Didactic transposition of concavity of functions: From scholarly knowledge to mathematical knowledge to be taught in school

Heidi Strømskag[1] and Yves Chevallard[2]

[1]Norwegian University of Science and Technology, Department of Mathematical Sciences, Trondheim, heidi.stromskag@ntnu.no

[2]Aix-Marseille University, Marseille, France, y.chevallard@free.fr

*In an institution **I**, a praxeology $\wp$ is generally a modification of a praxeology $\wp^*$ coming from a collective of institutions **I***, where the modification is conceptualised by the phenomenon of institutional transposition. This paper presents a praxeological analysis of the concept of concavity of functions as expressed in a mathematics textbook for Norwegian upper secondary school. The analysis shows how the institutional (here, didactic) transposition has "moved" the mathematics presented in upper secondary school away from the mathematics taught at the university and how this transposition has resulted in a poor* logos *block of the mathematics to be taught.*

Keywords: Transition to, across and from university mathematics; concavity; didactic transposition; praxeology; teaching and learning of analysis and calculus.

INTRODUCTION

The theory of didactic transposition (Chevallard, 1991) was introduced in 1985 by Yves Chevallard. The didactic transposition process refers to the transformations an object of knowledge undergoes from the moment it is produced by scholars, to the time it is selected and designed by noospherians to be taught, until it is actually taught (and studied) in a given educational institution (Chevallard & Bosch, 2014). When doing didactic transposition analyses, the empirical unit is enlarged to encompass data from outside of the mathematics classroom. This reflects the insight that to study teaching and learning of mathematics in the classroom, it is not enough to study what students and teachers are thinking and doing: the mathematics taught becomes itself an object of study. The researcher studies transformations between the following instances: the scholarly mathematical knowledge as it is produced by mathematicians; the mathematical knowledge to be taught as officially formulated in curriculums and as presented in textbooks; the mathematical knowledge as it is actually taught by teachers in classrooms; and the mathematical knowledge as it is actually learned by students (Bosch & Gascón, 2006). The didactic transposition process taking place between the mentioned instances is illustrated in Figure 1.

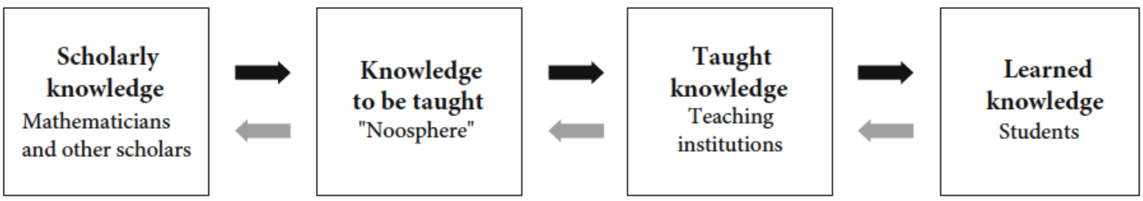

Figure 1. Didactic Transposition Processes (adapted from Chevallard & Bosch, 2014, p. 171)

In the research presented here, we have studied the didactic transposition of concavity of functions from scholarly knowledge to mathematical knowledge to be taught in secondary school. We have analysed transformations between the mathematical organisation of concavity of functions (in particular, its *logos* block) as expressed in a university textbook on calculus (Lindstrøm, 2016) and the mathematical organisation of the same theme as expressed in a mathematics textbook for Grade 12 (Kalvø et al., 2021). As asserted by Winsløw (2022), the calculus presented in mathematics courses at the university is indeed the result of a didactic transposition of the calculus of the 18th century. So, the mathematics presented in the university textbook analysed here is itself a body of transposed knowledge. An analysis of the transformations that this knowledge has undergone from the scholarly mathematical knowledge is however beyond the scope of this paper.

The transformations of concavity of functions that have taken place between the university textbook and the school textbook have been studied through a praxeological analysis. Generally, praxeological analyses, together with analyses of didactic transposition processes that specific knowledge objects have undergone, help us understand which mathematics is taught in school, and why it has become so. Our study centres on the following research question: *What are the transformations that the notion of concavity of functions has undergone during the didactic transposition process from the knowledge taught at the university to the knowledge to be taught in Norwegian upper secondary school?*

THEORETICAL TOOLS

The study reported here has been conducted in the framework of the anthropological theory of the didactic (ATD; Chevallard, 2019). A praxeology of a body of knowledge is in the ATD a *model* of this knowledge. This model is a unit composed of four components: T, τ, θ and Θ (sometimes referred to as "the four t-s"), where T is a *type* of tasks, τ is a technique (or a set of techniques) to solve the tasks, θ is a *technology*, that is, a discourse describing and explaining the techniques, and Θ is a theory, that is, a discourse justifying θ. T and τ belong to the *praxis* block of the praxeology, whereas θ and Θ belong to the *logos* block. A praxeology $\wp$ is written: $\wp = [T / \tau / \theta / \Theta]$.

A praxeology $\wp$ is usually the product of the activity of an institution or a collective of institutions $\boldsymbol{I}$. It is often the case that this "product" is the result of an *institutional transposition* of a praxeology $\wp^*$ living in a collective of institutions $\boldsymbol{I}^*$ to a praxeology $\wp$ that has to live within $\boldsymbol{I}$ and thus has to satisfy a set of conditions and constraints specific to $\boldsymbol{I}$ (Chevallard, 2020). This is the case when $\boldsymbol{I}$ is a collective of "didactic" institutions, that is, institutions declaring to teach some bodies of knowledge, such as secondary schools for example. This is referred to as *didactic transposition* of $\boldsymbol{I}^*$ into $\boldsymbol{I}$. Often, in this case, it is observed that $\wp$ is a "simplification" of $\wp^*$ through various processes. For example, it may be that a certain type of tasks T in $\boldsymbol{I}^*$ becomes useless in $\boldsymbol{I}$. It may be that a particular technique is inefficient, or that it leads the average user to make many mistakes. Moreover, in the process of transposition, it is likely that $\wp^*$

has been greatly simplified and thus distorted so the technology does not really justify the proposed technique. Finally, the theoretical elements are often implicit, repressed, or taken for granted. Therefore, for those who want to analyse a praxeology living in a given institution, the theoretical component is hard to bring to light. This is shown in the analysis section below.

METHODICAL APPROACH

The methodical approach is essentially that of *didactic transposition analysis* (Chevallard, 1991). The didactic transposition analysis of the concerned body of knowledge $\mathscr{k}$ presented here involves a comparison of praxeological analyses of two different "copies" (i.e., "transposed" versions) of $\mathscr{k}$ as they appear in two different institutions. The data are the mentioned textbooks (in Norwegian)[1]: The first is *Kalkulus*, an introductory textbook on calculus for the university, published in 2016. It is written by Tom Lindstrøm, professor of mathematics at the university of Oslo. The second is *Mønster* [Patterns]: *Mathematics R1*, a Grade 12 mathematics textbook for a theoretical programme at upper secondary school, preparing for university studies in science, technology, engineering, and mathematics. It is part of a textbook series for the national curriculum since 2020, written by Tove Kalvø, Jens C. L. Opdahl, Knut Skrindo, and Øystein J. Weider, all serving as teachers in mathematics at (different) upper secondary schools. The reasons for the choice of these books are: *Kalkulus* is an introductory textbook used in the first calculus course taken by students enrolled in teacher education programmes for Grade 8–13 at several Norwegian universities; *Mønster* is part of a brand-new textbook series for the theoretical programme; it is not a revised version of an old series as are two other textbook series for the same programme (i.e., Borgan et al., 2021; Oldervoll et al., 2021).

ANALYSIS OF A DIDACTIC TRANSPOSITION PROCESS

We present here an analysis of *concavity of functions* as treated in the textbook *Mønster* (Kalvø et al., 2021, pp. 208–224) and compare it with the treatment of the same topic in the university textbook *Kalkulus* (Lindstrøm, 2016, pp. 313–321)—which we regard as closer to scholarly knowledge. The aim is to bring to light the didactic changes this knowledge object, as presented in *Mønster*, has been subjected to.

The *Logos* Block of Concavity of Functions in *Kalkulus*

In Chapter 6.4 of *Kalkulus*, with heading "Discussion of Curves", there is a section entitled "Convex and Concave Functions" (pp. 283–288).[2] The author starts with a geometrical definition of the concepts of *convex function* and *concave function*:

> **6.4.5 Definition** The function f is called *convex* on the interval I if every time we select two points $a, b \in I$, then no point on the line segment between $(a, f(a))$ and $(b, f(b))$ will be

[1] Quotations from these textbooks have been translated into English by the first author.

[2] For functions, being convex and concave is synonymous with being "concave up" and "concave down", respectively (as used by e.g. Adams & Essex, 2018).

below the graph of $y = f(x)$ (see Figure 2).[3] We say that f is *concave* on I if every time we select two points $a, b \in I$, then no points on the line segment between $(a, f(a))$ and $(b, f(b))$ will be above the function graph (see Figure 3). (Lindstrøm, 2016, p. 314)

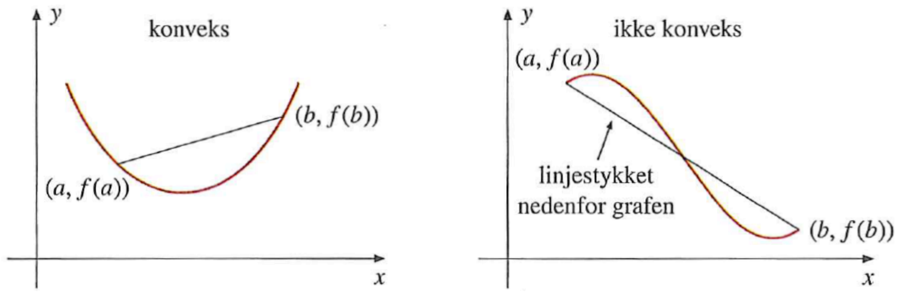

Figure 2. Convexity of a Function (taken from Lindstrøm, 2016, p. 314)

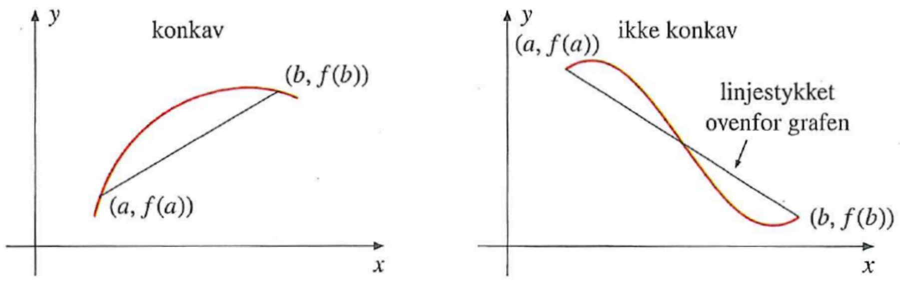

Figure 3. Concavity of a Function (taken from Lindstrøm, 2016, p. 314)

The author continues to build up elements in the *logos* block, which allows him to deduce a connection between concavity and the second derivative of a function twice differentiable. To be able to use the mean value theorem in the proof of the theorem that establishes the sought relationship, the following lemma using difference quotients is presented (p. 315):

6.4.6 Lemma A function is convex on an interval I if and only if the following applies. For all points $a, b, c \in I$ such that $a < c < b$, we have

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c}. \qquad (1)$$

Correspondingly, f is concave on I if and only if

$$\frac{f(c)-f(a)}{c-a} \geq \frac{f(b)-f(c)}{b-c}. \qquad (2)$$

The author writes that the two statements can be proved the same way and presents a proof for the convexity part of the lemma:

Proof: Assume first that f is convex. Then the point $(c, f(c))$ cannot be above the segment connecting $(a, f(a))$ and $(b, f(b))$, and we must have the situation shown in Figure 4a.

When we compare the slopes k, k_1, and k_2 of the three segments in the figure, we see that $k_1 \leq k \leq k_2$, which means that

---

[3] Figures taken from *Kalkulus* used in this paper are given titles by the authors (figures are untitled in the source). Moreover, we have renumbered them to have continuous numbering of figures.

$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(c)}{b-c}$.

If we omit the middle part, we get (1). (Lindstrøm, 2016, p. 315)

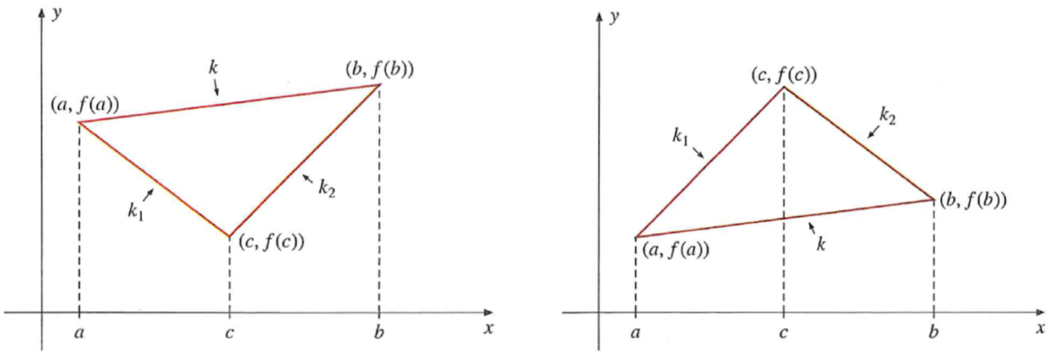

Figure 4. Convexity (left image) and Non-convexity (right image) (taken from Lindstrøm, 2016, pp. 315–316)

The next step of the author is assuming that *f* is *not* convex and showing, consequently, that (1) does not hold:

> Since *f* is not convex, we can find points a, b, $c \in I$ so that $a < c < b$ and $(c, f(c)$ is above the segment connecting $(a, f(a))$ and $(b, f(b))$; this means that we have the situation shown in Figure 4b.
>
> We now see that the ratio between the slopes is $k_1 > k > k_2$ – in other words
>
> $\frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-a} > \frac{f(b)-f(c)}{b-c}$.
>
> If we omit the middle part, we get the inverse inequality of (1). ■ (Lindstrøm, 2016, pp. 315–316)

Then everything is ready to state the theorem that is the central theoretical element in the author's mathematical organisation of the *logos* part of concavity of functions:

> **6.4.7 Theorem** Assume that f is continuous on an interval I and that $f''(x) \geq 0$ for all inner points $x \in I$. Then f is convex on I. If instead $f''(x) \leq 0$ for all inner points of I, then f is concave on I. (Lindstrøm, 2016, p. 316)

The proof addresses the convexity part of the theorem, using the above lemma:

> **Proof**: Choose three points a, b, $c \in I$ so that $a < c < b$. According to Lemma 6.4.6, it suffices to prove that $\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c}$. By the mean value theorem, there exist two numbers $c_1 \in (a, c)$ and $c_2 \in (c, b)$ so that $\frac{f(c)-f(a)}{c-a} = f'(c_1)$ and $\frac{f(b)-f(c)}{b-c} = f'(c_2)$. Since $f''(x) \geq 0$, f' is increasing and, consequently, $f'(c_2) \geq f'(c_1)$ [because $c_1 < c < c_2$]. Hence $\frac{f(c)-f(a)}{c-a} = f'(c_1) \leq f'(c_2) = \frac{f(b)-f(c)}{b-c}$. ■ (p. 316)

After this, two examples are given that discuss convexity/concavity. The second example introduces the notion of *inflection point* with this formulation: "a is an inflection point for f if f is continuous at a and there exists an $\varepsilon > 0$ so that f is convex on one of the intervals $(a - \varepsilon, a)$, $(a, a + \varepsilon)$ and concave on the other" (p. 318). An inflection point is a point where a function changes from being concave to being convex or vice versa. This is succeeded by 18 tasks that address appearances of curves more broadly.

The above is a brief account of the "scientific" treatment of the concept of concavity in *Kalkulus*, which describes the constituent parts of the *logos* block of concavity: 1) a *definition* of convex/concave function; 2) the *mean value theorem* (proved in a previous section) used in the proof of a lemma to be used in the proof of a central theorem with respect to the concept at stake; 3) the mentioned *lemma* (with proof); 4) the central *theorem* (with proof) declaring a connection between the sign of the second derivative and the concavity/convexity of a function; 5) a *definition* of inflection point.

The Mathematical Organisation of Concavity of Functions in *Mønster*

Here, we analyse the treatment of concavity of functions in *Mønster* (Kalvø et al., 2021) One remarkable point is that the section devoted to concavity issues is entitled "The Second Derivative". The question of the concavity of functions is thus presented as an application of the notion of second derivative. The words concave and concavity do not appear in the textbook: they are replaced by the expressions "hollow side" (*hul side*)[4]—the side that faces either down or up—and "curvature" (*krumning*), respectively. We will see that this is a "symptom" of the treatment of concavity by the given textbook. These notions appear in the following passage:

> We compare this with the graph of f and see the following:
> • When f'' is negative, f' is decreasing and the graph of f turns its *hollow side* down.
> • When f'' is positive, f' is increasing and the graph of f turns its *hollow side* up.
> A function with a graph turning its hollow side up or down is not linear. We say that the graph curves, and we mark the *curvature* of the graph with an arc below the sign line (see Figure 5). (Kalvø et al., 2021, pp. 211–212)

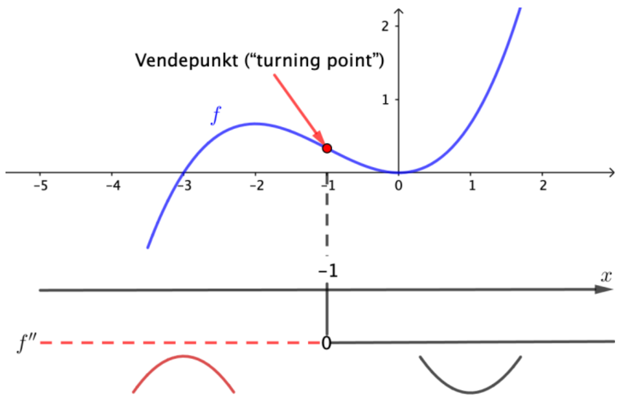

Figure 5. Sign Line for the Second Derivative (adapted from Kalvø et al., 2021, p. 211)

In this way, words that are traditional in mathematics (concave/convex, concavity/convexity) but which, a priori, mean nothing to the students, are replaced by expressions (hollow side facing down/up, curvature) that make sense in everyday

---

[4] Throughout the paper, italicized words in parentheses refer to Norwegian words used in *Mønster*.

language, which in this case have a metaphorical value, and which are used here as *definitions*. The same is true for *inflection point*, where the authors use the notion "turning point" (*vendepunkt*).

> The point where the graph goes from facing the hollow side up to [facing] the hollow side down (or vice versa), is called the *turning point*. At the turning point, the sign of the derivative changes. (p. 211)

The *technology* of this technique (using a sign line for the second derivative) is in fact reduced to a minimum. The authors of the textbook have adopted a "naturalistic" approach to functions. They do so by considering a specimen function regarded as generic, in this case the function defined by $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2$ for all $x \in \mathbb{R}$. The graphs of f and $f'(x) = \frac{1}{2}x^2 + x$ are shown in Figure 6 (adapted from Kalvø et al., 2021, p. 211). Looking at the graph of f, we "see" that f first increases, reaches a maximum at a point that appears to be -2, then decreases and reaches a minimum at $x = 0$, before increasing again. Let us then try to determine "visually" the intervals in which f is either concave or convex.

The function f is first concave, up to a value x_0 somewhere between -2 and 0; then it becomes convex after x_0. How can we determine x_0?

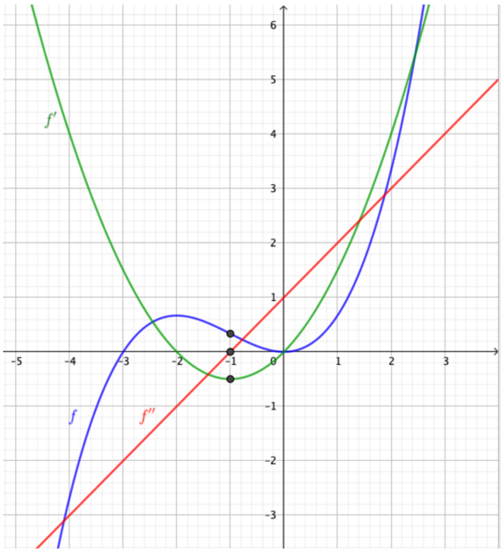

Figure 6. The Graphs of *f*, *f'*, and *f''*

To do this, we need to look not at the values of the derivative, but at how the derivative varies—that is, how the slope of the tangent to the graph of f' changes.

Instead of examining the graphs of f and f', it is technically more concise to simply examine the graph of f'' (see Figure 6). Here, $f''(x) = x + 1$. The second derivative f'' is therefore represented by a straight line with slope 1. It is strictly negative when $x < -1$, zero for $x = x_0 = -1$ and strictly positive when $x > -1$. The reader can examine two animated GIFs, where the first GIF (first link) highlights the *values* (positive, negative) of f' while the second GIF (second link) highlights the fact that f' is *increasing* or *decreasing*.

> https://commons.wikimedia.org/w/index.php?title=File:Tangent_function_animation.gif&oldid=507127692
>
> https://upload.wikimedia.org/wikipedia/commons/7/78/Animated_illustration_of_inflection_point.gif.

If we look at the treatment of concavity in *Mønster* as a certain ***praxeology***, we can analyse it as explained in the following paragraphs.

The *type of tasks* T studied is formulated more allusively than explicitly. A task t of type T consists in determining the curvature of the graph of a given function f and finding its possible turning point(s). This involves determining intervals of $\mathbb{R}$ on which f is either "concave down" or "concave up" by examining a "sign line" as shown in Figure 5.

The notion of concavity is *hinted at*, rather than properly defined. This is made possible, among other things, by a linguistic "manipulation" which is one of the keys to the didactic transposition carried out by the authors: the words *concavity* and *inflection point* do not appear. "Concavity" is replaced by the expression "hollow side down/up" associated with the expression "curvature"; "inflection point" is replaced by "turning point". While the words "concavity" and "inflection point" are relatively opaque words in ordinary language, and therefore require comments, if not a precise definition, the expressions by which they are replaced belong to everyday language and are known to all, which authorizes the authors not to say more about them.

The *technique* τ to perform a task $t \in T$ consists in calculating the second derivative f'' (differentiating a function: type of tasks T_1) and studying its sign (determining the sign of a function: type of tasks T_2). In essence, both T_1 and T_2 are assumed to have been studied beforehand and to be now largely routinised. The only new feature is that, given the function f, the derivatives f' and f'' must be calculated successively.

The *technology* θ of the technique τ is reduced to next to nothing. One would expect that when f is concave down, the authors would point out to their readers that *the slope of the tangent decreases*. Instead, they invite them to observe, on *the graph of f'*, that f' is decreasing. Even more so, they do not care to mention the equivalence of various properties such as

- the slope of the tangent decreases;
- the curve is below its tangents;
- for any point a on an interval I on which f is defined, the function $r_a: x \mapsto r_a(x) = \frac{f(x)-f(a)}{x-a}$ decreases. (In Figure 7, we have for example: $r_a(x_1) > f'(a) > r_a(x_2)$.)

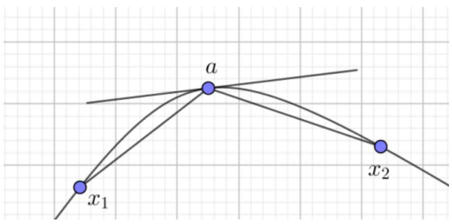

Figure 7. Chords and Tangent for a Function *f*

The third of these properties corresponds to Lemma 6.4.6 in *Kalkulus* (explained in the previous section).

Finally, about the praxeology of concavity in *Mønster*, we have uncovered that there is no *theory* Θ justifying the technology θ. This can be explained by two factors: first, θ is almost non-existent; second, the authors make no real attempt to justify what little exists of θ. We would like to state here what the theory (according to the ATD) would be: a system of statements (definitions, axioms, lemmas, theorems, corollaries…) from which we can derive a justification of θ. Let us suppose, for example, that we want to justify the fact that, when the derivative f' decreases, "the curve is below its tangents". We have (see Figure 8): $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon f'(a) - [f(a + \varepsilon) - f(a)]$. According to the mean value theorem, there exists $\gamma \in (0, 1)$ such that $f(a + \varepsilon) - f(a) = \varepsilon f'(a + \gamma\varepsilon)$. We thus have: $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon f'(a) - \varepsilon f'(a + \gamma\varepsilon) = \varepsilon[f'(a) - f'(a + \gamma\varepsilon)]$. Since f' decreases, $f'(a) > f'(a + \gamma\varepsilon)$ and therefore $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon[f'(a) - f'(a + \gamma\varepsilon)] > 0$. ■

In that case, we could look for the mathematical "principles" that justify the mean value theorem and the tools used to establish it (e.g., Rolle's theorem).

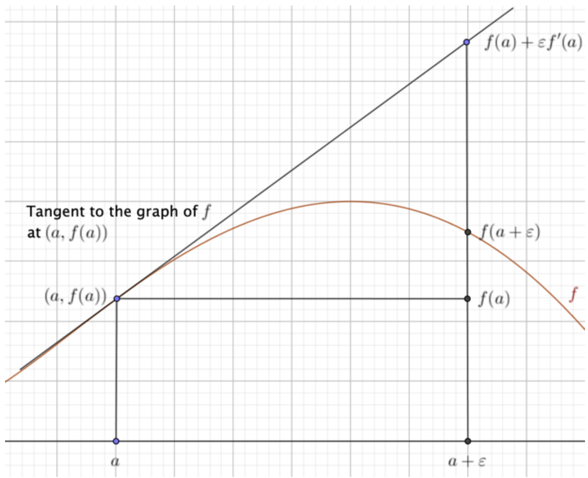

Figure 8. Decreasing Derivative

DISCUSSION

The presentation of concavity of functions in the secondary school textbook is but a "technical notice" expressed in a casual way, with as little mathematical "*logos*" as possible, most likely to make it accessible to a wider range of students. This contrasts with the presentation of the same theme in the university textbook, where we found a *logos* block consisting of definitions and proved results (theorems, lemma). In the school textbook, the notion of concavity has been substituted by an *application* of the notion of second derivative and, consequently, there is an exclusion of questions where concavity could have come into play. There are two other mathematics textbook series for the theoretical programme in upper secondary school in Norway: one is written by Borgan et al. (2021), the other by Oldervoll et al. (2021). They have a very similar treatment of concavity of functions, using exactly the same notions as the textbook analysed here.

How can we summarise the effect of didactic transposition on the notion of concavity of a function as it manifests itself here? The main fact is that, while in the university presentation, the *graphical* notion of concavity is *mathematised*, in secondary school textbooks it remains non-mathematised: concavity is *to be seen* on the graph of the function. At best, authors simply translate this *visual* property by saying that the slope of the tangent to the curve decreases or increases. This visually established property is then translated mathematically by the sign of the second derivative. The crucial gain is obvious: the subtle work required to mathematise the graphical notion of concavity is avoided, so that its presentation is accessible to a wider audience.

Another gain stems from an "iron law" of curriculum crafting: a new item benefits from appearing as an "application" of an established item—here the notion of second derivative. Is there a loss? Yes, there is. Whereas, at university, under appropriate regularity conditions, one can *prove* that, if a function is concave down, its second derivative is negative, and *conversely*, at the secondary level, for lack of a *mathematical* definition of concavity, students will miss this particular opportunity for a simple, founding experience in their mathematics education: tackling a theorem, and then its reciprocal. Didactic transposition thus surreptitiously makes its mark, and sometimes takes its toll, on students' and teachers' *praxis* and *logos* by distorting and, often, damaging the mathematical equipment which is available to them.

REFERENCES

Adams, R., & Essex, C. (2018). *Calculus: A complete course* (9th ed.). Toronto: Pearson.

Borgan, Ø., Borge, I. C., Engeseth, J., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2021). *Matematikk R1* [Mathematics for the theoretical programme, Grade 12]. (4th ed.). Oslo: Aschehoug.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51–63.

Chevallard, Y. (2019). Introducing the Anthropological Theory of the Didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114.

Chevallard, Y. (with Bosch, M.). (2020). A short (and somewhat subjective) glossary of the ATD. In M. Bosch, Y. Chevallard, F. J. Garcia, & J. Monaghan (Eds.), *Working with the anthropological theory of the didactic in mathematics education: A comprehensive casebook* (pp. xviii–xxxviii). London: Routledge.

Chevallard, Y. (with Johsua, M.-A.). (1991). *La transposition didactique. Du savoir savant au savoir enseigné* (2nd ed.). Grenoble: La pensée sauvage.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Dordrecht: Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Kalvø, T., Opdahl, J. C. L., Skrindo, K., & Weider, Ø. J. (2021). *Mønster: Matematikk R1* [Patterns: Mathematics for the theoretical programme, Grade 12]. Oslo: Gyldendal.

Lindstrøm, T. (2016). *Kalkulus* [Calculus] (4th ed.). Oslo: Universitetsforlaget.

Oldervoll, T., Svorstøl, O., Gustafsson, E., & Jacobsen, R. B. (2021). *Sinus R1: Matematikk* [Sinus: Mathematics for the theoretical programme, Grade 12]. (4th ed.). Oslo: Cappelen Damm.

Winsløw, C. (2022). Mathematical analysis at university. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & N. Ruiz-Munzón (Eds.), *Advances in the anthropological theory of the didactic* (pp. 295–305). Cham: Springer.