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F. Wiener's Trick and an Extremal Problem for H^p

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Abstract

For $0 < p \leq \infty$, let H^p denote the classical Hardy space of the unit disc. We consider the extremal problem of maximizing the modulus of the kth Taylor coefficient of a function $f \in H^p$ which satisfies $\|f\|_{H^p} \leq 1$ and $f(0) = t$ for some $0 \leq t \leq 1$. In particular, we provide a complete solution to this problem for $k = 1$ and $0 < p < 1$. We also study F. Wiener's trick, which plays a crucial role in various coefficient-related extremal problems for Hardy spaces.

Keywords Hardy spaces · Extremal problems · Coefficient estimates

Mathematics Subject Classification Primary 30H10; Secondary 42A05

1 Introduction

Let H^p denote the classical Hardy space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that k is a positive integer. For $0 < p \leq \infty$ and $0 \leq t \leq 1$, consider the extremal problem

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$$\Phi_k(p,t) = \sup\left\{\operatorname{Re}\frac{f^{(k)}(0)}{k!} \,:\, \|f\|_{H^p} \leq 1 \text{ and } f(0) = t\right\}. \tag{1}$$

By a standard normal families argument, there are extremals $f \in H^p$ attaining the supremum in (1) for every $k \geq 1$ and every $0 \leq t \leq 1$. A general framework for a class of extremal problems for H^p which includes (1) has been developed by Havinson [8], Kabaila [9], Macintyre and Rogosinski [11] and Rogosinski and Shapiro [14]. A particular consequence of this theory is that the structure of the extremals is well-known (see Lemma 4 below).

For our extremal problem, it can be deduced directly from Parseval's identity that $\Phi_k(2,t) = \sqrt{1-t^2}$ and that the unique extremal is $f(z) = t + \sqrt{1-t^2}\,z^k$. Similarly, the Schwarz–Pick inequality (see e.g. [15, VII.17.3]) shows that $\Phi_1(\infty,t) = 1-t^2$ and that the unique extremal is $f(z) = (t+z)/(1+tz)$. This served as the starting point for Beneteau and Korenblum [1], who studied the extremal problem (1) in the range $1 \leq p \leq \infty$. We will enunciate their results in Sects. 4 and 5, but for now we present a brief account of their approach.

The first step in [1] is to compute $\Phi_1(p,t)$ and identify an extremal function. This is achieved by interpolating between the two cases $p = 2$ and $p = \infty$ mentioned above, facilitated by the inner-outer factorization of H^p functions. It follows from the argument that the extremal function thusly obtained is unique.

The second step in [1] is to show that $\Phi_k(p,t) = \Phi_1(p,t)$ for every $k \geq 2$ using a trick attributed to F. Wiener [2], which we shall now recall. Set $\omega_k = \exp(2\pi i/k)$ and suppose that $f(z) = \sum_{n\geq 0} a_n z^n$. F. Wiener's trick is based on the transform

$$W_k f(z) = \frac{1}{k}\sum_{j=0}^{k-1} f(\omega_k^j z) = \sum_{n=0}^{\infty} a_{kn} z^{kn}. \tag{2}$$

The triangle inequality yields that $\|W_k f\|_{H^p} \leq \|f\|_{H^p}$ for $f \in H^p$ if $1 \leq p \leq \infty$. Hence, if f_1 is an extremal function for $\Phi_1(p,t)$, then $f_k(z) = f_1(z^k)$ is an extremal function for $\Phi_k(p,t)$ and consequently $\Phi_k(p,t) = \Phi_1(p,t)$. Note that this argument does not guarantee that the extremal f_k is unique for $\Phi_k(p,t)$.

We are interested in the extremal problem (1) for $0 < p < 1$ and whether the extremal identified using F. Wiener's trick above for $1 \leq p \leq \infty$ is unique. We shall obtain the following general result, which may be of independent interest.

Theorem 1 *Fix $k \geq 2$ and suppose that $0 < p \leq \infty$. Let W_k denote the F. Wiener transform (2). The inequality*

$$\|W_k f\|_{H^p} \leq \max\left(k^{1/p-1}, 1\right)\|f\|_{H^p}$$

is sharp. Moreover, equality is attained if and only if

(a) $f \equiv 0$ *when* $0 < p < 1$,
(b) $W_k f = f$ *when* $1 < p < \infty$.

The upper bound in the estimate is easily deduced from the triangle inequality. Hence, the novelty of Theorem 1 is that the inequality is sharp for $0 < p < 1$, and the statements (a) and (b). In Sect. 3, we also present examples of functions in H^1 and H^∞ which attain equality in Theorem 1, but for which $W_k f \neq f$. However, we will conversely establish that if both f and $W_k f$ are inner functions, then $f = W_k f$.

To illustrate the role played by the F. Wiener transform in various coefficient related extremal problems, we first recall that the estimate $\|W_k f\|_\infty \leq \|f\|_\infty$ was originally used by F. Wiener to resolve a problem posed by Bohr [2] and compute the so-called Bohr radius for H^∞. We also know from [12, Sect. 1.7] that the Krzyż conjecture on the maximal magnitude of the kth coefficient in the power series expansion of a non-vanishing function with $\|f\|_\infty = 1$ is equivalent to the assertion that if f is an extremal for the corresponding extremal problem, then $f = W_k f$. As far as we are aware, the Krzyż conjecture remains open for $k \geq 6$.

Theorem 1 shows that the extremal for $\Phi_k(p,t)$ is unique when $1 < p < \infty$. We shall see in Sect. 5 that the extremal problem $\Phi_k(p,t)$ with $k \geq 2$ and $1 \leq p \leq \infty$ has a unique extremal except for when $p = 1$ and $0 \leq t < 1/2$.

In the range $0 < p < 1$ with $k = 1$, the extremal problem (1) has been studied by Connelly [4, Sect. 4], who resolved the problem in the cases $0 \leq t < 2^{-1/p}$ and $2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2} < t \leq 1$. Connelly also states conjectures on the behavior of $\Phi_1(p,t)$ in the range $2^{-1/p} \leq t \leq 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$. The conjectures are based on numerical analysis (see [4, Sect. 5]).

In Sect. 4, we will extend Connelly's result to the full range $0 \leq t \leq 1$. Our result demonstrates that for each $0 < p < 1$ there is a unique $0 < t_p < 1/2$ such that the extremal for $\Phi_1(p, t_p)$ is not unique, thereby confirming the above-mentioned conjectures.

Brevig and Saksman [3] have recently studied the extremal problem

$$\Psi_k(p) = \sup\left\{\operatorname{Re}\frac{f^{(k)}(0)}{k!} \,:\, \|f\|_{H^p} \leq 1\right\}$$

for $0 < p < 1$. It is observed in [3, Sect. 5.3] that $\Psi_k(p) = \max_{0\leq t\leq 1}\Phi_k(p,t)$. In particular, the maxima of $\Phi_1(p,t)$ for $0 \leq t \leq 1$ is

$$\Psi_1(p) = \left(1 - \frac{p}{2}\right)^{1/p}\frac{2}{\sqrt{p(2-p)}}$$

and this is attained for $t = (1-p/2)^{1/p}$. From the main result in [1], it is easy to see that $t \mapsto \Phi_1(p,t)$ is a decreasing function from $\Phi_1(p,0) = 1$ to $\Phi_1(p,1) = 0$ when $1 \leq p \leq \infty$. Similarly, our main result shows that $\Phi_1(p,t)$ is increasing from $\Phi_1(p,0) = 1$ to the maxima mentioned above, then decreasing to $\Phi_1(p,1) = 0$. Figure 1 contains the plot of $t \mapsto \Phi_1(p,t)$ for several values $0 < p \leq \infty$, which illustrates this difference between $0 < p < 1$ and $1 \leq p \leq \infty$.

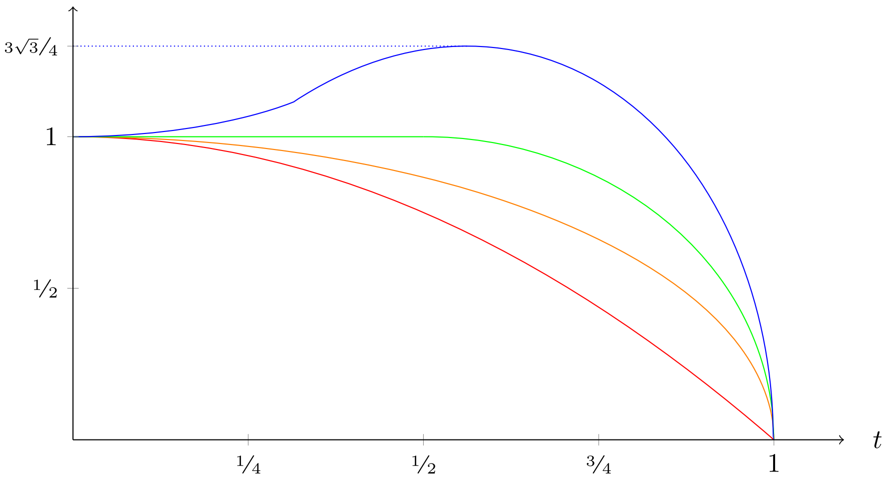

Fig. 1 Plot of the curves $t \mapsto \Phi_1(p,t)$ for $p = 1/2$, $p = 1$, $p = 2$ and $p = \infty$

Another difference between $0 < p < 1$ and $1 \leq p \leq \infty$ appears when we consider $k \geq 2$. Recall that in the latter case, we have $\Phi_k(p,t) = \Phi_1(p,t)$ for every $k \geq 2$ and every $0 \leq t \leq 1$. In the former case, we only get from Theorem 1 that

$$\Phi_1(p,t) \leq \Phi_k(p,t) \leq k^{1/p-1}\Phi_1(p,t). \tag{3}$$

Theorem 1 also shows that the upper bound in (3) is attained if and only if $t = 1$, since trivially $\Phi_1(p,1) = 0$ for every $0 < p \leq \infty$. However, by adapting an example due to Hardy and Littlewood [7], it is easy to see that if $0 < p < 1$ and $0 \leq t < 1$ are fixed, then the exponent $1/p - 1$ in (3) cannot be improved as $k \to \infty$. In the final section of the paper, we present some evidence that the lower bound in (3) can be attained for sufficiently large t, if $k \geq 2$ and $0 < p < 1$ are fixed.

Organization

The present paper is organized into five additional sections and one appendix. In Sect. 2, we collect some preliminary results pertaining to H^p and the structure of extremals for $\Phi_k(p,t)$. Section 3 is devoted to F. Wiener's trick and the proof of Theorem 1. A complete solution to the extremal problem $\Phi_1(p,t)$ for $0 < p \leq \infty$ and $0 \leq t \leq 1$ is presented in Sect. 4. In Sect. 5, we consider $\Phi_k(p,t)$ for $k \geq 2$ and $1 \leq p \leq \infty$ and study when the extremal is unique. Section 6 contains some remarks on $\Phi_k(p,t)$ for $k \geq 2$ and $0 < p < 1$. "Appendix A" contains the proof of a crucial lemma needed to resolve the extremal problem $\Phi_1(p,t)$ for $0 < p < 1$.

2 Preliminaries

Recall that for $0 < p < \infty$, the Hardy space H^p consists of the analytic functions f in $\mathbb{D}$ for which the limit of integral means

$$\|f\|_{H^p}^p = \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi}$$

is finite. H^∞ is the space of bounded analytic functions in $\mathbb{D}$, endowed with the norm $\|f\|_{H^\infty} = \sup_{|z|<1} |f(z)|$. It is well-known (see e.g. [6]) that H^p is a Banach space when $1 \leq p \leq \infty$ and a quasi-Banach space when $0 < p < 1$.

In the Banach space range $1 \leq p \leq \infty$, the triangle equality is

$$\|f + g\|_{H^p} \leq \|f\|_{H^p} + \|g\|_{H^p}. \tag{4}$$

The Hardy space H^p is strictly convex when $1 < p < \infty$, which means that it is impossible to attain equality in (4) unless $g \equiv 0$ or $f = \lambda g$ for a non-negative constant λ. H^p is not strictly convex for $p = 1$ and $p = \infty$, so in this case there are other ways to attain equality in (4). In the range $0 < p < 1$, the triangle inequality takes the form

$$\|f + g\|_{H^p}^p \leq \|f\|_{H^p}^p + \|g\|_{H^p}^p, \tag{5}$$

so here H^p is not even locally convex [5]. Our first goal is to establish that the triangle inequality (5) is not attained unless $f \equiv 0$ or $g \equiv 0$. This result is probably known to experts, but we have not found it in the literature.

If $f \in H^p$ for some $0 < p \leq \infty$, then the boundary limit function

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) \tag{6}$$

exists for almost every θ. Moreover, $f^* \in L^p = L^p([0, 2\pi])$ and

$$\|f\|_{H^p} = \|f^*\|_{L^p} = \left(\int_0^{2\pi} \left| f^*(e^{i\theta}) \right|^p \, \frac{d\theta}{2\pi} \right)^{1/p}$$

if $0 < p < \infty$ and $\|f\|_{H^\infty} = \operatorname{ess\,sup}_\theta |f^*(e^{i\theta})|$. For simplicity, we henceforth omit the asterisk and write $f^* = f$ with the limit (6) in mind.

Lemma 2 *Fix $0 < p < 1$ and suppose that $f, g \in H^p$. If*

$$\|f + g\|_{H^p}^p = \|f\|_{H^p}^p + \|g\|_{H^p}^p$$

then either $f \equiv 0$ or $g \equiv 0$.

Proof We begin by looking at equality in the triangle inequality for L^p in the range $0 < p < 1$. Here we have

$$\begin{aligned}\|f+g\|_{L^p}^p &= \int_0^{2\pi} \left|f(e^{i\theta})+g(e^{i\theta})\right|^p \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} |f(e^{i\theta})|^p + |g(e^{i\theta})|^p \frac{d\theta}{2\pi} = \|f\|_{L^p}^p + \|g\|_{L^p}^p.\end{aligned}$$

We used the elementary estimate $|z+w|^p \leq |z|^p + |w|^p$ for complex numbers z, w and $0 < p < 1$. It is easily verified that this estimate is attained if and only if $zw = 0$. Consequently,

$$\|f+g\|_{L^p}^p = \|f\|_{L^p}^p + \|g\|_{L^p}^p$$

if and only if $f(e^{i\theta})g(e^{i\theta}) = 0$ for almost every θ. It is well-known (see [6, Thm. 2.2]) that the only function $h \in H^p$ whose boundary limit function (6) vanishes on a set of positive measure is $h \equiv 0$. Hence we conclude that either $f \equiv 0$ or $g \equiv 0$. □

Let us next establish a standard result on the structure of the extremals for the extremal problem (1). The first step is the following basic result.

Lemma 3 *If $f \in H^p$ is extremal for $\Phi_k(p,t)$, then $\|f\|_{H^p} = 1$.*

Proof Suppose that $f \in H^p$ is extremal for $\Phi_k(p,t)$ but that $\|f\|_{H^p} < 1$. For $\varepsilon > 0$, set $g(z) = f(z) + \varepsilon z^k$. Note that $g(0) = f(0) = t$ for any $\varepsilon > 0$. If $1 \leq p \leq \infty$, then

$$\|g\|_{H^p} \leq \|f\|_{H^p} + \varepsilon < 1$$

for sufficiently small $\varepsilon > 0$. If $0 < p < 1$, then

$$\|g\|_{H^p}^p \leq \|f\|_{H^p}^p + \varepsilon^p < 1,$$

again for sufficiently small $\varepsilon > 0$, so $\|g\|_{H^p} < 1$. In both cases we find that

$$\frac{g^{(k)}(0)}{k!} = \frac{f^{(k)}(0)}{k!} + \varepsilon,$$

which contradicts the extremality of f for $\Phi_k(p,t)$. □

Let $(n_j)_{j=1}^k$ denote a sequence of distinct non-negative integers and let $(w_j)_{j=1}^k$ denote a sequence of complex numbers. A special case of the Carathéodory–Fejér problem is to determine the infimum of $\|f\|_{H^p}$ over all $f \in H^p$ which satisfy

$$\frac{f^{(n_j)}}{n_j!}(0) = w_j, \tag{7}$$

for $j = 1, \ldots, k$. Set $k = \max_{1\le j\le k} n_j$. If f is an extremal for the Carathéodory–Fejér problem (7), then there are complex numbers $|\lambda_j| \le 1$ for $j = 1, \ldots, k$ and a constant C such that

$$f(z) = C \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \prod_{j=1}^{k} \left(1 - \overline{\lambda_j} z\right)^{2/p} \tag{8}$$

for some $0 \le l \le k$, and the strict inequality $|\lambda_j| < 1$ holds for $0 < j \le l$. In (8) and in similar formulas to follow, we adopt the convention that in the case $l = 0$ the first product is empty and considered to be equal to 1.

For $1 \le p \le \infty$, this result is independently due to Macintyre and Rogosinski [11] and Havinson [8], while in the range $0 < p < 1$ the result is due to Kabaila [9]. An exposition of these results can be found in [6, Ch. 8] and [10, pp. 82–85], respectively.

Using Lemma 3, we can establish that the extremals of the extremal problem $\Phi_k(p, t)$ have to be of the same form.

Lemma 4 *If $f \in H^p$ is extremal for $\Phi_k(p, t)$, then there are complex numbers $|\lambda_j| \le 1$ for $j = 1, \ldots, k$ and a constant C such that*

$$f(z) = C \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \prod_{j=1}^{k} \left(1 - \overline{\lambda_j} z\right)^{2/p}.$$

for some $0 \le l \le k$, and the strict inequality $|\lambda_j| < 1$ holds for $0 < j \le l$.

Proof Suppose that f is extremal for $\Phi_k(p, t)$ and consider the Carathéodory–Fejér problem with conditions

$$f(0) = t \qquad \text{and} \qquad \frac{f^{(k)}(0)}{k!} = \Phi_k(p, t). \tag{9}$$

We claim that f is an extremal for the Carathéodory–Fejér problem (9). If it is not, then there must be some $f \in H^p$ with $\|f\|_{H^p} < 1$ which satisfies (9). However, this contradicts Lemma 3. Hence the extremal is of the stated form by (8). □

3 F. Wiener's Trick

Recall from (2) that if $f(z) = \sum_{n\ge 0} a_n z^n$ and $\omega_k = \exp(2\pi i/k)$, then

$$W_k f(z) = \frac{1}{k} \sum_{j=0}^{k-1} f(\omega_k^j z) = \sum_{n=0}^{\infty} a_{kn} z^{kn}.$$

We begin by giving two examples showing that $\|W_k f\|_{H^p} = \|f\|_{H^p}$ may occur for f such that $W_k f \ne f$ when $p = 1$ or $p = \infty$.

Example 5 Let $k \geq 2$ and consider $f(z) = (1+z)^{2k}$ in H^1. By the binomial theorem, we find that

$$f(z) = \sum_{n=0}^{2k} \binom{2k}{n} z^n,$$
$$W_k f(z) = 1 + \binom{2k}{k} z^k + z^{2k}.$$

Note that $f \neq W_k f$ since $k \geq 2$. By another application of the binomial theorem and a well-known identity for the central binomial coefficient, we find that

$$\|f\|_{H^1} = \|f^{1/2}\|_{H^2}^2 = \sum_{n=0}^{k} \binom{k}{n}^2 = \binom{2k}{k}.$$

Moreover,

$$\binom{2k}{k} = \int_0^{2\pi} W_k f(e^{i\theta})\,\overline{e^{ik\theta}}\,\frac{d\theta}{2\pi} \leq \|W_k f\|_{H^1}$$

by the triangle inequality. Hence

$$\binom{2k}{k} \leq \|W_k f\|_{H^1} \leq \|f\|_{H^1} = \binom{2k}{k},$$

so $\|W_k f\|_{H^1} = \|f\|_{H^1}$.

Example 6 Let $k \geq 2$ and consider $f(z) = (1+z^k)^2 - z(1-z^k)^2$ in H^∞. It is clear that $W_k f(z) = (1+z^k)^2 \neq f(z)$ since $k \geq 2$. Moreover $\|W_k f\|_{H^\infty} = 4$. The supremum is attained for $z = \omega_k^j$ for $j = 0, 1, \ldots, k-1$. We next compute

$$f(e^{i\theta}) = \left(1 + e^{ik\theta}\right)^2 - e^{i\theta}\left(1 - e^{ik\theta}\right)^2 = 4e^{ik\theta}\left(\cos^2\left(\frac{k\theta}{2}\right) + e^{i\theta}\sin^2\left(\frac{k\theta}{2}\right)\right).$$

Consequently, $\|f\|_{H^\infty} = 4$ and here the supremum is attained for $z = \omega_{2k}^j$ for $j = 0, 1, \ldots, 2k-1$.

Proof of Theorem 1 It follows from the triangle inequality (4) that

$$\|W_k f\|_{H^p} \leq \|f\|_{H^p} \tag{10}$$

for every $f \in H^p$ if $1 \leq p \leq \infty$. In the range $0 < p < 1$, we get from the triangle inequality (5) the estimate

$$\|W_k f\|_{H^p} \leq k^{1/p-1}\|f\|_{H^p} \tag{11}$$

for every $f \in H^p$. Combining (10) and (11), we have established that

$$\|W_k f\|_{H^p} \le \max\left(k^{1/p-1}, 1\right) \|f\|_{H^p}.$$

This is trivially attained for $f(z) = z^k$ when $1 \le p \le \infty$. We need to show that the upper bound $k^{1/p-1}$ cannot be improved when $0 < p < 1$ to finish proof of the first part of the theorem.

Let $\varepsilon > 0$ and consider $f_\varepsilon(z) = (z - (1+\varepsilon))^{-1/p}$. Clearly $\|f_\varepsilon\|_{H^p} \to \infty$ as $\varepsilon \to 0^+$. Moreover

$$\begin{aligned}
\|f_\varepsilon\|_{H^p}^p &= \int_0^{2\pi} \frac{1}{|e^{i\theta} - (1+\varepsilon)|} \frac{d\theta}{2\pi} \\
&\le \int_{|\theta|<\pi/k} \frac{1}{|e^{i\theta} - (1+\varepsilon)|} \frac{d\theta}{2\pi} + \int_{|\theta|\ge\pi/k} \frac{6}{\theta^2} \frac{d\theta}{2\pi} \\
&\le \int_{|\theta|<\pi/k} \frac{1}{|e^{i\theta} - (1+\varepsilon)|} \frac{d\theta}{2\pi} + \frac{6k}{\pi^2},
\end{aligned}$$

from which we conclude that

$$\|f_\varepsilon\|_{H^p}^p = \int_{|\theta|<\pi/k} \frac{1}{|e^{i\theta} - (1+\varepsilon)|} \frac{d\theta}{2\pi} + O(1). \tag{12}$$

Furthermore,

$$\begin{aligned}
\|W_k f_\varepsilon\|_{H^p}^p &= \sum_{j=0}^{k-1} \int_{|\theta-2\pi j/k|<\pi/k} \left|\sum_{l=0}^{k-1} \frac{f_\varepsilon\left(e^{i(\theta+2\pi l/k)}\right)}{k}\right|^p \frac{d\theta}{2\pi} \\
&\ge k^{-p} \sum_{j=0}^{k-1} \left(\int_{|\theta-2\pi j/k|<\pi/k} \left|f_\varepsilon\left(e^{i(\theta+2\pi j/k)}\right)\right|^p \frac{d\theta}{2\pi} - \frac{6k^2}{\pi^2}\right) \\
&= k^{-p+1} \int_{|\theta|<\pi/k} \frac{1}{|e^{i\theta} - (1+\varepsilon)|} \frac{d\theta}{2\pi} - \frac{6k^{-p+3}}{\pi^2}.
\end{aligned}$$

By (12) we find that

$$\lim_{\varepsilon\to 0^+} \frac{\|W_k f_\varepsilon\|_{H^p}^p}{\|f_\varepsilon\|_{H^p}^p} \ge k^{1-p}.$$

Hence, the constant $k^{1/p-1}$ in (11) cannot be replaced by any smaller quantity.

We next want to show that (a) and (b) holds. For a function $f \in H^p$, define $f_j(z) = f(\omega_k^j z)$ for $j = 0, 1, \ldots, k-1$ and recall that $\|f\|_{H^p} = \|f_j\|_{H^p}$.

We begin with (a). Suppose that $\|W_k f\|_{H^p} = k^{1/p-1}\|f\|_{H^p}$, which we can reformulate as

$$\|f_0 + f_1 + \cdots + f_{k-1}\|_{H^p}^p = \|f_0\|_{H^p}^p + \|f_1\|_{H^p}^p + \cdots + \|f_{k-1}\|_{H^p}^p.$$

By Lemma 2, the triangle inequality can be attained if and only if at least $k-1$ of the k functions f_j are identically equal to zero. Evidently this is possible if and only if $f \equiv 0$.

For (b), we suppose that $f \in H^p$ is such that $\|W_k f\|_{H^p} = \|f\|_{H^p}$. We need to prove that $W_k f = f$. If $f \equiv 0$ there is nothing to do. As in the proof of (a), we note that $\|W_k f\|_{H^p} = \|f\|_{H^p}$ can be reformulated as

$$\|f_0 + f_1 + \cdots + f_{k-1}\|_{H^p} = \|f_0\|_{H^p} + \|f_1\|_{H^p} + \cdots + \|f_{k-1}\|_{H^p}.$$

Viewing H^p as a subspace of L^p, the strict convexity of the latter implies that there are non-negative constants λ_j for $j = 1, 2, \ldots, k-1$ such that

$$f = f_0 = \lambda_1 f_1 = \ldots = \lambda_{k-1} f_{k-1}.$$

We shall only look at $f = \lambda_1 f_1$ which for $f(z) = \sum_{n\geq 0} a_n z^n$ is equivalent to

$$\sum_{n=0}^{\infty} a_n z^n = \lambda_1 \sum_{n=0}^{\infty} a_n \omega_k^n z^n.$$

Using W_k on this identity we get

$$\sum_{n=0}^{\infty} a_{kn} z^{kn} = \lambda_1 \sum_{n=0}^{\infty} a_{kn} z^{kn}.$$

This is only possible if $\lambda_1 = 1$ or $W_k f \equiv 0$. The latter implies that $f \equiv 0$ since $\|W_k f\|_{H^p} = \|f\|_{H^p}$ by assumption. Therefore we can restrict our attention to the case $\lambda_1 = 1$. For all integers n that are not a multiple of k, we now find that

$$a_n = \lambda_1 \omega_k^n a_n \qquad \Longrightarrow \qquad a_n = 0,$$

since $\lambda_1 = 1$ and $\omega_k^n \neq 1$. Hence $W_k f = f$ as desired. □

Recall that a function $f \in H^p$ is called inner if $|f(e^{i\theta})| = 1$ for almost every θ. We shall require the following simple result later on.

Lemma 7 *If both f and $W_k f$ are inner functions, then $f = W_k f$.*

Proof Since $|W_k f(e^{i\theta})| = |f(e^{i\theta})| = 1$ for almost every θ, we get from (2) that

$$1 = \left|W_k f(e^{i\theta})\right| = \left|\frac{1}{k}\sum_{j=0}^{k-1} f_j(e^{i\theta})\right| = \frac{1}{k}\sum_{j=0}^{k-1} |f_j(e^{i\theta})|, \tag{13}$$

where $f_j(z) = f(\omega_k^j z)$. The equality on the right hand side of (13) is possible if and only if

$$f(e^{i\theta}) = f_1(e^{i\theta}) = \ldots = f_{k-1}(e^{i\theta})$$

for almost every θ. As in the proof of Theorem 1 (b), we find that $f = W_k f$. □

4 The Extremal Problem $\Phi_1(p, t)$ for $0 < p \leq \infty$

In the present section, we resolve the extremal problem (1) in the case $k = 1$ completely. We begin with the case $1 \leq p \leq \infty$ which has been solved by Beneteau and Korenblum [1]. We give a different proof of their result based on Lemma 4, mainly to illustrate the differences between the cases $0 < p < 1$ and $1 \leq p \leq \infty$.

Theorem 8 (Beneteau–Korenblum) *Fix $1 \leq p \leq \infty$ and consider (1) with $k = 1$.*

(i) *If $0 \leq t < 2^{-1/p}$, let α denote the unique real number in the interval $0 \leq \alpha < 1$ such that $t = \alpha(1+\alpha^2)^{-1/p}$. Then*

$$\Phi_1(p, t) = \frac{1}{\left(1+\alpha^2\right)^{1/p}}\left(1 + \left(\frac{2}{p} - 1\right)\alpha^2\right),$$

and the unique extremal is

$$f(z) = \frac{\alpha + z}{1 + \alpha z}\frac{(1+\alpha z)^{2/p}}{\left(1+\alpha^2\right)^{1/p}}.$$

(ii) *If $2^{-1/p} \leq t \leq 1$, let β denote the unique real number in the interval $0 \leq \beta \leq 1$ such that $t = (1+\beta^2)^{-1/p}$. Then*

$$\Phi_1(p, t) = \frac{1}{\left(1+\beta^2\right)^{1/p}}\frac{2\beta}{p},$$

and the unique extremal is

$$f(z) = \frac{(1+\beta z)^{2/p}}{\left(1+\beta^2\right)^{1/p}}.$$

Proof Note that since $k = 1$, there are only two possibilities for the extremals in Lemma 4. They are

$$f_1(z) = \frac{\alpha + z}{1 + \alpha z}\frac{(1+\alpha z)^{2/p}}{\left(1+\alpha^2\right)^{1/p}}, \qquad 0 \leq \alpha < 1, \tag{14}$$

$$f_2(z) = \frac{(1+\beta z)^{2/p}}{\left(1+\beta^2\right)^{1/p}}, \qquad 0 \leq \beta \leq 1. \tag{15}$$

Here we have made $\alpha, \beta \geq 0$ by rotations. Note that if $p = \infty$, then f_2 does not depend on β. Moreover,

$$t = f_1(0) = \frac{\alpha}{(1+\alpha^2)^{1/p}}, \tag{16}$$

$$t = f_2(0) = \frac{1}{(1+\beta^2)^{1/p}}. \tag{17}$$

For $1 \leq p \leq \infty$ it is easy to verify that the function

$$\alpha \mapsto \frac{\alpha}{(1+\alpha^2)^{1/p}} \tag{18}$$

is strictly increasing on $0 \leq \alpha < 1$ and maps $[0, 1)$ to $[0, 2^{-1/p})$. Similarly, for $1 \leq p < \infty$ we find that the function

$$\beta \mapsto \frac{1}{(1+\beta^2)^{1/p}} \tag{19}$$

is strictly decreasing on $0 \leq \beta \leq 1$ and maps $[0, 1]$ to $[2^{-1/p}, 1]$. Consequently, if $0 \leq t < 2^{-1/p}$, then the unique extremal is (14) with α given by (16), and if $2^{-1/p} \leq t \leq 1$, then the unique extremal is (15) with β given by (17). The proof is completed by computing

$$f_1'(0) = \frac{1}{\left(1+\alpha^2\right)^{1/p}} \left(1 + \alpha^2\left(\frac{2}{p} - 1\right)\right) = t\left(\frac{1}{\alpha} + \alpha\left(\frac{2}{p} - 1\right)\right), \tag{20}$$

$$f_2'(0) = \frac{1}{(1+\beta^2)^{1/p}} \frac{2\beta}{p} = t\frac{2\beta}{p}, \tag{21}$$

to obtain the stated expressions for $\Phi_1(p, t)$ in (i) and (ii), respectively. □

Define α and β as functions of t implicitly through (16) and (17). Then α is increasing on $0 \leq t < 2^{-1/p}$ and β is decreasing on $2^{-1/p} \leq t \leq 1$. Inspecting the left hand side of (20) and (21), we extract the following result.

Corollary 9 *If* $1 \leq p \leq \infty$, *then the function* $t \mapsto \Phi_1(p, t)$ *is decreasing and takes the values* $[0, 1]$.

In the range $0 < p < 1$ a more careful analysis is required. This is due to the fact that the function (18) is increasing on the interval $0 \leq \alpha \leq \alpha_2$ and decreasing on the interval $\alpha_2 \leq \alpha < 1$, where

$$\alpha_2 = \sqrt{\frac{p}{2-p}}. \tag{22}$$

Inspecting (16), we conclude that for each $2^{-1/p} < t < 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$ there are two possible α-values which give the same $t = f_1(0)$. Let α_1 denote the unique real number in the interval (0, 1) such that

$$1 + \alpha_1^2 = 2\alpha_1^p. \tag{23}$$

Note that α_1 gives the value $t = 2^{-1/p}$ in (16).

Lemma 10 *If $\alpha_1 < \alpha < \alpha_2$ and $\alpha_2 < \widetilde{\alpha} < 1$ produce the same $t = f_1(0)$ in (16), then the quantity $f_1'(0)$ from (20) is maximized by α.*

Proof Since α and $\widetilde{\alpha}$ give the same $t = f_1(0)$ in (20), we only need to prove that

$$\frac{1}{\alpha} + \frac{\alpha}{\alpha_2^2} > \frac{1}{\widetilde{\alpha}} + \frac{\widetilde{\alpha}}{\alpha_2^2}. \tag{24}$$

Fix $\alpha_1 < \alpha < \alpha_2$. The unique number $\alpha_2 < \xi < 1$ such that

$$\frac{1}{\alpha} + \frac{\alpha}{\alpha_2^2} = \frac{1}{\xi} + \frac{\xi}{\alpha_2^2}$$

is $\xi = \alpha_2^2/\alpha$. Since the function

$$x \mapsto \frac{1}{x} + \frac{x}{\alpha_2^2}$$

is increasing for $x > \alpha_2$ it is sufficient to prove that $\xi > \widetilde{\alpha}$ to obtain (24). Since

$$x \mapsto \frac{x}{(1+x^2)^{1/p}}$$

is decreasing for $x > \alpha_2$, we see that $\xi > \widetilde{\alpha}$ if and only if

$$\frac{\widetilde{\alpha}}{(1+\widetilde{\alpha}^2)^{1/p}} > \frac{\xi}{(1+\xi^2)^{1/p}} \qquad \Longleftrightarrow \qquad \frac{\alpha}{(1+\alpha^2)^{1/p}} > \frac{\frac{\alpha_2^2}{\alpha}}{\left(1+\left(\frac{\alpha_2^2}{\alpha}\right)^2\right)^{1/p}}.$$

Here we used that α and $\widetilde{\alpha}$ give the same $t = f_1(0)$ in (16) on the left hand side and the identity $\xi = \alpha_2^2/\alpha$ on the right hand side. We now substitute $\alpha = \alpha_2\sqrt{x}$ for $0 < x < 1$ to obtain the equivalent inequality

$$\frac{x}{(1+\alpha_2^2 x)^{1/p}} > \frac{1}{\left(1+\frac{\alpha_2^2}{x}\right)^{1/p}}. \tag{25}$$

Actually, we only need to consider $(\alpha_1/\alpha_2)^2 < x < 1$, but the same proof works for $0 < x < 1$. We raise both sides of (25) to the power p, multiply by x^{1-p} and rearrange to get the equivalent inequality $F(x) > 0$ where

$$F(x) = \left(x - x^{1-p}\right) + \alpha_2^2 \left(1 - x^{2-p}\right).$$

Recalling that $\alpha_2^2 = p/(2-p)$, we compute

$$F'(x) = \left(1 - (1-p)x^{-p}\right) - px^{1-p} \quad \text{and} \quad F''(x) = p(1-p)x^{-p-1} - p(1-p)x^{-p}.$$

Since $F(1) = F'(1) = 0$, we get from Taylor's theorem that for every $0 < x < 1$ there is some $x < \eta < 1$ such that

$$F(x) = \frac{F''(\eta)}{2}(x-1)^2 = \frac{p(1-p)}{2}\eta^{-p}\left(\eta^{-1} - 1\right)(x-1)^2 > 0,$$

which completes the proof. □

By Lemma 10, we now only need to compare $f_1'(0)$ from (20) for $\alpha_1 \leq \alpha \leq \alpha_2$ with $f_2'(0)$ from (21) for β such that $f_1(0) = t = f_2(0)$. Inspecting (16) and (17), we find that

$$\frac{\alpha}{\left(1+\alpha^2\right)^{1/p}} = \frac{1}{\left(1+\beta^2\right)^{1/p}} \qquad \Longleftrightarrow \qquad \beta = \sqrt{\frac{1+\alpha^2}{\alpha^p} - 1}. \tag{26}$$

Next, we consider the equation $f_1'(0) = f_2'(0)$ with β as in (26). Inspecting (20) and (21) and dividing by t, we get the equation

$$\frac{1}{\alpha} + \alpha\left(\frac{2}{p} - 1\right) = \frac{2\beta}{p} = \frac{2}{p}\sqrt{\frac{1+\alpha^2}{\alpha^p} - 1}. \tag{27}$$

We square both sides, multiply by p^2 and rearrange to find that (27) is equivalent to the equation $F_p(\alpha) = 0$, where

$$F_p(\alpha) = p^2\alpha^{-2} + 2p(2-p) + (2-p)^2\alpha^2 - 4\left(\alpha^{-p} + \alpha^{2-p} - 1\right). \tag{28}$$

Suppose that $\alpha_1 \leq \alpha \leq \alpha_2$. If

- $F_p(\alpha) > 0$, then f_1 from (14) is the unique extremal for $\Phi_1(p, t)$.
- $F_p(\alpha) = 0$, then f_1 from (14) and f_2 from (15) are extremals for $\Phi_1(p, t)$.
- $F_p(\alpha) < 0$, then f_2 from (15) is the unique extremal for $\Phi_1(p, t)$.

Note that any solutions of $F_p(\alpha) = 0$ with $0 < \alpha < \alpha_1$ are of no interest since this implies that $\beta > 1$ by (26). Similarly, any solutions of $F_p(\alpha) = 0$ with $\alpha_2 < \alpha < 1$ can be ignored due to Lemma 10. The following result shows that there is only one solution, which is in the pertinent range.

Lemma 11 *Let F_p be as in (28). The equation $F_p(\alpha) = 0$ has a unique solution, denoted α_p, on the interval $(0, 1)$. Moreover,*

(a) *if $0 < \alpha < \alpha_p$, then $F_p(\alpha) > 0$.*
(b) *if $\alpha_p < \alpha < 1$, then $F_p(\alpha) < 0$.*
(c) *$\alpha_1 < \alpha_p < \alpha_2$ where α_1 and α_2 are from (23) and (22), respectively.*

The proof of Lemma 11 is a rather laborious calculus exercise, which we postpone to "Appendix A" below. Let α_p be as in Lemma 11 and define

$$t_p = \frac{\alpha_p}{\left(1+\alpha_p^2\right)^{1/p}}. \tag{29}$$

Note that $2^{-1/p} < t_p < 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$ by the fact that $\alpha_1 < \alpha_p < \alpha_2$. By the analysis above, Lemma 10 and Lemma 11, we obtain the following version of Theorem 8 in the range $0 < p < 1$.

Theorem 12 *Fix $0 < p < 1$ and consider (1) with $k = 1$. Let t_p be as in (29) and set $\alpha_2 = \sqrt{p/(2-p)}$.*

(i) *If $0 \le t \le t_p$, let α denote the unique real number in the interval $0 \le \alpha < \alpha_2$ such that $t = \alpha(1+\alpha^2)^{-1/p}$. Then*

$$\Phi_1(p,t) = \frac{1}{\left(1+\alpha^2\right)^{1/p}}\left(1+\left(\frac{2}{p}-1\right)\alpha^2\right),$$

and an extremal is

$$f(z) = \frac{\alpha+z}{1+\alpha z}\frac{(1+\alpha z)^{2/p}}{\left(1+\alpha^2\right)^{1/p}}.$$

(ii) *If $t_p \le t \le 1$, let β denote the unique real number in the interval $0 \le \beta \le 1$ such that $t = (1+\beta^2)^{-1/p}$. Then*

$$\Phi_1(p,t) = \frac{1}{\left(1+\beta^2\right)^{1/p}}\frac{2\beta}{p},$$

and an extremal is

$$f(z) = \frac{(1+\beta z)^{2/p}}{\left(1+\beta^2\right)^{1/p}}.$$

The extremals are unique for $0 \le t \ne t_p \le 1$. The only extremals for $\Phi_1(p, t_p)$ are the functions given in (i) and (ii).

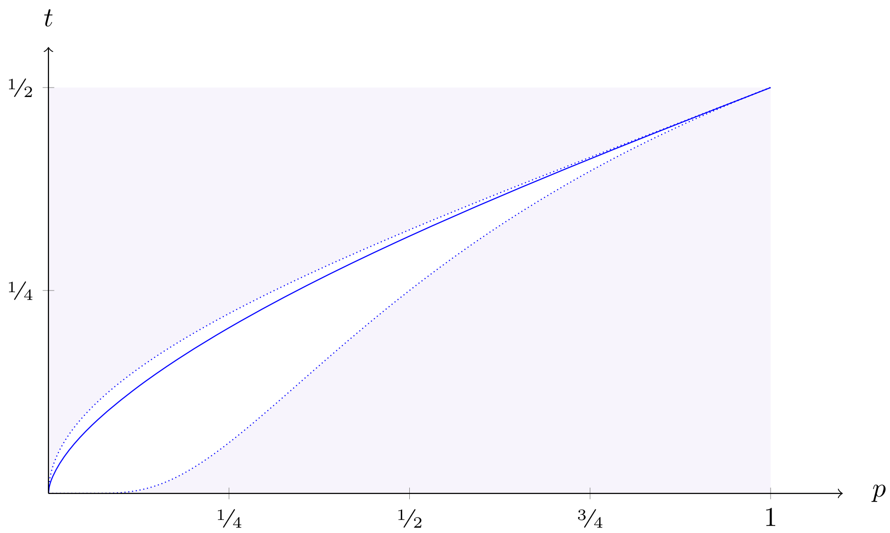

Fig. 2 Plot of the curve $p \mapsto t_p$. Points (p, t) above and below the curve correspond to the cases (i) and (ii) of Theorem 12, respectively. The estimates $2^{-1/p} < t_p < 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$ are represented by dotted curves. In the shaded area and in the range $1/2 \le t \le 1$, Theorem 12 is originally due to Connelly [4]

Theorem 12 extends [4, Theorem 4.1] to general $0 \le t \le 1$. The analysis in [4] is similar to ours, and we are able to also identify the extremals in the range

$$2^{-1/p} \le t \le 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$$

due to Lemma 10 and Lemma 11. It is also demonstrated in [4, Thm. 4.1] that when $p = 1/2$ there must exist at least one value of $0 < t < 1$ for which the extremal is not unique. Theorem 12 shows that there is precisely one such t and that this observation is not specific to $p = 1/2$, but in fact holds for any $0 < p < 1$. Figure 2 shows the value t_p for which the extremal is not unique as a function of p.

Inspecting Theorem 12, we get the following result similarly to how we extracted Corollary 9 from Theorem 8.

Corollary 13 *If* $0 < p < 1$, *then the function* $t \mapsto \Phi_1(p, t)$ *is increasing from* $\Phi_1(p, 0) = 1$ *to*

$$\Phi_1\left(p, \left(1 - \frac{p}{2}\right)^{1/p}\right) = \left(1 - \frac{p}{2}\right)^{1/p} \frac{2}{\sqrt{p(2-p)}}$$

and then decreasing to $\Phi_1(p, 1) = 0$.

5 The Extremal Problem $\Phi_k(p, t)$ for $k \geq 2$ and $1 \leq p \leq \infty$

We begin by recalling how F. Wiener's trick was used in [1] to obtain the solution to the extremal problem $\Phi_k(p, t)$ for $k \geq 2$ from Theorem 8.

Theorem 14 (Benetau–Korenblum) *Let $k \geq 2$ be an integer. For every $1 \leq p \leq \infty$ and every $0 \leq t \leq 1$,*

$$\Phi_k(p, t) = \Phi_1(p, t).$$

If f_1 is the extremal function for $\Phi_1(p, t)$, then $f_k(z) = f_1(z^k)$ is an extremal function for $\Phi_k(p, t)$.

Proof Suppose that f is an extremal for $\Phi_k(p, t)$. Since $\|W_k f\|_{H^p} \leq \|f\|_{H^p}$,

$$f(0) = W_k f(0) \qquad \text{and} \qquad \frac{f^{(k)}(0)}{k!} = \frac{(W_k f)^{(k)}(0)}{k!},$$

we conclude that $W_k f$ is also an extremal for $\Phi_k(p, t)$. Thus we may restrict our attention to extremals $\widetilde{f}_k$ of the form $\widetilde{f}_k(z) = \widetilde{f}(z^k)$ for $\widetilde{f} \in H^p$. The stated claims now follow at once from Theorem 8, since $\|\widetilde{f}_k\|_{H^p} = \|\widetilde{f}\|_{H^p}$. □

The purpose of the present section is to answer the following question. For which trios $k \geq 2$, $1 \leq p \leq \infty$ and $0 \leq t \leq 1$ is the extremal for $\Phi_k(p, t)$ unique? Note that while Theorem 14 provides an extremal $f_k(z) = f_1(z^k)$ where f_1 denotes the extremal from (the statement of) Theorem 8, it might not be unique.

In the case $1 < p < \infty$ it follows at once from Theorem 1 (b) that this extremal is unique, although it is perhaps easier to use the strict convexity of H^p and Lemma 3 directly. Since H^p is not strictly convex for $p = 1$ and $p = \infty$, these cases require further analysis. Note that the case (a) below is certainly known to experts as a consequence of the general theory developed in [8, 11, 14].

Theorem 15 *Consider the extremal problem (1) for $k \geq 2$ and $1 \leq p \leq \infty$.*

(a) *If $1 < p \leq \infty$, then the unique extremal is $f_k(z) = f_1(z^k)$.*
(b) *If $p = 1$ and $1/2 \leq t \leq 1$, then the unique extremal is $f_k(z) = f_1(z^k)$.*
(c) *If $p = 1$ and $0 \leq t < 1/2$, then the extremals are the functions of the form*

$$f(z) = C \prod_{j=1}^{k} \left(\lambda_j - z\right)\left(1 - \overline{\lambda_j} z\right)$$

with $|\lambda_j| \leq 1$ such that $\|f\|_{H^1} = 1$, $f(0) = t$ and $f^{(k)}(0) > 0$.

Proof of Theorem 15(a) In view of the discussion above, we need only consider the case $p = \infty$. By Lemma 4, we know that any extremal must be of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{l} \frac{\lambda_j - z}{1 - \overline{\lambda_j} z} \tag{30}$$

for some $0 \leq l \leq k$, constants $\lambda_j \in \mathbb{D}$ and $\theta \in \mathbb{R}$. If f is extremal for $\Phi_k(\infty, t)$, then so is $W_k f$ by Theorem 14. Consequently, $W_k f$ is also of the form (30). In particular, since both f and $W_k f$ are inner, we get from Lemma 7 that $f = W_k f$. From the definition of W_k, we know that $f(z) = W_k f(z) = g(z^k)$ for some analytic function g. This shows that the only possibility in (30) is

$$f(z) = e^{i\theta} \frac{\lambda - z^k}{1 - \overline{\lambda} z^k}$$

for some $\lambda \in \mathbb{D}$ and $\theta \in \mathbb{R}$. The unique extremal has $\theta = \pi$ and $\lambda = -t$. □

Proof of Theorem 15(b) Suppose that f is extremal for $\Phi_k(1, t)$. By rotations, we extend our scope to functions f such that $|f(0)| = t$. In this case, we can use Lemma 4 and write $f = gh$ for

$$g(z) = C \prod_{j=1}^{l} (z + \alpha_j) \prod_{j=l+1}^{k} (1 + \overline{\alpha_j} z),$$
$$h(z) = C \prod_{j=1}^{k} (1 + \overline{\alpha_j} z).$$

The constant $C > 0$ satisfies

$$\frac{1}{C^2} = \sum_{j=0}^{k} \left| \sum_{j_1 + j_2 + \cdots + j_k = j} \alpha_1^{j_1} \alpha_2^{j_2} \cdots \alpha_k^{j_k} \right|^2,$$

where $j_1, j_2, \ldots, j_k$ take only the values 0 and 1. Evidently $\|g\|_{H^2} = \|h\|_{H^2} = 1$. Set $A_l = |\alpha_1 \cdots \alpha_l|$ and $B_l = |\alpha_{l+1} \cdots \alpha_k|$. By keeping only the terms $j = 0$ and $j = k$ we obtain the trivial estimate

$$\frac{1}{C^2} \geq 1 + |\alpha_1 \alpha_2 \cdots \alpha_k|^2 = 1 + A_l^2 B_l^2. \tag{31}$$

We will adapt an argument due to F. Riesz [13] to get some additional information on the relationship between g and h. Write

$$f(z) = \sum_{j=0}^{2k} a_j z^j, \qquad g(z) = \sum_{j=0}^{k} b_j z^j \qquad \text{and} \qquad h(z) = \sum_{j=0}^{k} c_j z^j$$

and note that $|b_0| = t/|c_0| = t/C$. By the Cauchy product formula we find that

$$a_k = \sum_{j=0}^{k} b_j c_{k-j} = t \frac{c_k}{C} \frac{b_0}{|b_0|} + \sum_{j=1}^{k} b_j c_{k-j}. \tag{32}$$

Suppose that $\widetilde{g} \in H^2$ satisfies $|\widetilde{g}(0)| = t/C$ and $\|\widetilde{g}\|_{H^2} \leq 1$. Define $\widetilde{f} = \widetilde{g}h$. The Cauchy–Schwarz inequality shows that $\|\widetilde{f}\|_{H^1} \leq 1$, so the extremality of f implies that $|\widetilde{a}_k| \leq |a_k|$. Inspecting (32) and using the Cauchy–Schwarz inequality, we find that the optimal g must therefore satisfy

$$g(z) = \frac{t}{C}\frac{\overline{c_k}}{|c_k|} + \sqrt{\frac{1-\frac{t^2}{C^2}}{1-|c_k|^2}}\sum_{j=1}^{k}\overline{c_{k-j}}z^j, \tag{33}$$

where we used that $\|h\|_{H^2} = 1$. Using that $c_0 = C$, we compare the coefficients for z^k in (33) with the definition of g, to find that

$$\sqrt{\frac{1-\frac{t^2}{C^2}}{1-|c_k|^2}}C = C\prod_{j=l+1}^{k}\overline{\alpha_j} \qquad \Longrightarrow \qquad \frac{1-\frac{t^2}{C^2}}{1-|c_k|^2} = B_l^2.$$

Next we insert $t = C^2A_l$ from the definition of $f = gh$ and $|c_k|^2 = C^2A_l^2B_l^2$ from the definition of h to obtain

$$\frac{1-C^2A_l^2}{1-C^2A_l^2B_l^2} = B_l^2 \qquad \Longleftrightarrow \qquad \frac{(1-B_l^2)(1-C^2A_l^2(1+B_l^2))}{1-C^2A_l^2B_l^2} = 0. \tag{34}$$

The additional information we require is encoded in the equation on the right hand side of (34).

Suppose that $l \geq 1$. Evidently $A_l < 1$, since $|\alpha_j| < 1$ for $j = 1, \ldots, l$ by Lemma 4. It follows that the second factor on the right hand side of (34) can never be 0, since the trivial estimate (31) implies that

$$C^2 \leq \frac{1}{1+A_l^2B_l^2} < \frac{1}{A_l^2(1+B_l^2)}. \tag{35}$$

From the right hand side of (34) we thus find that $B_l = 1$, which shows that $C^2 < 1/(2A_l^2)$ by (35). Since $t = C^2A_l$, we conclude that $0 \leq t < 1/2$.

By the contrapositive, we have established that if $1/2 \leq t \leq 1$, then the extremal for $\Phi_k(1,t)$ has $l = 0$. In this case $A_0 = 1$ by definition, which shows that $C = \sqrt{t}$. The right hand side of (34) becomes

$$\frac{(1-B_0^2)(1-t(1+B_0^2))}{1-tB_0^2} = 0,$$

so either $B_0 = 1$ or $B_0^2 = 1/t - 1$. Returning to the definition of h we find that $|c_0|^2 = t$ and $|c_k|^2 = tB_0^2$. Consequently,

$$1 = \|h\|_{H^2}^2 = t(1+B_0^2) + \sum_{j=1}^{k-1}|c_j|^2.$$

Since $1/2 \leq t \leq 1$, we find that both $B_0 = 1$ and $B_0^2 = 1/t - 1$ will imply that $c_j = 0$ for $j = 1, \ldots, k-1$. Thus $h(z) = \sqrt{t} + \sqrt{1-t}\, z^k$. When $l = 0$ we have $g = h$, which shows that the unique extremal is

$$f(z) = \left(\sqrt{t} + \sqrt{1-t}\, z^k\right)^2,$$

which is of the form $f_k(z) = f_1(z^k)$ as claimed. □

Proof of Theorem 15(c) In the case $0 \leq t < 1/2$, we know from Theorem 8 and Theorem 14 that $\Phi_k(1, t) = 1$. See also Figure 1. The stated claim follows from Exercise 3 on page 143 of [6] by scaling and rotating the function

$$f(z) = C \prod_{j=1}^{k} \left(\lambda_j - z\right)\left(1 - \overline{\lambda_j} z\right)$$

to satisfy the conditions $\|f\|_{H^1} = 1$, $f(0) > 0$ and $f^{(k)}(0) > 0$. If the resulting function satisfies $f(0) = t$, then it is an extremal for $\Phi_k(p, t)$ and every extremal is obtained in this way. (This can be established similarly to the case (b) above.) □

6 The Extremal Problem $\Phi_k(p, t)$ for $k \geq 2$ and $0 < p < 1$

The purpose of this final section is to record some observations pertaining to the extremal problem (1) in the unresolved case $k \geq 2$ and $0 < p < 1$.

Suppose that $k \geq 0$ and consider the related extremal problem

$$\Psi_k(p) = \sup \left\{ \operatorname{Re} \frac{f^{(k)}(0)}{k!} \; : \; \|f\|_{H^p} \leq 1 \right\}.$$

Evidently, $\Psi_0(p) = 1$ for every $0 < p \leq \infty$ and the unique extremal is $f(z) = 1$. Recall (from [3] or [9]) that the extremals for Ψ_k satisfy a structure result identical to Lemma 4. Note that the parameter l in Lemma 4 describes the number of zeroes of the extremal in $\mathbb{D}$. Conjecture 1 from [3, Sect. 5] states that the extremal for $\Psi_k(p)$ does not vanish in $\mathbb{D}$ when $0 < p < 1$. The conjecture has been verified in the cases $k = 0, 1, 2$ and for $(k, p) = (3, 2/3)$.

Let us now suppose that $k \geq 1$. There are two obvious connections between the extremal problems Φ_k and Ψ_k. Namely,

$$\Phi_k(p, 0) = \Psi_{k-1}(p) \qquad \text{and} \qquad \max_{0 \leq t \leq 1} \Phi_k(p, t) = \Psi_k(p).$$

Assume that the above-mentioned conjecture from [3] holds. This assumption yields that the extremal for $\Phi_k(p, 0)$ has precisely one zero in $\mathbb{D}$ and the extremal for the t which maximizes $\Phi_k(p, t)$ does not vanish in $\mathbb{D}$. Note that the extremal for $\Phi_k(p, 1)$, which is $f(z) = 1$, does not vanish in $\mathbb{D}$.

Question 1 Suppose that $0 < p < 1$. Is it true that the extremal for $\Phi_k(p,t)$ has at most one zero in $\mathbb{D}$?

We have verified numerically that the question has an affirmative answer for $k = 2$. Note that for $1 < p \leq \infty$, the extremal for $\Phi_k(p,t)$ either has 0 or k zeroes in $\mathbb{D}$ by Theorem 15 (a). In the case $p = 1$, the extremal may have anywhere from 0 to k zeroes by Theorem 15 (b) and (c).

As mentioned in the introduction, Theorem 1 yields the estimates

$$\Phi_1(p,t) \leq \Phi_k(p,t) \leq k^{1/p-1}\Phi_1(p,t).$$

The upper bound is only attained if $\Phi_1(p,t) = 0$ which happens if and only if $t = 1$. Of course, since $\Phi_1(p,1) = 0$ the lower bound is also attained.

Question 2 Fix $k \geq 2$ and $0 < p < 1$. Is there some t_0 such that $\Phi_k(p,t) = \Phi_1(p,t)$ holds for every $t_0 \leq t \leq 1$?

By a combination of numerical and analytical computations, we have strong evidence that the question has an affirmative answer for $k = 2$ and that in this case

$$t_0 = \left(1 + \left(\frac{p}{2-p}\right)^2\right)^{1/p}.$$

Let us close by briefly explaining our reasoning. We began by considering the case $l = 0$ in Lemma 4. Setting

$$\widetilde{f} = \widetilde{g}h^{2/p-1}$$

and arguing as in the proof of Theorem 15 (b) (see also [3]), we found if $t \geq t_0$, then the only possible extremal for $\Phi_2(p,t)$ with $l = 0$ is of the form $f_2(z) = f_1(z^2)$ where f_1 is the corresponding extremal for $\Phi_1(p,t)$. Next, if $l = 2$ then (as in the case $k = 1$) we can only obtain t-values in the range $0 \leq t \leq 2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2}$. However, since

$$2^{-1/p}\sqrt{p}(2-p)^{1/p-1/2} < t_0$$

for $0 < p < 1$ we can ignore the case $l = 2$. The case $l = 1$ was excluded by numerical computations.

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Appendix A: Proof of Lemma 11

We will frequently appeal to the following corollary of Rolle's theorem: Suppose that f is continuously differentiable on $[a, b]$ and that $f'(x) = 0$ has precisely n solutions on (a, b). Then $f(x) = 0$ can have at most $n + 1$ solutions on $[a, b]$.

We are interested in solutions of the equation $F_p(\alpha) = 0$ on the interval $(0, 1)$, where we recall from (28) that

$$F_p(\alpha) = p^2\alpha^{-2} + 2p(2-p) + (2-p)^2\alpha^2 - 4\left(\alpha^{-p} + \alpha^{2-p} - 1\right).$$

The initial step in the proof of Lemma 11 is to identify the critical points of F_p on the interval $0 < \alpha < 1$. It turns out that there is only one.

Lemma 16 *Fix $0 < p < 1$ and let F_p be as in (28). The equation $F_p'(\alpha) = 0$ has the unique solution*

$$\alpha = \alpha_2 = \sqrt{\frac{p}{2-p}}$$

on $0 < \alpha < 1$.

Proof We begin by computing

$$F_p'(\alpha) = -2p^2\alpha^{-3} + 2(2-p)^2\alpha + 4p\alpha^{-p-1} - 4(2-p)\alpha^{1-p}.$$

The solutions of the equation $F_p'(\alpha) = 0$ on $0 < \alpha < 1$ do not change if we multiply both sides by $\alpha^{1+p}/(4-2p)$. Hence, we consider the equation $G_p(\alpha) = 0$, where

$$G_p(\alpha) = \frac{\alpha^{1+p}}{2(2-p)}F_p'(\alpha) = -\frac{p^2}{2-p}\alpha^{p-2} + (2-p)\alpha^{2+p} + \frac{2p}{2-p} - 2\alpha^2.$$

Evidently,

$$G_p'(\alpha) = \alpha\left(p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4\right),$$

and the sign of $G_p'(\alpha)$ is the same as the sign of $p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4$. Since

$$\frac{d}{d\alpha}\left(p^2\alpha^{p-4} + (4-p^2)\alpha^p - 4\right) = 0 \qquad \Longleftrightarrow \qquad \alpha = \sqrt[4]{\frac{4p-p^2}{4-p^2}},$$

and since $G_p'(1) = 0$, we conclude that G_p' changes sign at most once on $0 < \alpha < 1$. Since $G_p(0) = -\infty$, this means that $G_p(\alpha) = 0$ can have at most two solutions on $(0, 1]$. Hence $F_p'(\alpha) = 0$ can have at most two solutions on $(0, 1]$. It is easy to verify that these solutions are

$$\alpha = \sqrt{\frac{p}{2-p}} \qquad \text{and} \qquad \alpha = 1,$$

and hence the proof is complete. □

We next want to demonstrate that $F_\alpha(\alpha_1) > 0$ and $F_\alpha(\alpha_2) < 0$ where α_1 and α_2 are from (23) and (22), respectively.

Lemma 17 *Fix* $0 < p < 1$. *If* $\alpha_2 = \sqrt{p/(2-p)}$, *then* $F_p(\alpha_2) < 0$.

Proof We begin reformulating the inequality $F_p(\alpha_2) < 0$ as $H(p) > 0$, for

$$H(p) = -\frac{2-p}{4}\alpha_2^p F_p(\alpha_2) = 2 - \left(1 + 2p - p^2\right) p^{p/2}(2-p)^{(2-p)/2}.$$

Since we have $H(0) = H(1) = 0$, it is sufficient to prove that the function H has precisely one critical point on $0 < p < 1$ and that it is strictly positive for some $0 < p < 1$. We first check that

$$H(1/2) = \frac{16 - 7 \cdot 3^{3/4}}{8} > 0.$$

We then compute

$$H'(p) = -p^{p/2}(2-p)^{(2-p)/2}\left(2(1-p) + \frac{(1+2p-p^2)}{2}\log\left(\frac{p}{2-p}\right)\right).$$

The first factor is non-zero, so we therefore need to check that the equation $I(p) = 0$ has only one solution on $0 < p < 1$, where

$$I(p) = \frac{4(1-p)}{1+2p-p^2} + \log\left(\frac{p}{2-p}\right).$$

We compute

$$I'(p) = \frac{-4\left(3 - 2p + p^2\right)}{\left(1+2p-p^2\right)^2} + \frac{2}{p(2-p)} = \frac{2(1-p)^2\left(3p^2 - 6p + 1\right)}{p(2-p)\left(1+2p-p^2\right)^2}.$$

Hence $I'(p) = 0$ has the unique solution $p_0 = 1 - \sqrt{2/3}$ on the interval $0 < p < 1$. Noting that $I(0) = -\infty$ and $I(1) = 0$, we conclude by verifying that

$$I(p_0) = \sqrt{6} + \log\left(5 - 2\sqrt{6}\right) > 0$$

which demonstrates that $I(p) = 0$ has a unique solution on $0 < p < 1$. □

Lemma 18 *Fix* $0 < p < 1$. *Let* α_1 *denote the unique solution of the equation* $1 - 2\alpha^p + \alpha^2 = 0$ *on the interval* $(0, 1)$. *Then* $F_p(\alpha_1) > 0$.

Proof Using the equation defining α_1, we see that $\alpha_1^{-p} + \alpha_1^{2-p} - 1 = 1$. Hence,

$$\begin{aligned} F_p(\alpha_1) &= \frac{p^2}{\alpha_1^2} + 2p(2-p) + (2-p)^2\alpha_1^2 - 4 \\ &= \left(\frac{p}{\alpha_1} + \alpha_1(2-p) + 2\right)\left(\frac{1}{\alpha_1} - 1\right)(p - \alpha_1(2-p)). \end{aligned}$$

The first two factors are strictly positive for every $0 < \alpha_1 < 1$ and every $0 < p < 1$. Consequently, $F_p(\alpha_1) > 0$ if and only if $\alpha_1 < p/(2-p)$. The function

$$J_p(\alpha) = 1 - 2\alpha^p + \alpha^2$$

satisfies $J_p(0) = 1$ and $J_p(1) = 0$. Moreover, J_p is strictly decreasing on $(0, p^{2-p})$ and strictly increasing on $(p^{2-p}, 1)$. Since α_1 is the unique solution to $J_p(\alpha) = 0$ for $0 < \alpha < 1$, the desired inequality $\alpha_1 < p/(2-p)$ is equivalent to

$$0 > J_p\left(\frac{p}{2-p}\right) = 1 - 2\left(\frac{p}{2-p}\right)^p + \left(\frac{p}{2-p}\right)^2.$$

In order to establish this inequality, we multiply by $(2-p)^2/2$ on both sides to get the equivalent inequality $K(p) < 0$, where

$$K(p) = 2 - 2p + p^2 - p^p(2-p)^{2-p}.$$

Our plan is to use Taylor's theorem to write

$$K(p) = K(1) + K'(1)(p-1) + \frac{K''(\eta)}{2}(p-1)^2$$

where $0 < p < \eta < 1$. The claim will follow if we can prove that $K(1) = K'(1) = 0$ and $K''(p) < 0$ for $0 < p < 1$. Hence we compute

$$\begin{aligned} K'(p) &= -2 + 2p - p^p(2-p)^{2-p}\log\left(\frac{p}{2-p}\right), \\ K''(p) &= 2 - p^p(2-p)^{2-p}\left(\log^2\left(\frac{p}{2-p}\right) + \frac{2}{p(2-p)}\right). \end{aligned}$$

Evidently, $K(1) = K'(1) = K''(1) = 0$. Hence we are done if we can prove that K'' is strictly increasing on $0 < p < 1$. This will follow once we verify that both

$$p^p(2-p)^{2-p} \qquad \text{and} \qquad \log^2\left(\frac{p}{2-p}\right) + \frac{2}{p(2-p)}$$

are strictly positive and strictly decreasing on $0 < p < 1$. Strict positivity is obvious. The first function is strictly decreasing since

$$\frac{d}{dp}\left(p^p(2-p)^{2-p}\right) = p^p(2-p)^{2-p}\log\left(\frac{p}{2-p}\right)$$

and $\log(p/(2-p)) < 0$ for $0 < p < 1$. For the second function, we check that

$$\frac{d}{dp}\left(\log^2\left(\frac{p}{2-p}\right) + \frac{2}{p(2-p)}\right) = \frac{4}{p^2}\left(\frac{p}{2-p}\log\left(\frac{p}{2-p}\right) + \frac{p-1}{(2-p)^2}\right) < 0,$$

where for the final inequality we have again used that $\log(p/(2-p)) < 0$. □

We can finally wrap up the proof of Lemma 11.

Proof of Lemma 11 By Lemma 16 we know that $F_p'(\alpha) = 0$ has precisely one solution for $0 < \alpha < 1$. Since $F_p(0) = \infty$ and $F_p(1) = 0$, this implies that the equation $F_p(\alpha) = 0$ can have at most one solution on the interval $(0, 1)$. Lemma 17 shows that there is exactly one solution, since $F_p(\alpha_2) < 0$. Let α_p denote this solution. Inspecting the endpoints again, we find that $F_p(\alpha) > 0$ for $0 < \alpha < \alpha_p$ and $F_p(\alpha) < 0$ for $\alpha_p < \alpha < 1$. Using Lemma 17 again we conclude that $\alpha_p < \alpha_2$, while the inequality $\alpha_1 < \alpha_p$ follows similarly from Lemma 18. □

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