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Study programme driven engineering education: interplay between mathematics and engineering subjects

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This paper presents a report from an early phase of the project Mathematics as a Thinking Tool. The primary goal of this project is to increase engineering students' perception of the relevance of mathematics by developing a close connection between mathematics and engineering subjects, and to develop an approach driven by the learning goals of the study programme rather than the learning goals in specific subjects. We will present examples from electrical engineering and chemistry to show how mathematics may play along with the other fields in order to develop deep conceptual understanding. Some preliminary results from student surveys are also included.

1. Introduction

The Norwegian University of Science and Technology (NTNU) is the largest university in Norway, and by far the country's largest institution for engineering education, covering a wide range of engineering fields, with study programmes at both bachelor and master level. The Master of Technology (MT) programmes are organized as integrated 5-year programmes, admitting students coming directly from upper secondary school provided they have studied mathematics to the highest level at school, and obtained a certain grade in mathematics. There is an opening for a small number of students to enter the fourth year after having completed a 3-year bachelor programme, but most students follow the complete 5-year track.

These students take a minimum of four mathematics courses and one statistics course, each of 7.5 ECTS, during their first 2 years of study. The mathematics courses cover standard topics like univariate and multivariate calculus, linear algebra, differential equations, Fourier analysis and numerical methods. Students at computer science programmes learn discrete mathematics instead of multivariate calculus, and some programmes, including electrical engineering, learn complex analysis at the expense of some numerical methods. Apart from this, all students are trained in the same topics and sit for the same exams. Engineering students are, however, taught mathematics separately from students in mathematics. Therefore, some engineering applications are included in the courses, but not particularly targeted to specific engineering programmes.

Textbooks that are used are for example Adams & Essex (2021) and Kreyzig (2011). The book by Adams and Essex is a general calculus textbook, also used for 'general calculus courses', whereas the book by Kreyzig is more specifically targeted to engineering education and contains numerous examples of applications. The statistics course is common to all MT programmes, but this will not be further discussed in this paper. Each year approximately 1700 new students are admitted to the 17 different 5-year MT programmes.

With so many students, there will always be a considerable number that drift between programmes, either because they after some time realize that the first choice was not what they really wanted, or because they did not get their preferred choice from the beginning because of strong competition. Then, applying for a transfer to another programme for example after 1 year is a possibility. The uniformity of the package of mathematics courses makes such transitions easier, which is a strong argument for the current structure. Another argument is that mathematics *should* indeed be general. By studying mathematics, students are expected to acquire general problem-solving competencies. However, this structure gives little or no space for examples and applications targeted to specific areas of engineering. A long-lasting critique of mathematics in engineering education has been that mathematics is taught with too little emphasis on applications and being too focused on purely mathematical concepts. This critique still persists (Loch & Lamborn, 2016). Teaching the same mathematics to all engineering students may result in less emphasis on applications, in particular applications that require knowledge in one specific branch of engineering, to make sense to the learners.

There is also a long-lasting tradition in engineering education to teach the 'basics', like mathematics, early in the studies, and postponing the applications in the engineering subjects until later (Winkelman, 2009). In recent years, one has observed a turn towards a more targeted mathematics education for engineers at many universities, with mathematics taught separately to specific study programmes, thus giving the opportunity to include programme specific examples and applications (Alpers, 2008; Enelund *et al.*, 2011; Klingbeil & Bourne, 2014). Another development in engineering education is a turn towards a *programme driven approach*. This means that it is the needs of the study programme that form the basis for the design of the programme. For a service subject like mathematics, this implies that the design and content of the mathematics courses to a larger extent will have to be based on the needs of each study programme. Closely connected to a programme driven design is the idea of *contextual learning*, meaning that concepts are presented in the context of their use. These ideas are central in the so-called Conceive, Design, Implement, Operate (CDIO) Initiative (Crawley *et al.*, 2014). Both a programme driven approach and an approach based on contextual learning will challenge the traditional model of teaching mathematics as a context free subject in the first years, with applications to engineering appearing later (Winkelman, 2009).

Inspired by the ideas of a programme driven approach and contextual learning, a major project has been going on at NTNU to redesign the technological study programmes. This project is called *Technology Studies for the Future (FTS*[1]*)* and has recently submitted its final report with recommendations (Fremtidens teknologistudier, 2022). Within FTS, the project MARTA—Mathematics as a Tool for

[1] In Norwegian: Fremtidens TeknologiStudier.

Thinking[2]—has been chosen as one of nine pilot projects. MARTA is a collaboration between three departments: Department of Mathematical Sciences, Department of Electronic Systems and Department of Chemistry. In this paper, we will report on some of the ideas and first experiences from this project. Although situated at NTNU, we claim that the ideas both in FTS and MARTA are of more general interest, as they reflect ideas that are widespread and currently seem to be gaining momentum in engineering education internationally (see e.g. Crawley *et al.*, 2014).

2. Background

In engineering education, the tension between theory and practice, between academic and professional aims, has roots going far back in time (Edström, 2018), and recent studies show that this tension persists (Carvalho & Oliveira, 2018). Although mathematics has from early on been regarded as an important subject in engineering, part of the tension concerns precisely the role and perceived relevance of mathematics (Flegg *et al.*, 2012; Gueudet & Quéré, 2018). Also, the question of what kind of mathematics should be taught to engineers and who should teach it has a long history (Bajpaj, 1985; Alpers, 2020).

According to Edström (2018) engineering education in the United States before 1920 was highly practical, but from then on, a gradual change can be observed. The development towards a more theoretical approach progressed rapidly after the Second World War, and in a report commissioned by Massachusetts Institute of Technology (known as the Lewis report), some warnings are issued that not only has the separation from practice gone too far, but there are also warnings against too much routine learning:

> [M]any students seem to be able to graduate from the Institute[3] on the basis of routine learning, and . . . though fully equipped with knowledge of standard procedures . . . , they lack the critical judgement, the creative imagination, the competence in handling unique situations. (Lewis, 1949, pp. 28–29)

Jumping 50 years ahead in time, it seems that the issues raised by Lewis (1949) are still not resolved. The CDIO Initiative, launched in 2000, can be seen in connection with these issues (Crawley *et al.*, 2014). The CDIO approach has three overall goals: to educate students who are able to

1. Master a deeper working knowledge of technical fundamentals
2. Lead in the creation and operation of new products, processes and systems
3. Understand the importance and strategic impact of research and technical development on society (Crawley *et al.*, 2014, p. 13)

A crucial point in the CDIO approach is that conceptual understanding, rather than memorization of facts and definitions, or the simple application of a principle, should be the aim of the education. In addition, the CDIO approach values *contextual learning,* in the sense that new concepts should be presented in situations that students recognize as important to their current and future lives (Crawley *et al.*, 2014, pp. 32–33).

In the project MARTA, the aim is to develop a close connection between mathematics and engineering subjects early in the study programme, while maintaining conceptual understanding and deep learning (Marton & Säljö, 1976) both in mathematics and in the individual engineering fields. Developing the connection early is a contrast to the traditional model of first teaching the basics (like mathematics) and

2 In Norwegian: MAtematikk som Redskap for TAnken.
3 Massachusetts Institute of Technology.

later use the basics in applications (Winkelman, 2009). The overarching idea of the project MARTA is formulated as developing mathematics as a 'Tool for Thinking', which is coherent with the three CDIO goals quoted above (Crawley *et al.*, 2014, p. 13). The close coherence between MARTA/FTS and the CDIO principles lead us to argue that our ideas and experiences are of value also outside of NTNU.

Ideas about mathematics as a tool for thinking in engineering are not new. Already in 1985, Scanlan, in a talk about mathematics in engineering education, concluded by stating that mathematics should be an essential part of the students' formation and 'not a set of "tools" to be acquired before proceeding to the "important" part of the course' (Scanlan, 1985, p. 449). However, it is well documented that being able to apply mathematics that students are supposed to have learned, when they need it in their engineering courses is a big challenge (Harris *et al.*, 2015; Carvalho & Oliveira, 2018).

3. The project MARTA at NTNU

The ideas in MARTA represent a break with the existing model at NTNU, where mathematics is taught with largely the same content for all engineering programmes. The project aims at developing a closer connection between mathematics and each engineering programme, following ideas of a programme driven approach and contextual learning (Crawley *et al.*, 2014; Fremtidens teknologistudier, 2022). Although the goal is to connect mathematics and the engineering sciences, mathematics should still be taught with fundamental respect for its own logical structure. The idea is not that the engineering subjects may pick bits and scraps of mathematics whenever needed, as this would be contradictory to the principle of *conceptual understanding* emphasized in the CDIO approach (Crawley *et al.*, 2014). The main features of the revised mathematics courses are

1. stronger emphasis on engineering applications and
2. a shift of order and emphasis of the mathematical topics taught, according to the specific needs of the individual engineering programmes.

From the perspective of the engineering programmes, there is an expectation that the teachers of engineering will demonstrate how and why mathematics is important for the specific engineering field by actively incorporating examples from the engineering field where they can explicitly point to the role and importance of mathematical knowledge. It is expected that this approach will make the students better see the relevance of mathematics for their engineering specialization.

In the first part of the project, the only engineering programme involved has been *Electronic System Design and Innovation (MTELSYS)*. In MTELSYS, as in the other MT programmes, there is already from the beginning (at least) one course included each semester representing the special profile of the particular programme. This principle is referred to as the 'engineering string', ensuring that the students already from the beginning get a feeling for what is special about the study programme they have chosen. In the reformed version of MTELSYS, four new mathematics courses (one for each of the first four semesters) have been developed and they run in parallel with five courses that are specific to the MTELSYS programme. Thus, there is always at least one engineering course and one mathematics course taught in the same semester. In addition, the students have other, not programme specific courses, like physics and ICT and programming, At the time of writing, the first cohort is in the second year, and a new cohort has been admitted to the first year. The project MARTA can be seen as a successor to earlier projects, the last one called 'ACT! ACTive learning in core courses in mathematics and statistics for engineering education', running from 2018–2020. An important aim of ACT! was to modernize both content and form of core courses in mathematics and statistics serving the MT programmes at NTNU by incorporating principles of active learning and use of technology (Rønning, 2017, 2019, 2021a).

One component of ACT! was to establish a *user panel*, consisting of scientific staff and students from MT programmes, representatives from NTNU's Executive Committee for Engineering Education, in addition to scientific staff from the Department of Mathematical Sciences. The user panel played the role of a discussion forum for the design and description of reformed learning outcomes in the core courses in mathematics and statistics. Discussions in the user panel led to questions like 'what if we tried to make mathematics more closely connected to the engineering programmes?' Out of these discussions grew a close collaboration between the third and the fourth author of this paper, representing electronic systems and mathematics, respectively. After a decision in NTNU's Executive Committee for Engineering Education, the programme MTELSYS was allowed to follow the revised curriculum in mathematics, starting autumn 2020, then as a part of ACT! The ideas of a programme driven approach were developed further into the project MARTA, starting in 2021.

As of autumn 2022, the MT programmes *Chemical Engineering and Biotechnology (MTKJ),* as well as *Cybernetics and Robotics (MTTK)*, will also be included in the project. Through close connections and discussions between teachers in mathematics and the engineering programmes MTELSYS and MTKJ, it turns out that these two engineering programmes have many common interests regarding what mathematics they want their students to learn, and at what point in the study programme they want the different mathematical topics to be presented. It may not be obvious that electronic systems and chemistry should have so much in common that it is natural for them to have a shared approach to mathematics, but in this paper we will show, through suitably chosen examples, that this could indeed be the case. (There are of course many other examples that we will not have space for in this paper.) We hope that these examples can serve as an inspiration for other fields of engineering to look for commonalities in order to develop an approach to mathematics, which provides a higher relevance of the subject of mathematics. From a practical and economical point of view, it will not be feasible to provide separate courses in mathematics for each individual engineering programme (at NTNU, 17 different programmes), so identifying clusters of engineering programmes with common interests in terms of emphasis and sequencing of mathematical topics will be an important task when implementing a programme driven approach. The three programmes MTELSYS, MTKJ and MTTK comprise around 350 students, which amounts to 20% of the total cohort. This could be a reasonably sized cluster.

An important prerequisite for MARTA is the attitude that integrated teaching and learning of mathematics for engineers is a shared responsibility between those who teach mathematics and those who teach engineering applications. The planning and execution of MARTA is a collaboration between teachers from the three collaborating departments. Joint planning of teaching and learning activities is done in advance for all involved courses, and involved teachers are continuously communicating during the work. MARTA started as discussions between teachers from different fields who are very enthusiastic about the potential in collaboration. It is important to recognize that for the ideas of the project to be sustainable, they have to be manageable with teachers with varying degree of enthusiasm. Another requirement for sustainability is that the topics covered in the courses should not differ too much between the various engineering programmes because there is a limit to how many different profiles of mathematics that can be developed. Also, migration of students between study programmes will still exist, and it is important not to make the transition too complicated.

In this paper, we will discuss how chemistry and electrical engineering can benefit from a close collaboration with mathematics, and vice versa. The discussion is based on three cases, and it will bring to the fore both affordances and challenges involved when implementing a programme driven approach. It is expected that increased knowledge about how mathematics and engineering subjects can interact will be helpful when designing and sustaining mathematics courses for different clusters of engineering programmes.

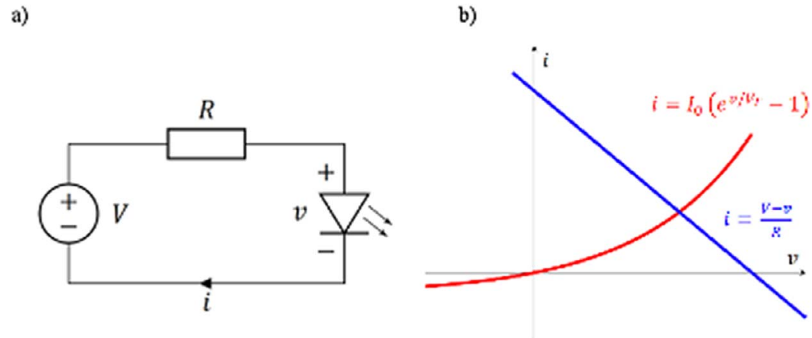

FIG. 1. (a) Circuit with LED. (b) Graphical solution to set of equations (1) and (2).

4. Three cases

A fundamental idea in the project MARTA is that learning of both engineering and mathematics will benefit from a coordinated approach. We will now present three cases from each of the fields of electrical engineering and chemistry. The study programme MTELSYS is called *electronic systems*, but this programme contains topics that are not necessarily from electronics but from electrical engineering more generally. Therefore, we will use the term *electrical engineering* to cover the examples from MTELSYS. The cases are labelled *non-linear behaviour*, *linear systems* and *Fourier analysis.* We shall treat the first two cases more in detail than the third case, although Fourier analysis is an important topic both in electronic systems and in chemistry. The collaboration with cybernetics has only recently started, so we have no experiences from this yet.

4.1. Case 1. Non-linear behaviour

4.1.1. Electrical engineering. Circuit theory is traditionally taught early in study programmes in electrical engineering. It is a topic of strong traditions, with limited variation in content, teaching and learning activities, both over time and across universities. Traditionally, emphasis has been placed on linear circuits, and hence most problems have been solvable using elementary algebra or calculus. As a consequence, all problems have analytic solutions, in accordance with a young student's expectation of how a mathematical problem should behave.

In reality, most engineering problems include some non-linear behaviour, and analytic solutions often cannot be found. We have tried to introduce such problems early, in order to develop the students' mental habits so that they can cope with realistic situations more easily. Figure 1(a) shows the first circuit that the students get familiar with in the circuit theory course, a light emitting diode (LED) powered by a voltage source and containing a resistor.

Given the voltage V and the resistor with resistance R, the problem is to determine the current i through the circuit. To solve this problem, an expression for the diode current as a function of voltage v is needed. An adequate model is given by Shockley's diode model (Shockley, 1949, p. 454),

$$i(v) = I_0\left(e^{v/V_T} - 1\right), \tag{1}$$

where I_0 and V_T are constants characteristic for the LED. The same current passes through the resistor, and Ohm's law gives

$$i(v) = \frac{V - v}{R} \tag{2}$$

From the monotonicity properties of the two expressions for $i(v)$ and the values at $v = 0$ it follows that there exists a unique choice of i and v that simultaneously solve the two equations, illustrated in Figure 1 b). However, closed expressions for i and v cannot be found, and in the mathematics course this example is used as an example of how even simple problems may require numerical solution techniques.

4.1.2. Chemistry. Chemical equilibrium is an important concept in chemistry, usually taught in General Chemistry courses to first-year students. Chemical equilibrium is a state in which the concentrations of reactants and products in a reversible chemical reaction are no longer changing in time. On a molecular scale, chemical reactions are still taking place, but the forward and backward rate of the reaction is such that no net change in concentrations is observed. The chemical equilibrium for a chemical equation can be characterized by an equilibrium constant that essentially contains information about the extent to which the equilibrium favours reactants or products. The equilibrium constant is given as the ratio between the product of the concentrations of the product compounds and the product of the concentrations of the reactant compounds, each concentration raised to the power of its stoichiometric coefficient (see equation (4) below). Knowing the value of the equilibrium constant and the stoichiometric relations of the reaction, it is possible to calculate the concentration of reactants and products of a reaction after equilibrium is established. As an example, we will consider the gas phase reaction between carbon dioxide (CO_2) and hydrogen gas (H_2), which reversibly generates carbon monoxide (CO) and water vapour (H_2O), represented by equation (3) below

$$CO_2 + H_2 \rightleftharpoons CO + H_2O \tag{3}$$

The task could for example be to find the equilibrium concentration of the gases, given the equilibrium constant K_c and some initial conditions. The usual approach taught to students is based on cumbersome procedures, involving substitution of variables. However, this approach will not work for cases where several chemical reactions occur, which will be the case in many realistic situations. Here we illustrate the simplicity in using numerical methods to sets of equations to solve the problem.

The concentrations of the involved reactants and products (in this case gases) must obey the linear relationship (3) reflecting the stoichiometry (mass conservation) of the reaction as well as the non-linear equation (4) for the equilibrium constant K_c,

$$K_c = \frac{[CO]\left[H_2O\right]}{\left[CO_2\right]\left[H_2\right]}. \tag{4}$$

Given initial concentrations (i.e. which gases one chooses to mix and in which concentrations), one may set up equations describing the relations between concentrations of reactants and products at equilibrium. For the reaction equation (3), and choosing to start with only CO_2 and H_2, we see that the concentration of CO and H_2O ([CO] and [H_2O]) will be equal due to stoichiometry. Further, the concentration of CO_2 and H_2 ([CO_2] and [H_2]) will be given by their initial concentrations modified

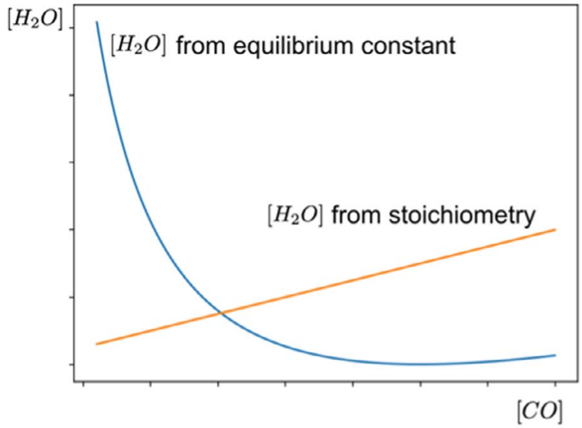

FIG. 2. Graphical solution to the set of equations (3) and (4).

by how much is converted to the products. Given constraints on the initial concentrations and the fact that concentrations cannot be negative, there is a unique equilibrium concentration of the gases, which obey the non-linear and linear equations simultaneously. This may easily be represented graphically (Fig. 2). The intersection between the curves indicates the concentration of H_2O ([H_2O]) for which both criteria for the equilibrium are fulfilled. Here the equilibrium concentration appears as the intersection between the graph of the non-linear equation (4) for the known equilibrium constant and the linear mass conservation relationship (3).

Using numerical methods for these processes introduces the student to the use of mathematical tools for handling more complex chemical equilibria where analytic solutions are not possible to obtain.

4.2. Case 2. Linear systems

4.2.1. Electrical engineering. In circuit theory, the superposition principle states that in a linear circuit any voltage or current can be decomposed in different components, one for each independent source in the circuit. In a standard textbook on electric circuits, this is explained in the following way:

> A linear system obeys the principle of **superposition,** which states that whenever a system is excited, or driven, by more than one independent source of energy, the total response is the sum of the individual responses (Nilsson & Riedel, 2011, p. 144).

The component corresponding to one source is calculated by setting all other sources equal to zero, and the complete solution is then obtained by adding the currents. This result is usually just stated, without further justification. From a mathematical point of view, this result follows from the linearity of the system. Each circuit with one source of energy can be modelled with a linear equation, and hence the full circuit can be modelled by a system of linear equations, algebraic equations, differential equations or a combination. The statement from Nilsson and Riedel, 'the total response is the sum of the individual responses' is therefore just a consequence of the linearity of the system. However, elementary circuit theory is often taught before linear algebra, and one can therefore not assume that all students are familiar with the necessary mathematics for the justification, which follows from the linear nature of Ohm's and Kirchoff's laws, etc. In our project we have deliberately introduced linear algebra very early, so that the

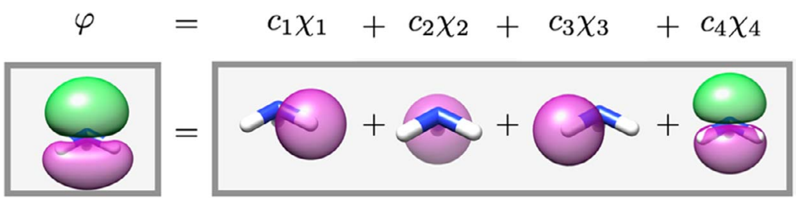

FIG. 3. Illustration of how LCAO is used for chemical intuition.

teacher in the circuit theory course can actively use results from linear algebra to explain the mathematical foundation of the superposition principle. This offers the possibility for the students to understand *why* the principle actually works, in accordance with the principle of conceptual understanding (Crawley *et al.*, 2014, p. 13).

4.2.2. Chemistry. The electronic Schrödinger equation is an eigenvalue equation in which the eigenfunctions contain information about the electronic structure of the molecules, and the eigenvalues represent electron energies. However, this equation cannot be analytically solved except in the case of the hydrogen atom, and therefore approximations must be made. A common approach in chemistry is to introduce a basis $\{\chi_n\}$ for a solution to the atomic problem in terms of atomic orbitals, and then expand the unknown molecular orbitals φ as linear combinations of the elements in the known basis (5). Variational theory on the electronic energy is then used to determine the expansion coefficients c_n that determine the shape of the molecular orbitals. This approach is commonly referred to as LCAO—Linear Combination of Atomic Orbitals.

$$\varphi = \sum_n c_n \chi_n. \tag{5}$$

For electronic-structure theory, this essentially turns the problem of solving the Schrödinger equation into an algebraic problem, where linear algebra is used to get quantitative predictions on properties of molecules. This is a topic the students usually meet in later year Physical Chemistry courses. However, the students meet the concepts of LCAO and molecular orbital theory in several courses before meeting electronic-structure theory in Physical Chemistry. The concepts of molecular orbital theory work as a qualitative tool providing students powerful chemical intuition on e.g. predicting molecular geometries and reactive properties through visual effects (see Fig. 3). For this reason, LCAO and molecular orbital theory is introduced already in the first semester General Chemistry course and is subsequently used frequently both in organic and inorganic chemistry. However, the fact that the principle is based on linear algebra is often not explained to students, and a lot of fundamental understanding is lost. Furthermore, despite students having touched upon LCAO in most of their previous basic courses in chemistry, they are at a loss when linear algebra is employed for more quantitative molecular orbital theory in Physical Chemistry. Connecting the LCAO expansion to linear algebra already in introductory courses has a two-fold effect: (1) The linear structure of the expansion allows for simple interpretations of constructive and destructive interference between atomic orbitals based on their phases and the sign in the expansion. This is important for interpreting resulting orbitals in terms of bonding or anti-bonding interactions between atoms in the molecule. (2) If the connection between LCAO and linear algebra is lacking from the beginning, it will be difficult to build on the students' knowledge of conceptual LCAO to quantitative

LCAO, which they meet in the Physical Chemistry course (Atkins *et al.*, 2017). Without the early connection, the students must re-learn the LCAO approach in the, for the students, new linear algebra context. Figure 3 shows how LCAO is used for chemical intuition, with specification of a particular molecular orbital of ammonia (NH_3) as a linear combination of atomic orbitals for the three hydrogen atoms and the nitrogen atom.

Introducing linear algebra to chemistry students at an early stage will therefore make it easier to make the connection to LCAO. This has the potential to enhance the students' understanding on *why* molecular orbital theory works, and not just *how* it works. This is again in accordance with the principle of conceptual understanding (Crawley *et al.*, 2014).

4.3. *Case 3. Fourier analysis*

4.3.1. Electrical engineering. Fourier analysis is a topic with high relevance for the electronic systems designer as well as for other types of electrical engineers. Consequently, this is part of the mathematical foundation for electrical engineering programmes all over the world. What often happens, however, is that students do not recognize the concepts from mathematics when the context is switched to engineering, where notation and vocabulary may be entirely different (Rønning, 2021b). When, in addition, the two contexts are separated in time, possibly by years, engineering instructors often end up by teaching Fourier analysis from scratch once more 'in their own way', without reference to students' earlier experiences in mathematics courses, thus obliterating any potential connections. Although Fourier series is an important topic both in mathematics and in electrical engineering, their motivation is slightly different in the two fields. In mathematics, the main interest is in representing periodic functions in a Fourier series and study convergence properties of this series, whereas in electrical engineering, the main interest is in the Fourier coefficients (the spectrum of the signal), and not so much in the series (Rønning, 2021b).

In the project MARTA, Fourier analysis is taught in parallel in mathematics and engineering courses. In the second semester, Fourier series are taught in the mathematics course while impedances, filters and frequency responses are taught in the engineering course. In contrast to earlier approaches, the connections to linear algebra, projections and inner products are explained. The relevance of Parseval's identity is demonstrated during a design project in which students have to generate a sinusoidal signal with a particular frequency. Students can identify and quantify the magnitude of unwanted harmonic components by a spectrum analyser and use Parseval's identity to compute the *Signal-to-Distortion-Ratio* as a quantitative measure of the quality of their design.

4.3.2. Chemistry. Molecules are dynamic entities and even at a theoretical absolute zero of temperature, the nuclei in a molecule will vibrate, i.e. move relative to each other. This is taught to students in Physical Chemistry courses where the connection between molecular properties and the rules of quantum mechanics are elucidated. The vibration of molecules at absolute zero results from the molecule being inherently quantum mechanical, and hence the position and momenta of the nuclei obey Heisenberg's uncertainty relation. Heisenberg's uncertainty relation says that there is an uncertainty associated with determining the position and the momentum of a quantum mechanical particle, and this stems from the non-commuting behaviour of the position and momentum operators.

For chemistry students, this can be a difficult concept to grasp. The use of Fourier analysis from mathematics could help in this thought process. A precisely defined position requires a function that is a Dirac delta function. The Dirac delta function, $\delta(t)$, is sometimes referred to as a *generalized function.*

It has the property that $\delta(t) = 0$ for $t \neq 0$, and $\int_{-\infty}^{\infty} \delta(t)dt = 1$. It has a physical interpretation of the density associated with a particle of unit mass at the point $t = 0$ on the t-axis (Kaplan, 1981, p. 200). The Dirac delta function can be represented in a Fourier series in the following way (Kaplan, 1981, p. 141):

$$\delta(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos nt.$$

However, the use of Fourier analysis shows that to generate the Dirac delta function, one needs to sum up (or rather, integrate up) all possible momentum eigenfunctions. Hence, a certainty in localization in space for a quantum mechanical particle requires full uncertainty in the momentum as the wave function will contain components from all possible momenta. Less uncertainty in momentum, obtained by not including all possible momenta in the integration, will raise uncertainty in position as the position is then no longer described by the Dirac delta function.

5. Discussion

In Case 1, we may formulate the tasks to be solved in electrical engineering and chemistry (T_E and T_C) in the following way:

T_E: Find the current in an electric circuit.
$T_{C:}$ Find the equilibrium concentration of the gases in a chemical reaction.

The mathematical techniques that are appropriate for solving the two tasks are the same: numerical methods for solving algebraic equations. Without these techniques, the solution process would either be very cumbersome, or one would have to deal with only very simple examples, or maybe just rely on reading off from a graph. For the numerical methods to work and to be reliable, it is important to know that a unique solution exists. In the example with the electric circuit, once the equations shown in Fig. 1(b) have been established, one may apply mathematical theory to justify that a unique solution exists, based on properties of the functions involved. In electrical engineering, the justifications are of a more empirical nature, 'we know that a solution exists' because of laws of nature and the idea of a deterministic universe. With sufficient data, the behaviour of a system will be uniquely determined from the relevant laws of nature. In the example from chemistry, we have argued, based on chemical principles, that there is a unique equilibrium concentration of the gases, which obey the non-linear and linear equations simultaneously. Knowing that a solution exists, one may apply mathematical methods, in this case numerical techniques, to find the solution. Seen from the point of view of mathematics, it is important that numerical methods are introduced at an appropriate stage for the engineering subjects to be able to draw on this knowledge. The engineering subjects will then be able to handle more complex situations where analytic solutions are not possible.

In Case 2, there is an interplay on the theoretical level of both mathematics and engineering. Linear algebra is an axiomatically constructed mathematical theory with a rich conceptual apparatus involving concepts like linear dependence/independence, linear combinations, basis, kernel, range, subspace and many others, giving rise to a number of techniques that can be justified based on the underlying theory. In electrical engineering, the superposition principle is stated as a *principle*, a technique without justification. The justification is based in mathematical theory with reference to the linearity of the system. This allows the addition of solutions. In chemistry, also concepts from mathematics, e.g. basis, are used to represent properties of a molecule as a linear combinations of entities characterizing the individual atoms in the molecule. Then techniques from linear algebra are used to get quantitative predictions on properties of molecules. Applying a mathematical theory (linear algebra) in these situations adds to the understanding of the theory in electrical engineering as well as in chemistry.

Also in Case 3 an interplay between the subjects can be observed. In the case of electrical engineering this is further elaborated in Rønning (to appear), where it is described, using the Antropological Theory of the Didactic (Bosch & Gascón, 2014) how analysis of an electric circuit requires an interplay between the mathematics and electrical engineering. Although the generating question for the problem, as well as the answer, came from electrical engineering, subquestions, works to be studied and partial answers (Chevallard, 2020), from both fields were involved.

In the examples we have presented, we have shown that an adequate treatment of an engineering problem based on conceptual understanding cannot be obtained within the engineering discipline itself. We have seen that to solve tasks from electrical engineering as well as from chemistry, it has been helpful to invoke not only mathematical techniques, but also justifications and theories. However, not only mathematical justifications are used. In Case 1 from chemistry, chemical principles ascertained the existence of a unique equilibrium concentration of the gases.

The main lesson to be learnt from these cases is that an interplay between the fields may lead to cross-fertilization having the potential for obtaining deep conceptual understanding.

6. Results from student surveys

The project MARTA is still in an early phase, and there is much to do in terms of collecting and analysing student data. However, we have performed two student surveys, from the first ($n = 40$) and second ($n = 40$) semester of Year 1 for the first cohort of MTELSYS students. (Chemistry students were not yet included in the project.) In the two surveys, some questions were identical and some questions were addressing issues specific to the semester in which they were given. In this section we will report on some of the findings from these surveys, and also compare the results to results from surveys done earlier of the whole group of MT students.

Two other surveys, from 2013 ($n = 662$) and from 2019 ($n = 314$), both given in the first mathematics course to Year 1 students from all 5-year MT programmes are used for comparison. The students were asked to indicate on a four-point Likert scale to what extent they agreed with the statements (1) ‘I am eager to understand concepts and underlying principles in the course’ and (2) ‘I have an understanding of why mathematics will be important for me later in my education’.

Comparing the survey from 2013 with the one from the pilot project, only small differences can be observed. In 2013, 92% of the respondents indicated that they somewhat or totally agreed with statement (1), compared to 85% and 88% for the first and second semester of the pilot project, respectively. For statement (2), 85% of the respondents somewhat or totally agreed in 2013, compared to 100% and 98% for the first and second semester of the pilot project, respectively.

Being presented with the statement ‘I have understood why mathematics will be important to me in my later studies’, in the surveys from the pilot project, 64% completely agreed to this and 36% partly agreed after the first semester. After the second semester, 65% completely agreed and 33% partly agreed. Hence 100% and 98% either completely or partly agree to this statement after the first and second semester, respectively. The same question was asked to the whole cohort of MT students in the traditional setting earlier, giving complete or partial agreement from 85% (2013) and 77% (2019).

The high number of respondents that agreed with the statements in 2013 makes it difficult to see clear effects of pilot project. As none of the students have experienced both types of mathematics courses, it is difficult to know what kind of benchmark and priors they are using when answering the questions. A

TABLE 1. *Perceived importance of topics in the first semester*

	Very important	Rather important	Not so important	Not important
Linear algebra (Matrices and systems of equations)	35%	53%	7%	5%
Complex numbers	68%	22%	5%	5%
Functions, including differentiation and integration	68%	27%	0%	5%
Sequences and series	28%	35%	35%	2%
Numerical methods and programming	60%	28%	7%	5%
Differential equations and Laplace transform	58%	35%	2%	5%

TABLE 2. *Perceived importance of topics in the second semester*

	Very important	Rather important	Not so important	Not important
Fourier series	70%	23%	5%	2%
Eigenvalues and eigenvectors	18%	60%	20%	2%
Vector space	20%	40%	35%	5%
Inner product space	8%	42%	48%	2%
Systems of differential equations/second-order differential equations	73%	27%	0%	0%
Numerical methods for differential equations	58%	35%	7%	0%
Wave equation	18%	51%	28%	3%

different approach, for example interviews or reflective diaries, is needed to further explore the effect of the pilot project. In further work with the project, we will administer surveys to MT students within the project as well as to students not covered by the project.

In the first semester the mathematics course was paired up with the course Introduction to Analog and Digital Electronics and in the second semester the mathematics course was paired up with the course Electronic System Design and Analysis, part 1. Each semester the students were asked ‘To what extent did you find mathematics useful in your work with this course?’, where ‘this course’ refers to one of the two courses mentioned above. In the first semester, 50% said it was useful ‘to a high degree’ and 48% ‘to some degree’. In the second semester, the percentages were 33% and 47%, respectively. The lower reported usefulness can possibly be connected to the lower perceived importance of the mathematics in the second semester. The topics covered in the mathematics courses in semester 1 and 2 can be seen in Tables 1 and 2, respectively.

When asked to assess the importance for them as a student at MTELSYS of the various topics in the first and second mathematics courses, the students answered as shown in Tables 1 and 2. It is clear that there are fewer topics in the second semester reported as ‘very important’, which together with the lower degree of reported usefulness might reveal a stronger integration of mathematics in the engineering course in the first semester than the second.

In the tables we have highlighted in bold face the topics that we have treated in the three cases earlier in the paper. It should be emphasized that the cases were chosen on the basis of what the *teachers* in the engineering subjects perceived as relevant mathematical topics. It can be noticed that both numerical methods and programming as well as Fourier series get a high score on importance from the students. We note that Linear algebra gets a not so high score. However, in open-ended answers about the importance of topics the students explicitly mentioned ‘differential equations for inductors and capacitors’ and it was useful ‘to understand where the Superposition principle comes up and solutions with Ax=b (matrix)’.

7. Conclusion

The preliminary experiences from the pilot project seem to indicate that there is much to be gained by seeing mathematics and engineering subjects in connection, and that this connection to a large extent can be obtained by a mutual adjustment of the subjects in question, without compromising their distinctive characters. Student surveys also indicate that the students are able to pinpoint particular topics from the subjects that are mutually supportive. However, there is still very limited evidence about the effects of the project. Up to now, only one study programme has been involved, and due to the Covid-19 pandemic, it has not been possible to perform interviews with students.

With the extension of the project to cover also the programmes in chemistry and in cybernetics, the basis for getting student data will be larger. This extension will also bring new challenges to the project, and will give some experiences with adapting to a greater variation of study programmes. At NTNU, there are in total 17 Master of Technology study programmes, so a crucial question is how many different versions of mathematics are needed to cover all programmes in a good way, balancing wishes for contextual learning and restrictions due to resources.

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