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Interval Observers for Discrete-Time Linear Systems with Uncertainties

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Abstract: In this paper, we consider the problem involved when designing the interval observer for the system described by a linear discrete-time model under external disturbances and measurement noises. To solve this problem, we used the reduced order model of the initial system, which is insensitive or has minimal sensitivity to the disturbances. The relations involved in designing the interval observer, which has minimal dimensions and estimates the prescribed linear function of the original system state vector, were obtained. The theoretical results were illustrated by a practical example.

Keywords: linear models; estimation; interval observers; identification canonical form

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1. Introduction

The problem of estimating the system state vector is critical in many practical applications. The main problems involved in designing an estimator are the system complexity and different uncertainties (external disturbances, measurement noises, and unknown parameters). Sliding mode observers can solve this problem [1–3] in some cases; however, under uncertainties, the estimation error is never equal to zero. This problem has recently been solved based on interval observers, which are used to evaluate the dynamic system state. One advantage of interval observers is that they can take into account many types of uncertainties in the system under consideration.

Different kinds of observers have been developed for many types of models: for continuous-time linear and non-linear [4–11], discrete-time [12,13], time delay [14,15], switched system [16,17], and singular [14]; the stability of interval observers was studied in [18]. Moreover, they have been successfully applied to solve many practical problems [19–21]. Exhaustive reviews are in [15,22,23].

It should be noted that all of the above-mentioned papers consider the full-state vector interval estimation problem. Unlike these papers, the uniqueness of the present paper is that the interval observers were constructed for estimating the prescribed linear function of the original system state vector. As a result, the suggested approach has fewer computational complexities than those considered in the above-mentioned papers. Such a solution may be useful in some practical applications where only the prescribed linear function of the state vector is necessary. Our approach is close to that of functional interval observers considered in [24–27], which enable estimating specified linear functions of the vector of state.

The main contributions of this paper are as follows: (i) unlike [15,23], where the interval observer was designed based on the original system, the reduced order model of the system was used to design the observer that allows accelerating the measurement results processing; (ii) unlike [15,23], where the full-state vector was estimated, the suggested approach allows for estimating the prescribed components of the state vector, which may be useful in some practical applications; (iii) the reduced order model is invariant with respect to the disturbance or has minimum sensitivity that allows reducing the interval width and increasing estimation accuracy; (iv) finally, identifying the canonical form (to design the interval observers) enabled obtaining simple designing procedures.

2. The Main Models

Consider the linear system described by the difference equations

$$\begin{array}{rcl} x(t+1) & = & Fx(t) + Gu(t) + L\rho(t), \\ y(t) & = & Hx(t) + v(t). \end{array} \tag{1}$$

Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ are vectors of state, control, and output; $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times l}$, $H \in \mathbb{R}^{l\times n}$, and $L \in \mathbb{R}^{n\times p}$ are known constant matrices; $\rho(t) \in \mathbb{R}^p$ is the disturbance, one assumes that $\rho(t)$ is an unknown bounded function, and $\max_i |\rho_i(t)| \le \rho_*$; $v(t)$ is the measurement noise; one assumes that $v(t) \in \mathbb{R}^l$ is a bounded unknown function and $\max_i |v_i(t)| \le v_*$.

The problem is to design an interval observer of minimal dimensions generating two functions $\underline{z}(t)$ and $\overline{z}(t)$, such that $\underline{z}(t) \le z(t) \le \overline{z}(t)$ for all $t \ge 0$ where $z(t)$ is determined by a known matrix $M \in \mathbb{R}^{s\times n}$ as $z(t) = Mx(t)$. For two vectors, x_1, x_2, and matrices, A_1, A_2, the inequalities $x_1 \le x_2$ and $A_1 \le A_2$ are understood element-wise.

The solution is based on the reduced-order model

$$\begin{array}{rcl} x_*(t+1) & = & F_*x_*(t) + G_*u(t) + J_*Hx(t) + L_*\rho(t), \\ z(t) & = & H_zx_*(t) + Qy_0(t), \end{array} \tag{2}$$

where $x_* \in \mathbb{R}^k$ is the state vector, F_*, G_*, J_*, L_*, Q, and H_z are matrices of appropriate dimensions to be determined, the variable y_0 is defined below.

Remark 1. *Model (2) is essentially part of system (1); therefore, we used the term $J_*Hx(t)$ other than $J_*y(t)$ to take into account measurement noise due to $y(t) = Hx(t) + v(t)$. The term $J_*y(t)$ will be used in the interval observer (12).*

The best solution from the interval width point of view is when the disturbance $\rho(t)$ does not affect the model. Clearly, the variable y_0 in (2) must be insensitive to $\rho(t)$ as well. To satisfy the last demand, consider the matrix L_0 with a maximal number of rows, such that $L_0L = 0$. Then the vector $x' = L_0x$ is insensitive to $\rho(t)$ and $y_0 = N_1x' = N_1L_0x$ with some matrix N_1. On the other hand, y_0 is a part of the output vector y, then $y_0 = N_2y(t) = N_2Hx$ with some matrix N_2. Then one has the equation $N_1L_0 = N_2H$ with a solution, if

$$\operatorname{rank}\begin{pmatrix} L_0 \\ H \end{pmatrix} < \operatorname{rank}(L_0) + \operatorname{rank}(H).$$

If this condition is satisfied, the equation $N_1L_0 = N_2H$ in the form

$$(N_1 \quad -N_2)\begin{pmatrix} L_0 \\ H \end{pmatrix} = 0$$

has a solution with the matrices N_1 and N_2 of maximal rank, and one may set $y_0(t) := N_2Hx(t) = N_2y(t)$.

One assumes that there exists the matrix $\Phi \in \mathbb{R}^{k\times n}$, such that $x_*(t) = \Phi x(t)$. It is known [3,28] that this matrix satisfies the equations

$$\begin{aligned} \Phi F &= F_*\Phi + J_* H, \\ G_* &= \Phi G, \\ L_* &= \Phi L. \end{aligned} \tag{3}$$

The second equation in (2) with $z(t) = Mx(t)$ can be presented as

$$M = H_z\Phi + QN_2H = (H_z \;\; Q)\begin{pmatrix} \Phi \\ N_2H \end{pmatrix}. \tag{4}$$

The equation has a solution when

$$\text{rank}\begin{pmatrix} \Phi \\ N_2H \end{pmatrix} = \text{rank}\begin{pmatrix} \Phi \\ N_2H \\ M \end{pmatrix}. \tag{5}$$

3. The Reduced Order Model Design

We construct the model invariant with respect to the disturbance $\rho(t)$ when $L_* = \Phi L = 0$. Based on (3) and (4), one may obtain conditions that allow checking whether such a solution exists. Since L_0 is such that $L_0 L = 0$, then $\Phi = NL_0$ for matrix N. The first condition is of the form [3,28]

$$\text{rank}\begin{pmatrix} L_0F \\ H \\ L_0 \end{pmatrix} < \text{rank}(L_0F) + \text{rank}\begin{pmatrix} H \\ L_0 \end{pmatrix}. \tag{6}$$

To obtain the second one, replace (2) Φ with NL_0 and transform it:

$$M = (H_zN \;\; Q)\begin{pmatrix} L_0 \\ N_2H \end{pmatrix}.$$

The equation is solvable when

$$\text{rank}\begin{pmatrix} L_0F \\ N_2H \end{pmatrix} = \text{rank}\begin{pmatrix} L_0F \\ N_2H \\ M \end{pmatrix}. \tag{7}$$

If conditions (6) and (7) are satisfied, one can design the model invariant with respect to the disturbance. If (7) is not satisfied, one has to analyze the rows of the matrix M based on (7) and compose from them matrix M_0, satisfying the condition (7). Then the interval observer invariant (with respect to the disturbance that estimates the variable $z_0(t) = M_0x$) is designed. The rest of the rows of matrix M are composed in matrix M_*, and the robust interval observer estimating the variable $z_*(t) = M_*x(t)$ is constructed based on the methods described in Section 6. If (6) is not satisfied, one has to use the robust solution as well.

To design the reduced order model, one specifies the matrix $F_* \in \mathbb{R}^{k\times k}$ in the identification canonical form (ICF)

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{8}$$

Note that the main requirement for an observer is stability. Since the matrix (8) has zero eigenvalues, the stability of the discrete-time linear model with (8) is achieved without any feedback. It is known for the discrete-time system [15] that to design the interval observer, the matrix F_* should be stable and nonnegative; therefore, ICF (8) is preferable since it satisfies both conditions.

A solution insensitive to the disturbance is based on Equation [29,30]

$$\begin{pmatrix} \Phi_1 & -J_{*1} & \dots & -J_{*k} \end{pmatrix}(V^{(k)}\ L^{(k)}) = 0, \tag{9}$$

where

$$V^{(k)} = \begin{pmatrix} F^k \\ HF^{k-1} \\ \dots \\ H \end{pmatrix},$$
$$L^{(k)} = \begin{pmatrix} L & FL & \dots & F^{k-1}L \\ 0 & HL & \dots & HF^{k-2}L \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

the matrix $V^{(k)}$ allows designing model (2), $L^{(k)}$ provides insensitivity to the disturbance. Equation (9) has a nonzero solution, if

$$\operatorname{rank}(V^{(k)}\ L^{(k)}) < n + lk. \tag{10}$$

To design the model, minimal k is determined from (10), the row $(\Phi_1\ -J_{*1}\ \dots\ -J_{*k})$ from (9), and then based on the relations

$$\begin{aligned} \Phi_i F &= \Phi_{i+1} + J_{*i}H, \quad i = 1, \dots, k-1, \\ \Phi_k F &= J_{*k}H, \end{aligned} \tag{11}$$

obtained from (8) and (3), the matrix Φ is found; here, Φ_i and J_{*i} are the ith rows of the matrix Φ and J_*, $i = 1, \dots, k$, respectively. Finally, condition (5) is checked. If it is satisfied, the matrices H_z and Q are found from (4) and G_* from (3). If (5) is not satisfied, one has to find another solution of (9) with the former or incremented k.

4. Interval Observer Design

Model (2) is the basis to design the observer, which is specified in the form

$$\begin{aligned} \underline{x}_*(t+1) &= F_*\underline{x}_*(t) + G_*u(t) + J_*y(t) - |J_*|E_k v_*, \\ \overline{x}_*(t+1) &= F_*\overline{x}_*(t) + G_*u(t) + J_*y(t) + |J_*|E_k v_*, \\ \underline{z}(t) &= H_z\underline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) &= H_z\overline{x}_*(t) + Qy_0(t), \\ \underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}, \end{aligned} \tag{12}$$

where $E_k = (1\ 1\ \dots\ 1)^T \in \mathbb{R}^{k\times 1}$, the elements of the matrix $|A|$ are absolute values of the corresponding elements of A; it is assumed that $x_*(0) \in [\underline{x}_{*0}, \overline{x}_{*0}]$ for some known $\underline{x}_{*0}, \overline{x}_{*0} \in \mathbb{R}^k$.

Theorem 1. *If $H_z \geq 0$ and $\underline{x}_*(0) \leq x_*(0) \leq \overline{x}_*(0)$, then for the observer (12) for $t \geq 0$, it follows*

$$\begin{aligned} \underline{x}_*(t) &\leq x_*(t) \leq \overline{x}_*(t), \\ \underline{z}(t) &\leq z(t) \leq \overline{z}(t). \end{aligned} \tag{13}$$

Proof. Consider the estimation errors

$$\begin{aligned} \underline{e}_*(t) &= x_*(t) - \underline{x}_*(t), \quad \overline{e}_*(t) = \overline{x}_*(t) - x_*(t), \\ \underline{e}_z(t) &= z(t) - \underline{z}(t), \quad \overline{e}_z(t) = \overline{z}(t) - z_*(t). \end{aligned} \tag{14}$$

It follows from (2) and (12)

$$\begin{array}{rcl} \underline{e}_*(t+1) & = & F_*\underline{e}_*(t) + J_*(Hx(t) - y(t)) + |J_*|E_k v_* = F_*\underline{e}_*(t) - J_* v(t) + |J_*|E_k v_*, \\ \overline{e}_*(t+1) & = & F_*\overline{e}_*(t) + J_*(y(t) - Hx(t)) + |J_*|E_k v_* = F_*\overline{e}_*(t) + J_* v(t) + |J_*|E_k v_*. \end{array} \tag{15}$$

Since $\underline{x}_*(0) \leq x_*(0) \leq \overline{x}_*(0)$, then $\underline{e}_*(0) \geq 0$ and $\overline{e}_*(0) \geq 0$. Note that in (15) $|J_*|E_k v_* \pm J_* v(t) \geq 0$ for all $t \geq 0$ and $F_* \geq 0$. As a result, solutions of (15) under $\underline{e}_*(0), \overline{e}_*(0) \geq 0$ are nonnegative element-wise, which is, for all $t \geq 0$ one has $\underline{e}_*(t), \overline{e}_*(t) \geq 0$ [15]. It follows from (14) that for all $t \geq 0$ $\underline{x}_*(t) \leq x_*(t) \leq \overline{x}_*(t)$. If $H_z \geq 0$, then the relation $z(t) = H_z x_*(t) + Qy_0(t)$ and the observer (12) yield

$$\begin{array}{rcl} \underline{e}_z(t) & = & z(t) - \underline{z}(t) = H_z x_*(t) + Qy_0(t) - (H_z \underline{x}_*(t) + Qy_0(t)) = H_z \underline{e}_*(t), \\ \overline{e}_z(t) & = & \overline{z}(t) - z_*(t) = H_z \overline{x}_*(t) + Qy_0(t) - (H_z x_*(t) + Qy_0(t)) = H_z \overline{e}_*(t). \end{array}$$

Taking into account $\underline{e}_*(t), \overline{e}_*(t) \geq 0$, and $H_z \geq 0$, we obtain from the last equations $\underline{e}_z(t)$ and $\overline{e}_z(t) \geq 0$, which are equivalent to $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$. The theorem has been proved. □

Remark 2. *When $H_z \leq 0$, then*

$$\begin{array}{rcl} \underline{z}(t) & = & H_z \overline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) & = & H_z \underline{x}_*(t) + Qy_0(t). \end{array} \tag{16}$$

It follows from (16)

$$\begin{array}{rcl} \underline{e}_z(t) & = & z(t) - \underline{z}(t) = H_z x_*(t) + Qy_0(t) - (H_z \overline{x}_*(t) + Qy_0(t)) = -H_z \overline{e}_*(t), \\ \overline{e}_z(t) & = & \overline{z}(t) - z_*(t) = H_z \underline{x}_*(t) + Qy_0(t) - (H_z x_*(t) + Qy_0(t)) = -H_z \underline{e}_*(t). \end{array}$$

Taking into account $H_z \leq 0$, we obtain $\underline{e}_z(t), \overline{e}_z(t) \geq 0$.

If H_z is an oscillating matrix, the main result is retained but relations become more complicated. Let H_z be a row matrix; we assume without loss of generality that the first p elements of H_z are positive and the rest of them are negative: $H_z = (H_z^+ \;\; H_z^-)$ where $H_z^+ \geq 0$ and $H_z^- \leq 0$. In this case

$$\underline{z}(t) = H_z^+ \underline{x}_{*p}(t) + H_z^- \overline{x}_*^{k-p}(t) + Qy_0(t),$$

where $\underline{x}_{*p}(t)$ and $\overline{x}_*^{k-p}(t)$ are the sub-vectors of vectors $\underline{x}_*(t)$ and $\overline{x}_*(t)$ containing the first p and the last $k-p$ elements, respectively. Then

$$\begin{array}{rcl} \underline{e}_z(t) & = & z(t) - \underline{z}(t) = H_z^+ x_{*p}(t) + H_z^- x_*^{k-p}(t) + Qy_0(t) - (H_z^+ \underline{x}_{*p}(t) + H_z^- \overline{x}_*^{k-p}(t) + Qy_0(t)) \\ & = & H_z^+ \underline{e}_{*p}(t) - H_z^- \overline{e}_*^{k-p}(t). \end{array}$$

Since $H_z^+ \geq 0$ and $H_z^- \leq 0$, then $\underline{e}_z(t) \geq 0$.

Let H_z be of the form

$$H_z = \begin{pmatrix} H_z^+ \\ H_z^- \end{pmatrix},$$

where $H_z^+ \geq 0$ and $H_z^- \leq 0$. In this case,

$$\underline{z}(t) = \begin{pmatrix} H_z^+ \underline{x}_*(t) \\ H_z^- \overline{x}_*(t) \end{pmatrix} + Qy_0(t).$$

Then

$$\underline{e}_z(t) = \begin{pmatrix} H_z^+ \\ H_z^- \end{pmatrix} x_*(t) + Qy_0(t) - \left(\begin{pmatrix} H_z^+ \underline{x}_*(t) \\ H_z^- \overline{x}_*(t) \end{pmatrix} + Qy_0(t) \right) = \begin{pmatrix} H_z^+ \underline{e}_*(t) \\ H_z^- \overline{e}_*(t) \end{pmatrix} \geq 0.$$

If H_z contains rows H_{zj} of the form $H_{zj} = (H_{zj}^+ \ \ H_{zj}^-)$, the corresponding formulas are combinations of two considered cases.

Remark 3. *Note that the inequality $\underline{x}_*(0) \le x_*(0) \le \overline{x}_*(0)$ for system (15) yields the relations $\underline{e}_*(t) \ge 0, \overline{e}_*(t) \ge 0$ for all $t \ge 0$. The properties of the matrix F_* allow us to state that these relations will be true, even though $\underline{x}_*(0) \le x_*(0) \le \overline{x}_*(0)$ is not true since system (15) with the matrix F_* "forgets" the initial conditions for $t \ge k$. Really, denote $v_0(t) = |J_*|E_k v_* \pm J_* v(t) \ge 0$ and consider the first equation in (15); solution (15) can be represented as*

$$\underline{e}_*(t) = F_*^t \underline{e}_*(0) + \sum_{i=0}^{t-1} F_*^{t-i-1} v_0(i).$$

Since $F_*^k = 0$, then the value $\underline{e}_*(t)$ for $t \ge k$ depends on the term $\sum_{i=0}^{t-1} F_*^{t-i-1} v_0(i)$, which is nonnegative by structure. As a result, $\underline{e}_*(t) \ge 0$ for all $t \ge k$. It can be shown by the analogy for the relation $\overline{e}_*(t) \ge 0$ for all $t \ge k$.

5. Robust Solution

If condition (10) is not true for all $k < n$, the model invariant with respect to the disturbance cannot be designed. The model having minimal sensitivity can be constructed as follows.

The contribution of the disturbance in the model (2) can be evaluated by the norm $\|\Phi L\|_F$ of the matrix ΦL, one can present it as $\|(\ \Phi_1 \ \ -J_{*1} \ \ \dots \ \ -J_{*k} \ )L^{(k)}\|_F$ [28]. We will minimize the norm $\|(\ \Phi_1 \ \ -J_{*1} \ \ \dots \ \ -J_{*k} \ )L^{(k)}\|_F$ under the condition

$$(\ \Phi_i \ \ -J_{*1} \ \ \dots \ \ -J_{*k} \ )V^{(k)} = 0. \tag{17}$$

The problem will be solved if one finds several linearly independent solutions (17) and collects them in the matrix

$$B = \begin{pmatrix} \Phi_1^{(1)} & -J_{*1}^{(1)} & \dots & -J_{*k}^{(1)} \\ & \dots & & \\ \Phi_1^{(n_*)} & -J_{*1}^{(n_*)} & \dots & -J_{*k}^{(n_*)} \end{pmatrix}, \tag{18}$$

the number of all solutions for some k is denoted by n_*. If $w = (w_1 \ \dots \ w_{n_*})$ is an arbitrary vector of weight coefficients, then wB is a solution as well. One has to find the vector w under the condition $\|w\| = 1$, such that the norm $\|wBL^{(k)}\|_F$ is minimal.

The problem is solved by a singular value decomposition of the matrix $BL^{(k)} : BL^{(k)} = U_L \Sigma_L V_L$ [31]. One has to use the first transposed column of the matrix U_L for a vector of coefficients $w = (w_1 \ \dots \ w_{n_*})$. A singular value decomposition implies that the norm $\|wBL^{(k)}\|_F$ is equal to the minimum singular value σ_1.

Finally, one has to find the row $(\Phi_1 \ \ -J_{*1} \ \ \dots \ \ -J_{*k}) = wB$, then the matrix Φ based on (11), and set $G_* := \Phi G$, $L_* := \Phi L$.

Due to the addend $L_*\rho$ in (2), the interval observer under $v \ne 0$, $\rho \ne 0$, and $H_z \ge 0$ becomes

$$\begin{aligned} \underline{x}_*(t+1) &= F_*\underline{x}_*(t) + G_*u(t) + J_*y(t) - |J_*|E_k v_* - |L_*|E_k \rho_*, \\ \overline{x}_*(t+1) &= F_*\overline{x}_*(t) + G_*u(t) + J_*y(t) + |J_*|E_k v_* + |L_*|E_k \rho_*, \\ \underline{z}(t) &= H_z\underline{x}_*(t) + Qy_0(t), \\ \overline{z}(t) &= H_z\overline{x}_*(t) + Qy_0(t), \\ \underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}. \end{aligned}$$

Equation (15) is corrected as well:

$$\begin{aligned} \underline{e}_*(t+1) &= F_*\underline{e}_*(t) - J_*v(t) + |J_*|E_k v_* + L_*\rho + |L_*|E_k \rho_*, \\ \overline{e}_*(t+1) &= F_*\overline{e}_*(t) + J_*v(t) + |J_*|E_k v_* - L_*\rho + |L_*|E_k \rho_*. \end{aligned}$$

Clearly, the relations (13) follow from Theorem 1 and relation $|L_*|E_k\rho_* \pm L_*\rho \geq 0$.

6. Interval Estimation of the Vector $x(t)$

The suggested approach to the interval estimation of the variable $z(t)$ can be used to obtain the estimate of the full vector $x(t)$ as follows. We assume that matrix H is of full row–rank and

$$H = (H_0 \ \ 0), \quad y(t) = H_0 x^{(1)}(t) + v(t), \quad x(t) = \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{pmatrix},$$

H_0 is a nonsingular matrix. Let us define

$$\begin{aligned} \underline{y}(t) &= y(t) - E_l v_*, \quad \overline{y}(t) = y(t) + E_l v_*, \\ \underline{x}^{(1)}(t) &= H_0^{-1}\underline{y}(t), \quad \overline{x}^{(1)}(t) = H_0^{-1}\overline{y}(t). \end{aligned} \tag{19}$$

Then

$$\begin{aligned} \underline{e}^{(1)}(t) &= x^{(1)}(t) - \underline{x}^{(1)}(t) = H_0^{-1}(y(t) - v(t)) - H_0^{-1}\underline{y}(t) = H_0^{-1}(E_l v_* - v(t)), \\ \overline{e}^{(1)}(t) &= \overline{x}^{(1)}(t) - x^{(1)}(t) = H_0^{-1}\overline{y}(t) - H_0^{-1}(y(t) - v(t)) = H_0^{-1}(E_l v_* + v(t)). \end{aligned}$$

Assuming that $H_0^{-1} \geq 0$, it follows from $E_l v_* \pm v(t) \geq 0$ that $\underline{e}^{(1)}(t) \geq 0$, $\overline{e}^{(1)}(t) \geq 0$ and $\underline{x}^{(1)}(t) \leq x^{(1)}(t) \leq \overline{x}^{(1)}(t)$. As a result, the variable $x^{(1)}(t)$ under the condition $H_0^{-1} \geq 0$ is estimated based on (19) and the disturbance $\rho(t)$ does not affect this estimate.

Remark 4. *The condition $H_0^{-1} \geq 0$ is satisfied in practical important cases when the vector $x^{(1)}$ is measured by sensors and $H_0 = H_0^{-1} = I_l$.*

The estimate of the variable $x^{(2)}$ is given by the observer (12). Set $z(t) := x^{(2)}(t) = M^{(2)}x(t)$ for matrix $M^{(2)}$ and checks the condition (5) to clarify if it is possible to design the observer invariant with respect to the disturbance. Then depending on the results, one constructs an insensitive or robust observer.

7. Example

Consider the discrete-time model of the electric servo-actuator of the manipulator:

$$\begin{aligned} x_1(t+1) &= k_1 x_2(t) + x_1(t), \\ x_2(t+1) &= k_2 x_3(t) + x_2(t) + \rho(t), \\ x_3(t+1) &= k_3 x_2(t) + k_4 x_3(t) + k_5 u(t), \\ y_1(t) &= x_1(t) + v_1(t), \\ y_2(t) &= x_3(t) + v_2(t), \end{aligned} \tag{20}$$

where the coefficients $k_1 \div k_5$ describe the sampling time and the servo-actuator parameters; the addend $\rho(t)$ is explained by the external loading moment. The following matrices describe this model:

$$F = \begin{pmatrix} 1 & k_1 & 0 \\ 0 & 1 & k_2 \\ 0 & k_3 & k_4 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ k_5 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Construct the interval observer estimating the variable $x_2(t)$ with $M = (0\ 1\ 0)$. Since the disturbance enters in the equation for $x_2(t)$, the model will be sensitive to the disturbance, and one may set in (9) $L^{(k)} = 0$ and obtain with $k = 1$ the equation

$$(\Phi \quad -J_*)\begin{pmatrix} 1 & k_1 & 0 \\ 0 & 1 & k_2 \\ 0 & k_3 & k_4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Its solution is $\Phi = (1/k_1 \quad -1\ \ 0)$ and $J_* = (1/k_1 \quad -k_2)$ that gives $G_* = 0$ and $L_* = -1$. It can be shown that condition (5) is satisfied and $H_z = -1$, $Q = (1/k_1\ \ 0)$.

The model is given by

$$\begin{aligned} x_*(t+1) &= (1/k_1)H_1 x(t) - k_2 H_2 x(t) - \rho(t), \\ z(t) &= -x_*(t) + (1/k_1)y_1(t). \end{aligned}$$

The interval observer is based on this model and is described as follows:

$$\begin{aligned} \underline{x}_*(t+1) &= (1/k_1)y_1(t) - k_2 y_2(t) - (1/k_1)v_{*1} - k_2 v_{*2} - \rho_*, \\ \overline{x}_*(t+1) &= (1/k_1)y_1(t) - k_2 y_2(t) + (1/k_1)v_{*1} + k_2 v_{*2} + \rho_*, \\ \underline{z}(t) &= -\overline{x}_*(t) + (1/k_1)y_1(t), \\ \overline{z}(t) &= -\underline{x}_*(t) + (1/k_1)y_1(t). \end{aligned} \tag{21}$$

The variables $x_1(t)$ and $x_3(t)$ can be estimated according to (19):

$$\begin{aligned} \underline{x}_1(t) &= y_1(t) - v_{*1},\ \ \underline{x}_3(t) = y_2(t) - v_{*2}, \\ \overline{x}_1(t) &= y_1(t) + v_{*1},\ \ \overline{x}_3(t) = y_2(t) + v_{*2}. \end{aligned}$$

Comparing the obtained results with those that were previously obtained for this example in [15] and other similar papers, one can conclude that the suggested approach gives a simpler observer and an interval with a smaller width.

Consider for the simulation, the model (20), and the observer (21); the control $u(t) = 0.2sin(t/100)$, the measurement noises $v_1(t)$, $v_2(t)$, and $\rho(t)$ are random processes evenly distributed on $[-0.01, 0.01]$. Set for simplicity $k_1 = k_2 = k_5 = 1, k_3 = k_4 = -1; v_{*1} = v_{*2} = \rho_* = 0.01$. Simulation results are shown in Figures 1 and 2, where the functions $x_2(t)$ and its estimates $\underline{z}(t)$ and $\overline{z}(t)$ are presented for $x_*(0) = 0$. In Figure 1, the initial conditions are $\underline{x}_*(0) = -0.05$ and $\overline{x}_*(0) = 0.05$, in Figure 2 $\underline{x}_*(0) = 0.05$ and $\overline{x}_*(0) = -0.05$. Crossing two graphs in Figure 2 corresponds to Remark 3.

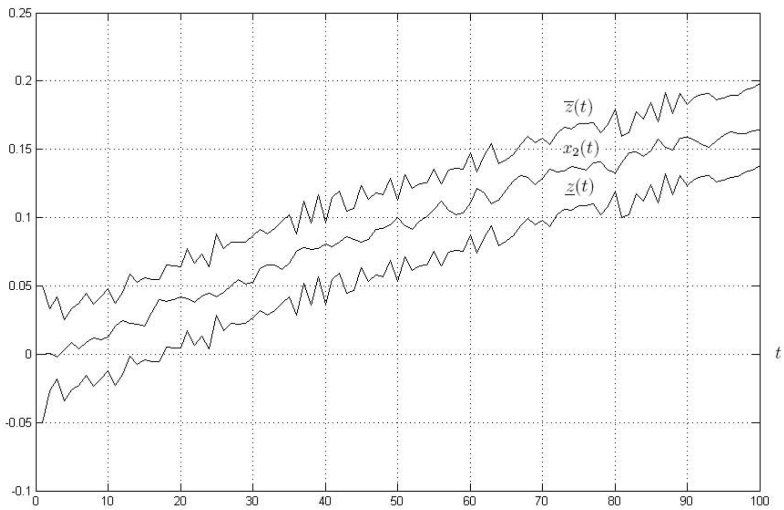

Figure 1. Graphs of the variable $x_2(t)$ and its estimates $\underline{z}(t)$ and $\overline{z}(t)$ with $\underline{x}_*(0) = -0.05$ and $\overline{x}_*(0) = 0.05$.

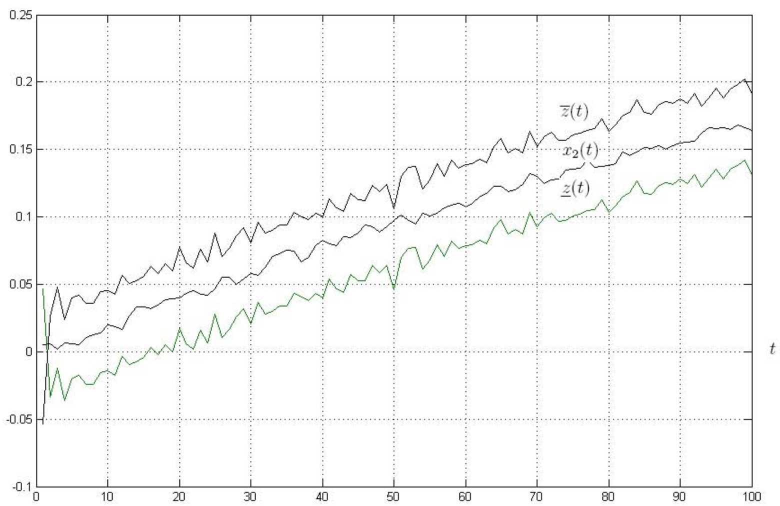

Figure 2. Graphs of the variable $x_2(t)$ and its estimates $\underline{z}(t)$ and $\overline{z}(t)$ with $\underline{x}_*(0) = 0.05$ and $\overline{x}_*(0) = -0.05$.

8. Conclusions

The problem surrounding the interval observer design for dynamic systems (described by linear discrete-time models under disturbance and measurement noises) was studied in this paper. To solve the problem, the reduced order model of the initial system, which is invariant (or has minimal sensitivity) to the disturbance was used. Such a model is realized in the identification canonical form. The relations to design the interval observer of minimal dimension, which estimates the prescribed linear function of the state vector of the initial system, were obtained.

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Abbreviation

The following abbreviation is used in this manuscript:

ICF & identification canonical form

References

1. Edwards, C.; Spurgeon, S.; Patton, R. Sliding mode observers for fault detection and isolation. *Automatica* **2000**, *36*, 541–553. [CrossRef]
2. Fridman, L.; Levant, A.; Davila, J. Observation of linear systems with unknown inputs via high-order sliding-modes. *Int. J. Syst. Sci.* **2007**, *38*, 773–791. [CrossRef]
3. Zhirabok, A.; Zuev, A.; Seriyenko, O.; Shumsky, A. Fault identificaition in nonlinear dynamic systems and their sensors based on sliding mode observers. *Autom. Remote Control* **2022** *83*, 214–236. [CrossRef]
4. Chebotarev, S.; Efimov, D.; Raissi, T.; Zolghadri, A. Interval observers for continous-time LPV systems with L_1/L_2 performance. *Automatica* **2015**, *58*, 82–89. [CrossRef]
5. Degue, K.; Efimov, D.; Richard, J. Interval observers for linear impulsive systems. *IFAC-PapersOnLine* **2016**, *49-18*, 867–872. [CrossRef]
6. Dinh, T.; Mazenc, F.; Niculescu, S. Interval observer composed of observers for nonlinear systems. In Proceedings of the 2014 European Control Conference (ECC), Strasbourg, France, 24–27 June 2014; pp. 660–665.
7. Kolesov, N.; Gruzlikov, A.; Lukoyanov, E. Using fuzzy interacting observers for fault diagnosis in systems with parametric uncertainty. In Proceedings of the XII-th International Symposium Intelligent Systems (INTELS'16), Moscow, Russia, 5–7 October 2016; pp. 499–504.
8. Mazenc, F.; Bernard, O. Interval observers for linear time-invariant systems with disturbances. *Automatica* **2011**, *47*, 140–147. [CrossRef]
9. Raissi, T.; Efimov, D.; Zolghadri, A. Interval state estimation for a class of nonlinear systems. *IEEE Trans. Autom. Control* **2012**, *57*, 260–265. [CrossRef]
10. Zheng, G.; Efimov, D.; Perruquetti, W. Interval state estimation for uncertain nonlinear systems. In Proceedings of the IFAC Nolcos 2013, Toulouse, France, 4–6 September 2013.
11. Zhirabok, A.; Zuev, A.; Kim Chung, I. A Method to design interval observers for linear time-invariant systems. *Comput. Syst. Sci. Int.* **2022**, *61*, 485–495. [CrossRef]
12. Efimov, D.; Perruquetti, W.; Raissi, T.; Zolghadri, A. Interval observers for time-varying discrete-time systems. *IEEE Trans. Autom. Control* **2013**, *58*, 3218–3224.
13. Mazenc, F.; Dinh, T.; Niculescu, S. Interval observers for discrete-time systems. *Inter. J. Robust Nonlinear Control* **2014**, *24*, 2867–2890. [CrossRef]
14. Efimov, D.; Polyakov, A.; Richard, J. Interval observer design for estimation and control of time-delay descriptor systems. *Eur. J. Control* **2015**, *23*, 26–35. [CrossRef]
15. Efimov, D.; Raissi, T. Design of interval state observers for uncertain dynamical systems. *Autom. Remote Control* **2015**, *77*, 191–225. [CrossRef]
16. Marouani, G.; Dinh, T.; Raissi, T.; Wang, X.; Messaoud, H. Unknown input interval observers for discrete-time linear switched systems. *European J. Control* **2021**, *59*, 165–174. [CrossRef]

17. Zammali, C.; Gorp, J.; Wang, Z.; Raissi, T. Sensor fault detection for switched systems using interval observer with L_∞ performance. *Eur. J. Control* **2020**, *57*, 147–156. [CrossRef]
18. Alives, J.; Moreno, J.; Davila, J.; Becerra, G.; Flores, F.; Chavez, C.; Marques, C. Stability radii-based interval observers for discrete-time nonlinear systems. *IEEE Access* **2022**, *10*, 3216–3227.
19. Blesa, J.; Rotondo, D.; Puig, V. FDI and FTC of wind turbines using the interval observer approach and virtual actuators/sensors. *Control Eng. Pract.* **2014**, *24*, 138–155. [CrossRef]
20. Rotondo, D.; Fernandez-Canti, R.; Tornil-Sin, S. Robust fault diagnosis of proton exchange membrane fuel cells using a Takagi-Sugeno interval observer approach. *Int. J. Hydrogen Energy* **2016**, *41*, 2875–2886. [CrossRef]
21. Zhang, K.; Jiang, B.; Yan, X.; Edwards, C. Interval sliding mode based fault accommodation for non-minimal phase LPV systems with online control application. *Intern. J. Control* **2019**. [CrossRef]
22. Khan, A.; Xie, W.; Zhang, B.; Liu, L. Design and applications of interval observers for uncertain dynamical systems. *IET Circuits Devices Syst.* **2020**, *14*, 721–740. [CrossRef]
23. Khan, A.; Xie, W.; Zhang, B.; Liu, L. A survey of interval observers design methods and implementation for uncertain systems. *J. Frankl. Inst.* **2021**, *358*, 3077–3126. [CrossRef]
24. Gu, D.; Liu, L.; Duan, G. Functional interval observer for the linear systems with disturbances. *IET Control Theory Appl.* **2018**, *12*, 2562–2568. [CrossRef]
25. Haochi, C.; Jun, H.; Xudong, Z.; Xiang, M.; Ning, X. Functional interval observer for discrete-time systems with disturbances. *Appl. Math. Comput.* **2020**, *383*, 125352.
26. Liu, L.; Xie, W.; Khan, A.; Zhang, L. Finite-time functional interval observer for linear systems with uncertainties. *IET Control Theory Appl.* **2020**, *14*, 2868–2878. [CrossRef]
27. Meyer, L. Robust functional interval observer for multivariable linear systems. *J. Dyn. Syst. Meas. Control* **2019**, *141*, 094502. [CrossRef]
28. Zhirabok, A.; Shumsky, A.; Solyanik, S.; Suvorov, A. Fault detection in nonlinear systems via linear methods. *Int. J. Appl. Math. Comput. Sci.* **2017**, *27*, 261–272. [CrossRef]
29. Zhirabok, A. Disturbance decoupling problem: Logic-dynamic approach-based solution. *Symmetry* **2019**, *11*, 555. [CrossRef]
30. Zhirabok, A. The problem of invariance in nonlinear discrete-time dynamic systems. *Symmetry* **2020**, *12*, 1241. [CrossRef]
31. Low, X.; Willsky, A.; Verghese, G. Optimally robust redundancy relations for failure detection in uncertain systems. *Automatica* **1996**, *22*, 333–344.