

ON (CO)ENDS IN ∞ -CATEGORIES

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ABSTRACT. In this short note we prove that two definitions of (co)ends in ∞ -categories, via twisted arrow ∞ -categories and via ∞ -categories of simplices, are equivalent. We also show that weighted (co)limits, which can be defined as certain (co)ends, can alternatively be described as (co)limits over left and right fibrations, respectively.

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1. INTRODUCTION

Ends and coends were first introduced by Yoneda [Yon60] and play an important role in the theory of both ordinary and enriched categories. Indeed, the calculus of (co)ends can be viewed as a central organizing principle in category theory; we refer the reader to the book [Lor19a] for further discussion and many applications of (co)ends.

Ends and coends can also be defined in the context of ∞ -categories, though it is not immediately clear how to do this if we use what seems to be the most common way of defining (co)ends, in terms of so-called “extranatural transformations”. Luckily, it is not actually necessary to introduce this notion, as there are two easy ways to define the end of a functor $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ as an ordinary limit:

- (A) Let $\text{Tw}^{\ell}(\mathcal{C})$ denote the (left) twisted arrow category of \mathcal{C} . This has morphisms in \mathcal{C} as objects, with a morphism from $f: x \rightarrow y$ to $f': x' \rightarrow y'$ given by a commutative diagram

$$\begin{array}{ccc} x & \longleftarrow & x' \\ f \downarrow & & \downarrow f' \\ y & \longrightarrow & y'. \end{array}$$

Taking the source and target of morphisms gives a functor

$$p: \text{Tw}^{\ell}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C},$$

and the *end* of F is just the limit of the composite functor

$$F \circ p: \text{Tw}^{\ell}(\mathcal{C}) \rightarrow \mathcal{D}.$$

- (B) If \mathcal{D} has products, the end of F can be expressed as the (reflexive) equalizer of the two morphisms

$$(1) \quad \prod_{x \in \mathcal{C}} F(x, x) \rightrightarrows \prod_{f: x \rightarrow y} F(x, y),$$

given on the factor indexed by f by projecting to $F(y, y)$ and $F(x, x)$ and composing with f in the first and second variable, respectively. We can reinterpret this (and get rid of the assumption on \mathcal{D}) using the *category of simplices* $\Delta_{/\mathcal{C}}$ of \mathcal{C} . An object here is a functor $[n] \rightarrow \mathcal{C}$ with $[n]$ in Δ (i.e. a sequence of n composable morphisms in \mathcal{C} for $n \geq 0$), and a morphism is a commutative triangle

$$\begin{array}{ccc} [n] & \xrightarrow{\phi} & [m] \\ & \searrow F & \swarrow G \\ & \mathcal{C} & \end{array}$$

for some morphism $\phi: [n] \rightarrow [m]$ in Δ . We then have a functor

$$q: \Delta_{/\mathcal{C}} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$$

that takes $F: [n] \rightarrow \mathcal{C}$ to $(F(0), F(n))$ and a morphism as above to

$$(G(0) \rightarrow G(\phi(0)) = F(0), F(n) = G(\phi(n)) \rightarrow G(m)).$$

The end of F can then be defined as the limit of the composite

$$F \circ q: \Delta_{/\mathcal{C}} \rightarrow \mathcal{D}.$$

We can compute this in two stages by first taking the right Kan extension along the projection $\Delta_{/\mathcal{C}} \rightarrow \Delta$ and then taking the limit of the resulting cosimplicial diagram; for ordinary categories the inclusion of the subcategory of Δ containing just the two coface maps $[0] \rightrightarrows [1]$ is coinitial¹, and this recovers the equalizer (1) we started with.²

Both of these definitions have natural extensions to ∞ -categories, and our main goal in this paper is to show that these definitions of ∞ -categorical ends are in fact equivalent:

Theorem 1.1. *For a functor of ∞ -categories $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$, the limits of the composites*

$$\text{Tw}^\ell(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}, \quad \Delta_{/\mathcal{C}} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$$

are (naturally) equivalent if either exists.

We introduce the ∞ -category of simplices $\Delta_{/\mathcal{C}}$ and define the functor to $\mathcal{C}^{\text{op}} \times \mathcal{C}$ in §2. The definition of (co)ends in terms of twisted arrow ∞ -categories was previously discussed in [Gla16, §2] and [GHN17]; we review the definition in §3 and then prove the comparison.

For functors of ∞ -categories $W: \mathcal{C} \rightarrow \mathcal{S}$ and $\phi: \mathcal{C} \rightarrow \mathcal{D}$, we can define the limit $\lim_{\mathcal{C}}^W \phi$ of ϕ *weighted* by W as the end of the functor

$$\phi^W: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D},$$

where $\phi(c)^{W(c')}$ denotes the limit over $W(c')$ of the constant diagram with value $\phi(c)$, provided \mathcal{D} admits such limits. In §4 we will prove an alternative description of such weighted limits:

¹We say a functor F of ∞ -categories is *cofinal* if the map on colimits induced by F is always an equivalence, and *coinitial* if F^{op} is cofinal, i.e. the induced map on limits is always an equivalence.

²Pulling back this coinitial map along the right fibration $\Delta_{/\mathcal{C}} \rightarrow \Delta$, we see that it is enough to consider the subcategory of $\Delta_{/\mathcal{C}}$ consisting of functors $[n] \rightarrow \mathcal{C}$ with $n = 0, 1$ and morphisms that lie over the two coface maps. This recovers the description of an end as a limit discussed in [ML98, §IX.5].

Theorem 1.2. *Let $p: \mathcal{W} \rightarrow \mathcal{C}$ be the left fibration corresponding to the functor W . Then there is an equivalence*

$$\lim_{\mathcal{C}}^W \phi \simeq \lim_{\mathcal{W}} \phi \circ p,$$

provided either limit exists in \mathcal{D} .

As a consequence of this result, the definition of weighted limits in terms of ends agrees with that introduced by Rovelli [Rov19]. Moreover, this identifies the (co)end of a functor $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ as the (co)limit of F weighted by the mapping space functor $\text{Map}_{\mathcal{C}}(-, -)$. We also identify the weighted colimit functor $\text{colim}_{\mathcal{C}^{\text{op}}}^W: \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{S}$ as the unique colimit-preserving functor extending $W: \mathcal{C} \rightarrow \mathcal{S}$, while for \mathcal{D} a presentable ∞ -category the weighted colimit functor can be identified with the presentable tensor product of this with $\text{id}_{\mathcal{C}}$. This characterization also specializes to give a further description of coends.

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2. (CO)ENDS VIA THE ∞ -CATEGORY OF SIMPLICES

In this section we define the ∞ -category of simplices $\Delta_{/\mathcal{C}}$ of an ∞ -category \mathcal{C} , and prove that this has a canonical functor to $\mathcal{C}^{\text{op}} \times \mathcal{C}$, which allows us to give our first definition of (co)ends.

Definition 2.1. If \mathcal{C} is an ∞ -category, its ∞ -category of simplices $\Delta_{/\mathcal{C}}$ is defined by the pullback square

$$\begin{array}{ccc} \Delta_{/\mathcal{C}} & \longrightarrow & \text{Cat}_{\infty/\mathcal{C}} \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \text{Cat}_{\infty}, \end{array}$$

where the lower horizontal map is the usual embedding of Δ in Cat_{∞} (taking the ordered set $[n]$ to the corresponding category). Since $\text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_{\infty}$ is a right fibration, so is the projection $\Delta_{/\mathcal{C}} \rightarrow \Delta$.

Remark 2.2. The functor $\Delta^{\text{op}} \rightarrow \mathcal{S}$ corresponding to the right fibration $\Delta_{/\mathcal{C}} \rightarrow \Delta$ is given by

$$[n] \mapsto \text{Map}_{\text{Cat}_{\infty}}([n], \mathcal{C}).$$

Thus this simplicial space is the ∞ -category \mathcal{C} viewed as a (complete) Segal space.

Warning 2.3. If \mathcal{C} is an ordinary category, then $\Delta_{/\mathcal{C}}$ as we have defined it here is not quite the category of simplices we discussed in the introduction, but a variant where the morphisms are triangles that commute up to a specified natural isomorphism. In other words, for us $\Delta_{/\mathcal{C}} \rightarrow \Delta$ is the fibration corresponding to the functor that takes $[n]$ to the *groupoid* of functors $[n] \rightarrow \mathcal{C}$, rather than the *set* of such functors.

Proposition 2.4. *Suppose \mathcal{J} is a small ∞ -category and $\phi: \mathcal{J}^{\text{op}} \rightarrow \mathcal{S}$ is the presheaf corresponding to a right fibration $p: \mathcal{E} \rightarrow \mathcal{J}$. Then ϕ is the colimit of the composite functor*

$$\mathcal{E} \xrightarrow{p} \mathcal{J} \xrightarrow{y} \text{Fun}(\mathcal{J}^{\text{op}}, \mathcal{S}),$$

where y is the Yoneda embedding.

Proof. This is essentially [Lur09, Lemma 5.1.5.3], but we include a proof. Suppose ψ is another presheaf on \mathcal{J} , corresponding to a right fibration $\mathcal{F} \rightarrow \mathcal{J}$. Then unstraightening gives a natural equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{S})}(\phi, \psi) \simeq \mathrm{Map}_{/\mathcal{J}}(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Map}_{/\mathcal{E}}(\mathcal{E}, \mathcal{E} \times_{\mathcal{J}} \mathcal{F}).$$

Here $\mathcal{E} \times_{\mathcal{J}} \mathcal{F} \rightarrow \mathcal{E}$ is the right fibration for the composite functor $\mathcal{E}^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \mathcal{J}^{\mathrm{op}} \xrightarrow{\psi} \mathcal{S}$, and we can identify its ∞ -groupoid of sections with the limit of this functor by [Lur09, Corollary 3.3.3.4]. Thus we have a natural equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{S})}(\phi, \psi) \simeq \lim_{\mathcal{J}^{\mathrm{op}}} \psi \circ p^{\mathrm{op}} \simeq \lim_{\mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{S})}(\mathbf{y} \circ p, \psi),$$

where the second equivalence follows from the Yoneda lemma. This shows that ϕ has the universal property of the colimit $\mathrm{colim}_{\mathcal{J}} \mathbf{y} \circ p$, as required. \square

Corollary 2.5. *The ∞ -category \mathcal{C} is the colimit of the composite functor*

$$\Delta_{/e} \rightarrow \Delta \rightarrow \mathrm{Cat}_{\infty}.$$

Proof. By thinking of ∞ -categories as complete Segal spaces, we can view Cat_{∞} as a full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$, and the composite $\mathbf{y}: \Delta \rightarrow \mathrm{Cat}_{\infty} \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ is the Yoneda embedding. Since colimits in Cat_{∞} can be computed by taking colimits in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ and then localizing, it is enough to show that \mathcal{C} is the colimit of the composite

$$\Delta_{/e} \rightarrow \Delta \xrightarrow{\mathbf{y}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}),$$

which follows from Proposition 2.4. \square

Definition 2.6. Let Δ_* denote the category with objects pairs $([n], i)$ with $[n] \in \Delta$ and $i \in [n]$, and a morphism $([n], i) \rightarrow ([m], j)$ given by a morphism $\phi: [n] \rightarrow [m]$ in Δ such that $\phi(i) \leq j$. Let $\pi: \Delta_* \rightarrow \Delta$ be the obvious projection.

The functor π is the cocartesian fibration for the inclusion $\Delta \hookrightarrow \mathrm{Cat}_{\infty}$; the cocartesian morphisms are the morphisms over $\phi: [n] \rightarrow [m]$ of the form $([n], i) \rightarrow ([m], \phi(i))$. Thus the cocartesian fibration for the composite $\Delta_{/e} \rightarrow \Delta \rightarrow \mathrm{Cat}_{\infty}$ is

$$\pi e: \Delta_{/e,*} := \Delta_{/e} \times_{\Delta} \Delta_* \rightarrow \Delta_{/e}.$$

Corollary 2.7. *There is a natural equivalence of ∞ -categories*

$$\Delta_{/e,*}[\mathrm{cocart}^{-1}] \xrightarrow{\sim} \mathcal{C}.$$

Proof. This follows from the description of colimits in Cat_{∞} in [Lur09, §3.3.4]: the colimit of a functor $F: \mathcal{J} \rightarrow \mathrm{Cat}_{\infty}$ is given by inverting the cocartesian morphisms in the corresponding cocartesian fibration. \square

Definition 2.8. Let $l: \Delta \rightarrow \Delta_*$ be the section of π defined on objects by $l([n]) = ([n], n)$; since any map $\phi: [n] \rightarrow [m]$ satisfies $\phi(n) \leq m$ this makes sense. Note that $\pi l = \mathrm{id}$. Moreover, we have a natural isomorphism

$$\mathrm{Hom}_{\Delta_*}([n], i), l([m]) \cong \mathrm{Hom}_{\Delta}([n], [m]),$$

so that l is right adjoint to π . Next, we define $\lambda: \Delta_* \rightarrow \Delta$ by $\lambda([n], i) = \{0, \dots, i\}$; if $\phi: [n] \rightarrow [m]$ satisfies $\phi(i) \leq j$ then ϕ restricts to a map $\{0, \dots, i\} \rightarrow \{0, \dots, j\}$, which we define to be $\lambda(\phi)$. A map $([n], n) \rightarrow ([m], i)$ in Δ_* is determined by a map $[n] \rightarrow \lambda([m], i)$ in Δ , i.e. we have a natural isomorphism

$$\mathrm{Hom}_{\Delta_*}(l([n]), ([m], i)) \cong \mathrm{Hom}_{\Delta}([n], \lambda([m], i)),$$

and so λ is right adjoint to l . Note also that $\lambda l = \mathrm{id}$ and there is a natural transformation $\alpha: \lambda \rightarrow \pi$ given at the object $([n], i)$ by the inclusion $\{0, \dots, i\} \hookrightarrow [n]$. (This can also be defined as λ applied to the unit transformation $\mathrm{id} \rightarrow l\pi$.)

Lemma 2.9. *Let LV denote the set of last-vertex morphisms in Δ , i.e. the maps $\phi: [n] \rightarrow [m]$ such that $\phi(n) = m$. Then*

- (i) λ takes the π -cocartesian morphisms to morphisms in LV ,
- (ii) l takes morphisms in LV to π -cocartesian morphisms,
- (iii) the unit map $[n] \rightarrow \lambda l([n])$ is in LV (being in fact the identity of $[n]$),
- (iv) the counit map $l\lambda([n], i) = ([i], i) \rightarrow ([n], i)$ is π -cocartesian.

Moreover, the adjunction $l \dashv \lambda$ induces an adjoint equivalence

$$\Delta[\text{LV}^{-1}] \overset{\sim}{\rightleftarrows} \Delta_*[\text{cocart}^{-1}].$$

Proof. Properties (i)–(iv) are immediate from the definition of the π -cocartesian morphisms, and imply that the adjunction $l \dashv \lambda$ descends to the localized ∞ -categories where the unit and counit transformations become natural equivalences. \square

We now want to lift this equivalence to $\Delta_{/e}$; we first lift the adjoint triple:

Proposition 2.10. *The adjoint triple $\pi \dashv l \dashv \lambda$ induces for all ∞ -categories \mathcal{C} an adjoint triple of functors $\pi_e \dashv l_e \dashv \lambda_e$ between $\Delta_{/e,*}$ and $\Delta_{/e}$.*

Proof. Since $\pi l = \text{id}$, pulling back l gives a commutative diagram

$$\begin{array}{ccccc} \Delta_{/e} & \xrightarrow{l_e} & \Delta_{/e,*} & \xrightarrow{\pi_e} & \Delta_{/e} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \xrightarrow{l} & \Delta_* & \xrightarrow{\pi} & \Delta \end{array}$$

where both squares are cartesian. The unit transformation $\text{id} \rightarrow l\pi$ pulls back similarly, and the adjunction identities hold since they lie over equivalences in Δ and Δ_* and right fibrations are conservative.

We define $\lambda_e: \Delta_{/e,*} \rightarrow \Delta_{/e}$ by taking the cartesian pullback of $\alpha: \lambda \rightarrow \pi$, which gives a filler in the diagram

$$\begin{array}{ccc} \Delta_{/e,*} \times \{1\} & \xrightarrow{\pi_e} & \Delta_{/e} \\ \downarrow & \nearrow \alpha_e & \downarrow \\ \Delta_{/e,*} \times \Delta^1 & & \Delta_{/e} \\ \downarrow & & \downarrow \\ \Delta_* \times \Delta^1 & \xrightarrow{\alpha} & \Delta \end{array}$$

where the value of α_e at an object $X \in \Delta_{/e,*}$ over $([n], i)$ in Δ_* is the cartesian morphism in $\Delta_{/e}$ over $\alpha_{([n], i)}$ with target $\pi_e X$. Since α restricts to the identity transformation along l , it follows that $\lambda_e \circ l_e \simeq \text{id}$.

The natural transformation $l\alpha: l\lambda \rightarrow l\pi$ factors as the composite $l\lambda \rightarrow \text{id} \rightarrow l\pi$ of the counit transformation for $l \dashv \lambda$ and the unit transformation for $\pi \dashv l$. Hence $l_e \alpha_e$ factors as $l_e \lambda_e \rightarrow \text{id} \rightarrow l_e \pi_e$ where the second morphism is the unit for $\pi_e \dashv l_e$, since this is the unique transformation over $\text{id} \rightarrow l\pi$ with target $l_e \pi_e$, as $\Delta_{/e,*} \rightarrow \Delta_*$ is a right fibration. We claim that the transformation $l_e \lambda_e \rightarrow \text{id}$ is a

count. To see this consider the commutative diagrams

$$\begin{array}{ccccc} \mathrm{Map}_{\Delta_{/e}}(X, \lambda_e Y) & \longrightarrow & \mathrm{Map}_{\Delta_{/e,*}}(l_e X, l_e \lambda_e Y) & \longrightarrow & \mathrm{Map}_{\Delta_{/e,*}}(l_e X, Y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_{\Delta}([m], \lambda([n], i)) & \longrightarrow & \mathrm{Map}_{\Delta_*}([m], l\lambda([n], i)) & \longrightarrow & \mathrm{Map}_{\Delta_*}([m], ([n], i)). \end{array}$$

for objects X and Y lying over $[m]$ and $([n], i)$, respectively. Here the left square is cartesian since l_e is a pullback of l , and the right square is cartesian since $\Delta_{/e,*} \rightarrow \Delta_*$ is a right fibration (and hence the morphism $l_e \lambda_e Y \rightarrow Y$ is cartesian). The composite in the bottom row is an equivalence, as we know that l is left adjoint to λ , so this implies that the composite in the top row is an equivalence, as required. \square

Corollary 2.11. *Let LV_e denote the morphisms in $\Delta_{/e}$ that lie over last-vertex morphisms in Δ . Then*

- (1) λ_e takes π_e -cocartesian morphisms to morphisms in LV_e ,
- (2) l_e takes morphisms in LV_e to π_e -cocartesian morphisms,
- (3) the unit map $X \rightarrow \lambda_e l_e X$ is in LV_e (since it is an equivalence),
- (4) the counit map $l_e \lambda_e X \rightarrow X$ is π_e -cocartesian.

Moreover, the adjunction $l_e \dashv \lambda_e$ induces an adjoint equivalence

$$\Delta_{/e}[\mathrm{LV}^{-1}] \xrightarrow{\sim} \Delta_{/e,*}[\mathrm{cocart}^{-1}].$$

Proof. The π_e -cocartesian morphisms are precisely the morphisms in $\Delta_{/e,*}$ that lie over π -cocartesian morphisms in Δ_* , so this follows from Lemma 2.9 and the fact that the unit and counit for $l_e \dashv \lambda_e$ lie over the unit and counit for $l \dashv \lambda$. \square

Combining this with the equivalence of Corollary 2.7, we have proved:

Proposition 2.12. *There is a natural equivalence of ∞ -categories*

$$\Delta_{/e}[\mathrm{LV}^{-1}] \simeq \mathcal{C},$$

and hence a natural transformation $\mathfrak{L}_e: \Delta_{/e} \rightarrow \mathcal{C}$. \square

To obtain a natural map $\Delta_{/e} \rightarrow \mathcal{C}^{\mathrm{op}}$, we combine this with the order-reversing automorphism of Δ :

Definition 2.13. Let $\mathrm{rev}: \Delta \rightarrow \Delta$ be the order-reversing automorphism of Δ , i.e. $\mathrm{rev}([n]) = [n]$ but for $\phi: [n] \rightarrow [m]$ we have $\mathrm{rev}(\phi)(i) = m - \phi(n - i)$.

Lemma 2.14. *We have a natural pullback square*

$$\begin{array}{ccc} \Delta_{/e^{\mathrm{op}}} & \xrightarrow{\mathrm{rev}_e} & \Delta_{/e} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\mathrm{rev}} & \Delta. \end{array}$$

Since rev is an equivalence, so is rev_e .

Proof. The pullback of $\Delta_{/e} \rightarrow \Delta$ along rev is the right fibration for the composite

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{rev}^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{\mathfrak{L}_e} \mathcal{S}$$

and this composite is precisely the complete Segal space corresponding to $\mathcal{C}^{\mathrm{op}}$. \square

Under the equivalence rev , the last-vertex morphisms in LV correspond to the *initial-vertex morphisms* IV , i.e. the maps $\phi: [n] \rightarrow [m]$ such that $\phi(0) = 0$. We thus get:

Corollary 2.15. *There is a natural equivalence of ∞ -categories*

$$\Delta_{/c}[\mathrm{IV}_c^{-1}] \simeq \mathcal{C}^{\mathrm{op}},$$

where IV_c are the morphisms in $\Delta_{/c}$ that lie over IV . Hence there is a natural transformation $\mathcal{J}_c: \Delta_{/c} \rightarrow \mathcal{C}^{\mathrm{op}}$.

Proposition 2.16. *Suppose $L: \mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ is the localization of an ∞ -category \mathcal{C} at a collection of morphisms W . Then the functor L is coinitial and cofinal.*

Proof. Without loss of generality the morphisms in W are closed under composition and contain all equivalences; we can then let \mathcal{W} denote the subcategory of \mathcal{C} containing the morphisms in W . By definition of $\mathcal{C}[W^{-1}]$ we then have a pushout square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \|\mathcal{W}\| \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[W^{-1}], \end{array}$$

where $\|\mathcal{W}\|$ denotes the ∞ -groupoid obtained by inverting all morphisms in \mathcal{W} . By [Lur09, Corollary 4.1.2.6] the map $\mathcal{W} \rightarrow \|\mathcal{W}\|$ is cofinal, so by [Lur09, Corollary 4.1.2.7] the pushout $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ is also cofinal. To see that it is also coinitial, we apply the same argument on opposite ∞ -categories. \square

Corollary 2.17. *For any ∞ -category \mathcal{C} , the functors*

$$\mathcal{L}_c: \Delta_{/c} \rightarrow \mathcal{C}, \quad \mathcal{J}_c: \Delta_{/c} \rightarrow \mathcal{C}^{\mathrm{op}}$$

are both coinitial and cofinal. \square

That \mathcal{J}_c is coinitial was previously shown by Shah [Sha18, Proposition 12.4]. As a consequence, we obtain an ∞ -categorical version of the Bousfield–Kan formula for homotopy colimits, which has also previously appeared in work of Shah [Sha18, Corollary 12.5] and Mazel–Gee [MG19, Theorem 5.8].

Corollary 2.18 (Bousfield–Kan formula). *Let \mathcal{D} be a cocomplete ∞ -category. The colimit of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is equivalent to the colimit of a simplicial object $\Delta^{\mathrm{op}} \rightarrow \mathcal{D}$ given by*

$$[n] \mapsto \mathrm{colim}_{\alpha \in \mathrm{Map}([n], \mathcal{C})} F(\alpha(0)).$$

Proof. We can compute the colimit of F after composing with the cofinal map $\mathcal{J}_c^{\mathrm{op}}: \Delta_{/c}^{\mathrm{op}} \rightarrow \mathcal{C}$, which takes $\alpha: [n] \rightarrow \mathcal{C}$ to $\alpha(0)$. This colimit we can in turn compute in two stages, by first taking the left Kan extension along the projection $\Delta_{/c}^{\mathrm{op}} \rightarrow \Delta^{\mathrm{op}}$, which produces a simplicial object of the given form, and then taking the colimit of this simplicial object. \square

We are now in a position to define ends and coends:

Definition 2.19. Given a functor $F: \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}$, its *coend*³ $\int_c F$ is the colimit of the composite functor

$$\Delta_{/c}^{\mathrm{op}} \xrightarrow{(\mathcal{J}_c^{\mathrm{op}}, \mathcal{L}_c^{\mathrm{op}})} \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}.$$

Dually, the *end* $\int_c^* F$ of F is the limit of the composite functor

$$\Delta_{/c} \xrightarrow{(\mathcal{L}_c, \mathcal{J}_c)} \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}.$$

³We use the original notational convention of [Yon60] rather than the ‘‘Australian’’ convention, where the coend is denoted $\int^c F$ and the end is denoted $\int_c F$ — after all, it is the *coend* of F that is somewhat analogous to an integral, not the end.

Lemma 2.20. *If \mathcal{D} is a cocomplete ∞ -category, then the coend of a functor*

$$F: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$$

can be computed as the colimit of a simplicial object $\Delta^{\text{op}} \rightarrow \mathcal{D}$ given by

$$[n] \mapsto \text{colim}_{\alpha \in \text{Map}([n], \mathcal{C})} F(\alpha(0), \alpha(n)).$$

Proof. The colimit over $\Delta_{/\mathcal{C}}^{\text{op}}$ can be computed in two steps by first taking the left Kan extension along the projection $\Delta_{/\mathcal{C}}^{\text{op}} \rightarrow \Delta^{\text{op}}$, which gives the desired simplicial object, and then taking the colimit of this simplicial object. \square

A key property of ends is the ‘‘Fubini theorem’’ for iterated ends. This was proved for ∞ -categories by Loregian [Lor19b], using the definition of (co)ends via twisted arrows. We include a proof, as it is very easy to see using ∞ -categories of simplices:

Proposition 2.21 (‘‘Fubini’s Theorem’’). *Given a functor $F: (\mathcal{C} \times \mathcal{D})^{\text{op}} \times (\mathcal{C} \times \mathcal{D}) \rightarrow \mathcal{E}$, there are natural equivalences of ends*

$$\int_{\mathcal{C}}^* \int_{\mathcal{D}}^* F \simeq \int_{\mathcal{C} \times \mathcal{D}}^* F \simeq \int_{\mathcal{D}}^* \int_{\mathcal{C}}^* F.$$

Proof. Since unstraightening preserves limits, we have a natural equivalence

$$\Delta_{/\mathcal{C} \times \mathcal{D}} \simeq \Delta_{/\mathcal{C}} \times_{\Delta} \Delta_{/\mathcal{D}}.$$

This means we have a pullback square

$$\begin{array}{ccc} \Delta_{/\mathcal{C} \times \mathcal{D}} & \longrightarrow & \Delta_{/\mathcal{C}} \times \Delta_{/\mathcal{D}} \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \times \Delta, \end{array}$$

where the bottom horizontal arrow is coinitial, since Δ^{op} is sifted. Since the right vertical arrow is a right fibration, this implies that the top horizontal arrow is also coinitial. Moreover, the composite of this functor with the projection to $\Delta_{/\mathcal{C}}$ is the functor induced by the composition with the projection $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$, and similarly for $\Delta_{/\mathcal{D}}$. It follows that we also have a commutative triangle

$$\begin{array}{ccc} \Delta_{/\mathcal{C} \times \mathcal{D}} & \xrightarrow{\quad\quad\quad} & \Delta_{/\mathcal{C}} \times \Delta_{/\mathcal{D}} \\ \searrow_{(\mathcal{J}_{\mathcal{C} \times \mathcal{D}}, \mathcal{L}_{\mathcal{C} \times \mathcal{D}})} & & \swarrow_{(\mathcal{J}_{\mathcal{C}}, \mathcal{J}_{\mathcal{D}}, \mathcal{L}_{\mathcal{C}}, \mathcal{L}_{\mathcal{D}})} \\ & \mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}} \times \mathcal{C} \times \mathcal{D}. & \end{array}$$

Together with the description of limits over a product as iterated limits this implies the result. \square

3. COENDS VIA THE TWISTED ARROW ∞ -CATEGORY

In this section we recall the definition of twisted arrow ∞ -categories, and prove that we can equivalently define (co)ends as (co)limits using these ∞ -categories.

Definition 3.1. Let $\epsilon: \Delta \rightarrow \Delta$ be the endomorphism given by

$$[n] \mapsto [n]^{\text{op}} \star [n],$$

and write $\iota: \text{id} \rightarrow \epsilon$, $\rho: \text{rev} \rightarrow \epsilon$ for the natural transformations corresponding to the inclusions of the factors $[n]$ and $[n]^{\text{op}}$. For \mathcal{C} an ∞ -category, we define $\text{Tw}^{\ell}(\mathcal{C})$ as the simplicial space

$$[n] \mapsto \text{Map}([n]^{\text{op}} \star [n], \mathcal{C}),$$

i.e. $\epsilon^* \mathcal{C}$ if we view \mathcal{C} as a complete Segal space. Restricting along ι and ρ we get a projection

$$\eta_{\mathcal{C}}: \mathrm{Tw}^{\ell}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}.$$

We refer to $\mathrm{Tw}^{\ell}(\mathcal{C})$ as the (left) *twisted arrow ∞ -category* of \mathcal{C} , as is justified by the following result:

Proposition 3.2 ([HMS19, Proposition A.2.3], [Lur17, Proposition 5.2.1.3]). *If \mathcal{C} is an ∞ -category then $\mathrm{Tw}^{\ell}(\mathcal{C})$ is a complete Segal space, and the projection $\eta_{\mathcal{C}}: \mathrm{Tw}^{\ell}(\mathcal{C}) \rightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is a left fibration.* \square

Variation 3.3. If we instead consider the endofunctor of $\mathbf{\Delta}$ given by $[n] \mapsto [n] \star [n]^{\mathrm{op}}$, we get the *right twisted arrow ∞ -category* $\mathrm{Tw}^r(\mathcal{C}) := \mathrm{Tw}^{\ell}(\mathcal{C})^{\mathrm{op}}$, whose projection $\mathrm{Tw}^r(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right fibration.

Lemma 3.4. *There is a natural pullback square*

$$\begin{array}{ccc} \mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})} & \xrightarrow{\epsilon_{\mathcal{C}}} & \mathbf{\Delta}/\mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{\Delta} & \xrightarrow{\epsilon} & \mathbf{\Delta} \end{array}$$

Proof. The pullback of $\mathbf{\Delta}/\mathcal{C} \rightarrow \mathbf{\Delta}$ along ϵ is the right fibration for the composite

$$\mathbf{\Delta} \xrightarrow{\epsilon} \mathbf{\Delta} \xrightarrow{\mathcal{C}} \mathcal{S},$$

which is by definition $\mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})}$. \square

Proposition 3.5. *There is a natural commutative square*

$$(2) \quad \begin{array}{ccc} \mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})} & \xrightarrow{\epsilon_{\mathcal{C}}} & \mathbf{\Delta}/\mathcal{C} \\ \mathfrak{L}_{\mathrm{Tw}^{\ell}(\mathcal{C})} \downarrow & & \downarrow (\mathfrak{T}_{\mathcal{C}}, \mathfrak{L}_{\mathcal{C}}) \\ \mathrm{Tw}^{\ell}(\mathcal{C}) & \xrightarrow{\eta_{\mathcal{C}}} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C}. \end{array}$$

Proof. Observe that the definition of ϵ implies that we have $\epsilon(\mathrm{LV}) \subseteq \mathrm{IV} \cap \mathrm{LV}$. The composite

$$\mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})}^{\mathrm{op}} \xrightarrow{\epsilon_{\mathcal{C}}} \mathbf{\Delta}/\mathcal{C}^{\mathrm{op}} \xrightarrow{(\mathfrak{T}_{\mathcal{C}}, \mathfrak{L}_{\mathcal{C}})} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

hence takes the morphisms in $\mathrm{LV}_{\mathrm{Tw}^{\ell}(\mathcal{C})}$ to equivalences, and so this composite factors uniquely through the localization

$$\mathfrak{L}_{\mathrm{Tw}^{\ell}(\mathcal{C})}: \mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})}^{\mathrm{op}} \rightarrow \mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})}^{\mathrm{op}}[\mathrm{LV}^{-1}] \simeq \mathrm{Tw}^{\ell}(\mathcal{C}),$$

giving a natural commutative square

$$(3) \quad \begin{array}{ccc} \mathbf{\Delta}_{/\mathrm{Tw}^{\ell}(\mathcal{C})} & \xrightarrow{\epsilon_{\mathcal{C}}} & \mathbf{\Delta}/\mathcal{C} \\ \mathfrak{L}_{\mathrm{Tw}^{\ell}(\mathcal{C})} \downarrow & & \downarrow (\mathfrak{T}_{\mathcal{C}}, \mathfrak{L}_{\mathcal{C}}) \\ \mathrm{Tw}^{\ell}(\mathcal{C}) & \xrightarrow{v_{\mathcal{C}}} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C}. \end{array}$$

It remains to show that the induced functor $v_{\mathcal{C}}$ is naturally equivalent to $\eta_{\mathcal{C}}$.

Viewing the natural transformation ι as a functor $\mathbf{\Delta} \times \mathbf{\Delta}^1 \rightarrow \mathbf{\Delta}$, the projection $p: \mathrm{Tw}^{\ell}(\mathcal{C}) \rightarrow \mathcal{C}$ is described as a morphism of complete Segal spaces by $\iota^* \mathcal{C}: \mathbf{\Delta}^{\mathrm{op}} \times (\mathbf{\Delta}^1)^{\mathrm{op}} \rightarrow \mathcal{S}$, corresponding to the right fibration obtained as a pullback

$$(3) \quad \begin{array}{ccc} \iota^* \mathbf{\Delta}/\mathcal{C} & \xrightarrow{\iota_{\mathcal{C}}} & \mathbf{\Delta}/\mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{\Delta} \times \mathbf{\Delta}^1 & \xrightarrow{\iota} & \mathbf{\Delta} \end{array}$$

The composite functor $\iota^* \Delta_{/c} \rightarrow \Delta^1$ is a cartesian fibration, and corresponds to the functor $q: \Delta_{/\mathrm{Tw}^\ell(c)} \rightarrow \Delta_{/c}$ given by composition with p , which fits in a commutative square

$$\begin{array}{ccc} \Delta_{/\mathrm{Tw}^\ell(c)} & \xrightarrow{q} & \Delta_{/c} \\ \mathfrak{L}_{\mathrm{Tw}^\ell(c)} \downarrow & & \downarrow \mathfrak{L}_c \\ \mathrm{Tw}^\ell(c) & \xrightarrow{p} & c \end{array}$$

It therefore suffices to show that q is equivalent to ϵ_c ; to see this we observe that the pullback square (3) induces a commutative triangle

$$\begin{array}{ccc} \iota^* \Delta_{/c} & \xrightarrow{\quad} & \Delta_{/c} \times \Delta^1 \\ & \searrow & \swarrow \\ & \Delta^1 & \end{array}$$

where the diagonal functors are both cartesian fibrations and the horizontal functor preserves cartesian morphisms. We can straighten this to a commutative square of ∞ -categories

$$\begin{array}{ccc} \Delta_{/\mathrm{Tw}^\ell(c)} & \xrightarrow{q} & \Delta_{/c} \\ \epsilon_c \downarrow & & \parallel \\ \Delta_{/c} & \xlongequal{\quad} & \Delta_{/c} \end{array}$$

which implies that $q \simeq \epsilon_c$. A similar argument works for the projection $\mathrm{Tw}^\ell(c) \rightarrow c^{\mathrm{op}}$, which completes the proof. \square

The following is a special case of [Bar17, Proposition 2.1].

Proposition 3.6. $\epsilon: \Delta \rightarrow \Delta$ is coinitial.

Proof. By [Lur09, Theorem 4.1.3.1] it suffices to show that the pullback $\Delta \times_{\Delta} \Delta_{/[n]}$ along ϵ is weakly contractible for all objects $[n]$ in Δ . This pullback we can identify with $\Delta_{/\mathrm{Tw}^\ell[n]}$ by Lemma 3.4, and we have a cofinal functor $\mathfrak{L}_{\mathrm{Tw}^\ell[n]}: \Delta_{/\mathrm{Tw}^\ell[n]} \rightarrow \mathrm{Tw}^\ell[n]$ from Corollary 2.17. Since cofinal functors are in particular weak homotopy equivalences, it suffices to show that the category $\mathrm{Tw}^\ell[n]$ is weakly contractible. This category can be described as the partially ordered set of pairs (i, j) with $0 \leq i \leq j \leq n$, with partial ordering given by

$$(i, j) \leq (i', j') \iff i' \leq i \leq j \leq j'.$$

Here $(0, n)$ is a terminal object, and so this category is indeed weakly contractible. \square

Corollary 3.7. The functor $\epsilon_c: \Delta_{/\mathrm{Tw}^\ell(c)} \rightarrow \Delta_{/c}$ is coinitial.

Proof. From Proposition 3.6 and Lemma 3.4 we know that this functor is the pullback of the coinitial functor ϵ along the cartesian fibration $\Delta_{/c} \rightarrow \Delta$. It is therefore coinitial by (the dual of) [Lur09, Proposition 4.1.2.15]. \square

Corollary 3.8. Given a functor $F: c^{\mathrm{op}} \times c \rightarrow \mathcal{D}$, its end is given by the limit of the composite

$$\mathrm{Tw}^\ell(c) \xrightarrow{\eta_c} c^{\mathrm{op}} \times c \xrightarrow{F} \mathcal{D}.$$

Proof. In the commutative square (2) from Proposition 3.5, the functor $\epsilon_{\mathcal{C}}$ is coinital by Corollary 3.7 while the functor $\mathfrak{L}_{\mathrm{Tw}^\ell(\mathcal{C})}$ is coinital by Corollary 2.17. We therefore have natural equivalences

$$\begin{aligned} \lim_{\mathrm{Tw}^\ell(\mathcal{C})} F \circ \eta_{\mathcal{C}} &\simeq \lim_{\Delta_{/\mathrm{Tw}^\ell(\mathcal{C})}} F \circ \eta_{\mathcal{C}} \circ \mathfrak{L}_{\mathrm{Tw}^\ell(\mathcal{C})} \\ &\simeq \lim_{\Delta_{/\mathrm{Tw}^\ell(\mathcal{C})}} F \circ (\mathfrak{J}_{\mathcal{C}}, \mathfrak{L}_{\mathcal{C}}) \circ \epsilon_{\mathcal{C}} \\ &\simeq \lim_{\Delta_{/\mathcal{C}}} F \circ (\mathfrak{J}_{\mathcal{C}}, \mathfrak{L}_{\mathcal{C}}), \end{aligned}$$

where the latter is the end of F as we defined it above. \square

Remark 3.9. Dually, for a functor $F: \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}$, its coend can be computed either as the colimit of the composite

$$\Delta_{/\mathcal{C}}^{\mathrm{op}} \xrightarrow{(\mathfrak{J}_{\mathcal{C}}^{\mathrm{op}}, \mathfrak{L}_{\mathcal{C}}^{\mathrm{op}})} \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D},$$

or as the colimit of

$$\mathrm{Tw}^r(\mathcal{C}) \simeq \mathrm{Tw}^\ell(\mathcal{C})^{\mathrm{op}} \xrightarrow{\eta_{\mathcal{C}}^{\mathrm{op}}} \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}.$$

4. WEIGHTED (CO)LIMITS

Weighted (co)limits can be defined as certain (co)ends. Our goal in this section is to show that they can also be expressed as (co)limits over left and right fibrations, respectively. The latter description agrees with the definition of weighted (co)limits studied by Rovelli [Rov19] in terms of an explicit construction in quasicategories.

Given a presheaf $W: \mathcal{J}^{\mathrm{op}} \rightarrow \mathcal{S}$ and a functor $\phi: \mathcal{J} \rightarrow \mathcal{C}$, the *colimit of ϕ weighted by W* , denoted $\mathrm{colim}_{\mathcal{J}}^W \phi$, can be defined as the coend of the functor $W \times \phi: \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \rightarrow \mathcal{C}$, at least if \mathcal{C} admits colimits indexed by ∞ -groupoids. Similarly, for $\psi: \mathcal{J}^{\mathrm{op}} \rightarrow \mathcal{C}$ the *limit of ψ weighted by W* , denoted $\lim_{\mathcal{J}^{\mathrm{op}}}^W \psi$, can be defined as the end of the functor $\psi^W: \mathcal{J} \times \mathcal{J}^{\mathrm{op}} \rightarrow \mathcal{C}$, provided \mathcal{C} admits limits indexed by ∞ -groupoids. One can also characterize weighted limits and colimits in terms of universal properties, as we have

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}}^W \phi, c) &\simeq \lim_{\mathcal{J}^{\mathrm{op}}}^W \mathrm{Map}_{\mathcal{C}}(\phi, c), \\ \mathrm{Map}_{\mathcal{C}}(c, \lim_{\mathcal{J}^{\mathrm{op}}}^W \psi) &\simeq \lim_{\mathcal{J}^{\mathrm{op}}}^W \mathrm{Map}_{\mathcal{C}}(c, \psi). \end{aligned}$$

Since all weighted limits exist in \mathcal{S} , this also gives a definition of weighted (co)limits without any assumptions on \mathcal{C} .

The key property of weighted limits is that they describe mapping spaces in functor categories. We state this in the case of presheaves:

Theorem 4.1 (Glasman). *For presheaves $\phi, \psi \in \mathcal{P}(\mathcal{J})$ we have a natural equivalence*

$$\mathrm{Map}_{\mathcal{P}(\mathcal{J})}(\phi, \psi) \simeq \lim_{\mathcal{J}^{\mathrm{op}}}^{\phi} \psi.$$

Proof. This is a special case of [Gla16, Proposition 2.3] or [GHN17, Proposition 5.1]. \square

Remark 4.2. As a consequence, the Yoneda lemma implies that we can express any presheaf as a colimit weighted by itself:

$$(4) \quad \phi \simeq \mathrm{colim}_{\mathcal{J}}^{\phi} y_{\mathcal{J}}$$

for $\phi \in \mathcal{P}(\mathcal{J})$, where $y_{\mathcal{J}}: \mathcal{J} \rightarrow \mathcal{P}(\mathcal{J})$ is the Yoneda embedding. This follows from the equivalences

$$\mathrm{Map}_{\mathcal{P}(\mathcal{J})}(\mathrm{colim}_{\mathcal{J}}^{\phi} y_{\mathcal{J}}, \psi) \simeq \lim_{\mathcal{J}^{\mathrm{op}}}^{\phi} \mathrm{Map}_{\mathcal{P}(\mathcal{J})}(y_{\mathcal{J}}, \psi) \simeq \lim_{\mathcal{J}^{\mathrm{op}}}^{\phi} \psi \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{J})}(\phi, \psi).$$

Proposition 4.3. *Suppose $q: \mathcal{V} \rightarrow \mathcal{J}$ is the left fibration corresponding to a functor $V: \mathcal{J} \rightarrow \mathcal{S}$. Then for a functor $\psi: \mathcal{J} \rightarrow \mathcal{C}$ there is an equivalence*

$$(5) \quad \lim_{\mathcal{J}}^V \psi \simeq \lim_{\mathcal{V}} \psi \circ q,$$

provided either side exists.

Proof. By the universal mapping properties of the two sides it suffices to show there is an equivalence

$$\lim_{\mathcal{J}}^V \text{Map}_{\mathcal{C}}(c, \psi) \simeq \lim_{\mathcal{V}} \text{Map}_{\mathcal{C}}(c, \psi \circ q),$$

natural in $c \in \mathcal{C}$. In other words, it suffices to show there is a natural equivalence (5) for functors $\Psi: \mathcal{J} \rightarrow \mathcal{S}$.

Using Theorem 4.1 and the straightening equivalence, we can rewrite the left-hand side as

$$\lim_{\mathcal{J}}^V \Psi \simeq \text{Map}_{\text{Fun}(\mathcal{J}, \mathcal{S})}(V, \Psi) \simeq \text{Map}_{/\mathcal{J}}(\mathcal{V}, \mathcal{E}),$$

where $\mathcal{E} \rightarrow \mathcal{J}$ is the left fibration for Ψ . We now have an obvious equivalence

$$\text{Map}_{/\mathcal{J}}(\mathcal{V}, \mathcal{E}) \simeq \text{Map}_{/\mathcal{V}}(\mathcal{V}, q^* \mathcal{E}),$$

so our weighted limit is naturally equivalent to the space of sections of the left fibration $q^* \mathcal{E}$. Since pullback of left fibrations corresponds to composition of functors to \mathcal{S} , this is the left fibration for $\Psi \circ q$. Moreover, the space of sections of a left fibration is equivalent to the limit of the corresponding functor to \mathcal{S} by [Lur09, Corollary 3.3.3.4], so that we have

$$\text{Map}_{/\mathcal{V}}(\mathcal{V}, q^* \mathcal{E}) \simeq \lim_{\mathcal{V}} \Psi \circ q,$$

as required. \square

Corollary 4.4. *Suppose $p: \mathcal{W} \rightarrow \mathcal{J}$ is the right fibration corresponding to a presheaf $W: \mathcal{J}^{\text{op}} \rightarrow \mathcal{S}$. Then for a functor $\phi: \mathcal{J} \rightarrow \mathcal{C}$ there is an equivalence*

$$(6) \quad \text{colim}_{\mathcal{J}}^W \phi \simeq \text{colim}_{\mathcal{W}} \phi \circ p,$$

provided either side exists.

Proof. By Proposition 4.3 we have an equivalence

$$\lim_{\mathcal{J}^{\text{op}}}^W \text{Map}_{\mathcal{C}}(\phi, c) \simeq \lim_{\mathcal{W}^{\text{op}}} \text{Map}_{\mathcal{C}}(\phi p, c),$$

natural in $c \in \mathcal{C}$. This implies (6) by the universal mapping properties of the two objects. \square

Corollary 4.5. *The definition of weighted limit from [Rov19] agrees with the definition as an end.*

Proof. Combine Proposition 4.3 with [Rov19, Theorem D]. \square

As a special case of these descriptions of weighted (co)limits we get another characterization of (co)ends:

Corollary 4.6. *For a functor $F: \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ there are natural equivalences*

$$\int_{\mathcal{J}} F \simeq \text{colim}_{\mathcal{J} \times \mathcal{J}^{\text{op}}}^{\text{Map}_{\mathcal{J}}(-, -)} F, \quad \int_{\mathcal{J}}^* F \simeq \lim_{\mathcal{J}^{\text{op}} \times \mathcal{J}}^{\text{Map}_{\mathcal{J}}(-, -)} F,$$

provided either side exists in \mathcal{C} .

Proof. By [Lur17, Proposition 5.2.1.11] the mapping space functor $\text{Map}_{\mathcal{J}}(-, -)$ corresponds to the right fibration $\text{Tw}^r(\mathcal{J}) \xrightarrow{\eta_{\mathcal{J}}^{\text{op}}} \mathcal{J} \times \mathcal{J}^{\text{op}}$ and hence the left fibration $\text{Tw}^{\ell}(\mathcal{J}) \xrightarrow{\eta_{\mathcal{J}}} \mathcal{J}^{\text{op}} \times \mathcal{J}$. From Corollary 4.4 and Proposition 4.3 we therefore have natural equivalences

$$\text{colim}_{\mathcal{J} \times \mathcal{J}^{\text{op}}}^{\text{Map}_{\mathcal{J}}(-, -)} F \simeq \text{colim}_{\text{Tw}^r(\mathcal{J})} F \circ \eta_{\mathcal{J}}^{\text{op}}, \quad \lim_{\mathcal{J}^{\text{op}} \times \mathcal{J}}^{\text{Map}_{\mathcal{J}}(-, -)} F \simeq \lim_{\text{Tw}^{\ell}(\mathcal{J})} F \circ \eta_{\mathcal{J}}.$$

The claims then follow by combining these equivalences with those of Corollary 3.8 and Remark 3.9. \square

Remark 4.7. Cordier and Porter [CP97] introduced homotopy-coherent ends and coends in the context of simplicial categories using a version of the characterization in Corollary 4.6: they consider simplicially enriched (co)limits weighted by a cofibrant replacement of the mapping space functor in a simplicial category.

We can also give a characterization of weighted colimits in \mathcal{S} via the universal property of presheaves as a free cocompletion:

Proposition 4.8. *For $W: \mathcal{J} \rightarrow \mathcal{S}$, the unique colimit-preserving functor*

$$y_{\mathcal{J},!}W: \mathcal{P}(\mathcal{J}) \rightarrow \mathcal{S}$$

extending W is equivalent to the weighted colimit functor

$$\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W: \operatorname{Fun}(\mathcal{J}^{\text{op}}, \mathcal{S}) \rightarrow \mathcal{S}.$$

Proof. It suffices to show that $\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W$ preserves colimits, and that $\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W \circ y_{\mathcal{J}}$ is equivalent to W . The former follows from the equivalence

$$\operatorname{Map}_{\mathcal{S}}(\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W \phi, X) \simeq \operatorname{Map}_{\operatorname{Fun}(\mathcal{J}, \mathcal{S})}(W, \operatorname{Map}_{\mathcal{S}}(\phi, X))$$

provided by Theorem 4.1 (using that colimits in functor ∞ -categories are computed objectwise), while the latter follows from Remark 4.2. \square

Proposition 4.9. *Suppose \mathcal{C} is a presentable ∞ -category. Then for $W: \mathcal{J} \rightarrow \mathcal{S}$, the weighted colimit functor*

$$\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W: \operatorname{Fun}(\mathcal{J}^{\text{op}}, \mathcal{C}) \rightarrow \mathcal{C}$$

is equivalent to the tensor product

$$y_{\mathcal{J},!}W \otimes \operatorname{id}_{\mathcal{C}}: \mathcal{P}(\mathcal{J}) \otimes \mathcal{C} \rightarrow \mathcal{S} \otimes \mathcal{C}$$

in the ∞ -category $\operatorname{Pr}^{\text{L}}$ of presentable ∞ -categories.

Proof. The equivalence $\mathcal{P}(\mathcal{J}) \otimes \mathcal{C} \simeq \operatorname{Fun}(\mathcal{J}^{\text{op}}, \mathcal{C})$ from [Lur17, Proposition 4.8.1.17] is determined by a functor $\mathcal{P}(\mathcal{J}) \times \mathcal{C} \rightarrow \operatorname{Fun}(\mathcal{J}^{\text{op}}, \mathcal{C})$ that preserves colimits in each variable, by the universal property of the cocomplete tensor product. This corresponds to the functor $\mathcal{P}(\mathcal{J}) \times \mathcal{C} \times \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}$ given by

$$(\phi, c, i) \mapsto \phi(i) \otimes c,$$

using the tensoring of \mathcal{C} over \mathcal{S} (given by taking colimits of constant functors).

The functor $\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W$ preserves colimits by the same argument as in the proof of Proposition 4.8 and is therefore determined by the composite

$$\mathcal{P}(\mathcal{J}) \times \mathcal{C} \rightarrow \operatorname{Fun}(\mathcal{J}^{\text{op}}, \mathcal{C}) \rightarrow \mathcal{C}, \quad (\phi, c) \mapsto \operatorname{colim}_{\mathcal{J}^{\text{op}}}^W(\phi \otimes c).$$

Now using Theorem 4.1 we have natural equivalences

$$\begin{aligned} \operatorname{Map}_{\mathcal{C}}(\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W(\phi \otimes c), x) &\simeq \operatorname{Map}_{\operatorname{Fun}(\mathcal{J}, \mathcal{S})}(W, \operatorname{Map}_{\mathcal{C}}(\phi \otimes c, x)) \\ &\simeq \operatorname{Map}_{\operatorname{Fun}(\mathcal{J}, \mathcal{S})}(W, \operatorname{Map}_{\mathcal{S}}(\phi, \operatorname{Map}_{\mathcal{C}}(c, x))) \\ &\simeq \operatorname{Map}_{\mathcal{S}}(\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W \phi, \operatorname{Map}_{\mathcal{C}}(c, x)) \\ &\simeq \operatorname{Map}_{\mathcal{C}}((\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W \phi) \otimes c, x), \end{aligned}$$

which gives a natural equivalence

$$\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W(\phi \otimes c) \simeq (\operatorname{colim}_{\mathcal{J}^{\text{op}}}^W \phi) \otimes c$$

using the Yoneda Lemma. The right-hand side here is the functor

$$\mathcal{P}(\mathcal{J}) \times \mathcal{C} \rightarrow \mathcal{C}, \quad (\phi, c) \mapsto (y_{\mathcal{J},!}W)(\phi) \otimes c$$

by Proposition 4.8, or in other words the composite

$$\mathcal{P}(\mathcal{J}) \times \mathcal{C} \xrightarrow{y_{\mathcal{J},!}W \times \text{id}_{\mathcal{C}}} \mathcal{S} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

which corresponds to $(y_{\mathcal{J},!}W) \otimes \text{id}_{\mathcal{C}}$ composed with the equivalence $\mathcal{S} \otimes \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, as required. \square

Combining Corollary 4.6 with the last two results we get the description of the coend functor that is taken as the definition in [AL18]:

Corollary 4.10. *Let \mathcal{J} be a small ∞ -category.*

(i) *The coend functor*

$$\int_{\mathcal{J}}: \mathcal{P}(\mathcal{J}^{\text{op}} \times \mathcal{J}) \rightarrow \mathcal{S}$$

is equivalent to the unique colimit-preserving functor $y_{\mathcal{J}^{\text{op}} \times \mathcal{J},!} \text{Map}_{\mathcal{J}}(-, -)$ extending $\text{Map}_{\mathcal{J}}(-, -): \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{S}$.

(ii) *If \mathcal{C} is a presentable ∞ -category, then the coend functor*

$$\int_{\mathcal{J}}: \text{Fun}(\mathcal{J} \times \mathcal{J}^{\text{op}}, \mathcal{C}) \rightarrow \mathcal{C}$$

is equivalent to the tensor product

$$y_{\mathcal{J}^{\text{op}} \times \mathcal{J},!} \text{Map}_{\mathcal{J}}(-, -) \otimes \text{id}_{\mathcal{C}}: \mathcal{P}(\mathcal{J}^{\text{op}} \times \mathcal{J}) \otimes \mathcal{C} \rightarrow \mathcal{S} \otimes \mathcal{C}$$

in Pr^{L} . \square

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