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The Rayleigh integral is always positive in steadily operated combustors

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Abstract

The Rayleigh index has been used for decades by a large number of researchers as an indicator to determine if a flame is driving or damping thermoacoustic interaction mechanisms. The use of the Rayleigh criterion has found applications in rocket combustors, gas turbine combustion technology and basic combustion research. The global Rayleigh index or integral is obtained by integrating the product of heat release rate and pressure fluctuations over space and time. Depending on the phase between pressure oscillations and heat release rate response, the oscillations can be enhanced or damped. It is commonly assumed in literature that the sign of the Rayleigh index from steady state data can be used to determine if the thermoacoustic feedback loop is stabilizing or destabilizing. However, we show in this paper that under fairly general conditions, a correctly measured Rayleigh index is always positive if evaluated from statistically stationary data. This proves to be true even if the heat release rate response to pressure fluctuations is in phase opposition to those pressure fluctuations. This is shown in a straightforward manner by substituting the wave equation with a heat release rate source term into the Rayleigh index. This was verified experimentally on a fully premixed combustion system by measuring the flame chemiluminescence using a photo multiplier and pressure fluctuations using a microphone placed sufficiently close to the flame to ensure acoustic compactness for the frequency range of interest. A large range of operating conditions have been tested, spanning linearly stable and unstable stationary thermoacoustic states, respectively corresponding to resonance or a limit cycle driven by the inherent stochastic forcing from the turbulent combustion noise. The experimental results corroborated the analytic finding: the Rayleigh index is found to be positive for all frequencies and all operating conditions.

Keywords: Thermoacoustic; Combustion dynamics; Rayleigh index; Rayleigh integral; Acoustic energy balance

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1. Introduction

Lord Rayleigh famously laid the foundation of thermoacoustic analysis by defining that heat release and pressure oscillations have to be in phase to encourage their vibration [1]. Based on this insight the Rayleigh Index (RI) was introduced in the1950′s as an integral of the product of pressure and heat release fluctuations [2,3]. The classical Rayleigh criterion states that a positive RI drives a thermoacoustic instability [4], the extended Rayleigh criterion is based on an energy balance and states that the RI should exceed the acoustic losses in order to cause an instability [5]. Chu's disturbance energy included the effect non-isentropic temperature fluctutations which is discussed in detail in [6], however effect of the mean flow on the energy balance are ignored in these works. A rather general disturbance energy formulation that includes mean flow effects was derived by Myers [7] and was later extended to multi-component flow in [8]. The additional terms to include effect of mean flow and non isentropic conditions lead to the question how an appropriate stability criterion should be defined [9,10].

Despite these refinements, the classical Rayleigh index has extensively been used to determine whether combustion oscillations are damped or driven by the thermoacoustic interaction. The criterion is either obtained from measured data [2,11–27] from linearized models [3,18,26,28–35], or based on the results of non-steady reactive CFD simulations [36–42].

It will be made clear in this work that it is crucial to distinguish between a RI obtained from statistically stationary time traces and transient data. Nearly all reported measured Raleigh indices were obtained from steadily operated combustors [2,11–17,19–25,37–39,41,42], so the systems are either linearly unstable and in a limit cycle that is stochastically forced by the inherent turbulent component of the heat release rate fluctuations, or linearly stable in which case the acoustic fluctuations can be considered a linear response to this stochastic combustion noise.

Although the RI has extensively been used in the past seven decades, this article will demonstrate that under fairly general assumptions the RI is always positive and that it hence can never be used to determine if the combustion process contributes positively or negatively to the thermoacoustic feedback cycle.

2. Measurements of the Rayleigh index

The Rayleigh index can be obtained experimentally by measuring pressure fluctuations p' and heat release fluctuations q'. Pressure fluctuations are typically measured using microphones or piezoelectric transducers. In case of perfectly premixed combustion the chemiluminescence of the combustion process can be considered proportional to the heat release rate by the flame. The chemiluminescence can be measured using a photo multiplier or high speed camera with a suitable optical filter.

2.1. Definition of Rayleigh indices

In this work only fully premixed combustion systems are considered, i.e. it can be assumed that no equivalence ratio fluctuations are present. Note that if equivalence ratio fluctuations can not be ignored, the flame chemiluminescence is not proportional to the heat release rate, which greatly complicates determining the Rayleigh index experimentally. In absence of equivalence ratio fluctuations, entropy fluctuations can be assumed to be small which justifies the use of the classical Rayleigh index such as defined in e.g. [43]. We will thereby distinguish the global Rayleigh index RI from the instantaneous RI_t, the local RI_x and the frequency dependent RI_ω Rayleigh indices:

$$RI = \frac{1}{T}\int_T \int_V \frac{\gamma - 1}{\gamma \bar{p}} p' q' \, dv \, dt = \int_T RI_t \, dt \tag{1}$$

$$= \frac{T}{\pi}\int_0^{+\infty} \int_V \frac{\gamma - 1}{\gamma \bar{p}} \Re(\hat{p}^* \hat{q}) \, dv \, d\omega \tag{2}$$

$$= \int_0^{+\infty} \int_V \frac{\gamma - 1}{\gamma \bar{p}} \Re(S_{pq}) \, dv \, d\omega \tag{3}$$

$$= \frac{T}{\pi}\int_0^{+\infty} RI_\omega \, d\omega = \int_V RI_x \, dv \tag{4}$$

in which V is a volume covering the entire domain of heat release, T the time interval of integration, γ the ratio of specific heats, ω the angular frequency and $S_{pq}(\omega)$ the cross power spectral density between p' and q'. It is important to note that T is a finite duration, sufficiently long to consider the process statistically stationary. Primes, overbars and hats indicate fluctuations, mean value, and Fourier transform, respectively. Note that the relation between the time domain and frequency domain representations is given by Parseval's theorem [44]. When determining the global Rayleigh index from experimental data, the spatial distribution of the pressure fluctuations is usually not available. The spatial distribution of the heat release rate fluctuations may or may not be available, depending on whether a chemiluminescence imaging system or a photomultiplier is used. In any case, if the frequency is sufficiently low, such that the flame zone can be considered compact with respect to the acoustic wave length, Eq. (1) can be approximated as: $RI \approx (1/T)\int_T [(\gamma - 1)/(\gamma \bar{p})]\, p' Q' \, dt$ where Q'

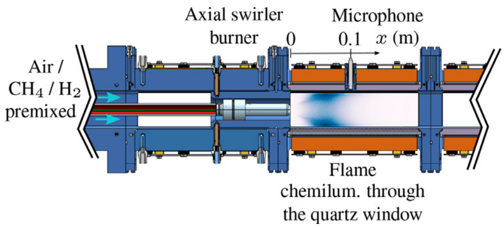

Fig. 1. Simultaneous acquisition of the acoustic pressure p' in the combustion chamber and the chemiluminescence I of the swirled turbulent flame using a photomultiplier and a high-speed intensified camera. The generic combustor is operated in fully premixed condition.

is the volume-integrated heat release rate fluctuations $\int_V q'\,dv$. In practice the value of $p'(t)$ is generally obtained from a microphone signal placed sufficiently close to the flame. Furthermore, for fully premixed flames proportionality between chemiluminescence and the heat the release rate, Q' can be assumed: $Q'(t) = (\bar{Q}/\bar{I})I'(t)$, where I is the measured global chemiluminescence intensity signal.

2.2. Experimental set-up

The experimental setup used in this paper is shown in Fig. 1. It is based on a water cooled combustor with a swirl-stabilized turbulent flame operated at atmospheric pressure. The combustion chamber and the upstream plenum have identical rectangular cross section. The inlet and outlet of the combustor have a variable geometry that that can be adjusted from almost fully reflecting to almost fully absorbing. The combustor was supplied with a perfectly premixed lean mixture of air and CH_4. It was equipped with four water cooled microphones placed flush-mounted on the combustion chamber walls, with the one located at 10 cm from the burner outlet shown in Fig. 1. A high speed intensified camera (LaVision HSSX and HS-IRO) equipped with a bandpass filter (Chroma, T > 70 % at 310 nm, FWHM 10 nm) and a photo-multiplier were used to collect the chemiluminescence of the OH^* radicals of the entire flame through the quartz windows. The time-averaged chemiluminescence of the flame is also shown in Fig. 1. An example of simultaneous record of the acoustic pressure p' at the flame and instantaneous chemiluminescence is shown in Fig. 2, for an operating condition corresponding to a linearly stable thermoacoustic state. The signal obtained by spatially integrating the camera images and the photomultiplier signal are overlapping, which justifies the use of the latter signal as a measure of Q' for the perfectly premixed configurations in this work. In what follows, p' and Q' were recorded at 20 kHz during 19 s, which is verified to be long enough to provide converged statistics of RI from RI_t.

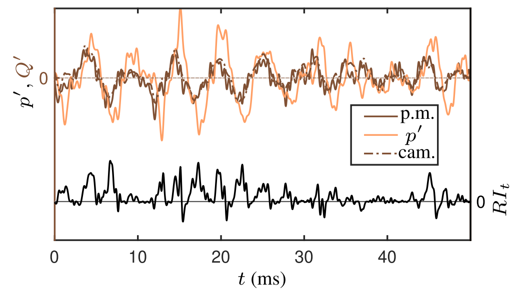

Fig. 2. Acoustic pressure p' recorded with the microphone shown in Fig. 1, spatially integrated chemiluminescence fluctuations of the flame $Q' = \int q'\,dv$ from the photo-multiplier (p.m.) and the high-speed camera (cam) (both are overlapping) and Rayleigh integral RI for a typical linearly stable condition of the thermoacoustic system.

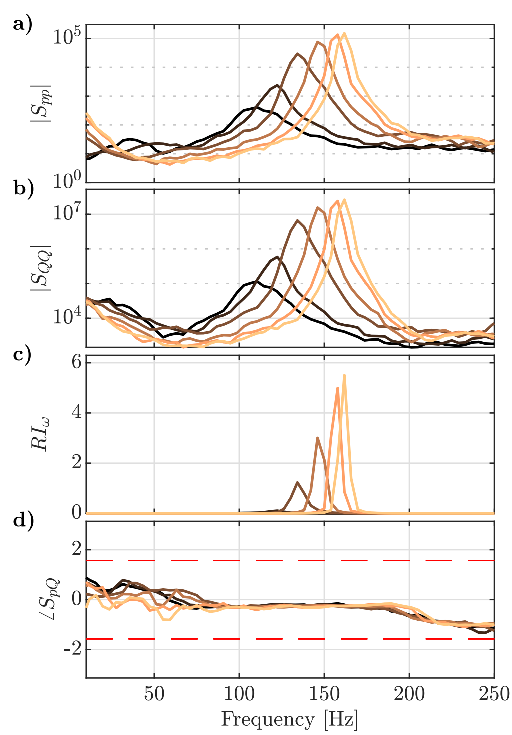

Fig. 3. a) Power spectral density of the acoustic pressure in Pa^2/Hz, and b) heat release rate in W^2/Hz, c) frequency dependent Rayleigh index in W. Note that RI_ω is positive for all cases and all frequencies. and d) phase of the cross spectral density between heat release and pressure in rad/s. The colors represent equivalence ratio ranging from 0.73 to 0.83 (from dark to light).

2.3. Experimental results

The results are shown in Figs. 3 and 4 for a variation of the equivalence ratio at constant massflow of air. Fig. 3a and b show a strong increase

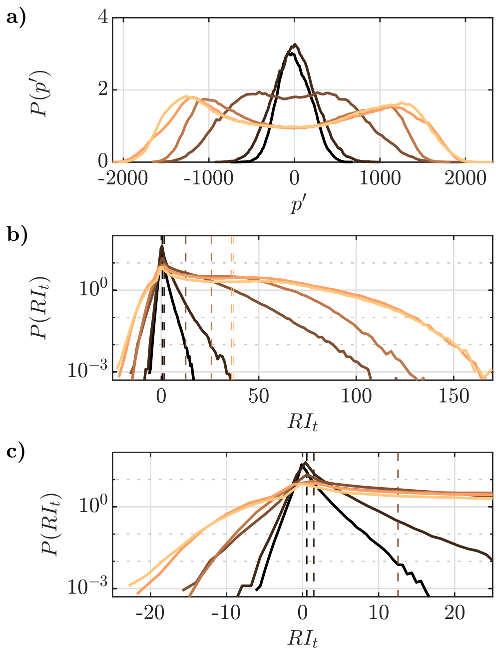

Fig. 4. a) PDF of the pressure fluctuations p'. b) PDF of the instantaneous Rayleigh index RI_t. c) zoom of x-axis of subfigure b). The colors represent equivalence ratio ranging from 0.73 to 0.83 (from dark to light). The dashed lines represent the mean value of the *RI* which appears to be always positive.

of acoustic and heat release amplitudes at a frequency corresponding to one of the first longitudinal modes of the combustor when increasing the equivalence ratio. Fig. 4a shows a clear transition of the probability density function (PDF) of the acoustic pressure from a uni-modal distribution to a bi-modal distribution. This change of the PDF clearly indicates that the cases for lower equivalence ratio were linearly stable, while thermoacoustic limit cycles are established at the highest equivalence ratios [45].

However, the Rayleigh index (Fig. 3c) shows positive values for all cases for all frequencies. Indeed, the average phase between pressure and heat release (Fig. 3d) is always contained between $-\pi/2$ and $+\pi/2$ in the considered frequency range. This is confirmed by Fig. 4b: it shows that although the instantaneous Rayleigh index can become negative, the time averaged value (dashed line) is always positive.

A possible explanation could be that the necessary conditions for linear instability are fulfilled for all cases (i.e. positive Rayleigh index), but that for the stable cases the damping is larger than the driving by the flame. Nevertheless, it might be somewhat surprising that the *RI* is positive for all frequencies and all operating conditions tested. It will be shown indeed in the subsequent section that the Rayleigh index will *always* be positive irrespective of the dynamics of the flame.

3. Analysis

In order to focus on the essentials, we will first consider an acoustic system without mean flow but with a distributed heat release rate source term q' and discuss the effects of non-zero Mach number, acoustic dissipation and other aspects later. Starting with the wave equation for the pressure:

$$\nabla \cdot c^2 \nabla p' - \frac{\partial^2 p'}{\partial t^2} = -(\gamma - 1)\frac{\partial q'}{\partial t}, \tag{5}$$

and its frequency domain equivalent, the Helmholtz equation:

$$\nabla \cdot c^2 \nabla \hat{p} + \omega^2 \hat{p} = -(\gamma - 1) i\omega \hat{q} \tag{6}$$

subject to a Robin boundary condition:

$$\hat{p} = Z \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = -Z \frac{1}{\bar{\rho}} \frac{1}{i\omega} \nabla \hat{p} \cdot \boldsymbol{n}, \tag{7}$$

in which Z is a frequency dependent, complex valued impedance, $\boldsymbol{n}$ the normal vector and $\bar{\rho}$ the mean local density and c the local speed of sound. The Rayleigh index can be expressed as a function of pressure only by substituting Eq. (6) into the integrand of Eq. (2):

$$RI_\omega = \frac{-1}{\gamma \bar{p}} \int_V \Re\left(\frac{1}{i\omega} \hat{p}^* \nabla \cdot c^2 \nabla \hat{p} - i\omega \hat{p}^* \hat{p} \right) dv \tag{8}$$

The integral in Eq. (2) is to be evaluated over real valued frequency ω. Therefore, because $\omega \in \mathbb{R}$ the second term under the integral in Eq. (8) vanishes because $\Re(i\omega\hat{p}^*\hat{p}) = \Im(\omega|\hat{p}|^2) = 0$. The first term can be re-written by making use of the chain rule: $\nabla \cdot (\hat{p}^* c^2 \nabla \hat{p}) = \nabla \hat{p}^* \cdot c^2 \nabla \hat{p} + \hat{p}^* \nabla \cdot c^2 \nabla \hat{p}$ and by making use of the divergence theorem:

$$RI_\omega = \frac{-1}{\gamma \bar{p}} \frac{1}{\omega} \left[\int_S \Im(\hat{p}^* c^2 \nabla \hat{p}) \cdot \boldsymbol{n} \, ds - \int_V \Im(\nabla \hat{p}^* \cdot c^2 \nabla \hat{p}) \, dv \right], \tag{9}$$

in which S is the bounding surface. The second integral of this equation will equally vanishes because $\Im(\nabla \hat{p}^* \cdot c^2 \nabla \hat{p}) = \Im(c^2 |\nabla \hat{p}|^2) = 0$. Substituting the boundary condition Eq. (7) into Eq. (9) the frequency dependent Rayleigh index is written as:

$$RI_\omega = \int_S \Re\left(\frac{1}{Z}\right) |\hat{p}|^2 \, ds \tag{10}$$

in which the relation $\bar{\rho} c^2 = \gamma \bar{p}$ was used. Because the real part of the impedance, i.e. the resistance, is non-negative for dissipative boundary conditions [46, Section 3.3.2] under the zero Mach number assumption, the RI_ω will always be positive for stationary cases (i.e. $\omega \in \mathbb{R}$). It should be noted that this statement is true irrespective of the nature of the coupling between heat release rate and the acoustic field and irrespective of the phase difference or time lags between pressure and heat release rate response. This is not necessarily true, however, for the local Rayleigh index. Although RI_x may locally be negative, its volume integral will always be positive. The Rayleigh index can be considered as the flux of energy supplied to the system by the flame. The right hand side of Eq. (10) represents the flux of acoustic energy dissipated on the boundaries. So, because the dissipated energy is always positive, the Rayleigh index is necessarily also positive. This is of course reflected in the classical balance of the system's acoustic energy E. The rate of change of the expected value of the acoustic energy can be written as:

$$\frac{d < E >_T}{dt} = \int_V \frac{\gamma - 1}{\gamma \bar{p}} < p'q' >_T \; dv - \int_S < p'\boldsymbol{u}' \cdot \boldsymbol{n} >_T \; ds, \qquad (11)$$

in which $< \cdot >_T$ denotes a time average over a duration T. In the limit of steady state conditions the time derivative on the left hand side (LHS) of Eq. (11) vanishes because T was chosen sufficiently long to ensure that the process is statistically stationary. Hence, the energy added by the heat release process (the first term on the RHS, which is equal to RI) is balanced by the acoustic losses (the second term on the LHS). Note that for illustrative purpose all acoustic losses in Eq. (11) are due to damping on the boundaries. In reality there will be additional damping terms (as discussed in Section 4.1), but this doesn't change the conclusion that the RI will balance the acoustic losses and is hence always positive. In this representation it seems almost trivial that the RI is equal to the acoustic losses and hence positive. However, it is paradoxical if one considers that the phase of the heat release rate response to an acoustic forcing can take any value depending of e.g. convective delay terms in the flame response. The paradox can easily be solved by considering the representation of a generic thermoacoustic system as a block diagram as shown in Fig. 5. The diagram represents a very general thermoacoustic feedback cycle defined by the flame response F which describes the (linear or non-linear) response of the heat release rate fluctuations Q'_r to axial velocity fluctuations right upstream of the flame u'_u, the function G describes the acoustic response of the pressure to the fluctuating heat release rate Q' of an acoustically compact flame. The admittance $-Z_u^{-1}$ characterises the entire acoustic system upstream of the flame. The overall heat release fluctuation Q' is the sum of the (linear or non-linear) response Q'_r and a stochastic heat release rate source term Q'_s. It is crucial to note that in an experiment or numerical simulation only Q' can be measured directly whereas Q'_r and

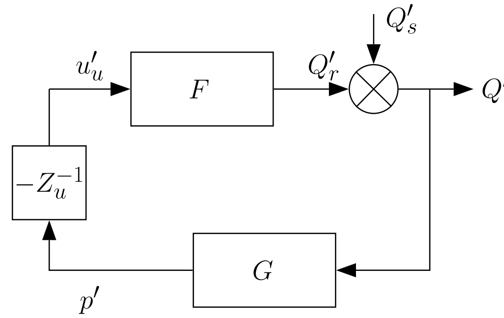

Fig. 5. Block diagram of a generic thermoacoustic feedback cycle in a longitudinal combustion system. The transfer function F causes the heat release response by the flame due to acoustic velocity excitation, whereas G causes the acoustic response p' due to excitation by the heat release Q'.

Q'_s can not be obtained in a direct manner. From this diagram it becomes clear that p' and Q' are deterministically related by G. The system G does not contain any source of acoustic energy it is therefore a passive system. A passive or dissipative, linear, time invariant system is characterised in the frequency domain by a transfer function with positive real part and in the time domain by a positive inner product of its input and output [44]. This explains why the Raleigh index is always positive in steady state cases. Note that Q'_r is deterministically related to p' via Z_u and F. The transfer function of F is not restricted to be positive real, and hence the phase between p' and Q'_r can take any value. So, it is crucial in this context to distinguish between Q' and Q'_r. A Rayleigh index based on Q' can not be used as a linear stability criterion or to determine if constructive or destructive interference takes place in the flame. A Rayleigh index based on Q'_r can be used to this end, but it is not possible to obtain Q'_r from experiment without additional excitation mechanisms. The relation between the Rayleigh index and acoustic dissipation on the boundaries can easily be shown for acoustically compact flames and longitudinal wave propagation. Expressing the spatial dependence of heat release fluctuations as a Dirac function: $q'(x) = Q'\delta(x - x_f)/S$ with x_f the axial position of the flame and S the cross section of the combustor, the Rayleigh integral is obtained as: $RI_\omega = \int_V [(\gamma - 1)/(\gamma \bar{p})]\,\Re(\hat{p}^* \hat{q}) dv = [(\gamma - 1)/(\gamma \bar{p})]\Re(\hat{p}^* \hat{Q})$. Relating the heat release to the pressure by means of the function $G(\omega)$:

$$RI_\omega = \frac{\gamma - 1}{\gamma \bar{p}} \Re\left(\frac{1}{G}\right) |\hat{p}|^2 \qquad (12)$$

Using the linearized Rankine-Hugoniot relations: $\hat{u}_d - \hat{u}_u = [(\gamma - 1)/(\gamma \bar{p})]\hat{Q}$ the reciprocal of the acoustic transfer function G can be expressed as:

$$\frac{1}{G} = \frac{\hat{Q}}{\hat{p}} = \frac{\gamma \bar{p}}{\gamma - 1}\left(\frac{\hat{u}_d}{\hat{p}} - \frac{\hat{u}_u}{\hat{p}}\right) \tag{13}$$

$$= \frac{\gamma \bar{p}}{\gamma - 1}\left(\frac{1}{Z_u} + \frac{1}{Z_d}\right) \tag{14}$$

and hence the Rayleigh index can be expressed as:

$$RI_\omega = \Re\left(\frac{1}{Z_u} + \frac{1}{Z_d}\right)|\hat{p}|^2, \tag{15}$$

which will be non-negative because the real part of the impedances are non-negative. It may at first sight be surprising that this integral does not seem to depend on F or Q'. However, these dependencies comes in via the dependence of $|\hat{p}|^2$ on F and Q'_s. Note that if the Rayleigh index would be calculated using Q'_r instead of Q', and if it can be assumed that amplitudes are sufficiently small such that F can be considered linear, the resulting expression would be:

$$RI_r = \Re\left(\frac{1}{Z_u}F\right)|\hat{p}|^2, \tag{16}$$

where RI_r is termed here the 'response Rayleigh index' which can be positive or negative depending on the phase of Z_u and F, however it can not be obtained experimentally.

Since the global Rayleigh index is always non-negative in steady state conditions, it is impossible to use it to determine if the flame provides constructive or destructive interference. It is important to stress that the local Rayleigh index can provide valuable information on what areas of the flame are driving and what areas are damping the flame-acoustic interaction.

To the knowledge of the authors it has not been reported before that the RI is positive even if the heat release rate response and acoustic pressure are out of phase. However, it should be mentioned that [47], Section 8.4.1] discusses the use of the RI in limit cycle operation and states that 'the overall result is unclear'.

However, in a linear analysis such as in [3,18,28–35] the RI can be useful to determine whether the flame is adding or subtracting acoustic energy from the system. This is because these systems do not represent a statically stationary state, but represent a transient response.

4. Discussion

It has been shown that under fairly general assumptions the Rayleigh index will always be positive. We will revisit these assumptions here and discuss under what conditions the RI can become negative.

4.1. Additional acoustic sinks and sources

In Eq. (6) it is assumed that all acoustic damping takes place on the boundaries, and that the fluctuating heat release is the only source of acoustic energy. However, in practical combustors damping within the volume due to conversion of acoustic motion into vorticity is generally significant. On the other hand, the unsteady vorticity may act as a source of acoustic energy. So, the question arises whether the finding that RI_ω is always positive will hold in cases where there are sources or sinks of acoustic energy due to vorticity. Such sources and sinks find their origin in the fluctuating component of the Lamb vector $\boldsymbol{L} = \boldsymbol{u} \times \nabla \times \boldsymbol{u}$ where $\boldsymbol{u}$ is the total velocity [48]. We include it here on the LHS of a simplified acoustic momentum equation: $\partial \boldsymbol{u}'_a/\partial t + (1/\bar{\rho})\nabla p' = \boldsymbol{L}'$ where $\boldsymbol{u}'_a$ is the irrotational component (acoustic) of the velocity. The Lamb vector will appear as an additional source term in the Helmholtz equation:

$$\nabla \cdot c^2 \nabla \hat{p} + \omega^2 \hat{p} = -(\gamma - 1)i\omega\hat{q} + \nabla \cdot \hat{\boldsymbol{L}} \tag{17}$$

The coherent part of the Lamb vector fluctuations can be expressed as a linear response to velocity fluctuations $\hat{\boldsymbol{L}} = \boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a$ in which $\boldsymbol{\Gamma}(\omega)$ is a frequency dependent tensor. The product $\bar{\rho}\hat{\boldsymbol{L}}$ is a force; the power exercised by this force on the acoustic field is $\hat{\boldsymbol{u}}_a^* \cdot \bar{\rho}\hat{\boldsymbol{L}} = \hat{\boldsymbol{u}}_a^* \cdot \bar{\rho}\,\boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a$. The work will be negative if it resists the fluid motion and hence acts as a damping term, i.e. the tensor $\boldsymbol{\Gamma}$ is negative definite. However, the Lamb vector can also act as a source of acoustic energy. In that case the work performed on the fluid will be locally positive. When using a similar derivation of the Rayleigh index as for Eq. (10), an additional term will be added to the integral: $(1/i\omega)\int \hat{p}^* \nabla \cdot \boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a\, dv$. By making use of the divergence theorem, this term can be written as $-(1/i\omega)[\int_V (\boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a) \cdot \nabla \hat{p}^* dv - \int_S \hat{p}^*(\boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a) \cdot \boldsymbol{n}\, ds]$. The second part of it will be zero if only sources in the volume are considered. Expressing the gradient of the pressure fluctuations as: $\nabla \hat{p} = (\boldsymbol{\Gamma}\,\hat{\boldsymbol{u}}_a - i\omega\hat{\boldsymbol{u}}_a)\bar{\rho}$, the additional term can be written as: $(\bar{\rho}/i\omega)\hat{\boldsymbol{u}}_a^*\boldsymbol{\Gamma}^*\boldsymbol{\Gamma}\hat{\boldsymbol{u}}_a - \bar{\rho}\hat{\boldsymbol{u}}_a^*\boldsymbol{\Gamma}\hat{\boldsymbol{u}}_a$. The first term is purely imaginary for real valued ω and will hence not contribute to the Rayleigh index. Hence, under a quasi-steady assumption the RI is expressed as:

$$RI_\omega = \int_S \Re\left(\frac{1}{Z}\right)|\hat{p}|^2\, ds - \bar{\rho}\int_V \hat{\boldsymbol{u}}_a^*\boldsymbol{\Gamma}\hat{\boldsymbol{u}}_a\, dv \tag{18}$$

The second term will be positive for negative definite $\boldsymbol{\Gamma}$. So, this means that as long as the Lamb vector acts as a net damping term, the Rayleigh index will always remain positive. It should be stressed, however, that under certain conditions this term can be negative. It will then act as a coherent source of acoustic energy (vortex sound).

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4.2. Non-Zero Mach number

Although the effect of mean flow on the wave propagation can be safely ignored in many cases of practical interest, this is not necessarily always the case. The real part of the impedance Z in Eq. (10) is not necessarily positive for sufficiently high Mach numbers. The effect of non-zero Mach number on wave propagation and the energy balance is discussed in detail in [28]. It will not be reproduced here, but the balance is similar to Eq. (11) with the difference that the flux term will depend on the Mach number. In analogy to the discussion about Eq. (11) it can directly be seen that because the source of energy will always balance the dissipation for steady state conditions, the Rayleigh integral will necessarily always be positive if no energy is added on the boundaries.

4.3. Conditions for a negative RI

There are certain instances when the global Rayleigh index may be evaluated as negative (and hence violate the slightly provocative statement in the title of this paper). These cases will be discussed in the following.

1. The combustor does not operate in (statistically) stationary conditions. Consider a fast change in the operating conditions such that the system transits from unstable to stable operation, say, through a sudden change in the fuel mass flow or by means of active control. In this case, the fluctuation energy decreases, and this is brought into effect through a negative *RI*. Note, however, that this is predominantly related to the coherent component and not to the stochastic part in the heat release rate. Furthermore, this effect would only be observed in an actual transient process, in other words, when the timescale of the change in operating conditions is short compared to the inverse of the decay rate. With a slow, quasi-static change in the operating conditions, the *RI* would remain positive throughout this change. In this case, the decrease in the coherent fluctuation energy would be small compared to the stochastic component.

2. Another instance in which the *RI* can be truly negative, even in steady operation, is when there is a significant source of sound other than the flame. This can be the case in the presence of an aeroacoustic instability, for example, associated with the flow over a side-branch or some other sort of cavity [49]. Here, the interaction between the unstable shear layer and the acoustic field acts as a source of sound, and the flame may act as a sink. In this case, the second integral on the right-hand-side of Eq. (18) would be positive and, hence, enable a negative *RI* despite the first integral, corresponding to the boundary losses, being positive.

Another significant source of sound may be present in partially premixed combustors when non-isentropic temperature waves are strongly accelerated at the outlet [50]. The flame may then, again, act as a sink for the acoustic energy, which would manifest in a negative *RI*. Similarly, in sequential combustors, one stage can act as a source of sound and the other as a sink. The latter would then exhibit a negative *RI*. All of this can apply for thermoacoustically unstable conditions or noise-driven oscillations. Finally, when forcing the flame acoustically, for flame response measurements, for example, the *RI* could be negative, with the acoustic power being supplied by the externally driven speaker.

3. A negative *RI* might manifest as a result of measurement inaccuracies. The most common way to evaluate the *RI* in a combustor rig is to collect the chemiluminescent light emission from the flame as a marker for the heat release rate and multiply this with a scalar pressure measurement, in an approximation of the integral in Eq. (1). The chemiluminescent light emission can be considered as a quantitative marker of the heat release rate only in premixed flames. Furthermore, care must be exercised in significantly non-adiabatic situations and strongly strained flames [51]. Also, as pointed out earlier, the local *RI* can, of course, be negative. Consequently, if the chemiluminescence measurement based on a photomultiplier does not capture the whole flame, a negative *RI* might be falsely obtained.

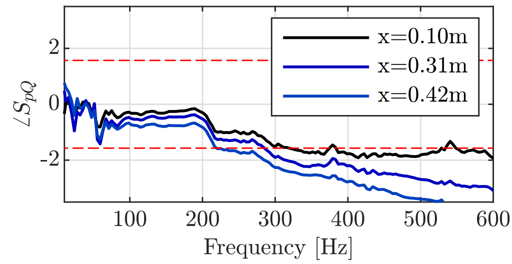

Fig. 6. Phase between pressure and heat release for an extended frequency range at different distance from the burner exit (x). The phase angle exceeds the $-\pi/2$ limit for higher frequencies and more distant microphones; indicating a violation of the compactness assumption.

Moreover, when the pressure signal is not collected directly at the flame but further downstream, which is often the case to protect the transducer from excessive thermal load, the phase relationship between pressure and heat release rate might be distorted from the true value. Fig. 6 illustrates this effect. When the phase between pressure and heat release rate fluctuations is calculated based on microphone signals further away from the flame, the phase angle moves out of the $-\pi/2$ to $\pi/2$ interval at higher frequencies. Finally, care must be taken when determining phase relationships between pressure and heat release rate via the corresponding Hilbert transforms in stable conditions.

In broadband or multi-frequency signals, this may generally lead to erroneous results and, hence, an incorrect sign of the RI.

5. Conclusion

Although the sign of Rayleigh index measured at non-transient conditions is commonly used to determine whether a flame is driving an instability or damping it [2,11–17,19–26,37–39,41,42,52], this work demonstrates that under fairly general conditions the Rayleigh index will always be positive. This is the case for both linearly stable and linearly unstable systems and holds irrespective of the phase of the flame transfer function. This result may seem surprising because the phase between the pressure and the heat release response can take any value depending e.g. on the time lag in the flame transfer function. However, the heat release response is not a quantity that can be measured directly. The measured overall heat release is the sum of the heat release response and a stochastic source term. It has been demonstrated that the phase between the overall heat release and the pressure is independent of the flame transfer function and is bounded between $-\pi/2$ and $+\pi/2$, hence, resulting in a positive Rayleigh index. It has been discussed that the Rayleigh index can be negative in case the measurement time is too short to ensure statistically stationary conditions, if a secondary acoustic source is present, for high Mach number flow, or due to measurement errors.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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