# ON THE ORDER OF MAGNITUDE OF SUDLER PRODUCTS II 

SIGRID GREPSTAD, MARIO NEUMÜLLER, AND AGAMEMNON ZAFEIROPOULOS


#### Abstract

We study the asymptotic behavior of Sudler products $P_{N}(\alpha)=\prod_{r=1}^{N} 2|\sin \pi r \alpha|$ for quadratic irrationals $\alpha \in \mathbb{R}$. In particular, we verify the convergence of certain perturbed Sudler products along subsequences, and show that $\liminf _{N} P_{N}(\alpha)=0$ and $\lim \sup _{N} P_{N}(\alpha) / N=\infty$ whenever the maximal digit in the continued fraction expansion of $\alpha$ exceeds 23. This generalizes known results for the period one case $\alpha=[0 ; \bar{a}]$.


## 1. Introduction and Main Results

1.1. Introduction. Let $\alpha \in \mathbb{R}$ and $N \geq 1$ be an integer. The Sudler product at stage $N$ and with parameter $\alpha$ is defined as

$$
P_{N}(\alpha)=\prod_{r=1}^{N}|2 \sin \pi r \alpha| .
$$

Sudler products have been studied extensively, as they bear connections with several areas of research; we mention partition theory, Padé approximants and dynamical systems, and refer to [12] and references therein for further examples and details. In the present paper our main focus will be on the asymptotic order of magnitude of $P_{N}(\alpha)$. This topic has received much attention in recent years, and we begin by briefly reviewing key results relevant to the main results of this paper. For a more detailed overview of the asymptotic behavior of $P_{N}(\alpha)$, we refer to the survey paper [7].

Erdős and Szekeres showed in [5] that $\liminf _{N \rightarrow \infty} P_{N}(\alpha)=0$ for almost all $\alpha$, and conjectured that this result is true for all values of $\alpha$. Lubinsky [10] later confirmed that $\lim \inf _{N \rightarrow \infty} P_{N}(\alpha)=0$ whenever $\alpha$ has unbounded partial quotients in its continued fraction expansion $\alpha=\left[0 ; a_{1}, a_{2}, \ldots\right]$.

More recently, Mestel and Verschueren [12] studied the behavior of Sudler products $P_{N}(\phi)$, where $\phi=[0 ; 1,1, \ldots]$ is the fractional part of the golden ratio. Their precise result was the following.

[^0]Theorem (Mestel, Verschueren): Let $\phi=(\sqrt{5}-1) / 2$ be the fractional part of the golden ratio and $\left(F_{n}\right)_{n=0}^{\infty}$ be the sequence of Fibonacci numbers. Then there exists a constant $C>0$ such that

$$
\lim _{n \rightarrow \infty} P_{F_{n}}(\phi)=C
$$

Moreover, for the same constant $C$ we have $\lim _{n \rightarrow \infty} \frac{P_{F_{n}-1}(\phi)}{F_{n}}=\frac{C \sqrt{5}}{2 \pi}$.
Here the appearance of the Fibonacci sequence is not at all surprising, as it is the sequence of denominators associated with the continued fraction expansion of $\phi$. The proof of the result relies on the specific continued fraction expansion $\phi=[0 ; 1,1, \ldots]$ and the algebraic properties of the sequence $\left(F_{n}\right)_{n=0}^{\infty}$. We now know that the convergence property for $P_{F_{n}}(\phi)$ is a special case of a phenomenon exhibited by all quadratic irrationals [8].

Theorem (Grepstad, Neumüller): Let $\alpha=\left[0 ; \overline{a_{1}, a_{2}, \ldots, a_{\ell}}\right]$ be a purely periodic quadratic irrational, where $\ell \geq 1$ and $a_{1}, \ldots, a_{\ell} \in \mathbb{N}$, and let $\left(q_{n}\right)_{n=1}^{\infty}$ be the sequence of denominators of convergents of $\alpha$. Then there exist constants $C_{1}, C_{2}, \ldots, C_{\ell}>0$ such that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} P_{q_{m \ell+k}}(\alpha)=C_{k}, \quad k=1,2, \ldots, \ell . \tag{1.1}
\end{equation*}
$$

Moreover if $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ is a quadratic irrational with the same periodic part as $\alpha$ in its continued fraction expansion, then

$$
\lim _{m \rightarrow \infty} P_{q_{h+m \ell+k}}(\beta)=C_{k}, \quad k=1,2, \ldots, \ell .
$$

for the same constants $C_{1}, \ldots, C_{\ell}$.
Later on, Grepstad, Kaltenböck and Neumüller employed the factorisation technique used in the proof of Mestel and Verschueren's result to show that $\liminf P_{N}(\phi)>0$ [6], finally disproving the conjecture in [5].

The proof of the lower bound on $P_{N}(\phi)$ given in [6] involved studying a perturbed Sudler product $\prod_{r=1}^{N} 2|\sin \pi(r \phi+\varepsilon)|$. A systematic treatment of such perturbed products was conducted in [2], where the result by Mestel and Verschueren was generalized to quadratic irrationals of the form $\beta=[0 ; b, b, \ldots]$ by an in-depth study of the product

$$
\begin{equation*}
P_{q_{n}}(\beta, \varepsilon)=\prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \beta+(-1)^{n} \frac{\varepsilon}{q_{n}}\right)\right| . \tag{1.2}
\end{equation*}
$$

In [2] it is shown that for each digit $b \geq 1$, the sequence of functions $P_{q_{n}}(\beta, \varepsilon)$ converges locally uniformly to an explicitly defined function $G_{b}(\varepsilon)$, and from this the authors deduce the following strong result on the asymptotic behavior of $P_{N}(\beta)$.

Theorem (Aistleitner, Technau, Zafeiropoulos): Let $\beta=[0 ; b, b, \ldots]$, where $b \geq 1$. The following holds.
(i) If $b \leq 5$, then $\liminf _{N \rightarrow \infty} P_{N}(\beta)>0$ and $\limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}<\infty$.
(ii) If $b \geq 6$, then $\liminf _{N \rightarrow \infty} P_{N}(\beta)=0$ and $\limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}=\infty$.

The theorem above gives a complete description of the asymptotic order of magnitude of $P_{N}(\beta)$ for irrationals $\beta=[0 ; b, b, \ldots]$. The main objective of this paper is to study the asymptotic behaviour of $P_{N}(\beta)$ for arbitrary quadratic irrationals $\beta$, that is irrationals whose continued fraction expansions are eventually periodic with some period length $\ell$. It turns out that for such $\beta$, the sequence of functions $P_{q_{n}}(\beta, \varepsilon)$ defined in 1.2) will converge along specific subsequences to $\ell$ explicitly defined functions $G_{k}(\beta, \varepsilon), 1 \leq k \leq \ell$ (see Theorem 1 below). This is, in some sense, the expected generalization of the result on $\beta=[0 ; b, b, \ldots]$ in [2]. As a consequence, we obtain a partial extension of the theorem above to arbitrary quadratic irrationals (Theorem 3).

As we shall explain later, the asymptotic behaviour of $P_{N}(\beta)$ for $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ is similar to that of $P_{N}(\alpha)$, where $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$. It turns out that purely periodic irrationals are quite easier to analyse in terms of their continued fraction expansions. Moreover, certain relations following from such an analysis are needed in the statement of our main results. Let us therefore briefly review certain basic properties for the convergents of purely periodic irrationals.
1.2. The irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$. The $n$-th convergent of $\alpha$ is the number $p_{n} / q_{n}$, where

$$
\begin{aligned}
p_{n+1} & =a_{n} p_{n}+p_{n-1}, & & p_{0}=1,
\end{aligned} p_{1}=0, ~ 子, ~ q_{0}=0, \quad q_{1}=1 .
$$

We mention that the dependence of $p_{n}$ and $q_{n}$ on $\alpha$ is not explicitly stated, but if necessary we will write $p_{n}(\alpha)$ and $q_{n}(\alpha)$ to make this dependence explicit. The sequence of convergents satisfies

$$
\frac{p_{1}}{q_{1}}<\frac{p_{3}}{q_{3}}<\cdots<\alpha<\cdots<\frac{p_{4}}{q_{4}}<\frac{p_{2}}{q_{2}}
$$

and

$$
\frac{1}{2 q_{n+1} q_{n}}<\left|\alpha-\frac{p_{n}}{q_{n}}\right|<\frac{1}{q_{n+1} q_{n}}, \quad n \geq 1 .
$$

We use the notation of [8] and set

$$
\begin{align*}
& c(\alpha)=c=q_{\ell+1}+p_{\ell}, \\
& a(\alpha)=a=\frac{c(\alpha)+\sqrt{c(\alpha)^{2}+4(-1)^{\ell-1}}}{2},  \tag{1.3}\\
& b(\alpha)=b=\frac{c(\alpha)-\sqrt{c(\alpha)^{2}+4(-1)^{\ell-1}}}{2} .
\end{align*}
$$

The sequence $\left(q_{n}\right)_{n=1}^{\infty}$ of denominators satisfies the additional recursive relation

$$
\begin{equation*}
q_{n+\ell}=c(\alpha) q_{n}+(-1)^{\ell-1} q_{n-\ell}, \quad n \geq 2 \ell \tag{1.4}
\end{equation*}
$$

For $k=0,1, \ldots, \ell-1$ we set

$$
\begin{equation*}
c_{k}=\frac{q_{\ell+k}-b q_{k}}{a-b} \quad \text { and } \quad e_{k}=(-1)^{k-1} \frac{\left|a q_{k}-q_{\ell+k}\right|}{q_{\ell}} . \tag{1.5}
\end{equation*}
$$

For notational convenience we extend the definitions of $c_{k}$ and $e_{k}$ to all integers $k \geq 0$ periodically modulo $\ell$, so that in particular we have $c_{\ell}=c_{0}$ and $e_{\ell}=e_{0}$. We also make use of the following relations (for more details see e.g. [8]):

$$
\begin{gather*}
c_{k}>0, \\
\Lambda_{m \ell+k}:=q_{m \ell+k} \alpha-p_{m \ell+k}=e_{k} b^{m}=(-1)^{m \ell+k+1}\left|e_{k} b^{m}\right|, \\
\frac{1}{q_{m \ell+k}}=\mathcal{O}\left(|b|^{m}\right), \quad m \rightarrow \infty  \tag{1.6}\\
q_{m \ell+k}|b|^{m}=c_{k}+\mathcal{O}\left(b^{2 m}\right), \quad m \rightarrow \infty .
\end{gather*}
$$

When studying the irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$, it is useful to consider two families of permutations on $\ell$-tuples of positive integers $\boldsymbol{a}=\left(a_{1}, \ldots, a_{\ell}\right)$. For $k=0,1, \ldots, \ell-1$ we define the permutation operator $\tau_{k}: \mathbb{N}^{\ell} \rightarrow \mathbb{N}^{\ell}$ by

$$
\tau_{k}(\boldsymbol{a})=\left(a_{k+1}, \ldots, a_{\ell}, a_{1}, \ldots, a_{k}\right)
$$

Likewise, we define the permutations $\sigma_{k}: \mathbb{N}^{\ell} \rightarrow \mathbb{N}^{\ell}$ for $k=2, \ldots, \ell$ by

$$
\sigma_{k}(\boldsymbol{a})=\left(a_{k-1}, \ldots, a_{1}, a_{\ell}, \ldots, a_{k}\right)
$$

while for $k=1$ we set $\sigma_{1}(\boldsymbol{a})=\left(a_{\ell}, \ldots, a_{1}\right)$. We can define $\tau_{k}$ and $\sigma_{k}$ for all $k \geq 1$ by extending the definitions above periodically modulo $\ell$. Given a purely periodic irrational $\alpha$ with period $\boldsymbol{a}$, the corresponding purely periodic irrationals with periods $\tau_{k}(\boldsymbol{a})$ and $\sigma_{k}(\boldsymbol{a})$ will be denoted by

$$
\begin{equation*}
\alpha_{\tau_{k}}=\left[0 ; \overline{a_{k+1}, \ldots, a_{\ell}, a_{1}, \ldots, a_{k}}\right] \quad \text { and } \quad \alpha_{\sigma_{k}}=\left[0 ; \overline{a_{k-1}, \ldots a_{1}, a_{\ell}, \ldots, a_{k}}\right] . \tag{1.7}
\end{equation*}
$$

The significance of the permutations $\tau_{k}$ and $\sigma_{k}$ when studying the approximation properties of $\alpha$ is indicated by the following relations, which hold for any index $k=0,1, \ldots, \ell-1$ :

$$
\begin{gather*}
c(\alpha)=c\left(\alpha_{\tau_{k}}\right)=c\left(\alpha_{\sigma_{k}}\right), \\
q_{\ell}\left(\alpha_{\tau_{k}}\right)=q_{\ell}\left(\alpha_{\sigma_{k}}\right), \\
p_{\ell}\left(\alpha_{\tau_{k}}\right)=q_{\ell-1}\left(\alpha_{\sigma_{k}}\right) \text { and } p_{\ell}\left(\alpha_{\sigma_{k}}\right)=q_{\ell-1}\left(\alpha_{\tau_{k}}\right), \\
\frac{q_{\ell+1}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}=a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)},  \tag{1.8}\\
\left|c_{k} e_{k}\right|=\frac{q_{\ell}\left(\alpha_{\tau_{k}}\right)}{c\left(\alpha_{\tau_{k}}\right)-2 b} .
\end{gather*}
$$

1.3. Main Results. We are now equipped to state our main results. As alluded to above, our first goal is to generalize the convergence result of [2] on perturbed products to irrationals $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ with $\ell \geq 2$. For any $\varepsilon \in \mathbb{R}$ we define

$$
\begin{equation*}
P_{q_{n}}(\beta, \varepsilon):=\prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \beta+(-1)^{n+1} \frac{\varepsilon}{q_{n}}\right)\right| . \tag{1.9}
\end{equation*}
$$

In view of the aforementioned theorem by Grepstad and Neümuller in [8], one would expect the perturbed products to converge along specific subsequences. We show that this is indeed the case. For the sake of convenience, we introduce the notation

$$
\begin{equation*}
u_{k}(t)=2\left(\frac{t}{\left|e_{k} c_{k}\right|}-\left\{t \alpha_{\sigma_{k}}\right\}+\frac{1}{2}\right), \quad t=1,2, \ldots \tag{1.10}
\end{equation*}
$$

for each $k=1, \ldots, \ell$, where $c_{k}, e_{k}$ are as in (1.5) and $\alpha_{\sigma_{k}}$ as in (1.7), all referring to the purely periodic irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$.
Theorem 1. Let $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ where $h \geq 0, \ell, a_{1}, \ldots, a_{\ell} \in \mathbb{N}$ and $P_{q_{n}}(\beta, \varepsilon)$ be the sequence of perturbed Sudler products defined in (1.9). Then for each $k=1, \ldots, \ell$ the subsequence $P_{q_{h+m \ell+k}}(\beta, \varepsilon)$ converges locally uniformly to a function $G_{k}(\beta, \varepsilon)$. The limit function satisfies

$$
\begin{align*}
G_{k}(\beta, \varepsilon)= & \left|1+\frac{\varepsilon}{\left|c_{k} e_{k}\right|}\right|\left(1+\frac{1}{|b|^{2}}\right)^{\frac{1}{c-2}} \frac{1}{(c!)^{1 /(c-2)}} \times \\
1.11) & \times \prod_{t=1}^{\infty}\left|\left(1-\frac{\left(1+\frac{2 \varepsilon}{\left|e_{k} c_{k}\right|}\right)^{2}}{u_{k}(t)^{2}}\right)\left(1-\frac{\left(1+\frac{2}{|b|^{2}}\right)^{2}}{u_{k}(t)^{2}}\right)^{\frac{1}{c-2}} \prod_{s=1}^{c-1}\left(1-\frac{(1+2 s)^{2}}{u_{k}(t)^{2}}\right)^{-\frac{1}{c-2}}\right| \tag{1.11}
\end{align*}
$$

when $\ell$ is even, and

$$
\begin{equation*}
G_{k}(\beta, \varepsilon)=\frac{\left|1+\frac{\varepsilon}{\left|c_{k} e_{k}\right|}\right|}{\prod_{s=1}^{c}|s-a|^{\frac{1}{c}}} \times \prod_{t=1}^{\infty}\left|\left(1-\frac{\left(1+\frac{2 \varepsilon}{\left|e_{k} c_{k}\right|}\right)^{2}}{u_{k}(t)^{2}}\right) \prod_{s=0}^{c-1}\left(1-\frac{\left(1+2\left(s-\frac{1}{|b|}\right)\right)^{2}}{u_{k}(t)^{2}}\right)^{-\frac{1}{c}}\right| \tag{1.12}
\end{equation*}
$$

when $\ell$ is odd. Here the sequence $\left(u_{k}(t)\right)_{t=1}^{\infty}$ is given in 1.10) and the constants a, b, $c$, $c_{k}$ and $e_{k}$ are defined in Section 1.2, all corresponding to the purely periodic irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$. In both cases, the functions $G_{k}(\beta, \cdot), k=1, \ldots, \ell$, are continuous and $C^{\infty}$ on every interval where they are non-zero.

The formulae (1.11) and 1.12 ) in Theorem 1 imply that the limit functions $G_{k}(\beta, \varepsilon)$ only depend on the periodic part of the continued fraction expansion of $\beta$; the digits $b_{1}, \ldots, b_{h}$ in the pre-periodic part do not play any role at all.

Remark 1. Note that we have altered the definition of $P_{q_{n}}(\alpha, \varepsilon)$ compared to [2], i.e. we use $(-1)^{n+1}$ instead of $(-1)^{n}$. This relates to the fact that in [8] the denominator of the $n$-th convergent was defined as $q_{n+1}$ while in [2] the denominator of the $n$-th convergent is $q_{n}$.

Remark 2. An alternative proof of Theorem 1 has recently appeared in 1]. There the limit function is given in a different form and additionally an explicit approximation error is obtained.

Since the constants $C_{1}, \ldots, C_{\ell}$ in (1.1) satisfy $C_{k}=G_{k}(\beta, 0),(1 \leq k \leq \ell)$, Theorem 1 allows us to explicitly calculate their values.

Corollary 1. Let $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ and $C_{1}, \ldots, C_{\ell}>0$ be the constants in (1.1). Then for $k=1,2, \ldots, \ell$ we have

$$
\begin{equation*}
C_{k}=\left(\frac{1+a^{2}}{c!}\right)^{\frac{1}{c-2}} \prod_{t=1}^{\infty}\left(1-\frac{1}{u_{k}(t)^{2}}\right)\left|1-\frac{\left(1+2 a^{2}\right)^{2}}{u_{k}(t)^{2}}\right|^{\frac{1}{c-2}} \prod_{s=1}^{c-1}\left|1-\frac{(1+2 s)^{2}}{u_{k}(t)^{2}}\right|^{-\frac{1}{c-2}} \tag{1.13}
\end{equation*}
$$

when $\ell$ is even, and

$$
\begin{equation*}
C_{k}=\frac{1}{\prod_{s=1}^{c}|s-a|^{\frac{1}{c}}} \prod_{t=1}^{\infty}\left(1-\frac{1}{u_{k}(t)^{2}}\right) \prod_{s=0}^{c-1}\left|1-\frac{(1+2 s-2 a)^{2}}{u_{k}(t)^{2}}\right|^{-\frac{1}{c}} \tag{1.14}
\end{equation*}
$$

when $\ell$ is odd.

Our next result relates the asymptotic size of $P_{N}(\beta)$ with the size of the constants $C_{1}, \ldots, C_{\ell}$ in (1.1). This is the analogue of Lemma 1 in [2], and the proof is nearly identical. Nevertheless, we include the proof later in the text for the sake of completeness.
Theorem 2. Let $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ and $\left(C_{k}\right)_{k=1}^{\ell}$ be the constants as in 1.1). If $C_{k_{0}}<1$ for some index $1 \leq k_{0} \leq \ell$ then

$$
\begin{equation*}
\liminf _{N \rightarrow \infty} P_{N}(\beta)=0 \quad \text { and } \quad \limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}=\infty \tag{1.15}
\end{equation*}
$$

Remark 3. By the aforementioned Theorem of Grepstad and Neumüller, the values of $C_{1}, \ldots, C_{k}$ only depend on the periodic part of the quadratic irrational $\beta$. Combined with Theorem 2, this explains why it suffices to consider only purely periodic irrationals when trying to detect those irrationals $\beta$ for which the Sudler product $P_{N}(\beta)$ satisfies 1.15).

Theorem 2 tells us that as long as one of the constants $C_{k}(1 \leq k \leq \ell)$ defined in (1.1) is less than 1 , the Sudler product corresponding to the irrational $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ satisfies 1.15). This raises the question of which out of the $\ell$ constants $C_{k}$ associated with $\alpha=\left[0 ; \overline{a_{1}, \ldots a_{\ell}}\right]$ is expected to be minimal.


Figure 1. Plots of the limit functions $G_{2}(\alpha, \varepsilon)$ for the stated values of $\alpha=\left[0 ; \overline{1, a_{2}}\right]$. It appears that $C_{2}=G_{2}(\alpha, 0)<1$ whenever $a_{2} \geq 4$.

The plots in Figures 1 and 2 show graphs of the functions $G_{k}(\alpha, \varepsilon)$ for specific choices of $\alpha=\left[0 ; \overline{a_{1}, a_{2}}\right]$ and $k \in\{1,2\}$. Since $C_{k}=G_{k}(\alpha, 0)$, the value of $C_{k}$ is the ordinate of the point of intersection of the graph with the vertical axis. These graphs seem to suggest that the bigger the digit $a_{k}$ is, the smaller the constant $C_{k}$ becomes.

In spite of the hints provided by the plots, it remains to verify rigorously that $C_{k}=G_{k}(\alpha, 0)$ decreases with increasing values for the digit $a_{k}$. Moreover, it should be pointed out that the given plots provide no information on the significance of the period length $\ell$ in the continued fraction expansion of $\alpha$. The period length can be chosen arbitrarily large, and it might be that the size of $\ell$ has an impact on the overall sizes of the constants $C_{k}$. Moving forward, we will thus focus on three main questions:

- Is the phenomenon implied by the graphs in Figures 1 and 2 indeed true, i.e. for any index $1 \leq k \leq \ell$, is $C_{k}$ decreasing as a function of the digit $a_{k}$ ?
- Suppose we fix some period length $\ell \geq 2$. Does there exist an integer $K=K_{\ell} \geq 1$ such that for any irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ with $\max _{1 \leq i \leq \ell} a_{i} \geq K$ the Sudler product $P_{N}(\alpha)$ satisfies (1.15)?


Figure 2. Plots of the limit functions $G_{2}(\alpha, \varepsilon)$ for the stated values of $\alpha=\left[0 ; \overline{2, a_{2}}\right]$. It appears that $C_{2}=G_{2}(\alpha, 0)<1$ whenever $a_{2} \geq 5$.

- If such an integer exists, can it be chosen independently of the period length $\ell$ ?

By a careful analysis of the product formulas established in Corollary 1 we find that when $k$ is the index corresponding to the maximal digit $a_{k}$ in $\alpha=\left[0 ; \overline{a_{1}, \ldots a_{\ell}}\right]$, then $C_{k}$ is bounded above by an expression which is indeed decreasing as a function of $a_{k}$. In turn, this leads to the following result, which provides a positive answer to questions two and three.
Theorem 3. Let $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ be a quadratic irrational with period length $\ell \geq 2$, and say $a_{k}=\max _{j} a_{j}$. Then

$$
\liminf _{N \rightarrow \infty} P_{N}(\beta)=0 \quad \text { and } \quad \limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}=\infty
$$

whenever $a_{k} \geq 23$.
Remark 4. Recall that it was shown by Lubinsky that $\lim \inf P_{N}(\alpha)=0$ whenever $\alpha$ has unbounded coefficients in its continued fraction expansion [10]. In fact, Lubinsky made the more striking observation that there exists a cutoff value $a_{k} \geq K$ for which Theorem 3 is true, not only for quadratic irrationals but for any irrational $\alpha$. Note, however, that Lubinsky's approach merely tells us that $K \approx e^{800}$ will suffice. Theorem 3 is thus a significant improvement of the best known cutoff value $K$ for quadratic irrationals.

Theorem 3 can be seen as a partial analogue of the second part of the aforementioned theorem by Aistleitner, Technau and Zafeiropoulos. We will not attempt to imitate the first part of their result, stating that $\liminf P_{N}(\alpha)>0$ for sufficiently small values of $\max _{i} a_{i}$. It will be clear from the proof of Theorem 3 that the role played by the period length $\ell$ is not fully understood, and in light of this we leave the following open problems for further discussion.

Questions. Let $\alpha=\left[0 ; \overline{a_{1}, a_{2}, \ldots, a_{\ell}}\right]$ be a quadratic irrational with $a_{k}=\max _{i} a_{i}$.

- According to Theorem 3, there exists an integer $K=K_{\ell} \geq 1$ such that (1.15) holds whenever $a_{k} \geq K_{\ell}$, and $K_{\ell} \leq 23$ for all period lengths $\ell$. However, if we fix $\ell$, what is then the optimal value of $K_{\ell}$ ? We will see in the proof (see Section 6) that for odd periods $\ell$, Theorem 3 holds for $K=22$. Moreover, for the special case when $\alpha=\left[0 ; \overline{1, a_{2}}\right]$, Theorem 3 holds for $K=21$, and the plots in Figure 1 and 2 suggest that we can actually do much better. This brings us to the following question.
- Is it possibly true that if $a_{k} \geq 6$, then $\liminf _{N \rightarrow \infty} P_{N}(\alpha)=0$ and $\limsup _{N \rightarrow \infty} \frac{P_{N}(\alpha)}{N}=\infty$ ? In other words, is $K_{\ell} \leq 6$ for all $\ell \geq 1$ ? Numerical evidence seems to suggest that the answer is positive, and that the threshold value $K=6$ established for irrationals $\beta=[0 ; b, b, \ldots]$ in [2] might in fact be a universal bound for all quadratic irrationals.

Finally, we point out that Aistleitner and Borda have shown the following duality in [1]: for any badly approximable $\alpha$, we have

$$
\liminf _{N \rightarrow \infty} P_{N}(\alpha)=0 \quad \text { if and only if } \quad \limsup _{N \rightarrow \infty} \frac{P_{N}(\alpha)}{N}=\infty
$$

Thus for a fixed period length $\ell \geq 2$, giving a complete characterisation of the quadratic irrationals $\alpha$ for which (1.15) holds also determines those irrationals $\alpha$ for which $\liminf _{N \rightarrow \infty} P_{N}(\alpha)>0$ and $\limsup _{N \rightarrow \infty} P_{N}(\alpha) / N<\infty$.
1.4. Oragnization of the paper. The remainder of the paper is organized as follows. Theorems 1 and 2 are proved in Sections 2 and 3, respectively. In Section 4, we analyse the product

$$
G(x)=\prod_{t=1}^{\infty}\left(1-\frac{x^{2}}{u_{k}(t)^{2}}\right),
$$

with $u_{k}(t)$ as defined in 1.10 . This product plays a crucial role in the expressions for $C_{k}$ in Corollary 1. Note that the sequence $\mathcal{U}=\left(u_{k}(t)\right)_{t \in \mathbb{N}}$ can be viewed as a perturbation of
the arithmetic progression $\left(2 t /\left|e_{k} c_{k}\right|\right)_{t \in \mathbb{N}}$, so it is natural to compare $G(x)$ to the product

$$
\prod_{t=1}^{\infty}\left(1-\frac{x^{2}}{\left(2 t /\left|e_{k} c_{k}\right|\right)^{2}}\right)
$$

In doing so, we obtain Theorem 4, which tells us that

$$
\begin{equation*}
K_{1} \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \leq|G(x)| \leq K_{2} \frac{1}{|x|} \tag{1.16}
\end{equation*}
$$

for appropriate constants $K_{1}$ and $K_{2}$.
In Section 5 we use Corollary 1 and Theorem 4 to find an upper bound on $C_{k}$ in (1.1) for both odd and even periods $\ell$. It turns out that $C_{k}$ can be bounded by expressions which clearly decrease to zero as $a_{k} \rightarrow \infty$ (see Theorems 5and 6). The speed of decay depends on the constants $K_{1}$ and $K_{2}$ in (1.16), and for this reason we treat separately the case $\alpha=\left[0 ; \overline{a_{1}, a_{2}}\right]$ where either $a_{1}=1$ or $a_{2}=1$ (as better constants $K_{1}$ and $K_{2}$ can then be found).

Finally, in Section 6 we use the bounds obtained for $C_{k}$ in Section 5 to show that $C_{k}<1$ whenever $\alpha=\left[0 ; \overline{a_{1}, a_{2}, \ldots, a_{k}}\right]$ with $\max _{j} a_{j} \geq 23$. By Theorem 2, this proves Theorem 3 .

## 2. Proof of Theorem 1

Theorem 1 states that when $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$, the perturbed Sudler products $P_{q_{n}}(\beta, \varepsilon)$ in (1.9) converge along subsequences $\left(q_{h+m \ell+k}\right)_{m=1}^{\infty}$ to explicit limit functions

$$
G_{k}(\beta, \varepsilon)=\lim _{m \rightarrow \infty} P_{q_{h+m \ell+k}}(\beta, \varepsilon), \quad k=1, \ldots, \ell
$$

We first present the proof of Theorem 1 for the purely periodic quadratic irrational $\alpha=$ $\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ (i.e. when $h=0$ ). We then briefly explain how the proof can be generalised for arbitrary quadratic irrationals $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$.

Our first observation is that $P_{q_{n}}(\alpha, \varepsilon)$ can be decomposed into a product of three factors.
Lemma 1. Let $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ and let $q_{n}$ denote the denominator of the $n$-th convergent of $\alpha$. Then for any $\varepsilon \in \mathbb{R}$,

$$
P_{q_{n}}(\alpha, \varepsilon)=A_{n}(\alpha, \varepsilon) \cdot B_{n}(\alpha) \cdot C_{n}(\alpha, \varepsilon),
$$

where

$$
\begin{aligned}
A_{n}(\alpha, \varepsilon) & =2 q_{n}\left|\sin \pi\left(\Lambda_{n}+(-1)^{n+1} \frac{\varepsilon}{q_{n}}\right)\right| \\
B_{n}(\alpha) & =\left|\prod_{t=1}^{q_{n}-1} \frac{s_{n}(t)}{2 \sin \left(\pi t / q_{n}\right)}\right|,
\end{aligned}
$$

$$
C_{n}(\alpha, \varepsilon)=\prod_{t=1}^{q_{n}-1}\left(1-\frac{s_{n}^{2}(0, \varepsilon)}{s_{n}^{2}(t)}\right)^{\frac{1}{2}}
$$

and

$$
s_{n}(0, \varepsilon)=2 \sin \pi\left(\frac{\Lambda_{n}}{2}+(-1)^{n+1} \frac{\varepsilon}{q_{n}}\right), \quad s_{n}(t)=2 \sin \pi\left(\frac{t}{q_{n}}-\left|\Lambda_{n}\right|\left(\left\{\frac{t q_{n-1}}{q_{n}}\right\}-\frac{1}{2}\right)\right) .
$$

Lemma 1 is the natural analogue of Lemma 5.1 of [12] for the product $P_{q_{n}}(\phi)$ and Lemma 4.2 in [8] for $P_{q_{n}}(\alpha)$. We omit the proof since it is nearly identical, the only difference being that it involves an additional term within the argument of the sine. We continue by analysing the behaviour of each of the three factors $A_{n}(\alpha, \varepsilon), B_{n}(\alpha)$ and $C_{n}(\alpha, \varepsilon)$.

The factor $B_{n}=B_{n}(\alpha)$ is independent of the perturbation argument $\varepsilon$, and it is shown in [8] that for each $k=1,2, \ldots, \ell$ the limit

$$
B^{(k)}=\lim _{m \rightarrow \infty} B_{m \ell+k}
$$

exists. Regarding the factor $A_{m \ell+k}(\alpha, \varepsilon)$, we have

$$
A_{m \ell+k}(\alpha, \varepsilon)=2 \pi| | c_{k} e_{k}|+\varepsilon|+\mathcal{O}\left(b^{2 m}\right), \quad m \rightarrow \infty
$$

therefore $A_{m \ell+k}(\alpha, \varepsilon)$ converges to $2 \pi\left|\left|c_{k} e_{k}\right|+\varepsilon\right|$. Finally we need to establish convergence for the factor $C_{n}(\alpha, \varepsilon)$. Here we can argue as in [12, Section 6], taking into account that the factor $s_{n}(0, \varepsilon)$ depends on the parameter $\varepsilon$ and satisfies

$$
\left|s_{m \ell+k}(0, \varepsilon)\right| \sim \pi\left|\left|e_{k} b^{m}\right|+2 \varepsilon\right|+\mathcal{O}\left(b^{2 m}\right), \quad m \rightarrow \infty
$$

and also

$$
s_{m \ell+k}(t)=\pi\left|e_{k} b^{m}\right| u_{k}(t)+\mathcal{O}\left(|b|^{m / 5}\right), \quad m \rightarrow \infty .
$$

The same arguments as in [12] imply that for any $\varepsilon \in \mathbb{R}$,

$$
\lim _{m \rightarrow \infty} C_{m \ell+k}(\alpha, \varepsilon)=\prod_{t=1}^{\infty}\left|1-\frac{\left(1+\frac{2 \varepsilon}{\left|e_{k} c_{k}\right|}\right)^{2}}{u_{k}(t)^{2}}\right|
$$

In view of Lemma 1 we deduce that for $k=1,2, \ldots, \ell$ the limiting function $G_{k}$ satisfies

$$
\begin{equation*}
G_{k}(\alpha, \varepsilon)=2 \pi| | e_{k} c_{k}|+\varepsilon| B^{(k)} \prod_{t=1}^{\infty}\left|1-\frac{\left(1+\frac{2 \varepsilon}{\left|e_{k} c_{k}\right|}\right)^{2}}{u_{k}(t)^{2}}\right|, \tag{2.1}
\end{equation*}
$$

where $u_{k}(t)$ is defined in (1.10). Arguing as in the proof of Theorem 1 in [2] we can show that the convergence is locally uniform. Now we fix a value of $k=1, \ldots, \ell$ and consider
indices $n=m \ell+k, m=1,2, \ldots$ In order to determine the formula of $G_{k}$ we will use relation $(1.4)$. We distinguish two cases depending on the parity of the period length $\ell$.

- If $\ell \equiv 0(\bmod 2)$, then $(1.4)$ gives $c(\alpha) q_{n}=q_{n+\ell}+q_{n-\ell}$, and thus

$$
\begin{equation*}
P_{c q_{n}}(\alpha)=P_{q_{n-\ell}+q_{n+\ell}}(\alpha) . \tag{2.2}
\end{equation*}
$$

The left hand side in $(2.2)$ is

$$
\begin{aligned}
P_{c q_{n}}(\alpha) & =\prod_{r=1}^{c q_{n}} 2|\sin \pi r \alpha|=\prod_{s=0}^{c-1} \prod_{r=1+s q_{n}}^{(s+1) q_{n}} 2|\sin \pi r \alpha|=\prod_{s=0}^{c-1} \prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \alpha+s q_{n} \alpha\right)\right| \\
& \stackrel{\text { 1.6 }}{=} \prod_{s=0}^{c-1} \prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \alpha+(-1)^{n+1} s\left|e_{k} b^{m}\right|\right)\right|=\prod_{s=0}^{c-1} P_{q_{n}}\left(\alpha, s q_{n}\left|e_{k} b^{m}\right|\right),
\end{aligned}
$$

while the right hand side is

$$
\begin{aligned}
P_{q_{n-\ell}+q_{n+\ell}}(\alpha) & =\prod_{r=1}^{q_{n+\ell}+q_{n-\ell}} 2|\sin \pi r \alpha|=\prod_{r=1}^{q_{n-\ell}} 2|\sin \pi r \alpha| \cdot \prod_{r=1+q_{n-\ell}}^{q_{n+\ell}+q_{n-\ell}} 2|\sin \pi r \alpha| \\
& =P_{q_{n-\ell}}(\alpha) \cdot \prod_{r=1}^{q_{n+\ell}} 2\left|\sin \pi\left(r \alpha+q_{n-\ell} \alpha\right)\right| \\
& \stackrel{\boxed{1.6}}{=} P_{q_{n-\ell}}(\alpha) \cdot \prod_{r=1}^{q_{n+\ell}} 2\left|\sin \pi\left(r \alpha+(-1)^{n+\ell+1}\left|e_{k} b^{m-1}\right|\right)\right| \\
& =P_{q_{n-\ell}}(\alpha) P_{q_{n+\ell}}\left(\alpha, q_{n+\ell}\left|e_{k} b^{m-1}\right|\right) .
\end{aligned}
$$

Since the functions $P_{q_{n}}(\alpha, \varepsilon)$ converge locally uniformly, letting $m \rightarrow \infty$ in 2.2 and taking (1.6) into account we obtain

$$
\begin{equation*}
\prod_{s=0}^{c-1} G_{k}\left(\alpha, s\left|c_{k} e_{k}\right|\right)=G_{k}(\alpha, 0) G_{k}\left(\alpha, \frac{\left|c_{k} e_{k}\right|}{|b|^{2}}\right) \tag{2.3}
\end{equation*}
$$

Substituting $G_{k}$ from (2.1) in (2.3) we obtain (1.11).

- If $\ell \equiv 1(\bmod 2)$, then $(1.4)$ becomes $q_{n+\ell}=c(\alpha) q_{n}+q_{n-\ell}$, so

$$
\begin{aligned}
\frac{P_{q_{n+\ell}}(\alpha)}{P_{q_{n-\ell}}(\alpha)} & =\prod_{r=1}^{q_{n+\ell}-q_{n-\ell}} 2\left|\sin \pi\left(r \alpha+q_{n-\ell} \alpha\right)\right|=\prod_{r=1}^{c(\alpha) q_{n}} 2\left|\sin \pi\left(r \alpha+q_{n-\ell} \alpha\right)\right| \\
& =\prod_{s=0}^{c(\alpha)-1} \prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \alpha+s q_{n} \alpha+q_{n-\ell} \alpha\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
& \text { 1.6) } \prod_{s=0}^{c(\alpha)-1} \prod_{r=1}^{q_{n}} 2\left|\sin \pi\left(r \alpha+(-1)^{n+1} s\left|e_{k} b^{m}\right|-(-1)^{n+1}\left|e_{k} b^{m-1}\right|\right)\right| \\
& =\prod_{s=0}^{c(\alpha)-1} P_{q_{n}}\left(\alpha, q_{n}\left|e_{k} b^{m}\right|\left(s-\frac{1}{|b|}\right)\right) .
\end{aligned}
$$

By the main result of [8] we know that the sequence $P_{q_{n}}(\alpha)$ converges to a limit $C_{k}>0$, hence letting $m \rightarrow \infty$ in the above equality we get

$$
\prod_{s=0}^{c(\alpha)-1} G_{k}\left(\alpha,\left|c_{k} e_{k}\right|\left(s-\frac{1}{|b|}\right)\right)=1
$$

Substituting this into (2.1) we obtain (1.12). This completes the proof of Theorem 1 when $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{k}}\right]$ is a purely periodic quadratic irrational.

We now deal with quadratic irrationals for which the length of the pre-period is $h \geq 1$. If we consider the irrational $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$ and $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ is as before, we can still find a factorisation

$$
P_{q_{n}}(\beta, \varepsilon)=A_{n}(\beta, \varepsilon) \cdot B_{n}(\beta) \cdot C_{n}(\beta, \varepsilon),
$$

where the three factors are defined similarly to the purely periodic case; the only difference which appears, is that the parameters $e_{k}$ and $c_{k}$ are replaced by different parameters $c_{h, k}$ and $e_{h, k}$. The definition of these parameters is given in the proof of Corollary 1.3 in [8].

The limits of the three factors of $P_{q_{n}}(\beta, \varepsilon)$ are:

$$
\begin{aligned}
\lim _{m \rightarrow \infty} A_{h+m \ell+k}(\beta, \varepsilon) & =2 \pi| | c_{h, k} e_{h, k}|+\varepsilon|, \\
\lim _{m \rightarrow \infty} B_{h+m \ell+k}(\beta) & =B^{(h, k)}, \quad \text { and } \\
\lim _{m \rightarrow \infty} C_{h+m \ell+k}(\beta, \varepsilon) & =\prod_{t=1}^{\infty}\left|1-\frac{\left(1+\frac{2 \varepsilon}{\left|e_{h, k} c_{h, k}\right|}\right)^{2}}{u_{k}(t)^{2}}\right| .
\end{aligned}
$$

We therefore deduce that for $k=1,2, \ldots, \ell$, the subsequence $P_{q_{h+\ell_{m+k}}}(\beta, \varepsilon)$ converges to some limit function $G_{k}(\beta, \varepsilon)$. We now invoke the fact that

$$
\left|c_{h, k} e_{h, k}\right|=\left|c_{k} e_{k}\right|, \quad k=1, \ldots, \ell
$$

(established in the proof of Corollary 1.3 in [8]) to deduce that for all $k=1, \ldots, \ell$,

$$
\lim _{m \rightarrow \infty} P_{q_{h+m \ell+k}}(\beta, \varepsilon)=G_{k}(\beta, \varepsilon)=\lambda_{k} G_{k}(\alpha, \varepsilon),
$$

for some constant $\lambda_{k}$. Finally, since we know from [8, Corollary 1.3] that $G_{k}(\beta, 0)=$ $G_{k}(\alpha, 0)$ for each $k=1, \ldots, \ell$, it follows that $\lambda_{k}=1$ for every $k$. This shows that adding a
pre-period to the continued fraction expansion of $\alpha$ leaves the limit functions $G_{k}$ unchanged, completing the proof of Theorem 1 .

## 3. Proof of Theorem 2

We now prove Theorem 2, namely show that if for the irrational $\beta=\left[0 ; b_{1}, \ldots, b_{h}, \overline{a_{1}, \ldots, a_{\ell}}\right]$, we have $G_{k}(\beta, 0)=C_{k}<1$ for some $1 \leq k \leq \ell$, then

$$
\liminf _{N \rightarrow \infty} P_{N}(\beta)=0 \quad \text { and } \quad \limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}=\infty
$$

Let $k_{0}$ denote the index for which $C_{k_{0}}<1$, and fix some $\lambda$ such that $C_{k_{0}}<\lambda<1$. Since $G_{k_{0}}$ is continuous at 0 and $G_{k_{0}}(\beta, 0)=C_{k_{0}}<\lambda$, there exists $\eta>0$ such that $G_{k_{0}}(\beta, \varepsilon)<\lambda$ for all $|\varepsilon|<\eta$. Consider a subsequence $\left(m_{i}\right)_{i=1}^{\infty}$ of $\left(q_{h+m \ell+k_{0}}\right)_{m=1}^{\infty}$ such that
(i) $m_{i+1} \geq 2 m_{i}, \quad i=1,2, \ldots \quad$ and
(ii) $\left\|m_{i+1} \beta\right\|<\frac{\eta}{4 m_{i}}, \quad i=1,2, \ldots$
where $\|x\|=\min \{|x-k|: k \in \mathbb{Z}\}$ denotes the distance of $x \in \mathbb{R}$ to the nearest integer. We set

$$
N_{i}=m_{i}+\ldots+m_{1} \quad \text { and } \quad M_{j}=N_{i}-N_{j}, \quad i \geq 1, \quad j=1, \ldots, i .
$$

Then

$$
\begin{aligned}
P_{N_{i}}(\beta) & =\prod_{r=1}^{N_{i}} 2|\sin \pi r \beta|=\prod_{j=1}^{i} \prod_{r=1}^{m_{j}} 2\left|\sin \pi\left(r+M_{j}\right) \beta\right| \\
& =\prod_{j=1}^{i} \prod_{r=1}^{m_{j}} 2\left|\sin \pi\left(r \beta+\frac{\varepsilon_{j}}{m_{j}}\right)\right|=\prod_{j=1}^{i} P_{m_{j}}\left(\beta,(-1)^{\delta_{j}} \varepsilon_{j}\right),
\end{aligned}
$$

where $\varepsilon_{j}= \pm m_{j}\left\|M_{j} \beta\right\|$ and $\delta_{j}=0$ or 1 . We see that

$$
\left|\varepsilon_{j}\right| \leq m_{j}\left(\left\|m_{j+1} \beta\right\|+\ldots+\left\|m_{k} \beta\right\|\right)<\frac{\eta}{2}
$$

By the choice of $\eta$, this implies that $P_{m_{j}}\left(\beta,(-1)^{\delta_{j}} \varepsilon_{j}\right)<\lambda$ for all $j$ large enough, hence $P_{N_{i}}(\beta) \ll \lambda^{i}, i \rightarrow \infty$. This shows that $\liminf _{N \rightarrow \infty} P_{N}(\beta)=0$.

Now set also $T_{i}=m_{i+1}-\left(N_{i}+1\right), i \geq 1$, so that

$$
\begin{equation*}
P_{T_{i}}(\beta)=\frac{P_{m_{i+1}-1}(\beta)}{\prod_{r=N_{i}+1}^{m_{i}+1-1} 2|\sin \pi r \beta|}=\frac{P_{m_{i+1}-1}(\beta)}{\prod_{j=1}^{i} \prod_{r=1}^{m_{j}} 2\left|\sin \pi\left(r+M_{j}-m_{i+1}\right) \beta\right|} . \tag{3.1}
\end{equation*}
$$

At this point we need to point out a simple fact which follows from the proof of Theorem 1.1 in [8] but is not explicitly stated in the text. If $\left(C_{k}\right)_{k=1}^{\ell}$ are the constants in (1.1), then for each $k=1, \ldots, \ell$ we have

$$
\lim _{m \rightarrow \infty} \frac{P_{q_{h+m \ell+k}-1}(\beta)}{q_{h+m \ell+k}}=\frac{C_{k}}{2 \pi\left|c_{k} e_{k}\right|},
$$

where $\left(c_{k}\right)_{k=1}^{\ell}$ and $\left(e_{k}\right)_{k=1}^{\ell}$ are defined in 1.5). Armed with this observation we deduce that the numerator in (3.1) is $P_{m_{i+1}-1}(\beta) \asymp m_{i+1} \asymp T_{i}, i \rightarrow \infty$, while for the denominator in (3.1) we can show arguing as in the previous step that

$$
\liminf _{i \rightarrow \infty} \prod_{j=1}^{i} \prod_{r=1}^{m_{j}} 2\left|\sin \pi\left(r+M_{j}-m_{i+1}\right) \beta\right|=0
$$

Therefore

$$
\limsup _{N \rightarrow \infty} \frac{P_{N}(\beta)}{N}=\limsup _{i \rightarrow \infty} \frac{P_{T_{i}}(\beta)}{T_{i}}=\infty
$$

## 4. A perturbed sinc product

In Sections 5 and 6, our aim will be to determine when the limit $C_{k}$ in (1.1) is guaranteed to be less than one. Corollary 1 suggests that we will need to differentiate between two cases, depending on the parity of the period length $\ell$. Common to both cases is the need for appropriate upper and lower bounds on the function

$$
\begin{equation*}
G(x)=\prod_{t=1}^{\infty}\left(1-\frac{x^{2}}{u_{k}(t)^{2}}\right), \quad x \in \mathbb{R} \tag{4.1}
\end{equation*}
$$

where we recall from (1.10) that

$$
u_{k}(t)=2\left(\frac{t}{\left|e_{k} c_{k}\right|}-\left\{t \alpha_{\sigma_{k}}\right\}+\frac{1}{2}\right)
$$

Let us now set

$$
\begin{equation*}
A=2\left|e_{k} c_{k}\right|^{-1} \quad \text { and } \quad \delta_{t}=1-2\left\{t \alpha_{\sigma_{k}}\right\} \tag{4.2}
\end{equation*}
$$

so that $u_{k}(t)=A t+\delta_{t}$. The function $G(x)$ in (4.1) can then be seen as a perturbed version of the well-known product

$$
\frac{\sin \left(\pi A^{-1} x\right)}{\pi A^{-1} x}=\prod_{t=1}^{\infty}\left(1-\frac{x^{2}}{(A t)^{2}}\right)
$$

It is not difficult to show that if the perturbations $\delta_{t}$ satisfy $\left|\delta_{t}\right| \leq \delta<A / 4(t \in \mathbb{N})$ for some $\delta>0$, then the function $G$ obeys the bounds

$$
C_{1} \frac{\operatorname{dist}(x, \mathcal{U})}{|x|^{1+4 \delta / A}} \leq|G(x)| \leq C_{2} \frac{1}{|x|^{1-4 \delta / A}},
$$

for constants $C_{1}$ and $C_{2}$, where $\operatorname{dist}(x, \mathcal{U})=\min \left\{\left|x-A t-\delta_{t}\right|: t \in \mathbb{N}\right\}$. This is related to Kadec's $1 / 4$-rule [9, and can e.g. be seen as a consequence of [3, Lemma 4]. Due to the low-discrepancy property of Kronecker sequences, the sequence $\left(\delta_{t}\right)_{t}$ satisfies a much stronger condition, which in turn enables us to establish stronger bounds on $G$.
Theorem 4. Let $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$, and let $k \leq \ell$ be an index satisfying $a_{k}=\max _{j} a_{j}$. Then for $|x| \geq A / 2=\left|e_{k} c_{k}\right|^{-1}$, the function $G$ in (4.1) satisfies

$$
\begin{equation*}
\frac{2}{\pi} e^{-f\left(a_{k}\right)}\left(1-\frac{2}{3 A m}\right)\left(1-\frac{1}{A m}\right)^{2} \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \leq|G(x)| \leq \frac{14 A}{9} e^{f\left(a_{k}\right)} \frac{1}{|x|} \tag{4.3}
\end{equation*}
$$

where the positive integer $m=m(x) \geq 1$ is such that $|A m-x|=\min \{|A n-x|: n \in \mathbb{N}\}$ and

$$
\begin{equation*}
f\left(a_{k}\right)=\frac{13.7}{a_{k}}+\frac{1}{20 \log a_{k}}+\frac{1}{100}+\frac{2}{a_{k}^{2}} . \tag{4.4}
\end{equation*}
$$

For $|x|<A / 2$, we have the bound

$$
\begin{equation*}
\frac{2}{\pi} e^{-f\left(a_{k}\right)} \leq|G(x)| \leq 1 \tag{4.5}
\end{equation*}
$$

Remark 5. Notice that since $f\left(a_{k}\right) \rightarrow 0.01$ as $a_{k} \rightarrow \infty$, equation (4.3) reads

$$
K_{1} \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \leq|G(x)| \leq K_{2} \frac{1}{|x|}
$$

with $K_{1} \approx 2 / \pi$ and $K_{2} \approx 14 A / 9$ whenever $a_{k}$ and $x$ are large. Similar bounds can be established when $k$ is not the index of a maximal continued fraction coefficient of $\alpha$, but the size of the constants $K_{1}$ and $K_{2}$ will then depend both on $a_{k}$ and on the size of $\max _{j} a_{j}$.

We will see that the following is an immediate consequence of the proof of Theorem 4 .
Corollary 2. Suppose that we have $m(x)=1$ in Theorem 4, that is $A / 2 \leq x \leq 3 A / 2$. Then $G(x)$ in (4.1) satisfies

$$
\frac{2}{\pi} e^{-g\left(a_{k}\right)}\left(1-\frac{2}{3 A}\right)\left(1-\frac{1}{A}\right)^{2} \frac{\operatorname{dist}(x, \mathcal{U})}{|x|} \leq|G(x)| \leq \frac{14 A}{9} e^{g\left(a_{k}\right)} \frac{1}{|x|}
$$

where

$$
\begin{equation*}
g\left(a_{k}\right)=\frac{3.3}{a_{k}}+\frac{1}{80 \log a_{k}}+\frac{1}{400}+\frac{2}{a_{k}^{2}} . \tag{4.6}
\end{equation*}
$$

Before we embark on the proof of Theorem 4, we establish two preliminary results. The first concerns the size of $A=2\left|e_{k} c_{k}\right|^{-1}$.

Lemma 2. Let $1 \leq k \leq \ell$. We have

$$
\begin{equation*}
\frac{1}{\left|c_{k} e_{k}\right|}=a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)} . \tag{4.7}
\end{equation*}
$$

Proof. By (1.8) we obtain

$$
\frac{1}{\left|c_{k} e_{k}\right|}=\frac{q_{\ell+1}\left(\alpha_{\tau_{k}}\right)+p_{\ell}\left(\alpha_{\tau_{k}}\right)-2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}=a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)} .
$$

Corollary 3. For any $k=1,2, \ldots, \ell$ we have

$$
a_{k}<\frac{1}{\left|c_{k} e_{k}\right|}<a_{k}+2 .
$$

By Corollary 3, it immediately follows that

$$
\begin{equation*}
2 a_{k}<A<2\left(a_{k}+2\right) \tag{4.8}
\end{equation*}
$$

The second result concerns the perturbations $\delta_{t}$. We state it without proof, as it is an easy consequence of [11, Corollary 3].

Lemma 3. Let $\alpha=\left[0 ; a_{1}, a_{2}, \ldots\right]$ be an irrational with bounded continued fraction coefficients. Then for any fixed $n \in \mathbb{N}$, we have

$$
\left|\sum_{t=n+1}^{n+N}\left(\frac{1}{2}-\{t \alpha\}\right)\right| \leq \log N\left(\frac{a}{8 \log a}+6\right)+\frac{a}{8}+\frac{23}{4}
$$

for all $N \geq 1$, where $a=\max _{j} a_{j}$.

Recall from (4.2) that $\delta_{t}=1-2\left\{t \alpha_{\sigma_{k}}\right\}$, and thus by Lemma 3 it follows that

$$
\begin{equation*}
\left|\sum_{t=n+1}^{n+N} \delta_{t}\right| \leq \min \left\{N, \log N\left(\frac{a_{k}}{4 \log a_{k}}+12\right)+\frac{a_{k}}{4}+\frac{23}{2}\right\}, \tag{4.9}
\end{equation*}
$$

where $a_{k}=\max _{j} a_{j}$ for the quadratic irrational $\alpha$. Note that

$$
\log N\left(\frac{a_{k}}{4 \log a_{k}}+12\right)+\frac{a_{k}}{4}+\frac{23}{2} \geq 12 \log N+\frac{23}{2}>N \quad \text { for all } \quad N \leq 60
$$

regardless of the value of $a_{k}$. Accordingly, we will use the bound $\sum_{n+1}^{n+N} \delta_{t} \leq N$ whenever $N \leq 60$.

We are now equipped to prove Theorem 4.

Proof of Theorem 4. Since $G$ in (4.1) is an even function, it suffices to consider $x \geq 0$. Let $m=m(x) \geq 0$ be the non-negative integer satisfying

$$
|x-A m|=\min \{|x-A n|: n=0,1,2, \ldots\},
$$

and let us first assume that $m \geq 1$, meaning that $x \geq A / 2$. Excluding the case of the golden ratio, we may safely assume that $a_{k}=\max _{j} a_{j} \geq 2$, and thus by (4.8) we have $A \geq 4$. It follows that

$$
\begin{equation*}
|x-A m| \leq \frac{A}{2}, \quad\left|x-A m-\delta_{m}\right| \leq \frac{A}{2}+1 \leq \frac{3 A}{4} \tag{4.10}
\end{equation*}
$$

and

$$
|x-A t| \geq \frac{A}{2} \quad \text { and } \quad\left|x-A t-\delta_{t}\right| \geq \frac{A}{2}-1 \geq \frac{A}{4}
$$

for any $t \neq m$.
The function $G$ may be split into three products $G(x)=\Pi_{1}(x) \Pi_{2}(x) \Pi_{3}(x)$, where

$$
\begin{aligned}
\Pi_{1}(x) & =1-\frac{x^{2}}{\left(A m+\delta_{m}\right)^{2}} \\
\Pi_{2}(x) & =\prod_{\substack{t \geq 1 \\
t \neq m}}\left(1-\frac{x^{2}}{\left(A t+\delta_{t}\right)^{2}}\right)\left(1-\frac{x^{2}}{(A t)^{2}}\right)^{-1} \\
\Pi_{3}(x) & =\prod_{\substack{t \geq 1 \\
t \neq m}}\left(1-\frac{x^{2}}{(A t)^{2}}\right)
\end{aligned}
$$

For the first product, we observe that

$$
\Pi_{1}(x)=1-\frac{x^{2}}{\left(A m+\delta_{m}\right)^{2}}=\frac{\left(A m+\delta_{m}-x\right)\left(A m+\delta_{m}+x\right)}{\left(A m+\delta_{m}\right)^{2}},
$$

and thus by 4.10 we get

$$
\begin{equation*}
\operatorname{dist}(x, \mathcal{U}) \frac{A m+\delta_{m}+x}{\left(A m+\delta_{m}\right)^{2}} \leq\left|\Pi_{1}(x)\right| \leq \frac{3 A}{4} \frac{A m+\delta_{m}+x}{\left(A m+\delta_{m}\right)^{2}} \tag{4.11}
\end{equation*}
$$

We then consider the second factor $\Pi_{2}(x)=\prod_{t \neq m} Q_{t}(x)$, where

$$
Q_{t}(x)=\left(1-\frac{x^{2}}{\left(A t+\delta_{t}\right)^{2}}\right)\left(1-\frac{x^{2}}{(A t)^{2}}\right)^{-1}, \quad t \geq 1
$$

We have

$$
Q_{t}(x)=\frac{1-\frac{x}{A t+\delta_{t}}}{1-\frac{x}{A t}} \cdot \frac{1+\frac{x}{A t+\delta_{t}}}{1+\frac{x}{A t}}=\frac{1+\frac{\delta_{t}}{A t-x}}{1+\frac{\delta_{t}}{A t}} \cdot \frac{1+\frac{\delta_{t}}{A t+x}}{1+\frac{\delta_{t}}{A t}}
$$

$$
=\exp \left\{\log \left(1+\frac{\delta_{t}}{A t-x}\right)+\log \left(1+\frac{\delta_{t}}{A t+x}\right)-2 \log \left(1+\frac{\delta_{t}}{A t}\right)\right\}
$$

Thus if we employ the inequality

$$
x-x^{2}<\log (1+x)<x \quad \text { for all } x>-\frac{1}{2}
$$

we obtain
(4.12) $\exp \left\{\delta_{t} s_{t}-\delta_{t}^{2}\left(\frac{1}{(A t-x)^{2}}+\frac{1}{(A t+x)^{2}}\right)\right\}<Q_{t}(x)<\exp \left\{\delta_{t} s_{t}+\frac{2 \delta_{t}^{2}}{(A t)^{2}}\right\}$,
where we define

$$
\begin{equation*}
s_{t}=\frac{1}{A t-x}+\frac{1}{A t+x}-\frac{2}{A t} . \tag{4.13}
\end{equation*}
$$

The factors in (4.12) contributing significantly to $\Pi_{2}(x)=\prod_{t \neq m} Q_{t}$ are the first-order terms $\delta_{t} s_{t}$. For the second-order terms, we observe that the contribution on the left hand side is larger (in absolute value) than that on the right hand side, and a straightforward calculation verifies that

$$
\sum_{t \neq m}\left(\frac{1}{(A t+x)^{2}}+\frac{1}{(A t-x)^{2}}\right)<\frac{8}{A^{2}} .
$$

We thus conclude that

$$
\begin{equation*}
\Pi_{2}(x)=\prod_{t \neq m} Q_{t}(x)=\exp \left(\sum_{t \neq m} \delta_{t} s_{t}+E\right), \tag{4.14}
\end{equation*}
$$

where $s_{t}$ is given in (4.13) and $|E|<\frac{8}{A^{2}}$.
It remains to find an appropriate bound for

$$
\left|\sum_{t \neq m} \delta_{t} s_{t}\right| \leq\left|\sum_{t<m} \delta_{t} s_{t}\right|+\left|\sum_{t>m} \delta_{t} s_{t}\right| .
$$

We first consider the final term on the right hand side above. Summation by parts yields

$$
\sum_{t=m+1}^{m+M} \delta_{t} s_{t}=s_{m+M} \sum_{t=m+1}^{m+M} \delta_{t}+\sum_{t=m+1}^{m+M-1}\left(s_{t}-s_{t+1}\right) \sum_{k=m+1}^{t} \delta_{k},
$$

for any $M \geq 1$. We observe that

$$
\left|s_{t+1}-s_{t}\right| \leq \frac{1}{A t-x}-\frac{1}{A(t+1)-x} \leq \frac{1}{A\left(t-m-\frac{1}{2}\right)\left(t-m+\frac{1}{2}\right)},
$$

and

$$
\left|s_{m+M}\right| \leq \frac{1}{A(m+M)-x} \leq \frac{1}{A\left(M-\frac{1}{2}\right)}
$$

Combining this with (4.9), we find that

$$
\left|\sum_{t=m+1}^{m+M} \delta_{t} s_{t}\right| \leq \varepsilon(M)+\sum_{t=m+1}^{m+M-1} \frac{\min \{t-m, K \log (t-m)+C\}}{A\left((t-m)^{2}-\frac{1}{4}\right)},
$$

where $K=12+a_{k} /\left(4 \log a_{k}\right), C=23 / 2+a_{k} / 4$, and where $\varepsilon(M) \rightarrow 0$ as $M \rightarrow \infty$. Letting $M \rightarrow \infty$, we thus find

$$
\begin{aligned}
\left|\sum_{t>m} \delta_{t} s_{t}\right| & \leq \frac{1}{A}\left(\sum_{t=1}^{60} \frac{t}{\left(t^{2}-\frac{1}{4}\right)}+K \sum_{t=61}^{\infty} \frac{\log t}{\left(t^{2}-\frac{1}{4}\right)}+C \sum_{t=61}^{\infty} \frac{1}{\left(t^{2}-\frac{1}{4}\right)}\right) \\
& \leq \frac{1}{A}(5.1+0.1 K+0.02 C),
\end{aligned}
$$

and inserting values of $C$ and $K$, and recalling that $A>2 a_{k}$, we get

$$
\left|\sum_{t>m} \delta_{t} s_{t}\right| \leq \frac{3.3}{a_{k}}+\frac{1}{80 \log a_{k}}+\frac{1}{400} .
$$

By an analogous argument, one can show that

$$
\left|\sum_{t<m} \delta_{t} s_{t}\right| \leq \frac{10.4}{a_{k}}+\frac{3}{80 \log a_{k}}+\frac{3}{400},
$$

and thus combined we have

$$
\left|\sum_{t \neq m} \delta_{t} s_{t}\right| \leq \frac{13.7}{a_{k}}+\frac{1}{20 \log a_{k}}+\frac{1}{100} .
$$

Inserting this in (4.14), we arrive at

$$
\begin{equation*}
e^{-f\left(a_{k}\right)} \leq\left|\Pi_{2}(x)\right| \leq e^{f\left(a_{k}\right)}, \tag{4.15}
\end{equation*}
$$

with $f$ defined as in (4.4).
Finally, we observe that

$$
\Pi_{3}(x)=\frac{A^{2} m^{2}}{(A m-x)(A m+x)} \cdot \frac{\sin \left(\pi x A^{-1}\right)}{\pi x A^{-1}}
$$

and since $\left|\pi x A^{-1}-\pi m\right| \leq \frac{\pi}{2}$ we get

$$
\frac{2}{A} \leq\left|\frac{\sin \left(\pi x A^{-1}\right)}{A m-x}\right| \leq \frac{\pi}{A}
$$

This implies that

$$
\begin{equation*}
\frac{2 A^{2} m^{2}}{\pi(A m+x) x} \leq\left|\Pi_{3}(x)\right| \leq \frac{A^{2} m^{2}}{(A m+x) x} . \tag{4.16}
\end{equation*}
$$

Combining the bounds (4.11), (4.15) and (4.16), we find that

$$
|G(x)| \geq \frac{2}{\pi} e^{-f\left(a_{k}\right)} \cdot \operatorname{dist}(x, \mathcal{U}) \cdot \frac{1}{|x|} \cdot \frac{(A m)^{2}}{\left(A m+\delta_{m}\right)^{2}} \cdot \frac{A m+\delta_{m}+x}{A m+x}
$$

and

$$
|G(x)| \leq \frac{3 A}{4} e^{f\left(a_{k}\right)} \cdot \frac{1}{|x|} \cdot \frac{(A m)^{2}}{\left(A m+\delta_{m}\right)^{2}} \cdot \frac{A m+\delta_{m}+x}{A m+x}
$$

The common factor

$$
\frac{(A m)^{2}}{\left(A m+\delta_{m}\right)^{2}} \cdot \frac{A m+\delta_{m}+x}{A m+x}=\left(1-\frac{\delta_{m}}{A m+\delta_{m}}\right)^{2}\left(1+\frac{\delta_{m}}{A m+x}\right)
$$

will necessarily tend to 1 as $m(x) \rightarrow \infty$. Only the rate of convergence from below will be important to us. We therefore apply the rough upper bound

$$
\frac{(A m)^{2}}{\left(A m+\delta_{m}\right)^{2}} \cdot \frac{A m+\delta_{m}+x}{A m+x} \leq\left(\frac{4}{3}\right)^{2}\left(\frac{7}{6}\right),
$$

and the more precise lower bound

$$
\frac{(A m)^{2}}{\left(A m+\delta_{m}\right)^{2}} \cdot \frac{A m+\delta_{m}+x}{A m+x} \geq\left(1-\frac{1}{A m}\right)^{2}\left(1-\frac{2}{3 A m}\right)
$$

Inserting these bounds in the inequalities for $|G(x)|$ completes the proof of Theorem 4 in the case $|x| \geq A / 2$.

Finally, we consider the case $0 \leq x<A / 2$. As an upper bound, we use

$$
|G(x)| \leq 1
$$

For the lower bound, we again split $G$ into the subproducts $\Pi_{1}, \Pi_{2}$ and $\Pi_{3}$. Note that in this case, the product $\Pi_{1}$ is empty, and the product $\Pi_{3}$ is simply a sinc function bounded by

$$
\frac{2}{\pi} \leq \Pi_{3}(x) \leq 1 .
$$

For the product $\Pi_{2}$, we may use the bound (4.15) established for the case $x \geq A / 2$ (in fact we can do better, as will be argued below). Combined we get

$$
\frac{2}{\pi} e^{-f\left(a_{k}\right)} \leq|G(x)| \leq 1
$$

and this completes the proof of Theorem 4.

Proof of Corollary 2. Retracing the proof of Theorem 4, we arrive at (4.14), and note that for $m(x)=1$ we have

$$
\Pi_{2}(x)=\prod_{t \neq m} Q_{t}(x)=\exp \left(\sum_{t>m} \delta_{t} s_{t}+\frac{8}{A^{2}}\right) \leq \exp \left(\frac{3.3}{a_{k}}+\frac{1}{80 \log a_{k}}+\frac{1}{400}+\frac{2}{a_{k}^{2}}\right)
$$

since the product $\sum_{t<m} \delta_{t} s_{t}$ is empty. Apart from this, the proof remains unchanged.

Remark 6. Note that this bound on $\Pi_{2}(x)$ is clearly also valid for $x \leq A / 2$. Thus, the lower bound on $G(x)$ in 4.5 may be improved to

$$
\frac{2}{\pi} e^{-g\left(a_{k}\right)} \leq|G(x)| \leq 1, \quad x \leq A / 2,
$$

with $g$ given in 4.6).
Remark 7. The lower bound for $G(x)$ in (4.3) is used in the following sections to determine an upper bound for the constants $C_{k}$. In turn, this upper bound gives a threshold value $K>1$ such that $a_{k}=\max _{1 \leq i \leq \ell} a_{i} \geq K$ implies that $C_{k}<1$.

We believe that an improvement for the threshold value $K=23$ in Theorem 3 might be obtained as follows: in the estimates for $\Pi_{2}(x)$, we have bounded $\left|\sum_{t \neq m(x)} \delta_{t} s_{t}(x)\right|$ from above uniformly for all $x>0$. In the proof of Theorem 3 we are actually interested in the quantity $G(1) G(3) \cdots G(2 c-1)$, which means that we need a bound for the product $\prod_{x=1}^{c} \Pi_{2}(2 x-1)$. Rather than using a uniform bound for all terms in this product, we could seek an upper bound for the double sum

$$
\left|\sum_{\substack{1 \leq x \leq c \\ t \neq m(2 x-1)}} \sum_{\substack{ \\t \geq 1}} \delta_{t} s_{t}(2 x-1)\right| .
$$

This would take the specific range of $x$-values into account, and possibly provide a substantial improvement in the lower bound on $\prod_{x=1}^{c} \Pi_{2}(2 x+1)$. We leave this task to the interested reader.

## 5. Upper bounds for the constants $C_{k}$

With Theorem 4 established, let us now revisit Corollary 1 and carefully analyse the expressions for $C_{k}$ provided in (1.13) and (1.14). Recall that we have to differentiate between the case of even and odd period length $\ell$. To ease the analysis, it will be useful to make an assumption on the size of $\max _{j} a_{j}$ given a quadratic irrational $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$. Presuming a priori that we cannot do better for general $\ell$ than for the $\ell=1$ case (see [2, Theorem 6]), we assume throughout this section that $\max _{j} a_{j} \geq 6$.

We will show the following.
Theorem 5. Let $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ for some even period length $\ell$, and assume $a_{k}=$ $\max _{j} a_{j} \geq 6$. Then the limit $C_{k}=\lim _{m \rightarrow \infty} P_{\ell m+k}(\alpha)$ obeys the bound

$$
C_{k}^{\frac{c-2}{c}} \leq \frac{\pi}{2 a_{k}} e^{1+f\left(a_{k}\right)}\left(200 e^{2.4} c^{2}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_{k}}}
$$

where $q_{\ell}=q_{\ell}\left(\alpha_{\sigma_{k}}\right), c$ is defined in (1.3) and we recall from (4.4) that

$$
f\left(a_{k}\right) \leq \frac{13.7}{a_{k}}+0.1, \quad a_{k} \geq 6
$$

Throughout this section, when there is no danger of confusion, we shall write for abbreviation $q_{\ell}=q_{\ell}\left(\alpha_{\sigma_{k}}\right)$ and $q_{\ell+1}=q_{\ell+1}\left(\alpha_{\sigma_{k}}\right)$.

For the special case $q_{\ell}=1$, we have an improved bound; note that this only occurs if $\ell=2$ and $\alpha=\left[0 ; \overline{a_{1}, a_{2}}\right]$ with either $a_{1}=1$ or $a_{2}=1$.

Corollary 4. Let $\alpha=\left[0 ; \overline{a_{1}, a_{2}}\right]$, and assume $a_{k}=\max \left\{a_{1}, a_{2}\right\} \geq 6$ and $\min \left\{a_{1}, a_{2}\right\}=1$. Then the limit $C_{k}=\lim _{m \rightarrow \infty} P_{2 m+k}(\alpha)$ obeys the bound

$$
C_{k}^{\frac{c-2}{c}} \leq \frac{\pi}{a_{k}} e^{1+g\left(a_{k}\right)}\left(6.2\left(a_{k}+2\right)^{4}\right)^{\frac{1}{a_{k}+2}}
$$

where we recall from (4.6) that

$$
g\left(a_{k}\right) \leq \frac{3.3}{a_{k}}+0.1, \quad a_{k} \geq 6
$$

For the odd period case, we will establish the following.
Theorem 6. Let $\alpha=\left[0 ; \overline{a_{1}, \ldots, a_{\ell}}\right]$ for some odd period length $\ell$, and assume $a_{k}=$ $\max _{j} a_{j} \geq 6$. Then the limit $C_{k}=\lim _{m \rightarrow \infty} P_{\ell m+k}(\alpha)$ obeys the bound

$$
C_{k} \leq \frac{\pi}{2 a_{k}} e^{1+f\left(a_{k}\right)}\left(40 c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot a_{k}^{\frac{5}{2 a_{k}}}
$$

where $f$ is given in (4.4).
5.1. Bounding $C_{k}$ when $\ell \equiv 0(\bmod 2)$. Considering first even period lengths $\ell$, we recall from (1.13) that

$$
\begin{equation*}
C_{k}^{c-2}=\frac{1+a^{2}}{c!}\left|\frac{G(1)^{c-1} G\left(1+2 a^{2}\right)}{G(1) G(3) G(5) \cdots G(2 c-1)}\right| \tag{5.1}
\end{equation*}
$$

with $a$ and $c$ as given in (1.3), and the function $G$ defined in 4.1). The term $|G(1)|$ in (5.1) is bounded above by 1 , and the term $G\left(1+2 a^{2}\right)$ can be bounded by the upper bound in Theorem 4. Keeping the expression for $C_{k}^{c-2}$ in mind, we will rather give a bound for $\left(1+a^{2}\right) G\left(1+2 a^{2}\right)$. By Theorem 4 we have

$$
\begin{equation*}
\left(1+a^{2}\right) G\left(1+2 a^{2}\right) \leq \frac{14 A}{9} \cdot e^{f\left(a_{k}\right)} \cdot \frac{\left(1+a^{2}\right)}{\left(1+2 a^{2}\right)} \leq \frac{4}{5} A e^{f\left(a_{k}\right)} \tag{5.2}
\end{equation*}
$$

where for the last inequality we have used that $a \geq 5$ whenever $a_{k} \geq 6$.

Now let us find a lower bound on

$$
\begin{equation*}
G(1) G(3) \cdots G(2 c-1)=\prod_{s=0}^{c-1} G(2 s+1) \tag{5.3}
\end{equation*}
$$

In view of Theorem 4, some of the factors of (5.3) will be bounded using (4.3) while others will be bounded using (4.5). Since $2 a_{k}<A<2\left(a_{k}+2\right)$, the integers $j$ satisfying $j<\frac{A}{2}$ are $j=1,2, \ldots, a_{k}$ and possibly also $a_{k}+1$. Thus the factors of (5.3) with $0 \leq s \leq\left\lfloor\frac{a_{k}}{2}\right\rfloor-1$ will be bounded using (4.5) and those with $s \geq\left\lfloor\frac{a_{k}}{2}\right\rfloor+1$ will be bounded using (4.3). We get

$$
\begin{equation*}
\prod_{s=0}^{\left\lfloor\frac{\left.a_{k}\right\rfloor-1}{2}\right\rfloor-1} G(2 s+1) \geq\left(\frac{2}{\pi} e^{-f\left(a_{k}\right)}\right)^{\left\lfloor\frac{a_{k}}{2}\right\rfloor} \tag{5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1} G(2 s+1) \geq \prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1} \frac{2}{\pi} e^{-f\left(a_{k}\right)}\left(1-\frac{2}{3 A m_{s}}\right)\left(1-\frac{1}{A m_{s}}\right)^{2} \cdot \frac{\operatorname{dist}(2 s+1, \mathcal{U})}{2 s+1} \tag{5.5}
\end{equation*}
$$

with $\mathcal{U}=\left(u_{k}(t)\right)_{t=1}^{\infty}$ given in (1.10) and $m_{s} \in \mathbb{N}$ as defined in Theorem 4 .
The factor $G\left(a_{k}+1\right)$ appears in (5.3) only when $a_{k}$ is even and $s=a_{k} / 2$. For this factor, it is not clear which of the two bounds (4.3) and (4.5) apply. However, under the restriction that $a_{k} \geq 6$, we clearly have $\operatorname{dist}\left(a_{k}+1, \mathcal{U}\right)>1$, and by combining the bounds (4.3) and (4.5) (keeping all terms that are below 1), we get the universal bound

$$
G\left(2\left\lfloor\frac{a_{k}}{2}\right\rfloor+1\right) \geq \frac{2}{\pi} e^{-f\left(a_{k}\right)}\left(1-\frac{2}{3 A}\right)\left(1-\frac{1}{A}\right)^{2} \frac{1}{2\left\lfloor\frac{a_{k}}{2}\right\rfloor+1},
$$

which holds regardless of the parity of $a_{k}$, and of whether $a_{k}+1<A / 2$ or $a_{k}+1 \geq A / 2$. Combining this bound with (5.4) and (5.5), we finally get

$$
\begin{equation*}
\prod_{s=0}^{c-1} G(2 s+1) \geq\left(\frac{2}{\pi} e^{-f\left(a_{k}\right)}\right)^{c} \cdot \frac{\Pi_{1} \cdot \Pi_{2}}{\Pi_{3}} \tag{5.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\Pi_{1}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}\left(1-\frac{2}{3 A m_{s}}\right)\left(1-\frac{1}{A m_{s}}\right)^{2} \tag{5.7}
\end{equation*}
$$

$$
\begin{equation*}
\Pi_{2}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1} \operatorname{dist}(2 s+1, \mathcal{U}) \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\Pi_{3}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}(2 s+1) \tag{5.9}
\end{equation*}
$$

We proceed by bounding the three product terms $\Pi_{1}, \Pi_{2}$ and $\Pi_{3}$ separately. In the following, we will make use of the inequalities

$$
\begin{gather*}
\sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n} \leq n!\leq e \sqrt{n}\left(\frac{n}{e}\right)^{n} \\
\sqrt{2 \pi n}\left(\frac{2 n}{e}\right)^{n} \leq(2 n)!!\leq e \sqrt{n}\left(\frac{2 n}{e}\right)^{n}  \tag{5.10}\\
\frac{\sqrt{4 \pi}}{e}\left(\frac{2 n}{e}\right)^{n} \leq(2 n-1)!!\leq \frac{e}{\sqrt{\pi}}\left(\frac{2 n}{e}\right)^{n}
\end{gather*}
$$

which are valid for all $n \in \mathbb{N}$.
Starting with the first and simplest of the three, we observe that

$$
\Pi_{1} \geq \prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}\left(1-\frac{1}{A m_{s}}\right)^{3}
$$

Recall that $m_{s}$ is the unique positive integer for which $\left|A m_{s}-(s+1)\right|$ is minimized. As $s$ runs through the values $\left\lfloor a_{k} / 2\right\rfloor, \ldots, c-1$, the integer $m_{s}$ runs through the values $1, \ldots, q_{\ell}$, and each integer occurs at most $a_{k}+1$ times. It follows that

$$
\Pi_{1} \geq \prod_{m=1}^{q_{\ell}}\left(1-\frac{1}{A m}\right)^{3\left(a_{k}+1\right)}
$$

Using that

$$
x-x^{2}<\log (1+x)<x \quad \text { for all } x>-\frac{1}{2}
$$

it is straightforward to show that

$$
\prod_{m=1}^{q_{\ell}}\left(1-\frac{1}{A m}\right) \geq \exp \left(-\frac{1}{A}\left(1+\log q_{\ell}\right)-\frac{\pi^{2}}{6 A^{2}}\right) \geq \exp \left(-\frac{1}{A}\left(1.14+\log q_{\ell}\right)\right)
$$

where for the final inequality we have used that $a_{k} \geq 6$, and thus $A>12$. Inserting this in the expression for $\Pi_{1}$ above, we get

$$
\begin{equation*}
\Pi_{1} \geq\left(\frac{1}{e^{1.14} q_{\ell}}\right)^{\frac{3\left(a_{k}+1\right)}{A}} \geq\left(\frac{1}{e^{1.14} q_{\ell}}\right)^{\frac{3\left(a_{k}+1\right)}{a_{k}}} \geq \frac{1}{e^{2} q_{\ell}^{2}} \tag{5.11}
\end{equation*}
$$

Let us now consider $\Pi_{3}$, for which we need to find an upper bound. From (5.10) it follows that

$$
\Pi_{3}=\frac{(2 c-1)!!}{\left(2\left\lfloor\frac{a_{k}}{2}\right\rfloor-1\right)!!} \leq \frac{e^{2}}{2 \pi}\left(\frac{2 c}{e}\right)^{c}\left(\frac{2\left\lfloor\frac{a_{k}}{2}\right\rfloor}{e}\right)^{-\left\lfloor\left\lfloor\frac{a_{k}}{2}\right\rfloor\right.} .
$$

We observe that

$$
\begin{equation*}
\left(\frac{2\left\lfloor\frac{a_{k}}{2}\right\rfloor}{e}\right)^{\left\lfloor\frac{a_{k}}{2}\right\rfloor} \geq\left(\frac{a_{k}-1}{e}\right)^{\frac{a_{k}-1}{2}} \geq \frac{1}{2}\left(\frac{e}{a_{k}}\right)^{\frac{1}{2}}\left(\frac{a_{k}}{e}\right)^{\frac{a_{k}}{2}} \tag{5.12}
\end{equation*}
$$

where in the last step we have used that $(1-1 / n)^{\frac{n}{2}} \geq \frac{1}{2}$ for $n \geq 2$. Inserting this in the bound above we find that

$$
\begin{equation*}
\Pi_{3} \leq \frac{e^{\frac{3}{2}} \sqrt{a_{k}}}{\pi}\left(\frac{e}{a_{k}}\right)^{\frac{a_{k}}{2}}\left(\frac{2 c}{e}\right)^{c} \tag{5.13}
\end{equation*}
$$

The estimation of the product $\Pi_{2}$ is far more intricate. When we analyse the size of the factors comprising $\Pi_{2}$, the numbers

$$
\begin{equation*}
R_{t}=\left\{\frac{t p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right\}+t\left(\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}-\alpha_{\sigma_{k}}\right)-\frac{2 b t}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}, \quad t=1, \ldots, q_{\ell}-1 \tag{5.14}
\end{equation*}
$$

appear naturally. The size of the product of $\left\|R_{t}\right\|$ and its relation with $\Pi_{2}$ is stated in the following lemmas, the proofs of which are postponed for Section 5.3.

Lemma 4. When $\ell$ is even, the numbers $R_{t}, t=1, \ldots, q_{\ell}-1$ given in (5.14) satisfy

$$
\begin{equation*}
\prod_{t=1}^{q_{\ell}-1}\left\|R_{t}\right\| \geq \frac{\sqrt{q_{\ell}}}{2(2 e)^{q_{\ell}+1}} . \tag{5.15}
\end{equation*}
$$

Lemma 5. The product $\Pi_{2}$ defined in (5.8) is bounded below by

$$
\begin{equation*}
\Pi_{2} \geq \frac{2}{5 c} \cdot 2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor} \cdot\left\lfloor\frac{a_{k}}{2}\right\rfloor!\cdot\left[\left(a_{k}-1\right)!\right]^{q_{\ell}-1} \cdot \prod_{t=1}^{q_{\ell}-1}\left\|R_{t}\right\| . \tag{5.16}
\end{equation*}
$$

We now find a bound for $\Pi_{2}$ using Lemmas 4 and 5. By (5.10) we have

$$
\left\lfloor\frac{a_{k}}{2}\right\rfloor!\geq\left(\frac{a_{k}}{2 e}\right)^{\frac{a_{k}}{2}} \quad \text { and } \quad\left(a_{k}-1\right)!\geq \frac{2}{\sqrt{a_{k}}}\left(\frac{a_{k}}{e}\right)^{a_{k}}
$$

The former is obvious when $a_{k}$ is even, and follows by an argument similar to (5.12) when $a_{k}$ is odd and greater than 6 . Inserting these bounds in (5.16) and employing Lemma 4 , we get

$$
\begin{equation*}
\Pi_{2} \geq \frac{\sqrt{a_{k} q_{\ell}}}{20 e c} \cdot 2^{c-a_{k}}\left(\frac{1}{e \sqrt{a_{k}}}\right)^{q_{\ell}} \cdot\left(\frac{a_{k}}{e}\right)^{a_{k}\left(q_{\ell}-\frac{1}{2}\right)} . \tag{5.17}
\end{equation*}
$$

Combining the bounds established for $\Pi_{1}, \Pi_{2}$ and $\Pi_{3}$, we conclude as follows.

Lemma 6. Let $G$ be given in (4.1). Under the assumption that $a_{k} \geq 6$ we have

$$
\prod_{s=0}^{c-1} G(2 s+1) \geq\left(\frac{2 a_{k}}{\pi e^{f\left(a_{k}\right)} c}\right)^{c} \cdot \frac{2^{-a_{k}} \pi}{20 e^{\frac{9}{2}} q_{\ell}^{\frac{3}{2}} c} \cdot\left(\frac{e}{a_{k}^{5 / 2}}\right)^{q_{\ell}}
$$

Proof. Inserting the bounds established for $\Pi_{1}, \Pi_{2}$ and $\Pi_{3}$ in (5.11), (5.17) and (5.13) into (5.6), we get

$$
\begin{aligned}
\prod_{s=0}^{c-1} G(2 s+1) & \geq\left(\frac{2}{\pi c} e^{1-f\left(a_{k}\right)}\right)^{c} \cdot \frac{2^{-a_{k}} \pi}{20 e^{\frac{9}{2}} q_{\ell}^{\frac{3}{2}} c}\left(\frac{1}{e \sqrt{a_{k}}}\right)^{q_{\ell}}\left(\frac{a_{k}}{e}\right)^{a_{k} q_{\ell}} \\
& \geq\left(\frac{2 a_{k}}{\pi e^{f\left(a_{k}\right)} c}\right)^{c} \cdot \frac{2^{-a_{k}} \pi}{20 e^{\frac{9}{2}} q_{\ell}^{\frac{3}{2}} c}\left(\frac{e}{a^{\frac{5}{2}}}\right)^{q_{\ell}},
\end{aligned}
$$

where for the last inequality we have used that $c<\left(a_{k}+2\right) q_{\ell}$.

We are now fully equipped to prove Theorem 5 .

Proof of Theorem 5. We recall from (5.1) and (5.2) that

$$
C_{k}^{c-2}=\frac{1+a^{2}}{c!}\left|\frac{G(1)^{c-1} G\left(1+2 a^{2}\right)}{\prod_{s=1}^{c-1} G(2 s+1)}\right| \leq \frac{1}{c!} \cdot \frac{4 A e^{f\left(a_{k}\right)}}{5} \cdot \frac{1}{\left|\prod_{s=1}^{c-1} G(2 s+1)\right|}
$$

Using Lemma 6 and the bound on $c!$ in (5.10), we thus get

$$
C_{k}^{c-2} \leq \frac{16 A e^{\frac{9}{2}} q_{\ell}^{\frac{3}{2}} \sqrt{c} e^{f\left(a_{k}\right)}}{\pi \sqrt{2 \pi}}\left(\frac{\pi e^{1+f\left(a_{k}\right)}}{2 a_{k}}\right)^{c} \cdot 2^{a_{k}} \cdot\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{q_{\ell}}
$$

Recall that $A<2\left(a_{k}+2\right)$ and $q_{\ell}<c / a_{k}$. This allows us to bound the first term on the right hand side by

$$
\frac{32 e^{\frac{9}{2}}}{\pi \sqrt{2 \pi}} \frac{\left(a_{k}+2\right)}{a_{k}^{3 / 2}} e^{f\left(a_{k}\right)} c^{2} \leq \frac{32 e^{\frac{9}{2}}}{\pi \sqrt{2 \pi}} \frac{8}{6^{3 / 2}} e^{f(6)} c^{2} \leq 200 e^{2.4} c^{2}
$$

Inserting this in the expression above, we get

$$
C_{k}^{c-2} \leq\left(\frac{\pi e^{1+f\left(a_{k}\right)}}{2 a_{k}}\right)^{c} \cdot\left(200 e^{2.4} c^{2}\right) \cdot 2^{a_{k}} \cdot\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{q_{\ell}}
$$

Raising both sides to the power $1 / c$ and using again that $c>a_{k} q_{\ell}$ completes the proof of Theorem 5

Proof of Corollary 4. When $q_{\ell}=1$ we have $c=a_{k}+2$, and revisiting Lemma 6 we see that we can obtain an improved bound on $\prod G(2 s+1)$ in this case. From Corollary 2 and Remark 6 it follows that

$$
\prod_{s=0}^{c-1} G(2 s+1) \geq\left(\frac{2}{\pi} e^{-g\left(a_{k}\right)}\right)^{c} \cdot \frac{\Pi_{1} \cdot \Pi_{2}}{\Pi_{3}}
$$

with $\Pi_{1}, \Pi_{2}$ and $\Pi_{3}$ defined as in (5.7)-(5.9), respectively, and $g\left(a_{k}\right)$ given in (4.6). We keep the bounds for $\Pi_{1}$ and $\Pi_{3}$ established in (5.11) and (5.13), and note that the bound on $\Pi_{2}$ in 5.16) simplifies to

$$
\Pi_{2} \geq \frac{2}{5 c} \cdot 2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor} \cdot\left\lfloor\frac{a_{k}}{2}\right\rfloor!\geq \frac{8}{5 c}\left(\frac{a_{k}}{e}\right)^{\frac{a_{k}}{2}}
$$

Inserting all three bounds above, we get

$$
\begin{equation*}
\prod_{s=0}^{c-1} G(2 s+1) \geq \frac{8 \pi}{5 e^{\frac{7}{2}} c \sqrt{a_{k}}}\left(\frac{e^{1-g\left(a_{k}\right)}}{\pi c}\right)^{c}\left(\frac{a_{k}}{e}\right)^{a_{k}} \geq \frac{8 \pi}{5 e^{\frac{3}{2}} c^{\frac{7}{2}}}\left(\frac{e^{-g\left(a_{k}\right)} a_{k}}{\pi c}\right)^{c} \tag{5.18}
\end{equation*}
$$

where we have used that $c=a_{k}+2>a_{k}$.
Again we have that

$$
C_{k}^{c-2}=\frac{1+a^{2}}{c!}\left|\frac{G(1)^{c-1} G\left(1+2 a^{2}\right)}{\prod_{s=1}^{c-1} G(2 s+1)}\right| \leq \frac{1}{c!} \cdot \frac{4 A e^{f\left(a_{k}\right)}}{5} \cdot \frac{1}{\left|\prod_{s=1}^{c-1} G(2 s+1)\right|}
$$

Inserting $A<2\left(a_{k}+2\right)=2 c$ and the improved bound (5.18) on $\prod G(2 s+1)$, we get

$$
C_{k}^{c-2} \leq \frac{1}{c!} \cdot \frac{e^{f\left(a_{k}\right)+\frac{3}{2}} c^{\frac{9}{2}}}{\pi}\left(\frac{\pi e^{g\left(a_{k}\right)} c}{a_{k}}\right)^{c} \leq \frac{e^{f\left(a_{k}\right)+\frac{3}{2}}}{\pi \sqrt{2 \pi}} \cdot c^{4} \cdot\left(\frac{\pi e^{1+g\left(a_{k}\right)}}{a_{k}}\right)^{c}
$$

where for the last inequality we have used the lower bound on $c!$ in 5.10). The proof is completed by bounding $e^{f\left(a_{k}\right)}$ by $e^{f(6)}$ and raising both sides to the power $1 / c$.
5.2. Bounding $C_{k}$ when $\ell \equiv 1(\bmod 2)$. Now let us consider the case of odd period lengths $\ell$. By (1.14), the constant $C_{k}$ is then given by

$$
\begin{equation*}
C_{k}^{c}=\frac{G(1)^{c}}{\prod_{s=1}^{c}|s-a|} \prod_{s=0}^{c-1}|G(2 a-2 s-1)|^{-1} \tag{5.19}
\end{equation*}
$$

where $a$ and $c$ are given in (1.3) and $G$ is the function defined in (4.1). Our goal is again to derive a bound for $C_{k}$ in the case when $k$ is the index such that $\max _{j} a_{j}=a_{k}$.

The assumptions that $\ell$ is odd and $a_{k} \geq 6$ necessarily imply that $c \geq 13$; we make use of this inequality in the estimates that follow. The definitions of $a$ and $c$ in (1.3) imply that

$$
\frac{1}{a-c}=\frac{\sqrt{c^{2}+4}+c}{2}<c+\frac{1}{13} \quad \text { and } \quad|a-s|>c-s, \quad s=1, \ldots, c-1
$$

and therefore

$$
\begin{equation*}
\prod_{s=1}^{c}|s-a|^{-1} \leq \frac{c+\frac{1}{13}}{(c-1)!}=\frac{c\left(c+\frac{1}{13}\right)}{c!} \stackrel{\sqrt{5.10}}{\leq} \frac{\sqrt{c}\left(c+\frac{1}{13}\right)}{\sqrt{2 \pi}}\left(\frac{e}{c}\right)^{c} \tag{5.20}
\end{equation*}
$$

We now seek a lower bound for the product

$$
\begin{equation*}
\prod_{s=0}^{c-1}|G(2 a-2 s-1)|=\prod_{s=0}^{c-1}|G(2 s+1+2(a-c))|=\prod_{s=0}^{c-1}|G(2 s+1-2 b)| . \tag{5.21}
\end{equation*}
$$

We argue as in the case of even $\ell$, and use Theorem 4 to bound the terms of this product. We use (4.5) to bound the factors of (5.21) with $0 \leq s \leq\left\lfloor\frac{a_{k}}{2}\right\rfloor-1$, which gives

$$
\prod_{s=0}^{\left\lfloor\frac{a_{k}}{2}\right\rfloor-1}|G(2 s+1-2 b)| \geq\left(\frac{2}{\pi e^{f\left(a_{k}\right)}}\right)^{\left\lfloor\frac{a_{k}}{2}\right\rfloor}
$$

For the factors of (5.21) corresponding to $\left\lfloor\frac{a_{k}}{2}\right\rfloor+1 \leq s \leq c-1$, the bound (4.3) gives

$$
\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1}|G(2 s+1-2 b)| \geq \prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1} \frac{2}{\pi e^{f\left(a_{k}\right)}}\left(1-\frac{2}{3 A m_{s}}\right)\left(1-\frac{1}{A m_{s}}\right)^{2} \frac{\operatorname{dist}(2 s+1-2 b, \mathcal{U})}{2 s+1-2 b},
$$

where the integers $m_{s}$ are defined in Theorem 4. For the factor $\left|G\left(2\left\lfloor\frac{a_{k}}{2}\right\rfloor+1-2 b\right)\right|$ we use the bound

$$
\left|G\left(2\left\lfloor\frac{a_{k}}{2}\right\rfloor+1-2 b\right)\right| \geq \frac{2}{\pi} e^{-f\left(a_{k}\right)}\left(1-\frac{2}{3 A}\right)\left(1-\frac{1}{A}\right)^{2} \frac{1}{2\left\lfloor\frac{a_{k}}{2}\right\rfloor+1-2 b} .
$$

Combining the estimates above, we obtain

$$
\begin{equation*}
\prod_{s=0}^{c-1}|G(2 s+1-2 b)| \geq\left(\frac{2}{\pi e^{f\left(a_{k}\right)}}\right)^{c} \cdot \frac{\Pi_{1} \cdot \Pi_{2}^{\prime}}{\Pi_{3}^{\prime}} \tag{5.22}
\end{equation*}
$$

where $\Pi_{1}$ is the product defined in (5.7), while

$$
\begin{equation*}
\Pi_{2}^{\prime}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1} \operatorname{dist}(2 s+1-2 b, \mathcal{U}) \quad \text { and } \quad \Pi_{3}^{\prime}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}(2 s+1-2 b) \tag{5.23}
\end{equation*}
$$

Let us first find a bound for $\Pi_{3}^{\prime}$ by comparing it with $\Pi_{3}$ in (5.9). Note that the bound (5.13) on $\Pi_{3}$ does not depend on the parity of $\ell$. Since

$$
-2 b=\frac{4}{\sqrt{c^{2}+4}+c}<\frac{2}{c}
$$

we have

$$
\Pi_{3}^{\prime} \leq \prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}\left(2 s+1+\frac{2}{c}\right)=\Pi_{3} \cdot \prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1}\left(1+\frac{2}{(2 s+1) c}\right) .
$$

We find that

$$
\begin{aligned}
\prod_{s=\left\lfloor\left\lfloor\frac{a_{k}}{2}\right\rfloor\right.}^{c-1}\left(1+\frac{2}{(2 s+1) c}\right) & =\exp \left(\sum_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1} \log \left(1+\frac{2}{(2 s+1) c}\right)\right) \\
& <\exp \left(\frac{2}{c} \sum_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1} \frac{1}{2 s+1}\right)<\exp \left(\frac{\log (c-1)}{c}\right)<e^{\frac{1}{5}}
\end{aligned}
$$

so in view of (5.13) we deduce that

$$
\begin{equation*}
\Pi_{3}^{\prime} \leq \frac{e^{\frac{17}{10}} \sqrt{a_{k}}}{\pi}\left(\frac{e}{a_{k}}\right)^{\frac{a_{k}}{2}}\left(\frac{2 c}{e}\right)^{c} \tag{5.24}
\end{equation*}
$$

As for the even period case, the estimation of $\Pi_{2}^{\prime}$ is more elaborate. When analysing the factors in $\Pi_{2}^{\prime}$, the numbers $R_{t}+b$ appear naturally, where $R_{t}$ is defined as in (5.14). The size of products over $\left\|R_{t}+b\right\|$ and its relation to $\Pi_{2}^{\prime}$ is stated in the lemmas below. Note that these are analogues of Lemmas 4 and 5 for the even period case. The proofs are postponed to Section 5.3.

Lemma 7. When $\ell$ is odd, the numbers $R_{t}, t=1, \ldots, q_{\ell}-1$ defined in (5.14) satisfy

$$
\begin{equation*}
\prod_{t=1}^{q_{\ell}-1}\left\|R_{t}+b\right\| \geq \frac{4 \pi}{e^{3}(2 e)^{q_{\ell}}} \tag{5.25}
\end{equation*}
$$

Lemma 8. The product $\Pi_{2}^{\prime}$ defined in (5.23) is bounded below by

$$
\begin{equation*}
\Pi_{2}^{\prime} \geq 2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor-1} \cdot\left\lfloor\frac{a_{k}}{2}\right\rfloor!\cdot\left[\left(a_{k}-1\right)!\right]^{q_{\ell}-1} \cdot \prod_{t=1}^{q_{\ell}-1}\left\|R_{t}+b\right\| \tag{5.26}
\end{equation*}
$$

Let us now bound $\Pi_{2}^{\prime}$ using Lemmas 7 and 8. We bound the factorials in 5.26) using (5.10), and combined with Lemma 7 this gives

$$
\begin{equation*}
\Pi_{2}^{\prime} \geq \frac{3 \sqrt{a_{k}}}{20} \cdot 2^{c-a_{k}}\left(\frac{1}{e \sqrt{a_{k}}}\right)^{q_{\ell}} \cdot\left(\frac{a_{k}}{e}\right)^{a_{k}\left(q_{\ell}-\frac{1}{2}\right)} \tag{5.27}
\end{equation*}
$$

Combining the bounds for $\Pi_{1}, \Pi_{2}^{\prime}$ and $\Pi_{3}^{\prime}$ established in (5.11), (5.24) and (5.27), we get the following analogue of Lemma 6. The proof is omitted.
Lemma 9. Let $G$ be defined in 4.1). Under the assumption that $a_{k} \geq 6$ we have

$$
\prod_{s=0}^{c-1}|G(2 s+1-2 b)| \geq\left(\frac{2 a_{k}}{\pi e^{f\left(a_{k}\right)} c}\right)^{c} \cdot \frac{3 \pi \cdot 2^{-a_{k}}}{20 e^{19 / 5} q_{\ell}^{2}} \cdot\left(\frac{e}{a_{k}^{5 / 2}}\right)^{q_{\ell}}
$$

With Lemma 9 established, we are equipped to prove Theorem 6.

Proof of Theorem 6. Recall from (5.19) that

$$
C_{k}^{c} \leq\left(\prod_{s=1}^{c}|s-a| \prod_{s=0}^{c-1}|G(2 a-2 s-1)|\right)^{-1}
$$

where we have used that $|G(1)| \leq 1$. Using 5.20 and Lemma 9 , we get

$$
C_{k}^{c} \leq 40 c^{\frac{3}{2}}\left(\frac{\pi e^{1+f\left(a_{k}\right)}}{2 a_{k}}\right)^{c} \cdot 2^{a_{k}} \cdot q_{\ell}^{2}\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{q_{\ell}}
$$

where we have replaced $(c+1 / 13)$ in 5.20 by the upper bound $27 c / 26$. Raising both sides to the power $1 / c$ and recalling that $c>a_{k} q_{\ell}$, we get

$$
\begin{aligned}
C_{k} & \leq \frac{\pi}{2 a_{k}} e^{1+f\left(a_{k}\right)}\left(40 c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot q_{\ell}^{\frac{2}{a_{k} q_{\ell}}}\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_{k}}} \\
& \leq \frac{\pi}{2 a_{k}} e^{1+f\left(a_{k}\right)}\left(40 c^{\frac{3}{2}}\right)^{\frac{1}{c}} \cdot 2^{\frac{1}{q_{\ell}}} \cdot a_{k}^{\frac{5}{2 a_{k}}}
\end{aligned}
$$

where in the final step we have used that $q_{\ell}^{2 / q_{\ell}}<e$.
5.3. Estimation of $\Pi_{2}$ and $\Pi_{2}^{\prime}$. Let us now turn to the proofs of Lemmas 4-5 and $7-$ 8. which lay the foundation for the estimates of $\Pi_{2}$ and $\Pi_{2}^{\prime}$ above.
5.3.1. Proofs of Lemmas 4 and 7. We consider first Lemmas 4 and 7, which provide bounds on products of factors $\left\|R_{t}\right\|$ and $\left\|R_{t}+b\right\|$, with $R_{t}$ given in (5.14). Recall that these are analogous statements relevant to the cases of even and odd period lengths $\ell$, respectively.

We treat first the proof of Lemma 4, and begin by examining the size of each of the three terms appearing in the expression for $R_{t}$ in (5.14).

Lemma 10. When $\ell$ is even, we have

$$
-\frac{1}{q_{\ell} q_{\ell+1}}<\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-\frac{2 b}{q_{\ell}}<-\frac{a_{k}}{a_{k}+1} \frac{1}{q_{\ell} q_{\ell+1}}
$$

Proof. First we observe that setting $m=1$ and $k=0$ in (1.6) yields $p_{\ell}-q_{\ell} \alpha_{\sigma_{k}}=b$, and therefore

$$
\begin{equation*}
\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-\frac{2 b}{q_{\ell}}=-\frac{b}{q_{\ell}} \tag{5.28}
\end{equation*}
$$

When $\ell$ is even, the definition of $b$ in 1.3$)$ gives

$$
b<\frac{1}{c-1}=\frac{1}{q_{\ell+1}+p_{\ell}-1}<\frac{1}{q_{\ell+1}}
$$

and

$$
\begin{aligned}
b & >\frac{1}{c}=\frac{1}{q_{\ell+1}+p_{\ell}}=\frac{1}{q_{\ell+1}}\left(1+\frac{p_{\ell}}{q_{\ell+1}}\right)^{-1} \\
& >\frac{1}{q_{\ell+1}}\left(1+\frac{q_{\ell}}{q_{\ell+1}}\right)^{-1}>\frac{1}{q_{\ell+1}}\left(1+\frac{q_{\ell}}{a_{k} q_{\ell}}\right)^{-1}=\frac{1}{q_{\ell+1}} \frac{a_{k}}{a_{k}+1},
\end{aligned}
$$

whence the claim follows.
Lemma 11. Let $\ell$ be even. For each $t=1,2, \ldots, q_{\ell}-1$ there exists a unique integer $i=i_{t} \in\left\{1,2, \ldots, q_{\ell}-1\right\}$ such that

$$
\frac{i}{q_{\ell}}-\frac{1}{q_{\ell+1}} \leq\left\{\frac{t p_{\ell}}{q_{\ell}}\right\}+t\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-\frac{2 b t}{q_{\ell}} \leq \frac{i}{q_{\ell}}-\frac{6}{7 q_{\ell} q_{\ell+1}}
$$

Moreover, the correspondence $t \mapsto i_{t}$ is one-to-one.

Proof. In view of the condition $a_{k} \geq 6$, by Lemma 10 we get

$$
-\frac{1}{q_{\ell+1}}<t\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-\frac{2 b t}{q_{\ell}}<-\frac{6}{7 q_{\ell} q_{\ell+1}}, \quad t=1,2, \ldots, q_{\ell}-1 .
$$

Also since $\left(p_{\ell}, q_{\ell}\right)=1$, for each $t=1,2, \ldots, q_{\ell}-1$ there exists a unique integer $1 \leq i<$ $q_{\ell}$ such that $\left\{t p_{\ell} / q_{\ell}\right\}=i / q_{\ell}$, and the correspondence $t \mapsto i$ is one-to-one. The result follows.

Lemma 11 gives the following estimates for the values of $\left\|R_{t}\right\|, t=1,2, \ldots, q_{\ell}-1$.
Corollary 5. Let $\ell$ be even and $R_{t}$ be as in (5.14). Then for every $t=1,2, \ldots, q_{\ell}-1$ there exists an integer $j_{t} \in\left\{1,2, \ldots,\left\lfloor\frac{1}{2} q_{\ell}\right\rfloor\right\}$ such that

$$
\begin{equation*}
\left\|R_{t}\right\| \geq \frac{j_{t}}{q_{\ell}}-\frac{1}{q_{\ell+1}} \quad \text { or } \quad\left\|R_{t}\right\| \geq \frac{j_{t}}{q_{\ell}}+\frac{6}{7 q_{\ell} q_{\ell+1}} \tag{5.29}
\end{equation*}
$$

The first inequality in (5.29) holds when the residue of $\operatorname{tp}\left(\alpha_{\tau_{k}}\right)$ modulo $q_{\ell}\left(\alpha_{\sigma_{k}}\right)$ is one of $1,2, \ldots,\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor$ while the second holds when the residue of $t p_{\ell}\left(\alpha_{\tau_{k}}\right)$ modulo $q_{\ell}\left(\alpha_{\sigma_{k}}\right)$ is one of $\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor+1, \ldots, q_{\ell}\left(\alpha_{\tau_{k}}\right)-1$. Moreover, there exists at most one integer $t$ such that $j_{t}=\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor$, and the correspondence $t \mapsto j_{t}$ is "at most two-to-one".

With Corollary 5 established, we are equipped to complete the proof of Lemma 4.

Proof of Lemma 4. We consider first the case $q_{\ell} \geq 4$ and for abbreviation set $Q=\left\lfloor\frac{1}{2} q_{\ell}\right\rfloor$. From Corollary 5 it follows that

$$
\prod_{t=1}^{q_{\ell}-1}\left\|R_{t}\right\| \geq\left(\frac{1}{2}-\frac{1}{q_{\ell+1}}\right) \prod_{j=1}^{Q-1}\left(\frac{j}{q_{\ell}}-\frac{1}{q_{\ell+1}}\right)\left(\frac{j}{q_{\ell}}+\frac{6}{7 q_{\ell} q_{\ell+1}}\right)
$$

$$
\begin{aligned}
& \geq\left(\frac{1}{2}-\frac{1}{6}\right) \frac{1}{q_{\ell}}\left(1-\frac{q_{\ell}}{q_{\ell+1}}\right) \prod_{j=2}^{Q-1}\left(\frac{j}{q_{\ell}}-\frac{1}{q_{\ell+1}}\right) \prod_{j=1}^{Q-1} \frac{j}{q_{\ell}} \\
& \geq \frac{5}{18 q_{\ell}} \prod_{j=1}^{Q-2}\left(\frac{j}{q_{\ell}}+\frac{1}{2 q_{\ell}}\right) \prod_{j=1}^{Q-1} \frac{j}{q_{\ell}} \quad\left(\text { because } 1-\frac{q_{\ell}}{q_{\ell+1}}>\frac{5}{6}\right) .
\end{aligned}
$$

We may therefore continue to find

$$
\begin{aligned}
\prod_{t=1}^{q_{\ell}-1}\left\|R_{t}\right\| & \geq \frac{5}{18} \frac{(2 Q-3)!!}{\left(2 q_{\ell}\right)^{Q-1}} \frac{(Q-1)!}{\left(q_{\ell}\right)^{Q-1}}=\frac{5}{18} \frac{(2 Q-2)!}{\left(2 q_{\ell}\right)^{2(Q-1)}} \\
& \stackrel{(5.10)}{\geq} \frac{5}{18} \sqrt{2 \pi \cdot 2(Q-1)}\left(\frac{Q-1}{q_{\ell} e}\right)^{2(Q-1)} \\
& \geq \frac{5}{18} \sqrt{2 \pi\left(q_{\ell}-3\right)}\left(\frac{q_{\ell}-3}{2 q_{\ell} e}\right)^{q_{\ell}-2} \\
& \geq \frac{5}{18} \sqrt{2 \pi\left(q_{\ell}-3\right)}\left(\frac{1}{2 e}\right)^{q_{\ell}-2}\left(1-\frac{3}{q_{\ell}}\right)^{q_{\ell}-3}\left(1-\frac{3}{q_{\ell}}\right) \\
& \geq \frac{5}{18} \frac{\sqrt{2 \pi q_{\ell}}}{(2 e)^{q_{\ell}+1}} .
\end{aligned}
$$

In the last step we have used the bound $\frac{q_{\ell}-3}{q_{\ell}} \geq \frac{1}{4}$ as well as the inequality $\left(1+\frac{r}{n}\right)^{n}<e^{r}$ for $n>r$, which implies that $\left(1-\frac{r}{n}\right)^{n-r}=\left(\frac{n}{n-r}\right)^{-(n-r)}>e^{-r}$ whenever $n>r$.

We have now shown that (5.15) holds for $q_{\ell} \geq 4$, and proceed by considering smaller values of $q_{\ell}$. When $q_{\ell}=1$, the product in question is empty and hence by definition equals 1 . When $q_{\ell}=2$, the product consists only of the factor $\left\|R_{1}\right\|$; by (5.29) and the assumption $a_{k} \geq 6$ we have $\left\|R_{1}\right\| \geq 1 / 3$, so (5.15) is still valid. Finally, when $q_{\ell}=3$ again (5.29) implies $\left\|R_{1}\right\|\left\|R_{2}\right\| \geq 1 / 12$ which proves that (5.15) is true also in this case. This completes the proof of Lemma 4 .

We now turn to Lemma 7. We will not write out the proof of this result in full detail, as it is very similar to that of Lemma 4. We simply point out that it is a consequence of the following results, analogous to Lemmas 1011 and Corollary 5 above.
Lemma 12. When $\ell$ is odd, we have

$$
\begin{equation*}
\frac{a_{k}}{a_{k}+1} \frac{1}{q_{\ell} q_{\ell+1}}<\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-\frac{2 b}{q_{\ell}}<\frac{1}{q_{\ell} q_{\ell+1}} . \tag{5.30}
\end{equation*}
$$

Lemma 13. Let $\ell$ be odd. For each $t=1,2, \ldots, q_{\ell}-1$ there exists a unique integer $i=i_{t} \in\left\{1,2, \ldots, q_{\ell}-1\right\}$ such that

$$
\frac{i}{q_{\ell}}-\frac{1}{q_{\ell+1}} \leq\left\|R_{t}+b\right\| \leq \frac{i}{q_{\ell}}+\frac{1}{7 q_{\ell+1}} .
$$

Moreover, the correspondence $t \mapsto i_{t}$ is one-to-one.
Corollary 6. Let $\ell$ be odd and $R_{t}$ be as in (5.14). Then for every $t=1,2, \ldots, q_{\ell}-1$ there exists an integer $j_{t} \in\left\{1,2, \ldots,\left\lfloor\frac{1}{2} q_{\ell}\right\rfloor\right\}$ such that

$$
\begin{equation*}
\left\|R_{t}+b\right\| \geq \frac{j_{t}}{q_{\ell}}-\frac{1}{q_{\ell+1}} \quad \text { or } \quad\left\|R_{t}+b\right\| \geq \frac{j_{t}}{q_{\ell}}-\frac{1}{7 q_{\ell+1}} . \tag{5.31}
\end{equation*}
$$

The first inequality in (5.31) holds when the residue of $\operatorname{tp}\left(\alpha_{\tau_{k}}\right)$ modulo $q_{\ell}\left(\alpha_{\sigma_{k}}\right)$ is one of $1,2, \ldots,\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor$ while the second holds when the residue of $t p_{\ell}\left(\alpha_{\tau_{k}}\right)$ modulo $q_{\ell}\left(\alpha_{\sigma_{k}}\right)$ is one of $\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor+1, \ldots, q_{\ell}\left(\alpha_{\tau_{k}}\right)-1$. Moreover, there exists at most one integer $t$ such that $j_{t}=\left\lfloor\frac{1}{2} q_{\ell}\left(\alpha_{\tau_{k}}\right)\right\rfloor$, and the correspondence $t \mapsto j_{t}$ is "at most two-to-one".
5.3.2. Proofs of Lemmas 5 and 8. Now let us treat Lemmas 5 and 8 . Recall that these provide lower bounds on the factors $\Pi_{2}$ and $\Pi_{2}^{\prime}$, given in (5.8) and (5.23). As in the previous subsection, we will provide a full proof of Lemma [5, and then simply sketch the proof of Lemma 8 .

Proof of Lemma 5. Recall from (5.8) that $\Pi_{2}$ is defined as

$$
\Pi_{2}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor}^{c-1} \operatorname{dist}(2 s+1, \mathcal{U}), \quad u_{k}(t)=\frac{2 t}{\left|e_{k} c_{k}\right|}-2\left\{t \alpha_{\sigma_{k}}\right\}+1
$$

For each $s \geq 0$ let $t_{s} \geq 1$ be the unique positive integer such that

$$
\begin{equation*}
\operatorname{dist}(2 s+1, \mathcal{U})=\left|2 s+1-u_{k}\left(t_{s}\right)\right| \tag{5.32}
\end{equation*}
$$

Then

$$
\begin{equation*}
\Pi_{2}=\prod_{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1}^{c-1}\left|2 s+1-u_{k}\left(t_{s}\right)\right|=2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor-1} \prod_{t=1}^{\infty} \prod_{\substack{s=\left\lfloor\frac{a_{k}}{k}\right\rfloor+1 \\ t_{s}=t}}^{c-1} \frac{1}{2}\left|u_{k}(t)-(2 s+1)\right| . \tag{5.33}
\end{equation*}
$$

We now analyze the factors appearing in (5.33). For any $t \geq 1$ and $s=0,1, \ldots, c-1$,

$$
\begin{aligned}
\frac{1}{2}\left(u_{k}(t)-(2 s+1)\right) & =\frac{t}{\left|c_{k} e_{k}\right|}-\left\{t \alpha_{\sigma_{k}}\right\}-s \\
& \stackrel{44.7}{=} t a_{k}+t\left(\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)-\left\{t \alpha_{\sigma_{k}}\right\}-s \\
& =\underbrace{t a_{k}+\left\lfloor\frac{t p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right\rfloor+\left\lfloor t \alpha_{\sigma_{k}}\right\rfloor-s}_{\in \mathbb{Z}}+R_{t},
\end{aligned}
$$

where the terms $R_{t}$ are as in (5.14), with this definition extended to all values $t \geq 1$. When $t=q_{\ell}$ and $s=c-1$ the right hand side of (5.34) is equal to

$$
\begin{aligned}
& a_{k} q_{\ell}+p_{\ell}+\left\lfloor q_{\ell} \alpha_{\sigma_{k}}\right\rfloor-(c-1)+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-2 b= \\
&=a_{k} q_{\ell}+q_{\ell-1}+\left\lfloor q_{\ell}\left(\alpha_{\sigma_{k}}-\frac{p_{\ell}}{q_{\ell}}\right)\right.\left.+p_{\ell}\right\rfloor-(c-1)+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-2 b= \\
&=q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-2 b \stackrel{\sqrt{5.28}}{=}-b \in\left(-\frac{1}{q_{\ell+1}}, 0\right) .
\end{aligned}
$$

This observation implies that $t_{c-1}=q_{\ell}$. Since $t_{s}$ is increasing in $s, t=q_{\ell}$ is the maximum value of $t$ for which the product in (5.33) is non-empty, and we may write

$$
\begin{equation*}
\Pi_{2}=2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor-1} \prod_{t=1}^{q_{\ell}} \prod_{\substack{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1 \\ t_{s}=t}}^{c-1} \frac{1}{2}\left|u_{k}(t)-(2 s+1)\right| . \tag{5.35}
\end{equation*}
$$

We now want to find lower bounds for the factors in the innermost product in (5.35) for each $1 \leq t \leq q_{\ell}$. When $t=1$, the values of $s$ appearing in the product

$$
\prod_{\substack{s=\frac{a_{k}}{2} J+1 \\ t_{s}=1}}^{c-1} \frac{1}{2}\left|u_{k}(1)-(2 s+1)\right|
$$

are the integers $s \geq 1+\left\lfloor\frac{a_{k}}{2}\right\rfloor$ for which $2 s+1<\frac{1}{2}\left(u_{k}(1)+u_{k}(2)\right)$, or equivalently

$$
1+\left\lfloor\frac{a_{k}}{2}\right\rfloor \leq s<\frac{3 a_{k}}{2}+\frac{3}{2}\left(\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)-\frac{\left\{\alpha_{\sigma_{k}}\right\}+\left\{2 \alpha_{\sigma_{k}}\right\}}{2} .
$$

The number of integers $s$ in any interval of the form $[n, \kappa)$, where $n$ is an integer, is $1+\lfloor\kappa\rfloor-n$. In our case we have $n=1+\left\lfloor\frac{a_{k}}{2}\right\rfloor$ and

$$
\begin{aligned}
\kappa & =\frac{3 a_{k}}{2}+\frac{3}{2}\left(\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)-\frac{\left\{\alpha_{\sigma_{k}}\right\}+\left\{2 \alpha_{\sigma_{k}}\right\}}{2} \\
& \geq \frac{3 a_{k}}{2}+\frac{3}{2}\left(\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)-\frac{3}{2}\left\{\alpha_{\sigma_{k}}\right\} \\
& =\frac{3 a_{k}}{2}+\frac{3}{2}(\underbrace{\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}-\alpha_{\sigma_{k}}}_{\frac{1}{2_{\ell} q_{\ell+1}} \leq \cdots \leq \frac{1}{q_{\ell} \ell_{\ell+1}}}+\underbrace{\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}}_{>0}),
\end{aligned}
$$

so there are at least $a_{k}$ values of $s$ for which $t_{s}=1$. Consider now the corresponding factors in the innermost product in (5.35). From (5.34) and Lemma 11 it follows that the minimal factor is equal to $\left\|R_{1}\right\|$, whereas the remaining factors can be bounded below by the values $1,2, \ldots, a_{k}-1$. Therefore

$$
\begin{equation*}
\prod_{\substack{\left.s=\frac{a_{k}}{2}\right\rfloor+1 \\ t_{s}=1}}^{c-1} \frac{1}{2}\left|u_{k}(1)-(2 s+1)\right| \geq\left\|R_{1}\right\| \cdot\left(a_{k}-1\right)! \tag{5.36}
\end{equation*}
$$

We then examine the innermost product of (5.35) for $1<t<q_{\ell}$. The factors appearing are those corresponding to integers $s$ such that

$$
\frac{1}{2}\left(u_{k}(t-1)+u_{k}(t)\right)<2 s+1<\frac{1}{2}\left(u_{k}(t)+u_{k}(t+1)\right),
$$

or equivalently

$$
\frac{2 t-1}{\left|c_{k} e_{k}\right|}-\left\{(t-1) \alpha_{\sigma_{k}}\right\}-\left\{t \alpha_{\sigma_{k}}\right\}+1<2 s+1<\frac{2 t+1}{\left|c_{k} e_{k}\right|}-\left\{t \alpha_{\sigma_{k}}\right\}-\left\{(t+1) \alpha_{\sigma_{k}}\right\}+1 .
$$

Observe that the number of integers in an interval $(\alpha, \beta)$ is at least $\lfloor\beta-\alpha\rfloor$. Here the length of the interval of possible values of $s$ is

$$
\begin{aligned}
\left(\frac{t+\frac{1}{2}}{\left|c_{k} e_{k}\right|}-\frac{\left\{t \alpha_{\sigma_{k}}\right\}-\left\{(t+1) \alpha_{\sigma_{k}}\right\}}{2}\right) & -\left(\frac{t-\frac{1}{2}}{\left|c_{k} e_{k}\right|}-\frac{\left\{t \alpha_{\sigma_{k}}\right\}-\left\{(t-1) \alpha_{\sigma_{k}}\right\}}{2}\right)= \\
& =\frac{1}{\left|c_{k} e_{k}\right|}+\frac{\left\{(t-1) \alpha_{\sigma_{k}}\right\}+\left\{(t+1) \alpha_{\sigma_{k}}\right\}}{2} \\
& \geq a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\alpha_{\sigma_{k}} \\
& \geq a_{k},
\end{aligned}
$$

so there exist at least $a_{k}$ values of $s$. The minimal factor $\frac{1}{2}\left|u_{k}(t)-(2 s+1)\right|$ is equal to $\left\|R_{t}\right\|$, while the remaining ones can be bounded from below by $1,2, \ldots, a_{k}-1$. We thus have

$$
\begin{equation*}
\prod_{\substack{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1 \\ t_{s}=t}}^{c-1} \frac{1}{2}\left|u_{k}(t)-(2 s+1)\right| \geq\left\|R_{t}\right\| \cdot\left(a_{k}-1\right)!, \quad t=2, \ldots, q_{\ell}-1 \tag{5.37}
\end{equation*}
$$

Finally we deal with the innermost product in (5.35) when $t=q_{\ell}$. The values of $s$ appearing in the product are those for which

$$
\begin{aligned}
2 c-1 \geq 2 s+1> & \frac{1}{2}\left(u_{k}\left(q_{\ell}\right)+u_{k}\left(q_{\ell}-1\right)\right) \\
= & 2(c-2 b)-\left(a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right) \\
& \quad-\left\{q_{\ell} \alpha_{\sigma_{k}}\right\}-\left\{\left(q_{\ell}-1\right) \alpha_{\sigma_{k}}\right\}+1,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& c-1 \geq s>c- \frac{1}{2} \\
&\left(a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right) \\
&-\frac{\left\{q_{\ell} \alpha_{\sigma_{k}}\right\}+\left\{\left(q_{\ell}-1\right) \alpha_{\sigma_{k}}\right\}}{2}-2 b .
\end{aligned}
$$

For any $\kappa>1$, the interval $(c-\kappa, c-1\rfloor$ contains precisely $\lfloor\kappa\rfloor$ integers. Here we need to apply this observation with

$$
\begin{aligned}
\kappa & =\frac{1}{2}\left(a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)+\frac{\left\{q_{\ell} \alpha_{\sigma_{k}}\right\}+\left\{\left(q_{\ell}-1\right) \alpha_{\sigma_{k}}\right\}}{2}+2 b \\
& \geq \frac{1}{2}\left(a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right)+\frac{\left(1-\frac{1}{q_{\ell+1}}\right)+\left(1-\frac{1}{q_{\ell+1}}-\alpha_{\sigma_{k}}\right)}{2}+2 b \\
& \geq \frac{1}{2}\left(a_{k}+\frac{p_{\ell}\left(\alpha_{\sigma_{k}}\right)}{q_{\ell}\left(\alpha_{\sigma_{k}}\right)}-\alpha_{\sigma_{k}}+\frac{p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}-\frac{2 b}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}+2-\frac{2}{q_{\ell+1}}+4 b\right)
\end{aligned}
$$

hence there must be at least $1+\left\lfloor\frac{a_{k}}{2}\right\rfloor$ such factors.
When $t=q_{\ell}$ and $s=c-1$, we saw already that

$$
\frac{1}{2}\left|u_{k}\left(q_{\ell}\right)-2 c+1\right|=\left|q_{\ell} a_{k}+p_{\ell}\left(\alpha_{\tau_{k}}\right)+p_{\ell}\left(\alpha_{\sigma_{k}}\right)-c+R_{q_{\ell}}\right|=\left|R_{q_{\ell}}\right|=b .
$$

For the remaining values of $s$ with $t_{s}=q_{\ell}$, we bound the factors $\frac{1}{2}\left|u_{k}\left(q_{\ell}\right)-(2 s+1)\right|$ from below by

$$
r-\frac{1}{q_{\ell+1}}, \quad r=1,2, \ldots,\left\lfloor\frac{a_{k}}{2}\right\rfloor
$$

Consequently,

$$
\begin{align*}
\prod_{\substack{\left\lfloor a_{k} / 2\right\rfloor+1 \leq s<c-1 \\
t_{s}=q_{\ell}}} \frac{1}{2}\left|u_{k}\left(q_{\ell}\right)-(2 s-1)\right| & \geq b \prod_{r=1}^{\left\lfloor a_{k} / 2\right\rfloor}\left(r-\frac{1}{q_{\ell+1}}\right) \\
& =b\left\lfloor\frac{a_{k}}{2}\right\rfloor!\prod_{r=1}^{\left\lfloor a_{k} / 2\right\rfloor}\left(1-\frac{1}{r q_{\ell+1}}\right) \\
& \geq \frac{1}{c}\left\lfloor\frac{a_{k}}{2}\right\rfloor!\left(1-\sum_{r=1}^{\left\lfloor a_{k} / 2\right\rfloor} \frac{1}{r q_{\ell+1}}\right) \\
& \geq \frac{4}{5 c}\left\lfloor\frac{a_{k}}{2}\right\rfloor!.
\end{align*}
$$

On combining (5.35) with (5.36), (5.37) and (5.38), we obtain inequality (5.16). This completes the proof of Lemma 5 .

Let us now give an outline of the proof of Lemma 8 . Recall that this result gives a bound on $\Pi_{2}^{\prime}$ defined in (5.23), relevant to the case of odd period lengths $\ell$. Using the same arguments that gave the bound for $\Pi_{2}$ above, we find that

$$
\Pi_{2}^{\prime}=2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor-1} \prod_{t=1}^{q_{\ell}} \prod_{\substack{s=\left\lfloor\frac{a_{k}}{2}\right\rfloor+1 \\ t_{s}=t}}^{c-1} \frac{1}{2}\left|u_{k}(t)-(2 s+1-2 b)\right|,
$$

where $t_{s}$ is defined in 5.32 . We now proceed as before, and find lower bounds on the factors in the innermost product for each $1 \leq t \leq q_{\ell}$. According to (5.34), the factors appearing in the product are

$$
\begin{equation*}
\frac{1}{2}\left(u_{k}(t)-(2 s+1-2 b)\right)=\underbrace{t a_{k}+\left\lfloor\frac{t p_{\ell}\left(\alpha_{\tau_{k}}\right)}{q_{\ell}\left(\alpha_{\tau_{k}}\right)}\right\rfloor+\left\lfloor t \alpha_{\sigma_{k}}\right\rfloor-s}_{\in \mathbb{Z}}+R_{t}+b, \tag{5.39}
\end{equation*}
$$

with $R_{t}$ given in (5.14).
When $\ell$ is odd, $t=q_{\ell}$ and $s=c-1$, the right hand side of (5.39) is equal to

$$
\begin{aligned}
& a_{k} q_{\ell}+p_{\ell}+\left\lfloor q_{\ell} \alpha_{\sigma_{k}}\right\rfloor-c+1+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-b= \\
& =a_{k} q_{\ell}+q_{\ell-1}+\left\lfloor q_{\ell}\left(\alpha_{\sigma_{k}}-\frac{p_{\ell}}{q_{\ell}}\right)+p_{\ell}\right\rfloor-c+1+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-b \\
& =q_{\ell+1}+p_{\ell}-\left(q_{\ell+1}+p_{\ell}\right)+1+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-b \\
& \quad=1+q_{\ell}\left(\frac{p_{\ell}}{q_{\ell}}-\alpha_{\sigma_{k}}\right)-b \stackrel{\stackrel{5.28}{=}}{=} .
\end{aligned}
$$

It follows that

$$
\prod_{\substack{\left.s=\frac{a_{k}}{2}\right\rfloor+1 \\ t_{s}=q_{\ell}}}^{c-1} \frac{1}{2}\left|u_{k}\left(q_{\ell}\right)-(2 s+1-2 b)\right| \geq\left\lfloor\frac{a_{k}}{2}\right\rfloor!.
$$

For $t=1, \ldots, q_{\ell}-1$, we argue as for the even period case in Lemma 5, and obtain

$$
\prod_{\substack{\left.s=\left\lfloor a_{k}\right\rfloor \\ \frac{a_{2}}{2}\right\rfloor \\ t_{s}=t}}^{c-1} \frac{1}{2}\left|u_{k}(t)-(2 s+1-2 b)\right| \geq\left\|R_{t}+b\right\| \cdot\left(a_{k}-1\right)!, \quad t=1, \ldots, q_{\ell}-1
$$

Combining these two estimates, we deduce that

$$
\Pi_{2}^{\prime} \geq 2^{c-\left\lfloor\frac{a_{k}}{2}\right\rfloor-1} \cdot\left[\left(a_{k}-1\right)!\right]^{q_{\ell}-1} \cdot\left\lfloor\frac{a_{k}}{2}\right\rfloor!\cdot \prod_{t=1}^{q_{\ell}-1}\left\|R_{t}+b\right\|
$$

which confirms Lemma 8 ,

## 6. Proof of Theorem 3

With Theorems 5 and 6 established, let us now prove Theorem 3. As explained in Remark 3, it suffices to study quadratic irrationals $\alpha$ with purely periodic continued fraction expansion.

Recall that by Theorem 2, Theorem 3 is verified if we can show that

$$
C_{k}=\lim _{m \rightarrow \infty} P_{q_{m \ell+k}}(\alpha)<1
$$

for every $\alpha=\left[0 ; \overline{a_{1}, a_{2}, \ldots, a_{\ell}}\right]$ with $a_{k}=\max _{j} a_{j} \geq 23$.
Assume first that the period length $\ell$ is odd. Then $q_{\ell}>1$, and from Theorem 6 it follows that

$$
C_{k} \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(40 c^{\frac{3}{2}}\right)^{\frac{1}{c}} a_{k}^{\frac{5}{2 a_{k}}}
$$

The factor $\left(40 c^{3 / 2}\right)^{1 / c}$ is decreasing in $c$ (within the range of relevant values of $c$ ), so we may safely use the fact that $c>a_{k} q_{\ell}$ to obtain

$$
\begin{aligned}
C_{k} & \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(40 a_{k}^{\frac{3}{2}} q_{\ell}^{\frac{3}{2}}\right)^{\frac{1}{a_{k} a_{\ell}}} a_{k}^{\frac{5}{2 a_{k}}} \\
& \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(40 a_{k}^{\frac{3}{2}}\right)^{\frac{1}{a_{k} a_{\ell}}} 2^{\frac{1}{a_{k}}} a_{k}^{\frac{5}{2 a_{k}}}
\end{aligned}
$$

where for the last inequality we have used that $q_{\ell}^{3 / 2 q_{\ell}}<2$. This expression is decreasing in $q_{\ell}$, so we insert $q_{\ell}=2$ to obtain

$$
C_{k} \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(160 a_{k}^{\frac{13}{2}}\right)^{\frac{1}{2 a_{k}}}
$$

The right hand side is a decreasing function of $a_{k}$, and it can be easily verified that $C_{k} \leq 1$ whenever $a_{k} \geq 22$. This proves Theorem 3 for odd period lengths $\ell$.

Now assume that the period length $\ell$ is even, and consider first the case $q_{\ell}>1$. By Theorem 5 we then have

$$
C_{k}^{\frac{c-2}{c}} \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(200 e^{2.4} c^{2}\right)^{\frac{1}{c}}\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_{k}}}
$$

The factor $\left(200 e^{2.4} c^{2}\right)^{1 / c}$ is again decreasing in $c$, so we replace $c$ by $a_{k} q_{\ell}$ to obtain

$$
\begin{aligned}
C_{k}^{\frac{c-2}{c}} & \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(200 e^{2.4} a_{k}^{2} q_{\ell}^{2}\right)^{\frac{1}{a_{k} q_{\ell}}}\left(\frac{a_{k}^{\frac{5}{2}}}{e}\right)^{\frac{1}{a_{k}}} \\
& \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(200 e^{2.4} a_{k}^{2}\right)^{\frac{1}{a_{k} q_{\ell}}} \cdot a_{k}^{\frac{5}{2 a_{k}}}
\end{aligned}
$$

where we have used that $q_{\ell}^{2 / q_{\ell}}<e$. This expression is decreasing in $q_{\ell}$, so we insert $q_{\ell}=2$ to obtain

$$
C_{k}^{\frac{c-2}{c}} \leq \frac{\pi}{\sqrt{2} a_{k}} e^{1+f\left(a_{k}\right)}\left(200 e^{2.4} a_{k}^{7}\right)^{\frac{1}{2 a_{k}}}
$$

One can again verify that the right hand side is below one whenever $a_{k} \geq 23$. This proves Theorem 3 for even $\ell$ in the case $q_{\ell}>1$.

Finally, we consider the case $q_{\ell}=1$. Recall that this can only occur if $\ell=2$ and $\alpha=$ [ $\left.0 ; \overline{a_{1}, a_{2}}\right]$, with either $a_{1}=1$ or $a_{2}=1$. By Corollary 4, we then have the bound

$$
C_{k}^{\frac{c-2}{c}} \leq \frac{\pi}{a_{k}} e^{1+g\left(a_{k}\right)}\left(6.2\left(a_{k}+2\right)^{4}\right)^{\frac{1}{a_{k}+2}}
$$

where $g\left(a_{k}\right) \leq 3.3 / a_{k}+0.1$ and one can again verify that the right hand side is below one whenever $a_{k} \geq 21$. This verifies Theorem 3 when $q_{\ell}=1$, and completes the proof.

## References

[1] C. Aistleitner, B. Borda, Quantum invariants of hyperbolic knots and extreme values of trigonometric products, Math. Z. (2022) https://doi.org/10.1007/s00209-022-03086-5
[2] C. Aistleitner, N. Technau, A. Zafeiropoulos, On the order of magnitude of Sudler products. To appear in Am. J. Math. (2022). pre-print: arXiv:2002.06602
[3] S. A. Avdonin, On Riesz bases of exponentials in $L^{2}$, Vestnik Leningrad Univ. Math. 7 (1979), 203-211.
[4] M. Einsiedler, T. Ward, Ergodic Theory: with a view towards Number Theory, Springer Verlag, London, 2011
[5] P. Erdős, G. Szekeres, On the product $\prod_{k=1}^{n}\left(1-z^{a_{k}}\right)$. Acad. Serbe Sci. Publ. Inst. Math., 13 (1959), 29-34.
[6] S. Grepstad, L. Kaltenböck, M. Neumüller, A positive lower bound for $\liminf _{N \rightarrow \infty} \prod_{r=1}^{N} 2|\sin \pi r \phi|$. Proc. Amer. Math. Soc. 147 (2019), 4863-4876.
[7] S. Grepstad, L. Kaltenböck, M. Neumüller, On the asymptotic behaviour of the sine product $\prod_{r=1}^{n} 2|\sin \pi r \alpha|$. D. Bilyk, J. Dick, F. Pillichshammer (Eds.) Discrepancy theory, Radon Series on Computational and Applied Mathematics 26 (2020), 103-115.
[8] S. Grepstad, M. Neumüller, Asymptotic behaviour of the Sudler product of sines for quadratic irrationals. J. Math. Anal. Appl. 465 (2017), no. 2, 928-960.
[9] M. I. Kadec, The exact value of the Paley-Wiener constant. Soviet Math. Dokl., 5 (1964), 559-561.
[10] D. Lubinsky, The size of $(q ; q)_{n}$ for $q$ on the unit circle. J. Number Theory 76 (1999), no. 2, 217-247.
[11] C. Pinner, On sums of fractional parts $\{n \alpha+\gamma\}$. J. Number Theory 65 (1997), 48-73.
[12] P. Verschueren, B. Mestel, Growth of the Sudler product of sines at the golden rotation number. J. Math. Anal. Appl. 433 (2016), 200-226.

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway; E-mail address: sigrid.grepstad@ntnu.no

Johannes Kepler Universität, Linz, Austria; E-mail address: mario.neumueller@jku.at

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway; E-mail address: agamemnon.zafeiropoulos@ntnu.no


[^0]:    Keywords: Sudler Products, quadraric irrationals. Math. Subject Classification Number: 11J70, 11 J71.
    SG and AZ are supported by Grant 275113 of the Research Council of Norway.
    MN is funded by FWF projects F5509-N26, F5512-N26 and P29910-N35.

