blocking alternatives that may preserve projections properties. Using recommended blocking schemes, not even the 2_{VI}^{6-1} design gives a P = 3 screen when arranged in two blocks.

The 2_{VI}^{6-1} design with generator F = ABCDE is a P = 5 design, which is clearly not obtainable when it is blocked. It can be constructed from the 2_{IV}^{16-11} design by removing all factors assigned to three-factor interaction columns. Blocked the recommended way in two blocks, it becomes a P = 2 design and even a P = 1 design when arranged in four. The word in its defining relation is of even length, and strategy S1 can be used. The 5040 blocking alternatives found to have the lowest confounding with two-factor interactions for the 2_{IV}^{16-11} design provided us with both (32,6,3,2), (32,6,4,2) and $\left(32,6,5_3,2\right)$ screens, and as (32,6,4,2) screens they are all equally good. Some preferences of blocking can be made for the $\left(32,6,5_3,2\right)$ screen if also four factor interactions are of concern. Details can be found in Hamre (2018). Strategy S1 was also used for arranging the 2_{VI}^{6-1} design in four blocks. Considering projections onto three, four and five factors separately, one may choose different block defining contrasts in each case. Out of the many alternatives (Hamre 2018), we have chosen to present two block defining contrasts with reasonably good performance for five active factors or less. The block defining contrasts for arranging the design in two and four blocks are given in Table A.1 in Appendix A, and they may be obtained as follows:

B_2^*= 0.25AB+0.25AC+0.25BD+0.25BE+0.25CD+0.25CE+0.5DE-0.25ABDE-0.25ACDE-0.5BCDE

B_{41}^*= -0.5AC-0.25AE-0.5BD-0.25BE-0.25CE-0.25DE+0.25ABCE+0.25ABDE+0.25ACDE+0.25BCDE

B_{42}^*= 0.25AB+0.25AC+0.25AE-0.25BD-0.25CD-0.5CE-0.25DE-0.25ABCE-0.5ACDE+0.25BCDE

The corresponding $D_{s,eff}$ values are given in Table 5.

The projection properties of the minimum aberration 2_{IV}^{7-2}, 2_{IV}^{8-3} and 2_{IV}^{9-4} designs when arranged in two and four blocks the recommended ways, are given in Table 2. When blocked in two blocks, the projectivity is two for the 2_{IV}^{7-2} and the 2_{IV}^{8-3} designs, and it is further reduced to one when arranged in four. The 2_{IV}^{9-4} design is even a (32, 8, 1, 2) screen. None of the designs have a defining relation consisting of only even words. Strategy S3 is therefore a possible alternative trying to obtain good blocking schemes. We have used two 32 run Hadamard matrices, had.32.t1 and had.32.t3, from the web page ”A library of Hadamard matrices” (Sloane 2018), hereafter referred to as M_1 and M_2, respectively. For both matrices all rows with a -1 in the first column of M_1 and M_2 were multiplied with -1, and thereafter the rows were rearranged such that the principle factor columns A,B,C,D and E could be identified among the 32 columns. The additional factor columns were generated by their generators:

F = ABC and G = ABDE for the 2_{IV}^{7-2} design

F = ABC, G = ABD and H = ACDE for the 2_{IV}^{8-3} design

F = ABC, G =ABD, H =ACD and J = BCDE for the 2_{IV}^{9-4} design.

The remaining 26 columns in the rearranged Hadamard matrices were then tested as potential block generators. For all three designs, M_1 yielded good results for arranging in two blocks, while M_2 provided good results for arranging in four. Block generators orthogonal to all main effects contrasts were found, and projection properties were preserved. Several equally good alternatives exist, Hamre (2018). One for each design and each blocking scheme is given in Table A.2 in Appendix A. $D_{s,eff}$ values are given in Table 5.

5.3. 64 Run Designs

The 2_{IV}^{32-26} design can be constructed by doubling the 2_{IV}^{16-11} design given in Table 3. Hence arranging the 2_{IV}^{32-26} design in two and four blocks can be done using strategy S2, exploiting the preferred ways of blocking the 2_{IV}^{16-11} design. All the 5040 blocks that turned out equally good using strategy S1 for the 2_{IV}^{16-11} design, were tested for both configurations $\boldsymbol{D}_{2n}^{*}$ and $\boldsymbol{D}_{2n}^{**}$. With the same configuration, all the tested blocks gave the same distributions of $D_{s,eff}$ with

$$\boldsymbol{D}_{2n}^{**} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_2 & -\boldsymbol{B}_2 \\ \hline \boldsymbol{B}_1 & -\boldsymbol{B}_1 \\ \boldsymbol{B}_2 & \boldsymbol{B}_2 \end{bmatrix}$$

being slightly better. Two possible alternatives for $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are given in Table A.1 in Appendix A. $D_{s,eff}$ values are given in Table 5.

For arranging the 2_{IV}^{16-11} design in four blocks there are 40320 equally good alternatives as measured by D_s-efficiency. All of these could then be tested for all the 96 possible configurations and for all 4960 possible projections onto three dimensions. We chose to only use 10 of these blocks.

For all these only the configuration mattered and four were better than the others. One of these is given below with blocks $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}_3$ and $\boldsymbol{B}_4$ taken from Table A.1. $D_{s,eff}$-values are given in Table 5.

$$\left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_1 \\ \boldsymbol{B}_4 & \boldsymbol{B}_4 \\ \hline \boldsymbol{B}_4 & \textbf{-}\boldsymbol{B}_4 \\ \boldsymbol{B}_3 & \boldsymbol{B}_3 \\ \hline \boldsymbol{B}_3 & \textbf{-}\boldsymbol{B}_3 \\ \boldsymbol{B}_2 & \textbf{-}\boldsymbol{B}_2 \\ \hline \boldsymbol{B}_2 & \boldsymbol{B}_2 \\ \boldsymbol{B}_1 & \textbf{-}\boldsymbol{B}_1 \end{array}\right].$$

The 2_V^{8-2} design with generators G = ABCD and H = ABEF is of odd resolution. Strategy S3 was used to block the design in 2, 4 and 8 blocks. The 64 run Hadamard matrices used to obtain orthogonal main effects blocking were the ones obtained by doubling the 32 run Hadamard matrices M1, M2 and had32.t3 (Sloane 2018) after rearranging. All of them contained the 6 principal factor columns A, B, C, D, E and F. The 57 remaining columns were all tested as block alternatives, assuring that main effects were orthogonal to the blocks. Several possibilities turned out to perform equally well, see Hamre (2018). Table A.3 in Appendix A provide ways to block the design in 2, 4 and 8 blocks. The $D_{s,eff}$ values are given in Table 5.

Table 5 summarizes the projection properties and the minimum, maximum and average value of the $D_{s,eff}$ criterion for the designs that are investigated. We observe that in many cases the projection properties are preserved when these designs are blocked. In others the projection properties obtained are much better than when the recommended way of blocking is used. The price one has to pay is a slight decrease in $D_{s,eff}$.

Table 5. Projection properties and values of $D_{s,eff}$ for the design blocked

Design	Strategy	Screen	Min $D_{s,eff}$	Max $D_{s,eff}$	Mean $D_{s,eff}$
2_{IV}^{8-4}	S1 + all possibilities	(16,8,3,2)	0.917	1	0.929
2_V^{5-1}	S3 + all possibilities	(16,5,3,2)	0.917	1	0.934
2_V^{5-1}	S3 + all possibilities	$(16,5,4_{1+3},2)$	0.814	1	0.952
2_{IV}^{16-11}	S1	(32,16,3,2)	0.917	1	0.970
2_{IV}^{16-11}	S2	(32,16,3,2)	0.917	1	0.967
2_{IV}^{16-11}	S1	(32,16,3,4)	0.834	1	0.908
2_{VI}^{6-1}	S1	(32,6,3,2)	0.943	0.983	0.971

2_{VI}^{6-1}	S1	(32,6,4,2)	0.917	0.982	0.959
2_{VI}^{6-1}	S1	$(32,6,5_3,2)$	0.948	0.963	0.958
2_{VI}^{6-1}	S1	$(32,6,5_{3+2},2)$	0.906	0.966	0.939
2_{VI}^{6-1}	S1 + S4	(32,6,3,4)	0.865	0.949	0.911
2_{VI}^{6-1}	S1	$(32,6,4_3,4)$	0.863	0.919	0.893
2_{VI}^{6-1}	S1	$(32,6,5_3,4)$	0.866	0.866	0.866
2_{IV}^{7-2}	S3	(32,7,3,2)	0.917	1	0.982
2_{IV}^{7-2}	S3	(32,7,3,4)	0.917	1	0.939
2_{IV}^{8-3}	S3	(32,8,3,2)	0.917	1	0.981
2_{IV}^{8-3}	S3	(32,8,3,4)	0.853	1	0.929
2_{IV}^{9-4}	S3	(32,9,3,2)	0.917	1	0.982
2_{IV}^{9-4}	S3	(32,9,3,4)	0.853	1	0.925
2_{IV}^{32-26}	S2	(64,32,3,2)	0.917	1	0.987
2_{IV}^{32-26}	S2	(64,32,3,4)	0.913	1	0.987
2_{V}^{8-2}	S3	(64,8,3,2)	0.965	1	0.992
2_{V}^{8-2}	S3	(64,8,4,2)	0.958	1	0.986
2_{V}^{8-2}	S3	(64,8,3,4)	0.931	1	0.982
2_{V}^{8-2}	S3	(64,8,4,4)	0.917	1	0.966
2_{V}^{8-2}	S3	(64,8,3,8)	0.834	1	0.918
2_{V}^{8-2}	S3	(64,8,4,8)	0.808	0.917	0.887

The decrease in $D_{s,eff}$ is related to an increase in the standard deviations of the effect estimators. For a given projection onto P factors, the covariance matrix for all coefficient estimators (block effect(s) included) is given by $\sigma^2\left[\mathbf{X}^T\mathbf{X}\right]^{-1}$. If $\hat{\beta}_i$ is the effect coefficient estimator with the largest variance given by $\sigma^2\left(\mathbf{X}^T\mathbf{X}\right)_{ii}^{-1}$ and $\hat{\beta}_j$ the one with the smallest variance given by $\sigma^2\left(\mathbf{X}^T\mathbf{X}\right)_{jj}^{-1}$, the ratio of their standard deviations is $SD_e\text{-}ratio=\sqrt{\frac{\left(\mathbf{X}^T\mathbf{X}\right)_{ii}^{-1}}{\left(\mathbf{X}^T\mathbf{X}\right)_{jj}^{-1}}}$. Similarly, if $\hat{\beta}_b$ is the block defining contrast with the largest variance, one may define $SD_b\text{-}ratio=\sqrt{\frac{\left(\mathbf{X}^T\mathbf{X}\right)_{bb}^{-1}}{\left(\mathbf{X}^T\mathbf{X}\right)_{jj}^{-1}}}$.

In Table 6 these ratios are computed for each design and for two projections, one giving the smallest $D_{s,eff}$ and one giving the largest. Since main effects columns are orthogonal to the block defining contrasts, their corresponding estimators always attain the smallest variance. These ratios therefore give the increase in standard deviations due to some interactions being partially confounded with the block defining contrast(s). As observed, this increase is normally larger for projections having the smallest $D_{s,eff}$ and then often in the range 20-40%. As intended, the increase in standard deviations is smaller for the factor effects than for the block effects in most cases.

Table 6. Standard deviation ratios for projections with smallest and largest $D_{s,eff}$

Design	Screen	Min $D_{s,eff}$	SD_e^{min}-*ratio*	SD_b^{min}-*ratio*	Max $D_{s,eff}$	SD_e^{max}-*ratio*	SD_b^{max}-*ratio*
2_{IV}^{8-4}	(16,8,3,2)	0.917	1.2247	1.4142	1	1	1
2_{V}^{5-1}	(16,5,3,2)	0.917	1.2247	1.4142	1	1	1
2_{V}^{5-1}	$(16,5,4_{1+3},2)$	0.814	1.4142	2	1	1	1
2_{IV}^{16-11}	(32,16,3,2)	0.917	1.2247	1.4142	1	1	1
2_{IV}^{16-11}	(32,16,3,2)	0.917	1.2247	1.4142	1	1	1
2_{IV}^{16-11}	(32,16,3,4)	0.834	1.3904	1.5275	1	1	1
2_{VI}^{6-1}	(32,6,3,2)	0.943	1.128	1.206	0.983	1.035	1.069
2_{VI}^{6-1}	(32,6,4,2)	0.917	1.4142	2	0.982	1.0408	1.1547
2_{VI}^{6-1}	$(32,6,5_3,2)$	0.948	1.4142	2	0.963	1.2910	1.6330
2_{VI}^{6-1}	$(32,6,5_{3+2},2)$	0.906	2.2361	4	0.966	1.2910	1.6330
2_{VI}^{6-1}	(32,6,3,4)	0.865	1.318	1.318	0.949	1.0742	1.0742
2_{VI}^{6-1}	$(32,6,4_3,4)$	0.863	1.512	1.604	0.919	1.291	1.4142
2_{VI}^{6-1}	$(32,6,5_3,4)$	0.866	1.8257	2.4495	0.866	1.8257	2.4495
2_{IV}^{7-2}	(32,7,3,2)	0.917	1.2247	1,4142	1	1	1
2_{IV}^{7-2}	(32,7,3,4)	0.917	1.2247	1,4142	1	1	1
2_{IV}^{8-3}	(32,8,3,2)	0.917	1.2247	1,4142	1	1	1
2_{IV}^{8-3}	(32,8,3,4)	0.853	1.2247	1,4142	1	1	1
2_{IV}^{9-4}	(32,9,3,2)	0.917	1,2247	1,4142	1	1	1
2_{IV}^{9-4}	(32,9,3,4)	0.853	1.2247	1,4142	1	1	1
2_{IV}^{32-26}	(64,32,3,2)	0.917	1.2247	1,4142	1	1	1

2_{IV}^{32-26}	(64,32,3,4)	0.913	1.1704	1.2163	1	1	1
2_{V}^{8-2}	(64,8,3,2)	0.965	1.1547	1.1547	1	1	1
2_{V}^{8-2}	(64,8,4,2)	0.958	1.2247	1.4142	1	1	1
2_{V}^{8-2}	(64,8,3,4)	0.931	1.1547	1.1547	1	1	1
2_{V}^{8-2}	(64,8,4,4)	0.917	1.2247	1.4142	1	1	1
2_{V}^{8-2}	(64,8,3,8)	0.834	1.5492	1.5492	1	1	1
2_{V}^{8-2}	(64,8,4,8)	0.808	1.5492	1.5492	0.917	1,2247	1.4142

6. Discussion and Comparison of Various Ways of Evaluating Blocked Regular Designs

Most traditional blocking schemes for regular design are evaluated and ranked from the two wordlength patterns W and W_b. Common for all the blocking schemes is that the designs are at least of resolution III, and that main effects are unconfounded with block effects. Assuming that three-factor and higher order interactions are negligible, a rather common assumption when blocking schemes are evaluated, the relevant terms to consider are A_3, A_4 and A_{2b}.

Aliased effects can be partitioned into alias sets. For instance, from the word ABC three alias sets can be constructed, A and BC, B and AC and C and AB. Let n_{mb} be the number of two-factor interactions that are either aliased with main effects or confounded with blocks and let $M_1,\ldots,M_f$ be the alias sets that do not contain such effects. With k main effects the number of two-factor interactions in these alias sets sums to $\binom{k}{2}-n_{mb}$. Distributing these two-factor interactions as uniformly as possible over $M_1,\ldots,M_f$ is related to minimizing A_4 (Cheng et al. 1999). A small A_4 is also necessary in order to have many clear two-factor interactions. Minimization of $3A_3+A_{2b}$ amounts to minimizing n_{mb}.

Sequentially minimizing the sequence $3A_3, A_4, A_{2b}$ (Cheng and Wu 2002) might then support what we have denoted the recommended way of blocking, emphasizing many clear two-factor interactions, while the same sequential operation on $3A_3+A_{2b}, A_4$ fits well into the way of blocking two-level designs proposed in Cheng and Mukerjee (2001), and mentioned in the introduction. It is interesting to note that these sequential operations lead to two different blocking schemes for a 2^{5-1} design arranged in two blocks. The recommended one, which is a

2_V^{5-1} design with defining relation I=ABCDE and the two-factor interaction AB used as block factor, and one which is a 2_{IV}^{5-1} design with defining relation I=ABCE with ABD used as block factor. The first scheme has 9 clear two-factor interactions and projectivity $P=1$. The second scheme has only 4 clear two-factor interactions but projectivity $P=3_2$.

It is not obvious how blocking schemes with only partial confounding can be fairly evaluated with criteria constructed from W and W_b. Cheng and Mukerjee (2001) pointed out that their criterion could be tied to maximum estimation capacity, see also Chen and Cheng (1999) and Cheng et al. (1999). For a blocked two-level design d with k factors, let $E_u\left(d\right)$ be the number of models with k main effects and u two-factor interactions that can be estimated by d. Then an estimation capacity sequence $\left(E_1\left(d\right), E_2\left(d\right), \ldots\right)$ can be constructed and used for comparison of blocked designs. For two designs d_1 and d_2, if $E_u\left(d_1\right) \geq E_u\left(d_2\right)$ for all u with strict inequality for some, then d_1 dominates d_2. Let the two blocking schemes for the 2^{5-1} designs be denoted $2_V^{5-1,AB}$ and $2_{IV}^{5-1,ABD}$ respectively. A comparison of the estimation capacity sequence for these two schemes and our blocking scheme, denoted $2_V^{5-1,B_b^*}$ is given in Table 7.

Table 7. Estimation capacity sequence for three blocking schemes for 2^{5-1} designs.

Design	$E_1\left(d\right)$	$E_2\left(d\right)$	$E_3\left(d\right)$	$E_4\left(d\right)$	$E_5\left(d\right)$	$E_6\left(d\right)$	$E_7\left(d\right)$	$E_8\left(d\right)$	$E_9\left(d\right)$
$2_{IV}^{5-1,ABD}$	10	42	96	129	102	44	8	0	0
$2_V^{5-1,AB}$	9	36	84	126	126	84	36	9	1
$2_V^{5-1,B_b^*}$	10	45	120	209	246	195	100	30	4

The $2_{IV}^{5-1,ABD}$ dominates the $2_V^{5-1,AB}$ with respect to estimation capacity for $u \leq 4$, but for $u \geq 5$, it is opposite. However, both these schemes are clearly dominated by the $2_V^{5-1,B_b^*}$. The reason why is easily explainable. Let us just compare the $2_V^{5-1,AB}$ and the $2_V^{5-1,B_b^*}$. There are enough degrees of freedom for estimating nine two-factor interactions together with five main effects, the intercept and the block effect. For the $2_V^{5-1,AB}$ we then get $E_u\left(d\right)=\binom{9}{u}$, $1 \leq u \leq 9$. B_b^* is constructed from four two-factor interaction columns and partially confounded with the corresponding effects, which means that only three of them can be estimated at the same time. Leaving out one, say AD, we have a set of nine two-factor interactions to freely choose from. For given $1 \le u \le 9$ this gives $\binom{9}{u}$ different models that can be estimated. In addition, we also need to count the ones where AD is forced to be in the model.

Words of length four in the defining relation give alias sets consisting of more than one two-factor interaction. Let us assume there are a such alias sets and let A_c be the collection of these. Further let c be the number of clear two-factor interactions. When possible and needed, the recommended way of blocking uses one or more two-factor interaction column(s) corresponding to effects in A_c as block defining contrast(s) to have as many clear two-factor interactions as possible. To get a better understanding of how this affects the estimation capacity sequence, let us look at the 2_{IV}^{7-2} design run in four blocks. The design has a = 3 with two members in each and c = 15. Blocked the recommended way, ACD and BCD are used as block factors. These are chosen such that their interaction effect is a two-factor interaction in one of the alias sets in A_c, leaving only $a-1$=2 alias sets to consider for estimation capacity. For a given u, let $n_{(u|y,z)}$ be the number of models that can be estimated using y alias sets in A_c and z of the clear two-factor interactions. Further define $n_{(0|\cdot,\cdot)} = 1$. Then to obtain $n_{(u|2,15)}$ we may count the ones when one of the clear two-factor interactions is taken out, and then add the ones we get when this interaction is forced to be in the model. Hence, we have $n_{(u|2,15)} = n_{(u|2,14)} + n_{(u-1|2,14)}$. If instead the two block factors had been chosen such that their interaction effect was one of the 15 clear two-factor interactions as in Xu and Lau (2006), the number of models that can be estimated had become $n_{(u|3,14)} = n_{(u|2,14)} + 2n_{(u-1|2,14)} > n_{(u|2,15)}$. Hence the recommended way of blocking, giving priority to having as many clear two-factor interactions as possible, does not necessarily give a blocking scheme with the best estimation capacity.

Blocking the 2_{IV}^{7-2} design in four blocks the way we propose, one of the block factors is constructed from four clear two-factor interaction columns. The two others can be expressed as a linear combination of two-factor interaction and higher order interaction columns, a situation not uncommon when strategy S3 is used. Leaving out one clear two-factor interaction, say AD, we get that the number of estimated models for a given u equals $n_{(u|3,14)}$ + the ones we get when AD is forced to be in the models. Hence it dominates both the schemes given above.

We have checked all our proposed blocking schemes. The one for the 2_{VI}^{6-1} design stands out as an exception. Blocking this design in two and four blocks the recommended way make use of three-factor interaction columns. Arranged in four blocks one of the two-factor interactions will be fully confounded with a block effect. We have used strategy S1 which for this design, as explained in Section 4, will cause block factor effects to be partially confounded with two-factor interaction, and for our scheme 10 such for each block factor. As a result, the recommended way of blocking dominates our schemes with respect to estimation capacity for $u \geq 10$ when the design is run in two blocks. Arranged in four blocks, our way of blocking will as a maximum allow the estimation of 12 two-factor interactions together with 6 main effects, while the recommended way will allow up to 14. Another strategy, for instance S3, might have given blocking schemes with better estimation capacities. In all other cases our schemes were equally good or better than the recommended ones, even though estimation capacity was not a criterion for choosing scheme and strategy.

From what we have seen, different criteria may lead to different blocking schemes with different projection properties. Maximizing the number of clear two-factor interactions may come in conflict with maximizing the estimation capacity. Our blocking schemes were constructed with the purpose of being able to estimate as many effects as possible for a subset of factors up to a certain size, which differs from the motivation behind the recommended way of blocking and the method suggested by Cheng and Mukerjee (2001). It is therefore encouraging that our schemes in most cases seem to do that well with respect to estimation capacity and are comparable with the recommended way of blocking when it comes to the number of two-factor interactions that can be estimated in a model.

The partial confounding exploited in our blocking schemes impacts the efficiency with which the effects can be estimated for some models. When blocked in two blocks the recommended way, the 2_{VI}^{6-1} design is a $P = 6_2$ design, and the 2_{IV}^{7-2} and 2_{IV}^{8-3} designs both have projectivity $P = 3_2$. The 2_V^{8-2} design is a $P = 8_2$ design when blocked in two and four blocks. Hence, if the respective number of main effects and two-factor interactions are considered adequate for modelling the response, the recommended way of blocking represents a good alternative in these cases, having a $D_{s,eff}$ = 1 for the effects of interest. However, if factor sparsity is a reasonable assumption, research shows that the better projection properties a design has, the more robust the screening will be with respect to assumptions about the model, amount of noise and distortion of effects too small to be detected (Tyssedal and Chaudhry 2017, Chaudhry and Tyssedal 2019 and Chaudhry 2019).

7. Concluding Remarks

We have demonstrated that many regular two-level designs with 16, 32 and 64 runs can be blocked such that their projection properties are preserved or only weakly affected. Different strategies have been used depending on how the designs are constructed. These include letting mirror image pair runs be in the same block, exploiting that blocking of a *n* run design can be utilized to block a design in 2*n* runs and taking advantage of the rich selection of different Hadamard matrices that exists. We have restricted our blocking to cases with an equal number of runs in the same block and kept the block defining contrast(s) orthogonal to main effect contrasts. Blocking alternatives have been assessed using the $D_{s,eff}$ criterion. Following these strategies has made it possible to obtain blocking schemes with better projection properties than can be achieved using higher order interactions columns from regular designs. Unintended, our blocking schemes also seem to have good estimation capacity, sometimes better than other schemes, and are comparable with the blocking schemes given in Wu and Hamada (2009) when it comes to how many two-factor interactions that can be estimated in a model.

When the number of runs increases there are enormously many possible blocking alternatives and investigating all of them becomes almost infeasible in time. For the two 16 run designs we were able to examine all possible blocking arrangements. It was encouraging that strategy S1 led us directly to the best ones for the 2_{IV}^{8-4} design, according to our criteria, by only considering less than 1% of the alternatives. The best way of blocking may also depend on how well one wants to estimate the effects when the design is projected onto different dimensions.

Each choice of dimension may prefer different block defining contrast(s). It is our experience, however, that searching strategies, such as S2, that only consider a rather small subset of all blocking candidates, come up with almost equally good blocking alternatives as when investigating more numerous ways of blocking. We believe that the reason why S1 and S2 perform that well is because alias reduction is inherent in both.

The use of Hadamard matrices, strategy S3, represent an alternative way of blocking that has a potential to be successful in preserving projection properties when blocking all regular designs, and particularly for those that does not fit into the design structures for using S1 and S2. An exhaustive search is also here close to infeasible when n increases, due to the combinatorial explosion of Hadamard matrices that occurs for n = 32. Fortunately, it seems like high values for minimum and average $D_{s,eff}$ can be obtained by only searching through a few Hadamard matrices.

Preserving projection properties when blocking has a price. Some effects will be estimated with less efficiency due to partial confounding of interactions with block generators. We think that for a factor-based screening this drawback is well compensated for by the possibility to separate estimates of interactions and block effects. We also believe that the methods presented here can be useful for experimental work with other restrictions on randomization than blocking, such as split-plot experimentation. Especially full confounding between subplot interactions and whole plot interactions can be avoided assigning whole plot factors to columns here used as block defining contrasts.

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Appendix A

Table A.1. Possible ways to arrange the 2_{IV}^{16-11} and the 2_{VI}^{6-1} designs in two and four blocks in order to have good projection properties.

2_{IV}^{16-11}					2_{VI}^{6-1}		
B_2^*	Blocks	B_{41}^*	B_{42}^*	Blocks	B_2^*	B_{41}^*	B_{42}^*
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	1	$\boldsymbol{B}_3$	1	1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	1
1	$\boldsymbol{B}_1$	-1	1	$\boldsymbol{B}_3$	1	-1	-1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	1
1	$\boldsymbol{B}_1$	1	-1	$\boldsymbol{B}_2$	1	-1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	-1
1	$\boldsymbol{B}_1$	1	1	$\boldsymbol{B}_1$	-1	-1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	-1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
-1	$\boldsymbol{B}_2$	-1	1	$\boldsymbol{B}_3$	-1	1	-1
-1	$\boldsymbol{B}_2$	-1	1	$\boldsymbol{B}_3$	1	-1	1
-1	$\boldsymbol{B}_2$	-1	1	$\boldsymbol{B}_3$	1	-1	1
-1	$\boldsymbol{B}_2$	-1	1	$\boldsymbol{B}_3$	-1	1	-1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	-1
1	$\boldsymbol{B}_1$	1	1	$\boldsymbol{B}_1$	-1	-1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	-1
1	$\boldsymbol{B}_1$	1	-1	$\boldsymbol{B}_2$	1	-1	1
-1	$\boldsymbol{B}_2$	1	1	$\boldsymbol{B}_1$	-1	1	1
-1	$\boldsymbol{B}_2$	1	-1	$\boldsymbol{B}_2$	-1	1	1
1	$\boldsymbol{B}_1$	-1	1	$\boldsymbol{B}_3$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	1
1	$\boldsymbol{B}_1$	-1	1	$\boldsymbol{B}_3$	1	1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1
1	$\boldsymbol{B}_1$	-1	-1	$\boldsymbol{B}_4$	1	-1	-1

Table A.2. Possible ways to run the 2_{IV}^{7-2}, 2_{IV}^{8-3} and the 2_{IV}^{9-4} designs in two and four blocks such that projection properties are preserved.

2_{IV}^{7-2}			2_{IV}^{8-3}			2_{IV}^{9-4}		
B_2^*	B_{41}^*	B_{42}^*	B_2^*	B_{41}^*	B_{42}^*	B_2^*	B_{41}^*	B_{42}^*
-1	-1	1	1	1	1	-1	-1	1
-1	1	-1	1	-1	-1	1	1	-1
1	1	1	1	1	-1	-1	-1	1
1	1	-1	-1	1	1	-1	1	1
1	1	1	-1	1	-1	1	-1	-1
1	-1	-1	-1	-1	1	1	1	-1
-1	-1	1	1	1	1	-1	-1	-1
-1	-1	-1	-1	1	-1	1	1	1
1	1	1	1	-1	1	1	1	1
1	-1	1	1	-1	1	1	-1	-1
-1	-1	-1	1	1	1	1	1	1
-1	-1	-1	-1	-1	1	-1	-1	1
-1	-1	1	-1	-1	-1	1	1	-1
-1	1	1	-1	-1	-1	-1	-1	-1
1	1	-1	1	1	-1	-1	1	-1
1	1	-1	-1	-1	-1	-1	-1	1
1	-1	-1	-1	-1	-1	1	1	1
1	1	1	-1	1	1	-1	-1	-1
1	1	-1	-1	-1	1	1	-1	-1
1	1	1	1	-1	-1	1	1	-1
-1	1	-1	1	-1	1	-1	-1	1
-1	-1	1	1	1	-1	-1	1	1
-1	-1	-1	-1	-1	-1	1	1	-1
-1	-1	1	1	-1	1	-1	-1	1
-1	1	-1	-1	1	-1	-1	-1	1
-1	-1	-1	-1	1	-1	-1	1	-1
-1	-1	1	-1	-1	-1	-1	1	-1
-1	-1	1	1	1	-1	1	-1	-1
1	-1	-1	1	1	1	-1	1	1
1	1	-1	1	1	1	1	-1	1
1	1	1	-1	-1	1	1	-1	-1
1	1	1	1	1	1	1	1	1

Table A.3. Possible blocking generators for blocking the 2_V^{8-2} design in two, four and eight blocks

Row	B_2^*		B_{41}^*		B_{42}^*		B_{81}^*		B_{82}^*		B_{83}^*	
1/33	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1
2/34	-1	1	1	1	-1	1	1	-1	1	1	1	1
3/35	-1	-1	-1	-1	-1	1	-1	-1	-1	1	1	-1
4/36	1	-1	1	-1	1	1	1	-1	-1	-1	1	1
5/37	1	1	-1	1	1	1	1	1	1	-1	-1	1
6/38	1	-1	1	1	1	-1	1	-1	-1	-1	-1	-1
7/39	1	1	-1	-1	1	1	1	1	1	-1	1	-1
8/40	-1	1	1	-1	-1	1	1	-1	1	-1	-1	-1
9/41	1	1	1	-1	-1	1	1	-1	-1	1	1	1
10/42	1	-1	-1	-1	1	1	1	1	-1	1	-1	1
11/43	1	1	1	1	1	-1	1	-1	1	1	1	1
12/44	-1	1	-1	1	1	1	1	1	-1	1	1	-1
13/45	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1
14/46	-1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	1
15/47	-1	-1	1	1	-1	1	1	-1	-1	-1	-1	-1
16/48	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1
17/49	-1	1	1	1	1	1	-1	1	-1	-1	-1	-1
18/50	1	1	1	-1	1	-1	1	1	1	-1	1	-1
19/51	1	-1	-1	1	1	1	-1	1	1	1	-1	-1
20/52	1	1	-1	-1	-1	1	1	1	1	-1	-1	1
21/53	1	-1	1	1	-1	-1	-1	1	1	1	1	1
22/54	-1	-1	1	-1	-1	1	-1	-1	-1	1	1	-1
23/55	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1
24/56	-1	-1	-1	-1	1	-1	-1	-1	-1	1	-1	1
25/57	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1
26/58	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1
27/59	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	1
28/60	-1	-1	1	1	-1	-1	-1	1	-1	-1	-1	-1
29/61	-1	1	-1	-1	-1	1	1	1	-1	1	1	-1
30/62	1	1	-1	1	1	1	-1	1	-1	-1	1	1
31/63	1	-1	1	-1	1	-1	1	1	-1	1	-1	1
32/64	1	1	1	1	1	1	-1	1	1	1	1	1