Epistemic Potentials and Challenges with Digital Collaborative Concept Maps in Undergraduate Linear Algebra

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This paper aims to investigate epistemological potentials and challenges of digital concept mapping in collaborative activities of pre-service teachers regarding conceptualization in undergraduate linear algebra. Design experiments were undertaken within a larger design-based project with pre-service mathematics teachers for upper secondary school in Germany to look at students' connections and translations between three modes of representations and thinking of concepts such as matrices and determinants. Besides testifying that concept maps have the potential to foster students' organization of the concepts, the results also show how collaborative digital mapping can support three kinds of transitions and students' experiences: (1) *within a digital CmapTools,* (2) *across a digital and a physical medium and* (3) *beyond a single digital resource by integrating DGS and CmapTools, which gained importance since the pandemic outrage.*

Keywords: Pre-service mathematics teachers, Concept mapping, Design research, Linear algebra, Digital tools.

Introduction

While undergraduate linear algebra for engineering and science students is based on the vector space definition and other axiomatic definitions of the concepts, courses about didactics of linear algebra for pre-service secondary school mathematics teachers usually treat the concepts in a non-axiomatic manner in Germany (Donevska-Todorova, 2018a, 2018b). Such approaches may result in the lack of *cognitive flexibility* (Alves Dias & Artigue, 1995) where students perceive many of the concepts disconnected from one to another and prevent them from getting a wider picture of what constitutes a typical axiomatic-structural undergraduate linear algebra curriculum. Hence, pre-service teachers may gain disorganized, fragmented knowledge with little connections between many of the concepts or insufficient conceptual understanding without recognizing the concepts' meanings (Donevska-Todorova, 2016, 2017).

Preventing such development has become even more challenging since the global COVID-19 pandemic. Remote teaching has challenged educators to search for adequate novel tools that may quickly and effectively transform traditional mathematics classrooms and simultaneously involve learners in resource designs that maximize their activities and engagement. It seems to us that collaborative concept mapping in the virtual space has the potential to respond to this challenge. Thus, the purpose of using collaborative digital concept maps in our teaching approach is to enable future teachers to systematically connect different modes of representation and thinking in linear algebra in an organized structure, which is in line with Papert's notion about digital tools being "vehicles" for development of "Mathematical Ways of Thinking" (Papert, 1983). We focus on the following research question: what are the epistemological potentials and challenges of collaborative digital mapping through algebraic, geometric, and axiomatic transitions and experiences with linear algebra concepts by pre-service mathematics teachers?

Theoretical framework

Regarding the representation of mathematical objects in linear algebra, Hillel (2000) addressed three modes (p. 192): the *abstract* mode, the *algebraic* mode, and the *geometric* mode. The abstract mode is referring to axiomatic definitions, structures and language of vector spaces, subspaces, linear combination, and linear independence of vectors in linear algebra. An algebraic mode refers to the use of algebra of (specifically) properties of $\mathbb{R}^n$, like algebra with *n*–tuples, the interplay between matrices and associated systems of linear equations. A geometric mode refers to the language of *2D* and *3D* geometry and geometric vectors, such as parallelogram rule with vectors, lines, planes, their intersection(s) etc. The interplay between these modes is often necessary for cognitive flexibility (Alves Dias & Artigue, 1995).

However, this is not an easy and trivial task. As addressed by Sierpinska (2000), students tend to think practically rather than theoretically. Practical thinking can be described as thinking *locally* and an effort to reason that is changed by the action itself, while theoretical thinking is about generalising and reflecting on the situation through linking the action and associated mathematical objects by using different procedures. In other words, theoretical thinking concerns concepts (i.e., definitions, the use of set theory etc.), while practical thinking is limited to the results and/or experiences in the involved action. From an epistemological point of view, Sierpinska (2000) characterizes three thinking modes referring to practical and theoretical thinking: synthetic-geometric thinking mode, analytic-arithmetic thinking mode and analytic-structural thinking mode. In the synthetic-geometric thinking mode, the learner refers to geometric properties of the action (possibly) based on 'practical' observations but does not refer to thinking on how observed mathematical objects are created. The analytic-arithmetic mode is associated with referring to algebraic features of the objects, for example, thinking and reasoning with *n*–tuples, coordinates of objects in Cartesian geometry, the system of linear equations and associated matrix algebra. The analytic-structural thinking mode requires a synthesis of progressive reasoning on different situations and mathematical objects and thinking of them as (a part of) a conceptual system. We note that the three thinking modes are in parallel to Hillel's (2000) three modes of representation.

Concept mapping in mathematics education

Several resources (e.g., Brinkmann, 2003) point out that concept maps were first introduced by Novak and Gowin (1984) as research tools for structuring an individual's knowledge. The initial intention was to use the concept maps as graphical representations of one's knowledge for research purposes in science. Other authors define the concept maps as advanced organizers with a meaningful and practical structured approach (Willerman & Mac Harg, 1991) or as an aid to instruction in science and mathematics, new teaching strategies that will enhance the understanding of those concepts which are common for both disciplines (Malone & Dekkers, 1984). The concept maps are often referred to as instruments, tools, techniques, methods, graphical displays, or networks with a very wide range of aims.

The historical development of concept mapping follows the order of their implementation in research, teaching and learning. The later tendencies lead to the implementation of concept mapping in investigations on how learners learn. Qualitative analysis of students' concept mapping has been in expansion for different purposes: suggesting teaching approaches that help students integrate new knowledge and build upon their existing naive concepts, learning by illustrating patterns of conceptual development (Kinchin, Hay & Adams, 2000), assessing conceptual understanding (Williams, 1998; Varghese, 2009). Concept maps can be used to organize information on a topic, to facilitate meaningful learning, to identify students' knowledge structures, especially misconceptions or alternative conceptions, to serve as a memory aid, to revise a topic and to design instructional materials (Brinkmann, 2003). The "theory underlying concept maps" (Novak & Cañas, 2008) points out two major important foundations of concept maps:

- *Psychological,* related to learning processes like discovery learning, rote learning, meaningful learning, etc., thus they serve as a kind of template or scaffold to help organization of knowledge and to structure it, and
- *Epistemological,* so serving new knowledge creation as a constructive process involving both previous knowledge and emotions or the drive to create new meanings and new ways to represent these meanings (Novak & Cañas, 2008, p. 9).

Related to the epistemological aspect, McGowen and Davis (2019) analysed a sequence of concept maps and corresponding schematic diagrams and together with quantitative and qualitative data found out that there are students with low gain in undergraduate mathematics who seem unable to productively integrate new knowledge into an existing structure and that they reveal radically different processes of knowledge construction and organization.

Digital concept maps as tools for dynamic synchronous systematic and structural organization of concepts through collaboration

Previous research (in the previous sub-Section) shows the variety of potential that concept mapping offers in an organization, structuring and consolidation of mathematical knowledge, yet what do their digital forms have to extend or promote differently? The following Table 1 offers answers to some insights to this question.

Table 1: Comparison of characteristics of physical and digital concept maps

Concept map	Individual's map	Collective/ Collaborative map
Physical (non-digital, e.g., paper-pencil, flipchart-marker, flipchart-stickers, board-chalk, board-stickers)	Single	Static
Digital (e g., Miro, CmapTools)	Multiple	Dynamic, synchronous, integrates e-content, DGS files, shareable, extensive, adaptable

Table 1 shows a comparison of characteristics of physical and digital concept maps including examples of digital maps such as Miro and CmapTools. Further aspects of the question are explained below in the section Results and Discussion.

Concept mapping in the teaching and learning of linear algebra

An example of a concept map for systems of linear equations in two unknowns showing lower stage algebraic methods and geometric method with lines can be found in Brinkmann (2003). Lapp, Nyman and Berry (2010) examined the connections of linear algebra concepts at the undergraduate level. They have developed two techniques for qualitative analysis of student-constructed concept maps and showed that eigenvalues and eigenvectors seem to be the most disconnected concepts from the concepts as basis and dimension in the conceptual network. Another research (Stewart, 2008) aimed to discover students' difficulties in understanding some linear algebra concepts and to suggest possible ways for their prevention, through students' involvements in tests, interviews, and concept maps. In the light of the obtained results, we hypothesized that (digital) concept maps would be a key tool to connect abstract notions belonging to linear algebra context, such as linear independence, matrix algebra and determinants etc. In other words, we focused on whether digital concept maps would coordinate geometric, algebraic, and abstract representation and associated thinking modes.

Methods

During the initial design experiments in a first design-based research (DBR) cycle with pre-service teachers at a large university in Germany, physical concept mapping was applied in a course about linear algebra and analytic geometry at the beginning and the end of the second semester. The collected data, scans of flip charts were stored, qualitatively analysed and the results suggested that the time distance of the mapping activities should be reduced, and their frequency should be increased to enable students to capture the vast number of new concepts completely and structurally.

In a second DBR cycle, in addition to physical maps, CmapTools were implemented according to the previous findings. Besides regular weekly exercises, a group of 15 students were asked to create digital concept maps with an opportunity to update them every week and finally submit them in three-time slots during the semester. They were also encouraged to collaborate in small groups of up to three students during the digital mapping, edit and advance their maps by linking a variety of resources at any time. The created maps were collected via the course in the Learning Management System Moodle where students reflected and engaged themselves in further discussions in a Forum.

Results and Discussion

Instead of reporting on quantitative data, this section represents a qualitative analysis of a case related to the research question. It offers a collection of exemplary concept maps of one group consisting of three students. The maps show students' connections of different modes of representations and thinking modes. For example, Figure 1 shows students' work regarding the link between 2×2 and 3×3 square matrices and determinants.

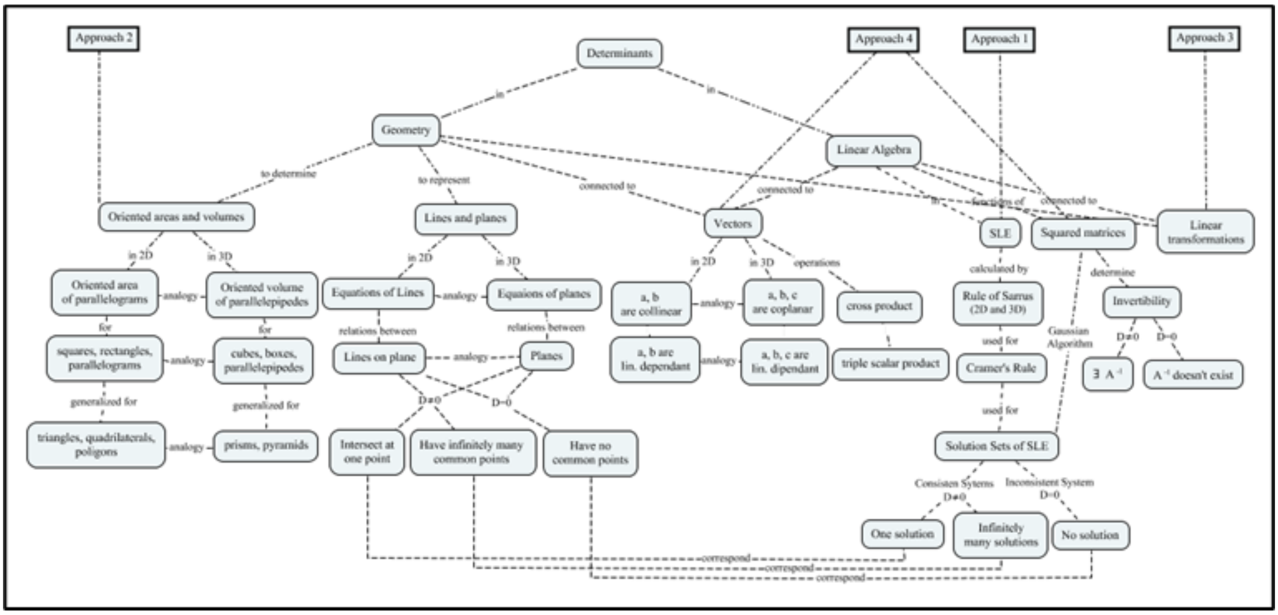

Figure 1: Students' concept map about determinants of square matrices in dimensions 2 and 3 with CmapTools

Related to the *psychological* foundation of the maps according to the "theory underlying concept maps" (Novak & Cañas, 2008) which was mentioned above, this map shows scaffolding and structuring knowledge hierarchically in seven layers of nodes-concepts and established arcs-processes of thinking. Further, related to the *epistemological* foundation, Figure 1 shows students' rich network of concepts related to determinants of matrices and their mathematical meanings. Moreover, it shows relations and transitions between Hillel's (2000) algebraic representations and modes of thinking of concepts such as systems of linear equations, linear transformations and geometric representations and modes of thinking of concepts as oriented areas of parallelograms in 2D and oriented volume of a parallelepiped in 3D geometry. Finally, this concept map represents a structured and dynamic network of epistemological connections of linear algebra concepts (1) *within a digital tool* (CmapTools).

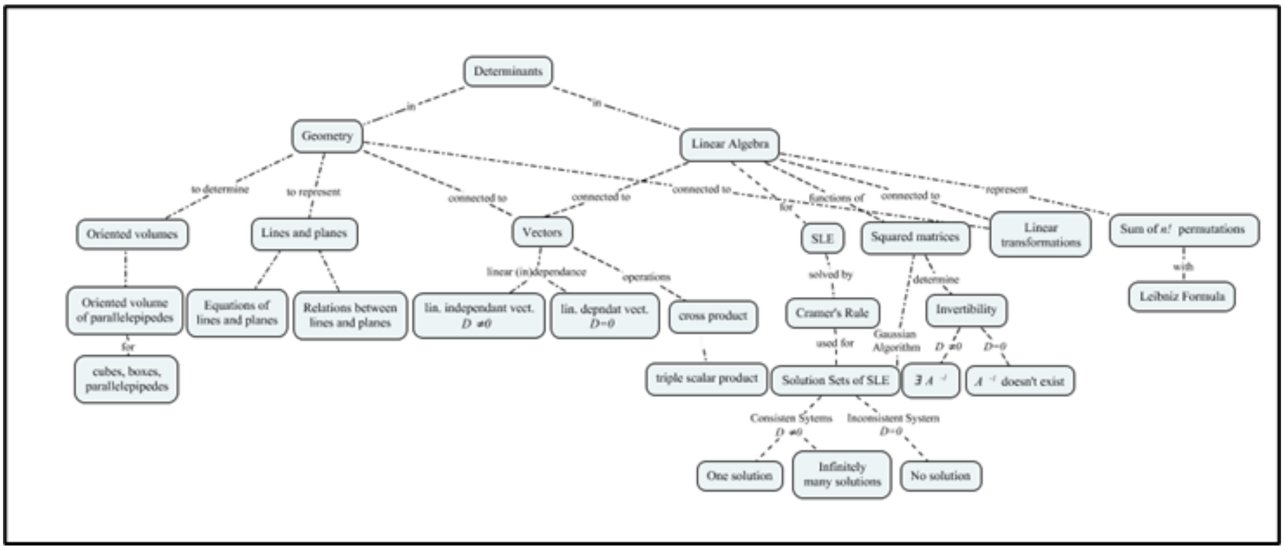

Figure 2: Students' concept map of determinants of matrices in dimension *n* with CmapTools

Likewise, the concept map in Figure 1, the one in Figure 2 created by the same group of students, shows the connections between the algebraic and the geometric representations of the concepts, yet in a generalized form for the dimension *n*. It enabled this group of students to complete their initial static flipchart-marker map, used as a sketch, with the missing concepts and relations in the digital map. The initial static map was later incorporated as a JPG file in the digital form offering possibilities to reflect on the advances and enrichment of the network. These are not only *epistemic* values of the collaborative digital concept map but also didactical and confirm the characteristics of the digital maps for easy adaptations and editing (given in Table 1).

Figure 3 describes exemplary cases regarding matrices and associated concepts, which illustrate Sierpinska's (2000) notion of students thinking of prototypes of concepts locally. This concept map is a specification and reduction of the concept in dimensions 2 and 3.

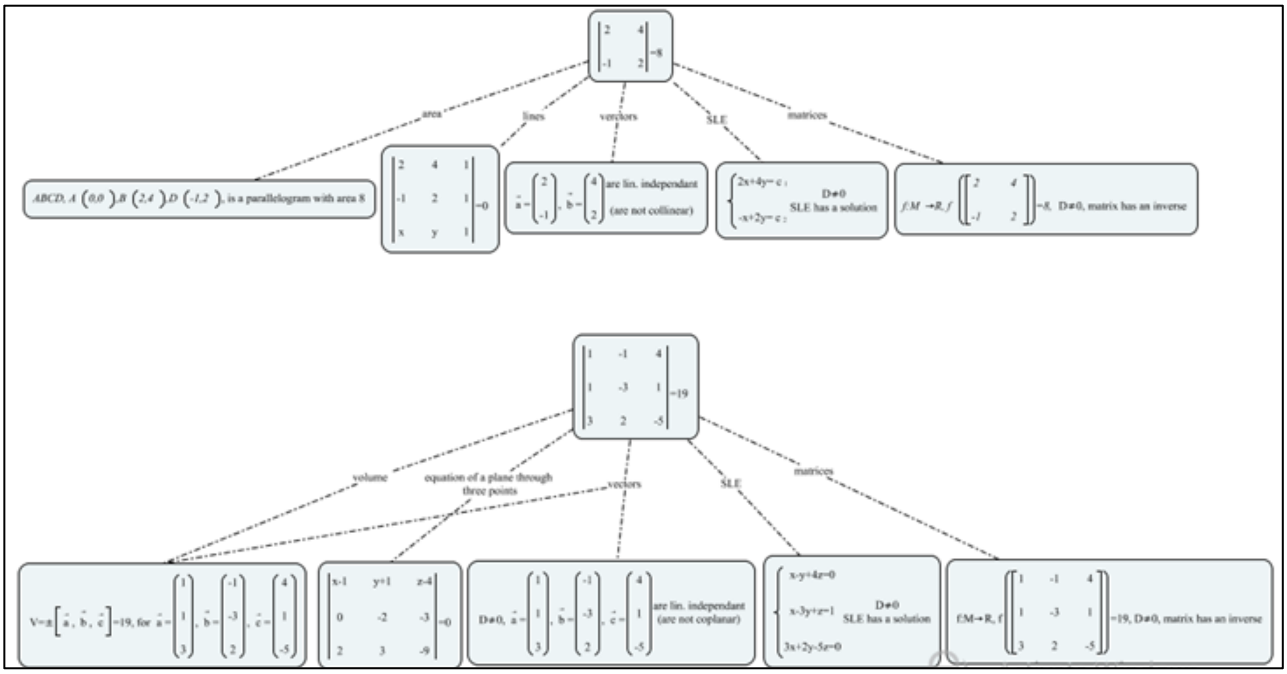

Figure 3: Students' concept maps with examples of determinants of square matrices in dimension 2 and 3 with CmapTools

These empowered students to create DGS GeoGebra files and directly link them with the map. These files enabled students' translations between geometric, algebraic, and axiomatic structural modes of representation and thinking of determinants of matrices. All three concept maps represent a collection of collaboratively created digital resources for the teaching and learning of linear algebra.

In reference to the characteristics of the physical and the digital maps presented in Table 1, some *challenges* are worth mentioning. Students' practical work showed how individual maps differ from collective ones. The synchronous engagement and the integration of the additional technology-enhanced resources were not constantly smooth to maintain. These affordances required more time and effort invested in *often adaptations*. We further consider that *'zoom in'* into the depth of each of the nodes-concepts (e.g., providing 'on click' definitions or dynamic visualizations of the concepts with other digital tools, e.g., a GeoGebra file ((3) *beyond a single digital resource of use*) in the concept maps may potentially bring one more dimension and meaningfulness of the maps.

Conclusions and further research

This paper tried to respond to the Call of the CERME12 TWG14 by tackling challenges when implementing novel approaches in the teaching and learning of undergraduate linear algebra with pre-service teachers. Based on the "theory underlying concept maps" (Novak & Cañas, 2008) we identified *psychological* and *epistemological* potentials and challenges of innovative collaborative digital concept maps in supporting conceptual understanding by connecting three modes of representations in linear algebra: algebraic, geometric, and abstract (Hillel, 2000) and the corresponding thinking modes analytic-arithmetic, synthetic-geometric and analytic-structural (Sierpinska, 2000). Concerning the research question, we have undertaken design experiments in two DBR cycles with physical concept maps and the digital tool CmapTools. Through a qualitative analysis by comparing the collected data we found a case of a group of students providing a collection of concept maps bringing into forth the *potentials* of the maps in showing connections (1) *within a digital tool* (Figure 1), (2) *across a digital and a physical medium* (Figure 2) and (3) *beyond a single digital resource of use* by integrating CmapTools and DGS (Figure 3) which enabled students to reflect, consolidate and reorganize their knowledge. To summarize, we have investigated how digital collaborative concept maps enable learners to construct, scaffold and consolidate an individual's knowledge and how they contribute to meaningful learning, effectiveness in conceptual understanding in linear algebra, negotiation of the meaning of mathematical concepts while establishing connections between them, assigning them appropriate placement in a structured and hierarchical network of concepts through its nodes and arcs. Often requiring adaptations of the maps and 'zoom in' options in the nodes leave room for further research.

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