Characterisation of fraction representation transformations of Norwegian preservice teachers

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Learning to use a diverse set of representations to support teaching and understanding is an important and integrated part of Norwegian mathematics teacher education. This study uses a thematic analysis and Duval's theory of semiotic representations to characterise representation transformations of 53 preservice teachers' answers to three additive fraction problems. From the analysis, three themes characterising the transformations between representation registers emerged: connected, assimilated and inconsistent. The three themes are exemplified with student work, and their didactical implications are discussed.

Keywords: representations, fractions, transformations, preservice teachers, teacher education

Introduction and theoretical background

Unique to mathematics as a field of science is the ontological position that mathematical objects in and of themselves do not exist in the real world; the only way to access a mathematical object is through representations of that object. For example, a fraction can be represented symbolically as $\frac{a}{b}$ or as a point on a number line. The different representations of an object are not a form of decorative illustration but offer their own unique perspectives of what the object *is* and how it can be manipulated. Together, the representations form an amalgamation of the mathematical object. Hence, the role of representations in both doing and learning mathematics is significant (NCTM, 2000; Kilpatrick et al., 2001; Duval, 2006). Representations, in particular the idea of linking different representations and recognising what is involved in using a particular representation, are also emphasised as an important part of teachers' professional knowledge, for example, within mathematical knowledge for teaching (Ball et al., 2008). As teacher education is a primary arena for acquiring such specialised professional knowledge, it is important to investigate preservice teachers' (PSTs') knowledge about connections within and between mathematical representations.

Representations, their use and how they are connected are especially important for the learning of fractions, a subject that can be challenging for both students and preservice teachers (Ni & Zhou, 2005; Newton, 2008). There is empirical evidence that fraction knowledge predicts later mathematical academic success (Siegler et al., 2012), making fractions an essential object of study in mathematics education and teacher education. Therefore, it has been suggested that teachers should have an understanding of the range of representations, how they are used and how they relate to the concept of fractions so that they can better teach the subject (Siegler et al., 2010). These aspects of representations are encompassed in the theory of semiotic representations (Duval, 2006). Within this theory, mathematical activities consist of the transformations between representation registers (e.g., symbolic, diagrammatic, natural language and mathematical language). These transformations are divided into *treatments*, which are transformations within the same representation register, such as computing a fraction addition problem using symbolic notation, and *conversions*, a transformation from a source register to a (different) target register without changing the denoted object, such as drawing a number-line representation of a fraction addition problem that has been given in symbolic notation (Duval, 2006). In this theory, comprehension is the ability to coordinate or mobilise (at least) two representation registers simultaneously. Here, coordination is understood as the ability to use both transformations effortlessly and transform them into a suitable register for the task at hand.

Regarding research on preservice teachers' comprehension of fractions, it has focused predominantly on multiplication and division (Olanoff et al., 2014). This is also true for studies looking at transformations between representations (Son & Lee, 2016; Jansen & Hohensee, 2016), even though there are exceptions that focus on all arithmetical operations at the same time (Rosli et al., 2013). Generally, this body of research shows that PSTs often have difficulties with representing fractions (Olanoff et al., 2014). However, because the multiplication and division of fractions is more difficult than addition and subtraction (Newton, 2008), it is not fully known if their difficulties are related to performing a more complex operation or to the coordination of representation registers. Addressing this question, we investigate the conversions that a group of PSTs make between different representation registers for addition and subtraction of fractions. We state the following research question: What characterises the conversions between representation registers in Norwegian preservice teachers' written answers to additive fraction problems at the end of their first course in mathematics education?

Methods and analysis

A set of problems regarding the different representations of fractions was devised for 151 PSTs as part of the final mandatory assignment of their introductory course in mathematics education at the end of their first year of study. This topic had previously been covered in course lectures. The course description states that "…we will thoroughly analyse the foundational understanding of concepts in fractions," and one of the described learning outcomes of the course is that the student "..has knowledge of different representations, and the effects the use of representations can have on pupils' learning."

In the present paper, we analyse solutions to three additive fraction problems (Figure 1). The problems

Problem 1

Add or subtract the following fractions using an area model, a number line model and a set model.

a) $\frac{1}{4}+\frac{1}{2}$ b) $1\frac{1}{2}-\frac{3}{4}$ c) $\frac{2}{3}+\frac{3}{5}$

Figure 1. The three tasks (translated from Norwegian)

were presented symbolically using three fraction models (i.e., representation registers): the area model (A), the number line model (NL) and the set model (S). These registers are denominated as *diagrammatic registers* as opposed to the *discursive registers* of mathematical symbols and text in natural language.

A subset (N = 53) of the PST answers were selected purposively (the 53 first when sorting alphabetically) and analysed in the current study. In order to find new categories or themes within the PSTs' conversions, the analysis of the data was guided by a thematic analysis (Braun & Clarke, 2006). During the first phase, the PSTs' answers were imported to the software package NVivo for processing. Because Duval's theory of semiotic representations functions as an analytic framework in the current study, some preliminary ideas for codes were already noted. While familiarising ourselves further with the data corpus, more codes were generated. These codes were mostly semantic. The second phase of the coding process was inductive, deductive and nonsequential. The coding was done by one researcher. Eventually, saturation of the codes was achieved. Three themes of conversions between representations were identified. These themes were checked against the coding data, and subthemes were identified (often corresponding to some of the codes used).

Because the 53 PSTs were given three tasks, there was a total of 159 tasks to analyse. Each of the tasks asked for three representations; however, some PSTs only provided a single representation to each task, resulting in a total of 431 coded representations. These representations were distributed in the following registers: 145 area models, 144 number line models and 142 set models. The discursive registers were not counted because they often permeated throughout the answers.

There is not a one-to-one correspondence between the written product of the PSTs and the cognitive processes leading to that product. Therefore, a central premise for the analysis is that to some extent, the PSTs' work reflects their thinking or the communication of their mathematical knowledge.

Results

Three main themes characterised the PSTs' conversions between representation registers: assimilation of the different representation registers, inconsistency between the different representation registers and connected use of representation registers (see Table 1).

These themes encapsulated the answers from most of the PSTs (51 out of 53) and their representations (344 of 431), providing a functional way to analyse the relationships between representation registers in these types of fraction problems. Answers from the two PSTs not encapsulated by these themes included only a single representation for each task so that the relationship between models could not be evaluated and showed no sign of assimilation or inconsistency.

Table 1. Characteristics of the PSTs' conversions from symbolic to diagrammatic representations

Themes and *subthemes*	Representations	PSTs
Assimilation of different registers	**219**	**44 (83%)**
Symbolic treatment of diagrammatic register	190	42 (79%)
Number line model treated as area model	17	11 (21%)
Set model treated as area model	6	3 (6%)
Isomorphic representations	6	2 (4%)
Inconsistency between registers		
Discursive vs. diagrammatic	19	11 (21%)
Diagrammatic vs. diagrammatic	4	3 (6%)
Connected use of different diagrammatic registers	**22**	**12 (23%)**
All three registers	9	5 (9%)
Two registers	13	9 (17%)

Assimilation of different registers

In a large proportion of the representations, we found evidence of assimilation of two or more representation registers: either (i) discursive and diagrammatic or (ii) diagrammatic and diagrammatic. By assimilation, we mean that some elements of one representation register are brought into the other register during the conversion between them. Assimilation occurred most frequently in the conversion from the symbolic source register and then into either of the diagrammatic target registers (see Table 1).

Assimilation from symbolic registers to diagrammatic registers. Figure 2 shows two examples of how elements from the symbolic register, such as the equal or plus sign, have been brought into the diagrammatic register, where they have no denoted meaning by themselves and where there is no supporting text explaining what the symbols mean in the current context.

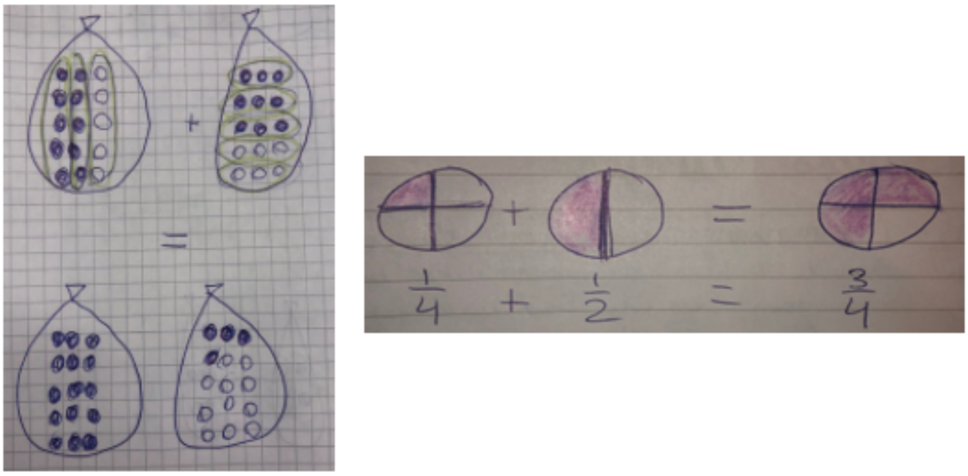

Figure 2. Typical examples of symbolic elements in the set model and area model

This type of assimilation may seem innocuous because the correct numerical answer is obtained and the PST may have taken the symbols as implicitly defined or understood. However, unless the unit and meaning of the signs are explicitly defined, such drawings can be confusing to pupils and contribute to misconceptions such as adding denominators. Indeed, assimilation also led to idiosyncratic representations in some of the PSTs' answers as exemplified in Figure 3.

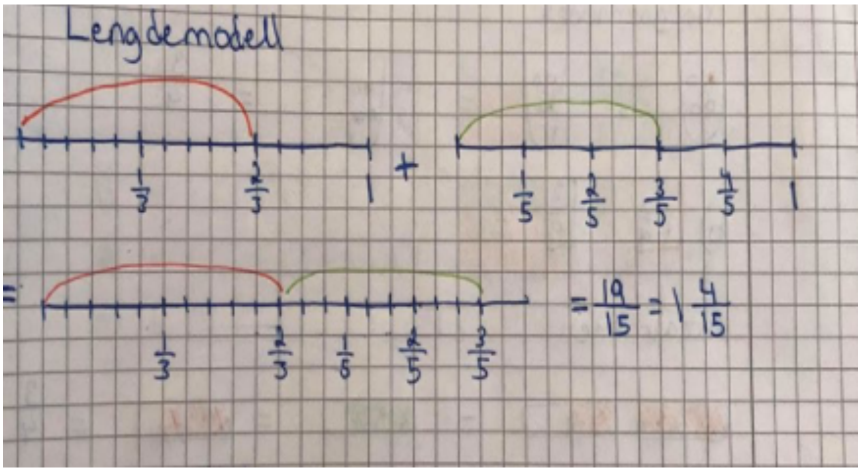

Figure 3. Addition is implicitly defined as "gluing together" two number lines, leading to a representation where one-fifth is greater than two-thirds. The numerical answer is disconnected from the diagrammatic representation

Assimilation between diagrammatic representations. In addition to the assimilation of a discursive and diagrammatic register, assimilation of two diagrammatic registers was also observed. In Figure

4 (left), we see an example where the elements from the area model were brought into the number line model. The unit has been implicitly defined as a filled area rather than, for example, a point on the number line.

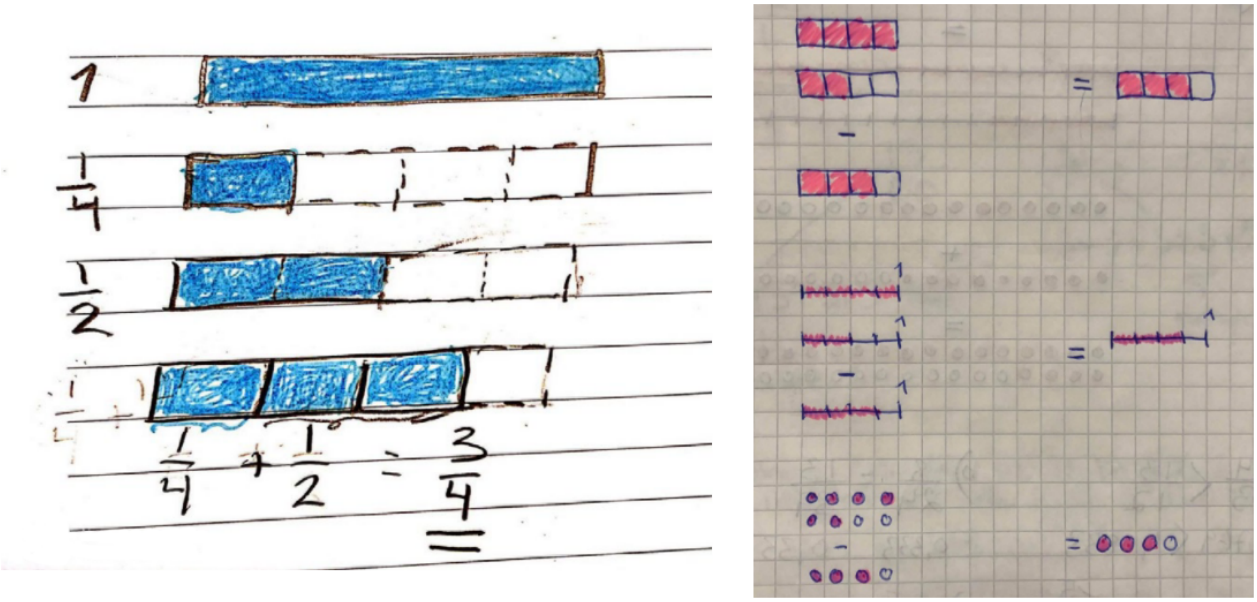

Figure 4. Left: A representation presented as a number line but treated as an area model. Right: Three representations that are isomorphic

Isomorphic representations. In the most extreme cases of assimilation, all diagrammatic registers were more or less collapsed into one register. An example is shown in Figure 4 (right), where none of the representations offers distinct or additional insights into the concept, and the conversion between representations can be described as a one-to-one mapping between different shapes. A hybrid of the set model and area model seems to dominate the expression of what a fraction is. Notably, the representations also have elements of the symbolic source register.

All of the subthemes contain naïve transformations between representation registers. By naïve, we mean that the relationship between the source and target representation is treated as congruent; the transformation between two registers is essentially reduced to an encoding process of symbols (Duval, 2006). Assimilation rarely led to an incorrect answer, potentially because the answer was found symbolically and directly translated to the diagrammatic representation. However, assimilation misses an opportunity to communicate the unique properties and uses of the different representations that can aid pupils' mathematical reasoning and problem solving in other contexts.

Inconsistency between registers

In some answers, we observed a lack of consistency between treatments in the different representations. An inconsistency between registers emerged as two subthemes: (i) an inconsistency between two diagrammatic registers, which was relatively uncommon, and (ii) an inconsistency between a diagrammatic register and a discursive register, which was relatively common.

In the inconsistency between different diagrammatic registers, the different diagrammatic representations used to represent the solution to the problem represented different answers. The most dramatic interpretation of this is that there is no connection (and therefore no conversion) between the different registers; the registers are independent of each other regarding the concept they are representing. Of course, it could also be the result of a simple mistake, but one would expect that the PSTs would notice or comment on the contradicting results.

The most common type of inconsistency of the latter subtheme was observed in the subtraction problem, in which the minuend is greater than one. The inconsistency observed in this problem has diagrammatic representations showing the correct subtrahend (Figure 5), whereas in the (self-provided) text, the subtrahend incorrectly described the problem as removing three-fourths of one and a half: ”We have a rope that is one and a half metres long. We are going to cut off three-fourths of the rope, how much rope will there be left?” In some sense, this appears to be a variant of the referent unit error (Lee, 2017) but only in an additive context. Although seemingly a subtle point, it is a fundamental one because in any practical context, the teacher will have to supplement the diagrammatic register with a natural language register. An inconsistency in what is verbally described and diagrammatically depicted is likely to confuse the learner in a classroom setting.

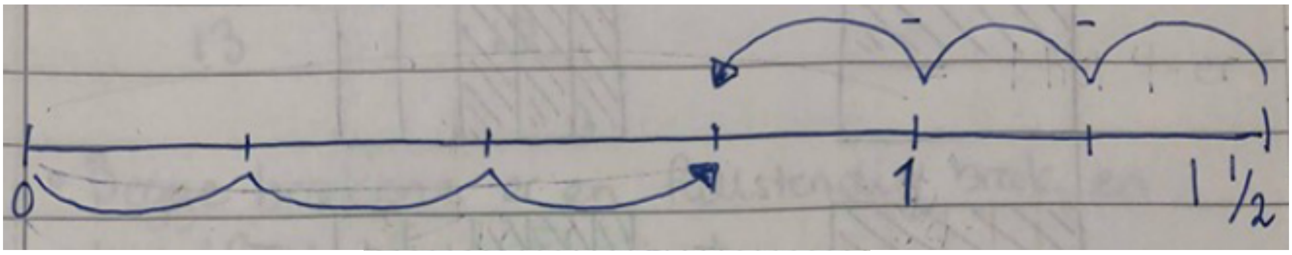

Figure 5. Correct representation of subtracting three-fourths from one and a half on the number line

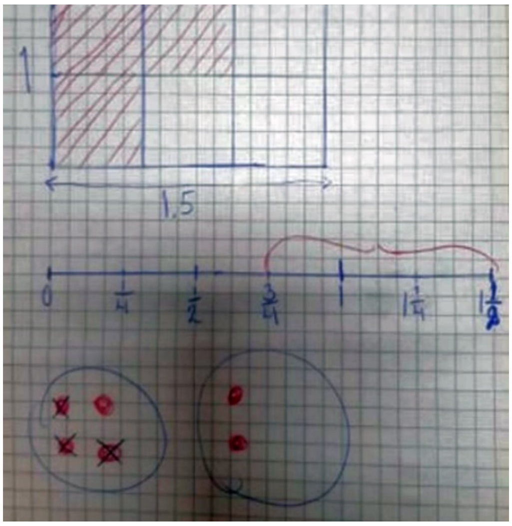

Figure 6. Connected use of the different registers. Note that these representations were also supplied with a text giving more context (e.g., viewing the area model as a lawn)

Connected use of different diagrammatic registers

The final theme can be understood as the ideal scenario. Here, the registers are treated as different entities but conceptually bound through the conversion between the representation registers. In this theme, there is evidence of transformational fluency, and each of the representations offers a distinct perspective of the fraction task. An example is shown in Figure 6, where the unit is explicitly stated or explained using supplementing text. Often, the representations are also framed in a real-life context (e.g., thermometer or cases of strawberries). Furthermore, there is a distinct difference between each of the diagrammatic representations, yielding a more diverse view of the concept of fractions. Only nine answers across five PSTs were coded as fully connected uses of different registers. In addition to these, 13 answers across nine PSTs were coded as successfully connecting two of the three diagrammatic registers. Note that some of these PSTs were also included in the other themes because for instance, there was assimilation in the area model but not in the other models.

Concluding remarks

Through a thematic analysis of 53 PSTs' answers to three additive fraction problems, three themes regarding the transformations between representation registers were identified. Going deeper into the transformations described in Duval's theory of semiotic representations, these themes were found to correspond to three different types of conversions between representation registers: naïve conversion (assimilation), inconsistent or lacking conversion between some registers (inconsistency) and fluent conversion (connected).

Each representation register has its own unique qualities. The PSTs that show a connected use of the representation registers are able to leverage the strengths and limitations of each register to communicate their thinking about a fraction problem. That teachers clearly distinguish between different representation registers themselves is essential if they are to guide their pupils towards making meaningful connections between mathematical subdomains and between mathematics and the real world. Therefore, these results are consistent with the idea that developing PSTs' appreciation of the unique strengths and limitations of different representations and representation registers should be a prioritized aspect of teacher education (e.g., Lamon, 2012). To better understand how to mitigate the assimilation of and inconsistencies between representation registers, investigating how PSTs think when they assimilate registers will be an important future endeavour.

Finally, because these are first-year PSTs at the end of their first course in mathematics education, their answers might reflect their previous education to a larger extent than their teacher training. Therefore, an analysis of later year PSTs would be of high interest to assess whether a larger proportion of PSTs acquire fluency with conversions in additive fraction problems during the course of teacher training.

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