arXiv:2008.12722v2 [math.AP] 1 Mar 2022

ENHANCED EXISTENCE TIME OF SOLUTIONS TO EVOLUTION EQUATIONS OF WHITHAM TYPE

MATS EHRNSTRÖM AND YUEXUN WANG

Abstract. We show that Whitham type equations $u_t + uu_x - \mathcal{L}u_x = 0$, where L is a general Fourier multiplier operator of order $\alpha \in [-1, 1]$, $\alpha \neq 0$, allow for small solutions to be extended beyond their ordinary existence time. The result is valid for a range of quadratic dispersive equations with inhomogenous symbols in the dispersive regime given by the parameter α.

1. Introduction

The enhanced existence time of small solutions to weakly dispersive water wave equations has gained a lot of attention. Going back to Shatah's normal form [33] and the subsequent work of Delort and collaborators on the Klein–Gordon equation [7], it has obtained renewed momentum both through the work on the Burgers–Hilbert equation [16], but more generally through the analysis related to global well-posedness for the water-wave problem [2, 14, 21, 35], as well as in dedicated papers on improved lifespan of solutions to the same problem [5, 17, 23].

The problem concerns the extension of small data beyond the typical hyperbolic existence time: for quadratic equations this means existence times of orders $|u_0|^{-2}$ in the appropriate norm, also sometimes called cubic lifespan by the terms arising in the energy estimates. Such estimates are of natural interest where singularity formation is present, and general well-posedness in an entire regular space is not possible. At the centre lies the interaction of frequencies via multiple wave interactions. The use of the normal form transform gives rise to generally multidimensional symbols that feature singularities where frequency directions interact; by controlling these by means of multilinear estimates in the appropriate norm one obtains the necessary energy estimate.

These questions have been investigated in several settings in the water-wave problem, including gravitational [17] and pure surface tension [20] waves on infinite depth, arbitrary lifespan on restricted sets of conditions for periodic waves [4], extended life span for all parameter values in the

2010 *Mathematics Subject Classification.* 76B15, 76B03, 35S30, 35A20.

Key words and phrases. Enhanced life span, Whitham type, dispersive equations.

Both authors acknowledge the support by grant nos. 231668 and 250070 from the Research Council of Norway. The second author also acknowledges the support of the ANR project ANuI, and the partial support of grant no. 830018 from China.

periodic gravity-capillary waves [5], and more. While the water-wave problem in its full formulation is involved and hardly treatable without exact reformulations using its Hamiltonian, holomorphic or paralinear structure, several model equations display some of the same frequency and qualitative behaviour. The Burgers–Hilbert equation [16] and more generally the fractional Korteweg–de Vries (fKdV) equation $\partial_t u + u\partial_x u - |D|^\alpha \partial_x u = 0$, $\alpha \in \mathbb{R}$, are examples of this [26]. The Burgers–Hilbert equation is the case $\alpha = -1$ in the fKdV scale, the inviscid Burgers equation is $\alpha = 0$, and the gravity and capillary water wave problems on infinite depth correspond to fractional values $\alpha = \pm\frac{1}{2}$ in terms of modelling and linear dispersion [25]. More generally the fKdV equation is a good measure of the balance between nonlinearity and dispersion arising in water wave equations.

In [12] we showed that the main results from [16] and [18] for the Burgers–Hilbert equation extend to set $\alpha \in [-1,1] \setminus \{0\}$ in the fKdV scale. The operator $|D|$ in this equation relates to the case of infinite depth h in the linear dispersion term $\xi \tanh(h\xi)$. On unit depth, the symbol is

$$\left(\frac{(1+\beta\xi^2)\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \tag{1.1}$$

where $\beta = 0$ in the gravity case, and $\beta > 0$ for capillary-gravity waves. The corresponding inhomogeneous quadratic equations are called the Whitham and capillary-gravity Whitham equations, and are part of a class of very weakly dispersive nonlinear equations [9,15]. In this investigation, we extend the results from [12] to that more general class of inhomogeneous model equations, with the goal of providing simple but still general assumptions on the dispersive operator. We do so in the spirit of the setup in [9], considering the family of Whitham-type equations

$$\partial_t u + u\partial_x u - \mathcal{L}\partial_x u = 0, \tag{1.2}$$

where the operator $\mathcal{L}$ is a Fourier multiplier with symbol p,

$$\mathcal{F}(\mathcal{L}f)(\xi) = p(\xi)\hat{f}(\xi).$$

Here, $\mathcal{F}f(\xi) = \int f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x$, so that $\partial_x \sim \mathrm{i}\xi$. We shall use $\mathrm{D} = -i\partial_x \sim \xi$ and $|\mathrm{D}| \sim |\xi|$, and solutions $u\colon [0,T]\times\mathbb{R} \mapsto \mathbb{R}$ will be considered to be regular. What we need are the following assumptions:

(A1) The symbol $p\colon \mathbb{R}\to\mathbb{R}$ is C^2, even and strictly monotone on $[0,+\infty)$;

(A2) at the far field,

$$|p^{(i)}(\xi)| \eqsim |\xi|^{\alpha-i},\ |\xi| \gg 1, \qquad i = 0,1,2, \tag{1.3}$$

where $\alpha \in [-1,1]\setminus\{0\}$;

(A3) and one has a local expansion

$$p(\xi) = p(0) + \xi^{2j_*}\tilde{p}_{2j_*}(\xi), \qquad |\xi| \ll 1, \tag{1.4}$$

where $j_* \in \mathbb{N}$[1] and $\tilde{p}_{2j_*}$ is locally Lipschitz with $\tilde{p}_{2j_*}(0) \neq 0$.

In difference to [9] we allow for both negative- and positive-order operators, displaying decay, respectively growth, in (1.3). The above assumptions describe the results in terms of regularity, the behaviour of the dispersive symbol at infinity, and the order of its local extremum at the origin. This is in line with a research program to describe nonlocal dispersive equations in terms of a quantifiable properties of their nonlocalities and nonlinearities, see for example [10, 11, 27].

As mentioned above, a famous instance of a symbol p satisfying our assumptions is the linear dispersion for uni-directional water waves (1.1), with $\beta = 1$ and $\beta = 0$ corresponding to $\alpha = 1/2$ and $\alpha = -1/2$, respectively, with $j_* = 1$ near the origin. One can build any even, inhomogeneous, one-sided monotone function with a local extremum at the origin and sublinear decay or growth at infinity to fit the assumptions. Here, the order $j_* \geq 1$ of the local extremum at the origin is related to regularity of solutions. Our main theorem is the following.

Theorem 1.1. *Let $\alpha \in [-1, 1] \setminus \{0\}$ and $N \geq \max\{3, 2j_* - 1\}$. Assume that*

$$\|u_0\|_{H^N(\mathbb{R})} \leq \varepsilon,$$

for some sufficiently small constant $\varepsilon > 0$ which depends only on α and N. Then there exist a positive number $T \gtrsim \varepsilon^{-2}$ and a unique solution u in $C([-T, T]; H^N(\mathbb{R})) \cap C^1([-T, T]; H^{N-2}(\mathbb{R}))$ of (1.2) with $u(0, x) = u_0(x)$ such that

$$\|u\|_{L^\infty([-T,T];H^N(\mathbb{R}))} \lesssim \varepsilon.$$

The proof is built upon a normal form formulation and the modified energy approach from [18], and is a generalisation and simplification of the proof devised in [12]. Note that for $\alpha \leq \frac{1}{2}$ the equations are quasilinear and cannot be solved with standard contraction techniques [29]. Although the role of the parameter α is not visible in our main theorem, an inspection of the proofs shows that values of $|\alpha|$ close to the origin yield worse estimates than $|\alpha|$ close to unit size for both positive and negative values of α, even though $\alpha = -1$ is the current threshold for the method. As in [12], we follow the method of proof from [18] up to the point of Lemma 2.2 in the paper at hand. We formulate in Lemma 3.1 and Corollary 3.2 the general properties of the multiplier needed for the energy estimates later in the paper. These results are similar to other ones in the literature, see for example Lemma 2.3 in [13], and are a fundamental part of the paper. The class of equations has no general integrability structure, and singularities at 'low' frequencies and surplus growth at 'high' frequencies both have to be dealt with. Explained further in [12], we emphasise the commutator introduced in (5.8), which is handled in Proposition 5.7. Similar commutators appear in other works

[1] We use $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(see for example Section 4 in [22]), and are related to cancellations in the resonant set. In our case integration by parts and splitting of the frequency domain reduce the energy estimates required in the last step, while the final commutator is built on a global transformation of variables in Fourier space, serving for a reduction of two orders of differentiation. We have not been able to find whether the expansion in small variables $(\xi-\eta)/\eta$ and $(\sigma-\eta)/\eta$ that we use in high frequencies has been applied elsewhere. We naturally expect that the methods could be used also on other equations with similar forms of multipliers, see for example [30].

Finally, our results are in line with what is known for these types of equations in the same parameter range [24, 26, 28]. Local well-posedness follows by classical methods [1], whereas it is known to exist (i) small and medium-sized travelling waves that are either smooth [3,8,34] or non-smooth [6, 11] and existing for all times, and (ii) time-dependent solutions that are not small, but break down in finite time [19, 32].

The outline of the paper is as follows. In Section 2 we will introduce the pseudoproduct which yields the normal form for the modified energy. In Section 3 we analyse the multiplier which constitutes the main problem after the transformation, and in Section 4 use this to prove the equivalence of norms. Section 5, finally, is the heart of the paper. It is almost entirely carried out in Fourier space, and contains the quartic energy estimates necessary to close the argument via a continuity argument.

2. The modified energy

Standard theory [31] can be used to show that there exists a positive number $T \gtrsim \varepsilon^{-1}$ and a unique solution $u \in C([0,T]; H^N(\mathbb{R}))$ of (1.2). Therefore, to prove Theorem 1.1, we need only to prove an a priori $H^N(\mathbb{R})$-bound for the classical solution $u \in C([0,T]; H^N(\mathbb{R}))$. For this we define the modified energy by

$$E^{(k)}(t) = \|\partial_x^k u\|_{L^2}^2 + 2\big(\partial_x^k u, \partial_x^k B(u,u)\big)_2, \tag{2.1}$$

where the bilinear form B defined as a pseudo-product

$$\mathcal{F}(B(f_1,f_2))(\xi) = \int_{\mathbb{R}} m(\xi-\eta,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)\,\mathrm{d}\eta, \tag{2.2}$$

and the multiplier m is given by

$$m(\xi-\eta,\eta) = \frac{\xi}{2\big[p(\xi-\eta)(\xi-\eta)+p(\eta)\eta-p(\xi)\xi\big]}. \tag{2.3}$$

Thus, the meaning of m for two general variables (a,b) is

$$m(a,b) = \frac{a+b}{2\big[p(a)a+p(b)b-p(a+b)(a+b)\big]}.$$

The modified energy (2.1) removes all the cubic terms from the equation and itself satisfies a quartic equation.

Lemma 2.1. *We have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t) = \big(\partial_x^k(-u\partial_x u), \partial_x^k B(u,u)\big)_2 + 2\big(\partial_x^k u, \partial_x^k B(-u\partial_x u, u)\big)_2. \tag{2.4}$$

Proof. We first claim that

$$-u\partial_x u - \mathcal{L}\partial_x B(u,u) + B(\mathcal{L}\partial_x u, u) + B(u, \mathcal{L}\partial_x u) = 0. \tag{2.5}$$

Indeed, taking Fourier transform on (2.5) yields

$$\begin{aligned}
&\mathcal{F}\big\{-u\partial_x u - \mathcal{L}\partial_x B(u,u) + B(\mathcal{L}\partial_x u, u) + B(u, \mathcal{L}\partial_x u)\big\}(\xi)\\
&= \mathrm{i}\int_{\mathbb{R}}\Big(-\frac{\xi}{2} + m(\xi-\eta,\eta)\left[-p(\xi)\xi + p(\xi-\eta)(\xi-\eta) + p(\eta)\eta\right]\Big)\hat{u}(\xi-\eta)\hat{u}(\eta)\,\mathrm{d}\eta\\
&= 0,
\end{aligned}$$

where we have used the definition (2.3) of m. From the definition of the modified energy (2.1) and equation (1.2), one calculates that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t)\\
&= \big(\partial_x^k\partial_t u, \partial_x^k u\big)_2 + \big(\partial_x^k\partial_t u, \partial_x^k B(u,u)\big)_2 + \big(\partial_x^k u, \partial_x^k\partial_t B(u,u)\big)_2\\
&= \big(\partial_x^k(\mathcal{L}\partial_x u - u\partial_x u), \partial_x^k u\big)_2 + \big(\partial_x^k(\mathcal{L}\partial_x u - u\partial_x u), \partial_x^k B(u,u)\big)_2\\
&\quad + 2\big(\partial_x^k u, \partial_x^k B(\mathcal{L}\partial_x u - u\partial_x u, u)\big)_2\\
&= -\big(\partial_x^k u, \partial_x^k(u\partial_x u)\big)_2 - \big(\partial_x^k u, \partial_x^k(\mathcal{L}\partial_x B(u,u))\big)_2\\
&\quad + \big(\partial_x^k(-u\partial_x u), \partial_x^k B(u,u)\big)_2 + 2\big(\partial_x^k u, \partial_x^k B(\mathcal{L}\partial_x u - u\partial_x u, u)\big)_2,
\end{aligned} \tag{2.6}$$

Since the multiplier m is symmetric in $\xi-\eta$ and η, so is the bilinear form $B(f_1, f_2)$ on f_1 and f_2. Thus $B(\mathcal{L}\partial_x u, u) = B(u, \mathcal{L}\partial_x u)$. Finally inserting (2.5) into (2.6) gives the quartic equation for the evolution of the modified energy.

□

Our main two tasks are, on the one hand, to show that the modified energy $E^{(k)}(t)$ is almost equivalent to the Sobolev energy provided the solution is small in $H^N(\mathbb{R})$, on the other hand, to obtain a quartic-type a priori estimate on $E^{(k)}(t)$. More precisely, we will prove the following two lemmas:

Lemma 2.2. *Let $\alpha \in [-1,1]\setminus\{0\}$. For any $N \geq \max\{2, 2j_*-1\}$, one has*

$$\sum_{k=2j_*-1}^{N} E^{(k)}(t) + \|u\|_{L^2}^2 = \big(1 + \mathcal{O}(\|u\|_{H^2})\big)\|u\|_{H^N}^2.$$

Lemma 2.3. *Let $\alpha \in [-1,1]\setminus\{0\}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t) \lesssim \big(\|u\|_{H^2}\|u\|_{H^3} + \|u\|_{H^k}^2\big)\|u\|_{H^k}^2, \quad k \geq 2j_*-1.$$

We will first show how to prove Theorem 1.1 by using Lemma 2.2 and Lemma 2.3 and postpone their proofs to Section 4 and Section 5 respectively.

Proof of Theorem 1.1. In view of Lemma 2.2, summing over k from $2j_*-1$ to N, one has

$$\sum_{k=2j_*-1}^{N} E^{(k)}(t) \lesssim \sum_{k=2j_*-1}^{N} E^{(k)}(0) + \int_0^t \|u(s,\cdot)\|_{H^N}^4 \,\mathrm{d}s,$$

which in turn yields

$$\sum_{k=2j_*-1}^{N} E^{(k)}(t) + \|u\|_{L^2}^2 \lesssim \sum_{k=2j_*-1}^{N} E^{(k)}(0) + \|u_0\|_{L^2}^2 + \int_0^t \|u(s,\cdot)\|_{H^N}^4 \,\mathrm{d}s. \tag{2.7}$$

Here, we have used the L^2-conservation of solutions to (1.2). According to Lemma 2.3, we on the other hand have

$$\sum_{k=2j_*-1}^{N} E^{(k)}(t) + \|u\|_{L^2}^2 \eqsim \frac{1}{2}\|u\|_{H^N}^2 \tag{2.8}$$

for all $t \geq 0$ and all $\|u\|_{H^N} < \varepsilon$ sufficiently small. We conclude from (2.7) and (2.8) that

$$\|u\|_{H^N}^2 \lesssim \|u_0\|_{H^N}^2 + \int_0^t \|u(s,\cdot)\|_{H^N}^4 \,\mathrm{d}s,$$

which finishes the proof by applying Grönwall's inequality or an analogous continuity argument. □

3. The bound of the multiplier m

In this section we study the singularities and growth of the multiplier m at 'low' and 'high' frequencies, respectively.

Lemma 3.1. *Let $\varphi(a,b) = p(a)a + p(b)b - p(a+b)(a+b)$. Then*

$$\varphi(a,b) = \varphi(b,a), \quad \varphi(-a,-b) = -\varphi(a,b), \quad \varphi(a,b) = \varphi(-(a+b),b),$$

and

$$|\varphi(a,b)| \eqsim \frac{|ab(a+b)|}{r^2} \min(r^{2j_*}, 1 + r^\alpha),$$

where $r = \sqrt{a^2+b^2}$. In particular, $a = 0$, $b = 0$ and $a = -b$ are the only zeros of φ.

Note that $|ab(a+b)|r^{-2}$ is proportional to the smallest of $|a|$, $|b|$ and $|a+b|$, which is always present in φ. Apart from this factor, it is r^{2j_*} that determines φ locally, and $1 + r^\alpha$ in the far field.

Proof of Lemma 3.1. As p is even, it is immediate to verify that $a = 0$, $b = 0$ and $a = -b$ are zeros of φ. Similarly, the symmetries $(a,b) \leftrightarrow (b,a)$ and $(-a,b) \leftrightarrow (a-b,b)$, as well as the anti-symmetry $(-a,-b) \leftrightarrow (a,b)$, follow directly from the definition.

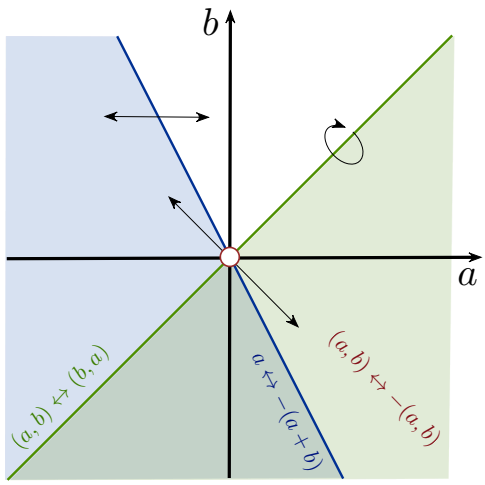

FIGURE 1. The symmetries of the function φ.

To see that there are no other zeros, note first that, by the three above symmetries, it is enough to investigate the region

$$a = \lambda b, \qquad -\frac{1}{2} < \lambda < 1, \quad \lambda \neq 0,$$

in the upper half-plane where $b > 0$, see Figure 1. There, we have

$$\varphi(a,b) = b\underbrace{\left[p(\lambda b)\lambda + p(b) - p((1+\lambda)b)(1+\lambda)\right]}_{(*)}$$

Recall that p is even and strictly monotone on the half-line. In case it is increasing, we have

$$(*) \lesseqgtr (1+\lambda)\left[p(b) - p((1+\lambda)b)\right] \lesseqgtr 0,$$

where the inequalities vary with the sign of $\lambda \in (-\frac{1}{2}, 0) \cup (0, 1)$. Again, if p is instead decreasing, the signs reverse. Hence, p has no additional zeros, and if we take a $\frac{\mathrm{d}}{\mathrm{d}a}$-derivative for any fixed $b > 0$ at the zero $a = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}a}\varphi(a,b)|_{a=0} = -p'(b)b - [p(b) - p(0)] \neq 0,$$

because of the strict monotonicity of p. Hence, by the three symmetries, all zeros of φ outside the origin are simple, and φ may be expressed as

$$\varphi(a,b) = ab(a+b)\tilde{\varphi}(a,b), \tag{3.1}$$

where $\tilde{\varphi}$ is continuous and non-vanishing for $(a,b) \neq (0,0)$.

We now consider three cases. Let $r = \sqrt{a^2+b^2}$, and consider separately the cases

(i) $0 < r \ll 1$,
(ii) $|a| \lesssim 1$ with $r \gg 1$,
(iii) $|r| \gg 1$ with $|a|, |b|, |a+b| \gg 1$.

Note that the intermediate cases are already covered by (3.1), and that (ii) covers the case when either b or $a+b$ is substituted for a in view of the symmetries of φ. Exactly one, or all, of a, b and $a+b$ can be small at the same time.

In the case of (i) all terms are small. We assume by symmetry that a is the smallest in the triad $a, b, a+b$, and perform the calculation for the variables a and r, noting that r is the biggest of them. From the local Lipschitz continuity of the Taylor coefficient p_{2j_*} we have an expansion

$$\begin{aligned}
\varphi(a,b) &= p(a)a + p(b)b - p(a+b)(a+b) \\
&= a[p(0) + a^{2j_*}p_{2j_*}(a)] + b[p(0) + b^{2j_*}p_{2j_*}(b)] \\
&\quad - (a+b)[p(0) + (a+b)^{2j_*}p_{2j_*}(a+b)] \\
&= a^{2j_*+1}\left(p_{2j_*}(0) + o_r(1)\right) + b^{2j_*+1}p_{2j_*}(b) \\
&\quad - \sum_{j=0}^{2j_*+1}\binom{2j_*+1}{j}a^j b^{2j_*+1-j}(p_{2j_*}(b) + a\, o_r(1)) \\
&= p_{2j_*}(0)\Big(-\sum_{j=1}^{2j_*}\binom{2j_*+1}{j}a^{j-1}b^{2j_*+1-j} + o(r^{2j_*})\Big)a \\
&\eqsim -p_{2j_*}(0)ar^{2j_*}.
\end{aligned}$$

It is possible to explicitly extract also b and $a+b$ in the above expression, but it is not necessary, as we know that the orders of of a, b and $a+b$ are identical. Thus, we have

$$|\varphi(a,b)| \eqsim |ab(a+b)|r^{2j_*-2}, \qquad r \ll 1,$$

for small frequencies.

In the case of (ii), we instead expand $p(\cdot + a)$ in a, yielding

$$\begin{aligned}
\varphi(a,b) &= p(a)a + p(b)b - p(a+b)(a+b) \\
&= p(a)a + p(b)b - \left[p(b) + a\int_0^1 p'(sa+b)\,\mathrm{d}s\right](a+b) \\
&= a\left[p(a) - p(b) - (a+b)\int_0^1 p'(sa+b)\,\mathrm{d}s\right].
\end{aligned} \tag{3.2}$$

As p is monotone and a is assumed bounded and b large, the terms $p(a)-p(b)$ and $-(a+b)\int_0^1 p'(sa+b)\,\mathrm{d}s$ are of the same sign. By monotonicity and the far-field assumption (1.3), the distance between p at the origin and at infinity is bounded and nonzero for $\alpha < 0$, and infinite for $\alpha > 0$. More precisely,

$$|p(a) - p(b)| \eqsim 1 + r^\alpha,$$

while

$$\left| -(a+b)\int_0^1 p'(sa+b)\,\mathrm{d}s\right| \eqsim r^\alpha.$$

Therefore,

$$|\varphi(a,b)| \eqsim |a|(1+r^\alpha),$$

when $|b| \eqsim r \gg 1$ and $|a| \lesssim 1$.

If, as in the case of (iii), none of the variables a, b and $a+b$ are bounded, the difference $p(a)-p(b)$ need not be bounded away from zero. One can see from (3.2) that the expression $|a|(1+r^\alpha)$ becomes

$$|\varphi(a,b)| \eqsim \min(|a|,|b|,|a+b|)\max(|a|^\alpha,|b|^\alpha,|a+b|^\alpha),$$

where we note that the max is determined by different variables depending on the sign of α. By combining (ii) and (iii) we thus get the global behaviour

$$|\varphi(a,b)| \eqsim \frac{|ab(a+b)|}{r^2}\left((1+|a|)^\alpha+(1+|b|)^\alpha\right), \qquad r \gtrsim 1,$$

for 'large' frequencies (either a, b or $a+b$ could still be small). □

The following corollary is immediate from Lemma 3.1. It describes the low-frequency singularities and high-frequency derivatives of the symbol m.

Corollary 3.2. *The symbol m satisfies*

$$|m(\xi-\eta,\eta)| \eqsim \frac{1}{|\eta(\xi-\eta)|}\left[\frac{1}{r^{2j_*-2}}+\frac{r^2}{(1+|\eta|)^\alpha+(1+|\xi-\eta|)^\alpha}\right],$$

where r^2 could be replaced by any two-term sum of ξ^2, η^2 and $(\xi-\eta)^2$. In particular,

$$|m(\xi-\eta,\eta)| \lesssim \underbrace{\frac{1}{|\eta(\xi-\eta)\xi^{2j_*-2}|}}_{m_1(\xi-\eta,\eta)}+\underbrace{1+|\xi|\left(1+\frac{1}{|\eta|}+\frac{1}{|\xi-\eta|}\right)}_{m_2(\xi-\eta,\eta)}.$$

Note that although Corollary 3.2 gives a precise equivalence for m, the estimate that follows from it is in fact enough to treat all the cases $\alpha \in (-1,0)\cup(0,1)$ in a uniform way. The singularities in the symbol still provide some difficulties, and Sections 4 and 5 are mainly devoted to treat these in the energy estimates.

4. The proof of Lemma 2.2

This section is devoted to the proof of the equivalent energy norm defined in Lemma 2.2. Let $k \geq 2j_*-1$. We will be done if we can show that

$$\left(\partial_x^k u, \partial_x^k B(u,u)\right)_2 \lesssim \|u\|_{H^2}\|u\|_{H^k}^2. \tag{4.1}$$

Using the symmetry of B one may write

$$\begin{aligned}
&\left(\partial_x^k u, \partial_x^k B(u,u)\right)_2\\
&= 2\int_{\mathbb{R}^2} m(\xi-\eta,\eta)\hat{u}(\xi-\eta)(\mathrm{i}\eta)^k\hat{u}(\eta)\overline{(\mathrm{i}\xi)^k\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi\\
&\quad+\sum_{j=j_*}^{k-1} c_{k,j}\int_{\mathbb{R}^2} m(\xi-\eta,\eta)(\mathrm{i}(\xi-\eta))^j\hat{u}(\xi-\eta)(\mathrm{i}\eta)^{k-j}\hat{u}(\eta)\overline{(\mathrm{i}\xi)^k\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}$$

$$=: 2A_0 + \sum_{j=j_*}^{k-1} c_{k,j} A_j,$$

where $c_{k,j}$ are binomial coefficients. The term A_0 possesses a singularity at low frequencies and an extra derivative at high frequencies, and we handle it first. We will use integration by parts to eliminate the worst part of A_0 and thus reduce it. Note that on the Fourier side, integration by parts corresponds to formula $-\mathrm{i}\xi = -(\mathrm{i}(\xi-\eta)+\mathrm{i}\eta)$, hence

$$\begin{aligned} A_0 = &\underbrace{-\int_{\mathbb{R}^2} m(\xi-\eta,\eta)\mathrm{i}(\xi-\eta)\hat{u}(\xi-\eta)(\mathrm{i}\eta)^k\hat{u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi}_{A_0^1\text{: good part}} \\ &\underbrace{-\int_{\mathbb{R}^2} m(\xi-\eta,\eta)\hat{u}(\xi-\eta)(\mathrm{i}\eta)^{k+1}\hat{u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi}_{A_0^2\text{: bad part}}, \end{aligned} \tag{4.2}$$

in which we have utilised the fact $\overline{\hat{u}(\xi)} = \hat{u}(-\xi)$ since u is real and m is invariant under the map $(\xi,\eta)\mapsto -(\xi,\eta)$. We next perform the change of variables $\xi \leftrightarrow \eta$ on A_0^2 and note that the solution u is real, to find that

$$A_0^2 = -\int_{\mathbb{R}^2} m(\eta-\xi,\xi)\hat{u}(\xi-\eta)(\mathrm{i}\eta)^{k-1}\hat{u}(\eta)\overline{(\mathrm{i}\xi)^{k+1}\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Therefore

$$\begin{aligned} A_0 + A_0^2 = &\int_{\mathbb{R}^2} \eta^{-1}\big[m(\xi-\eta,\eta)\eta + m(\eta-\xi,\xi)\xi\big] \\ &\times \hat{u}(\xi-\eta)(\mathrm{i}\eta)^k\hat{u}(\eta)\overline{(\mathrm{i}\xi)^k\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi = 0, \end{aligned} \tag{4.3}$$

where we have applied the equality

$$m(\xi-\eta,\eta)\eta + m(\eta-\xi,\xi)\xi = 0.$$

Inserting (4.3) to (4.2), we finally may express

$$A_0 = \frac{1}{2}A_0^1. \tag{4.4}$$

Hence, it is sufficient to estimate the terms A_0^1 and A_j, $j = 1,\cdots,k-1$. In view of Corollary 3.2 the multiplier m is controlled by $m_1 + m_2$. On the one hand, the factor $(\xi-\eta)\eta^k\xi^{k-1}$ appearing in A_0^1 and A_j eliminates the low-frequency singularities of $m_1(\xi-\eta,\eta)$ because of $k \geq 2j_* - 1$; on the other hand, the multiplier m_1 does not contain any derivatives (high-frequency). We let $A_0^1(m_1)$ denote the part controlled by m_1 of the integral, and similarly for other integrals to come. Consequently we may estimate

$$\begin{aligned} A_0^1(m_1) &\lesssim \big\||D|^{-1}\partial_x u\big\|_{L^2}\big\||D|^{-1}\partial_x^k u\big\|_{H^1}\big\||D|^{-(2j_*-2)}\partial_x^{k-1}u\big\|_{L^2} \\ &\lesssim \|u\|_{L^2}\|u\|_{H^k}\|u\|_{H^{k-2j_*+1}} \leq \|u\|_{L^2}\|u\|_{H^k}^2, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} A_j(m_1) &\lesssim \big\| |D|^{-1}\partial_x^j u\big\|_{L^2} \big\| |D|^{-1}\partial_x^{k-j} u\big\|_{H^1} \big\| |D|^{-(2j_*-2)}\partial_x^k u\big\|_{L^2} \\ &\lesssim \|u\|_{H^{j-1}}\|u\|_{H^{k-j}}\|u\|_{H^{k-2j_*+2}} \le \|u\|_{H^k}^3. \end{aligned} \tag{4.6}$$

For the parts $A_0^1(m_2)$ and $A_j(m_2)$, there is also no singularity at low frequencies in the total symbol, thus we need only to focus on the high-frequency derivatives. The estimate on $A_0^1(m_2)$ is straightforward:

$$\begin{aligned} A_0^1(m_2) \lesssim& \big\|\partial_x u\big\|_{H^1}\big\|\partial_x^k u\big\|_{L^2}\big\|\partial_x^{k-1} u\big\|_{L^2} \\ &+\big\||D|^{-1}\partial_x u\big\|_{H^1}\big\|\partial_x^k u\big\|_{L^2}\big\||D|\partial_x^{k-1} u\big\|_{L^2} \\ &+\big\|\partial_x u\big\|_{L^2}\big\||D|^{-1}\partial_x^k u\big\|_{H^1}\big\||D|\partial_x^{k-1} u\big\|_{L^2} \\ \lesssim& \|u\|_{H^2}\|u\|_{H^k}^2. \end{aligned} \tag{4.7}$$

To handle $A_j(m_2)$, we shall distribute the derivative in the total symbol to its correct position. Before going ahead, we make a general rehearsal for convenience hereafter.

Lemma 4.1.

$$T(\hat f,\hat g)(a) := \int_{\mathbb{R}} m_2(a-b,b)\hat f(a-b)\hat g(b)\,\mathrm{d}b.$$

Then

$$\begin{aligned} \|T(\hat f,\hat g)\|_{L^2} \lesssim& \|(1+|D|)f\|_{L^2}\|(1+|D|^{-1})g\|_{H^1} \\ &+\|(1+|D|^{-1})f\|_{H^1}\|(1+|D|)g\|_{L^2}. \end{aligned}$$

Proof. The desired result follows from

$$\begin{aligned} m_2(a-b,b) &= 1+\frac{|a-b|}{|b|}+\frac{|b|}{|a-b|} \\ &\lesssim (1+|a-b|)(1+\frac{1}{|b|})+(1+|b|)(1+\frac{1}{|a-b|}), \end{aligned}$$

and

$$\Big\|\int \big|\hat f(a-b)\widehat g(b)\big|\,\mathrm{d}b\Big\|_{L^2_a} \le \int\Big(\int \big|\hat f(a-b)\big|^2\,\mathrm{d}a\Big)^{1/2}\big|\widehat g(b)\big|\,\mathrm{d}b \lesssim \|f\|_{L^2}\|g\|_{H^1}.$$

□

From Lemma 4.1, it follows that

$$\begin{aligned} A_j(m_2) \lesssim& \big\|\partial_x^k u\big\|_{L^2}\Big(\big\|(1+|D|)\partial_x^j u\big\|_{L^2}\big\|(1+|D|^{-1})\partial_x^{k-j} u\big\|_{H^1} \\ &+\big\|(1+|D|^{-1})\partial_x^j u\big\|_{H^1}\big\|(1+|D|)\partial_x^{k-j} u\big\|_{L^2}\Big) \\ \lesssim& \|u\|_{H^k}^3. \end{aligned} \tag{4.8}$$

We conclude (4.1) from (4.5)-(4.8), which proves Lemma 2.2.

5. The proof of Lemma 2.3

We will first perform some basic algebraic manipulations to reduce the evolution (2.4) of the modified energy $E^{(k)}(t)$ to certain higher-order terms which posses extra derivatives at high frequencies. Pointwise monomial estimates like the ones in Section 4 however cannot be used directly to deal with the higher-order terms left (in fact, they are untrue in that formulation if one wants a bound in $H^k(\mathbb{R})$). To tackle the problem, we therefore perform global transformations which include integration by parts and a series of change of variables in spectral space, which finally reduce the most difficult part of the higher-order terms to a commutator that has two orders of gain in the required Fourier variables via its difference structure. The main difficulty is that, to find the correct commutator, we need to divide the Fourier space $\mathbb{R}^3$ into different parts and then make several changes of variables in some symmetric domains far away from the low frequencies to avoid the singularities.

5.1. **Reduction of $\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t), k \geq 2j_* - 1$.** Note that there are two more derivatives on the right-hand side of (2.4) compared to the Sobolev norm in Lemma 2.2. We first introduce a new bilinear form to find a cancellation, which reduces the terms to those contain at most one extra derivative. Let $Q = \partial_x^{-1}B$ and denote by n the symbol of Q so that

$$n(\xi-\eta,\eta) = \frac{-\mathrm{i}}{2\big[p(\xi-\eta)(\xi-\eta)+p(\eta)\eta-p(\xi)\xi\big]}, \tag{5.1}$$

and

$$|n(\xi-\eta,\eta)| \lesssim \frac{1}{|\xi|^{2j_*-1}|\xi-\eta||\eta|} + (1+\frac{1}{|\xi|})(1+\frac{1}{|\xi-\eta|}+\frac{1}{|\eta|}). \tag{5.2}$$

We decompose (2.4) into its highest-order and remainder terms, as

$$\begin{aligned}
&\big(\partial_x^k B(u,u), \partial_x^k(-u\partial_x u)\big)_2 \\
&= \big(\partial_x^{k+1}Q(u,u), \partial_x^k(-u\partial_x u)\big)_2 = 2\big(\partial_x^k Q(u,\partial_x u), \partial_x^k(-u\partial_x u)\big)_2 \\
&= 2\underbrace{\big(Q(u,\partial_x^{k+1}u), \partial_x^k(-u\partial_x u)\big)_2}_{F_0:\text{ highest-order term}} + 2\sum_{j=1}^{k} c_{k,j}\underbrace{\big(Q(\partial_x^j u, \partial_x^{k+1-j}u), \partial_x^k(-u\partial_x u)\big)_2}_{F_j:\text{ remainder terms}},
\end{aligned}$$

and, using integration by parts,

$$\begin{aligned}
&2\big(\partial_x^k B(u,-u\partial_x u), \partial_x^k u\big)_2 = 2\big(\partial_x^k Q(u,u\partial_x u), \partial_x^{k+1}u\big)_2 \\
&= 2\underbrace{\big(Q(u,\partial_x^k(u\partial_x u)), \partial_x^{k+1}u\big)_2}_{G_0:\text{ highest-order term}} + 2\sum_{j=1}^{k} c_{k,j}\underbrace{\big(Q(\partial_x^j u, \partial_x^{k-j}(u\partial_x u)), \partial_x^{k+1}u\big)_2}_{G_j:\text{ remainder term}}.
\end{aligned}$$

Although the two highest-order terms F_0 and G_0 posses the extra derivatives and singularities, the summation of them completely cancel each other

out. Indeed, an easy calculation shows

$$F_0 = -\int_{\mathbb{R}^2} \overline{n(\eta-\xi,\xi)}\hat{u}(\xi-\eta)\mathcal{F}(\partial_x^k(u\partial_x u))(\eta)\overline{\mathcal{F}(\partial_x^{k+1}u)(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi,$$

and then

$$\begin{aligned} F_0 + G_0 = \int_{\mathbb{R}^2} &\big[n(\xi-\eta,\eta) - \overline{n(\eta-\xi,\xi)}\big]\hat{u}(\xi-\eta) \\ &\times \mathcal{F}(\partial_x^k(u\partial_x u))(\eta)\overline{\mathcal{F}(\partial_x^{k+1}u)(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi = 0, \end{aligned} \tag{5.3}$$

where we have used the equality $n(\xi-\eta,\eta) = \overline{n(\eta-\xi,\xi)}$ in view of (5.1).

In the reminder terms, F_j, G_j for $j = 2,\dots,k-1$ are easy terms. In fact, using integration by parts, one rewrites

$$F_j = \frac{1}{2}\big(B(\partial_x^j u, \partial_x^{k+1-j}u), \partial_x^k u^2\big)_2, \quad G_j = -\frac{1}{2}\big(B(\partial_x^j u, \partial_x^{k+1-j}u^2), \partial_x^k u\big)_2,$$

and then apply Lemma 4.1 together with the Gagliardo–Nirenberg inequality (for details, see the calculation in [12]) to obtain

$$|\sum_{j=2}^{k-1} F_j| + |\sum_{j=2}^{k-1} G_j| \lesssim \|u\|_{H^k}^4. \tag{5.4}$$

By the symmetry of Q and integration by parts, one calculates

$$\begin{aligned} G_k &= \frac{1}{2}(Q(\partial_x u^2, \partial_x^k u), \partial_x^{k+1}u)_2 \\ &= \frac{1}{2}\int_{\mathbb{R}^2} n(\xi-\eta,\eta)(\xi-\eta)\eta^k\xi^{k+1}\widehat{u^2}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= \frac{1}{2}\int n(\xi-\eta,\eta)(\xi-\eta)^2\eta^k\xi^k\widehat{u^2}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} n(\xi-\eta,\eta)(\xi-\eta)\eta^{k+1}\xi^k\widehat{u^2}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^2} n(\xi-\eta,\eta)(\xi-\eta)^2\eta^k\xi^k\widehat{u^2}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\quad - \frac{1}{2}(Q(\partial_x u^2, \partial_x^k u), \partial_x^{k+1}u)_2, \end{aligned}$$

where in the last equaility we have taken advantage of the anti-symmetry of n in $(\xi-\eta,\eta)$, and the fact that u is real whereas n is imaginary. Thus

$$G_k = \frac{1}{2}\int_{\mathbb{R}^2} n(\xi-\eta,\eta)(\xi-\eta)^2\eta^k\xi^k\widehat{u^2}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Therefore via (5.2) we get

$$|G_k| \lesssim \|u\|_{H^2}\|u\|_{H^3}\|u\|_{H^k}^2. \tag{5.5}$$

The following lemma summarises (5.3)–(5.5).

Lemma 5.1. *We have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t) \approx F_1 + G_1 + \mathcal{O}(\|u\|_{H^2}\|u\|_{H^3} + \|u\|_{H^k}^2)\|u\|_{H^k}^2, \quad k \geq 2j_* - 1.$$

Note here that the earlier $F_k = F_1$ by the symmetry of B and Q, so we need only to focus on F_1 and G_1. However, there is still one extra derivative in both F_1 and G_1, with which we shall perform commutator estimates to treat in the next subsection.

5.2. **Higher-order estimates: F_1 and G_1.** Just as in [12] we shall divide frequency space into several symmetric regions. The first of these are

$$\mathcal{A}_1 = \{(\xi,\eta,\sigma) \in \mathbb{R}^3 \colon \min\{|\xi|,|\eta|,|\sigma|\} < 1\},$$

and it complement $\mathcal{A}_1^c$, in which $|\xi|,|\eta|,|\sigma| \geq 1$. Also, for convenience and to standardise the calculations, we shall use the measure notation

$$\mathrm{d}M(\hat{u}) = \hat{u}(\xi-\eta)\hat{u}(\eta-\sigma)\hat{u}(\sigma)\overline{\hat{u}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma,$$

for the quartic factor that will appear in many estimates. An estimate that will be used frequently for both $\mathcal{A}_1$ and some later sets is

$$|\xi| + |\eta| + |\sigma| \lesssim 1 + |\xi-\eta| + |\eta-\sigma|. \tag{5.6}$$

To employ commutator estimates to handle the terms F_1 and G_1, we first split F_1 and G_1 into a low-frequencies part (to eliminate the singularities) and a high-frequencies part (to distribute the derivatives) respectively, and then extract the lower-order parts from the high-frequencies part. By integration by parts, we decompose F_1 as

$$\begin{aligned}
F_1 &= -\int_{\mathbb{R}^3} n(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{(\mathrm{i}\xi)^k\hat{u}(\xi-\sigma)\widehat{\partial_x u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \\
&= \int_{\mathbb{R}^3} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi-\sigma)\widehat{\partial_x u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \\
&= \underbrace{\int_{\mathcal{A}_1} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi-\sigma)\widehat{\partial_x u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma}_{\mathcal{A}_1F_1} \\
&\quad + \underbrace{\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi-\sigma)\widehat{\partial_x u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma}_{\mathcal{A}_1^cF_1},
\end{aligned}$$

where we keep the derivative ξ^{k-1} for eliminating the possible low-frequency singularity at ξ in $m(\xi-\eta,\eta)$ of $\mathcal{A}_1F_1$, and then write the last term by

Leibniz's law as

$$\mathcal{A}_1^c F_1 = \underbrace{\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{\hat{u}(\xi-\sigma)\widehat{\partial_x^k u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma}_{F_{1,0}}$$

$$+\sum_{l=1}^{k-1} c_{k-1,l}\underbrace{\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{\widehat{\partial_x^l u}(\xi-\sigma)\widehat{\partial_x^{k-l} u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma}_{F_{1,l}\text{: lower-order term}},$$

where we have used $\xi^{k-1} = [(\xi-\sigma)+\sigma]^{k-1} = \sum_{l=1}^{k-1} c_{k-1,l}(\xi-\sigma)^l\sigma^{k-l}$.

For G_1, if $k=1$ (which means $j_*=1$), to eliminate the low-frequency singularity, we write

$$G_1 = \frac{1}{2}(Q(\partial_x u, \partial_x u^2), \partial_x^2 u)_2,$$

and then obtain

$$|G_1| \lesssim \|u\|_{H^1}^2\|u\|_{H^2}^2.$$

If $k\geq 2$ (which means $j_*\geq 2$), we instead decompose G_1 in the following manner:

$$\begin{aligned} G_1 &= \int_{\mathbb{R}^3} n(\xi-\eta,\eta)(\xi-\eta)\sigma\eta^{k-1}\xi^{k+1}\,\mathrm{d}M(\hat{u}) \\ &= -\mathrm{i}\int_{\mathbb{R}^3} m(\xi-\eta,\eta)(\xi-\eta)\sigma\eta^{k-1}\xi^{k}\,\mathrm{d}M(\hat{u}) \\ &= \underbrace{-\mathrm{i}\int_{\mathcal{A}_1} m(\xi-\eta,\eta)(\xi-\eta)\sigma\eta^{k-1}\xi^{k}\,\mathrm{d}M(\hat{u})}_{\mathcal{A}_1 G_1} \\ &\quad \underbrace{-\mathrm{i}\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)(\xi-\eta)\sigma\eta^{k-1}\xi^{k}\,\mathrm{d}M(\hat{u})}_{\mathcal{A}_1^c G_1}, \end{aligned}$$

and then express the last term further by Leibniz's law as

$$\mathcal{A}_1^c G_1 = \underbrace{-\mathrm{i}\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)(\xi-\eta)\eta\sigma^{k-1}\xi^{k}\,\mathrm{d}M(\hat{u})}_{G_{1,0}}$$

$$-\mathrm{i}\sum_{l=1}^{k-2} c_{k-1,l}\underbrace{\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)(\xi-\eta)\eta(\eta-\sigma)^l\sigma^{k-l-1}\xi^{k}\,\mathrm{d}M(\hat{u})}_{G_{1,l}\text{: lower-order term}},$$

where we have used $\eta^{k-2} = [(\eta-\sigma)+\sigma]^{k-2} = \sum_{l=1}^{k-2} c_{k-2,l}(\eta-\sigma)^l\sigma^{k-l-2}$.

Since the terms $F_{1,l}$ $(l=1,2,\cdots,k-1)$ and $G_{1,l}$ $(l=1,2,\cdots,k-2)$ have no extra derivative at high frequencies and no singularity at low frequencies,

it is straightforward to verify that

$$|\sum_{l=1}^{k-1} F_{1,l}| + |\sum_{l=1}^{k-2} G_{1,l}| \lesssim \|u\|_{H^2}\|u\|_{H^k}^3.$$

We are now left with four terms: $\mathcal{A}_1F_1, \mathcal{A}_1G_1$ and $F_{1,0}, G_{1,0}$. The former two terms involve low frequencies, the latter two terms only contain high frequencies. We first handle the terms involving low frequencies.

Lemma 5.2. *We have*

$$|\mathcal{A}_1F_1| + |\mathcal{A}_1G_1| \lesssim \|u\|_{H^1}\|u\|_{H^2}\|u\|_{H^k}^2.$$

Proof. We recall that

$$\mathcal{A}_1F_1 = \int_{\mathcal{A}_1} m(\xi-\eta,\eta)\widehat{\partial_x u}(\xi-\eta)\widehat{\partial_x^k u}(\eta)\overline{(\mathrm{i}\xi)^{k-1}\hat{u}(\xi-\sigma)\widehat{\partial_x u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma.$$

The part $\mathcal{A}_1F_1(m_1)$ is straightforward since the factor $(\xi-\eta)\eta^k\xi^{k-1}$ eliminates the low-frequency singularities. Considering instead $\mathcal{A}_1F_1(m_2)$, the worst case is that all the derivatives $\xi^{k-1} = \sigma^{k-1} + \sum_{l=1}^{k-1} c_{k-1,l}(\xi-\sigma)^l\sigma^{k-1-l}$ fall on $\widehat{\partial_x u}(\sigma)$, which yields a full $\widehat{\partial_x^k u}(\sigma)$. But in $\mathcal{A}_1$, the triangle inequality (5.6) allows us to move the one extra derivative ξ in m_2 to $\xi-\eta$ or $\xi-\sigma$ to get the desired bound.

As for

$$\mathcal{A}_1G_1 = -\mathrm{i}\int_{\mathcal{A}_1} m(\xi-\eta,\eta)(\xi-\eta)\sigma\eta^{k-1}\xi^k\,\mathrm{d}M(\hat{u}).$$

the difficult part is $\mathcal{A}_1G_1(m_2)$, with the worst situation when all the derivatives $\eta^{k-1}\sigma$ fall on $\widehat{u}(\sigma)$. This again gives a maximal $\widehat{\partial_x^k u}(\sigma)$. Noting the definition of $\mathrm{d}M(\hat{u})$, just as above we invoke (5.6) to move the one extra derivative ξ in m_2 to $\xi-\eta$ or $\eta-\sigma$ to obtain the bound of Lemma 5.2. □

We next make the following very useful observation.

Lemma 5.3. *One has*

$$F_{1,0} = G_{1,0} + \mathcal{O}(\|u\|_{H^2}\|u\|_{H^3}\|u\|_{H^k}^2).$$

Proof. Note that the set $\mathcal{A}_1^c$ is invariant under changes of variables among the variables ξ, η and σ. We now apply the changes of variables $\xi \leftrightarrow \sigma \leftrightarrow \eta$ to get

$$\begin{aligned} F_{1,0} &= \mathrm{i}\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)(\xi-\eta)\eta^k\sigma^k\hat{u}(\xi-\eta)\hat{u}(\eta)\overline{\hat{u}(\xi-\sigma)\hat{u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \\ &= \mathrm{i}\int_{\mathcal{A}_1^c} m(\eta-\sigma,\sigma)(\eta-\sigma)\sigma^k\xi^k\,\mathrm{d}M(\hat{u}), \end{aligned}$$

and $\eta \leftrightarrow \xi \leftrightarrow \sigma \leftrightarrow \eta$ to yield

$$G_{1,0} = -\mathrm{i}\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)\eta(\xi-\eta)\sigma^{k-1}\xi^k\,\mathrm{d}M(\hat{u})$$

$$= -\mathrm{i}\int_{\mathcal{A}_1^c} m(\sigma-\eta,\eta)\eta(\sigma-\eta)\xi^{k-1}\sigma^k\hat{u}(\sigma-\eta)\hat{u}(\eta-\xi)\hat{u}(\xi)\overline{\hat{u}(\sigma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma$$
$$= \mathrm{i}\int_{\mathcal{A}_1^c} m(\eta-\sigma,\sigma)\sigma(\eta-\sigma)\xi^{k-1}\sigma^k\hat{u}(\eta-\sigma)\hat{u}(\xi-\eta)\hat{u}(\sigma)\overline{\hat{u}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma.$$

In the last equality we have utilized that the fact that $G_{1,0}$ and u are real and the equality

$$m(\sigma-\eta,\eta)\eta(\sigma-\eta) = m(\eta-\sigma,\sigma)\sigma(\eta-\sigma).$$

We see that

$$F_{1,0} - G_{1,0} = \mathrm{i}\int_{\mathcal{A}_1^c} m(\eta-\sigma,\sigma)(\eta-\sigma)(\xi-\sigma)\xi^{k-1}\sigma^k\,\mathrm{d}M(\hat{u}).$$

The factor $(\xi-\sigma)$ has a difference structure and may be controlled by $|\xi-\eta|+|\eta-\sigma|$. In $\mathcal{A}_1^c$, the one extra derivative ξ in m_2 is then added to ξ^{k-1} to yield the $H^2\times H^3\times H^k\times H^k$-estimate. □

From now on we will use the expressions

$$F_{1,0} = \mathrm{i}\int_{\mathcal{A}_1^c} m(\eta-\sigma,\sigma)(\eta-\sigma)\sigma^k\xi^k\,\mathrm{d}M(\hat{u}),$$

and

$$G_{1,0} = -\mathrm{i}\int_{\mathcal{A}_1^c} m(\xi-\eta,\eta)(\xi-\eta)\eta\sigma^{k-1}\xi^k\,\mathrm{d}M(\hat{u}).$$

Lemma 5.3 means that we may write

$$F_{1,0} = \frac{1}{2}(F_{1,0}+G_{1,0}) + \mathcal{O}(\|u\|_{H^2}\|u\|_{H^3}\|u\|_{H^k}^2).$$

The terms $F_{1,0}+G_{1,0}$ and $F_{1,0}$ are thus formally equivalent, but we will for convenience use $F_{1,0}+G_{1,0}$ below to obtain a good commutator. We again split the remaining frequency space into two parts,

$$\mathcal{A}_2 = \{(\xi,\eta,\sigma)\in\mathcal{A}_1^c\colon \tfrac{1}{10}|z_2| < |z_1-z_2|+|z_2-z_3|, \\ \text{for some choice of } z_j=\xi,\eta,\sigma\},$$

and its complement $\mathcal{A}_2^c$, in which $\frac{1}{10}|z_2|\geq|z_1-z_2|+|z_2-z_3|$ for all choices of $z_j=\xi,\eta,\sigma$. As above, we let $\mathcal{A}_2F_{1,0}$ denote the restriction of the integral $F_{1,0}$ to the set $\mathcal{A}_2$, and similarly for other integrals and sets to come, so that

$$\mathcal{F}_{1,0}+G_{1,0} = \mathcal{A}_2(F_{1,0}+G_{1,0}) + \mathcal{A}_2^c(F_{1,0}+G_{1,0}).$$

Just as in $\mathcal{A}_1$, the triangle inequality (5.6) holds in $\mathcal{A}_2$, allowing us to move derivatives from ξ,η and σ to $\xi-\eta$ and $\eta-\sigma$, so the term $\mathcal{A}_2(F_{1,0}+G_{1,0})$ may be handled separately, with the following resulting estimate.

Lemma 5.4. *We have*

$$|\mathcal{A}_2F_{1,0}| + |\mathcal{A}_2G_{1,0}| \lesssim \|u\|_{H^2}\|u\|_{H^3}\|u\|_{H^k}^2.$$

We shall use symmetry to reduce the integrals further. To that aim, let

$$\mathcal{A}_{2,+}^c = \{(\xi,\eta,\sigma) \in \mathcal{A}_2^c \colon \xi,\eta,\sigma \geq 1\},$$

be the 'positive' part of $\mathcal{A}_{2,+}^c$. Then, in $\mathcal{A}_{2,+}^c$, we in place of the earlier triangle inequality (5.6) obtain the equivalence

$$\xi \eqsim \sigma \eqsim \eta \eqsim \xi - \eta + \sigma \gtrsim 1, \tag{5.7}$$

which again will help us to exchange derivatives. What remains is $\mathcal{A}_2^c(G_{1,0}+F_{1,0})$. Since both the multiplier m and the solution u are real, by the shift of variables $(\xi,\eta,\sigma) \to -(\xi,\eta,\sigma)$, one observes that

$$\mathcal{A}_2^c F_{1,0} = 2\mathcal{A}_{2,+}^c F_{1,0}, \quad \mathcal{A}_2^c G_{1,0} = 2\mathcal{A}_{2,+}^c G_{1,0}.$$

We then decompose $\mathcal{A}_2^c F_{1,0}$ as

$$\begin{aligned}
\mathcal{A}_{2,+}^c F_{1,0} &= \mathrm{i}\int_{\mathcal{A}_{2,+}^c} m(\eta-\sigma,\sigma)(\eta-\sigma)\sigma^k\xi^k \,\mathrm{d}M(\hat{u}) \\
&= \underbrace{\mathrm{i}\int_{\mathcal{A}_{2,+}^c} \frac{m(\eta-\sigma,\sigma)}{\eta}(\eta-\xi)(\eta-\sigma)\sigma^k\xi^k \,\mathrm{d}M(\hat{u})}_{\mathcal{A}_{2,+}^c F_{1,0}^{(\eta-\xi)}} \\
&\quad + \underbrace{\mathrm{i}\int_{\mathcal{A}_{2,+}^c} \frac{m(\eta-\sigma,\sigma)}{\eta}(\eta-\sigma)\sigma^k\xi^{k+1} \,\mathrm{d}M(\hat{u})}_{\mathcal{A}_{2,+}^c \tilde{F}_{1,0}},
\end{aligned}$$

and $\mathcal{A}_2^c G_{1,0}$ as

$$\begin{aligned}
\mathcal{A}_{2,+}^c G_{1,0} &= -\mathrm{i}\int_{\mathcal{A}_{2,+}^c} m(\xi-\eta,\eta)\eta(\xi-\eta)\sigma^{k-1}\xi^k \mathrm{d}M(\hat{u}) \\
&= \underbrace{-\mathrm{i}\int_{\mathcal{A}_{2,+}^c} m(\xi-\eta,\eta)(\eta-\sigma)(\xi-\eta)\sigma^{k-1}\xi^k \mathrm{d}M(\hat{u})}_{\mathcal{A}_{2,+}^c G_{1,0}^{(\eta-\sigma)}} \\
&\quad \underbrace{-\mathrm{i}\int_{\mathcal{A}_{2,+}^c} m(\xi-\eta,\eta)(\xi-\eta)\sigma^k\xi^k \mathrm{d}M(\hat{u})}_{\mathcal{A}_{2,+}^c \tilde{G}_{1,0}}.
\end{aligned}$$

Note that the symbol $\frac{m(\eta-\sigma,\sigma)}{\eta}$ is bounded when considering $\mathcal{A}_{2,+}^c F_{1,0}^{(\eta-\xi)}$. The one extra derivate ξ in $\mathcal{A}_{2,+}^c G_{1,0}^{(\eta-\sigma)}$ may also be transmitted to $|\xi-\eta|+|\eta-\sigma|+\sigma$, so as to give us the following reduction.

Lemma 5.5.

$$\mathcal{A}_2^c(F_{1,0}+G_{1,0}) = \mathcal{A}_2^c(\tilde{F}_{1,0}+\tilde{G}_{1,0}) + \mathcal{O}(\|u\|_{H^2}^2\|u\|_{H^k}^2),$$

with $\mathcal{A}_2^c(\tilde{F}_{1,0}+\tilde{G}_{1,0})$ *given by the integral*

$$\mathrm{i}\int_{\mathcal{A}_{2,+}^c}\left[\frac{m(\eta-\sigma,\sigma)}{\eta}(\eta-\sigma)-\frac{m(\xi-\eta,\eta)}{\xi}(\xi-\eta)\right]\sigma^k\xi^{k+1}\,\mathrm{d}M(\hat{u}). \tag{5.8}$$

The commutator in (5.8) contains the cancellation we need, but in order to disclose it, we need to use a Fourier symmetry that is not present in the set $\mathcal{A}_{2,+}^c$. We therefore as a final division of $\mathbb{R}^3$ introduce the sets

$$\mathcal{B}_1=\{(\xi,\eta,\sigma)\in\mathbb{R}^3\colon \xi,\xi-\eta+\sigma,\sigma\geq 1\},$$

and

$$\begin{aligned}\mathcal{B}_2=\big\{(\xi,\eta,\sigma)\in\mathbb{R}^3\colon\ &|\sigma-\eta|+|\xi-\sigma|\leq\tfrac{1}{10}\xi,\\ &|\xi-\eta|+|\sigma-\xi|\leq\tfrac{1}{10}\sigma,\\ &|\eta-\sigma|+|\xi-\eta|\leq\tfrac{1}{10}(\xi-\eta+\sigma)\big\},\end{aligned}$$

and their intersection $\mathcal{B}=\mathcal{B}_1\cap\mathcal{B}_2$. While the triangle inequality (5.6) does not in general hold in $\mathcal{A}_{2,+}^c$, the set $\mathcal{B}$ has the advantage that in both $\mathcal{A}_{2,+}^c\setminus\mathcal{B}$ and $\mathcal{B}\setminus\mathcal{A}_{2,+}^c$, it holds, enabling us to exchange derivatives. Performing the change of variables $\eta\mapsto\xi-\eta+\sigma$ on the first term in (5.8), we may write

$$\begin{aligned}&\mathcal{A}_2^c(\tilde{F}_{1,0}+\tilde{G}_{1,0})\\ &=\underbrace{\mathrm{i}\int_{\mathcal{A}_{2,+}^c\cap\mathcal{B}}\left[\frac{m(\xi-\eta,\sigma)}{\xi-\eta+\sigma}-\frac{m(\xi-\eta,\eta)}{\xi}\right](\xi-\eta)\sigma^k\xi^{k+1}\,\mathrm{d}M(\hat{u})}_{[\mathcal{A}_{2,+}^c\cap\mathcal{B}](\tilde{F}_{1,0}+\tilde{G}_{1,0})\colon\text{ the main commutator}}\\ &\quad\underbrace{-\mathrm{i}\int_{\mathcal{A}_{2,+}^c\setminus\mathcal{B}}m(\xi-\eta,\eta)(\xi-\eta)\sigma^k\xi^k\,\mathrm{d}M(\hat{u})}_{[\mathcal{A}_{2,+}^c\setminus\mathcal{B}](\tilde{F}_{1,0}+\tilde{G}_{1,0})}\\ &\quad\underbrace{+\,\mathrm{i}\int_{\mathcal{B}\setminus\mathcal{A}_{2,+}^c}\frac{m(\xi-\eta,\sigma)}{\xi-\eta+\sigma}(\xi-\eta)\sigma^k\xi^{k+1}\mathrm{d}M(\hat{u})}_{[\mathcal{B}\setminus\mathcal{A}_{2,+}^c](\tilde{F}_{1,0}+\tilde{G}_{1,0})}.\end{aligned} \tag{5.9}$$

By using (5.6) exactly in the same way as before, we then arrive at our final expression. The reason for keeping $\mathcal{A}_{2,+}^c\cap\mathcal{B}$ is both that it contains the most difficult term in the analysis, but also that $\mathcal{A}_{2,+}^c\cap\mathcal{B}$ contains a symmetry in which *two* orders the terms in the commutator cancel. This will all be made clear in the proof of Prop. 5.7. For now, we have:

Proposition 5.6.

$$\mathcal{A}_{2,+}^c(\tilde{F}_{1,0}+\tilde{G}_{1,0})=[\mathcal{A}_{2,+}^c\cap\mathcal{B}](\tilde{F}_{1,0}+\tilde{G}_{1,0})+\mathcal{O}(\|u\|_{H^2}\|u\|_{H^3}\|u\|_{H^k}^2). \tag{5.10}$$

The main commutator. We are now at the position to estimate the last term $[\mathcal{A}_{2,+}^c\cap\mathcal{B}](\tilde{F}_{1,0}+\tilde{G}_{1,0})$ to complete the proof of Lemma 2.3. As mentioned above, this is a good commutator that will improve our estimates by

two orders of cancellations, allowing us to move derivatives in an advantageous way, and finally kill the additional derivative.

Proposition 5.7.

$$|[\mathcal{A}_{2,+}^c \cap \mathcal{B}](\tilde{F}_{1,0} + \tilde{G}_{1,0})| \lesssim \|u\|_{H^1}\|u\|_{H^2}\|u\|_{H^k}^2.$$

Proof. Let

$$N(\xi,\eta,\sigma) = \frac{m(\xi-\eta,\eta)}{\xi} - \frac{m(\xi-\eta,\sigma)}{\xi-\eta+\sigma}.$$

In view of (2.3) one then calculates that

$$\begin{aligned} N(\xi,\eta,\sigma) &= \frac{1}{2\big[p(\xi-\eta)(\xi-\eta)+p(\eta)\eta-p(\xi)\xi\big]} \\ &\quad - \frac{1}{2\big[p(\xi-\eta)(\xi-\eta)+p(\sigma)\sigma-p(\xi-\eta+\sigma)(\xi-\eta+\sigma)\big]} \\ &= \frac{p(\sigma)\sigma-p(\xi-\eta+\sigma)(\xi-\eta+\sigma)-\big[p(\eta)\eta-p(\xi)\xi\big]}{2\big[p(\xi-\eta)(\xi-\eta)+p(\eta)\eta-p(\xi)\xi\big]} \\ &\quad \times \frac{1}{\big[p(\xi-\eta)(\xi-\eta)+p(\sigma)\sigma-p(\xi-\eta+\sigma)(\xi-\eta+\sigma)\big]}. \end{aligned}$$

If we let

$$U(\xi,\eta,\sigma) = \big[p(\sigma)\sigma-p(\xi-\eta+\sigma)(\xi-\eta+\sigma)\big] - \big[p(\eta)\eta-p(\xi)\xi\big]$$

be the numerator above, it follows from the definition of m that

$$|N(\xi,\eta,\sigma)| \lesssim \frac{|m(\xi-\eta,\eta)m(\xi-\eta,\sigma)|}{\xi(\xi-\eta+\sigma)}|U(\xi,\eta,\sigma)|. \tag{5.11}$$

We first handle $U(\xi,\eta,\sigma)$. As we are in $\mathcal{A}_{2,+}^c$, we may write

$$\xi = (1+\mu)\eta \quad \text{and} \quad \sigma = (1+\nu)\eta,$$

where $|\mu|,|\nu| \le \frac{1}{10}$ are uniformly small. A Taylor expansion of the p-terms in U then reads

$$\begin{aligned} U(\xi,\eta,\sigma) &= \eta\Big[p((1+\nu)\eta)(1+\nu) - p((1+\mu+\nu)\eta)(1+\mu+\nu) \\ &\qquad - [p(\eta) - p((1+\mu)\eta)(1+\mu)]\Big] \\ &= -\eta\int_0^1\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}r}\Big[p((1+s\nu+r\mu)\eta)(1+s\nu+r\mu)\Big]\mathrm{d}r\,\mathrm{d}s \\ &= -\eta\int_0^1\int_0^1 \Big[p''((1+s\nu+r\mu)\eta)\mu\nu\eta^2(1+s\nu+r\mu) \\ &\qquad\qquad + 2p'((1+s\nu+r\mu)\eta)\mu\nu\eta\Big]\mathrm{d}r\,\mathrm{d}s, \end{aligned}$$

which according to the asymptotic assumptions on p may be estimated by

$$\begin{aligned} |U(\xi,\eta,\sigma)| &\lesssim \eta^3|\mu\nu|\eta^{\alpha-2} + \eta^2|\mu\nu|\eta^{\alpha-1} \\ &= |\xi-\eta||\sigma-\eta|\eta^{\alpha-1}, \end{aligned} \tag{5.12}$$

where we recall that $\eta \gtrsim 1$ in $\mathcal{A}^c_{2,+}$. In effect, two derivatives have been moved from $\xi \eqsim \sigma$ to $\xi - \eta$ and $\eta - \sigma$.

We next treat the quotient with m in the right-hand side of (5.11). Using the equivalence and positive size of ξ, σ, η and $\xi - \eta + \sigma$, the estimate for m in Lemma 2.2 directly yields

$$\begin{aligned}\frac{|m(\xi-\eta,\eta)m(\xi-\eta,\sigma)|}{\xi(\xi-\eta+\sigma)} &\eqsim \left|\frac{m(\xi-\eta,\eta)}{\xi}\right|^2\\ &\eqsim \left(\frac{1}{\xi^2|\xi-\eta|}\left[\frac{\xi^2}{(1+|\xi|)^\alpha+(1+|\xi-\eta|)^\alpha}\right]\right)^2\\ &\eqsim \frac{1}{|\xi-\eta|^2}\left[\frac{1}{1+|\xi|^{2\alpha}+|\xi-\eta|^{2\alpha}}\right].\end{aligned}$$

By multiplying with (5.12), and using $\xi \eqsim \eta$, we get the combined estimate

$$|N(\xi,\eta,\sigma)| \lesssim \frac{|\sigma-\eta|}{|\xi-\eta|}\min\left\{\xi^{-1\pm\alpha}\right\} \leq \frac{1}{\xi}\frac{|\sigma-\eta|}{|\xi-\eta|},$$

which is to be multiplied with $(\xi-\eta)\sigma^k\xi^{k+1}$ in (5.9). The combined upper bound $|\sigma-\eta|\sigma^k\xi^k$ on the multiplier in $[\mathcal{A}^c_{2,+}\cap\mathcal{B}](\tilde{F}_{1,0}+\tilde{G}_{1,0})$ then gives an $H^1\times H^2\times(H^k)^2$-estimate, and completes the proof of Proposition 5.7. □

Acknowledgment

The authors are grateful to the referee for the comments and suggestions that helped improve the exposition of the paper.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, *Nonlocal models for nonlinear, dispersive waves.*, Phys. D 40 (1989), no. 3, 360–392., (1989).

[2] T. Alazard and J.-M. Delort, *Sobolev estimates for two dimensional gravity water waves*, Astérisque, (2015), pp. viii+241.

[3] M. N. Arnesen, *Existence of solitary-wave solutions to nonlocal equations*, Discrete Contin. Dyn. Syst., 36 (2016), pp. 3483–3510.

[4] M. Berti and J.-M. Delort, *Almost global solutions of capillary-gravity water waves equations on the circle*, vol. 24 of Lecture Notes of the Unione Matematica Italiana, Springer, Cham; Unione Matematica Italiana, [Bologna], 2018.

[5] M. Berti, R. Feola, and L. Franzoi, *Quadratic life span of periodic gravity-capillary water waves*, Water Waves, 3 (2021), pp. 85–115.

[6] G. Bruell and R. Dhara, *Waves of maximal height for a class of nonlocal equations with homogeneous symbols*, Indiana Univ. Math. J. 70 (2021), pp. 711-742.

[7] J.-M. Delort and J. Szeftel, *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, Int. Math. Res. Not., (2004), pp. 1897–1966.

[8] V. Duchêne, D. Nilsson, and E. Wahlén, *Solitary wave solutions to a class of modified Green-Naghdi systems*, J. Math. Fluid Mech., 20 (2018), pp. 1059–1091.

[9] M. Ehrnström, M. D. Groves, and E. Wahlén, *On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type*, Nonlinearity, 25 (2012), pp. 1–34.

[10] M. Ehrnström, M. A. Johnson, O. I. H. Maehlen, and F. Remonato, *On the bifurcation diagram of the capillary-gravity Whitham equation*, Water Waves, 1 (2019), pp. 275–313.

[11] M. Ehrnström and E. Wahlén, *On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 36 (2019), pp. 1603–1637.

[12] M. Ehrnström and Y. Wang, *Enhanced existence time of solutions to the fractional Korteweg–de Vries equation*, SIAM J. Math. Anal., 51 (2019), pp. 3298–3323.

[13] R. Feola, B. Grébert, and F. Iandoli, *Long time solutions for quasi-linear hamiltonian perturbations of schrödinger and klein-gordon equations on tori*, 2020.

[14] P. Germain, N. Masmoudi, and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Ann. of Math. (2), 175 (2012), pp. 691–754.

[15] F. Hildrum, *Solitary waves in dispersive evolution equations of Whitham type with nonlinearities of mild regularity*, Nonlinearity, 33 (2020), pp. 1594–1624.

[16] J. K. Hunter and M. Ifrim, *Enhanced life span of smooth solutions of a Burgers-Hilbert equation*, SIAM J. Math. Anal., 44 (2012), pp. 2039–2052.

[17] J. K. Hunter, M. Ifrim, and D. Tataru, *Two dimensional water waves in holomorphic coordinates*, Comm. Math. Phys., 346 (2016), pp. 483–552.

[18] J. K. Hunter, M. Ifrim, D. Tataru, and T. K. Wong, *Long time solutions for a Burgers-Hilbert equation via a modified energy method*, Proc. Amer. Math. Soc., 143 (2015), pp. 3407–3412.

[19] V. M. Hur, *Wave breaking in the Whitham equation*, Adv. Math., 317 (2017), pp. 410–437.

[20] M. Ifrim and D. Tataru, *The lifespan of small data solutions in two dimensional capillary water waves*, Arch. Ration. Mech. Anal., 225 (2017), pp. 1279–1346.

[21] A. D. Ionescu and F. Pusateri, *Global solutions for the gravity water waves system in 2d*, Invent. Math., 199 (2015), pp. 653–804.

[22] ———, *Global regularity for 2D water waves with surface tension*, Mem. Amer. Math. Soc., 256 (2018), pp. v+124.

[23] A. D. Ionescu and F. Pusateri, *Long-time existence for multi-dimensional periodic water waves*, Geom. Funct. Anal., 29 (2019), pp. 811–870.

[24] C. Klein and J.-C. Saut, *A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation*, Phys. D, 295/296 (2015), pp. 46–65.

[25] D. Lannes, *The water waves problem*, Mathematical surveys and monographs, 188 (2013).

[26] F. Linares, D. Pilod, and J.-C. Saut, *Dispersive perturbations of Burgers and hyperbolic equations I: Local theory*, SIAM J. Math. Anal., 46 (2014), pp. 1505–1537.

[27] O. I. H. Maehlen, *Solitary waves for weakly dispersive equations with inhomogeneous nonlinearities*, Discrete Contin. Dyn. Syst., 40 (2020), pp. 4113–4130.

[28] L. Molinet, D. Pilod, and S. Vento, *On well-posedness for some dispersive perturbations of Burgers' equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 1719–1756.

[29] L. Molinet, J. C. Saut, and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal., 33 (2001), pp. 982–988.

[30] D. Nilsson, *Extended lifespan of the fractional BBM equation*, Asymptotic Analysis, vol. Pre-press, no. Pre-press, pp. 1–21, 2021.

[31] J.-C. Saut, *Sur quelques généralisations de l'équation de Korteweg-de Vries*, J. Math. Pures Appl. (9), 58 (1979), pp. 21–61.

[32] J.-C. Saut and Y. Wang, *The wave breaking for Whitham-type equations revisited*, arXiv:2006.03803, to appear in SIAM. J. Math. Anal.

[33] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math., 38 (1985), pp. 685–696.

[34] A. Stefanov and J. D. Wright, *Small Amplitude Traveling Waves in the Full-Dispersion Whitham Equation*, J. Dynam. Differential Equations, 32 (2020), pp. 85–99.

[35] S. Wu, *Almost global wellposedness of the 2-D full water wave problem*, Invent. Math., 177 (2009), pp. 45–135.

Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, 7491 Trondheim, Norway.
Email address: mats.ehrnstrom@ntnu.no

School of Mathematics and Statistics, Lanzhou University, 370000 Lanzhou, People's Republic of China.

Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405 Orsay, France.
Email address: yuexunwang@lzu.edu.cn