ON THE ORDER OF MAGNITUDE OF SUDLER PRODUCTS II

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ABSTRACT. We study the asymptotic behavior of Sudler products $P_N(\alpha) = \prod_{r=1}^N 2|\sin \pi r \alpha|$ for quadratic irrationals $\alpha \in \mathbb{R}$. In particular, we verify the convergence of certain perturbed Sudler products along subsequences, and show that $\liminf_N P_N(\alpha) = 0$ and $\limsup_N P_N(\alpha)/N = \infty$ whenever the maximal digit in the continued fraction expansion of α exceeds 23. This generalizes known results for the period one case $\alpha = [0; \overline{a}]$.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Let $\alpha \in \mathbb{R}$ and $N \geq 1$ be an integer. The Sudler product at stage N and with parameter α is defined as

$$P_N(\alpha) = \prod_{r=1}^N |2\sin \pi r\alpha|.$$

Sudler products have been studied extensively, as they bear connections with several areas of research; we mention partition theory, Padé approximants and dynamical systems, and refer to [12] and references therein for further examples and details. In the present paper our main focus will be on the asymptotic order of magnitude of $P_N(\alpha)$. This topic has received much attention in recent years, and we begin by briefly reviewing key results relevant to the main results of this paper. For a more detailed overview of the asymptotic behavior of $P_N(\alpha)$, we refer to the survey paper [7].

Erdős and Szekeres showed in [5] that $\liminf_{N\to\infty} P_N(\alpha) = 0$ for almost all α , and conjectured that this result is true for all values of α . Lubinsky [10] later confirmed that $\liminf_{N\to\infty} P_N(\alpha) = 0$ whenever α has unbounded partial quotients in its continued fraction expansion $\alpha = [0; a_1, a_2, \ldots]$.

More recently, Mestel and Verschueren [12] studied the behavior of Sudler products $P_N(\phi)$, where $\phi = [0; 1, 1, ...]$ is the fractional part of the golden ratio. Their precise result was the following.

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Theorem (Mestel, Verschueren): Let $\phi = (\sqrt{5}-1)/2$ be the fractional part of the golden ratio and $(F_n)_{n=0}^{\infty}$ be the sequence of Fibonacci numbers. Then there exists a constant C > 0 such that

$$\lim_{n \to \infty} P_{F_n}(\phi) = C.$$

Moreover, for the same constant C we have $\lim_{n\to\infty} \frac{P_{F_n-1}(\phi)}{F_n} = \frac{C\sqrt{5}}{2\pi}$.

Here the appearance of the Fibonacci sequence is not at all surprising, as it is the sequence of denominators associated with the continued fraction expansion of ϕ . The proof of the result relies on the specific continued fraction expansion $\phi = [0; 1, 1, ...]$ and the algebraic properties of the sequence $(F_n)_{n=0}^{\infty}$. We now know that the convergence property for $P_{F_n}(\phi)$ is a special case of a phenomenon exhibited by all quadratic irrationals [8].

Theorem (Grepstad, Neumüller): Let $\alpha = [0; \overline{a_1, a_2, \ldots, a_\ell}]$ be a purely periodic quadratic irrational, where $\ell \geq 1$ and $a_1, \ldots, a_\ell \in \mathbb{N}$, and let $(q_n)_{n=1}^{\infty}$ be the sequence of denominators of convergents of α . Then there exist constants $C_1, C_2, \ldots, C_\ell > 0$ such that

(1.1)
$$\lim_{m \to \infty} P_{q_{m\ell+k}}(\alpha) = C_k, \qquad k = 1, 2, \dots, \ell.$$

Moreover if $\beta = [0; b_1, \dots, b_h, \overline{a_1, \dots, a_\ell}]$ is a quadratic irrational with the same periodic part as α in its continued fraction expansion, then

$$\lim_{m \to \infty} P_{q_{h+m\ell+k}}(\beta) = C_k, \qquad k = 1, 2, \dots, \ell.$$

for the same constants C_1, \ldots, C_{ℓ} .

Later on, Grepstad, Kaltenböck and Neumüller employed the factorisation technique used in the proof of Mestel and Verschueren's result to show that $\liminf P_N(\phi) > 0$ [6], finally disproving the conjecture in [5].

The proof of the lower bound on $P_N(\phi)$ given in [6] involved studying a perturbed Sudler product $\prod_{r=1}^{N} 2|\sin \pi (r\phi + \varepsilon)|$. A systematic treatment of such perturbed products was conducted in [2], where the result by Mestel and Verschueren was generalized to quadratic irrationals of the form $\beta = [0; b, b, ...]$ by an in-depth study of the product

(1.2)
$$P_{q_n}(\beta,\varepsilon) = \prod_{r=1}^{q_n} 2 \Big| \sin \pi \Big(r\beta + (-1)^n \frac{\varepsilon}{q_n} \Big) \Big|.$$

In [2] it is shown that for each digit $b \ge 1$, the sequence of functions $P_{q_n}(\beta, \varepsilon)$ converges locally uniformly to an explicitly defined function $G_b(\varepsilon)$, and from this the authors deduce the following strong result on the asymptotic behavior of $P_N(\beta)$.

Theorem (Aistleitner, Technau, Zafeiropoulos): Let $\beta = [0; b, b, ...]$, where $b \ge 1$. The following holds.

(i) If
$$b \le 5$$
, then $\liminf_{N \to \infty} P_N(\beta) > 0$ and $\limsup_{N \to \infty} \frac{P_N(\beta)}{N} < \infty$.
(ii) If $b \ge 6$, then $\liminf_{N \to \infty} P_N(\beta) = 0$ and $\limsup_{N \to \infty} \frac{P_N(\beta)}{N} = \infty$.

The theorem above gives a complete description of the asymptotic order of magnitude of $P_N(\beta)$ for irrationals $\beta = [0; b, b, ...]$. The main objective of this paper is to study the asymptotic behaviour of $P_N(\beta)$ for arbitrary quadratic irrationals β , that is irrationals whose continued fraction expansions are eventually periodic with some period length ℓ . It turns out that for such β , the sequence of functions $P_{q_n}(\beta, \varepsilon)$ defined in (1.2) will converge along specific subsequences to ℓ explicitly defined functions $G_k(\beta, \varepsilon)$, $1 \leq k \leq \ell$ (see Theorem 1 below). This is, in some sense, the expected generalization of the result on $\beta = [0; b, b, \ldots]$ in [2]. As a consequence, we obtain a partial extension of the theorem above to arbitrary quadratic irrationals (Theorem 3).

As we shall explain later, the asymptotic behaviour of $P_N(\beta)$ for $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ is similar to that of $P_N(\alpha)$, where $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$. It turns out that purely periodic irrationals are quite easier to analyse in terms of their continued fraction expansions. Moreover, certain relations following from such an analysis are needed in the statement of our main results. Let us therefore briefly review certain basic properties for the convergents of purely periodic irrationals.

1.2. The irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$. The *n*-th convergent of α is the number p_n/q_n , where

$$p_{n+1} = a_n p_n + p_{n-1}, \qquad p_0 = 1, \quad p_1 = 0,$$

 $q_{n+1} = a_n q_n + q_{n-1}, \qquad q_0 = 0, \quad q_1 = 1.$

We mention that the dependence of p_n and q_n on α is not explicitly stated, but if necessary we will write $p_n(\alpha)$ and $q_n(\alpha)$ to make this dependence explicit. The sequence of convergents satisfies

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \dots < \alpha < \dots < \frac{p_4}{q_4} < \frac{p_2}{q_2}$$

and

$$\frac{1}{2q_{n+1}q_n} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}q_n}, \quad n \ge 1.$$

We use the notation of [8] and set

(1.3)
$$c(\alpha) = c = q_{\ell+1} + p_{\ell},$$
$$a(\alpha) = a = \frac{c(\alpha) + \sqrt{c(\alpha)^2 + 4(-1)^{\ell-1}}}{2},$$
$$b(\alpha) = b = \frac{c(\alpha) - \sqrt{c(\alpha)^2 + 4(-1)^{\ell-1}}}{2}.$$

The sequence $(q_n)_{n=1}^{\infty}$ of denominators satisfies the additional recursive relation

(1.4)
$$q_{n+\ell} = c(\alpha)q_n + (-1)^{\ell-1}q_{n-\ell}, \qquad n \ge 2\ell$$

For $k = 0, 1, \ldots, \ell - 1$ we set

(1.5)
$$c_k = \frac{q_{\ell+k} - bq_k}{a - b}$$
 and $e_k = (-1)^{k-1} \frac{|aq_k - q_{\ell+k}|}{q_\ell}$

For notational convenience we extend the definitions of c_k and e_k to all integers $k \ge 0$ periodically modulo ℓ , so that in particular we have $c_{\ell} = c_0$ and $e_{\ell} = e_0$. We also make use of the following relations (for more details see e.g. [8]):

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(1.6)

$$\Lambda_{m\ell+k} := q_{m\ell+k}\alpha - p_{m\ell+k} = e_k b^m = (-1)^{m\ell+k+1} |e_k b^m|,$$

$$\frac{1}{q_{m\ell+k}} = \mathcal{O}(|b|^m), \quad m \to \infty$$

$$q_{m\ell+k} |b|^m = c_k + \mathcal{O}(b^{2m}), \quad m \to \infty.$$

When studying the irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$, it is useful to consider two families of permutations on ℓ -tuples of positive integers $\mathbf{a} = (a_1, \ldots, a_\ell)$. For $k = 0, 1, \ldots, \ell - 1$ we define the permutation operator $\tau_k : \mathbb{N}^\ell \to \mathbb{N}^\ell$ by

$$au_k(\boldsymbol{a}) = (a_{k+1}, \ldots, a_\ell, a_1, \ldots, a_k).$$

Likewise, we define the permutations $\sigma_k : \mathbb{N}^\ell \to \mathbb{N}^\ell$ for $k = 2, \ldots, \ell$ by

$$\sigma_k(\boldsymbol{a}) = (a_{k-1}, \ldots, a_1, a_\ell, \ldots, a_k)$$

while for k = 1 we set $\sigma_1(\mathbf{a}) = (a_\ell, \ldots, a_1)$. We can define τ_k and σ_k for all $k \ge 1$ by extending the definitions above periodically modulo ℓ . Given a purely periodic irrational α with period \mathbf{a} , the corresponding purely periodic irrationals with periods $\tau_k(\mathbf{a})$ and $\sigma_k(\mathbf{a})$ will be denoted by

(1.7)
$$\alpha_{\tau_k} = [0; \overline{a_{k+1}, \dots, a_{\ell}, a_1, \dots, a_k}] \quad \text{and} \quad \alpha_{\sigma_k} = [0; \overline{a_{k-1}, \dots, a_1, a_{\ell}, \dots, a_k}].$$

The significance of the permutations τ_k and σ_k when studying the approximation properties of α is indicated by the following relations, which hold for any index $k = 0, 1, \ldots, \ell - 1$:

(1.8)

$$c(\alpha) = c(\alpha_{\tau_k}) = c(\alpha_{\sigma_k}),$$

$$q_{\ell}(\alpha_{\tau_k}) = q_{\ell}(\alpha_{\sigma_k}),$$

$$p_{\ell}(\alpha_{\tau_k}) = q_{\ell-1}(\alpha_{\sigma_k}) \text{ and } p_{\ell}(\alpha_{\sigma_k}) = q_{\ell-1}(\alpha_{\tau_k}),$$

$$\frac{q_{\ell+1}(\alpha_{\tau_k})}{q_{\ell}(\alpha_{\tau_k})} = a_k + \frac{p_{\ell}(\alpha_{\sigma_k})}{q_{\ell}(\alpha_{\sigma_k})},$$

$$|c_k e_k| = \frac{q_{\ell}(\alpha_{\tau_k})}{c(\alpha_{\tau_k}) - 2b}.$$

1.3. Main Results. We are now equipped to state our main results. As alluded to above, our first goal is to generalize the convergence result of [2] on perturbed products to irrationals $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ with $\ell \geq 2$. For any $\varepsilon \in \mathbb{R}$ we define

(1.9)
$$P_{q_n}(\beta,\varepsilon) := \prod_{r=1}^{q_n} 2 \left| \sin \pi \left(r\beta + (-1)^{n+1} \frac{\varepsilon}{q_n} \right) \right|.$$

In view of the aforementioned theorem by Grepstad and Neümuller in [8], one would expect the perturbed products to converge along specific subsequences. We show that this is indeed the case. For the sake of convenience, we introduce the notation

(1.10)
$$u_k(t) = 2\left(\frac{t}{|e_k c_k|} - \{t\alpha_{\sigma_k}\} + \frac{1}{2}\right), \quad t = 1, 2, \dots$$

for each $k = 1, ..., \ell$, where c_k, e_k are as in (1.5) and α_{σ_k} as in (1.7), all referring to the purely periodic irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$.

Theorem 1. Let $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ where $h \ge 0, \ell, a_1, \ldots, a_\ell \in \mathbb{N}$ and $P_{q_n}(\beta, \varepsilon)$ be the sequence of perturbed Sudler products defined in (1.9). Then for each $k = 1, \ldots, \ell$ the subsequence $P_{q_{h+m\ell+k}}(\beta, \varepsilon)$ converges locally uniformly to a function $G_k(\beta, \varepsilon)$. The limit function satisfies

$$G_{k}(\beta,\varepsilon) = \left|1 + \frac{\varepsilon}{|c_{k}e_{k}|}\right| \left(1 + \frac{1}{|b|^{2}}\right)^{\frac{1}{c-2}} \frac{1}{(c!)^{1/(c-2)}} \times (1.11) \times \prod_{t=1}^{\infty} \left|\left(1 - \frac{\left(1 + \frac{2\varepsilon}{|e_{k}c_{k}|}\right)^{2}}{u_{k}(t)^{2}}\right) \left(1 - \frac{\left(1 + \frac{2}{|b|^{2}}\right)^{2}}{u_{k}(t)^{2}}\right)^{\frac{1}{c-2}} \prod_{s=1}^{c-1} \left(1 - \frac{(1+2s)^{2}}{u_{k}(t)^{2}}\right)^{-\frac{1}{c-2}}\right|$$

when ℓ is even, and (1.12)

$$G_k(\beta,\varepsilon) = \frac{\left| 1 + \frac{\varepsilon}{|c_k e_k|} \right|}{\prod\limits_{s=1}^c |s-a|^{\frac{1}{c}}} \times \prod_{t=1}^\infty \left| \left(1 - \frac{\left(1 + \frac{2\varepsilon}{|e_k c_k|}\right)^2}{u_k(t)^2} \right) \prod_{s=0}^{c-1} \left(1 - \frac{\left(1 + 2(s - \frac{1}{|b|})\right)^2}{u_k(t)^2} \right)^{-\frac{1}{c}} \right|$$

when ℓ is odd. Here the sequence $(u_k(t))_{t=1}^{\infty}$ is given in (1.10) and the constants a, b, c, c_k and e_k are defined in Section 1.2, all corresponding to the purely periodic irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$. In both cases, the functions $G_k(\beta, \cdot), k = 1, \ldots, \ell$, are continuous and C^{∞} on every interval where they are non-zero.

The formulae (1.11) and (1.12) in Theorem 1 imply that the limit functions $G_k(\beta, \varepsilon)$ only depend on the periodic part of the continued fraction expansion of β ; the digits b_1, \ldots, b_h in the pre-periodic part do not play any role at all.

Remark 1. Note that we have altered the definition of $P_{q_n}(\alpha, \varepsilon)$ compared to [2], i.e. we use $(-1)^{n+1}$ instead of $(-1)^n$. This relates to the fact that in [8] the denominator of the *n*-th convergent was defined as q_{n+1} while in [2] the denominator of the *n*-th convergent is q_n .

Remark 2. An alternative proof of Theorem 1 has recently appeared in [1]. There the limit function is given in a different form and additionally an explicit approximation error is obtained.

Since the constants C_1, \ldots, C_ℓ in (1.1) satisfy $C_k = G_k(\beta, 0), (1 \le k \le \ell)$, Theorem 1 allows us to explicitly calculate their values.

Corollary 1. Let $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ and $C_1, \ldots, C_\ell > 0$ be the constants in (1.1). Then for $k = 1, 2, \ldots, \ell$ we have

$$(1.13) \quad C_k = \left(\frac{1+a^2}{c!}\right)^{\frac{1}{c-2}} \prod_{t=1}^{\infty} \left(1 - \frac{1}{u_k(t)^2}\right) \left|1 - \frac{(1+2a^2)^2}{u_k(t)^2}\right|^{\frac{1}{c-2}} \prod_{s=1}^{c-1} \left|1 - \frac{(1+2s)^2}{u_k(t)^2}\right|^{-\frac{1}{c-2}}$$

when ℓ is even, and

(1.14)
$$C_k = \frac{1}{\prod_{s=1}^c |s-a|^{\frac{1}{c}}} \prod_{t=1}^\infty \left(1 - \frac{1}{u_k(t)^2}\right) \prod_{s=0}^{c-1} \left|1 - \frac{(1+2s-2a)^2}{u_k(t)^2}\right|^{-\frac{1}{c}}$$

when ℓ is odd.

Our next result relates the asymptotic size of $P_N(\beta)$ with the size of the constants C_1, \ldots, C_ℓ in (1.1). This is the analogue of Lemma 1 in [2], and the proof is nearly identical. Nevertheless, we include the proof later in the text for the sake of completeness.

Theorem 2. Let $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ and $(C_k)_{k=1}^{\ell}$ be the constants as in (1.1). If $C_{k_0} < 1$ for some index $1 \le k_0 \le \ell$ then

(1.15)
$$\liminf_{N \to \infty} P_N(\beta) = 0 \quad and \quad \limsup_{N \to \infty} \frac{P_N(\beta)}{N} = \infty.$$

Remark 3. By the aforementioned Theorem of Grepstad and Neumüller, the values of C_1, \ldots, C_k only depend on the periodic part of the quadratic irrational β . Combined with Theorem 2, this explains why it suffices to consider only purely periodic irrationals when trying to detect those irrationals β for which the Sudler product $P_N(\beta)$ satisfies (1.15).

Theorem 2 tells us that as long as *one* of the constants C_k $(1 \le k \le \ell)$ defined in (1.1) is less than 1, the Sudler product corresponding to the irrational $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ satisfies (1.15). This raises the question of which out of the ℓ constants C_k associated with $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ is expected to be minimal. ON THE ORDER OF MAGNITUDE OF SUDLER PRODUCTS II

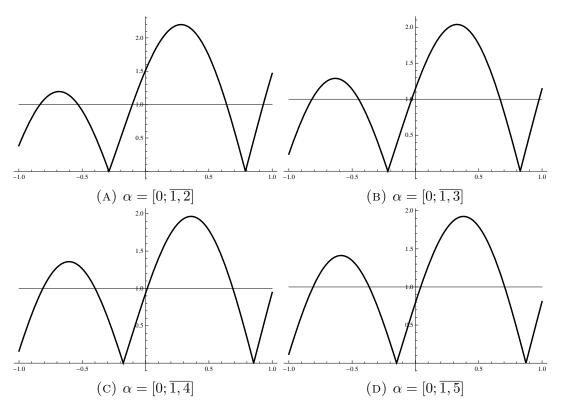


FIGURE 1. Plots of the limit functions $G_2(\alpha, \varepsilon)$ for the stated values of $\alpha = [0; \overline{1, a_2}]$. It appears that $C_2 = G_2(\alpha, 0) < 1$ whenever $a_2 \ge 4$.

The plots in Figures 1 and 2 show graphs of the functions $G_k(\alpha, \varepsilon)$ for specific choices of $\alpha = [0; \overline{a_1, a_2}]$ and $k \in \{1, 2\}$. Since $C_k = G_k(\alpha, 0)$, the value of C_k is the ordinate of the point of intersection of the graph with the vertical axis. These graphs seem to suggest that the bigger the digit a_k is, the smaller the constant C_k becomes.

In spite of the hints provided by the plots, it remains to verify rigorously that $C_k = G_k(\alpha, 0)$ decreases with increasing values for the digit a_k . Moreover, it should be pointed out that the given plots provide no information on the significance of the period length ℓ in the continued fraction expansion of α . The period length can be chosen arbitrarily large, and it might be that the size of ℓ has an impact on the overall sizes of the constants C_k . Moving forward, we will thus focus on three main questions:

- Is the phenomenon implied by the graphs in Figures 1 and 2 indeed true, i.e. for any index $1 \le k \le \ell$, is C_k decreasing as a function of the digit a_k ?
- Suppose we fix some period length $\ell \geq 2$. Does there exist an integer $K = K_{\ell} \geq 1$ such that for any irrational $\alpha = [0; \overline{a_1, \ldots, a_{\ell}}]$ with $\max_{1 \leq i \leq \ell} a_i \geq K$ the Sudler product $P_N(\alpha)$ satisfies (1.15)?

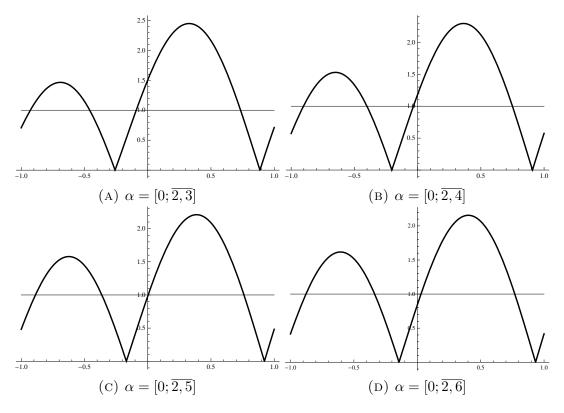


FIGURE 2. Plots of the limit functions $G_2(\alpha, \varepsilon)$ for the stated values of $\alpha = [0; \overline{2, a_2}]$. It appears that $C_2 = G_2(\alpha, 0) < 1$ whenever $a_2 \geq 5$.

• If such an integer exists, can it be chosen independently of the period length ℓ ?

By a careful analysis of the product formulas established in Corollary 1 we find that when k is the index corresponding to the maximal digit a_k in $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$, then C_k is bounded above by an expression which is indeed decreasing as a function of a_k . In turn, this leads to the following result, which provides a positive answer to questions two and three.

Theorem 3. Let $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ be a quadratic irrational with period length $\ell \geq 2$, and say $a_k = \max_j a_j$. Then

$$\liminf_{N \to \infty} P_N(\beta) = 0 \qquad and \qquad \limsup_{N \to \infty} \frac{P_N(\beta)}{N} = \infty,$$

whenever $a_k \geq 23$.

Remark 4. Recall that it was shown by Lubinsky that $\liminf P_N(\alpha) = 0$ whenever α has unbounded coefficients in its continued fraction expansion [10]. In fact, Lubinsky made the more striking observation that there exists a cutoff value $a_k \geq K$ for which Theorem 3 is true, not only for quadratic irrationals but for any irrational α . Note, however, that Lubinsky's approach merely tells us that $K \approx e^{800}$ will suffice. Theorem 3 is thus a significant improvement of the best known cutoff value K for quadratic irrationals. Theorem 3 can be seen as a partial analogue of the second part of the aforementioned theorem by Aistleitner, Technau and Zafeiropoulos. We will not attempt to imitate the first part of their result, stating that $\liminf P_N(\alpha) > 0$ for sufficiently small values of $\max_i a_i$. It will be clear from the proof of Theorem 3 that the role played by the period length ℓ is not fully understood, and in light of this we leave the following open problems for further discussion.

Questions. Let $\alpha = [0; \overline{a_1, a_2, \dots, a_\ell}]$ be a quadratic irrational with $a_k = \max_i a_i$.

- According to Theorem 3, there exists an integer $K = K_{\ell} \ge 1$ such that (1.15) holds whenever $a_k \ge K_{\ell}$, and $K_{\ell} \le 23$ for all period lengths ℓ . However, if we fix ℓ , what is then the optimal value of K_{ℓ} ? We will see in the proof (see Section 6) that for odd periods ℓ , Theorem 3 holds for K = 22. Moreover, for the special case when $\alpha = [0; \overline{1, a_2}]$, Theorem 3 holds for K = 21, and the plots in Figure 1 and 2 suggest that we can actually do *much* better. This brings us to the following question.
- Is it possibly true that if $a_k \ge 6$, then $\liminf_{N\to\infty} P_N(\alpha) = 0$ and $\limsup_{N\to\infty} \frac{P_N(\alpha)}{N} = \infty$? In other words, is $K_\ell \le 6$ for all $\ell \ge 1$? Numerical evidence seems to suggest that the answer is positive, and that the threshold value K = 6 established for irrationals $\beta = [0; b, b, \ldots]$ in [2] might in fact be a universal bound for all quadratic irrationals.

Finally, we point out that Aistleitner and Borda have shown the following duality in [1]: for any badly approximable α , we have

$$\liminf_{N \to \infty} P_N(\alpha) = 0 \quad \text{if and only if} \quad \limsup_{N \to \infty} \frac{P_N(\alpha)}{N} = \infty.$$

Thus for a fixed period length $\ell \geq 2$, giving a complete characterisation of the quadratic irrationals α for which (1.15) holds also determines those irrationals α for which $\liminf_{N\to\infty} P_N(\alpha) > 0$ and $\limsup_{N\to\infty} P_N(\alpha)/N < \infty$.

1.4. **Oragnization of the paper.** The remainder of the paper is organized as follows. Theorems 1 and 2 are proved in Sections 2 and 3, respectively. In Section 4, we analyse the product

$$G(x) = \prod_{t=1}^{\infty} \left(1 - \frac{x^2}{u_k(t)^2} \right),$$

with $u_k(t)$ as defined in (1.10). This product plays a crucial role in the expressions for C_k in Corollary 1. Note that the sequence $\mathcal{U} = (u_k(t))_{t \in \mathbb{N}}$ can be viewed as a perturbation of

the arithmetic progression $(2t/|e_kc_k|)_{t\in\mathbb{N}}$, so it is natural to compare G(x) to the product

$$\prod_{t=1}^{\infty} \left(1 - \frac{x^2}{\left(2t/|e_k c_k|\right)^2} \right).$$

In doing so, we obtain Theorem 4, which tells us that

(1.16)
$$K_1 \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \le |G(x)| \le K_2 \frac{1}{|x|},$$

for appropriate constants K_1 and K_2 .

In Section 5 we use Corollary 1 and Theorem 4 to find an upper bound on C_k in (1.1) for both odd and even periods ℓ . It turns out that C_k can be bounded by expressions which clearly decrease to zero as $a_k \to \infty$ (see Theorems 5 and 6). The speed of decay depends on the constants K_1 and K_2 in (1.16), and for this reason we treat separately the case $\alpha = [0; \overline{a_1, a_2}]$ where either $a_1 = 1$ or $a_2 = 1$ (as better constants K_1 and K_2 can then be found).

Finally, in Section 6 we use the bounds obtained for C_k in Section 5 to show that $C_k < 1$ whenever $\alpha = [0; \overline{a_1, a_2, \ldots, a_k}]$ with $\max_j a_j \ge 23$. By Theorem 2, this proves Theorem 3.

2. Proof of Theorem 1

Theorem 1 states that when $\beta = [0; b_1, \dots, b_h, \overline{a_1, \dots, a_\ell}]$, the perturbed Sudler products $P_{q_n}(\beta, \varepsilon)$ in (1.9) converge along subsequences $(q_{h+m\ell+k})_{m=1}^{\infty}$ to explicit limit functions

$$G_k(\beta,\varepsilon) = \lim_{m \to \infty} P_{q_{h+m\ell+k}}(\beta,\varepsilon), \quad k = 1, \dots, \ell.$$

We first present the proof of Theorem 1 for the purely periodic quadratic irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ (i.e. when h = 0). We then briefly explain how the proof can be generalised for arbitrary quadratic irrationals $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$.

Our first observation is that $P_{q_n}(\alpha, \varepsilon)$ can be decomposed into a product of three factors.

Lemma 1. Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ and let q_n denote the denominator of the *n*-th convergent of α . Then for any $\varepsilon \in \mathbb{R}$,

$$P_{q_n}(\alpha,\varepsilon) = A_n(\alpha,\varepsilon) \cdot B_n(\alpha) \cdot C_n(\alpha,\varepsilon),$$

where

$$A_n(\alpha,\varepsilon) = 2q_n \left| \sin \pi \left(\Lambda_n + (-1)^{n+1} \frac{\varepsilon}{q_n} \right) \right|$$
$$B_n(\alpha) = \left| \prod_{t=1}^{q_n-1} \frac{s_n(t)}{2\sin(\pi t/q_n)} \right|,$$

$$C_n(\alpha,\varepsilon) = \prod_{t=1}^{q_n-1} \left(1 - \frac{s_n^2(0,\varepsilon)}{s_n^2(t)}\right)^{\frac{1}{2}},$$

and

$$s_n(0,\varepsilon) = 2\sin\pi\left(\frac{\Lambda_n}{2} + (-1)^{n+1}\frac{\varepsilon}{q_n}\right), \quad s_n(t) = 2\sin\pi\left(\frac{t}{q_n} - |\Lambda_n|\left(\left\{\frac{tq_{n-1}}{q_n}\right\} - \frac{1}{2}\right)\right).$$

Lemma 1 is the natural analogue of Lemma 5.1 of [12] for the product $P_{q_n}(\phi)$ and Lemma 4.2 in [8] for $P_{q_n}(\alpha)$. We omit the proof since it is nearly identical, the only difference being that it involves an additional term within the argument of the sine. We continue by analysing the behaviour of each of the three factors $A_n(\alpha, \varepsilon), B_n(\alpha)$ and $C_n(\alpha, \varepsilon)$.

The factor $B_n = B_n(\alpha)$ is independent of the perturbation argument ε , and it is shown in [8] that for each $k = 1, 2, \ldots, \ell$ the limit

$$B^{(k)} = \lim_{m \to \infty} B_{m\ell+k}$$

exists. Regarding the factor $A_{m\ell+k}(\alpha,\varepsilon)$, we have

$$A_{m\ell+k}(\alpha,\varepsilon) = 2\pi \left| \left| c_k e_k \right| + \varepsilon \right| + \mathcal{O}(b^{2m}), \quad m \to \infty,$$

therefore $A_{m\ell+k}(\alpha, \varepsilon)$ converges to $2\pi ||c_k e_k| + \varepsilon|$. Finally we need to establish convergence for the factor $C_n(\alpha, \varepsilon)$. Here we can argue as in [12, Section 6], taking into account that the factor $s_n(0, \varepsilon)$ depends on the parameter ε and satisfies

$$s_{m\ell+k}(0,\varepsilon)| \sim \pi ||e_k b^m| + 2\varepsilon| + \mathcal{O}(b^{2m}), \quad m \to \infty,$$

and also

$$s_{m\ell+k}(t) = \pi |e_k b^m| u_k(t) + \mathcal{O}(|b|^{m/5}), \quad m \to \infty.$$

The same arguments as in [12] imply that for any $\varepsilon \in \mathbb{R}$,

$$\lim_{m \to \infty} C_{m\ell+k}(\alpha, \varepsilon) = \prod_{t=1}^{\infty} \left| 1 - \frac{\left(1 + \frac{2\varepsilon}{|e_k c_k|} \right)^2}{u_k(t)^2} \right|$$

In view of Lemma 1 we deduce that for $k = 1, 2, ..., \ell$ the limiting function G_k satisfies

(2.1)
$$G_k(\alpha,\varepsilon) = 2\pi \left| |e_k c_k| + \varepsilon \left| B^{(k)} \prod_{t=1}^{\infty} \right| 1 - \frac{\left(1 + \frac{2\varepsilon}{|e_k c_k|}\right)^2}{u_k(t)^2} \right|,$$

where $u_k(t)$ is defined in (1.10). Arguing as in the proof of Theorem 1 in [2] we can show that the convergence is locally uniform. Now we fix a value of $k = 1, ..., \ell$ and consider indices $n = m\ell + k, m = 1, 2, ...$ In order to determine the formula of G_k we will use relation (1.4). We distinguish two cases depending on the parity of the period length ℓ .

• If $\ell \equiv 0 \pmod{2}$, then (1.4) gives $c(\alpha)q_n = q_{n+\ell} + q_{n-\ell}$, and thus

(2.2)
$$P_{cq_n}(\alpha) = P_{q_{n-\ell}+q_{n+\ell}}(\alpha).$$

The left hand side in (2.2) is

12

$$P_{cq_n}(\alpha) = \prod_{r=1}^{cq_n} 2|\sin \pi r\alpha| = \prod_{s=0}^{c-1} \prod_{r=1+sq_n}^{(s+1)q_n} 2|\sin \pi r\alpha| = \prod_{s=0}^{c-1} \prod_{r=1}^{q_n} 2|\sin \pi (r\alpha + sq_n\alpha)|$$

$$\stackrel{(1.6)}{=} \prod_{s=0}^{c-1} \prod_{r=1}^{q_n} 2|\sin \pi (r\alpha + (-1)^{n+1}s|e_k b^m|)| = \prod_{s=0}^{c-1} P_{q_n}(\alpha, sq_n|e_k b^m|),$$

while the right hand side is

$$P_{q_{n-\ell}+q_{n+\ell}}(\alpha) = \prod_{r=1}^{q_{n+\ell}+q_{n-\ell}} 2|\sin \pi r\alpha| = \prod_{r=1}^{q_{n-\ell}} 2|\sin \pi r\alpha| \cdot \prod_{r=1+q_{n-\ell}}^{q_{n+\ell}+q_{n-\ell}} 2|\sin \pi r\alpha|$$
$$= P_{q_{n-\ell}}(\alpha) \cdot \prod_{r=1}^{q_{n+\ell}} 2|\sin \pi (r\alpha + q_{n-\ell}\alpha)|$$
$$\stackrel{(1.6)}{=} P_{q_{n-\ell}}(\alpha) \cdot \prod_{r=1}^{q_{n+\ell}} 2|\sin \pi (r\alpha + (-1)^{n+\ell+1}|e_k b^{m-1}|)|$$
$$= P_{q_{n-\ell}}(\alpha) P_{q_{n+\ell}}(\alpha, q_{n+\ell}|e_k b^{m-1}|).$$

Since the functions $P_{q_n}(\alpha, \varepsilon)$ converge locally uniformly, letting $m \to \infty$ in (2.2) and taking (1.6) into account we obtain

(2.3)
$$\prod_{s=0}^{c-1} G_k(\alpha, s|c_k e_k|) = G_k(\alpha, 0) G_k\left(\alpha, \frac{|c_k e_k|}{|b|^2}\right).$$

Substituting G_k from (2.1) in (2.3) we obtain (1.11).

• If $\ell \equiv 1 \pmod{2}$, then (1.4) becomes $q_{n+\ell} = c(\alpha)q_n + q_{n-\ell}$, so

$$\frac{P_{q_{n+\ell}}(\alpha)}{P_{q_{n-\ell}}(\alpha)} = \prod_{r=1}^{q_{n+\ell}-q_{n-\ell}} 2|\sin \pi (r\alpha + q_{n-\ell}\alpha)| = \prod_{r=1}^{c(\alpha)q_n} 2|\sin \pi (r\alpha + q_{n-\ell}\alpha)|$$
$$= \prod_{s=0}^{c(\alpha)-1} \prod_{r=1}^{q_n} 2|\sin \pi (r\alpha + sq_n\alpha + q_{n-\ell}\alpha)|$$

$$\stackrel{(1.6)}{=} \prod_{s=0}^{c(\alpha)-1} \prod_{r=1}^{q_n} 2|\sin \pi (r\alpha + (-1)^{n+1}s|e_k b^m| - (-1)^{n+1}|e_k b^{m-1}|)|$$

$$= \prod_{s=0}^{c(\alpha)-1} P_{q_n} \left(\alpha, q_n |e_k b^m| \left(s - \frac{1}{|b|} \right) \right).$$

By the main result of [8] we know that the sequence $P_{q_n}(\alpha)$ converges to a limit $C_k > 0$, hence letting $m \to \infty$ in the above equality we get

$$\prod_{s=0}^{c(\alpha)-1} G_k\left(\alpha, |c_k e_k| (s - \frac{1}{|b|})\right) = 1.$$

Substituting this into (2.1) we obtain (1.12). This completes the proof of Theorem 1 when $\alpha = [0; \overline{a_1, \ldots, a_k}]$ is a purely periodic quadratic irrational.

We now deal with quadratic irrationals for which the length of the pre-period is $h \ge 1$. If we consider the irrational $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$ and $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ is as before, we can still find a factorisation

$$P_{q_n}(\beta,\varepsilon) = A_n(\beta,\varepsilon) \cdot B_n(\beta) \cdot C_n(\beta,\varepsilon),$$

where the three factors are defined similarly to the purely periodic case; the only difference which appears, is that the parameters e_k and c_k are replaced by different parameters $c_{h,k}$ and $e_{h,k}$. The definition of these parameters is given in the proof of Corollary 1.3 in [8].

The limits of the three factors of $P_{q_n}(\beta, \varepsilon)$ are:

$$\lim_{m \to \infty} A_{h+m\ell+k}(\beta,\varepsilon) = 2\pi \left| |c_{h,k}e_{h,k}| + \varepsilon \right|,$$
$$\lim_{m \to \infty} B_{h+m\ell+k}(\beta) = B^{(h,k)}, \quad \text{and}$$
$$\lim_{m \to \infty} C_{h+m\ell+k}(\beta,\varepsilon) = \prod_{t=1}^{\infty} \left| 1 - \frac{\left(1 + \frac{2\varepsilon}{|e_{h,k}c_{h,k}|}\right)^2}{u_k(t)^2} \right|$$

We therefore deduce that for $k = 1, 2, ..., \ell$, the subsequence $P_{q_{h+\ell m+k}}(\beta, \varepsilon)$ converges to some limit function $G_k(\beta, \varepsilon)$. We now invoke the fact that

$$|c_{h,k}e_{h,k}| = |c_k e_k|, \qquad k = 1, \dots, \ell,$$

(established in the proof of Corollary 1.3 in [8]) to deduce that for all $k = 1, \ldots, \ell$,

$$\lim_{m \to \infty} P_{q_{h+m\ell+k}}(\beta, \varepsilon) = G_k(\beta, \varepsilon) = \lambda_k G_k(\alpha, \varepsilon),$$

for some constant λ_k . Finally, since we know from [8, Corollary 1.3] that $G_k(\beta, 0) = G_k(\alpha, 0)$ for each $k = 1, \ldots, \ell$, it follows that $\lambda_k = 1$ for every k. This shows that adding a

pre-period to the continued fraction expansion of α leaves the limit functions G_k unchanged, completing the proof of Theorem 1.

3. Proof of Theorem 2

We now prove Theorem 2, namely show that if for the irrational $\beta = [0; b_1, \ldots, b_h, \overline{a_1, \ldots, a_\ell}]$, we have $G_k(\beta, 0) = C_k < 1$ for some $1 \le k \le \ell$, then

$$\liminf_{N \to \infty} P_N(\beta) = 0 \quad \text{and} \quad \limsup_{N \to \infty} \frac{P_N(\beta)}{N} = \infty.$$

Let k_0 denote the index for which $C_{k_0} < 1$, and fix some λ such that $C_{k_0} < \lambda < 1$. Since G_{k_0} is continuous at 0 and $G_{k_0}(\beta, 0) = C_{k_0} < \lambda$, there exists $\eta > 0$ such that $G_{k_0}(\beta, \varepsilon) < \lambda$ for all $|\varepsilon| < \eta$. Consider a subsequence $(m_i)_{i=1}^{\infty}$ of $(q_{h+m\ell+k_0})_{m=1}^{\infty}$ such that

(i) $m_{i+1} \ge 2m_i$, $i = 1, 2, \dots$ and (ii) $||m_{i+1}\beta|| < \frac{\eta}{4m_i}$, $i = 1, 2, \dots$

where $||x|| = \min\{|x-k| : k \in \mathbb{Z}\}$ denotes the distance of $x \in \mathbb{R}$ to the nearest integer. We set

$$N_i = m_i + \ldots + m_1$$
 and $M_j = N_i - N_j$, $i \ge 1$, $j = 1, \ldots, i$

Then

$$P_{N_i}(\beta) = \prod_{r=1}^{N_i} 2|\sin \pi r\beta| = \prod_{j=1}^i \prod_{r=1}^{m_j} 2|\sin \pi (r+M_j)\beta|$$
$$= \prod_{j=1}^i \prod_{r=1}^{m_j} 2\left|\sin \pi \left(r\beta + \frac{\varepsilon_j}{m_j}\right)\right| = \prod_{j=1}^i P_{m_j}(\beta, (-1)^{\delta_j}\varepsilon_j),$$

where $\varepsilon_j = \pm m_j \|M_j\beta\|$ and $\delta_j = 0$ or 1. We see that

$$|\varepsilon_j| \leq m_j \left(\|m_{j+1}\beta\| + \ldots + \|m_k\beta\| \right) < \frac{\eta}{2}.$$

By the choice of η , this implies that $P_{m_j}(\beta, (-1)^{\delta_j} \varepsilon_j) < \lambda$ for all j large enough, hence $P_{N_i}(\beta) \ll \lambda^i, i \to \infty$. This shows that $\liminf_{N \to \infty} P_N(\beta) = 0$.

Now set also $T_i = m_{i+1} - (N_i + 1), i \ge 1$, so that

(3.1)
$$P_{T_i}(\beta) = \frac{P_{m_{i+1}-1}(\beta)}{\prod_{r=N_i+1}^{m_{i+1}-1} 2|\sin \pi r\beta|} = \frac{P_{m_{i+1}-1}(\beta)}{\prod_{j=1}^{i} \prod_{r=1}^{m_j} 2|\sin \pi (r+M_j-m_{i+1})\beta|}$$

At this point we need to point out a simple fact which follows from the proof of Theorem 1.1 in [8] but is not explicitly stated in the text. If $(C_k)_{k=1}^{\ell}$ are the constants in (1.1), then for each $k = 1, \ldots, \ell$ we have

$$\lim_{m \to \infty} \frac{P_{q_{h+m\ell+k}-1}(\beta)}{q_{h+m\ell+k}} = \frac{C_k}{2\pi |c_k e_k|} \,,$$

where $(c_k)_{k=1}^{\ell}$ and $(e_k)_{k=1}^{\ell}$ are defined in (1.5). Armed with this observation we deduce that the numerator in (3.1) is $P_{m_{i+1}-1}(\beta) \simeq m_{i+1} \simeq T_i$, $i \to \infty$, while for the denominator in (3.1) we can show arguing as in the previous step that

$$\liminf_{i \to \infty} \prod_{j=1}^{i} \prod_{r=1}^{m_j} 2|\sin \pi (r + M_j - m_{i+1})\beta| = 0.$$

Therefore

$$\limsup_{N \to \infty} \frac{P_N(\beta)}{N} = \limsup_{i \to \infty} \frac{P_{T_i}(\beta)}{T_i} = \infty.$$

4. A perturbed sinc product

In Sections 5 and 6, our aim will be to determine when the limit C_k in (1.1) is guaranteed to be less than one. Corollary 1 suggests that we will need to differentiate between two cases, depending on the parity of the period length ℓ . Common to both cases is the need for appropriate upper and lower bounds on the function

(4.1)
$$G(x) = \prod_{t=1}^{\infty} \left(1 - \frac{x^2}{u_k(t)^2} \right), \qquad x \in \mathbb{R},$$

where we recall from (1.10) that

$$u_k(t) = 2\left(\frac{t}{|e_k c_k|} - \{t\alpha_{\sigma_k}\} + \frac{1}{2}\right).$$

Let us now set

(4.2)
$$A = 2|e_k c_k|^{-1}$$
 and $\delta_t = 1 - 2\{t\alpha_{\sigma_k}\},$

so that $u_k(t) = At + \delta_t$. The function G(x) in (4.1) can then be seen as a perturbed version of the well-known product

$$\frac{\sin(\pi A^{-1}x)}{\pi A^{-1}x} = \prod_{t=1}^{\infty} \left(1 - \frac{x^2}{(At)^2}\right).$$

It is not difficult to show that if the perturbations δ_t satisfy $|\delta_t| \leq \delta < A/4$ $(t \in \mathbb{N})$ for some $\delta > 0$, then the function G obeys the bounds

$$C_1 \frac{\operatorname{dist}(x,\mathcal{U})}{|x|^{1+4\delta/A}} \le |G(x)| \le C_2 \frac{1}{|x|^{1-4\delta/A}},$$

for constants C_1 and C_2 , where dist $(x, \mathcal{U}) = \min\{|x - At - \delta_t| : t \in \mathbb{N}\}$. This is related to Kadec's 1/4-rule [9], and can e.g. be seen as a consequence of [3, Lemma 4]. Due to the low-discrepancy property of Kronecker sequences, the sequence $(\delta_t)_t$ satisfies a much stronger condition, which in turn enables us to establish stronger bounds on G.

Theorem 4. Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$, and let $k \leq \ell$ be an index satisfying $a_k = \max_j a_j$. Then for $|x| \geq A/2 = |e_k c_k|^{-1}$, the function G in (4.1) satisfies

(4.3)
$$\frac{2}{\pi}e^{-f(a_k)}\left(1-\frac{2}{3Am}\right)\left(1-\frac{1}{Am}\right)^2\frac{\operatorname{dist}(x,\mathcal{U})}{|x|} \le |G(x)| \le \frac{14A}{9}e^{f(a_k)}\frac{1}{|x|},$$

where the positive integer $m = m(x) \ge 1$ is such that $|Am - x| = \min\{|An - x| : n \in \mathbb{N}\}$ and

(4.4)
$$f(a_k) = \frac{13.7}{a_k} + \frac{1}{20\log a_k} + \frac{1}{100} + \frac{2}{a_k^2}.$$

For |x| < A/2, we have the bound

(4.5)
$$\frac{2}{\pi}e^{-f(a_k)} \le |G(x)| \le 1.$$

Remark 5. Notice that since $f(a_k) \to 0.01$ as $a_k \to \infty$, equation (4.3) reads

$$K_1 \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \le |G(x)| \le K_2 \frac{1}{|x|},$$

with $K_1 \approx 2/\pi$ and $K_2 \approx 14A/9$ whenever a_k and x are large. Similar bounds can be established when k is *not* the index of a maximal continued fraction coefficient of α , but the size of the constants K_1 and K_2 will then depend both on a_k and on the size of $\max_j a_j$.

We will see that the following is an immediate consequence of the proof of Theorem 4.

Corollary 2. Suppose that we have m(x) = 1 in Theorem 4, that is $A/2 \le x \le 3A/2$. Then G(x) in (4.1) satisfies

$$\frac{2}{\pi}e^{-g(a_k)}\left(1-\frac{2}{3A}\right)\left(1-\frac{1}{A}\right)^2\frac{\operatorname{dist}(x,\mathcal{U})}{|x|} \le |G(x)| \le \frac{14A}{9}e^{g(a_k)}\frac{1}{|x|},$$

where

(4.6)
$$g(a_k) = \frac{3.3}{a_k} + \frac{1}{80 \log a_k} + \frac{1}{400} + \frac{2}{a_k^2}.$$

Before we embark on the proof of Theorem 4, we establish two preliminary results. The first concerns the size of $A = 2|e_k c_k|^{-1}$.

Lemma 2. Let $1 \le k \le \ell$. We have

(4.7)
$$\frac{1}{|c_k e_k|} = a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})}$$

Proof. By (1.8) we obtain

$$\frac{1}{|c_k e_k|} = \frac{q_{\ell+1}(\alpha_{\tau_k}) + p_\ell(\alpha_{\tau_k}) - 2b}{q_\ell(\alpha_{\tau_k})} = a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})}$$

Corollary 3. For any $k = 1, 2, \ldots, \ell$ we have

$$a_k < \frac{1}{|c_k e_k|} < a_k + 2$$

By Corollary 3, it immediately follows that

 $(4.8) 2a_k < A < 2(a_k + 2).$

The second result concerns the perturbations δ_t . We state it without proof, as it is an easy consequence of [11, Corollary 3].

Lemma 3. Let $\alpha = [0; a_1, a_2, ...]$ be an irrational with bounded continued fraction coefficients. Then for any fixed $n \in \mathbb{N}$, we have

$$\Big|\sum_{t=n+1}^{n+N} \left(\frac{1}{2} - \{t\alpha\}\right)\Big| \le \log N\left(\frac{a}{8\log a} + 6\right) + \frac{a}{8} + \frac{23}{4},$$

for all $N \geq 1$, where $a = \max_j a_j$.

Recall from (4.2) that $\delta_t = 1 - 2\{t\alpha_{\sigma_k}\}$, and thus by Lemma 3 it follows that

(4.9)
$$\left|\sum_{t=n+1}^{n+N} \delta_t\right| \le \min\left\{N, \log N\left(\frac{a_k}{4\log a_k} + 12\right) + \frac{a_k}{4} + \frac{23}{2}\right\},\$$

where $a_k = \max_j a_j$ for the quadratic irrational α . Note that

$$\log N\left(\frac{a_k}{4\log a_k} + 12\right) + \frac{a_k}{4} + \frac{23}{2} \ge 12\log N + \frac{23}{2} > N \quad \text{for all} \quad N \le 60,$$

regardless of the value of a_k . Accordingly, we will use the bound $\sum_{n+1}^{n+N} \delta_t \leq N$ whenever $N \leq 60$.

We are now equipped to prove Theorem 4.

Proof of Theorem 4. Since G in (4.1) is an even function, it suffices to consider $x \ge 0$. Let $m = m(x) \ge 0$ be the non-negative integer satisfying

$$|x - Am| = \min\{|x - An| : n = 0, 1, 2, \ldots\}$$

and let us first assume that $m \ge 1$, meaning that $x \ge A/2$. Excluding the case of the golden ratio, we may safely assume that $a_k = \max_j a_j \ge 2$, and thus by (4.8) we have $A \ge 4$. It follows that

(4.10)
$$|x - Am| \le \frac{A}{2}, \qquad |x - Am - \delta_m| \le \frac{A}{2} + 1 \le \frac{3A}{4},$$

and

$$|x - At| \ge \frac{A}{2}$$
 and $|x - At - \delta_t| \ge \frac{A}{2} - 1 \ge \frac{A}{4}$,

for any $t \neq m$.

The function G may be split into three products $G(x) = \Pi_1(x)\Pi_2(x)\Pi_3(x)$, where

$$\Pi_{1}(x) = 1 - \frac{x^{2}}{(Am + \delta_{m})^{2}},$$

$$\Pi_{2}(x) = \prod_{\substack{t \ge 1 \\ t \ne m}} \left(1 - \frac{x^{2}}{(At + \delta_{t})^{2}}\right) \left(1 - \frac{x^{2}}{(At)^{2}}\right)^{-1},$$

$$\Pi_{3}(x) = \prod_{\substack{t \ge 1 \\ t \ne m}} \left(1 - \frac{x^{2}}{(At)^{2}}\right).$$

For the first product, we observe that

$$\Pi_1(x) = 1 - \frac{x^2}{(Am + \delta_m)^2} = \frac{(Am + \delta_m - x)(Am + \delta_m + x)}{(Am + \delta_m)^2},$$

and thus by (4.10) we get

(4.11)
$$\operatorname{dist}(x,\mathcal{U})\frac{Am+\delta_m+x}{(Am+\delta_m)^2} \le |\Pi_1(x)| \le \frac{3A}{4}\frac{Am+\delta_m+x}{(Am+\delta_m)^2} \cdot$$

We then consider the second factor $\Pi_2(x) = \prod_{t \neq m} Q_t(x)$, where

$$Q_t(x) = \left(1 - \frac{x^2}{(At + \delta_t)^2}\right) \left(1 - \frac{x^2}{(At)^2}\right)^{-1}, \quad t \ge 1.$$

We have

$$Q_t(x) = \frac{1 - \frac{x}{At + \delta_t}}{1 - \frac{x}{At}} \cdot \frac{1 + \frac{x}{At + \delta_t}}{1 + \frac{x}{At}} = \frac{1 + \frac{\delta_t}{At - x}}{1 + \frac{\delta_t}{At}} \cdot \frac{1 + \frac{\delta_t}{At + x}}{1 + \frac{\delta_t}{At}}$$

ON THE ORDER OF MAGNITUDE OF SUDLER PRODUCTS II

$$= \exp\left\{\log\left(1 + \frac{\delta_t}{At - x}\right) + \log\left(1 + \frac{\delta_t}{At + x}\right) - 2\log\left(1 + \frac{\delta_t}{At}\right)\right\}.$$

Thus if we employ the inequality

$$x - x^2 < \log(1 + x) < x$$
 for all $x > -\frac{1}{2}$

we obtain

(4.12)
$$\exp\left\{\delta_t s_t - \delta_t^2 \left(\frac{1}{(At-x)^2} + \frac{1}{(At+x)^2}\right)\right\} < Q_t(x) < \exp\left\{\delta_t s_t + \frac{2\delta_t^2}{(At)^2}\right\},\$$

where we define

where we define

(4.13)
$$s_t = \frac{1}{At - x} + \frac{1}{At + x} - \frac{2}{At}.$$

The factors in (4.12) contributing significantly to $\Pi_2(x) = \prod_{t \neq m} Q_t$ are the first-order terms $\delta_t s_t$. For the second-order terms, we observe that the contribution on the left hand side is larger (in absolute value) than that on the right hand side, and a straightforward calculation verifies that

$$\sum_{t \neq m} \left(\frac{1}{(At+x)^2} + \frac{1}{(At-x)^2} \right) < \frac{8}{A^2}.$$

We thus conclude that

(4.14)
$$\Pi_2(x) = \prod_{t \neq m} Q_t(x) = \exp\left(\sum_{t \neq m} \delta_t s_t + E\right),$$

where s_t is given in (4.13) and $|E| < \frac{8}{A^2}$.

It remains to find an appropriate bound for

$$\left| \sum_{t \neq m} \delta_t s_t \right| \le \left| \sum_{t < m} \delta_t s_t \right| + \left| \sum_{t > m} \delta_t s_t \right|.$$

We first consider the final term on the right hand side above. Summation by parts yields

$$\sum_{t=m+1}^{m+M} \delta_t s_t = s_{m+M} \sum_{t=m+1}^{m+M} \delta_t + \sum_{t=m+1}^{m+M-1} (s_t - s_{t+1}) \sum_{k=m+1}^t \delta_k,$$

for any $M \geq 1$. We observe that

$$|s_{t+1} - s_t| \le \frac{1}{At - x} - \frac{1}{A(t+1) - x} \le \frac{1}{A(t - m - \frac{1}{2})(t - m + \frac{1}{2})},$$

and

$$|s_{m+M}| \le \frac{1}{A(m+M) - x} \le \frac{1}{A(M - \frac{1}{2})}.$$

Combining this with (4.9), we find that

$$\left|\sum_{t=m+1}^{m+M} \delta_t s_t\right| \le \varepsilon(M) + \sum_{t=m+1}^{m+M-1} \frac{\min\{t-m, K \log(t-m) + C\}}{A\left((t-m)^2 - \frac{1}{4}\right)},$$

where $K = 12 + a_k/(4 \log a_k)$, $C = 23/2 + a_k/4$, and where $\varepsilon(M) \to 0$ as $M \to \infty$. Letting $M \to \infty$, we thus find

$$\begin{aligned} \left| \sum_{t>m} \delta_t s_t \right| &\leq \left| \frac{1}{A} \left(\sum_{t=1}^{60} \frac{t}{(t^2 - \frac{1}{4})} + K \sum_{t=61}^{\infty} \frac{\log t}{(t^2 - \frac{1}{4})} + C \sum_{t=61}^{\infty} \frac{1}{(t^2 - \frac{1}{4})} \right) \\ &\leq \left| \frac{1}{A} \left(5.1 + 0.1K + 0.02C \right) \right. \end{aligned}$$

and inserting values of C and K, and recalling that $A > 2a_k$, we get

$$\left| \sum_{t>m} \delta_t s_t \right| \le \frac{3.3}{a_k} + \frac{1}{80 \log a_k} + \frac{1}{400}.$$

By an analogous argument, one can show that

$$\left| \sum_{t < m} \delta_t s_t \right| \leq \frac{10.4}{a_k} + \frac{3}{80 \log a_k} + \frac{3}{400}$$

and thus combined we have

$$\left| \sum_{t \neq m} \delta_t s_t \right| \le \frac{13.7}{a_k} + \frac{1}{20 \log a_k} + \frac{1}{100}$$

Inserting this in (4.14), we arrive at

(4.15)
$$e^{-f(a_k)} \le |\Pi_2(x)| \le e^{f(a_k)},$$

with f defined as in (4.4).

Finally, we observe that

$$\Pi_3(x) = \frac{A^2 m^2}{(Am - x)(Am + x)} \cdot \frac{\sin(\pi x A^{-1})}{\pi x A^{-1}}$$

and since $|\pi x A^{-1} - \pi m| \le \frac{\pi}{2}$ we get

$$\frac{2}{A} \le \left| \frac{\sin(\pi x A^{-1})}{Am - x} \right| \le \frac{\pi}{A} \cdot$$

This implies that

(4.16)
$$\frac{2A^2m^2}{\pi(Am+x)x} \le |\Pi_3(x)| \le \frac{A^2m^2}{(Am+x)x}$$

Combining the bounds (4.11), (4.15) and (4.16), we find that

$$|G(x)| \ge \frac{2}{\pi} e^{-f(a_k)} \cdot \operatorname{dist}(x, \mathcal{U}) \cdot \frac{1}{|x|} \cdot \frac{(Am)^2}{(Am + \delta_m)^2} \cdot \frac{Am + \delta_m + x}{Am + x},$$

and

$$|G(x)| \le \frac{3A}{4}e^{f(a_k)} \cdot \frac{1}{|x|} \cdot \frac{(Am)^2}{(Am+\delta_m)^2} \cdot \frac{Am+\delta_m+x}{Am+x}$$

The common factor

$$\frac{(Am)^2}{(Am+\delta_m)^2} \cdot \frac{Am+\delta_m+x}{Am+x} = \left(1 - \frac{\delta_m}{Am+\delta_m}\right)^2 \left(1 + \frac{\delta_m}{Am+x}\right)$$

will necessarily tend to 1 as $m(x) \to \infty$. Only the rate of convergence from below will be important to us. We therefore apply the rough upper bound

$$\frac{(Am)^2}{(Am+\delta_m)^2} \cdot \frac{Am+\delta_m+x}{Am+x} \le \left(\frac{4}{3}\right)^2 \left(\frac{7}{6}\right),$$

and the more precise lower bound

$$\frac{(Am)^2}{(Am+\delta_m)^2} \cdot \frac{Am+\delta_m+x}{Am+x} \ge \left(1-\frac{1}{Am}\right)^2 \left(1-\frac{2}{3Am}\right).$$

Inserting these bounds in the inequalities for |G(x)| completes the proof of Theorem 4 in the case $|x| \ge A/2$.

Finally, we consider the case $0 \le x < A/2$. As an upper bound, we use

$$|G(x)| \le 1$$

For the lower bound, we again split G into the subproducts Π_1 , Π_2 and Π_3 . Note that in this case, the product Π_1 is empty, and the product Π_3 is simply a sinc function bounded by

$$\frac{2}{\pi} \le \Pi_3(x) \le 1.$$

For the product Π_2 , we may use the bound (4.15) established for the case $x \ge A/2$ (in fact we can do better, as will be argued below). Combined we get

$$\frac{2}{\pi}e^{-f(a_k)} \le |G(x)| \le 1,$$

and this completes the proof of Theorem 4.

Proof of Corollary 2. Retracing the proof of Theorem 4, we arrive at (4.14), and note that for m(x) = 1 we have

$$\Pi_2(x) = \prod_{t \neq m} Q_t(x) = \exp\left(\sum_{t > m} \delta_t s_t + \frac{8}{A^2}\right) \le \exp\left(\frac{3.3}{a_k} + \frac{1}{80 \log a_k} + \frac{1}{400} + \frac{2}{a_k^2}\right),$$

since the product $\sum_{t < m} \delta_t s_t$ is empty. Apart from this, the proof remains unchanged. \Box

Remark 6. Note that this bound on $\Pi_2(x)$ is clearly also valid for $x \leq A/2$. Thus, the lower bound on G(x) in (4.5) may be improved to

$$\frac{2}{\pi}e^{-g(a_k)} \le |G(x)| \le 1, \quad x \le A/2,$$

with g given in (4.6).

Remark 7. The lower bound for G(x) in (4.3) is used in the following sections to determine an upper bound for the constants C_k . In turn, this upper bound gives a threshold value K > 1 such that $a_k = \max_{1 \le i \le \ell} a_i \ge K$ implies that $C_k < 1$.

We believe that an improvement for the threshold value K = 23 in Theorem 3 might be obtained as follows: in the estimates for $\Pi_2(x)$, we have bounded $\left|\sum_{t \neq m(x)} \delta_t s_t(x)\right|$ from above uniformly for all x > 0. In the proof of Theorem 3 we are actually interested in the quantity $G(1)G(3) \cdots G(2c-1)$, which means that we need a bound for the product $\prod_{x=1}^{c} \Pi_2(2x-1)$. Rather than using a uniform bound for all terms in this product, we could seek an upper bound for the double sum

$$\Big|\sum_{\substack{1 \le x \le c \\ t \ne m(2x-1)}} \delta_t s_t (2x-1)\Big|.$$

This would take the specific range of x-values into account, and possibly provide a substantial improvement in the lower bound on $\prod_{x=1}^{c} \prod_{2} (2x+1)$. We leave this task to the interested reader.

5. Upper bounds for the constants C_k

With Theorem 4 established, let us now revisit Corollary 1 and carefully analyse the expressions for C_k provided in (1.13) and (1.14). Recall that we have to differentiate between the case of even and odd period length ℓ . To ease the analysis, it will be useful to make an assumption on the size of $\max_j a_j$ given a quadratic irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$. Presuming a priori that we cannot do better for general ℓ than for the $\ell = 1$ case (see [2, Theorem 6]), we assume throughout this section that $\max_j a_j \ge 6$.

We will show the following.

Theorem 5. Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ for some even period length ℓ , and assume $a_k = \max_j a_j \geq 6$. Then the limit $C_k = \lim_{m \to \infty} P_{\ell m + k}(\alpha)$ obeys the bound

$$C_k^{\frac{c-2}{c}} \le \frac{\pi}{2a_k} e^{1+f(a_k)} \left(200e^{2.4}c^2\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_\ell}} \cdot \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_k}}.$$

where $q_{\ell} = q_{\ell}(\alpha_{\sigma_k})$, c is defined in (1.3) and we recall from (4.4) that

$$f(a_k) \le \frac{13.7}{a_k} + 0.1, \quad a_k \ge 6.$$

Throughout this section, when there is no danger of confusion, we shall write for abbreviation $q_{\ell} = q_{\ell}(\alpha_{\sigma_k})$ and $q_{\ell+1} = q_{\ell+1}(\alpha_{\sigma_k})$.

For the special case $q_{\ell} = 1$, we have an improved bound; note that this only occurs if $\ell = 2$ and $\alpha = [0; \overline{a_1, a_2}]$ with either $a_1 = 1$ or $a_2 = 1$.

Corollary 4. Let $\alpha = [0; \overline{a_1, a_2}]$, and assume $a_k = \max\{a_1, a_2\} \ge 6$ and $\min\{a_1, a_2\} = 1$. Then the limit $C_k = \lim_{m \to \infty} P_{2m+k}(\alpha)$ obeys the bound

$$C_k^{\frac{c-2}{c}} \le \frac{\pi}{a_k} e^{1+g(a_k)} \left(6.2(a_k+2)^4\right)^{\frac{1}{a_k+2}},$$

where we recall from (4.6) that

$$g(a_k) \le \frac{3.3}{a_k} + 0.1, \quad a_k \ge 6.$$

For the odd period case, we will establish the following.

Theorem 6. Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ for some odd period length ℓ , and assume $a_k = \max_i a_i \geq 6$. Then the limit $C_k = \lim_{m \to \infty} P_{\ell m+k}(\alpha)$ obeys the bound

$$C_k \le \frac{\pi}{2a_k} e^{1+f(a_k)} \left(40c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_\ell}} \cdot a_k^{\frac{5}{2a_k}},$$

where f is given in (4.4).

5.1. Bounding C_k when $\ell \equiv 0 \pmod{2}$. Considering first even period lengths ℓ , we recall from (1.13) that

(5.1)
$$C_k^{c-2} = \frac{1+a^2}{c!} \left| \frac{G(1)^{c-1}G(1+2a^2)}{G(1)G(3)G(5)\cdots G(2c-1)} \right|,$$

with a and c as given in (1.3), and the function G defined in (4.1). The term |G(1)| in (5.1) is bounded above by 1, and the term $G(1+2a^2)$ can be bounded by the upper bound in Theorem 4. Keeping the expression for C_k^{c-2} in mind, we will rather give a bound for $(1+a^2)G(1+2a^2)$. By Theorem 4 we have

(5.2)
$$(1+a^2)G(1+2a^2) \le \frac{14A}{9} \cdot e^{f(a_k)} \cdot \frac{(1+a^2)}{(1+2a^2)} \le \frac{4}{5}Ae^{f(a_k)},$$

where for the last inequality we have used that $a \ge 5$ whenever $a_k \ge 6$.

Now let us find a lower bound on

(5.3)
$$G(1)G(3)\cdots G(2c-1) = \prod_{s=0}^{c-1} G(2s+1)$$

In view of Theorem 4, some of the factors of (5.3) will be bounded using (4.3) while others will be bounded using (4.5). Since $2a_k < A < 2(a_k + 2)$, the integers j satisfying $j < \frac{A}{2}$ are $j = 1, 2, \ldots, a_k$ and possibly also $a_k + 1$. Thus the factors of (5.3) with $0 \le s \le \lfloor \frac{a_k}{2} \rfloor - 1$ will be bounded using (4.5) and those with $s \ge \lfloor \frac{a_k}{2} \rfloor + 1$ will be bounded using (4.3). We get

(5.4)
$$\prod_{s=0}^{\lfloor \frac{a_k}{2} \rfloor - 1} G(2s+1) \ge \left(\frac{2}{\pi} e^{-f(a_k)}\right)^{\lfloor \frac{a_k}{2} \rfloor}$$

and

(5.5)
$$\prod_{s=\lfloor\frac{a_k}{2}\rfloor+1}^{c-1} G(2s+1) \ge \prod_{s=\lfloor\frac{a_k}{2}\rfloor+1}^{c-1} \frac{2}{\pi} e^{-f(a_k)} \left(1-\frac{2}{3Am_s}\right) \left(1-\frac{1}{Am_s}\right)^2 \cdot \frac{\operatorname{dist}(2s+1,\mathcal{U})}{2s+1},$$

with $\mathcal{U} = (u_k(t))_{t=1}^{\infty}$ given in (1.10) and $m_s \in \mathbb{N}$ as defined in Theorem 4.

The factor $G(a_k+1)$ appears in (5.3) only when a_k is even and $s = a_k/2$. For this factor, it is not clear which of the two bounds (4.3) and (4.5) apply. However, under the restriction that $a_k \ge 6$, we clearly have dist $(a_k + 1, \mathcal{U}) > 1$, and by combining the bounds (4.3) and (4.5) (keeping all terms that are below 1), we get the universal bound

$$G(2\lfloor \frac{a_k}{2} \rfloor + 1) \ge \frac{2}{\pi} e^{-f(a_k)} \left(1 - \frac{2}{3A}\right) \left(1 - \frac{1}{A}\right)^2 \frac{1}{2\lfloor \frac{a_k}{2} \rfloor + 1},$$

which holds regardless of the parity of a_k , and of whether $a_k + 1 < A/2$ or $a_k + 1 \ge A/2$. Combining this bound with (5.4) and (5.5), we finally get

(5.6)
$$\prod_{s=0}^{c-1} G(2s+1) \ge \left(\frac{2}{\pi}e^{-f(a_k)}\right)^c \cdot \frac{\Pi_1 \cdot \Pi_2}{\Pi_3},$$

where

(5.7)
$$\Pi_1 = \prod_{s=\lfloor \frac{a_k}{2} \rfloor}^{c-1} \left(1 - \frac{2}{3Am_s}\right) \left(1 - \frac{1}{Am_s}\right)^2,$$

(5.8)
$$\Pi_2 = \prod_{s=\lfloor \frac{a_k}{2} \rfloor+1}^{c-1} \operatorname{dist}(2s+1,\mathcal{U}),$$

and

(5.9)
$$\Pi_3 = \prod_{s=\lfloor \frac{a_k}{2} \rfloor}^{c-1} (2s+1).$$

We proceed by bounding the three product terms Π_1 , Π_2 and Π_3 separately. In the following, we will make use of the inequalities

(5.10)
$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \le n! \le e\sqrt{n} \left(\frac{n}{e}\right)^n,$$
$$\sqrt{2\pi n} \left(\frac{2n}{e}\right)^n \le (2n)!! \le e\sqrt{n} \left(\frac{2n}{e}\right)^n,$$
$$\frac{\sqrt{4\pi}}{e} \left(\frac{2n}{e}\right)^n \le (2n-1)!! \le \frac{e}{\sqrt{\pi}} \left(\frac{2n}{e}\right)^n$$

which are valid for all $n \in \mathbb{N}$.

Starting with the first and simplest of the three, we observe that

$$\Pi_1 \ge \prod_{s=\lfloor \frac{a_k}{2} \rfloor}^{c-1} \left(1 - \frac{1}{Am_s}\right)^3.$$

Recall that m_s is the unique positive integer for which $|Am_s - (s+1)|$ is minimized. As s runs through the values $\lfloor a_k/2 \rfloor, \ldots, c-1$, the integer m_s runs through the values $1, \ldots, q_\ell$, and each integer occurs at most $a_k + 1$ times. It follows that

$$\Pi_1 \ge \prod_{m=1}^{q_\ell} \left(1 - \frac{1}{Am} \right)^{3(a_k+1)}$$

•

Using that

$$x - x^2 < \log(1 + x) < x$$
 for all $x > -\frac{1}{2}$

it is straightforward to show that

$$\prod_{m=1}^{q_{\ell}} \left(1 - \frac{1}{Am} \right) \ge \exp\left(-\frac{1}{A} \left(1 + \log q_{\ell} \right) - \frac{\pi^2}{6A^2} \right) \ge \exp\left(-\frac{1}{A} (1.14 + \log q_{\ell}) \right),$$

where for the final inequality we have used that $a_k \ge 6$, and thus A > 12. Inserting this in the expression for Π_1 above, we get

(5.11)
$$\Pi_1 \ge \left(\frac{1}{e^{1.14}q_\ell}\right)^{\frac{3(a_k+1)}{A}} \ge \left(\frac{1}{e^{1.14}q_\ell}\right)^{\frac{3(a_k+1)}{2a_k}} \ge \frac{1}{e^2q_\ell^2}.$$

Let us now consider Π_3 , for which we need to find an upper bound. From (5.10) it follows that

$$\Pi_3 = \frac{(2c-1)!!}{\left(2\lfloor\frac{a_k}{2}\rfloor - 1\right)!!} \le \frac{e^2}{2\pi} \left(\frac{2c}{e}\right)^c \left(\frac{2\lfloor\frac{a_k}{2}\rfloor}{e}\right)^{-\lfloor\frac{-k}{2}\rfloor}$$

We observe that

(5.12)
$$\left(\frac{2\lfloor \frac{a_k}{2} \rfloor}{e}\right)^{\lfloor \frac{a_k}{2} \rfloor} \ge \left(\frac{a_k-1}{e}\right)^{\frac{a_k-1}{2}} \ge \frac{1}{2} \left(\frac{e}{a_k}\right)^{\frac{1}{2}} \left(\frac{a_k}{e}\right)^{\frac{a_k}{2}}$$

where in the last step we have used that $(1 - 1/n)^{\frac{n}{2}} \ge \frac{1}{2}$ for $n \ge 2$. Inserting this in the bound above we find that

(5.13)
$$\Pi_3 \le \frac{e^{\frac{3}{2}}\sqrt{a_k}}{\pi} \left(\frac{e}{a_k}\right)^{\frac{a_k}{2}} \left(\frac{2c}{e}\right)^c.$$

The estimation of the product Π_2 is far more intricate. When we analyse the size of the factors comprising Π_2 , the numbers

(5.14)
$$R_t = \left\{ \frac{tp_{\ell}(\alpha_{\tau_k})}{q_{\ell}(\alpha_{\tau_k})} \right\} + t \left(\frac{p_{\ell}(\alpha_{\sigma_k})}{q_{\ell}(\alpha_{\sigma_k})} - \alpha_{\sigma_k} \right) - \frac{2bt}{q_{\ell}(\alpha_{\sigma_k})}, \quad t = 1, \dots, q_{\ell} - 1$$

appear naturally. The size of the product of $||R_t||$ and its relation with Π_2 is stated in the following lemmas, the proofs of which are postponed for Section 5.3.

Lemma 4. When ℓ is even, the numbers $R_t, t = 1, \ldots, q_\ell - 1$ given in (5.14) satisfy

(5.15)
$$\prod_{t=1}^{q_{\ell}-1} \|R_t\| \ge \frac{\sqrt{q_{\ell}}}{2(2e)^{q_{\ell}+1}}$$

Lemma 5. The product Π_2 defined in (5.8) is bounded below by

(5.16)
$$\Pi_{2} \geq \frac{2}{5c} \cdot 2^{c - \lfloor \frac{a_{k}}{2} \rfloor} \cdot \left\lfloor \frac{a_{k}}{2} \right\rfloor! \cdot \left[(a_{k} - 1)! \right]^{q_{\ell} - 1} \cdot \prod_{t=1}^{q_{\ell} - 1} \|R_{t}\|.$$

We now find a bound for Π_2 using Lemmas 4 and 5. By (5.10) we have

$$\left\lfloor \frac{a_k}{2} \right\rfloor! \ge \left(\frac{a_k}{2e}\right)^{\frac{a_k}{2}}$$
 and $(a_k - 1)! \ge \frac{2}{\sqrt{a_k}} \left(\frac{a_k}{e}\right)^{a_k}$.

The former is obvious when a_k is even, and follows by an argument similar to (5.12) when a_k is odd and greater than 6. Inserting these bounds in (5.16) and employing Lemma 4, we get

(5.17)
$$\Pi_2 \ge \frac{\sqrt{a_k q_\ell}}{20ec} \cdot 2^{c-a_k} \left(\frac{1}{e\sqrt{a_k}}\right)^{q_\ell} \cdot \left(\frac{a_k}{e}\right)^{a_k(q_\ell - \frac{1}{2})}$$

Combining the bounds established for Π_1 , Π_2 and Π_3 , we conclude as follows.

Lemma 6. Let G be given in (4.1). Under the assumption that $a_k \ge 6$ we have

$$\prod_{s=0}^{c-1} G(2s+1) \ge \left(\frac{2a_k}{\pi e^{f(a_k)}c}\right)^c \cdot \frac{2^{-a_k}\pi}{20e^{\frac{9}{2}}q_\ell^{\frac{3}{2}}c} \cdot \left(\frac{e}{a_k^{5/2}}\right)^{q_\ell}.$$

Proof. Inserting the bounds established for Π_1 , Π_2 and Π_3 in (5.11), (5.17) and (5.13) into (5.6), we get

$$\begin{split} \prod_{s=0}^{c-1} G(2s+1) &\geq \left(\frac{2}{\pi c} e^{1-f(a_k)}\right)^c \cdot \frac{2^{-a_k}\pi}{20e^{\frac{9}{2}}q_\ell^{\frac{3}{2}}c} \left(\frac{1}{e\sqrt{a_k}}\right)^{q_\ell} \left(\frac{a_k}{e}\right)^{a_k q_\ell} \\ &\geq \left(\frac{2a_k}{\pi e^{f(a_k)}c}\right)^c \cdot \frac{2^{-a_k}\pi}{20e^{\frac{9}{2}}q_\ell^{\frac{3}{2}}c} \left(\frac{e}{a^{\frac{5}{2}}}\right)^{q_\ell}, \end{split}$$

where for the last inequality we have used that $c < (a_k + 2)q_\ell$.

We are now fully equipped to prove Theorem 5.

Proof of Theorem 5. We recall from (5.1) and (5.2) that

$$C_k^{c-2} = \frac{1+a^2}{c!} \left| \frac{G(1)^{c-1}G(1+2a^2)}{\prod_{s=1}^{c-1}G(2s+1)} \right| \le \frac{1}{c!} \cdot \frac{4Ae^{f(a_k)}}{5} \cdot \frac{1}{\left| \prod_{s=1}^{c-1}G(2s+1) \right|}.$$

Using Lemma 6 and the bound on c! in (5.10), we thus get

$$C_k^{c-2} \le \frac{16Ae^{\frac{9}{2}}q_\ell^{\frac{3}{2}}\sqrt{c}e^{f(a_k)}}{\pi\sqrt{2\pi}} \left(\frac{\pi e^{1+f(a_k)}}{2a_k}\right)^c \cdot 2^{a_k} \cdot \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{q_\ell}$$

Recall that $A < 2(a_k + 2)$ and $q_{\ell} < c/a_k$. This allows us to bound the first term on the right hand side by

$$\frac{32e^{\frac{9}{2}}}{\pi\sqrt{2\pi}}\frac{(a_k+2)}{a_k^{3/2}}e^{f(a_k)}c^2 \le \frac{32e^{\frac{9}{2}}}{\pi\sqrt{2\pi}}\frac{8}{6^{3/2}}e^{f(6)}c^2 \le 200e^{2.4}c^2.$$

Inserting this in the expression above, we get

$$C_k^{c-2} \le \left(\frac{\pi e^{1+f(a_k)}}{2a_k}\right)^c \cdot (200e^{2.4}c^2) \cdot 2^{a_k} \cdot \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{q_\ell}.$$

Raising both sides to the power 1/c and using again that $c > a_k q_\ell$ completes the proof of Theorem 5.

Proof of Corollary 4. When $q_{\ell} = 1$ we have $c = a_k + 2$, and revisiting Lemma 6 we see that we can obtain an improved bound on $\prod G(2s+1)$ in this case. From Corollary 2 and Remark 6 it follows that

$$\prod_{s=0}^{c-1} G(2s+1) \ge \left(\frac{2}{\pi} e^{-g(a_k)}\right)^c \cdot \frac{\Pi_1 \cdot \Pi_2}{\Pi_3},$$

with Π_1 , Π_2 and Π_3 defined as in (5.7)–(5.9), respectively, and $g(a_k)$ given in (4.6). We keep the bounds for Π_1 and Π_3 established in (5.11) and (5.13), and note that the bound on Π_2 in (5.16) simplifies to

$$\Pi_2 \ge \frac{2}{5c} \cdot 2^{c - \lfloor \frac{a_k}{2} \rfloor} \cdot \left\lfloor \frac{a_k}{2} \right\rfloor! \ge \frac{8}{5c} \left(\frac{a_k}{e}\right)^{\frac{a_k}{2}}.$$

Inserting all three bounds above, we get

(5.18)
$$\prod_{s=0}^{c-1} G(2s+1) \ge \frac{8\pi}{5e^{\frac{7}{2}}c\sqrt{a_k}} \left(\frac{e^{1-g(a_k)}}{\pi c}\right)^c \left(\frac{a_k}{e}\right)^{a_k} \ge \frac{8\pi}{5e^{\frac{3}{2}}c^{\frac{7}{2}}} \left(\frac{e^{-g(a_k)}a_k}{\pi c}\right)^c,$$

where we have used that $c = a_k + 2 > a_k$.

Again we have that

$$C_k^{c-2} = \frac{1+a^2}{c!} \left| \frac{G(1)^{c-1}G(1+2a^2)}{\prod_{s=1}^{c-1}G(2s+1)} \right| \le \frac{1}{c!} \cdot \frac{4Ae^{f(a_k)}}{5} \cdot \frac{1}{\left| \prod_{s=1}^{c-1}G(2s+1) \right|}$$

Inserting $A < 2(a_k + 2) = 2c$ and the improved bound (5.18) on $\prod G(2s + 1)$, we get

$$C_k^{c-2} \le \frac{1}{c!} \cdot \frac{e^{f(a_k) + \frac{3}{2}} c^{\frac{9}{2}}}{\pi} \left(\frac{\pi e^{g(a_k)} c}{a_k}\right)^c \le \frac{e^{f(a_k) + \frac{3}{2}}}{\pi \sqrt{2\pi}} \cdot c^4 \cdot \left(\frac{\pi e^{1+g(a_k)}}{a_k}\right)^c$$

where for the last inequality we have used the lower bound on c! in (5.10). The proof is completed by bounding $e^{f(a_k)}$ by $e^{f(6)}$ and raising both sides to the power 1/c.

5.2. Bounding C_k when $\ell \equiv 1 \pmod{2}$. Now let us consider the case of odd period lengths ℓ . By (1.14), the constant C_k is then given by

(5.19)
$$C_k^c = \frac{G(1)^c}{\prod\limits_{s=1}^c |s-a|} \prod\limits_{s=0}^{c-1} |G(2a-2s-1)|^{-1},$$

where a and c are given in (1.3) and G is the function defined in (4.1). Our goal is again to derive a bound for C_k in the case when k is the index such that $\max_j a_j = a_k$.

The assumptions that ℓ is odd and $a_k \ge 6$ necessarily imply that $c \ge 13$; we make use of this inequality in the estimates that follow. The definitions of a and c in (1.3) imply that

$$\frac{1}{a-c} = \frac{\sqrt{c^2 + 4 + c}}{2} < c + \frac{1}{13} \quad \text{and} \quad |a-s| > c-s, \quad s = 1, \dots, c-1$$

and therefore

(5.20)
$$\prod_{s=1}^{c} |s-a|^{-1} \le \frac{c+\frac{1}{13}}{(c-1)!} = \frac{c(c+\frac{1}{13})}{c!} \stackrel{(5.10)}{\le} \frac{\sqrt{c(c+\frac{1}{13})}}{\sqrt{2\pi}} \left(\frac{e}{c}\right)^{c}.$$

We now seek a lower bound for the product

(5.21)
$$\prod_{s=0}^{c-1} |G(2a-2s-1)| = \prod_{s=0}^{c-1} |G(2s+1+2(a-c))| = \prod_{s=0}^{c-1} |G(2s+1-2b)|.$$

We argue as in the case of even ℓ , and use Theorem 4 to bound the terms of this product. We use (4.5) to bound the factors of (5.21) with $0 \le s \le \lfloor \frac{a_k}{2} \rfloor - 1$, which gives

$$\prod_{s=0}^{\lfloor \frac{a_k}{2} \rfloor - 1} |G(2s + 1 - 2b)| \ge \left(\frac{2}{\pi e^{f(a_k)}}\right)^{\lfloor \frac{a_k}{2} \rfloor}.$$

For the factors of (5.21) corresponding to $\lfloor \frac{a_k}{2} \rfloor + 1 \leq s \leq c - 1$, the bound (4.3) gives

$$\prod_{s=\lfloor\frac{a_k}{2}\rfloor+1}^{c-1} |G(2s+1-2b)| \ge \prod_{s=\lfloor\frac{a_k}{2}\rfloor+1}^{c-1} \frac{2}{\pi e^{f(a_k)}} \left(1-\frac{2}{3Am_s}\right) \left(1-\frac{1}{Am_s}\right)^2 \frac{\operatorname{dist}(2s+1-2b,\mathcal{U})}{2s+1-2b},$$

where the integers m_s are defined in Theorem 4. For the factor $|G(2\lfloor \frac{a_k}{2} \rfloor + 1 - 2b)|$ we use the bound

$$|G(2\lfloor \frac{a_k}{2} \rfloor + 1 - 2b)| \ge \frac{2}{\pi} e^{-f(a_k)} \left(1 - \frac{2}{3A}\right) \left(1 - \frac{1}{A}\right)^2 \frac{1}{2\lfloor \frac{a_k}{2} \rfloor + 1 - 2b}.$$

Combining the estimates above, we obtain

(5.22)
$$\prod_{s=0}^{c-1} |G(2s+1-2b)| \ge \left(\frac{2}{\pi e^{f(a_k)}}\right)^c \cdot \frac{\Pi_1 \cdot \Pi_2'}{\Pi_3'},$$

where Π_1 is the product defined in (5.7), while

(5.23)
$$\Pi'_{2} = \prod_{s=\lfloor \frac{a_{k}}{2} \rfloor+1}^{c-1} \operatorname{dist}(2s+1-2b, \mathcal{U}) \quad \text{and} \quad \Pi'_{3} = \prod_{s=\lfloor \frac{a_{k}}{2} \rfloor}^{c-1} (2s+1-2b).$$

Let us first find a bound for Π'_3 by comparing it with Π_3 in (5.9). Note that the bound (5.13) on Π_3 does not depend on the parity of ℓ . Since

$$-2b = \frac{4}{\sqrt{c^2 + 4} + c} < \frac{2}{c},$$

we have

$$\Pi'_{3} \leq \prod_{s=\lfloor \frac{a_{k}}{2} \rfloor}^{c-1} \left(2s+1+\frac{2}{c} \right) = \Pi_{3} \cdot \prod_{s=\lfloor \frac{a_{k}}{2} \rfloor}^{c-1} \left(1+\frac{2}{(2s+1)c} \right).$$

We find that

$$\prod_{s=\lfloor\frac{a_k}{2}\rfloor}^{c-1} \left(1 + \frac{2}{(2s+1)c}\right) = \exp\left(\sum_{s=\lfloor\frac{a_k}{2}\rfloor}^{c-1} \log\left(1 + \frac{2}{(2s+1)c}\right)\right)$$
$$< \exp\left(\frac{2}{c}\sum_{s=\lfloor\frac{a_k}{2}\rfloor}^{c-1} \frac{1}{2s+1}\right) < \exp\left(\frac{\log(c-1)}{c}\right) < e^{\frac{1}{5}},$$

so in view of (5.13) we deduce that

(5.24)
$$\Pi'_{3} \leq \frac{e^{\frac{17}{10}}\sqrt{a_{k}}}{\pi} \left(\frac{e}{a_{k}}\right)^{\frac{a_{k}}{2}} \left(\frac{2c}{e}\right)^{c}.$$

As for the even period case, the estimation of Π'_2 is more elaborate. When analysing the factors in Π'_2 , the numbers $R_t + b$ appear naturally, where R_t is defined as in (5.14). The size of products over $||R_t + b||$ and its relation to Π'_2 is stated in the lemmas below. Note that these are analogues of Lemmas 4 and 5 for the even period case. The proofs are postponed to Section 5.3.

Lemma 7. When ℓ is odd, the numbers $R_t, t = 1, \ldots, q_\ell - 1$ defined in (5.14) satisfy

(5.25)
$$\prod_{t=1}^{q_{\ell}-1} \|R_t + b\| \ge \frac{4\pi}{e^3 (2e)^{q_{\ell}}}$$

Lemma 8. The product Π'_2 defined in (5.23) is bounded below by

(5.26)
$$\Pi'_{2} \ge 2^{c - \lfloor \frac{a_{k}}{2} \rfloor - 1} \cdot \lfloor \frac{a_{k}}{2} \rfloor! \cdot [(a_{k} - 1)!]^{q_{\ell} - 1} \cdot \prod_{t=1}^{q_{\ell} - 1} ||R_{t} + b||$$

Let us now bound Π'_2 using Lemmas 7 and 8. We bound the factorials in (5.26) using (5.10), and combined with Lemma 7 this gives

(5.27)
$$\Pi'_{2} \ge \frac{3\sqrt{a_{k}}}{20} \cdot 2^{c-a_{k}} \left(\frac{1}{e\sqrt{a_{k}}}\right)^{q_{\ell}} \cdot \left(\frac{a_{k}}{e}\right)^{a_{k}(q_{\ell}-\frac{1}{2})}$$

Combining the bounds for Π_1 , Π'_2 and Π'_3 established in (5.11), (5.24) and (5.27), we get the following analogue of Lemma 6. The proof is omitted.

Lemma 9. Let G be defined in (4.1). Under the assumption that $a_k \ge 6$ we have

$$\prod_{s=0}^{c-1} |G(2s+1-2b)| \ge \left(\frac{2a_k}{\pi e^{f(a_k)}c}\right)^c \cdot \frac{3\pi \cdot 2^{-a_k}}{20e^{19/5}q_\ell^2} \cdot \left(\frac{e}{a_k^{5/2}}\right)^{q_\ell}.$$

With Lemma 9 established, we are equipped to prove Theorem 6.

Proof of Theorem 6. Recall from (5.19) that

$$C_k^c \le \left(\prod_{s=1}^c |s-a| \prod_{s=0}^{c-1} |G(2a-2s-1)|\right)^{-1}$$

where we have used that $|G(1)| \leq 1$. Using (5.20) and Lemma 9, we get

$$C_k^c \le 40c^{\frac{3}{2}} \left(\frac{\pi e^{1+f(a_k)}}{2a_k}\right)^c \cdot 2^{a_k} \cdot q_\ell^2 \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{q_\ell},$$

where we have replaced (c+1/13) in (5.20) by the upper bound 27c/26. Raising both sides to the power 1/c and recalling that $c > a_k q_\ell$, we get

$$C_{k} \leq \frac{\pi}{2a_{k}} e^{1+f(a_{k})} \left(40c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot q_{\ell}^{\frac{2}{a_{k}q_{\ell}}} \left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_{k}}}$$
$$\leq \frac{\pi}{2a_{k}} e^{1+f(a_{k})} \left(40c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot a_{k}^{\frac{5}{2a_{k}}},$$

where in the final step we have used that $q_{\ell}^{2/q_{\ell}} < e$.

5.3. Estimation of Π_2 and Π'_2 . Let us now turn to the proofs of Lemmas 4 – 5 and 7 – 8, which lay the foundation for the estimates of Π_2 and Π'_2 above.

5.3.1. Proofs of Lemmas 4 and 7. We consider first Lemmas 4 and 7, which provide bounds on products of factors $||R_t||$ and $||R_t + b||$, with R_t given in (5.14). Recall that these are analogous statements relevant to the cases of even and odd period lengths ℓ , respectively.

We treat first the proof of Lemma 4, and begin by examining the size of each of the three terms appearing in the expression for R_t in (5.14).

Lemma 10. When ℓ is even, we have

$$-\frac{1}{q_{\ell}q_{\ell+1}} < \left(\frac{p_{\ell}}{q_{\ell}} - \alpha_{\sigma_k}\right) - \frac{2b}{q_{\ell}} < -\frac{a_k}{a_k + 1} \frac{1}{q_{\ell}q_{\ell+1}} \cdot$$

Proof. First we observe that setting m = 1 and k = 0 in (1.6) yields $p_{\ell} - q_{\ell}\alpha_{\sigma_k} = b$, and therefore

(5.28)
$$\left(\frac{p_{\ell}}{q_{\ell}} - \alpha_{\sigma_k}\right) - \frac{2b}{q_{\ell}} = -\frac{b}{q_{\ell}}.$$

When ℓ is even, the definition of b in (1.3) gives

$$b < \frac{1}{c-1} = \frac{1}{q_{\ell+1}+p_{\ell}-1} < \frac{1}{q_{\ell+1}}$$

and

$$b > \frac{1}{c} = \frac{1}{q_{\ell+1} + p_{\ell}} = \frac{1}{q_{\ell+1}} \left(1 + \frac{p_{\ell}}{q_{\ell+1}} \right)^{-1}$$

> $\frac{1}{q_{\ell+1}} \left(1 + \frac{q_{\ell}}{q_{\ell+1}} \right)^{-1} > \frac{1}{q_{\ell+1}} \left(1 + \frac{q_{\ell}}{a_k q_{\ell}} \right)^{-1} = \frac{1}{q_{\ell+1}} \frac{a_k}{a_k + 1} ,$
e claim follows.

whence the claim follows.

Lemma 11. Let ℓ be even. For each $t = 1, 2, ..., q_{\ell} - 1$ there exists a unique integer $i = i_t \in \{1, 2, ..., q_{\ell} - 1\}$ such that

$$\frac{i}{q_{\ell}} - \frac{1}{q_{\ell+1}} \leq \left\{ \frac{tp_{\ell}}{q_{\ell}} \right\} + t\left(\frac{p_{\ell}}{q_{\ell}} - \alpha_{\sigma_k} \right) - \frac{2bt}{q_{\ell}} \leq \frac{i}{q_{\ell}} - \frac{6}{7q_{\ell}q_{\ell+1}}$$

Moreover, the correspondence $t \mapsto i_t$ is one-to-one.

Proof. In view of the condition $a_k \ge 6$, by Lemma 10 we get

$$-\frac{1}{q_{\ell+1}} < t\left(\frac{p_{\ell}}{q_{\ell}} - \alpha_{\sigma_k}\right) - \frac{2bt}{q_{\ell}} < -\frac{6}{7q_{\ell}q_{\ell+1}}, \quad t = 1, 2, \dots, q_{\ell} - 1$$

Also since $(p_{\ell}, q_{\ell}) = 1$, for each $t = 1, 2, ..., q_{\ell} - 1$ there exists a unique integer $1 \leq i < q_{\ell}$ such that $\{tp_{\ell}/q_{\ell}\} = i/q_{\ell}$, and the correspondence $t \mapsto i$ is one-to-one. The result follows.

Lemma 11 gives the following estimates for the values of $||R_t||, t = 1, 2, ..., q_{\ell} - 1$.

Corollary 5. Let ℓ be even and R_t be as in (5.14). Then for every $t = 1, 2, \ldots, q_{\ell} - 1$ there exists an integer $j_t \in \{1, 2, \ldots, \lfloor \frac{1}{2}q_{\ell} \rfloor\}$ such that

(5.29)
$$||R_t|| \ge \frac{j_t}{q_\ell} - \frac{1}{q_{\ell+1}} \quad or \quad ||R_t|| \ge \frac{j_t}{q_\ell} + \frac{6}{7q_\ell q_{\ell+1}}$$

The first inequality in (5.29) holds when the residue of $tp_{\ell}(\alpha_{\tau_k})$ modulo $q_{\ell}(\alpha_{\sigma_k})$ is one of $1, 2, \ldots, \lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor$ while the second holds when the residue of $tp_{\ell}(\alpha_{\tau_k})$ modulo $q_{\ell}(\alpha_{\sigma_k})$ is one of $\lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor + 1, \ldots, q_{\ell}(\alpha_{\tau_k}) - 1$. Moreover, there exists at most one integer t such that $j_t = \lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor$, and the correspondence $t \mapsto j_t$ is "at most two-to-one".

With Corollary 5 established, we are equipped to complete the proof of Lemma 4.

Proof of Lemma 4. We consider first the case $q_{\ell} \ge 4$ and for abbreviation set $Q = \lfloor \frac{1}{2}q_{\ell} \rfloor$. From Corollary 5 it follows that

$$\prod_{t=1}^{q_{\ell}-1} \|R_t\| \geq \left(\frac{1}{2} - \frac{1}{q_{\ell+1}}\right) \prod_{j=1}^{Q-1} \left(\frac{j}{q_{\ell}} - \frac{1}{q_{\ell+1}}\right) \left(\frac{j}{q_{\ell}} + \frac{6}{7q_{\ell}q_{\ell+1}}\right)$$

$$\geq \left(\frac{1}{2} - \frac{1}{6}\right) \frac{1}{q_{\ell}} \left(1 - \frac{q_{\ell}}{q_{\ell+1}}\right) \prod_{j=2}^{Q-1} \left(\frac{j}{q_{\ell}} - \frac{1}{q_{\ell+1}}\right) \prod_{j=1}^{Q-1} \frac{j}{q_{\ell}} \\ \geq \frac{5}{18q_{\ell}} \prod_{j=1}^{Q-2} \left(\frac{j}{q_{\ell}} + \frac{1}{2q_{\ell}}\right) \prod_{j=1}^{Q-1} \frac{j}{q_{\ell}} \qquad \left(\text{because } 1 - \frac{q_{\ell}}{q_{\ell+1}} > \frac{5}{6}\right).$$

We may therefore continue to find

$$\prod_{t=1}^{q_{\ell}-1} \|R_{t}\| \geq \frac{5}{18} \frac{(2Q-3)!!}{(2q_{\ell})^{Q-1}} \frac{(Q-1)!}{(q_{\ell})^{Q-1}} = \frac{5}{18} \frac{(2Q-2)!}{(2q_{\ell})^{2(Q-1)}} \\
\stackrel{(5.10)}{\geq} \frac{5}{18} \sqrt{2\pi \cdot 2(Q-1)} \left(\frac{Q-1}{q_{\ell}e}\right)^{2(Q-1)} \\
\geq \frac{5}{18} \sqrt{2\pi(q_{\ell}-3)} \left(\frac{q_{\ell}-3}{2q_{\ell}e}\right)^{q_{\ell}-2} \\
\geq \frac{5}{18} \sqrt{2\pi(q_{\ell}-3)} \left(\frac{1}{2e}\right)^{q_{\ell}-2} \left(1-\frac{3}{q_{\ell}}\right)^{q_{\ell}-3} \left(1-\frac{3}{q_{\ell}}\right) \\
\geq \frac{5}{18} \frac{\sqrt{2\pi q_{\ell}}}{(2e)^{q_{\ell}+1}}.$$

In the last step we have used the bound $\frac{q_{\ell}-3}{q_{\ell}} \geq \frac{1}{4}$ as well as the inequality $\left(1+\frac{r}{n}\right)^n < e^r$ for n > r, which implies that $\left(1-\frac{r}{n}\right)^{n-r} = \left(\frac{n}{n-r}\right)^{-(n-r)} > e^{-r}$ whenever n > r.

We have now shown that (5.15) holds for $q_{\ell} \ge 4$, and proceed by considering smaller values of q_{ℓ} . When $q_{\ell} = 1$, the product in question is empty and hence by definition equals 1. When $q_{\ell} = 2$, the product consists only of the factor $||R_1||$; by (5.29) and the assumption $a_k \ge 6$ we have $||R_1|| \ge 1/3$, so (5.15) is still valid. Finally, when $q_{\ell} = 3$ again (5.29) implies $||R_1|| ||R_2|| \ge 1/12$ which proves that (5.15) is true also in this case. This completes the proof of Lemma 4.

We now turn to Lemma 7. We will not write out the proof of this result in full detail, as it is very similar to that of Lemma 4. We simply point out that it is a consequence of the following results, analogous to Lemmas 10–11 and Corollary 5 above.

Lemma 12. When ℓ is odd, we have

(5.30)
$$\frac{a_k}{a_k + 1} \frac{1}{q_\ell q_{\ell+1}} < \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k}\right) - \frac{2b}{q_\ell} < \frac{1}{q_\ell q_{\ell+1}} + \frac{1}{q_\ell q_\ell} + \frac{1}{q_\ell q_\ell} + \frac{1}{q_\ell q_\ell} + \frac{1}{q_\ell} + \frac{1}{q_\ell q_\ell} + \frac{1}{q_\ell} +$$

Lemma 13. Let ℓ be odd. For each $t = 1, 2, ..., q_{\ell} - 1$ there exists a unique integer $i = i_t \in \{1, 2, ..., q_{\ell} - 1\}$ such that

$$\frac{i}{q_{\ell}} - \frac{1}{q_{\ell+1}} \leq ||R_t + b|| \leq \frac{i}{q_{\ell}} + \frac{1}{7q_{\ell+1}} \cdot$$

Moreover, the correspondence $t \mapsto i_t$ is one-to-one.

Corollary 6. Let ℓ be odd and R_t be as in (5.14). Then for every $t = 1, 2, \ldots, q_{\ell} - 1$ there exists an integer $j_t \in \{1, 2, \ldots, \lfloor \frac{1}{2}q_{\ell} \rfloor\}$ such that

(5.31)
$$\|R_t + b\| \ge \frac{j_t}{q_\ell} - \frac{1}{q_{\ell+1}} \quad or \quad \|R_t + b\| \ge \frac{j_t}{q_\ell} - \frac{1}{7q_{\ell+1}}$$

The first inequality in (5.31) holds when the residue of $tp_{\ell}(\alpha_{\tau_k})$ modulo $q_{\ell}(\alpha_{\sigma_k})$ is one of $1, 2, \ldots, \lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor$ while the second holds when the residue of $tp_{\ell}(\alpha_{\tau_k})$ modulo $q_{\ell}(\alpha_{\sigma_k})$ is one of $\lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor + 1, \ldots, q_{\ell}(\alpha_{\tau_k}) - 1$. Moreover, there exists at most one integer t such that $j_t = \lfloor \frac{1}{2}q_{\ell}(\alpha_{\tau_k}) \rfloor$, and the correspondence $t \mapsto j_t$ is "at most two-to-one".

5.3.2. Proofs of Lemmas 5 and 8. Now let us treat Lemmas 5 and 8. Recall that these provide lower bounds on the factors Π_2 and Π'_2 , given in (5.8) and (5.23). As in the previous subsection, we will provide a full proof of Lemma 5, and then simply sketch the proof of Lemma 8.

Proof of Lemma 5. Recall from (5.8) that Π_2 is defined as

$$\Pi_{2} = \prod_{s=\lfloor \frac{a_{k}}{2} \rfloor}^{c-1} \operatorname{dist} \left(2s+1, \mathcal{U} \right), \qquad u_{k}(t) = \frac{2t}{|e_{k}c_{k}|} - 2\{t\alpha_{\sigma_{k}}\} + 1$$

For each $s \ge 0$ let $t_s \ge 1$ be the unique positive integer such that

(5.32)
$$\operatorname{dist}(2s+1,\mathcal{U}) = |2s+1-u_k(t_s)|$$

Then

(5.33)
$$\Pi_2 = \prod_{s=\lfloor \frac{a_k}{2} \rfloor+1}^{c-1} |2s+1-u_k(t_s)| = 2^{c-\lfloor \frac{a_k}{2} \rfloor-1} \prod_{t=1}^{\infty} \prod_{\substack{s=\lfloor \frac{a_k}{2} \rfloor+1\\t_s=t}}^{c-1} \frac{1}{2} |u_k(t)-(2s+1)|$$

We now analyze the factors appearing in (5.33). For any $t \ge 1$ and $s = 0, 1, \ldots, c - 1$,

$$\frac{1}{2}(u_{k}(t) - (2s+1)) = \frac{t}{|c_{k}e_{k}|} - \{t\alpha_{\sigma_{k}}\} - s$$

$$\stackrel{(4.7)}{=} ta_{k} + t\left(\frac{p_{\ell}(\alpha_{\sigma_{k}})}{q_{\ell}(\alpha_{\sigma_{k}})} + \frac{p_{\ell}(\alpha_{\tau_{k}})}{q_{\ell}(\alpha_{\tau_{k}})} - \frac{2b}{q_{\ell}(\alpha_{\tau_{k}})}\right) - \{t\alpha_{\sigma_{k}}\} - s$$

$$= \underbrace{ta_{k} + \left\lfloor \frac{tp_{\ell}(\alpha_{\tau_{k}})}{q_{\ell}(\alpha_{\tau_{k}})} \right\rfloor + \lfloor t\alpha_{\sigma_{k}} \rfloor - s}_{\in \mathbb{Z}} + R_{t},$$

where the terms R_t are as in (5.14), with this definition extended to all values $t \ge 1$. When $t = q_{\ell}$ and s = c - 1 the right hand side of (5.34) is equal to

$$\begin{aligned} a_k q_\ell + p_\ell + \lfloor q_\ell \alpha_{\sigma_k} \rfloor - (c-1) + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - 2b = \\ &= a_k q_\ell + q_{\ell-1} + \left\lfloor q_\ell \left(\alpha_{\sigma_k} - \frac{p_\ell}{q_\ell} \right) + p_\ell \right\rfloor - (c-1) + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - 2b = \\ &= q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - 2b \stackrel{(5.28)}{=} -b \in \left(-\frac{1}{q_{\ell+1}}, 0 \right). \end{aligned}$$

This observation implies that $t_{c-1} = q_{\ell}$. Since t_s is increasing in $s, t = q_{\ell}$ is the maximum value of t for which the product in (5.33) is non-empty, and we may write

(5.35)
$$\Pi_2 = 2^{c - \lfloor \frac{a_k}{2} \rfloor - 1} \prod_{t=1}^{q_\ell} \prod_{\substack{s = \lfloor \frac{a_k}{2} \rfloor + 1 \\ t_s = t}}^{c-1} \frac{1}{2} |u_k(t) - (2s+1)|.$$

We now want to find lower bounds for the factors in the innermost product in (5.35) for each $1 \le t \le q_{\ell}$. When t = 1, the values of s appearing in the product

$$\prod_{\substack{s=\lfloor\frac{a_k}{2}\rfloor+1\\t_s=1}}^{c-1} \frac{1}{2} |u_k(1) - (2s+1)|$$

are the integers $s \ge 1 + \lfloor \frac{a_k}{2} \rfloor$ for which $2s + 1 < \frac{1}{2} (u_k(1) + u_k(2))$, or equivalently

$$1 + \left\lfloor \frac{a_k}{2} \right\rfloor \leq s < \frac{3a_k}{2} + \frac{3}{2} \left(\frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) - \frac{\{\alpha_{\sigma_k}\} + \{2\alpha_{\sigma_k}\}}{2} \cdot$$

The number of integers s in any interval of the form $[n, \kappa)$, where n is an integer, is $1 + \lfloor \kappa \rfloor - n$. In our case we have $n = 1 + \lfloor \frac{a_k}{2} \rfloor$ and

$$\kappa = \frac{3a_k}{2} + \frac{3}{2} \left(\frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) - \frac{\{\alpha_{\sigma_k}\} + \{2\alpha_{\sigma_k}\}}{2} \\
\geq \frac{3a_k}{2} + \frac{3}{2} \left(\frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) - \frac{3}{2} \{\alpha_{\sigma_k}\} \\
= \frac{3a_k}{2} + \frac{3}{2} \left(\underbrace{\frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} - \alpha_{\sigma_k}}_{\frac{1}{2q_\ell q_{\ell+1}} + \frac{p_\ell(\alpha_{\tau_k})}{2} - \frac{2b}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right),$$

so there are at least a_k values of s for which $t_s = 1$. Consider now the corresponding factors in the innermost product in (5.35). From (5.34) and Lemma 11 it follows that the minimal factor is equal to $||R_1||$, whereas the remaining factors can be bounded below by the values $1, 2, \ldots, a_k - 1$. Therefore

(5.36)
$$\prod_{\substack{s=\lfloor\frac{a_k}{2}\rfloor+1\\t_s=1}}^{c-1} \frac{1}{2} |u_k(1) - (2s+1)| \ge ||R_1|| \cdot (a_k - 1)!$$

We then examine the innermost product of (5.35) for $1 < t < q_{\ell}$. The factors appearing are those corresponding to integers s such that

$$\frac{1}{2}\left(u_k(t-1)+u_k(t)\right) < 2s+1 < \frac{1}{2}\left(u_k(t)+u_k(t+1)\right),$$

or equivalently

$$\frac{2t-1}{|c_k e_k|} - \{(t-1)\alpha_{\sigma_k}\} - \{t\alpha_{\sigma_k}\} + 1 < 2s+1 < \frac{2t+1}{|c_k e_k|} - \{t\alpha_{\sigma_k}\} - \{(t+1)\alpha_{\sigma_k}\} + 1.$$

Observe that the number of integers in an interval (α, β) is at least $\lfloor \beta - \alpha \rfloor$. Here the length of the interval of possible values of s is

$$\begin{pmatrix} \frac{t+\frac{1}{2}}{|c_{k}e_{k}|} - \frac{\{t\alpha_{\sigma_{k}}\} - \{(t+1)\alpha_{\sigma_{k}}\}}{2} \end{pmatrix} - \begin{pmatrix} \frac{t-\frac{1}{2}}{|c_{k}e_{k}|} - \frac{\{t\alpha_{\sigma_{k}}\} - \{(t-1)\alpha_{\sigma_{k}}\}}{2} \end{pmatrix} = = \frac{1}{|c_{k}e_{k}|} + \frac{\{(t-1)\alpha_{\sigma_{k}}\} + \{(t+1)\alpha_{\sigma_{k}}\}}{2} \\ \ge a_{k} + \frac{p_{\ell}(\alpha_{\sigma_{k}})}{q_{\ell}(\alpha_{\sigma_{k}})} + \frac{p_{\ell}(\alpha_{\tau_{k}})}{q_{\ell}(\alpha_{\tau_{k}})} - \frac{2b}{q_{\ell}(\alpha_{\tau_{k}})} - \alpha_{\sigma_{k}} \\ \ge a_{k},$$

so there exist at least a_k values of s. The minimal factor $\frac{1}{2}|u_k(t) - (2s+1)|$ is equal to $||R_t||$, while the remaining ones can be bounded from below by $1, 2, \ldots, a_k - 1$. We thus have

(5.37)
$$\prod_{\substack{s=\lfloor\frac{a_k}{2}\rfloor+1\\t_s=t}}^{c-1} \frac{1}{2} |u_k(t) - (2s+1)| \ge ||R_t|| \cdot (a_k - 1)!, \qquad t = 2, \dots, q_\ell - 1.$$

Finally we deal with the innermost product in (5.35) when $t = q_{\ell}$. The values of s appearing in the product are those for which

$$2c - 1 \ge 2s + 1 > \frac{1}{2} (u_k(q_\ell) + u_k(q_\ell - 1))$$

= $2(c - 2b) - \left(a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})}\right)$
 $- \{q_\ell \alpha_{\sigma_k}\} - \{(q_\ell - 1)\alpha_{\sigma_k}\} + 1,$

which is equivalent to

$$c-1 \geq s > c-\frac{1}{2} \left(a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) - \frac{\{q_\ell\alpha_{\sigma_k}\} + \{(q_\ell - 1)\alpha_{\sigma_k}\}}{2} - 2b.$$

For any $\kappa > 1$, the interval $(c - \kappa, c - 1]$ contains precisely $\lfloor \kappa \rfloor$ integers. Here we need to apply this observation with

$$\kappa = \frac{1}{2} \left(a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) + \frac{\{q_\ell\alpha_{\sigma_k}\} + \{(q_\ell - 1)\alpha_{\sigma_k}\}}{2} + 2b$$

$$\geq \frac{1}{2} \left(a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} \right) + \frac{\left(1 - \frac{1}{q_{\ell+1}}\right) + \left(1 - \frac{1}{q_{\ell+1}} - \alpha_{\sigma_k}\right)}{2} + 2b$$

$$\geq \frac{1}{2} \left(a_k + \frac{p_\ell(\alpha_{\sigma_k})}{q_\ell(\alpha_{\sigma_k})} - \alpha_{\sigma_k} + \frac{p_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} - \frac{2b}{q_\ell(\alpha_{\tau_k})} + 2 - \frac{2}{q_{\ell+1}} + 4b \right)$$

hence there must be at least $1 + \left\lfloor \frac{a_k}{2} \right\rfloor$ such factors.

When $t = q_{\ell}$ and s = c - 1, we saw already that

$$\frac{1}{2}|u_k(q_\ell) - 2c + 1| = |q_\ell a_k + p_\ell(\alpha_{\tau_k}) + p_\ell(\alpha_{\sigma_k}) - c + R_{q_\ell}| = |R_{q_\ell}| = b.$$

For the remaining values of s with $t_s = q_\ell$, we bound the factors $\frac{1}{2}|u_k(q_\ell) - (2s+1)|$ from below by

$$r - \frac{1}{q_{\ell+1}}, \qquad r = 1, 2, \dots, \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Consequently,

$$\prod_{\substack{\lfloor a_k/2 \rfloor + 1 \leq s < c-1 \\ t_s = q_\ell}} \frac{1}{2} |u_k(q_\ell) - (2s-1)| \geq b \prod_{r=1}^{\lfloor a_k/2 \rfloor} \left(r - \frac{1}{q_{\ell+1}}\right) \\
= b \left\lfloor \frac{a_k}{2} \right\rfloor! \prod_{r=1}^{\lfloor a_k/2 \rfloor} \left(1 - \frac{1}{rq_{\ell+1}}\right) \\
\geq \frac{1}{c} \left\lfloor \frac{a_k}{2} \right\rfloor! \left(1 - \sum_{r=1}^{\lfloor a_k/2 \rfloor} \frac{1}{rq_{\ell+1}}\right) \\
\geq \frac{4}{5c} \left\lfloor \frac{a_k}{2} \right\rfloor! \quad .$$
(5.38)

On combining (5.35) with (5.36), (5.37) and (5.38), we obtain inequality (5.16). This completes the proof of Lemma 5. $\hfill \Box$

Let us now give an outline of the proof of Lemma 8. Recall that this result gives a bound on Π'_2 defined in (5.23), relevant to the case of odd period lengths ℓ . Using the same arguments that gave the bound for Π_2 above, we find that

$$\Pi'_{2} = 2^{c - \lfloor \frac{a_{k}}{2} \rfloor - 1} \prod_{t=1}^{q_{\ell}} \prod_{\substack{s = \lfloor \frac{a_{k}}{2} \rfloor + 1 \\ t_{s} = t}}^{c-1} \frac{1}{2} |u_{k}(t) - (2s + 1 - 2b)|,$$

where t_s is defined in (5.32). We now proceed as before, and find lower bounds on the factors in the innermost product for each $1 \leq t \leq q_{\ell}$. According to (5.34), the factors appearing in the product are

(5.39)
$$\frac{1}{2}\left(u_k(t) - (2s+1-2b)\right) = \underbrace{ta_k + \left\lfloor \frac{tp_\ell(\alpha_{\tau_k})}{q_\ell(\alpha_{\tau_k})} \right\rfloor + \lfloor t\alpha_{\sigma_k} \rfloor - s}_{\in \mathbb{Z}} + R_t + b,$$

with R_t given in (5.14).

When ℓ is odd, $t = q_{\ell}$ and s = c - 1, the right hand side of (5.39) is equal to

$$\begin{aligned} a_k q_\ell + p_\ell + \lfloor q_\ell \alpha_{\sigma_k} \rfloor - c + 1 + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - b &= \\ &= a_k q_\ell + q_{\ell-1} + \left\lfloor q_\ell \left(\alpha_{\sigma_k} - \frac{p_\ell}{q_\ell} \right) + p_\ell \right\rfloor - c + 1 + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - b \\ &= q_{\ell+1} + p_\ell - (q_{\ell+1} + p_\ell) + 1 + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - b \\ &= 1 + q_\ell \left(\frac{p_\ell}{q_\ell} - \alpha_{\sigma_k} \right) - b \stackrel{(5.28)}{=} 1. \end{aligned}$$

It follows that

$$\prod_{\substack{s=\lfloor\frac{a_k}{2}\rfloor+1\\t_s=q_\ell}}^{c-1} \frac{1}{2} |u_k(q_\ell) - (2s+1-2b)| \geq \left\lfloor \frac{a_k}{2} \right\rfloor!$$

For $t = 1, \ldots, q_{\ell} - 1$, we argue as for the even period case in Lemma 5, and obtain $\prod_{\ell=1}^{c-1} \frac{1}{2} |q_{\ell}(t) - (2q + 1 - 2h)| \ge ||P_{\ell} + h||_{\ell} (q_{\ell} - 1)|_{\ell} = t - 1 = q_{\ell} - 1$

$$\prod_{\substack{s=\lfloor\frac{a_k}{2}\rfloor+1\\t_s=t}} \frac{1}{2} |u_k(t) - (2s+1-2b)| \ge ||R_t + b|| \cdot (a_k - 1)!, \qquad t = 1, \dots, q_\ell - 1.$$

Combining these two estimates, we deduce that

$$\Pi'_{2} \geq 2^{c - \lfloor \frac{a_{k}}{2} \rfloor - 1} \cdot [(a_{k} - 1)!]^{q_{\ell} - 1} \cdot \lfloor \frac{a_{k}}{2} \rfloor! \cdot \prod_{t=1}^{q_{\ell} - 1} ||R_{t} + b||,$$

which confirms Lemma 8.

6. Proof of Theorem 3

With Theorems 5 and 6 established, let us now prove Theorem 3. As explained in Remark 3, it suffices to study quadratic irrationals α with purely periodic continued fraction expansion.

Recall that by Theorem 2, Theorem 3 is verified if we can show that

$$C_k = \lim_{m \to \infty} P_{q_{m\ell+k}}(\alpha) < 1$$

for every $\alpha = [0; \overline{a_1, a_2, \dots, a_\ell}]$ with $a_k = \max_j a_j \ge 23$.

Assume first that the period length ℓ is odd. Then $q_{\ell} > 1$, and from Theorem 6 it follows that

$$C_k \le \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} \left(40c^{\frac{3}{2}}\right)^{\frac{1}{c}} a_k^{\frac{5}{2a_k}}.$$

The factor $(40c^{3/2})^{1/c}$ is decreasing in c (within the range of relevant values of c), so we may safely use the fact that $c > a_k q_\ell$ to obtain

$$C_{k} \leq \frac{\pi}{\sqrt{2}a_{k}} e^{1+f(a_{k})} \left(40a_{k}^{\frac{3}{2}}q_{\ell}^{\frac{3}{2}}\right)^{\frac{1}{a_{k}q_{\ell}}} a_{k}^{\frac{5}{2a_{k}}}$$
$$\leq \frac{\pi}{\sqrt{2}a_{k}} e^{1+f(a_{k})} \left(40a_{k}^{\frac{3}{2}}\right)^{\frac{1}{a_{k}q_{\ell}}} 2^{\frac{1}{a_{k}}} a_{k}^{\frac{5}{2a_{k}}},$$

where for the last inequality we have used that $q_{\ell}^{3/2q_{\ell}} < 2$. This expression is decreasing in q_{ℓ} , so we insert $q_{\ell} = 2$ to obtain

$$C_k \le \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} \left(160a_k^{\frac{13}{2}}\right)^{\frac{1}{2a_k}}$$

The right hand side is a decreasing function of a_k , and it can be easily verified that $C_k \leq 1$ whenever $a_k \geq 22$. This proves Theorem 3 for odd period lengths ℓ .

Now assume that the period length ℓ is even, and consider first the case $q_{\ell} > 1$. By Theorem 5 we then have

$$C_k^{\frac{c-2}{c}} \le \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} (200e^{2.4}c^2)^{\frac{1}{c}} \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_k}}$$

The factor $(200e^{2.4}c^2)^{1/c}$ is again decreasing in c, so we replace c by a_kq_ℓ to obtain

$$C_k^{\frac{c-2}{c}} \leq \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} (200e^{2.4}a_k^2 q_\ell^2)^{\frac{1}{a_k q_\ell}} \left(\frac{a_k^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_k}}$$
$$\leq \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} (200e^{2.4}a_k^2)^{\frac{1}{a_k q_\ell}} \cdot a_k^{\frac{5}{2a_k}},$$

where we have used that $q_{\ell}^{2/q_{\ell}} < e$. This expression is decreasing in q_{ℓ} , so we insert $q_{\ell} = 2$ to obtain

$$C_k^{\frac{c-2}{c}} \le \frac{\pi}{\sqrt{2}a_k} e^{1+f(a_k)} \left(200e^{2.4}a_k^7\right)^{\frac{1}{2a_k}}.$$

One can again verify that the right hand side is below one whenever $a_k \ge 23$. This proves Theorem 3 for even ℓ in the case $q_\ell > 1$. Finally, we consider the case $q_{\ell} = 1$. Recall that this can only occur if $\ell = 2$ and $\alpha = [0; \overline{a_1, a_2}]$, with either $a_1 = 1$ or $a_2 = 1$. By Corollary 4, we then have the bound

$$C_k^{\frac{c-2}{c}} \le \frac{\pi}{a_k} e^{1+g(a_k)} \left(6.2(a_k+2)^4 \right)^{\frac{1}{a_k+2}},$$

where $g(a_k) \leq 3.3/a_k + 0.1$ and one can again verify that the right hand side is below one whenever $a_k \geq 21$. This verifies Theorem 3 when $q_\ell = 1$, and completes the proof.

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