ELEMENTARY ALGEBRA AS A MODELLING TOOL: A PLEA FOR A NEW CURRICULUM

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L'ALGÈBRE ÉLÉMENTAIRE COMME OUTIL DE MODÉLISATION : PLAIDOYER POUR UN NOUVEAU CURRICULUM

Résumé – L'algèbre élémentaire est la base sur laquelle s'élève l'édifice mathématique moderne. Mais le curriculum algébrique façonné par le processus de transposition didactique depuis plus d'un siècle n'est plus en mesure d'assurer cette fonction essentielle. Notre étude examine la notion de formule et la manière dont son évolution curriculaire apparaît comme un témoin et une cause de la dégradation de l'algèbre enseignée en tant qu'outil de modélisation du monde. Cet examen suppose des analyses minutieuses de faits curriculaires qui ne semblent pas avoir attiré l'attention des chercheurs, telle la déparamétrisation de l'algèbre ou les effets non prévus de sa rencontre avec l'analyse mathématique. En prenant appui sur la théorie anthropologique du didactique (TAD), cela conduit à esquisser la perspective d'une revigoration indispensable du curriculum algébrique.

Mots-Clés : en français sans majuscules séparés par des virgules.

EL ÁLGEBRA ELEMENTAL COMO HERRAMIENTA DE MODELIZACIÓN: ARGUMENTO PARA UN NUEVO CURRÍCULO

Resumen – El álgebra elemental es la base sobre la que se levanta todo el edificio matemático moderno. Pero el currículo algebraico modelado por el proceso de transposición didáctica desde hace más de un siglo, ya no puede asegurar esta función esencial. El presente estudio se centra en la noción de fórmula y en cómo su evolución curricular parece ser testigo y causa de la degradación del álgebra enseñada como herramienta de modelización. Este examen implica un análisis minucioso de hechos curriculares que parecen no haber atraído atención de los investigadores,

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como la desparametrización del álgebra o los efectos no previstos de su encuentro con el análisis matemático. Apoyándonos en la teoría antropológica de lo didáctico (TAD), esto permite esbozar la perspectiva de una imperiosa revigorización del currículo algebraico.

Palabras-claves: álgebra, currículo, fórmulas, parámetros, sistemas, transposición didáctica.

ABSTRACT

Elementary algebra is the foundation on which the entire modern mathematical edifice rises. But the algebraic curriculum fashioned by the process of didactic transposition for more than a century is no longer able to ensure this essential function. The present study focuses on the notion of formula and how its curricular evolution appears to be a witness and a cause of the degradation of taught algebra as a modelling tool. This examination involves careful analyses of curricular facts that seem not to have attracted the full attention of researchers, such as the deparametrization of algebra or the unplanned effects of its encounter with mathematical analysis. Drawing on the anthropological theory of the didactic (ATD), this allows to outline the perspective of an imperative reinvigoration of the algebraic curriculum.

Key words: algebra, curriculum, didactic transposition, formulas, parameters, systems.

INTRODUCTION

The passing of time changes curricular contents, often without the people involved—the whole “noosphere”[1]—being really aware of it. This affects all the subject areas traditionally taught or considered worthy to be taught at school. The subject area $\mathcal{A}$ we shall refer to here as elementary algebra is known to everyone thanks to secondary school. At the same time, it calls for a

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[1] In the *anthropological theory of the didactic* (ATD), the word noosphere designates the (fuzzy) set of persons and institutions that, be it on a part-time basis, “think”—the Greek word nóos (νόος) means “mind as used in resolving and purposing”, and “thought” (“Νόος”, 2020)—about the educational system (and here, more specifically, about the teaching of mathematics). Teachers, educators, researchers in education and in particular in didactics are thus (part-time) “noospherians”. In what follows, the meaning of words specific to the ATD will be clarified as we go along.

reexamination for the reason we have just mentioned: the process of "curricular ageing" (Chevallard & Strømskag, 2022). This universal process can lead any domain $\mathcal{D}$ (such as arithmetic, geometry, trigonometry, analysis, etc.) to lose a large part of its instrumental value, that is, what it allows us to do in our relation to the existing and oncoming world—and to deteriorate to the point that its study at school is but a rite of passage imposed on the younger generations, involving a few fetishized emblems (such as the letters x and y when $\mathcal{D} = \mathcal{A}$). More precisely, this rite of passage can be described as follows. The domain considered, here $\mathcal{A}$, serves as a "frontier" on which the younger generations come to settle temporarily, experiencing the difficulties of life on a frontier. Then, after a few years, they will definitely move away from it and forget almost everything they had to learn to survive there, without ever thinking that it could also apply to life away from the frontier.

Forgetting what has been studied is one of the biggest issues raised by the historical relationship between school and society. In this perspective, we will try to set out conditions favourable to making the algebra taught at school an effective tool for understanding, even after schooltime, a large number of "facts" of the world around us. As we shall see, the possibilities offered by elementary algebra have been greatly reduced by the evolution of the algebra curriculum over the last century. What goes on in a given classroom depends on events that occurred in the noosphere long before; we will see this in detail in the case of parameters. For this reason, teachers and students work under constraints that they often ignore. By contrast, we will highlight the potential of elementary algebra once some of these constraints are removed. Our analysis relies on 40 years of research studies conducted mainly within the framework of the ATD, of which, for lack of space, we can only mention a very few (Chevallard, 1990; Gascón, 1994; Bosch & Chevallard, 1999; Chevallard & Bosch, 2012; Bosch, 2015; Ruiz-Munzón, Bosch, & Gascón, 2020; Strømskag, 2020). Our methodology is essentially that of didactic transposition analysis (Chevallard, 1991), as explained in the section "An Epistemological and Methodological Interlude".

THE TAKE-OFF OF ALGEBRA: FROM RULES TO FORMULAS

Arithmetical Rules and the Introduction of Formulas

We shall first highlight the key points of the changes that have affected the algebra curriculum, in order to identify the *core requirements for revitalizing school algebra*. We will focus on an aspect little taken into account: the role played (or not played), in the algebra curriculum, by the notion of *formula*, seen both as a symptom and as a cause of the impoverishment of school algebra.

Here is an example: How can we find the area of a trapezoid? Here is a typical *rule* found on the Internet (How to Find the Area, n.d., Example Question #1 section):

> To find the area of a trapezoid, multiply one half (or 0.5, since we are working with decimals) by the sum of the lengths of its bases … and by its height.

Another formulation is the following (Area of a Trapezium, n.d.):[2] "Area$=\frac{1}{2}\times$Sum of parallel sides$\times$Distance between them." This rule uses symbols: it is in the process of being algebraized.[3] Once fully algebraized, it becomes a *formula*: "$A=\frac{1}{2}(b_1+b_2)h$", where b_1 and b_2 are the lengths of the parallel sides and h the distance between them.

A formula is "*A rule or principle expressed in algebraic language*" ("Formula", n.d., para. 3). In his *Elements of Algebra*, Sylvestre-François Lacroix (1765–1843) writes: "It is because the results in algebra are for the most part only an indication of the operations to be performed upon numbers in order to find others, that they are called in general *formulas*" (Lacroix, 1797/1831, p. 9). Lacroix illustrates the difference between "ordinary language" and "algebraic characters" with this problem:

> PROBLEM.
>
> To divide a number into three such parts, that the excess of the middle one above the least shall be a given number, and the excess

[2] Here we consider *trapezium* and *trapezoid* as equivalent: they both refer to any convex quadrilateral with (at least) two sides parallel.

[3] It is worth remembering that the signs +, –, ×, and so forth, were first used *in algebra* and were not introduced into arithmetic until the beginning of the 19th century—see Smith (1925/1958, p. 395).

> of the greatest above the middle one shall be another given number. (p. 9)

The "ordinary language" used here can be (partially) translated thus: "Let the number to be divided be denoted by *a*. The excess of the middle part above the least by *b*. The excess of the greatest above the middle one by *c*. The least part being *x*" (p. 9). The "excess of a number above another" is their difference: for example, the excess of 7 above 3 is $7-3=4$. The "middle term" is therefore $x+b$ and the greatest term is $(x+b)+c$, so that we have $a=x+x+b+x+b+c$ or $a=3x+2b+c$, and $x=\frac{a-2b-c}{3}$. As we shall see in the section "The Teacher's and the Student's Topos", this algebraic result can also be obtained, albeit more painfully, "by ordinary language" only.

An Epistemological and Methodological Interlude

In the framework of the ATD (Chevallard, 2019, 2020), the modelling of didactic phenomena rests on the notions of person, institution, and institutional position. All human individuals are persons. Any "instituted" reality is an institution: a family, a class, a couple, a school, a ministry, the Norwegian society, the French society are institutions. Every institution is organized into a set of institutional positions: In a classroom, there is the teacher position and the position of student; in the noosphere of mathematics education, there are the positions of textbook author, of "activist teacher", of "great mathematician", and so forth. An institutional position is occupied by persons who thus become "subjects" of the institution. Persons are shaped by the evolving set of institutional positions they occupy and have occupied (or even wish to occupy). Persons are thus singular representatives of this or that position to which they are subjected—as students, citizens, parents, friends, and so forth. In the opposite direction, persons can change the positions they occupy: there is thus a key dialectic of persons and institutions in the making of a society.

A pivotal consequence of this dialectic is that, to study any institutional position, one studies the persons who are or have been its subjects; and, conversely, in order to study a person, one studies the positions he or she occupies or has occupied (or wishes to occupy). This is what we have done above with persons (such as Lacroix) whose statements allow us to enlighten the historical evolution of the teacher and student positions regarding elementary algebra. A methodological consequence of this principle goes against beliefs that often pervade the academic world: Any statement of any person subjected or having been subjected to some position must be considered a testimony to that position. Such testimonies are not subject to a criterion of “prestige”, scientific or otherwise, about what is (or was) the position at such and such historical moment, in such and such type of institution: we must give equal prima facie weight to the statements of a student, a teacher, a textbook author, a researcher, and so forth. The essential criterion results from the institutional and personal analysis of the statements collected. If, for example, some person’s statement about the notion we are interested in seems different from what seems to be the “usual” response, we must investigate what other positions that person occupies at the same time (every person occupies a diversity of positions in a plurality of institutions). If, for example, some “informant” states that the roots of a quadratic equation $ax^2+bx+c=0$ are given by the formula $x=\frac{2c}{-b\pm\sqrt{b^2-4ac}}$, while the “usual” answer is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we will have to find out what other positions our informant is subjected to in order to give this answer. Such a person’s seemingly divergent view typically echoes a fact of institutional transposition that would remain to investigate. More generally, the facts we are looking into come within the competency of didactic transposition analysis, as expounded by Chevallard (1991).

Parameters: From Implicit to Explicit

To distinguish between “arithmetical rules” and “algebraic formulas”, we must keep in mind the key notion of *parameter*, understood in the classical sense of the term.[4] In the formula $A=l\times b$ for the area of a rectangle with length l and breadth b, the letters l and b are parameters specifying the rectangle.[5] Likewise,

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[4] This classical sense does not necessarily include the sense of boundary or limit. The Online Etymology Dictionary recalls that “parameter” was “a geometry term until late 1920s when it began to be extended to ‘measurable factor which helps to define a particular system,’ hence the common meaning (influenced by *perimeter*) of ‘boundary, limit, characteristic factor,’ common from 1950s” (“Parameter”, n.d.).

[5] Of course, the letter A is also a parameter. Generally speaking, in a formula $a=f(b,c,d)$, we regard a, b, c, and d as *parameters*. As will be explained in the section “The Vanishing of (Explicit) Parameters”, the parameter a is sometimes called the “subject” of the formula: in $A=l\times b$,

Lacroix's formula $x=\frac{a-2b-c}{3}$ contains three parameters a, b, and c. While a rule (in words) contains parameters *implicitly* (like "length" and "breadth"), a formula contains *explicit* parameters. We will refer here to the book *Algebra* written by Percival Abbott (1869–1954), which was in its time an epitome of elementary algebra.[6] Consider the following rule given by Abbott:

> The area of a rectangle in square metres is equal to the length in metres multiplied by the breadth in metres. (1942/1971, p. 13)

The author goes on to write (p. 14): "This rule is shortened in Algebra by employing letters as symbols, to represent the quantities." The letters l, b, and A representing the length, breadth, and area (in metres and square metres), "the above rule can now be written in the form: $A=l\times b$." A rule thus expressed is called a *formula*.

With this, a question arises. The "classical" doctrine on the arithmetic-algebra divide is expressed by Abbott as follows:

> In Arithmetic we employ *definite numbers*; we operate with these and obtain definite numerical results. Whereas in Algebra, while we may use definite numbers on occasions, we are, in the main, concerned with *general expression* and *general results*, in which letters or other symbols represent numbers not named or specified. (Abbott, 1942/1971, p. 13)

This statement is subtly contradictory to the notion of "arithmetical rule": In the rule for the rectangle, we have, *not* "definite numbers", but "*implicit* parameters", the *length* and the *breadth*, *not* represented by *letters*. How can this seeming discrepancy be explained?

The Teacher's and the Student's Topos

A quick answer to the above question is: In arithmetic, students are *given* a rule—they only have to *apply* it when the (implicit) parameters in the rule take definite numerical values. Here is an

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the letter A is the subject. The subject can be looked at as a *function* of any of the other parameters, for instance $a=\varphi_d(b,c)$, where the parameters b and c are now *variables*, which can be rewritten as $z=\varphi_d(x,y)$, with the function φ_d depending on the parameter d. Also the equality $a=f(b,c,d)$ can be regarded as an *equation* in the *unknown* b (for example): solving it for b leads to a new formula, $b=g(a,c,d)$, with b as subject.

[6] Abbott's *Algebra* was published in the famous "Teach Yourself" series (see "Teach Yourself", 2022).

example, where the calculation rule is expressed as a formula (Area of a Trapezium, n.d., Computing Area section):

> Find the area of a trapezium with parallel lines of 9 cm and 7 cm, and a height of 3 cm.
>
> $\text{Area}=\frac{1}{2}\times(9\text{ cm}+7\text{ cm})\times 3\text{ cm}=\frac{1}{2}\times(16\text{ cm})\times(3\text{ cm})=\frac{1}{2}\times 48\text{ cm}^2$
> $=24\text{ cm}^2.\square$

Deriving a formula (relying on some basic, given rules or formulas) is usually rather easy when done algebraically. In contrast, if you do it "arithmetically", it usually becomes more complex, and beyond the reach of beginners. Today's reader is no longer familiar with long arithmetical reasonings. This is why we have reproduced in extenso, below, so that everyone can judge, the rhetorical work achieved by Lacroix (1797/1831) to derive the arithmetical rule corresponding to the formula $x=\frac{a-2b-c}{3}$:

> The middle part will be the least, plus the excess of the mean above the least.
>
> The greatest part will be the middle one, plus the excess of the greatest above the middle one. The three parts will together form the number proposed.
>
> Whence the least part, plus the least part, plus the excess of the middle one above the least, plus also the least part, plus the excess of the middle one above the least, plus the excess of the greatest above the middle one, will be equal to the number to be divided.
>
> Whence three times the least part, plus twice the excess of the middle part above the least, plus also the excess of the greatest above the middle one, will be equal to the number to be divided.
>
> Whence three times the least part will be equal to the number to be divided, minus twice the excess of the middle part above the least, and minus also the excess of the greatest above the middle one.
>
> Whence in fine, the least part will be equal to a third of what remains after deducting from the number to be divided twice the excess of the middle part above the least, and also the excess of the greatest above the middle one. (pp. 9–10)

Here is now Lacroix's (1797/1831) algebraic translation of this piece of arithmetical work:

> The middle part will be $x+b$.
>
> The greatest will be $x+b+c$.
>
> Whence $x+x+b+x+b+c=a$.

$3x+2b+c=a$.

$3x=a-2b-c$.

$x=\frac{a-2b-c}{3}$. (pp. 9–10)

The fact that Lacroix's rhetorical work seems much beyond the reach of students can be linked to the notion of *topos* of an institutional position, that is the set of *types of tasks* that persons in that position may have to perform (Chevallard, 2019)—let us repeat: in arithmetic, the teacher *provided* the rules ready-made, and the students simply *applied* them to numerical values.

A Venerable Rule Goes Algebraic

The algebraization of arithmetic was a game-changer. Let us consider the "rule of false position" as presented by Francis Walkingame in his *Complete Practical Arithmetic* (1860):

> POSITION; OR, THE RULE OF FALSE
>
> Is a rule, that by false or supposed numbers, taken at pleasure, discovers the true one required. It is divided into two parts, SINGLE and DOUBLE.
>
> SINGLE POSITION
>
> Is by using one supposed number, and working with it as the true one, you find the real number required, by the following
>
> RULE.—As the total of the errors is to the true total, so is the supposed number, to the true one required.
>
> PROOF.—Add the several parts of them together, and if it agrees with the sum, it is right.
>
> EXAMPLE.
>
> (1) A school-master being asked how many scholars he had, said, if I had as many, half as many, and one quarter, as many more, I should have 88—how many had he? (Walkingame, 1860, p. 108)

If the "supposed" number is 40, the value of the required number is $40+20+10+40=110$. The true value x is to 40 as 88 is to 110: $\frac{x}{40}=\frac{88}{110}$. This gives $x=32$. The "proof" reduces to the equality:[7]

[7] Here the meaning of "proof" is not the one usual in mathematics today. Webster's 1828 dictionary defines it thus: "Trial; essay; experiment; any effort, process or operation that ascertains truth or fact. Thus the quality of spirit is ascertained by proof; the strength of gun-powder, of fire arms and of cannon is determined by proof; the correctness of operations in arithmetic is ascertained by *proof.*" ("Proof", 2020, para. 1).

32+16+8+32=88. By algebra, x being the true value, we have:[8] $x+\frac{x}{2}+\frac{1}{4}x+x=\frac{11}{4}x$, and thus $x=\frac{4}{11}\times 88=4\times 8=32$; which amounts to forming and solving a first degree equation. In his *History of Mathematics*, D. E. Smith (1925/1958) made this comment:

> To the student of today, having a good symbolism at his disposal, it seems impossible that the world should ever have been troubled by an equation like $ax+b=0$. Such, however, was the case, and in the solution of the problem the early writers, beginning with the Egyptians, resorted to a method known until recently as the Rule of False Position. (p. 437)

The algebraization of arithmetic was a huge step forward—historically, it was an unstoppable, winning process for the sciences. But there happened an almost surreptitious drawback that greatly reduced the power of the algebra *actually taught*.

A PYRRHIC VICTORY

The Vanishing of (Explicit) Parameters

The algebraic modelling of arithmetical rules might pertain to the student topos, as the exercise set of Chapter I of Abbott's *Algebra* (1942/1971) seems to show. Here is his first exercise (p. 20):

> 1. Write down expressions for:
>
> (1) The number of pence in £*x*.
>
> (2) ” ” pounds in *n* pence.

And now for the penultimate exercise (p. 21):

> 19. A train travels at v km/h. How far does it go in x hours and how long does it take to go y km?

The implicit parameters of arithmetical rules are here translated into explicit parameters, such as n, v, x, and y. However, in the history of school algebra, explicit parameters *will tend to disappear* to be replaced by "definite numbers". Here is a typical example in the chapter "Simple Equations" of Abbott's book:

> A motorist travels from town A to town B at an average speed of 64 km/h. On his return journey his average speed is 80 km/h. He takes 9 hours for the double journey (not including stops). How far is it from A to B? (p. 66)

[8] This is a "modern" reformulation of Walkingame (1860, p. 109).

In truth, calculations with parameters have not totally vanished. But they will soon become residual. Chapter VI of Abbott's book is entitled "Formulae". Its last section is entitled "Literal Equations". It opens with the following considerations:

> The operations employed in changing the subject of a formula are the same in principle as those used in the solution of equations. One essential difference from the equations dealt with in Chapter V is that whereas these were concerned with obtaining numerical values when solving the equations, in the formula the quantity which is the subject of the formula is expressed in terms of other quantities, and its numerical value is not determined, except when the numerical values of these quantities are known. (Abbott, 1942/1971, p. 75)

The author continues as follows:

> It is frequently necessary, however, to solve equations in which the values of the unknown quantities will be found in terms of letters which occur in the equation. Such equations are termed *literal equations*. The methods of solution are the same in principle as those employed in Chapter V. (Abbott, 1942/1971, p. 76)

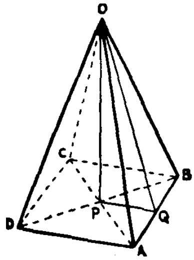
The examples given by Abbott are $5x-a=2x-b$, and $a(x-2)=5x-(a+b)$. The 14 equations in the related exercise set are quite similar to these, that is, they are quite *simple*, and nobody knows what they claim to model—they are an artificial, didactic device.

Transformation Versus Evaluation of Formulas

The paucity of the material thus presented is in striking contrast to the chapter's introduction, which begins with this promising statement (Abbott, 1942/1971):

> One of the most important applications of elementary Algebra is to the use of formulae. In every form of applied science and mathematics, … formulae are constantly employed, and their interpretation and manipulation are essential. (p. 69)

The author explains that "formulae involve three operations: (1) Construction; (2) manipulation; (3) evaluation" (p. 69). The construction of a formula does not start from scratch: it relies on formulas previously established, either theoretically or empirically. The first "worked example" given by Abbott is typical: "Find a formula for the total area (A) of the surface of a square pyramid as in Fig. 10 [see figure opposite] when AB=a and OQ=d"



(p. 70). The area A is the sum of the area A_1 of the base, and four times the area A_2 of a side, so that $A=A_1+4A_2=a^2+2ad=a(a+2d)$. The formulas involved in constructing the required formula are those for the areas of a square ($A_1=a^2$) and of a triangle ($A_2=\frac{1}{2}ad$). Here, the use of elementary algebra is genuine but minimalist.

The second example and all the exercises are just *evaluation* tasks, as in the exercise below (Abbott, 1942/1971):

> 2. The volume of a cone, V, is given by the formula $V=\frac{1}{3}\pi r^2 h$, where r=radius of base, h=height of cone. Find V when $r= 3{\cdot}5$, $h=12$, $\pi=\frac{22}{7}$. (p. 70)

The *manipulation* of formulas is only present in the section entitled "Transformation of Formulae". On this issue, the author seems to walk on eggshells, as if he were in danger of going too far. About the volume of a cone ($V=\frac{1}{3}\pi r^2 h$), Abbott writes:

> It may be necessary to express the height of the cone in terms of the volume and the radius of the base. In that case we would write the formula in the form: $h=\frac{3V}{\pi r^2}$, that is, the formula has been transformed. (Abbott, 1942/1971, pp. 71–72)

He then goes on like this:

> When one quantity is expressed in terms of others, as in [the formula] $V=\frac{1}{3}\pi r^2 h$, the quantity thus expressed, in this case V, is sometimes called the *subject of the formula*. When the formula was transformed into $h=\frac{3V}{\pi r^2}$, the subject of the formula is now h. This process of transformation has been termed by Prof. Sir Percy Nunn *"changing the subject of the formula"*. (p. 72)

About such a change, Abbott (1942/1971) adds this caveat:

> The transformation of formulae often requires skill and experience in algebraical manipulation; the following examples will help to illustrate the methods to be followed. (p. 72)

He then gives five "worked examples", such as, given $T=\frac{\pi f d^3}{16}$, to find f and d in terms of the other letters—one arrives at $f=\frac{16T}{\pi d^3}$ and $d=\sqrt[3]{\frac{16T}{\pi f}}$—; or, given $L=l+\frac{8d^2}{3l}$, to find d in terms of L and l—one gets $\sqrt{\frac{3l(L-l)}{8}}=\sqrt{\frac{3lL-3l^2}{8}}$. All the examples and exercises can be done easily in their head by anyone with a minimal command of elementary algebra.

The Limitations of Taught Algebra

In the case of the formula $L=l+\frac{8d^2}{3l}$, readers are *not* asked to find the expression of l in terms of L and d—the answer is $l=\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-\frac{8d^2}{3}}$—, which would require solving a quadratic equation *with* parameters. Here we reach the demarcation line drawn by the traditional didactic transposition of elementary algebra.

This line draws a curricular curiosity. Firstly, the quadratic equations with parameters considered have only *one* parameter. Secondly, students are *not* asked to give the expression of their solutions (which, in the general case, would include the parameter), but simply to specify, according to the value of the parameter, when they have 0, 1 or 2 roots. (It is only when they have one root—therefore when the parameter has a well-determined value—that the students may be asked to give the value of this root.)[9] This sudden change of didactic contract (Brousseau, 1997)—an equation is no longer "something to be solved" but to be "studied" or "discussed"—was (and still is) a source of difficulty for students.

In spite of this, the question of the "transformation of formulas" is *at the heart of what algebra can consist of*. In Abbott's Example 5 (p. 74), readers are asked to express the length l of a simple pendulum in terms of g and the period $t=2\pi\sqrt{\frac{l}{g}}$. (The answer is $l=\frac{gt^2}{4\pi^2}$.) In Exercise 13, No. 2 (p. 74), they are asked to express the radius r of a sphere in terms of its volume V. (The answer is

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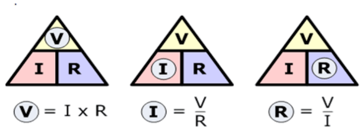
[9] For an illustration of the curricular "curio" referred to here, see for instance lecture notes on Precalculus by Marta Hidegkuti (2009)—published on her website: http://www.teaching.martahidegkuti.com. Note also that some of today's online calculators can solve quadratic equations *with* parameters: see for example, Wolfram Alpha Equation Solver calculator (https://www.wolframalpha.com/calculators/equation-solver-calculator).

$r=\sqrt[3]{\frac{3V}{4\pi}}$.) The usefulness of these transformations seems obvious. Now the big problem is that the type of tasks in question—"changing the subject of a formula", to use Abbott's (and Nunn's) words—has become marginalized in most secondary curriculums.

THE PITFALLS OF DIDACTIC TRANSPOSITION

The Marginalization of Formula Transformation

In what sense has "formula transformation" been marginalized? Let us consider the following exercise proposed by Abbott (1942/1971): "There is an electrical formula $I=\frac{V}{R}$. Express this (1) as a formula for V and (2) as a formula for R. Find I if V=2 and R=20." (p. 75). We are touching here the borderline that will separate what can be called "algebraic literacy" from "algebraic illiteracy". Certain teaching or occupational institutions choose to "spare" their subjects the algebraic "work" needed to go from the formula $I=\frac{V}{R}$ to the formulas $V=RI$ and $R=\frac{V}{I}$, respectively; that is to say, they choose illiteracy. One of the most widespread techniques, it seems, consists in substituting to the algebra needed a graphic "mnemonic trick" which takes the form of a triangle in which the parameters V, I, R are displayed. The example below is taken from the GuyHowto website (Nimar_geek, 2020).



Very often, students or professionals are supposed to know these three formulas *by heart*, the triangle having only for function to help them in their memorization effort. The triangle technique is pushed forward by *institutions*. This institutional implementation sends a message that can be decoded as: "You don't need to know algebraic calculation *at all*."

There is also another technique, which is implemented more by *people*—students, in particular—than by institutions. This technique consists in *avoiding* any *literal* calculation. Suppose we are given the formula $I=\frac{V}{R}$ and values for I and R, and are asked to calculate V. If $I=1.2$ and $R=20$, the formula gives rise to the equality $1.2=\frac{V}{20}$, which is a linear equation in V that the student can therefore easily solve: $V=20\times 1.2=24$. This technique therefore consists in *first* transforming a formula into a "simple" numerical equation.

The Return of the Repressed

In some cases, however, the resulting equation is less simple than expected. Here is an example taken from the Stack Exchange website (Coefficient of Friction, 2015). A student asks for help:

> Can someone describe to me the algorithm for solving the following example question?
>
> Mass=50 kg
>
> Forward Force Acting on Mass=100 N
>
> Acceleration of Mass=0.1m/s^2
>
> Coefficient of friction?

An expert in engineering and scientific computing, John Alexiou, gives the following answer (Alexiou, 2015):

> Do a free body diagram and you will find for a horizontal plane that
>
> $$F-\mu mg=ma$$
>
> $$(100)-\mu(50)(9.80665)=(50)(0.1)$$
>
> $$\boxed{\mu=\frac{(100)-(0.1)(50)}{(50)(9.80665)}=0.1937\ldots}$$

This solution seems to use the second, "economical" technique mentioned above. Solving the equality $F-\mu mg=ma$ for μ leads to the formula $\mu=\frac{F-ma}{mg}$, which gives $\mu=\frac{100-50\times 0.1}{50\times 9.80665}=0.193746\ldots$. This small piece of work later elicited the following acid comment from a certain David White, professedly standing on the "opposite side of the border":

> Ah yes. Immediately substitute values into the starting equation and avoid the algebra necessary to separate the unknown variable. THAT technique is why most U.S. high school students are practically algebraically illiterate. (Coefficient of Friction, 2015)

We will retain the rough estimate of American students' command of algebra given by White but add that the algebraic illiteracy White believes to perceive in American students certainly extends to many countries—in ATD terms, it appears to be a *civilizational* fact, and thus a fact not restricted to just one society. It should be added also that White immediately received the following rejoinder from Alexiou: "I actually didn't, since the final expression was already solved for μ and the substitution was the *last* step. The difference is if I carry the symbols vs. their value around." Alexiou is certainly not algebraically illiterate (John Alexiou, n.d., Profile) and seems to fully understand White's criticism.

A Turning Point in the Didactic Transposition Process

How has this demarcation line been drawn? The answer must involve the conditions and constraints that have historically determined the didactic transposition of elementary algebra. What is remarkable here is that we can observe two "noospherians" in action, namely Abbott and Nunn, taking part in this transpositive work. Let us dwell on the first exercise proposed by Abbott on the transformation of formulas: "The formula for the area (A) of a circle, in terms of its radius (r) is $A=\pi r^2$. Change the subject of the formula to express r in terms of the area." In his book *The Teaching of Algebra* (1914), touching the phrase "Change the subject of a formula", Nunn writes this:

> Since this phrase has already obtained a certain amount of currency, the author may be permitted to claim here the modest credit of its paternity. He believes that it was used for the first time in his lectures to teachers of mathematics in 1909. It was subsequently adopted in the Report on the Teaching of Algebra by the Committee of the Mathematical Association. (p. 78)

Why did Nunn introduce this way of saying, which was adopted by Abbott and others, when what is required is simply to "solve the equation $A=\pi r^2$ for r"? Nunn seems to have been quite aware of the change he wanted to popularize. He writes, for example, this fine remark about, precisely, the notion of formula:

> A formula can usually be regarded as stating both a mathematical relation (i.e. a numerical identity underlying diverse equivalent forms) and a rule of procedure. In most cases, however, one of these ways of looking at it is more natural than the other. Thus the formula $\tan a=\sin a/\cos a$ suggests most readily a fact of relationship, the formula $V=\pi r^2 h$ a practical rule. (Nunn, 1914,

p. 64)

In fact, Nunn begins by launching an attack against the position of strength given, in the didactic transposition of algebra, to equations:

> In history equations began as conundrums, and the school tradition has not lifted them to a much higher level of intellectual dignity. The pupil may become skilful in compelling "*x*" to reveal the value hidden in a symbolic statement of baffling complexity. … He may have gained but an imperfect idea either of the practical or of the scientific importance of processes which he has learnt to handle for merely artificial purposes. (Nunn, 1914, p. 77)

The twin notions of "subject of a formula" and "change the subject of a formula" ensure the distinction and the promotion of a subclass of equations that Nunn no longer designates under the name of equations, which leads to the coming apart of two distinct topics: equations and formulas. Nunn writes:

> Finally it should be noted that the word "equation" is avoided throughout Section I. There seem good reasons for withholding the term until … the introduction of directed numbers, [where] it becomes appropriate to use the typical form $f(x)=0$ and to associate a new technical name with it. (Nunn, 1914, p. 79)

Paradoxically, the class of formulas, considered as potential equations, and indispensable in many fields of science and technology, was going to be marginalized by their very "promotion".

The Rules and Oddities of Didactic Transposition

This detail of the didactic transposition process is due to a few great constraints. The first constraint is that of *simplicity*: the transposed content must be "simple" enough to offer students a *topos* that they can actually occupy. "Didactic transposers" (like Abbott and Nunn) look for types of tasks T such that there exists a core of tasks $T_0 \subset T$ that the students concerned can manage to perform after a reasonable amount of study time. This condition may be satisfied both by Abbott's "simple equations" *and* by the tasks of "changing the subject" in a formula.[10] But other conditions must be met.

The first one is that the teacher be able to produce tasks $t \in T_0$ *at will*, for didactic reasons of repetitive training. However,

[10] For an updated example where a formula is "solved for" one of the letters it contains, see Solving Literal Equations (n.d.).

because of their origin in specific domains (geometry, physics, technology, etc.), it seems that the list of formulas to be "solved" is limited. Note nevertheless that, if one accepts to break the link between a formula and what it models—as will be done in a very general way in the algebra curriculum—, it is easy to create formula transformation tasks: Nunn (1914) stresses that, when one has to transform a formula, "the meaning of the symbols need not be known" (p. 107). He goes so far as giving a pair of purely formal ("unexplained") examples of formula transformations:

> The most interesting thing about this process is that you do not have to know what a formula means in order to change its subject. … Take as examples the unexplained formulae $\mathrm{P}=\frac{a+b\mathrm{Q}}{n}$ and $\mathrm{P}=\frac{a}{b\mathrm{Q}-t}$ and change the subject to Q in each case.[11] (Nunn, 1914, p. 107)

Ignoring here the fact that it has had devastating effects (to which we will return in the section "The Cultural Fiasco of Elementary Algebra"), the breaking of the link between an algebraic formula and what it could model, makes it possible to satisfy a (second-order) condition often imposed on the didactic transposition process: there should also exist a set $T_1 \subset T$ of "more difficult" exercises, which enables teachers to devise and manage some special situations (selective exams, etc.).

In any case, we must note that the link between algebraic entities and the "systems" they can model, largely broken in the case of "simple equations", has not been totally broken in the case of formulas. Abbott (1942/1971) devotes the last section of his chapter on simple equations to the topic of "problems leading to simple equations", in which he draws on the traditional body of arithmetical problems. In his chapter on formulas, every formula that appears is explicitly attached to a reality that it is supposed to model—to break this link, he cannot but add to the chapter the already mentioned section entitled "Literal Equations".

All of this has an overall effect that deserves to be emphasized: In a general way, the constraints mentioned contribute to making the algebra taught a separate field, almost foreign to the other fields of mathematics and science. In fact, when physics (or technology) teachers call upon the triangle trick (described in the section "The Marginalization of Formula Transformation"), for example, they

[11] Note that P and Q are in roman in the original text.

seem to mean: "We are in a physics (or technology) class; therefore, not too much mathematics, please!" Of course, this helps to make the algebra taught simpler to learn, which is a priori commendable. But simplification may have unexpected, unfortunate consequences.

A major problem in designing a curriculum is this: if a school claims to teach a certain body of knowledge $\mathcal{K}$, we must first make sure that the knowledge taught $\mathcal{K}^*$ "conforms" to $\mathcal{K}$, that it is not a "denatured" version of $\mathcal{K}$. "Respecting the students" means, first of all, teaching them mathematics that is "true" mathematics, not some watered-down version of it. Pushing the paradox to the extreme, the French poet and thinker Charles Péguy (1873–1914) once harshly wrote: "Speaking rigorously, we can say that [the students] are made for the course, and that the course is not made for them, since it is made for the *object of the course* [emphasis added]" (Péguy, 1957, p. 399).[12] Speaking concretely, if a teacher's teaching seems to be well suited for the students, is it really algebra that is being taught? That is the question.

THE CULTURAL FIASCO OF ELEMENTARY ALGEBRA

Systems and Models: Algebra at Work

So, what is (elementary) algebra? In the following, we give an answer that focuses on the main criteria to analyse and assess the outputs of didactic transposition processes relating to elementary algebra (i.e., the algebra taught or to be taught in secondary school). To do this, we must first introduce two basic notions of the ATD: the notions of *system* and *model*. A system $\mathcal{S}$ is any entity subject to some laws of its own. For example, a (geometric) sphere is a system whose "laws" are generally called the *properties* of the sphere, such as the following: "A great circle … of a sphere is the intersection of the sphere and a plane that passes through the centre point of the sphere. A great circle is the largest circle that can be drawn on any given sphere" ("Great Circle", 2021). Any *formula* is a system as well; the formulas for the volume and the surface area of a sphere of radius r—$V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$—are systems in their own right, which themselves have properties (we have

[12] All quotations from the French and the Norwegian have been translated into English by the authors.

$V = A \times \frac{r}{3}$ or $A = \frac{3V}{r}$ or $3V - rA = 0$, etc.). Given a system $\mathcal{S}$, a system $\mathcal{S}'$ is said to be a *model* of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can answer certain *questions* Q about $\mathcal{S}$. In practice, given a question Q relating to $\mathcal{S}$ which one wants to answer, one tries to build up a model $\mathcal{S}'$ of $\mathcal{S}$ (or choose one already existing) whose study with respect to the question Q, is easier, safer, quicker than by a "direct" study of $\mathcal{S}$. For example, if the radius r of a sphere increases by 20%, the new surface area A' will become $4\pi r'^2 = 4\pi(1.2r)^2 = 1.44A$, so that the surface area will increase by 44%—a tricky result to obtain experimentally.

Algebraic Expressions Turned into Cadavers

The great catastrophe which historically disorganized and denatured elementary algebra resulted from the generalized rupture of the link between the systems $\mathcal{S}$ to be modelled algebraically and their algebraic models $\mathcal{S}'$ relating to some question Q about $\mathcal{S}$. In most textbooks, this link has disappeared entirely. Here is an example where there is still a trace of such a link. In his book *Algebra: An Elementary Text-Book for the Higher Classes of Secondary Schools and Colleges* (5th edition, 1904), George Chrystal (1851–1911) offers a host of formal, nonfunctional, algebraic calculation exercises; that is, exercises disconnected from any explicit need to analyse a given system, such as developing the following algebraic expressions (p. 56): $(x+y)(x-y)(x^2-y^2)(x^2+y^2)^2$ and $(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$. In one case, however, Chrystal puts the reader on the track of the modelled reality. He presents the following equalities:

> (21.) $(x^2-ay^2)(x'^2-ay'^2)=(xx'\pm ayy')^2-a(xy'\pm yx')^2$;
> $(x^2-ay^2)^3=(x^3+3axy^2)^2-a(3x^2y+ay^3)^2$;
> $(x^2-\mathrm{B}y^2-\mathrm{C}z^2+\mathrm{BC}u^2)(x'^2-\mathrm{B}y'^2-\mathrm{C}z'^2+\mathrm{BC}u'^2)=\{xx'+\mathrm{B}yy'\pm\mathrm{C}(zz'+\mathrm{B}uu')\}^2-\mathrm{B}\{xy'+x'y\pm\mathrm{C}(uz'+u'z)\}^2-\mathrm{C}\{xz'-\mathrm{B}yu'\pm(zx'-\mathrm{B}uy')\}^2+\mathrm{BC}\{yz'-xu'\pm(ux'-zy')\}^2$. (Chrystal, 1904, p. 57)

In relation to them, he mentions the name of Lagrange and adds this comment: "The theorems (21.) are of great importance in the theory of numbers; they show that the products and powers of numbers having a certain form are numbers of the same form" (p. 57).[13] But as soon as the system $\mathcal{S}$ modelled by $\mathcal{S}'$ and the

[13] In other words, these algebraic equalities model "systems" of numbers. Chrystal adds elsewhere (p. 82) a footnote that reads: "If each of two

question Q raised about $\mathcal{S}$ both disappear, $\mathcal{S}'$ *is no longer the model of anything*. The algebraic expressions that the students are required to work with then become "algebraic cadavers", formerly living creatures from which life has gone, abandoned human creations whose raison d'être has vanished.

On an example, let us travel the path backwards from cadaver to life. The "cadaver" is this identity:

$$3(ab+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2].$$[14]

What is (or was) its raison d'être? It shows in particular that, if the sum of three numbers a, b, and c is constant, then the product $ab+bc+ca$ is maximal when $a=b=c$ (see Strømskag & Chevallard, 2022). But what can this result be used for? One answer concerns the prices of diamonds, when assumed to be proportional to the *square* of their weight. If the price of a diamond of weight w is equal to kw^2, where $k>0$, and if a diamond is broken into three pieces of weight a, b, and c, respectively, the price of each of these pieces is ka^2, kb^2, and kc^2 while the price of the original diamond of weight w_0 was $kw_0{}^2=k(a+b+c)^2$. We have: $(a+b+c)^2-(a^2+b^2+c^2)=2(ab+bc+ca)>0$. The price of the original diamond is therefore greater than the sum of the prices of the three diamonds obtained. As a consequence of the equality $3(a+bc+ca)=(a+b+c)^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=w_0{}^2-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$, the loss of value caused by the breaking of the diamond into three parts is maximal when $a=b=c$, that is when the three pieces have the same weight.[15]

It is striking that this example, which is of a formerly traditional type (see e.g., The Value of Diamond, n.d.), involves three parameters a, b, and c. The vanishing of parameters from algebraic expressions goes together with the purely *formal* existence of such expressions, which consequently lose their *functional* role, that is,

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integers be the sum of two squares, their product can be exhibited in two ways as the sum of two integral squares." This seems to be the most famous result in number theory attached to the name of Lagrange.

[14] This identity and the following example of the price of diamonds are borrowed from André Combes (1961, pp. 39–40).

[15] The reader may be tempted, anachronistically, to solve this problem of maximum using differential calculus (i.e., calculating derivatives, etc.). In fact, more generally, many "problems of maxima and minima" have long been "solved by algebra" only (see e.g., Ramchundra, 1859).

the role of elements of a model of a system. Let us note in this respect that we commonly speak of algebraic *expression*s without specifying what these expressions do *express*.[16] So, what does such an algebraic expression express? Our answer is: An algebraic expression expresses a *calculation programme* (Chevallard, 2006). "Given two numbers, add them and divide the sum by 3" is a calculation programme expressed "in words", of which a possible *algebraic* expression is $\frac{x+y}{3}$. *This is the starting point of the algebraic adventure*. One must also observe that, not unexpectedly, to the "deparametrization" of algebra echoes, in the teaching of "scholarly" algebra, the primacy long given to polynomials in *one* indeterminate (see e.g., "Polynomial", 2021). This seems to be a typical "reverse" effect, that is, a "bottom-up" influence, instead of the usual "top-down" effect, of the didactic transposition of elementary algebra on the algebra taught in higher education: the deparametrization of the secondary curriculum contaminates higher teaching.

Algebra bumps into mathematical analysis

The example of the diamond cut into three pieces may have prompted the reader to use differential calculus to find the maximal loss. However, as we know, this can be done by using elementary algebra only. The problem of the diamond is a special case of a very general fact: elementary algebra has long been used, *to the exclusion of calculus*, to solve what were classically called "problems of maxima and minima".[17] A key theorem—called here Theorem θ—used for this was the following: Given nonnegative numbers $a_1, a_2, ..., a_n$, if the sum $S=a_1+a_2+...+a_n$ is constant, then the product $P=a_1a_2...a_n$ reaches a maximum when these numbers are equal (and therefore are equal to S/n).

Let us give an example of a bit sophisticated use of Theorem θ, taken from Natansón (1977, pp. 39–41): What is the maximum

[16] An algebraic expression is "an expression obtained by a finite number of the fundamental operations of algebra upon symbols representing numbers" ("Algebraic expression", n.d.). The operations involved are addition, subtraction, multiplication, division and raising to integral or fractional powers.

[17] The reader not familiar with this historical fact can skim through the old, classic book by Ramchundra (1859), aptly entitled *A Treatise on Problems of Maxima and Minima Solved by Algebra*.

volume of an open-top box x cm high made with a sheet of paper of width a and length b, with $a \leq b$? Of course, we must have $x<\frac{a}{2}$. The volume of the box is $V=x(a-2x)(b-2x)$. This volume is the product of three terms: x, $a-2x$, and $b-2x$, but the sum of these terms is *not* constant (it is equal to $a+b-3x$). However, we can observe that V will reach a maximum at the same time as $4V$, which is the product of the three terms $4x$, $a-2x$, and $b-2x$, whose sum is constant (it is equal to $a+b$). According to Theorem θ, the value of x we are looking for should verify $4x=a-2x=b-2x$. If $a=b$, this system of equations has the solution $x=\frac{a}{6}$ and the maximum volume is therefore $V=\frac{2a^3}{27}$. For example, if $a=20$, we thus arrive at $x=\frac{10}{3}$ $=3.333\ldots$ and $V=592.592592\ldots$.[18] When $a \neq b$, the system of equations $4x=a-2x=b-2x$ *has no solution*, because we cannot have the equality $a-2x=b-2x$. In order to have more leeway in working out the situation, we will therefore *enrich* it by introducing in the expression of V a parameter λ. Let us consider the *expression*[19] $\bar{V}=x(a-2x)(\lambda b-2\lambda x)$, with $\lambda>0$. $\bar{V}$ reaches a maximum at the same time as V. Once again, however, the sum of the terms is *not* constant, so that, instead of $\bar{V}$, we will consider the expression $2(1+\lambda)\bar{V}=[2(1+\lambda)x](a-2x)(\lambda b-2\lambda x)$: the sum of the three terms is now $a+\lambda b$. Can we find a value of λ such that the system of two equations $2(1+\lambda)x=a-2x$ and $2(1+\lambda)x=\lambda b-2\lambda x$ has a solution in x? The first equality is equivalent to $x=\frac{a}{2(2+\lambda)}$. The second equality is equivalent to $x=\frac{\lambda b}{2(1+2\lambda)}$. Is there a value of λ such that $\frac{a}{2(2+\lambda)}=\frac{\lambda b}{2(1+2\lambda)}$, that is, such that $b\lambda^2+2(b-a)\lambda-a=0$, with $\lambda>0$? This quadratic equation has only one positive solution, that is, $\lambda=\frac{\sqrt{(b-a)^2+ab}-(b-a)}{b}$. This leads to $x=\frac{1}{2}$ or as well to $x=\frac{1}{6}$

[18] This particular case is one of the preparatory exercises for the South Australian Mathematics Talent Quest (SAMTQ Senior Years 11–12, n.d., Resources section, para. 9). The algebraic solution proposed there does not use Theorem θ, at the cost of comparatively intricate considerations and calculations specific to the case studied.

[19] It must be stressed that $\bar{V}$ is an expression formally defined; it does not pretend to refer to the volume of anything.

$[(a+b)-\sqrt{(a-b)^2+ab}]$. (This formula is valid as well when $a=b$: we are then brought back to the equality $x=\frac{a}{6}$.) If, for example, $a=10$ and $b=16$, we arrive at the following values:
$x=\frac{1}{6}(26-\sqrt{36+160})=\frac{1}{6}(26-14)=2$ and $V=2(10-4)(16-4)=144$.
Thanks to the introduction of the parameter λ, we have a general formula giving x, which applies for all a and b, where $a\leq b$. Today, the technique used here has disappeared to make room for *particular* solutions based on *differential calculus*.[20]

An old Frontier to Reconquer and Revitalize

So, what might a revival in secondary education of taught algebra be made of? The first condition to be achieved is obviously to recover an elementary algebra with expressions in *several* indeterminates in order to be able to model a diversity of mathematical or extra-mathematical systems. Here is a simple example: It seems that many "nonmath" people wrongly believe that, if a quantity q increases by p% (with $p>0$) and then decreases by p%, the final quantity, q', will be *equal* to the initial quantity q. But let us do some algebra. We have:

$$q'=q\left(1+\frac{p}{100}\right)\left(1-\frac{p}{100}\right)=q\left(1-\frac{p^2}{10000}\right)=q-\frac{p^2q}{10000}<q.$$

If, for example, $p=50$, q will be diminished by 25%. The mistake made here by many non-scientific people is a profound one—especially when they say they are "surprised" to find that $q'<<q$.

A related error is that made a few years ago by a Norwegian journalist who stated correctly that "Our neighboring countries [i.e., Sweden, Denmark, and Finland] get 50 percent more police power out of each employee than police director Odd Reidar Humlegård does with his 8,000 police-trained officials" but wrongly summarized this statement with the following headline: "*Gets 50 percent less out of each employee*" (Helsingeng, 2013). But let us now consider a more sophisticated result, which, as we shall see, is not wrong—it is simply misleading.

[20] Concerning the last particular case considered, see for example, Steve M. (2016). Let us note here that the "traditional" proof of Theorem θ was flawed. But this flaw was ignored for a long time during the 19th century in secondary education, even though Augustin-Louis Cauchy had solved the problem in his 1821 *Cours d'Analyse* [Course of Analysis]. All this we have discussed elsewhere (Strømskag & Chevallard, 2022).

A French physicist and popularizer, Étienne Klein, published on 31 March 2020, a short study (Klein, 2020) in which he aimed to convince the reader that a screening test to detect a certain disease may be *completely illusory* in the sense that a person declared to be affected by the disease may have a very low risk (about 2%) of actually being ill. Let p be the relative frequency of the disease (Klein assumes that $p=10^{-3}$) and let f be the reliability of the test (Klein assumes that $f=95\%=0.95$). In a population of size N, the number of ill people is equal to pN; by assumption, the test will find them all positive. The number of non-ill people is equal to $N-pN=(1-p)N$. Among these, some are found positive: their number is $(1-f)(1-p)N$. The total number of positives is therefore: $pN+(1-f)(1-p)N=[p+(1-f)(1-p)]N$. The probability of being sick when positive for the test is therefore equal to $q=\frac{p}{p+(1-f)(1-p)}=\frac{p}{1-f(1-p)}=\frac{1}{f+\frac{1-f}{p}}$. We see that, for f fixed, when p *increases*, the fraction $\frac{1-f}{p}$ decreases, and so does the denominator $f+\frac{1-f}{p}$, so that the probability q *increases*. Solving the equality $q=\frac{1}{f+\frac{1-f}{p}}$ for p, we arrive at $p=\frac{1-f}{\frac{1}{q}-f}$. If you want to have $q=2\%$ (with $f=95\%$), take $p=\frac{0.05}{\frac{1}{0.02}-0.95}=\frac{0.05}{50-0.95}=\frac{0.05}{49.05}=\frac{5}{4905}=\frac{1}{981}$, that is, a little more than one in 1,000 cases. To get $q=50\%=0.5$, likewise, take $p=\frac{1-f}{\frac{1}{q}-f}=\frac{0.05}{2-0.95}=\frac{0.05}{1.05}=\frac{5}{105}=\frac{1}{21}$. To get $q=80\%$, take $p=\frac{1-f}{\frac{1}{q}-f}=\frac{0.05}{1.25-0.95}=\frac{0.05}{0.3}=\frac{1}{6}$. Thus 1 in 981, 1 in 21, and 1 in 6 cases of disease result in a probability of having the disease of 2%, 50%, and 80%, respectively, when tested positive. It is only the "clever" choice of the value of the parameter p that leads Klein (2020) to conclude that screening tests are illusory.

The above examples illustrate a massive fact: outside of scientific and technical communities, elementary algebra, which is the gateway to all modern sciences, has not been made a part of even the highest common culture. Today, algebra is a curricular frontier on which students stay for a while and from which they then withdraw—to never return, with a very few exceptions.

WHAT WILL BECOME OF ALGEBRA?

Taking Stock of the Current Situation

Can we save algebra today? In order to sketch out an answer to this question, we begin by drawing a portrait of the situation of algebra in Norwegian secondary school textbooks over the last decades. We shall focus on their treatment of formulas, mainly in relation to the first two of the three major operations distinguished by Abbott: (1) construction; (2) manipulation; (3) evaluation. Evaluation, which almost always appears as the obligatory final step in the "treatment" of a formula, will not be the subject of specific comments, given that the present state of techniques of numerical calculation allows us to ignore most of the difficulties that students had to face in this respect a few decades ago.

Constructing Formulas: A Very few Cases

Here is first an exercise proposed for a lower secondary school examination in 1886 (Røstad, 1961, p. 90):

> **385**. a) In a circle an arc of 60° has a length of 12.56 cm. How large is the circumference, how large is the radius and the area? How large is the area of the sector bounded by the given arc and the two radii to the endpoints of the arc?
>
> b) Solve the task again when the arc has length *a* cm.

Question (a) looks like a traditional question of arithmetic: the length of the circumference is $6 \times 12.56\,\text{cm} = 75.36\,\text{cm}$, the radius is $\frac{75.36\,\text{cm}}{2\pi} \approx 12\,\text{cm}$, and the area of the circle is $\pi \times 12^2\,\text{cm}^2 \approx 452\,\text{cm}^2$; the area of the sector is therefore $\frac{452\,\text{cm}^2}{6} \approx 75.3\,\text{cm}^2$. Question (b) requires the "construction" of a formula. The students can rely on their treatment of the first question so as to make explicit the parameters that have remained implicit until now: they will successively obtain the formulas $l=6a$, $r=\frac{3a}{\pi}$, $A=\frac{9a^2}{\pi}$, and $A_S=\frac{3a^2}{2\pi}$, where l is the length of the circle and A_S is the area of the sector.[21]

[21] Note that the numerical results of question (a) can be used to check the formulas obtained, and conversely.

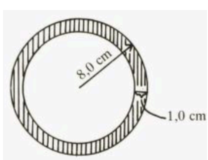
Generally speaking, "constructing a formula" is constructing an algebraic *model* of a certain system. In the previous case, the formulas to be constructed model this or that aspect of a system that is a circle—its length, area, and so forth. Note that, for such a modelling work, since it is necessary to have some knowledge of the kind of systems to be modelled, the systems considered are often, as here, geometric (rather than physical, or chemical, etc.).

Our inquiry into Norwegian textbooks has revealed very few exercises on formula construction. In the three volumes of the classic textbook *Matematikk for den Høgre Skolen* (Mathematics for Secondary Education) written by Anders Søgaard (1883–1964) and Ralph Tambs Lyche (1890–1991), widely used from 1940 to about 1970, and of a rather demanding level, we find this exercise (Søgaard & Tambs Lyche, 1939/1969, p. 104):

> ***Exercise 230***. A sphere with radius r is circumscribed by a cylinder. It connects as closely as possible to the sphere. Find a formula for the surface area a of the cylinder, expressed in terms of r. Evaluate it for $r=2$.

The answer is almost immediate: we have $a=2r\times 2\pi r=4\pi r^2$. Note that, here, this formula is constructed to be evaluated, not to be "transformed"—for example, to arrive at $r=\sqrt{\frac{a}{4\pi}}$.

In an exercise book, we find the following exam exercise given to 11th graders in 1980 (Erstad et al., 1984, p. 127):

> **1003** In this task you will need that a circle with radius r has area πr^2. The figure shows a cross section of a water pipe. The outer radius of the pipe is equal to 8.0 cm and the thickness of the pipe material is equal to 1.0 cm.
>
> 
>
>
> a) Calculate the area of the pipe material (shaded in the figure).
>
> Let the outer radius of the pipe be equal to R and let the thickness of the pipe be equal to x.
>
> b) Show that the area A of the pipe material is given by $A=2\pi Rx-\pi x^2$.
>
> c) Use the formula in b) and calculate the thickness x when $A=88\text{ cm}^2$ and $R=8.0\text{ cm}$. Use $\frac{22}{7}$ as an approximation of π.

It may seem strange to the "uninitiated" reader that students were reminded that the area of a circle with radius r is equal to πr^2; but it was customary to provide students with a formulary to ease the memory burden. Here students are required to construct a formula that is given to them explicitly (we have $A=\pi R^2-\pi(R-x)^2=\pi R^2-\pi R^2+2\pi Rx-x^2=2\pi Rx-\pi x^2$).

Note also that they are *not* required to solve for x the quadratic equation $A=2\pi Rx-\pi x^2$, that is, $x^2-2Rx+\frac{A}{\pi}=0$, which contains the parameters R and A (and also the constant π)—which would lead to $x=R\pm\sqrt{R^2-\frac{A}{\pi}}$. They are only asked to solve for x the "simple" (that is, *without* parameters) equation $x^2-16x+28=0$, which reduces to $(x-8)^2=36$, so that $x=8\pm6$ (for obvious reasons, we have $x=8-6=2$). Here we step again on the demarcation line created by the didactic transposition of elementary algebra—one can ask the students to solve a quadratic equation *without* parameters, but not with parameters. The "change of subject" (here, from A to x) being thus out of the question, one bypasses this prohibition by giving the parameters definite numerical values *before* calculating the corresponding value of x.

Manipulating Formulas: A Limited Perspective

In order to construct a formula, we can start from a formula and "transform" it by looking at it as an equation that we solve for one of its parameters, when this is not "forbidden", as in the case of the equation $x^2-2Rx+\frac{A}{\pi}=0$. Of course, in order to circumvent the solving of a quadratic equation with parameters, the formula $x=R-\sqrt{R^2-\frac{\text{A}}{\pi}}$ might as well be given ready-made to students. The hunt for examples of this kind in the textbooks reviewed proved rather disappointing. One of the rare specimens we came across, in a textbook for 11th graders, begins as follows (Erstad et al., 1984, pp. 79–80):

> **748** When a car with speed v brakes to stop, the braking distance d will be given by $d=kv^2$, where k is a constant that depends on the tires and the road surface. The braking distance is therefore proportional to the square of the speed.

As is customary, students are first asked to calculate d for various values of v (i.e., 10 m/s, 21 m/s, and 28 m/s), and for two values of k, that is, 0.065 s^2/m (dry asphalt) and 0.32 s^2/m (icy road). Then the distance d is taken to be $\frac{100}{3}$ m and students have to calculate the "maximum speed" of the car in order for it to stop after this distance, on dry asphalt and on icy road. In what we can call "parametric" algebra, we go from the formula $d=kv^2$ to the formula $v=\sqrt{\frac{d}{k}}$; and, more precisely, since $d=\frac{100}{3}\text{m}$, $v=\frac{10}{\sqrt{3k}}$. In "aparametric" algebra, we shall write $\frac{100}{3}\text{m}=0.065\,\text{s}^2/\text{m}\times v^2$, which leads to $v^2=\frac{100\text{m}}{0.195\,\text{s}^2/\text{m}}\approx 512.82\,\text{m}^2/\text{s}^2$, and therefore to $v\approx\sqrt{512.82\,\text{m}^2/\text{s}^2}\approx 22.646\,\text{m/s}$. Here we come across a borderline case: are students expected to transform the first formula $d=kv^2$ into the formula $v=\sqrt{\frac{d}{k}}$ or to give numerical values to the parameters d and k *before* determining v numerically?

In a very general way, what can be called "algebra in action" appears somehow unassertive. In a textbook for Grade 9 (Breiteig et al., 1998, p. 106), the whole of the book's treatment of the transformation of formulas boils down to transforming $C=\pi d$ into $\frac{C}{\pi}=d$ and $v=\frac{d}{t}$ into $d=vt$. In a textbook for Grade 11, in the section entitled "Calculations With Formulas", the authors consider the formula $v=\frac{d}{t}$ and explain how to "solve the formula with respect to the time t" as if the students were complete beginners in elementary algebra (Sandvold et al., 2006, p. 26). The same authors explain how to solve for c the (fabricated) formula $p=a+\frac{1}{2}bc^2$ to arrive at $c=\pm\sqrt{\frac{2-2a}{b}}$. Students are then proposed to solve for t the following formulas: $d=vt$; $d=\frac{1}{2}at^2$; $v=v_0+at$; $d=\frac{(v_0+v)t}{2}$. All this is a limited viaticum for a further journey into elementary algebra.

We will end this quick inventory with two examples touching in various places the frontier on which the didactic transposition of elementary algebra has stopped. In the three-volume textbook by Søgaard and Tambs Lyche (1939/1969), we find this exercise, which follows that on the sphere and the cylinder examined above:

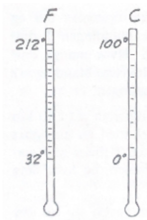


Fig. 17.

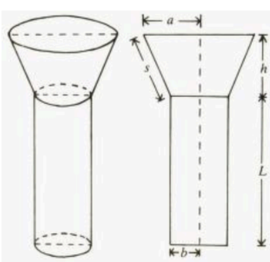
Exercise 231. Look at the figure and explain that we will get the Fahrenheit temperature F expressed in terms of the Celsius temperature C by the formula $F=\frac{180}{100}C+32$.

Using this formula, find C in terms of F.

Take $F=100$, 50, 41, 32, 14, 0, and calculate C in each case.

This exercise on the relationship between two temperature scales (Fahrenheit and Celsius) is a practically and culturally important question, and one should not be surprised to see it appear here. The first part of this exercise looks like a formula *construction* task: the goal is to find φ such that $F=\varphi(C)$. But here two essential pieces of information are given to the student: (1) φ is a linear function, so that there are numbers a and b such that, for any C, we have $F=aC+b$; and (2), the values of a and b are given. It is therefore sufficient to verify that, with these values, we have $\varphi(0)=32$ and $\varphi(100)=212$, which is immediate. The second part is a task of formula *transformation*, without parameters: the student must arrive at $C=\frac{100}{180}(F-32)=\frac{5}{9}(F-32)$. As usual, the last part is an evaluation task. Again, one can emphasize the unadventurous side of what is required of students.

The second and final example is borrowed from a 1984 exercise book already mentioned (Erstad et al., 1984, p. 172):



Figur 1 Figur 2

1157 A chute on a construction site consists of a long cylindrical pipe with a conical funnel on top. Figure 1 shows a drawing of the chute itself. Figure 2 shows a cross section through the central axis of the chute. The magnitudes a, b, s, h, and L are shown in Figure 2. The following measurements are given: $a=1.2$m, $b=0.5$m, $L=3.2$m, and $s=1.7$m.

a) Show that $h=\sqrt{s^2-(a-b)^2}$.

b) Calculate h.

c) The formula for the volume V of the circular cone funnel is given by $V=\frac{1}{3}\pi h(a^2+ab+b^2)$. Find the volume of the whole chute.

d) The surface area is denoted by A. A formula for A is given by

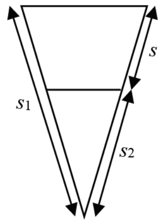
$$A=\frac{\pi a^2 s}{a-b}-\frac{\pi b^2 s}{a-b} \qquad (1)$$

Show by transformation of Formula 1 that it can also be written: $A=\pi(a+b)s$.

e) Calculate the total surface area of the chute.

Questions (a) and (b) are plane geometry questions and simply involve the Pythagorean Theorem. The formula for h is *given* (one arrives at $h\approx 1.549\,\mathrm{m}$), probably to ensure that students will not go astray from this stage onwards. Note that we can check each time a necessary (but not sufficient) condition of correctness of results by making $a=b$: in the figure above, if $a=b$, then $h=s$, in agreement with the fact that, if $a=b$, then $\sqrt{s^2-(a-b)^2}=\sqrt{s^2}=s$. In question (c), the formula giving the volume V of the funnel is given too: if $a=b$, we have $V=\frac{1}{3}\pi h(a^2+ab+b^2)=\frac{1}{3}\pi h(b^2+b^2+b^2)= \pi b^2 h$, which is correct. Students must "construct" a formula giving the volume $\bar{V}$ of the whole chute by adding to V that of a cylinder of radius b and height L, to arrive at $\bar{V}=\frac{1}{3}\pi h(a^2+ab+b^2)+ \pi b^2 L$. (We get $\bar{V}\approx 6.228\,\mathrm{m}^3$.) In the case where $a=b$, we arrive correctly at $\bar{V}=\pi b^2 h+\pi b^2 L=\pi b^2(h+L)$. In question (d), the test made so far ($a=b$) can no longer be formally done, but the algebraic simplification requested from the students—we have: $A=\frac{\pi a^2 s}{a-b}-\frac{\pi b^2 s}{a-b}$ $=\frac{\pi a^2 s-\pi b^2 s}{a-b}=\frac{\pi(a^2-b^2)s}{a-b}=\pi(a+b)s$—allows us to return to a form where it becomes possible again. If $a=b$, then $A=2\pi bs$, which, again, is correct.

The transition from the given expression of A to the requested expression can be seen as the last step in the construction of the resulting formula, which could start from the (given) formula for the surface area of a cone, that is, πrl, where r is the radius of the base and l is the apothem of the cone. The surface area A of the circular cone funnel is the difference between the surface area of the cone of radius $r_1=a$ and apothem s_1 and that of the cone of radius $r_2=b$ and apothem s_2 (where $a>b$): $A=\pi as_1-\pi as_2$ (see figure below).



By considering similar triangles, we have $\frac{s_1}{a}=\frac{s_2}{b}=\frac{s_1-s_2}{a-b}=\frac{s}{a-b}$ so that $s_1=\frac{as}{a-b}$ and $s_2=\frac{bs}{a-b}$ and finally $A=\frac{\pi as}{a-b}-\frac{\pi bs}{a-b}$. Once again, we have to admit that many meaningful algebraic tasks (as this one) are left out: the style of these textbooks is apparently one of algebraic minimalism.

The Pressure of Mathematical Analysis, Again

As already noted, the historical introduction of mathematical analysis produced important effects on the ecology of the mathematics curriculum. Within the algebraic "biome", the ecosystem of formulas was subjected to new conditions and constraints. This fact was early perceived by mathematicians. In 1831, in his treatise *The Elements of the Differential Calculus*, John Radford Young (1799–1885) wrote:

> In algebra we usually employ the first letters, *a*, *b*, *c*, &c. of the alphabet, to represent *known* quantities, and the latter letters, *z*, *y*, *x*, &c. as symbols of the unknown quantities; but, in the higher calculus, the early letters are adopted as the symbols of *constant* quantities, whether they be known or unknown, and the latter letters are used to represent variables. (p. 1)

He adds: "Any analytical expression composed of constants and variables, is said to be a *function* of the variables" (p. 1). We therefore have two types of entities: *equations*, which are part of algebra, and *functions*, which belong to analysis. A process of essentialization is underway: a letter seems now to *intrinsically* represent a known or an unknown quantity in an equation, while in the expression of a function, a letter *intrinsically* represents a constant or a variable. This will be completed by the introduction of the phrases "independent variable" and "dependent variable":

> We thus see, in these two examples, the effect produced on the function by changing the value of the variable, and, on account of this dependence of the value of the function upon that of the

> variable, the former, that is y, is called the *dependent* variable, and the latter, x, the *independent* variable. (Young, 1831, p. 2)

In this line of thought, when a "formula schema", say $a=f(b,c,d)$, is given, we can consider it either as representing a function (where a is the dependent variable, and b, c, and d are the independent variables) or as an equation, once we have chosen the unknown parameter x among a, b, c, and d (if $x=a$, the equation is already solved, if $x=b$, c, or d, we have to "change the subject" of the formula accordingly). This can normally be done because one wishes to use the formula (or one of its transforms) in the study of a certain question. By contrast, the current use of "independent variable" and "dependent variable" as described in a Wikipedia article (below), is indicative of the (faulty) tendency to view variables as *inherently* dependent or independent ("Dependent and Independent Variables", 2021):

> *Dependent* and *independent variables* are variables in mathematical modeling, statistical modeling and experimental sciences. Dependent variables receive this name because, in an experiment, their values are studied under the supposition or hypothesis that they depend, by some law or rule (e.g., by a mathematical function), on the values of other variables. Independent variables, in turn, are not seen as depending on any other variable in the scope of the experiment in question.

At this point a footnote is appended which emphasizes this point: "Even if the existing dependency is invertible (e.g., by finding the inverse function when it exists), the nomenclature is kept if the inverse dependency is not the object of study in the experiment." In the pre-calculus curriculum, a formula could be the source of an equation with its unknown and known quantities or of a function with its independent and dependent variables; but the formula *came first*. On the contrary, the submission of algebra to analysis tended to destroy—by cannibalizing it—the ecosystem of formulas, and, thus, elementary algebra.

This evolution is reflected in the most recent documents we have observed. In a tutorial booklet for teachers of lower secondary school entitled *Guidelines to Algebra*, Brekke et al. (2000) first explain the roles of letters as variables and "generalized numbers", before they go on to explain the roles of parameters and unknowns as follows:

> *Functional Relationships*
>
> These are most often given in an algebraic form, for example, all straight lines can be written in the form $y=ax+b$. Here the letters

> are used in *three* different ways.
>
> • a and b are what we call parameters. They stand for arbitrary numbers, but which are fixed in each case. If you insert different values for a and b, you get different lines.
>
> • Both x and y are variables, but they have different meanings in this expression. For the independent variable, x, one can insert arbitrary numbers and thus calculate the corresponding values of the *dependent* variable, y.
>
> *Equations*
>
> In the work with equations, the letters play a *fourth* role. Here they are no longer variables, but *unknown* numbers which one should find the value of.
>
> • In equations with two unknowns, one does not distinguish between the unknowns x and y. (pp. 9–10)

Here, the "parameters" (a and b) are, so to speak, of a different kind from x and y (they are assigned fixed values, while x can take any value in a range of values), whereas, for example, in the (fictional) formula $d=ac+b$, the letters a, b, c, and d have a priori the same "nature". Note in passing that parameters have vanished from equations. Nowhere is the reconceptualization of elementary algebra in terms of analysis more evident than in the following passage, entitled "Formulas", from a book published in 2020 for Grade 11 (Oldervoll et al., 2020):

> A formula gives us the value of a variable by help of the value of one or several other variables. The volume V of a sphere is given by $\frac{4}{3}\pi r^3$. In this formula, we find the value of V when we know the value of the variable r, which is the radius. The variable r we call the *independent variable* and V we call the *dependent variable*. We choose values for the independent variable and calculate the value of the dependent variable. The formula above also contains the constant $\pi\approx 3.14$. Sometimes we need values of two variables to calculate the third. The volume V of a cylinder is given by $V=\pi r^2 h$. … In this case we have two independent variables and one dependent variable. In most of the formulas we will work on in this book, we will have one independent and one dependent variable. (p. 24)

Here, formulas are fixed, rigid, immobile expressions. They have ceased to be the fuel of algebraic work.

Conditions for a More Authentic Algebra Curriculum

The apparent extinction of formulas as the driving force and the keystone of elementary algebra is all the more remarkable that, at the same time, the official recommendations concerning algebra were trying to promote a seemingly different point of view, if we believe a document published just before the advent of the 21st century (Ministry of Education, Research and Church Affairs, 1996):

> It is crucial for the development of insight into the target area of numbers and algebra that the work with variables and formulas takes place in meaningful contexts. … Increased awareness must be created about the concept of variable itself and about *what formulas and expressions can be used for* [emphasis added]. The topic requires particular attention because it to some extent breaks with previous modes of thinking. The formal side of algebra must have a basis in working with concrete examples. Algebra becomes a tool for solving problems, a language that can facilitate thinking and reasoning, and a source for *discovering new connections* [emphasis added]. (p. 156)

How can we pave the way towards a curricular reconstruction more in line with this official wish? We will answer here in a deliberately compact way, placing ourselves in the framework of the ATD, by stating the main theoretical-technological principles (in the sense of the ATD) on which to base the curricular development work to be undertaken. In a mathematics class, it is both legitimate and essential to study triples ($\mathcal{S}$, Q, $\mathcal{S}'$) composed of a mathematical or extra-mathematical system $\mathcal{S}$, a question Q raised about $\mathcal{S}$, and a model $\mathcal{S}'$ related to the pair ($\mathcal{S}$, Q) which contains mathematical elements playing a key role in the construction of the answer to Q.

Mathematics education is thus potentially concerned with *all* situations in which mathematics is or can be used to better understand and manage the situation in question. In this overarching perspective, it is worthwhile knowing that, from about 1600 to 1800, mathematics was divided into two branches, that of *pure* mathematics and that of *mixed* mathematics (the distinction between pure and *applied* mathematics, that sounds so familiar to us, only began to emerge in earnest in the 19th century). Here is how Francis Bacon (1561–1626) presented this distinction in *The Advancement of Learning* (1605/1901):

> Mathematics is either pure or mixed. To the pure belong the sciences employed about quantity, wholly abstracted from matter and physical axioms. This has two parts—geometry and

> arithmetic; the one regarding continued, and the other discrete quantity. … Mixed mathematics has for its subject axioms and the parts of physics, and considers quantity so far as may be assisting to illustrate, demonstrate, and actuate those; for without the help of mathematics many parts of nature could neither be sufficiently comprehended, clearly demonstrated, nor dexterously fitted for use. And of this kind are perspective, music, astronomy, cosmography, architecture, and mechanics. (pp. 172–174)

So, what key conditions should school algebra $\mathcal{A}^*$ meet to be faithful to scholarly (elementary) algebra $\mathcal{A}$? By way of a conclusion, we shall sum up the core of a more "authentic" study and use of algebra identified in the course of our inquiry:

1. The students—whoever they are—start from a system $\mathcal{S}$ and a question Q raised about it, whose adequate treatment seems to involve mathematical elements;

2. These students build up a model $\mathcal{S}'$ of $\mathcal{S}$, relative to the question Q, which will be built with elementary algebra (and will include as many parameters as seems useful!);

3. They work on $\mathcal{S}'$ to derive an answer deemed adequate to Q;

4. At the same time, prompted by this process of inquiring about $\mathcal{S}$, they discover the resources of algebra, study or restudy them in order to make an appropriate and efficient use of the tools thus garnered.

These points outline a *research and innovation programme* to which the present study is, in our view, a seminal contribution in order to help develop, in the decade to come, the full collaboration of researchers, teachers, and teacher educators.

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