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Breaches of the didactic contract as a driving force behind learning and non-learning: a story of flaws and wants

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In agreement with the main tenets of the anthropological theory of the didactic (ATD), this study uncovers dependencies between what students can learn, the established curriculum and the current state of mathematicians' mathematics ('scholarly mathematics'). One main result is that the mathematics taught, too often taken for granted by curriculum developers and teachers, is in fact problematic not only to students but also to teachers and curriculum developers and is sometimes a challenge even to current scholarly mathematics. The mathematics taught during a given historical period within a given institution contains flaws that, when they cease to go unnoticed, generate crises, in the form of breaches of the prevailing didactic contract. The resolution of these crises allows the institution and its actors—in particular students and teachers—to learn new contents and often also leads to the more or less damaging unlearning of old contents. This key phenomenon is illustrated, at the triple level of the classroom, the curriculum and scholarly mathematics, with regard to elementary algebra and mathematical analysis, most importantly in the case of maxima and minima problems.

1. Introduction

Our study is conducted within the framework of the *anthropological theory of the didactic* (ATD), of which we will use a minimum of concepts, notably the concept of institution, and some of its usual symbolic notations, in particular x for a student, y for a teacher, α for the author of a mathematics textbook, μ for a mathematics researcher, and ξ for a mathematics education researcher (Chevallard, 2019). In the ATD, an institution is anything 'instituted', that is, any created reality that is subject to rules specific to the type of institution considered and of which people can be members (permanent or temporary). A school is an institution, and so are a class at school, a couple, a scientific community, etc. Further, we use the concept of *didactic contract* from the theory of didactic situations, as conceptualized by Guy Brousseau (see Brousseau, 1997, pp. 31–32). In *any* institution, the didactic contract is the system of clauses (explicit or implicit, but supposedly known to all the members of the institution) that, at a given moment, regulate the study of a question (here, a question of mathematics). Let us stress that this latter notion applies not only in a classroom but also when the institution is the community of mathematicians, for example.

In this study we consider situations of the following general type. In a given institution I (which may be a class, but not only that, as we shall see), a certain activity is carried out in a routinized way using various means, both material and non-material. At a certain point in time, some actors in the institution I realize that the means used can be questioned in a certain way. They make this questioning known, which provokes a more or less important crisis, to which I reacts according to one of the following three strategies: Σ_1, where the institution I decides to ignore the problem and live with it; Σ_2, where I seeks to solve the problem, solves it and integrates this solution in an appropriate manner into the means of its activity; Σ_3, where I ignores the solution brought forth by Σ_2 and changes its system of means in a somewhat important way, thanks to the fact that a (necessarily different) system that eludes the problem turns out to be available at an acceptable cost.

In the background of our study, two main questions are present: that of the didactic transposition of mathematics and that of the treatment of (mathematical) existence and uniqueness problems. The didactic transposition of mathematics and its 'outputs' (Chevallard, 1981, 1989) deals with the mathematics curriculum, or rather the successive mathematics curriculums, with their changes over time and, above all, as we shall see, their curricular stases (i.e., epistemic stagnations). One cannot explicate 'student thinking', or more generally anyone's 'thinking', without elucidating the conditions and constraints under which it is formed: What 'mathematical thinking' students carried out when doing, say, algebra (not to mention mathematical analysis) was undoubtedly very different at the beginning of the 20th century, and even more so at the beginning of the 19th century, from what today's research on school algebra reports (e.g., Kieran, 2018; Arcavi *et al.*, 2016; Schmittau, 2011; Sutherland, 1999). In each historical period, what students have in mind depends largely on what they are taught and what they study—and the way they do it. The recent study entitled 'Algebraic thinking' (Hodgen *et al.*, 2018), which presents a review of 146 papers on this topic spanning two decades,[1] shows a dominance of psychological approaches to the teaching and learning of elementary algebra. At the same time, the study raises an important criticism against the lack of systematic analyses of mathematics tasks used in research on algebraic thinking. In this respect, the ATD, which takes into account every teaching and learning institution, and emphasizes the types of mathematics tasks and the ways of dealing with them in these institutions, can, we believe, allow for a more thorough consideration of what is classically called 'algebraic thinking'.

The second question addresses an aspect that has been little studied so far by mathematics education research, namely the question of the treatment (or the lack of treatment) of existence and uniqueness problems, which are so frequent in mathematics but, as we will see, much more rarely taken into consideration (e.g., Moreira & David, 2008; Vaiyavutjamai & Clements, 2006). More generally, given a mathematical property, Π, a person z can either be unaware of Π, or can be aware of it but take it for granted (that is, as obviously true), and thus ignore the possible need for a proof of Π. The person z can also recognize this need but decide to simply admit Π as true, or provide a proof of Π, which, to a certain observer w, will appear as erroneous or, on the contrary, as fully valid.

In what follows, we will focus on situations in which z (whether $z = x, y, \alpha, \mu$, or ξ) starts by ignoring Π to realize only later—if ever—that this property should at least be discussed. In the case where $z = x$,

[1] The reviewed papers were published from 1998 to 2017 in proceedings of congresses of the European Society for Research in Mathematics Education (CERME).

y or α, we shall see that such a relation to Π will sometimes prove to be a didactic 'help' for z, or, on the contrary, will sometimes turn into an epistemological obstacle that hinders teaching and jeopardizes learning.

2. From peaceful ignorance to irksome awareness

2.1. *A peaceful world*

In some classroom, the teacher y asks the students x_i to calculate the sum of 17 and 15. This task is performed under the presupposition that the requested sum (1) *exists* and (2) is *unique*. No one in the class—neither the students nor the teacher—can even imagine that it could be otherwise. The usual technique to calculate the sum 17 + 15 can be represented as follows: $17 + 15 = 10 + 10 + (7 + 5) = 10 + 10 + 12 = 10 + 10 + 10 + 2 = 30 + 2 = 32$. Of course, there are *many* other calculation strategies, for example these: $17 + 15 = (15 + 2) + 15 = 15 + 15 + 2 = 30 + 2 = 32$; $17 + 15 = (17 - 5) + (15 + 5) = 12 + 20 = 32$; $17 + 15 = (17 + 3) + (15 - 3) = 20 + 12 = 32$; $17 + 15 = (20 - 3) + (20 - 5) = 40 - 8 = 32$; etc. This multiplicity of paths to the 'desired' goal might also raise the question, 'But will we get the same result every time?' Such a situation might therefore raise the incongruous problem of the uniqueness of the sum. Note also that, by taking an unusual calculation path, another question might arise, 'Will we get anywhere by doing this?' This might lead one to raise the perhaps more hidden issue of the very existence of the sum. In contrast, if we always use the same calculation technique, as is traditional in primary education, such doubts are likely to be dispelled. We thus enter, and remain immersed in, a didactically facilitating mathematical universe where the entities we are talking about always exist and are always unique. This, it seems, is everyone's first and long-lasting experience of mathematics.

2.2. *Challenging the self-evident*

In the previous illustration, the underlying institution I was a primary school class. Things change when you change the institution, and, in the case at hand, when you take the mathematics community as institution I. In the history of mathematics, as in the history of curriculums, one can observe—we will admit this—that, for a given domain, there is an alternation between often long periods when the mathematical entities of this domain seem to be taken for granted by the actors concerned, and periods when these entities are questioned, at least by some actors, in their very reality. Thus, the obvious existence and uniqueness of the sum of two integers came to be questioned, in the nineteenth century, when mathematicians, in particular Richard Dedekind (1831–1916) and Giuseppe Peano (1858–1932), strove to develop an axiomatic theory of *natural numbers* (Mendelson, 2015, Chap. 3; Feferman, 1989, Chap. 3). Needless to say, this applies as well to the theory of *real* numbers (Feferman, 1989, Chap. 7). However, let us underline here this decisive fact: the lack of a 'mathematically validated' definition of numbers (in the sense of the closing 19th century) had fortunately *not* prevented the mathematicians of the previous centuries to do mathematics peacefully. This is an essential aspect of the dialectic of blissful ignorance and productive awareness that greatly determines, as the case may be, the advances in the mathematical sciences, the updating of curriculums, and what the teachers and students learn.

In truth, it should be noted here that the above-mentioned 'crisis' regarding the definition of numbers does not seem to have had any significant repercussions in the world of primary school mathematics. But let us add another example consisting of a simple 'thought experiment' (which the reader can carry out if they wish). Let us imagine that a mischievous mathematician μ says to one of his colleagues μ':

'Have you ever read, or heard of, a proof of the statement that a circle has a unique centre and radius?' Three hypotheses can be made at this point: (1) the answer of μ' will be 'No'; (2) this mathematician will feel a slight embarrassment in realizing that they have never asked themselves this question; (3) μ' will reassure themselves by thinking that the proof of this fact must be trivial. In the latter case, the passage from happy ignorance to mathematical doubt may constitute for μ' a somewhat unpleasant surprise, a short-lived personal 'crisis'. But this crisis will be easily settled: They will imagine, for example, to take three points A, B and C on the circle and will consider that the centre—*any* centre—of the circle lies on the perpendicular bisector of AB and on the perpendicular bisector of BC and is therefore the point of intersection of these two lines, so that this circle has indeed a unique centre. After that, it is likely that μ' (in the position of a 'mere mathematician') will forget this incidence completely. But if μ' is a member of a curriculum committee, it is likely that they will not insist on introducing this question into the secondary school curriculum. Similarly, if the above question is posed by a mathematics teacher y to another mathematics teacher y', it is also likely that y' will ignore this question in the class for which they are responsible.

2.3. *Coming to grips with the unsuspected*

In the above cases, the institution I concerned is not substantially modified once the crisis is over. The use of the strategy Σ_1 maintains the previous institutional world. On the other hand, the institution does not *learn* anything new: for good or evil, whether I is a class or the mathematics community, for example, there is a 'stasis' of the institution's curriculum.[2] In other cases, however, it is not possible to return to the happy ignorance of the good old days.

As far as the secondary school mathematics curriculum is concerned, one of the most striking experiences for students is undoubtedly the following. When we consider only equations of the *first* degree $ax + b = 0$ with $a \neq 0$, that is to say *linear* equations, we are immersed in a mathematical universe where an equation has always one and only one solution—there is existence and uniqueness of the solution. All changes when we move on to equations of the *second* degree, $ax^2 + bx + c = 0$ with $a \neq 0$. As we all know, a quadratic equation can have *zero* solution, or *one* only, or *two*. This is a mathematical fact, independent of anyone's will, and that every student must learn to accept, recognize, and manage—unless they flee away from the mathematics taught. In a class that encounters this fact for the first time, it typically causes a *breach of the didactic contract*, because, until now, the mathematical universe did not look 'like that'. This fact is therefore likely to hinder the reception and understanding by the class of a structurally new type of mathematical situations.

For the teaching and learning to be successful, such hitherto unseen fact—for the students, not for the teacher—will also have to be related to previously 'well known' mathematical facts. For example, the students concerned have known for a long time that, given a straight line and a circle, either the straight line does not intersect the circle, or is tangent to it, or intersects it at two distinct points.[3] In algebraic terms, this is expressed by the fact that the quadratic equation $(x - h)^2 + (ax + b - k)^2 - r^2 = 0$ has 0, 1, or 2 solutions, where $(x - h)^2 + (y - k)^2 = r^2$ (with $r > 0$) is the equation of the circle and $y = ax + b$ the

2 More exactly, an institution I has in general several curriculums. In the ATD, an institutional curriculum of I is defined as a sequence of positions, together with the knowledge equipment needed to validly occupy them, which some members of the institution can come to successively occupy (Chevallard, 2018).

3 Note that this is a property that μ' may have implicitly used in the reasoning we have attributed to them above: the two perpendicular bisectors intersect because the points A, B and C of the circle are not aligned, since a straight line intersects a circle in *at most two* distinct points.

equation of the line. Moreover, this algebraic modelling of an initially geometrical problem can result in old knowledge formulated in a quite new way. Here, the equation $(x-h)^2+(ax+b-k)^2-r^2=0$ can be written as follows: $(a^2+1)x^2+2(a(b-k)-h)x+h^2+(b-k)^2-r^2=0$. The reduced discriminant is $\Delta'=(a(b-k)-h)^2-(a^2+1)(h^2+(b-k)^2-r^2)=a^2(b-k)^2+h^2-2ah(b-k)-(a^2+1)h^2-a^2(b-k)^2-(b-k)^2+(a^2+1)r^2=-a^2h^2-2ah(b-k)-(b-k)^2+(a^2+1)r^2=-(ah+b-k)^2+(a^2+1)r^2$. We have:

$$\Delta'>0 \Leftrightarrow (ah+b-k)^2<(a^2+1)r^2 \Leftrightarrow \frac{(ah+b-k)^2}{a^2+1}<r^2 \Leftrightarrow \frac{|ah+b-k|}{\sqrt{a^2+1}}<r.$$

The expression $\frac{|ah+b-k|}{\sqrt{a^2+1}}$ is that of the distance of the centre of the circle to the line of equation $y=ax+b$ (see 'Distance From a Point to a Line', 2021), so that we can establish over again—algebraically—well-known geometric results.

More generally, the students will need to learn how to integrate quadratic equations into modelling activities. A historical example of this otherwise commonplace requirement is provided by an episode that took place on February 19, 1795, during the fourth lesson that Pierre-Simon de Laplace (1749–1827) gave at the ephemeral 'Normal School of the Year III',[4] when this great mathematician examined the following elementary physics problem: 'Two lights, one of which [B] is four times as bright as the other [A], being separated by an interval of three feet, determine on the line joining them the point [P] which they illuminate equally (see Fig. 1)' (Laplace, 1992, pp. 68–69).

Fig. 1. The two lights on a line.

It is supposed here that the 'strength' of the light emitted by a source decreases as x^{-2}, where x is the distance to the source. For $\overline{AB}=3$ and $\overline{AP}=x\in(0,3)$, we have the equality $\frac{1}{x^2}=\frac{4}{(3-x)^2}$ or $4x^2-(3-x)^2=0$, which is equivalent to $(2x-(3-x))(2x+(3-x)=0$, or $(3x-3)(x+3)=0$. We therefore arrive at $\overline{AP}=x=1$. But what about the second 'solution' to the equation obtained, that is $x=-3$? This solution corresponds to another point P′ on the other side of A, so that $\overline{P'A}=3$. It is undeniable that A and B equally illuminate P′ since $1/\overline{P'A}^2=1/3^2=1/9$ and $4/\overline{P'B}^2=4/(3+3)^2=4/36=1/9$. The point P′ is therefore a solution to the *physical* problem addressed that might have been overlooked without the use of an algebraic equation. Laplace (1992) concludes:

> These unexpected solutions show us the richness of the algebraic language, to the generality of which nothing escapes when one knows how to read it well. (p. 69).

This example may seem unremarkable. But it illustrates, at a relatively low level of the mathematical studies, a typical teaching and learning difficulty. To overcome such a difficulty—in this case, to accept that *not everything* can be represented by an equation of the *first* degree, and to accept the consequences

4 On this 'school' (i.e., the 'École normale de l'an III'), see https://parcoursrevolution.paris.fr/en/points-of-interest/71-ecole-normale-superieure-rue-d-ulm.

of this recognition (for example, the fact of having to learn to factor algebraic expressions)—leads every time to enter a different, new mathematical world, and this more or less slightly traumatic situation may incite students to withdraw from that unpleasant learning experience and simply give it up. What we will now suggest is that this can happen *as well* at the level of a curriculum, where not only students, but also teachers and textbook authors are concerned.

3. The case of elementary problems of maxima and minima

3.1. Solving problems by elementary algebra: two examples

Let us consider the following exercise (Natansón, 1977, p. 18):

> A 200 m long fence can be built with some available boards, using the wall of a factory as one of its sides (see Fig. 2). It is required to enclose a rectangular yard of maximum area.

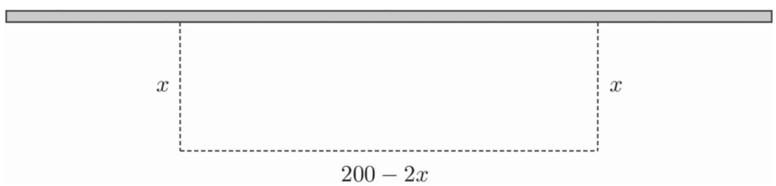

FIG. 2. The enclosed rectangular yard (adapted from Natansón, 1977, p. 19).

We must therefore maximize the product $P = x(200 - 2x)$ on the interval $(0, 100)$. Today's reader will be tempted to use differential calculus to find the maximum sought. But it must be stressed that, until late in the 19th century in France and early in the 20th century in Norway, even the first elements of differential calculus were *not* taught in secondary schools (Belhoste, 1995, p. 38; Birkeland, 1999, p. 38). In fact, this type of problems, of which the foregoing exercise is an easy specimen, was solved by elementary algebra.[5] A key theorem—denoted here by the Greek letter θ—used for this was the following:

> Given nonnegative numbers $a_1, a_2,..., a_n$, if the sum $S = a_1 + a_2 + ... + a_n$ is constant, then the product $P = a_1 a_2 ... a_n$ reaches a maximum when these numbers are equal (and therefore equal to S/n).

The students had to learn how to use Theorem θ appropriately. For example, here the sum of the factors of the product to be maximized, $A = x(200 - 2x)$, is *not* constant (it is equal to $200 - x$). But A is maximum at the same time as $\bar{A} = 2A = 2x(200 - 2x)$, whose sum of factors S is equal to 200. According to Theorem θ, the maximum is reached when $2x = 200 - 2x$, that is when $x = 50$.

Let us give another example, a little bit more sophisticated, also taken from Natansón (1977):

> Above the centre of a round table hangs a lamp (see Fig. 3). At what height h should the lamp be placed so that the luminous intensity L on the edge of the table is maximum? It is supposed that at point A, we have $L = k \dfrac{\sin\ \varphi}{l^2}$, where k is a constant.[6] (pp. 37–38).

5 The reader not familiar with this historical fact can skim through the old, classic book by Ramchundra (1859), aptly entitled *A Treatise on Problems of Maxima and Minima Solved by Algebra*.

6 Luminous intensity is defined as the quantity of visible light that is emitted in unit time per unit solid angle ('Luminous Intensity', n.d.).

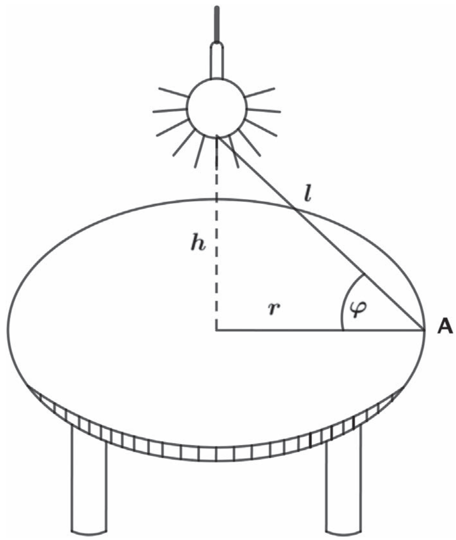

FIG. 3. The table with a lamp hanging over it (adapted from Natansón, 1977, p. 37).

Since $\cos\varphi = \dfrac{r}{l}$ we have $L = \dfrac{k}{r^2}\sin\varphi \times \cos^2\varphi$. L and $\overline{L} = \frac{r^4}{k^2} L^2$ are maximum at the same time. Since $\overline{L} = \sin^2\varphi \times \cos^4\varphi$, we have $\dfrac{1}{4}\,\overline{L} = (1-\cos^2\varphi)\,\dfrac{\cos^2\varphi}{2}\,\dfrac{\cos^2\varphi}{2}$. According to Theorem θ, $\dfrac{1}{4}\,\overline{L}$ is maximum when $1-\cos^2\varphi = \dfrac{\cos^2\varphi}{2}$, that is to say when $\cos\varphi = \sqrt{\dfrac{2}{3}}$ so that $\tan\varphi = \dfrac{\sqrt{2}}{2}$. We therefore have $h = r\tan\varphi = r\,\dfrac{\sqrt{2}}{2} \approx 0.7\,r$.

Today, the technique we have used here has disappeared to make room for solutions based on *differential calculus*: since $\cos\varphi = \dfrac{r}{l}$, we have $L = k\,\dfrac{\sin\varphi}{l^2} = \dfrac{k}{r^2}\sin\varphi\cos^2\varphi$. The derivative of the function defined on $[0, \pi/2]$ by $f(\varphi) = \sin\varphi\cos^2\varphi$ is $f'(\varphi) = \cos\varphi\,(3\cos^2\varphi - 2)$; therefore f' is positive on $[0, \varphi_m)$ and negative on $(\varphi_m, \pi/2]$, where $\cos\varphi_m = \sqrt{\dfrac{2}{3}}$. We can then conclude as above. The secondary education institution and, with it, the teachers and students involved have thus both *unlearned* an old way of doing things (using elementary algebra and Theorem θ), on the one hand, and *learned* basic calculus, on the other. What had happened?

3.2. *An unseen mathematical flaw belatedly disclosed*

The proof of Theorem θ that had become usual in the mathematics secondary curriculum proceeded, in our words, roughly as follows[7]:

7 This kind of (flawed) argument is still found in the 20th century: for example, see Dörrie (1965, p. 38).

Suppose that the numbers $a_1, a_2,..., a_n$ are not all equal. If $a_i \neq a_j$, let us replace a_i and a_j by the number $\frac{a_i + a_j}{2}$. The sum S is unchanged but the product P is increased by $\left(\frac{a_i + a_j}{2}\right)^2 - a_i\, a_j$. Since $\left(\frac{a_i + a_j}{2}\right)^2 - a_i\, a_j = \frac{1}{4}\,[(a_i + a_j)^2 - 4a_ia_j] = \frac{1}{4}\,(a_i - a_j)^2 > 0$, the new product is greater. Let K be the set of n-tuples $(a_1, a_2,..., a_n) \in \mathbb{R}_+^n$ such that $a_1 + a_2 +... + a_n = S$. If the tuple $(a_1, a_2,..., a_n)$ has two unequal elements, we can thus find another tuple $(a'_1, a'_2,..., a'_n)$ such that $a'_1a'_2... a'_n > a_1a_2... a_n$. This technique obviously does not work if all the elements of $(a_1, a_2,..., a_n)$ are equal: in this case, it does not allow to obtain a tuple $(a'_1, a'_2,..., a'_n)$ such that $a'_1a'_2... a'_n > a_1a_2... a_n$. 'Therefore' the maximum is reached when $a_1 = a_2 =... = a_n = S/n$.

This traditional proof had the merit of *not* disrupting the then established mathematical universe of secondary education because it used only elementary and familiar results. Where is the flaw in it? The skeleton of the argument can be described as follows:

Consider a set X and an element $x_0 \in X$. Let ϕ be a mapping from X to X and f a function from X into $\mathbb{R}$. To make the notation easier to follow, we will denote $\phi(x)$ by x'. If $x_0{}' = x_0$ (i.e., if x_0 is fixed point for ϕ) and if, for all $x \in X \setminus \{x_0\}, f(x') > f(x)$, then the maximum of f on X is not reached at any point $x \neq x_0$ and therefore is reached at x_0.

In the example of interest to us, we have $X = \{(a_1, a_2,..., a_n) \in \mathbb{R}_+^n \mid a_1 + a_2 +... + a_n = S\}$. Here, S is a given number, while $\phi(x) = \phi(a_1, a_2,..., a_n)$ can be taken to be $(a'_1, a'_2,..., a'_n)$ as defined above, where i and j are the least indices such that $a_i \neq a_j$, and $f(x) = f(a_1, a_2,..., a_n) = a_1a_2... a_n$. The argument would lead therefore to conclude that f reaches its maximum at $x_0 = (S/n, S/n, \ldots, S/n)$. In fact, as we are going to see, although the *result* is correct, its *derivation* is wrong.

What is the weak point here? In 1913, Oskar Perron (1880–1975) gave a counterexample that boiled down to this (see e.g., Cajori, 1991, p. 370): for all $x \in X = \mathbb{N}^* = \{1, 2, 3, \ldots\}$, let $x' = x^2$ and $f(x) = x$. If $x \neq 1, f(x') = f(x^2) = x^2 > x = f(x)$; if the above argument applied, we would conclude that the greatest natural number is ... 1. Something is surreptitiously present in the reasoning that we are examining: the unproven hypothesis that *there is a maximum*. Its conclusion should therefore be rephrased as follows: 'If $x_0{}' = x_0$ and if, for all $x \in X \setminus \{x_0\}, f(x') > f(x)$, *and if there exists a maximum of f in X*, then this maximum is not reached at any point $x \neq x_0$ and therefore is reached at x_0.' The existence assumption is satisfied in the case of the n-tuples $(a_1, a_2,..., a_n)$ with constant sum S—we will return to this below—but it is obviously wrong in Perron's counter-example.

To illustrate the 'traditional' error of reasoning, of which, for decades, teachers and textbook authors seem to have been wholly unaware, here is an example taken from a French algebra exercise book for secondary education by a respected author, Philippe André (1877); this example concerns the 'dual' of Theorem θ:

891. *Find the minimum of $x + y + z \ldots$, when $xyz \ldots = a$.*
We have seen (*Course of Algebra*, no **346**) that the minimum takes place when all the factors are equal. Here is again a proof of this theorem. Let m be the minimum sought: we can write from the exercise statement.
[1] $x + y + z \ldots = m$
[2] $xyz \ldots = a$.
Assuming in [1] x and y to be unequal, we can replace, without changing the product, these factors by the equal factors $\sqrt{xy}$, $\sqrt{xy}$; and we will then have.
[3] $\sqrt{xy} + \sqrt{xy} + z \ldots = m$.

> But this second sum is smaller than the first; for (*Course*, no **343**) we have $\frac{x+y}{2} > \sqrt{xy}$ and therefore $\frac{2(x+y)}{2} > 2\sqrt{xy}$, or $x+y > \sqrt{xy}+\sqrt{xy}$. So the minimum sought will occur when all the factors are equal. (André, 1877, p. 358).

Here, the logical fallacy is placidly present in the very conclusion, '*So* the minimum sought will occur when all the factors are equal.'

3.3. *Cauchy's long-established but unheeded proof*

Surprisingly enough, Theorem θ had long been proved, in essence, by Augustin-Louis Cauchy (1789–1857) in his famous *Cours d'Analyse* [Course of Analysis] published in 1821. It unpretentiously resulted from his Theorem XVII, which is nothing other than the Geometric–Arithmetic Mean theorem, and which, in the case when the numbers A, B, C, D, $\ldots$, are not all equal, Cauchy stated as follows: $\sqrt[n]{ABCD\ldots} < \frac{A+B+C+D+\ldots}{n}$ (see Bradley & Sandifer, 2009, pp. 306–307).[8] This gives $\sqrt[n]{P} = \sqrt[n]{a_1 a_2 \ldots a_n} < \frac{a_1+a_2+\cdots+a_n}{n} = \frac{S}{n}$ and thus $P < \left(\frac{S}{n}\right)^n$. When $a_1 = a_2 =\ldots= a_n = \frac{S}{n}$, we have $P = \left(\frac{S}{n}\right)^n$, which proves that P reaches a maximum if, and only if, $a_1 = a_2 =\ldots= a_n$. However, this proof was ignored and the old secondary mathematics curriculum remained unchanged on this point. Why was it so?

3.4. *An old case trivialized*

As an institution, secondary mathematics education thus refused the breach of didactic contract that would have allowed it (and, if appropriate, its teachers and students) to *learn* something new and striking—namely, that the reasoning that had become routine was definitively flawed. Instead of strategy Σ_1 ('the institution *I* decides to ignore the problem and live with it'), it could have adopted strategy Σ_2 ('*I* seeks to solve the problem, solves it and integrates this solution in an appropriate manner into the means of its activity'), possibly by finally deciding simply to explicitly *admit* a mathematical result—Theorem θ—originating in the world of 'scholarly' mathematics. This, it seems, did not happen.

One of the reasons for this phenomenon is in all likelihood the following. Far from there being a divergence between 'scholarly' mathematics and secondary school mathematics, there was a relative harmony between them on the point at issue, insofar as many mathematicians were unable to see the mistake in a logical-mathematical erroneous conclusion that they themselves shared. Perron, in fact, had proposed his counterexample in relation to this kind of mistake, made in 1841 by the great geometer Jacob Steiner (1796–1863), an episode that Morris Kline (1972) recounts thus:

> Another interesting theme pursued in the nineteenth century was the solution of maximum and minimum problems by purely geometric methods, that is, without relying upon the calculus of variations. Of the several theorems Jacob Steiner proved by using synthetic methods, the most famous result is the isoperimetric theorem: Of all plane figures with a given perimeter the circle bounds

8 Note also that Natansón (1977, pp. 29–33) accurately presents Cauchy's proof. See as well 'Inequality of Arithmetic and Geometric Means' (2021, Proof by Cauchy Using Forward–Backward Induction section).

> the greatest area. Steiner gave various proofs. ... Unfortunately, Steiner assumed that there exists a curve that does have maximum area. Dirichlet tried several times to persuade him that his proofs were incomplete on that account but Steiner insisted that this was self-evident. Once, however, he did write (in the first of the 1842 papers): '... and the proof is readily made if one assumes that there is a largest figure.' (p. 838).

Therefore, if the secondary education institution did not attempt to learn more on this point—just like Steiner seems to have done—one can see that the breach of contract necessary for some learning to happen was not at all undemanding even for first-rate mathematicians.

The subsequent history of the secondary mathematics curriculum illustrates what we have called strategy Σ_3 ('where I ignores the existing solution and changes its system of means in a somewhat important way, thanks to the fact that a (necessarily different) system that eludes the problem turns out to be available at an acceptable cost'). The introduction into the curriculum of secondary education of elements of *mathematical analysis* was to change the means of solving elementary maxima and minima problems. Although the subtle conceptual system of analysis was difficult to master, it offered powerful tools to tackle the problems traditionally solved using Theorem θ. Just to summarize a process that spanned several decades, let us here give the floor to Griffiths and Hilton (1970), who state what they humorously call 'the *mostest* theorem' as follows:

> If $f: X \to \mathbb{R}$ is continuous, and X is compact, then f has a maximum and a minimum in X (i.e., there exist points $a, b \in X$ such that $f(a)$ is the greatest and $f(b)$ the least value of f). (p. 427).

If $X = \{(a_1, a_2,..., a_n) \in \mathbb{R}^n_+ \mid a_1 + a_2 +... + a_n = S\}$, and the function g is defined on $\mathbb{R}^n_+$ by $g(x) = g(a_1, a_2,..., a_n) = a_1 + a_2 +... + a_n$, we have $X = g^{-1}(S)$; the mapping g being continuous, X is a closed set (Griffiths & Hilton, 1970, p. 422, para. 25.6.11). As X is bounded (since $a_i \leq S$ for $i = 1, 2, \ldots, n$), X is compact. The function f being continuous, in accordance with the 'mostest' theorem, it attains a maximum in the compact set X, so that the product $P = f(x) = f(a_1, a_2,..., a_n) = a_1 a_2 ... a_n$ has a maximum on X, which was to be proved.

The availability of elementary topology thus trivialized a long open problem. Incidentally, let us emphasize that the brutal encounter of elementary algebra with issues at the heart of modern analysis had the long-term effect of locking elementary algebra into an essentially *formal* small world, while its original *functional* power was, it seems, largely forgotten (see Strømskag & Chevallard, 2021).

4. Discussion and conclusion

A key lesson can be drawn from the foregoing study. As the example of the addition of whole numbers illustrates, the mathematical content to be studied is very often problematic. Far from the idealized image of the teacher who 'knows' (and for whom the mathematics to be taught is not problematic), such a problematic aspect is to be sought already, most often, either in the current scholarly mathematics, or in the outcomes of the didactic transposition of scholarly mathematics. Somewhat unexpectedly, the analyses we have conducted highlight the role played by questions of existence and uniqueness of the mathematical entities manipulated, at different levels of mathematical activity. The belief in the existence and uniqueness seems to be a primordial, *a priori* conviction, which either comes up against mathematical reality (e.g., a quadratic equation generally has two solutions, not just one), or comes up against that constitutive requirement of mathematics and science, to prove what one is claiming, even implicitly. This requirement of proof was felt all the more widely as mathematical activity had more powerful tools for mathematically 'treating' intramathematical or extramathematical reality. This is particularly the case with the development of (elementary) algebra, whose power was artificially reduced by the process of didactic transposition in secondary curriculums since at least the beginning of the 20th century.[9] Even when reduced in this way, algebra retained a remarkable power, albeit based, as it was later discovered, on a paralogism, the existence of a maximum (or a minimum) being assumed in circumstances where it was by no means self-evident. The breach of didactic contract that would have resulted from the recognition and treatment of this paralogism seems never to have been clearly assumed in secondary education, before mathematical analysis imposed its invasive presence: secondary school then learned calculus and unlearned elementary algebra.

Let us conclude. In any institution, it happens in a rather unpredictable way that some members of the institution one day unveil what they believe to be a flaw in the institution's knowledge as manifested through its didactic contract. The institution's handling of such a crisis can make it possible for the institution to change its curriculum, or to block all changes, thereby sparing the institution a learning gain that would destabilize it for a more or less long period. To learn or not to learn, that is the question. Whether we like it or not, the answer to this question is built, at all levels, through the breaches of contract either accepted or refused, which are the driving force behind the institution's evolution. This is a pivotal phenomenon at a time when the big issue facing us is that of an uncompromising redefinition of mathematics curriculums.

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9 See again Strømskag and Chevallard (2021). We gave above, in connection with the intersection of a circle and a line, an illustration of the power of algebra in the form of a small algebraic work involving 5 parameters (h, k, a, b and r). For a sharper and prestigious illustration, the interested reader can see Descartes' letter to Princess Elisabeth of Bohemia dated 17 November 1643 (Shapiro, 2007, pp. 73–77).

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