EXPLORING ENGINEERING STUDENTS' ENGAGEMENT WITH PROOF WITHOUT WORDS: THE CASE OF CALCULUS

F. Radmehr[1]
Norwegian University of Science and Technology
Trondheim, Norway
Ferdowsi University of Mashhad
Mashhad, Iran
https://orcid.org/0000-0002-0592-9148

M. Ezzati
Ferdowsi University of Mashhad
Mashhad, Iran

Conference Key Areas: *Mathematics at the heart of Engineering, Student Engagement*
Keywords: *task design, proof without words, mathematics, calculus, integral.*

ABSTRACT

Previous studies have reported that many engineering students struggle to develop a conceptual understanding of mathematics in university calculus courses. Engaging with mathematical proofs is one of the approaches to developing a conceptual understanding of mathematics; however, it is not always easy to use standard proofs for this purpose. In the present study, we focus on a type of mathematical proof known as Proof Without Words (PWW). A PWW typically consists of pictures or diagrams that help readers understand why a mathematical statement is true without

[1] *Corresponding Author*
F. Radmehr
f.radmehr65@gmail.com; farzad.radmehr@ntnu.no

doi:10.5821/conference-9788412322262.1223

providing verbal justifications. This study investigates how engineering students engage with PWW tasks related to calculus and also how they perceive its usefulness for teaching and learning calculus. Twenty undergraduate engineering students participated in semi-structured group interviews (in groups of two students), engaging with three PWW tasks related to calculus. Afterwards, students' perceptions of PWW tasks were explored using several open-ended questions. The findings indicate that many students engaged well with this type of activity. Furthermore, all students perceived that PWW tasks could positively impact their mathematical understanding, and many believed they could be used for teaching and learning calculus. Furthermore, several students highlighted that PWWs could also help with comprehending standards proofs and make mathematics learning more enjoyable.

1 INTRODUCTION

Engaging students with mathematical proofs could help students develop a conceptual understanding of mathematics [1]. However, past research has shown that students often have difficulties with constructing and comprehending formal proofs [2]. Therefore, mathematics educators focus on ways to improve the teaching and learning of mathematical proofs [2-3]. On the other hand, improving engineering students' conceptual understanding of mathematics has been the focus of several studies, and previous studies have reported that many engineering students struggle to develop a conceptual understanding of mathematics in calculus courses [4]. In this study, we focus on exploring how a proof type, proof without words, could be used to improve the teaching and learning of calculus to engineering students.

2 PROOF WITHOUT WORDS

There are different classifications for mathematical proofs. For example, Sigler et al. [5] have classified mathematical proofs into *standard proof*, *elegant proof*, and *proof without words (PWW)*. The focus of this paper, PWW, could be defined as "pictures or diagrams that help the reader see why a particular mathematical statement may be true, and also see how one might begin to go about proving it true" [6, p. 118]. Mathematics educators have provided several reasons why PWWs could be used in teaching and learning mathematics. For example, Bell [7] highlighted that using PWWs could help students understand the proof process, develop reasoning skills, and learn how to begin to construct a formal mathematical proof. PWWs could be a supplement for formal proofs and help students understand why a mathematical theorem is true in a more interesting way than some of the formal proofs [6]. Nelsen [8] highlighted this by quoting Martin Garnder, a famous mathematician, at the beginning of his third collection of PWWs: "A dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance" (p. vii). More recently, Kristiyajati & Wijaya [9] pointed out that many mathematics teachers also perceived that PWW tasks are interesting activities for students to engage with.

PWW tasks could be used in mathematics classrooms to develop visual thinking in all levels of schooling, from primary to tertiary levels [6]. Many PWWs have been constructed to prove theorems in different mathematical domains such as geometry, algebra, inequalities, and calculus (see, for example, [8]). The majority of previous research that explored the teaching and learning of PWW focused on school mathematics (e.g., [10]). Only a very few studies (e.g., [11]) have explored university students' understanding of PWW tasks and how PWW tasks could be used at the tertiary level. Our literature search indicates that such studies have not been conducted within the context of teaching and learning calculus, particularly for engineering students. Therefore, the present study explores how engineering students engage with PWW tasks and how they perceive its usefulness for teaching and learning calculus to explore the possibility of using such tasks in teaching and learning calculus to engineering students. We considered the following research questions to address this matter:

RQ1: *How do engineering students engage with calculus-related PWW tasks?*

RQ2: *How do engineering students perceive engagement with calculus-related PWW tasks?*

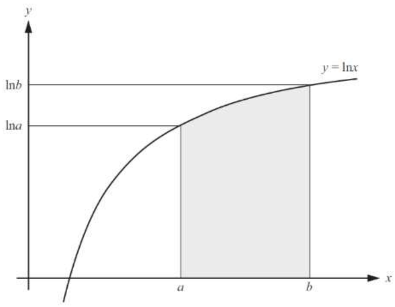
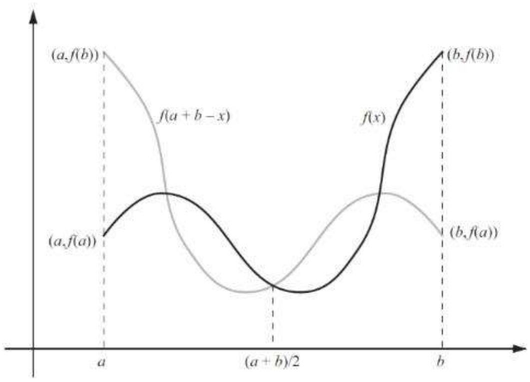
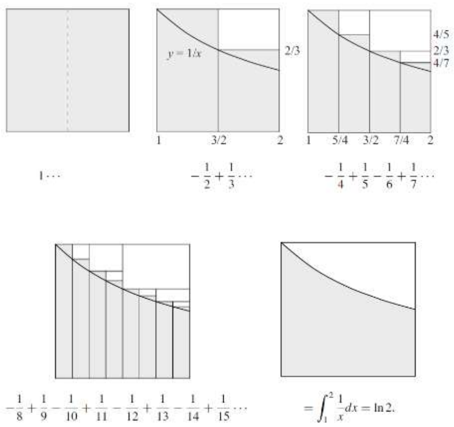
3 METHODOLOGY

In this qualitative study, we take a phenomenological approach where we describe "the lived experiences of individuals about a phenomenon as described by participants" ([12], p.14). The participants were 20 undergraduate engineering students from a major university in eastern Iran who participated in semi-structured group interviews in groups of two. Based on their calculus grades, the students are labelled as S1 to S20 from high to low performance to help readers identify any possible association between students' perception of PWW tasks and their achievement in response to RQ2. The interviews were conducted in Adobe Connect because of Covid 19 pandemic, and they were video recorded and transcribed. Students first engaged with three PWW tasks related to calculus, and then their perceptions of PWW tasks were explored individually using four open-ended questions (e.g., what are your opinions about using PWW tasks for teaching calculus?). The PWW tasks were adapted from Nelson (2015) and were related to (a) the integral area relationships in the context of the natural logarithm, (b) an integral transform, and (c) the relationship between harmonic series and integral. Each task has three parts, and students were given each part separately. In part A, students were asked to respond to the following question: *What mathematical concepts or procedures do you think the figure refers to?* In part B, they were asked to prove the given formula or equalities using the given figure(s). In part C, students were asked to prove the formula/equalities formally. These figures and formula/equalities are provided in Table 1. A correct solution to the tasks was discussed with students before they responded to the four opened-ended questions.

4 RESULTS

We first describe how engineering students engaged with the three PWW tasks in response to RQ1. Then, we discuss students' perceptions of PWW tasks to respond to RQ2.

Table 1. The figures and formulas/equalities in the three PWW tasks

Figure	Formula/equalities
	$\int_a^b ln(x)\,dx$ $= b\,lnb - a\,lna - \int_{ln\,a}^{ln\,b} e^y\,dy$ $= x\,lnx\|_a^b - (b-a) = (\,x\,lnx - x\,)\|_a^b$
	$\int_a^b f(x)\,dx$ $= \int_a^b f(a+b-x)\,dx$ $= \int_a^{\frac{a+b}{2}} [f(a+b-x)+f(x)]dx$ $= \int_{\frac{a+b}{2}}^b [f(a+b-x)+f(x)]dx$
	$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1} = ln\,2$

4.1 Students' engagement with the PWW tasks

Overall, students engaged relatively well with the first two tasks. In response to Task 1, Part A, 90% of groups referred to the concept of integral and mentioned that the shaded area is $\int_a^b \ln x dx$ when the figure was presented to them. For Task 2, Part A, similarly, 90% of groups referred to the concept of integral when the figure was presented to them. These groups also identified that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Furthermore, 80% of groups stated that $f(x)$ and $f(a+b-x)$ are symmetric with respect to the line $x = \frac{a+b}{2}$. Regarding Task 3, Part A, all groups mentioned that the figure is related to the concept of integral and series. Moreover, 80% of groups referred also to the integral-area relationship and how a series could be used to calculate the area of rectangles. Thirty percent referred explicitly to the concept of Riemann integral. One group (10%) was also able to construct $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$ based on the given series under the figures.

The overall success rate of groups for Parts B and C are summarised in Table 2. The findings indicate that many groups were able to prove the formulas using the given figures for Tasks 1 and 2. However, Task 3 was more challenging for students, and even with the interviewer's help, only 60% were able to prove the formula using the figures successfully. Furthermore, the success rate for formally proving the formulas in Part C of Tasks 2 and 3 was much lower compared to Part B. The provided help for Task 1, Part B was that $blnb$ is the area of the largest rectangle in the Figure and in Part C was what could be considered as U and dv in the integration by parts. Regarding Task 2, Part B, the interviewer help some groups to amend their drawing for $f(a+b-x)+f(x)$ in $[a,b]$. For Part C, the provided help was related to students' calculations; some students made a mistake when using the integration by substitution. They incorrectly calculated $dx = dt$ when they considered $a+b-x=t$, and the interviewer told them their mistake. Regarding Part B of Task 3, students realised that the given series under the figures are related to shaded areas; however, they struggled with justifying how the shaded area is exactly calculated with the given series. The interviewer helped them with that. Finally, for Part C, the help was that they could use geometric series to formally prove the given formula.

Table 2. *The overall success rate of groups for Parts B and C across the three PWW tasks*

	Part B			Part C		
	Without help	With the help of the interviewer	Total success rate	Without help	With the help of the interviewer	Total success rate
Task 1	7 (70%)	2 (20%)	9 (90%)	8 (80%)	1 (10%)	9 (90%)
Task 2	5 (50%)	5 (50%)	10 (100%)	2 (20%)	3 (30%)	5 (50%)
Task 3	0 (0%)	6 (60%)	6 (60%)	0 (0%)	3 (30%)	3 (30%)

4.2 Students' perceptions of PWW tasks

All students agreed that PWW tasks could help them with learning calculus. They also pointed out several benefits of using PWW tasks for learning calculus, such as helping comprehend mathematical proofs and learn mathematical concepts (Table

3). Some students also enjoyed engaging with PWW tasks, and a few found it helpful for keeping mathematical concepts longer in mind.

Table 3. The benefits of using proof without words

Themes	N	%	Sample responses
Helping students comprehend mathematical proofs	17 (S1-12, 15-19)	85	"PWWs are more interesting and attract students' attention more than standard proofs. They are presented with figures, whereas standards proofs have only a few symbols or parameters. One can understand the processes in the proof by looking at the figure; they are more understandable than standard proofs" (S4)
Helping students to prove standard proofs	12 (S1-4, 6, 7, 9, 10, 13, 15, 17, 18)	60	"When learning a standard proof, if we have seen a PWW of it, the figure helps us identify its relationship with the standard proof, and consequently a better understanding of the standard proof." (S9)
Improving students' attitudes towards mathematical proofs	12 (S1, 5, 6-10, 12, 14, 16, 19, 20)	60	"In general, PWWs are more interesting to follow, and therefore, it helps us to have a more positive attitude towards mathematical proofs." (S5)
Helping students to learn mathematical concepts	10 (S3, 5-7, 10, 11, 13, 16, 17, 19)	50	"In fact, it makes learning the main concepts easier for us. If we encounter an integral formula without a visual illustration, we would have to memorise formulas and not understand their applications. But in PWWs, using figures makes learning better." (S11)
Enjoyable activity for students	9 (S2, 4, 5, 9, 12, 14, 15, 17, 20)	45	"PWWs... are fun for me, and I enjoy engaging with them..." (S14)
Helping mathematical concepts last longer in students' minds.	4 (S1, 7, 12, 20)	20	"When I come across this series later, its figure will appear in front of my eyes and will last in my mind longer..." (S1)
More accessible to students that standards proofs and less likely that students make mistakes when engaging with them	2 (S9, 19)	10	"We are not experts in proving mathematical theorems; we may not notice if something goes wrong in standard proofs; however, we have figures in PWWs to help us check our work." (S19)

A few challenges of engaging with PWW tasks were also reported, such as some students perceived that comprehending PWWs could be challenging for learning advanced mathematical concepts such as those found in multivariable calculus (Table 4).

Table 4. *The challenges of using proof without words*

Themes	N	%	Sample responses
Comprehending PWWs could be challenging and require strong visual thinking	7 (S4, 5, 9, 12, 14, 15, 20)	35	"In some mathematical topics, it is easier to use standards proofs because if we want to prove using a figure, we need a multi-dimensional figure and paying attention to details requires good visual thinking skills..." (S15)
Analysing the information provided in PWWs could be challenging	3 (S1, 4, 20)	15	"Information in figures of PWWs could be difficult to analyse, and we might make a mistake when analysing them. For example, surfaces have three variables in multivariable calculus and mostly have complex relationships. If we have not mastered them, we might mistake in distinguishing surfaces in the figures." (S1)
Comprehending PWWs is time-consuming	2 (S12, 14)	10	"The teaching time in university courses is short, and the volume of content to teach is high while PWW requires a lot of time to comprehend." (S14)

5 DISCUSSION AND CONCLUSIONS

Helping students develop a conceptual understanding of calculus concepts is one of the main goals of engineering calculus courses [4]. In this study, we explored to what extent PWW tasks could be used to achieve this goal as previous studies reported that engaging with PWW tasks could be interesting for students and could contribute to improving students' mathematical learning in different ways [6-10]. This study also contributes to the existing literature in mathematics education regarding the affordances and constraints of using PWW tasks in teaching and learning of mathematics at the tertiary level, including for engineering students, as most studies related to using PWW tasks were conducted in the school context (e.g., [7, 9-10]).

The findings suggest that all engineering students perceived that engaging with PWW tasks is useful for learning calculus. The reasons they provided for such perceptions are in line with what is highlighted in the literature about the benefits of using PWW tasks. For instance, students perceived that engaging with PWW tasks could help them to have a better understanding of mathematical proofs and how to construct standard proofs. Similarly, Bell [7] highlighted using PWWs in teaching mathematics "can help students improve their ability to reason when asked to explain an illustration, and this heightened reasoning can lead to understanding how to begin a formal proof." (p. 690). Furthermore, many students enjoyed engaging with PWW tasks, as pointed out in [9]. In terms of their engagement with the PWW tasks, students performed relatively well on the first two tasks; however, they faced more challenges when engaging with the third task as they needed to associate their knowledge of integral with their knowledge of series. In line with the previous studies regarding comprehending and constructing proofs [2], several students faced difficulties constructing standards proofs and were more successful in the PWW tasks. Overall, the study findings suggest that PWW tasks could be used in teaching calculus courses to engineering students to provide more opportunities for students to develop a conceptual understanding of mathematics and contribute to making calculus courses more interesting to engineering students.

REFERENCES

[1] Hanna, G. (2020), Mathematical Proof, Argumentation, and Reasoning, Encyclopedia of Mathematics Education, S. Lerman, Springer, Cham, pp. 120-128.

[2] Authors (2020), Teaching Mathematics and its Applications: An International Journal of the IMA.

[3] Weber, K. (2015), Effective proof reading strategies for comprehending mathematical proofs, *International Journal of Research in Undergraduate Mathematics Education*, Vol. 1, No. 3, pp. 289-314.

[4] Rivera-Figueroa, A., Guevara-Basaldúa, V. (2018), On conceptual aspects of calculus: a study with engineering students from a Mexican university, *International Journal of Mathematical Education in Science and Technology,* Vol. 50, No. 6, pp. 883-894.

[5] Sigler, A., Segal, R., Stupel, M. (2016), The standard proof, the elegant proof, and the proof without words of tasks in geometry, and their dynamic investigation, *International Journal of Mathematical Education in Science and Technology*, Vol. 47, No. 8, pp. 1228-1229.

[6] Alsina, C., Nelsen, R. (2010), An Invitation to Proofs Without Words, *European Journal of Pure and Applied mathematics,* Vol. 3, No. 1, pp. 118-127.

[7] Bell, C. (2011), Proofs without words: A visual application of reasoning and proof. *The Mathematics Teacher*, Vol. 104, No. 9, pp. 690-695.

[8] Nelsen, R., B. (2015), Proofs without words III: Further exercises in visual thinking, The Mathematical Association of America, Washington.

[9] Kristiyajati, A., Wijaya A. (2018), Teachers' perception on the use of "proof without words(PWWs)" visualization of arithmetic sequences, *Journal of Physics: Conference Series*, Vol. 1097, No. 1.

[10] Polat, K, Akgun L (2020), Examining the processes of high school students to do proof without words, *Education Reform Journal*, Vol. 5, No. 1, pp. 8-26.

[11] Bardelle, C (2009), Visual proofs: An experiment, Proc. of CERME 6, V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello, Lyon, WG2, pp. 251-259.

[12] Creswell, J. W., Research design: Qualitative, quantitative, and mixed methods approaches, Sage, Los Angeles.