

ON STRATIFICATION FOR SPACES WITH NOETHERIAN MOD p COHOMOLOGY

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ABSTRACT. Let X be a topological space with Noetherian mod p cohomology and let $C^*(X; \mathbb{F}_p)$ be the commutative ring spectrum of \mathbb{F}_p -valued cochains on X . The goal of this paper is to exhibit conditions under which the category of module spectra over $C^*(X; \mathbb{F}_p)$ is stratified in the sense of Benson, Iyengar, and Krause, providing a classification of all its localizing subcategories. We establish stratification in this sense for classifying spaces of a large class of topological groups including Kac–Moody groups as well as whenever X admits an H -space structure. This in particular identifies the corresponding Balmer spectrum and proves the generalized telescope conjecture in all of these cases. More generally, using Lannes’ theory we prove that stratification for X is equivalent to a condition that generalizes Chouinard’s theorem for finite groups.

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INTRODUCTION

Background and motivation. In a series of papers [BIK08, BIK11b] culminating in [BIK11c], Benson, Iyengar, and Krause classified all modular representations of a finite group G up to a certain equivalence relation: Two representations are considered equivalent if they can be constructed from each other via the tensor-triangulated structure available on the stable module category $\text{StMod}_{\mathbb{F}_p G}$ of G . More precisely, they show that the localizing tensor-ideals of $\text{StMod}_{\mathbb{F}_p G}$ are parametrized by subsets of the projective variety of the group cohomology ring $H^*(G; \mathbb{F}_p)$.

Earlier, a similar classification theorem was obtained by Neeman [Nee92] for the derived category of a Noetherian commutative ring A ; in this case, localizing subcategories are parametrized by the Zariski spectrum of A . Both classification problems admit a common formulation in the setting of stable homotopy theory, as noticed first by Hovey, Palmieri, and Strickland [HPS97]: Given a commutative ring spectrum R , we may ask when the localizing subcategories of the category Mod_R of R -module spectra are parametrized by the Zariski spectrum $\text{Spec}(\pi_* R)$ via certain support functions. When this is the case, we follow Benson, Iyengar, and Krause and say that Mod_R is stratified or simply that stratification holds for R . It turns out that, in general, stratification is only reasonable to expect when R is Noetherian, i.e., when $\pi_* R$ is a graded Noetherian ring, so we will work under this assumption from now on.

Date: May 29, 2019.

Neeman’s theorem is then the special case that $R = HA$, the Eilenberg–Mac Lane ring spectrum of a Noetherian commutative ring A . Another important source of Noetherian commutative ring spectra comes from unstable homotopy theory, given by commutative ring spectra of \mathbb{F}_p -valued cochains $C^*(X; \mathbb{F}_p)$ on a topological space X with Noetherian mod p cohomology. In this language, the theorem of Benson, Iyengar, and Krause establishes stratification for cochains on $X = BG$, the classifying space of a finite group G , see [BIK11a]. Their work has subsequently been extended by Benson and Greenlees [BG14] and more recently in [BCHV17] for classifying spaces of certain topological groups as well as by Shamir [Sha12] for certain spaces constructed from iterated spherical fibrations. This motivates the following guiding question:

Question. *For which spaces X with Noetherian mod p cohomology ring is the category $\text{Mod}_{C^*(X; \mathbb{F}_p)}$ of module spectra over the commutative ring spectrum of cochains $C^*(X; \mathbb{F}_p)$ stratified?*

The goal of the present paper is to establish stratification in this sense for cochains on a large class of topological groups as well as for H -spaces with Noetherian mod p cohomology, and to provide a systematic approach to answering this question in general.

Main results. We first consider commutative ring spectra of cochains on classifying spaces of topological groups, extending our previous work in [BCHV17]. By reducing to the case of compact Lie groups via work of Broto and Kitchloo [BK02], we in particular establish stratification for Kac–Moody groups.

Theorem A (Theorem 3.17). *If K is a Kac–Moody group, then $\text{Mod}_{C^*(BK; \mathbb{F}_p)}$ is stratified.*

Hopf spaces with Noetherian mod p cohomology form the second class of examples. Our proof of stratification in this case relies crucially on several deep results in unstable homotopy theory stemming from the resolution of the Sullivan conjecture, as outlined below. In particular, we make use of a structure theorem for H -spaces due to Broto, Crespo, and Saumell [BCS01] and Castellana, Crespo, and Scherer [CCS07].

Theorem B (Theorem 4.17). *If X is a connected H -space with Noetherian mod p cohomology, then $\text{Mod}_{C^*(X; \mathbb{F}_p)}$ is stratified.*

Finally, we formulate an approach to answering the above question for any space with Noetherian mod p cohomology, based again on Lannes’ theory [Lan92] and its consequences. This allows us to reduce this categorical classification problem to a condition that suitably generalizes Chouinard’s theorem [Cho76] about projective representations for finite group; the terminology will be introduced more carefully in the next subsection.

Theorem C (Theorem 5.12). *Let X be a p -good connected space with Noetherian mod p cohomology, then $\text{Mod}_{C^*(X; \mathbb{F}_p)}$ is stratified if and only if X satisfies Chouinard’s condition.*

In particular, we verify Chouinard’s condition for H -spaces with Noetherian mod p cohomology, thereby recovering Theorem B as an instance of Theorem C.

Whenever Mod_R is stratified we can in fact deduce more consequences than just a classification of the localizing subcategories of Mod_R , see Theorems 1.9 and 1.10 below. For example, stratification implies that the thick subcategories of compact objects in Mod_R are in bijection with specialization closed subsets of $\text{Spec}^h(\pi_* R)$, and hence identifies the Balmer spectrum [Bal05] of Mod_R as

$$\text{Spc}(\text{Mod}_R) \cong \text{Spec}^h(\pi_* R).$$

In particular, we thus determine the Balmer spectrum of $\text{Mod}_{C^*(X; \mathbb{F}_p)}$ whenever X is a connected H -space with Noetherian mod p cohomology, or when X is the classifying space of a Kac–Moody group.

In addition, stratification implies a version of Ravenel’s telescope conjecture [Rav92] for Mod_R , see [Bar19] for a recent survey. In our context, the generalized telescope conjecture amounts to the statement that the subcategory of acyclics of any localization functor on Mod_R which preserves filtered colimits is compactly generated. We thus obtain the generalized telescope conjecture for all the aforementioned types of spaces.

Outline of the methods. From our perspective, stable homotopy theory provides a convenient context for developing an abstract descent theory for stratifications that places the results mentioned above in a uniform setting. Consider thus a morphism $f: R \rightarrow S$ of Noetherian commutative ring spectra, i.e., homotopically coherent ring spectra with Noetherian graded commutative coefficient rings. Inspired by [BIK11c] and [BG14], in [BCHV17] we isolated two key conditions on f that guarantee that stratification descends from Mod_S to Mod_R along f :

- (1) Induction and coinduction along f give conservative functors $\text{Mod}_R \rightarrow \text{Mod}_S$, i.e., they reflect isomorphisms. In this case, we say that f is biconservative.
- (2) f satisfies a support theoretic condition called Quillen lifting.

Geometrically, these conditions correspond, roughly speaking, to the surjectivity and injectivity of the induced map on homogeneous Zariski spectra $\text{res}_f: \text{Spec}^h(\pi_* S) \rightarrow \text{Spec}^h(\pi_* R)$.

In [BCHV17], we showed that Conditions (1) and (2) are sufficient to descent stratification from Mod_S to Mod_R . The key novelty in this result was that f was not assumed to satisfy any additional finiteness conditions, in contrast to the situation studied in the aforementioned works [BIK11c, BG14]. This opened up the possibility to apply such descent techniques to many new examples.

Our first aim in this paper is to characterize those biconservative morphisms $f: R \rightarrow S$ that allow descent of stratification. In particular, we complement [BCHV17] by showing that Condition (2) is also necessary for descent; in fact, we exhibit several equivalent characterizations of this condition. Informally speaking, this shows that our descent theorem is optimal:

Theorem D (Theorem 2.4 and Corollary 2.6). *Let $f: R \rightarrow S$ be a biconservative morphism of Noetherian commutative ring spectra and assume Mod_S is stratified, then the following are equivalent:*

- (1) Mod_R is stratified.
- (2) The morphism f satisfies Quillen lifting.
- (3) Every module $M \in \text{Mod}_R$ satisfies the following Avrunin–Scott identities:

$$\text{supp}_S(f^* M) = \text{res}_f^{-1} \text{supp}_R(M) \quad \text{and} \quad \text{cosupp}_S(f^! M) = \text{res}_f^{-1} \text{cosupp}_R(M),$$

where f^* and $f^!$ denote induction and coinduction along f , respectively.

Given a commutative ring spectrum R , this theorem thus reduces the problem of stratifying Mod_R to the construction of a suitable morphism $f: R \rightarrow S$. By analogy with Chouinard’s theorem for discrete groups, for the remainder of this introduction we will say that a morphism $f: R \rightarrow S$ satisfies Chouinard’s condition¹ whenever f is biconservative.

To prove Theorem D, we introduce and study a stronger version of Quillen lifting called simple Quillen lifting that in practice is easier to check for a given example. Furthermore, we relate both Chouinard’s condition as well as simple Quillen lifting to properties of the induced morphism

$$\text{res}_f: \text{Spec}^h(\pi_* S) \longrightarrow \text{Spec}^h(\pi_* R).$$

Our main theorems are then an application of this general descent theorem; however, in each case additional input from unstable homotopy theory is required. Theorem A follows the line of arguments in our previous paper [BCHV17] and earlier work of Benson and Greenlees [BG14] on

¹Note that this terminology is reserved for a more restrictive situation in the main body of the paper.

stratification for topological groups G . The idea is to descent stratification along a morphism of commutative ring spectra

$$\rho_G: C^*(BG; \mathbb{F}_p) \longrightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE; \mathbb{F}_p),$$

where the product is indexed on a set of representatives of conjugacy classes of elementary abelian p -subgroups of G . By virtue of Theorem D, it suffices to show that ρ_G satisfies Quillen lifting and Chouinard’s condition. In the case of a Kac–Moody group, the first property follows from a generalization of Quillen’s F -isomorphism to Kac–Moody groups due to Broto and Kitchloo [BK02], see Theorem 3.10. The verification of Chouinard’s condition for ρ_G uses the construction of an appropriate compactly generated G -space X equipped with specified isotropy, which may be thought of as a suitable replacement of the morphism $BG \rightarrow BU(n)$ induced by a unitary embedding of a compact Lie group G . In fact, our methods apply more generally to spaces occurring in the Broto–Kitchloo hierarchy of compact Lie groups [KM98], see Theorem 3.14.

We would like to mimic this approach for an arbitrary connected space X with Noetherian mod p cohomology. However, we are immediately confronted with the problem of finding a homotopical analogue of the morphism ρ_G for X , which in turn relies on finding a replacement of the notion of elementary abelian subgroup for X . We resolve this problem by making use of Lannes’ [Lan87, Lan92] and Miller’s [Mil84] work on the Sullivan conjecture to construct a morphism of commutative ring spectra

$$\rho_X: C^*(X; \mathbb{F}_p) \longrightarrow \prod_{(E, \varphi) \in \mathcal{E}(X)} C^*(BE; \mathbb{F}_p).$$

The product in the target is indexed on homotopy classes of maps $\varphi: BE \rightarrow X$ which induce finite algebra morphisms in mod p cohomology.² For $X = BG$ the classifying space of a finite p -group, ρ_X specializes to the restriction functor featuring in the work of Chouinard, Quillen, as well as Benson–Iyengar–Krause. We thus view ρ_X as a suitable catalyst for our descent techniques; in particular, we say that X satisfies Chouinard’s condition if ρ_X does.

Lannes’ theory provides a bijection between the set $\mathcal{E}(X)$ appearing in the target of ρ_X and isomorphism classes of objects in the Rector category of $H^*(X; \mathbb{F}_p)$, a category which generalizes Quillen’s category for $X = BG$. An elaboration of this observation combined with Rector’s work [Rec84] then proves that ρ_X satisfies Quillen lifting for any p -good connected space X with Noetherian mod p cohomology, see Proposition 5.8. In light of Theorem D, this establishes Theorem C.

We then demonstrate this framework in the case of H -spaces X with Noetherian mod p cohomology. By Theorem C, in order to prove stratification for $\text{Mod}_{C^*(X; \mathbb{F}_p)}$, it remains to show that ρ_X satisfies Chouinard’s condition. To this end, we make use of a theorem due to Broto, Crespo, and Saumell [BCS01] and Castellana, Crespo, and Scherer [CCS07], according to which any H -space X with Noetherian mod p cohomology may be “decomposed”, after p -completion, into the classifying space of an abelian compact Lie group and a nilpotent space with finite mod p -cohomology. Combined with work of Henn, Lannes, and Schwartz [HLS93], we arrive at:

Theorem E (Theorem 5.15). *Any connected H -space X with Noetherian mod p cohomology satisfies Chouinard’s condition, i.e., induction and coinduction along ρ_X are conservative.*

Theorem B is now an immediate consequence of Theorem C and Theorem E. In fact, we also give a proof of Theorem B that does not rely on Theorem E, but instead applies Theorem C directly to the fiber sequence provided by the deconstruction of H -spaces, see the proof of Theorem 4.17. In particular, our result applies to $S^3\langle 3 \rangle$, the 3-connected cover of S^3 , to provide an example of a commutative ring spectrum for which we deduce stratification, but are currently unable to prove costratification.

²For simplicity, we omit p -completions throughout the introduction.

Organization of the paper. The first two sections deal with the interaction between the stratification theory of Benson–Iyengar–Krause and morphisms of commutative ring spectra, culminating in the proof of our main descent theorem. Section 1 starts with a review of support theory and then studies base-change and conservativity in this context. The subject of Section 2 is the proof of Theorem D. Along the way, we establish the main properties of Quillen lifting and simple Quillen lifting, augmented by some examples.

From Section 3 onwards, we specialize to cochains on spaces with Noetherian mod p cohomology. Theorems A, B, and C are proven in Sections 3, 4, and 5, respectively. Each section begins with a review of the necessary background material and concludes with explicit examples.

Notation and conventions.

- Let R be a commutative ring spectrum and write Mod_R for the category of module spectra over R , where the symmetric monoidal structure is given by the relative smash product $\otimes = \otimes_R$ with unit R . If \mathcal{S} is a set of objects in Mod_R , then the smallest thick subcategory of Mod_R containing \mathcal{S} will be denoted by $\text{Thick}_R(\mathcal{S})$, and the smallest localizing subcategory of Mod_R containing \mathcal{S} will be denoted $\text{Loc}_R(\mathcal{S})$. If R is clear from context, we sometimes omit the corresponding subscript.
- We write $H^*(X)$ for the mod p cohomology of a space X .
- The p -completion of a space X is always meant in the sense of Bousfield–Kan [BK72], and is denoted X_p^\wedge . A space X is called p -complete if the p -completion map $X \rightarrow X_p^\wedge$ is a homotopy equivalence, and is called \mathbb{F}_p -finite if $H^*(X)$ is finite. A space is p -good if the natural completion map $X \rightarrow X_p^\wedge$ induces an isomorphism on mod p cohomology.
- We write $\text{res}_f: \text{Spec}^h(\pi_*B) \rightarrow \text{Spec}^h(\pi_*A)$ for the restriction induced by a morphism of commutative ring spectra $f: A \rightarrow B$, where Spec^h denotes the homogeneous spectrum of prime ideals.
- A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be conservative if it detects equivalences; if F is an exact functor between stable categories, then this is equivalent to F detecting the zero object, i.e., $F(X) \simeq 0$ implies $X \simeq 0$ for any $X \in \mathcal{C}$.

Acknowledgements. We would like to thank Jesper Grodal and Julia Pevtsova for useful conversations. The first author was partially supported by the DNRF92 and the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 751794, the second author was partially supported by FEDER-MEC grant MTM2016-80439-P and acknowledges financial support from the Spanish Ministry of Economy and Competitiveness through the “María de Maeztu” Programme for Units of Excellence in R&D (MDM-2014-0445), the third author was partially supported by the SFB 1085 “Higher Invariants”, and the fourth author would like to thank the Max Planck Institute for Mathematics for its hospitality. The first and second author would furthermore like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Homotopy Harnessing Higher Structures*, where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

1. SUPPORT AND BASE-CHANGE

In this section we review the support theory introduced by Benson, Iyengar, and Krause [BIK08], specialized to the setting of structured ring spectra, and show how this support theory interacts with a morphism $f: R \rightarrow S$ of ring spectra.

1.1. Support theory and stratification. Let R be a Noetherian commutative ring spectrum and fix a homogeneous prime ideal $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$. Associated to \mathfrak{p} , Benson, Iyengar, and Krause [BIK08, BIK12] have constructed local cohomology and local homology functors on Mod_R ,

denoted $\Gamma_{\mathfrak{p}}$ and $\Lambda^{\mathfrak{p}}$ respectively. In the context of structured ring spectra, these have been studied in detail in [BHV18, BCHV17], from where we briefly recall their definition. For $\mathfrak{r} \in \mathrm{Spec}^h(\pi_*R)$, one first constructs Koszul objects $R//\mathfrak{r}$ [BCHV17, Section 3], the homotopical quotient of R by \mathfrak{r} . This object depends on a choice of generators of \mathfrak{r} , but the thick subcategory $\mathrm{Thick}_R(R//\mathfrak{r})$ generated by $R//\mathfrak{r}$ does not. Consider then the specialization closed set $\mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}^h(\pi_*R) \mid \mathfrak{q} \supseteq \mathfrak{p}\}$. The inclusion of the localizing subcategory $\mathrm{Loc}_R(R//\mathfrak{r} \mid \mathfrak{r} \in \mathcal{V}(\mathfrak{p}))$ into Mod_R has a colimit preserving right adjoint $\Gamma_{\mathcal{V}(\mathfrak{p})}$. Moreover, there is a localization functor $L_{\mathcal{Z}(\mathfrak{p})}$ on Mod_R characterized by the property $\pi_*L_{\mathcal{Z}(\mathfrak{p})}M \cong (\pi_*M)_{\mathfrak{p}}$ for all $M \in \mathrm{Mod}_R$.

Definition 1.1. For $\mathfrak{p} \in \mathrm{Spec}^h(\pi_*R)$ we define the local cohomology functor at \mathfrak{p} as

$$\Gamma_{\mathfrak{p}} = \Gamma_{\mathcal{V}(\mathfrak{p})}L_{\mathcal{Z}(\mathfrak{p})}.$$

The functor $\Gamma_{\mathfrak{p}}$ admits a right adjoint, the local homology functor $\Lambda^{\mathfrak{p}}$ at \mathfrak{p} .

Remark 1.2. The functor $\Gamma_{\mathfrak{p}}$ is smashing, that is, $\Gamma_{\mathfrak{p}}M \simeq \Gamma_{\mathfrak{p}}R \otimes M$ for all R -modules M . In what follows we will denote the essential image of $\Gamma_{\mathfrak{p}}$ by $\Gamma_{\mathfrak{p}}\mathrm{Mod}_R$, and likewise for $\Lambda^{\mathfrak{p}}$.

We refer the reader to the aforementioned references for a more detailed construction and further properties. These functors give rise to a good notion of support and cosupport for R -modules as follows.

Definition 1.3. Let M be an R -module, then we define the support and cosupport of M as

$$\mathrm{supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}^h(\pi_*R) \mid \Gamma_{\mathfrak{p}}M \neq 0\}, \quad \mathrm{cosupp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}^h(\pi_*R) \mid \Lambda^{\mathfrak{p}}M \neq 0\}.$$

The next theorem provides some evidence for the usefulness of these functors, see [BIK11b, Theorem 7.2] and [BIK12, Remark 8.8].

Theorem 1.4 (Benson–Iyengar–Krause). *If R is Noetherian, then the local-to-global principle holds for R , i.e., for every $M \in \mathrm{Mod}_R$ we have*

$$\mathrm{Loc}_R(M) = \mathrm{Loc}_R(\Gamma_{\mathfrak{p}}M \mid \mathfrak{p} \in \mathrm{Spec}^h(\pi_*R)), \quad \mathrm{Coloc}_R(M) = \mathrm{Coloc}_R(\Lambda^{\mathfrak{p}}M \mid \mathfrak{p} \in \mathrm{Spec}^h(\pi_*R)).$$

In particular, M is trivial if and only if $\mathrm{supp}_R(M) = \emptyset$ if and only if $\mathrm{cosupp}_R(M) = \emptyset$.

We will use the fact that both support and cosupport detect zero objects implicitly throughout the remainder of this paper.

Support theory is used to give a classification of the localizing subcategories of Mod_R . We note that there is a pair of natural morphisms

$$\left\{ \begin{array}{c} \text{Localizing subcategories} \\ \text{of } \mathrm{Mod}_R \end{array} \right\} \begin{array}{c} \xrightarrow{\mathrm{supp}_R} \\ \xleftarrow{\mathrm{supp}_R^{-1}} \end{array} \left\{ \text{Subsets of } \mathrm{Spec}^h(\pi_*R) \right\}, \quad (1.5)$$

where for a localizing subcategory $\mathcal{S} \subseteq \mathrm{Mod}_R$ and a subset $\mathcal{U} \subseteq \mathrm{Spec}^h(\pi_*R)$ we set

$$\mathrm{supp}_R(\mathcal{S}) = \bigcup_{M \in \mathcal{S}} \mathrm{supp}_R(M) \quad \text{and} \quad \mathrm{supp}_R^{-1}(\mathcal{U}) = \{M \in \mathrm{Mod}_R \mid \mathrm{supp}_R(M) \subseteq \mathcal{U}\}.$$

Benson, Iyengar, and Krause [BIK11b] exhibit conditions to ensure that the maps in (1.5) are bijections. To state their definition, recall that a localizing subcategory \mathcal{D} of Mod_R is said to be minimal if \mathcal{D} has no proper localizing subcategories.

Definition 1.6. Let R be a Noetherian commutative ring spectrum. We say that Mod_R is (canonically) stratified if for each $\mathfrak{p} \in \mathrm{Spec}^h(\pi_*R)$, the localizing subcategory $\Gamma_{\mathfrak{p}}\mathrm{Mod}_R$ is minimal.

Remark 1.7. This is slightly different from the definition for triangulated categories given in [BIK11b], where they require that the local to global principle holds and that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is minimal or non-zero. As seen in Theorem 1.4 the local to global principle always holds for Mod_R when R is Noetherian, while in [BCHV17, Proposition 2.13(1)] we show that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is always non-zero. Since we will only consider the canonical action of $\pi_* R$ on Mod_R , we will from now on omit the adjective canonical when referring to stratification.

The next result demonstrates the usefulness of this definition, see [BIK11b, Theorem 4.2] for the proof.

Theorem 1.8 (Benson–Iyengar–Krause). *If Mod_R is stratified, then the maps (1.5) are inclusion preserving bijections.*

When Mod_R is stratified, there are a number of further consequences one can deduce, see [BIK11b, Theorems 1.2, 1.3 and 1.6].

Theorem 1.9 (Benson–Iyengar–Krause). *If Mod_R is stratified, then the following results hold:*

- (1) *There is a bijection between thick subcategories of compact objects in Mod_R , and specialization closed subsets of $\text{Spec}^h(\pi_* R)$.*
- (2) *For any $M, N \in \text{Mod}_R$, we have*

$$\text{supp}_R(M \otimes_R N) = \text{supp}_R(M) \cap \text{supp}_R(N)$$

and if M and N are additionally compact R -modules

$$\text{supp}_R(\text{Hom}_R(M, N)) = \text{supp}_R(M) \cap \text{supp}_R(N).$$

As a further consequence, the telescope conjecture hold for Mod_R , see [BIK11b, Theorem 6.3] and [BIK11c, Theorem 11.13].

Theorem 1.10 (Benson–Iyengar–Krause). *Suppose Mod_R is stratified by $\pi_* R$. Let \mathcal{U} be a localizing subcategory of Mod_R . Then the following conditions are equivalent.*

- (1) *The localizing subcategory \mathcal{U} is smashing, i.e., the associated localizing endofunctor on Mod_R preserves colimits.*
- (2) *The localizing subcategory \mathcal{U} is generated by compact objects in Mod_R .*
- (3) *The support of \mathcal{U} is specialization closed, i.e., it is a union of Zariski closed subsets of $\text{Spec}^h(\pi_* R)$.*

1.2. Base-change and conservative maps of ring spectra. A remarkable result proven by Chouinard [Cho76] is that the projectivity of a module over the group ring of a finite group G can be tested at all elementary abelian p -subgroup of G combined. This can be interpreted as a way to detect trivial objects when working in the stable module category of G . In this section, we study homotopical analogues of this result in the setting of commutative ring spectra. We do so by expressing the detection of trivial objects as a conservativity property of certain functors that we introduce hereunder.

Let $f: R \rightarrow S$ be a morphism of commutative ring spectra and view S as an R -module via f . Forgetting along f induces a restriction functor $f_* = \text{Res}: \text{Mod}_S \rightarrow \text{Mod}_R$ which admits both a left adjoint f^* and a right adjoint $f^!$, given by induction (or extension of scalars) along f ,

$$f^* = \text{Ind} = S \otimes_R (-): \text{Mod}_R \longrightarrow \text{Mod}_S,$$

and coinduction, defined as

$$f^! = \text{Coind} = \text{Hom}_R(S, -): \text{Mod}_R \longrightarrow \text{Mod}_S.$$

Note that restriction detects equivalences, i.e., a map $\alpha: N_1 \rightarrow N_2$ between S -modules is an equivalence if and only if $f_*(\alpha)$ is. It follows that f^* or $f^!$ are conservative if and only if the

composites f_*f^* or $f_*f^!$ are, respectively. We capture this property by introducing the following terminology:

Definition 1.11. A map $f: R \rightarrow S$ of commutative ring spectra is called conservative (resp. co-conservative) if the associated induction functor $f^*: \text{Mod}_R \rightarrow \text{Mod}_S$ (resp. coinduction $f^!$) is conservative. If a map is both conservative and coconservative, it will be called biconservative.

In [BCHV17] we studied the base-change properties of support and cosupport along a morphism of ring spectra $f: R \rightarrow S$. We single out the following results, where we let

$$\text{res}_f: \text{Spec}^h(\pi_*S) \longrightarrow \text{Spec}^h(\pi_*R)$$

denote the map induced by f on $\text{Spec}^h(\pi_*(-))$. The first two items of the next result are proven in [BCHV17, Corollary 3.9 and Proposition 3.12], while the third one strengthens [BCHV17, Lemma 3.11].

Theorem 1.12. *Let $f: R \rightarrow S$ be a map of commutative ring spectra, $M \in \text{Mod}_R$ and $N \in \text{Mod}_S$, then we have:*

- (1) $\text{res}_f \text{supp}_S(f^*M) \subseteq \text{supp}_R(M)$, and equality holds if f is conservative.
- (2) $\text{supp}_R(f_*N) = \text{res}_f \text{supp}_S(N)$ and $\text{cosupp}_R(f_*N) = \text{res}_f \text{cosupp}_S(N)$.
- (3) $\text{res}_f \text{cosupp}_S(f^!M) \subseteq \text{cosupp}_R(M)$, and equality holds if f is coconservative.

Proof. In light of the references provided above, it remains to verify the last claim. The inclusion “ \subseteq ” holds unconditionally, so it suffices to establish “ \supseteq ” when f is coconservative. Let $M \in \text{Mod}_R$ and $\mathfrak{p} \in \text{cosupp}_R(M)$. By adjunction, there are equivalences of S -modules

$$0 \neq f^! \Lambda^{\mathfrak{p}} M \simeq f^! \text{Hom}_R(R, \Lambda^{\mathfrak{p}} M) \simeq f^! \text{Hom}_R(\Gamma_{\mathfrak{p}} R, M) \simeq \text{Hom}_S(f^* \Gamma_{\mathfrak{p}} R, f^! M).$$

The last equivalence can be found, for example, in Proposition 2.15 of [BDS16], see their Equation (2.19). Let \mathcal{V} and \mathcal{W} be specialization closed subsets of $\text{Spec}^h(\pi_*R)$ such that $\mathcal{V} \setminus \mathcal{W} = \{\mathfrak{p}\}$ and write $\tilde{\mathcal{V}} = \text{res}_f^{-1}(\mathcal{V})$ and $\tilde{\mathcal{W}} = \text{res}_f^{-1}(\mathcal{W})$; note in particular that $\tilde{\mathcal{V}} \setminus \tilde{\mathcal{W}} = \text{res}_f^{-1}(\mathfrak{p})$. It then follows from Corollary 3.8 in [BCHV17] and adjunction that

$$\text{Hom}_S(f^* \Gamma_{\mathfrak{p}} R, f^! M) \simeq \text{Hom}_S(L_{\tilde{\mathcal{W}}} \Gamma_{\tilde{\mathcal{V}}} f^* R, f^! M) \simeq \text{Hom}_S(S, \Lambda^{\tilde{\mathcal{V}}} \Delta^{\tilde{\mathcal{W}}} f^! M) \simeq \Lambda^{\tilde{\mathcal{V}}} \Delta^{\tilde{\mathcal{W}}} f^! M.$$

By Lemma 3.4 of [BCHV17], the cosupport of this expression can thus be computed to be

$$\emptyset \neq \text{cosupp}_S(f^! \Lambda^{\mathfrak{p}} M) = \text{cosupp}_S(\Lambda^{\tilde{\mathcal{V}}} \Delta^{\tilde{\mathcal{W}}} f^! M) = \text{res}_f^{-1}(\mathfrak{p}) \cap \text{cosupp}_S(f^! M).$$

This implies that $\mathfrak{p} \in \text{res}_f \text{cosupp}_S(f^! M)$ as desired. \square

Lemma 1.13. *Fix a prime ideal $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$. If $f: R \rightarrow S$ is a map of commutative ring spectra such that induction $f^*: \text{Mod}_R \rightarrow \text{Mod}_S$ considered as a functor on $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is conservative (resp. coinduction as a functor on $\Lambda^{\mathfrak{p}} \text{Mod}_R$ is conservative), then \mathfrak{p} is in the image of $\text{res}_f: \text{Spec}^h(\pi_*S) \rightarrow \text{Spec}^h(\pi_*R)$.*

Proof. Using Theorem 1.12(1) and (2), we get

$$\text{supp}_R(f_* f^* \Gamma_{\mathfrak{p}} R) = \text{res}_f \text{supp}_S(f^* \Gamma_{\mathfrak{p}} R) \subseteq \text{supp}_R(\Gamma_{\mathfrak{p}} R) = \{\mathfrak{p}\}.$$

Now suppose that \mathfrak{p} is not in the image of res_f , then $f_* f^* \Gamma_{\mathfrak{p}} R \simeq 0$. Therefore, $f^* \Gamma_{\mathfrak{p}} R \simeq 0$, so $f^*: \Gamma_{\mathfrak{p}} \text{Mod}_R \rightarrow \text{Mod}_S$ is not conservative. The version for coinduction follows similarly from Theorem 1.12(3). \square

Under the assumption that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is a minimal localizing subcategory of Mod_R , the converse holds as well, as we will show after some preparation in Lemma 1.17 below.

Lemma 1.14. *Let $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$ such that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is minimal. Then, for all $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$ and all $N \in \text{Mod}_R$,*

$$\text{supp}_R(M \otimes N) = \text{supp}_R(M) \cap \text{supp}_R(N).$$

Dually, for all $M \in \text{Mod}_R$ and all $N \in \Lambda^{\mathfrak{p}} \text{Mod}_R$

$$\text{cosupp}_R(\text{Hom}_R(M, N)) = \text{supp}_R(M) \cap \text{cosupp}_R(N).$$

Proof. Since $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$, we have

$$\text{supp}_R(M \otimes N) \subseteq \text{supp}_R(M) \cap \text{supp}_R(N) \subseteq \{\mathfrak{p}\}.$$

Under the minimality assumption on $\Gamma_{\mathfrak{p}} \text{Mod}_R$, the same argument as in the proof of [BIK11b, Theorem 7.3] implies that if $\text{supp}_R(M) \cap \text{supp}_R(N) = \{\mathfrak{p}\}$ then $\text{supp}_R(M \otimes N) = \{\mathfrak{p}\}$, which proves the first claim.

Likewise, the second equality follows from the proof of [BIK12, Theorem 9.5]. \square

The next result is a local analogue of [BCHV17, Proposition 3.14].

Proposition 1.15. *Let $f: R \rightarrow S$ be a morphism of Noetherian commutative ring spectra and suppose that $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$ is such that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is minimal. Then, for all $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$,*

$$\text{supp}_S(f^*M) = \text{res}_f^{-1} \text{supp}_R(M), \quad \text{while} \quad \text{cosupp}_S(f^!N) = \text{res}_f^{-1} \text{cosupp}_R(N),$$

for all $N \in \Lambda^{\mathfrak{p}} \text{Mod}_R$.

Proof. Let $\mathfrak{q} \in \text{Spec}^h(\pi_*S)$ with $\mathfrak{p} = \text{res}_f(\mathfrak{q})$. First, we can use the projection formula [BCHV17, Lemma 2.2] to write

$$f_*\Gamma_{\mathfrak{q}}f^*M \simeq f_*(\Gamma_{\mathfrak{q}}S \otimes_S f^*M) \simeq f_*(\Gamma_{\mathfrak{q}}S) \otimes_R M.$$

By Theorem 1.12 and [BCHV17, Proposition 2.13(1)] we have

$$\text{supp}_R(f_*\Gamma_{\mathfrak{q}}S) = \text{res}_f \text{supp}_S(\Gamma_{\mathfrak{q}}S) = \{\text{res}_f(\mathfrak{q})\} = \{\mathfrak{p}\}. \quad (1.16)$$

Since $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$, Lemma 1.14 applies in this situation:

$$\text{supp}_R(f_*(\Gamma_{\mathfrak{q}}S) \otimes_R M) = \text{supp}_R(f_*\Gamma_{\mathfrak{q}}S) \cap \text{supp}_R(M) \subseteq \{\mathfrak{p}\}.$$

It follows that $f_*\Gamma_{\mathfrak{q}}f^*M \not\cong 0$ if and only if $\mathfrak{p} \in \text{supp}_R(M)$. But f_* is conservative, so we conclude that $\mathfrak{q} \in \text{supp}_S(f^*M)$ if and only if $\mathfrak{p} \in \text{supp}_R(M)$.

For the second claim, note that by adjunction there are equivalences of R -modules

$$\begin{aligned} \Lambda^{\mathfrak{q}}f^!N &\simeq \text{Hom}_S(S, \Lambda^{\mathfrak{q}}f^!N) \\ &\simeq \text{Hom}_S(\Gamma_{\mathfrak{q}}S, f^!N) \\ &\simeq \text{Hom}_R(f_*\Gamma_{\mathfrak{q}}S, N). \end{aligned}$$

But we are assuming that $N \in \Lambda^{\mathfrak{p}} \text{Mod}_R$, so Lemma 1.14 yields

$$\text{cosupp}_R(\text{Hom}_R(f_*\Gamma_{\mathfrak{q}}S, N)) = \text{supp}_R(f_*\Gamma_{\mathfrak{q}}S) \cap \text{cosupp}_R(N) = \{\mathfrak{p}\} \cap \text{cosupp}_R(N),$$

where we used (1.16). We conclude that $\mathfrak{q} \in \text{cosupp}_S(f^!N)$ if and only if $\mathfrak{p} \in \text{cosupp}_R(N)$. \square

Lemma 1.17. *Fix a prime ideal $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$, let $f: R \rightarrow S$ be a map of Noetherian commutative ring spectra, and assume that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is minimal. Then the following three conditions are equivalent:*

- (1) *Induction $f^*: \text{Mod}_R \rightarrow \text{Mod}_S$ restricted to $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is conservative.*
- (2) *The prime ideal \mathfrak{p} is in the image of*

$$\text{res}_f: \text{Spec}^h(\pi_*S) \longrightarrow \text{Spec}^h(\pi_*R).$$

- (3) *Coinduction $f^!: \text{Mod}_R \rightarrow \text{Mod}_S$ restricted to $\Lambda^{\mathfrak{p}} \text{Mod}_R$ is conservative.*

Proof. Lemma 1.13 gives one implication, so consider $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$ with $f^*(M) \simeq 0$. By Proposition 1.15, this implies that

$$\emptyset = \text{supp}_S(f^*M) = \text{res}_f^{-1} \text{supp}_R(M).$$

Since \mathfrak{p} is in the image of res and $\text{supp}_R(M) \subseteq \{\mathfrak{p}\}$, it follows that $\text{supp}_R(M) = \emptyset$, hence $M \simeq 0$ by Theorem 1.4. The dual equivalence between (2) and (3) is proven similarly. \square

Remark 1.18. Note that the assumption in the previous lemmas are slightly asymmetric: even though we deduce statements about induction and coinduction, we only assume that $\Gamma_{\mathfrak{p}} \text{Mod}_R$ (and not $\Lambda^{\mathfrak{p}} \text{Mod}_R$) is minimal.

We can combine the previous two local lemmas in a global statement, which says that conservativity for a morphism f of ring spectra is a strengthened form of surjectivity of res_f .

Proposition 1.19. *Suppose $f: R \rightarrow S$ is a map of commutative ring spectra and consider the following two conditions:*

- (1) f is conservative (resp. coconservative).
- (2) $\text{res}_f: \text{Spec}^h(\pi_*S) \rightarrow \text{Spec}^h(\pi_*R)$ is surjective.

Then (1) implies (2), and the reverse implication holds provided R is Noetherian and Mod_R is stratified.

Proof. We will only prove the claim about conservativity of f , leaving the modifications for the dual proof to the reader. If f is conservative, then $f^*: \Gamma_{\mathfrak{p}} \text{Mod}_R \subseteq \text{Mod}_R \rightarrow \text{Mod}_S$ is conservative for all $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$, so $\text{Spec}^h(\pi_*R)$ is in the image of res_f by Lemma 1.13. This establishes the implication (1) \implies (2).

Conversely, assume that Mod_R is stratified and that res_f is surjective. Let $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$ and $M \in \text{Mod}_R$ with $f^*(M) \simeq 0$. This implies that

$$f^*(\Gamma_{\mathfrak{p}}M) \simeq f^*(\Gamma_{\mathfrak{p}}R) \otimes_S f^*(M) \simeq 0.$$

Since res_f is surjective, Lemma 1.17 yields $\Gamma_{\mathfrak{p}}M \simeq 0$. But \mathfrak{p} was arbitrary, so it follows by the local-to-global principle Theorem 1.4 that $M \simeq 0$. \square

Remark 1.20. The base-change formula in Theorem 1.12 applied to $\text{supp}_R(f_*S)$ yields

$$\text{supp}_R(f_*S) = \text{res}_f(\text{supp}_S S) = \text{res}_f(\text{Spec}^h(\pi_*S)).$$

Thus, res_f being surjective is the same as having $\text{supp}_R(f_*S) = \text{Spec}^h(\pi_*R)$.

Lemma 1.21. *Let $f: R \rightarrow S$ and $g: S \rightarrow T$. If both are conservative (resp. coconservative) then so is their composite. Conversely, if $g \circ f$ is conservative (resp. coconservative) then so is f ; if f^* is additionally essentially surjective, then g is also conservative (resp. coconservative).*

Proof. We verify the last claim about conservativity and leave the proof of the remaining statements to the reader. Consider $N \in \text{Mod}_S$ with $g^*(N) \simeq 0$. By assumption, there exists $M \in \text{Mod}_S$ such that $f^*(M) \simeq N$, hence $(gf)^*(M) \simeq g^*(f^*(M)) \simeq 0$. Since $(gf)^*$ is conservative, $M \simeq 0$, so $N \simeq 0$. \square

Lemma 1.22. *If $R \in \text{Loc}_R(S)$, then f is biconservative. In particular this holds if $f: R \rightarrow S$ admits an R -module retract.*

Proof. Indeed, given $M \in \text{Mod}_R$ such that $f^*(M) \simeq S \otimes_R M \simeq 0$, it follows that $N \otimes_R M \simeq 0$ for any $N \in \text{Loc}_R(S)$. In particular, $M \simeq R \otimes_R M \simeq 0$. The argument for coconservativity is similar. Finally, if f admits an R -module retract, then $R \in \text{Loc}_R(S)$. \square

We end this section with a formal base-change property for (co)conservativity of maps that will turn out to be useful later.

Lemma 1.23. *Given a pushout diagram of commutative ring spectra*

$$\begin{array}{ccc} R & \xrightarrow{i} & S \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{j} & B \end{array}$$

such that f is conservative (resp. coconservative), then g is conservative (resp. coconservative) as well.

Proof. Both statements are consequences of the Beck–Chevalley condition and its dual, which we briefly recall for the convenience of the reader. The natural equivalence $i_*g_* \simeq f_*j_*$ has a mate $\zeta: f^*i_* \rightarrow j_*g^*$ given by the composite

$$f^*i_* \xrightarrow{\eta_g} f^*i_*g_*g^* \simeq f^*f_*j_*g^* \xrightarrow{\epsilon_f} j_*g^*,$$

where η_g is the unit of the adjunction $(g^* \dashv g_*)$ and ϵ_f is the counit of the adjunction $(f^* \dashv f_*)$. In order to show that ζ is an equivalence, it suffices to evaluate it on S and restrict to Mod_R . The claim then follows from the equivalence $A \otimes_R S \simeq B$ of R -modules. Passing to right adjoints yields natural equivalences $i^!f_* \simeq g_*j^!$ and, by symmetry, $j_*g^! \simeq f^!i_*$.

Suppose now that $N \in \text{Mod}_S$ such that $g^*(N) \simeq 0$, then there are equivalences of A -modules $0 \simeq j_*g^*(N) \simeq f^*i_*(N)$. Therefore, $i_*(N) \simeq 0$ by assumption on f^* and thus $N \simeq 0$. Similarly, if $f^!$ is conservative, then $0 \simeq j_*g^!(N) \simeq f^!i_*(N)$ implies $N \simeq 0$. \square

2. QUILLEN LIFTING AND STRATIFICATION

In this section we study stratification in the sense of [BIK11b] for structured ring spectra. We review and extend our previous work [BCHV17], showing how to descend stratification along a morphism of ring spectra. To do so, we introduce the notion of simple Quillen lifting, which is a stronger form of the Quillen lifting property considered in [BCHV17].

2.1. Descent for stratifications. In this section we isolate a condition for descending stratification along a morphism of Noetherian commutative ring spectra $f: R \rightarrow S$.

Proposition 2.1. *Let $f: R \rightarrow S$ be a morphism of Noetherian commutative ring spectra. For a fixed prime ideal $\mathfrak{p} \in \text{Spec}^h(\pi_*R)$, the following conditions are equivalent:*

- (1) $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is minimal and $\mathfrak{p} \in \text{Im}(\text{res}: \text{Spec}^h(\pi_*S) \rightarrow \text{Spec}^h(\pi_*R))$.
- (2) If $M, N \in \Gamma_{\mathfrak{p}} \text{Mod}_R$ are non-zero modules, then $\text{Hom}_S(f^*M, f^!N) \not\cong 0$.

If we assume additionally that Mod_S is stratified, then both conditions are equivalent to:

- (3) If $M, N \in \Gamma_{\mathfrak{p}} \text{Mod}_R$ are non-zero modules, then

$$\text{supp}_S(f^*M) \cap \text{cosupp}_S(f^!N) \neq \emptyset.$$

Proof. We first establish the equivalence between Conditions (1) and (2). By [BIK11b, Lemma 4.1], $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is a minimal localizing subcategory of Mod_R if and only if $\text{Hom}_R(M, N) \not\cong 0$ for all non-zero $M, N \in \Gamma_{\mathfrak{p}} \text{Mod}_R$. Assume (1), then Lemma 1.17 shows that $f_*f^*(M) \simeq 0$ if and only if $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$ is zero. This implies that $\text{supp}_R(f_*f^*M) = \text{supp}_R(M)$ for all $M \in \Gamma_{\mathfrak{p}} \text{Mod}_R$. Therefore, for non-zero $M, N \in \Gamma_{\mathfrak{p}} \text{Mod}_R$, we have

$$0 \not\cong \text{Hom}_R(f_*f^*M, N) \simeq \text{Hom}_S(f^*M, f^!N).$$

Conversely, (2) says $0 \not\cong \text{Hom}_R(f_*f^*M, N)$ for non-zero $M, N \in \Gamma_{\mathfrak{p}} \text{Mod}_R$, hence $0 \not\cong \text{Hom}_R(M, N)$ as well as, since $f_*f^*M \in \text{Loc}_R(M)$. Therefore, $\Gamma_{\mathfrak{p}} \text{Mod}_R$ is a minimal localizing subcategory of Mod_R . Furthermore, (2) implies that f^* and $f^!$ are conservative when restricted to $\Gamma_{\mathfrak{p}} \text{Mod}_R$.

Lemma 1.13 then shows $\mathfrak{p} \in \text{Im}(\text{res}_f: \text{Spec}^h(\pi_*S) \rightarrow \text{Spec}^h(\pi_*R))$.

The equivalence of Conditions (2) and (3) is a direct consequence of the formula

$$\operatorname{cosupp}_S(\operatorname{Hom}_S(X, Y)) = \operatorname{supp}_S(X) \cap \operatorname{cosupp}_S(Y)$$

for $X, Y \in \operatorname{Mod}_S$, which by [BIK12, Theorem 9.5(2)] holds whenever Mod_S is stratified. \square

Corollary 2.2. *Let $f: R \rightarrow S$ be a conservative (or coconservative) morphism of Noetherian commutative ring spectra and assume Mod_S is stratified, then Mod_R is stratified if and only if the following condition holds:*

- (*) *For any homogeneous prime ideal $\mathfrak{p} \in \operatorname{Spec}^h(\pi_* R)$ and non-zero modules $M, N \in \Gamma_{\mathfrak{p}} \operatorname{Mod}_R$ we have $\operatorname{supp}_S(f^* M) \cap \operatorname{cosupp}_S(f^! N) \neq \emptyset$.*

Proof. Assume first that (*) holds. In order to show that Mod_R is stratified, we must show that the localizing subcategories $\Gamma_{\mathfrak{p}} \operatorname{Mod}_R$ are minimal for all $\mathfrak{p} \in \operatorname{Spec}^h(\pi_* R)$, which follows by the implication (3) \implies (1) of Proposition 2.1. Conversely, suppose Mod_R is stratified. Since f is conservative (or coconservative), $\operatorname{res}_f: \operatorname{Spec}^h(\pi_* S) \rightarrow \operatorname{Spec}^h(\pi_* R)$ is surjective by Proposition 1.19, so (*) follows from (1) \implies (3) of Proposition 2.1. \square

In [BCHV17] we introduced the concept of Quillen lifting for a morphism of ring spectra $f: R \rightarrow S$, which we recall below. From the previous corollary, we can then deduce that Quillen lifting is not only sufficient, but in fact also necessary for descent of stratification along a conservative and coconservative morphism of Noetherian ring spectra.

Definition 2.3. A morphism of Noetherian commutative ring spectra $f: R \rightarrow S$ is said to satisfy Quillen lifting if for any two modules $M, N \in \operatorname{Mod}_R$ such that there is

$$\mathfrak{p} \in \operatorname{res}_f(\operatorname{supp}_S(f^* M)) \cap \operatorname{res}_f(\operatorname{cosupp}_S(f^! N)),$$

there exists a prime ideal $\mathcal{Q}_f(\mathfrak{p}) := \mathfrak{q} \in \operatorname{res}_f^{-1}(\mathfrak{p})$ with $\mathfrak{q} \in \operatorname{supp}_S(f^* M) \cap \operatorname{cosupp}_S(f^! N)$. In this case, we will refer to $\mathcal{Q}(\mathfrak{p}) = \mathcal{Q}_f(\mathfrak{p})$ as a Quillen lift of \mathfrak{p} along f .

Theorem 2.4. *Let $f: R \rightarrow S$ be a biconservative morphism of Noetherian commutative ring spectra and assume Mod_S is stratified, then the following are equivalent:*

- (1) *Mod_R is stratified.*
- (2) *f satisfies Condition (*).*
- (3) *f satisfies Quillen lifting.*

Proof. The equivalence of (1) and (2) is the content of Corollary 2.2. Next, we will prove that Quillen lifting implies Condition (*). Since f is biconservative, Theorem 1.12 implies that for all $M, N \in \operatorname{Mod}_R$ there are equalities

$$\operatorname{res}_f \operatorname{supp}_S(f^* M) = \operatorname{supp}_R(M) \quad \text{and} \quad \operatorname{res}_f \operatorname{cosupp}_S(f^! N) = \operatorname{cosupp}_R(N).$$

Given $\mathfrak{p} \in \operatorname{Spec}^h(\pi_* R)$, if $M, N \in \Gamma_{\mathfrak{p}} \operatorname{Mod}_R$ are non-zero modules, then $\operatorname{supp}_R(M) = \{\mathfrak{p}\} = \operatorname{supp}_R(N)$, hence also $\mathfrak{p} \in \operatorname{cosupp}_R(N)$ by [BIK12, Theorem 4.13]. This puts us in the situation of Quillen lifting, hence the Quillen lift of \mathfrak{p} along f gives $\mathcal{Q}(\mathfrak{p}) \in \operatorname{supp}_S(f^* M) \cap \operatorname{cosupp}_S(f^! N)$.

It remains to show that (1) \implies (3), which will be proven in Lemma 2.16 below using the notion of simple Quillen lifting. \square

Remark 2.5. If f is finite, then f^* is conservative if and only if $f^!$ is conservative, see [BCHV17, Lemma 3.17]. Further, if there exists a natural equivalence $f^* \simeq f^!$, then f is finite. Indeed, this equivalence implies that $f^!$ preserves colimits, and hence the left adjoint f_* preserves compact objects. This for example is the situation occurring for the stable module category of a finite group in [BIK11c], where induction and coinduction coincide.

Finally, we relate [Theorem 2.4](#) to the notion of Avrunin–Scott stratification introduced by Hovey and Palmieri in [\[HP01\]](#). Their Definition 5.5 axiomatizes the Avrunin–Scott theorem in modular representation theory, see [\[AS82\]](#), [\[BCR96, Theorem 10.7\]](#), and [\[BIK11c\]](#). The Avrunin–Scott identities appearing in the next result form a suitable generalization to a morphism $f: R \rightarrow S$ for which f^* is not necessarily equivalent to $f^!$.

Corollary 2.6. *Under the assumptions of [Theorem 2.4](#), the equivalent Conditions (1)–(3) hold if and only if the Avrunin–Scott identities are satisfied for all $M \in \text{Mod}_R$:*

$$\text{supp}_S(f^*M) = \text{res}_f^{-1} \text{supp}_R(M) \quad \text{and} \quad \text{cosupp}_S(f^!M) = \text{res}_f^{-1} \text{cosupp}_R(M).$$

Proof. Suppose Mod_R is stratified, then [\[BCHV17, Proposition 3.14\]](#) implies that the Avrunin–Scott identities are satisfied for all R -modules M . Conversely, assume that these two identities hold for $M, N \in \text{Mod}_R$. Let \mathfrak{p} be a homogeneous prime ideal with

$$\mathfrak{p} \in \text{res}_f(\text{supp}_S(f^*M)) \cap \text{res}_f(\text{cosupp}_S(f^!N)) \subseteq \text{supp}_R(M) \cap \text{cosupp}_R(N).$$

It follows that

$$\emptyset \neq \text{res}_f^{-1}(\mathfrak{p}) \subseteq \text{res}_f^{-1}(\text{supp}_R(M)) \cap \text{res}_f^{-1}(\text{cosupp}_R(N)) = \text{supp}_S(f^*M) \cap \text{cosupp}_S(f^!M),$$

where the last equivalence uses the Avrunin–Scott identities for M and N , respectively. This verifies Quillen lifting for the morphism f , so the equivalent Conditions (1)–(3) of [Theorem 2.4](#) hold. \square

2.2. Retractive descent. The goal of this subsection is to exhibit sufficient conditions that guarantee descent of stratification along a morphism of ring spectra. In particular, we will show that the stratification property is closed under retracts.

Proposition 2.7. *Let $f: R \rightarrow S$ be as above and suppose there exists a map of commutative Noetherian ring spectra $g: S \rightarrow T$ satisfying the following two conditions:*

- (1) gf is biconservative.
- (2) gf satisfies Quillen lifting.

If Mod_S is stratified, then Mod_R is stratified.

Proof. By [Lemma 1.21](#), f is biconservative. In light of [Theorem 2.4](#), it thus remains to show that f satisfies Quillen lifting. To this end, let $M, N \in \text{Mod}_R$ be two modules such that there exists $\mathfrak{p} \in \text{res}_f(\text{supp}_S(f^*M)) \cap \text{res}_f(\text{cosupp}_S(f^!N))$. Since gf is biconservative, [Theorem 1.12](#) shows that

$$\text{res}_f \text{supp}_S(f^*M) \subseteq \text{supp}_R(M) = \text{res}_{gf} \text{supp}_T((gf)^*M)$$

and

$$\text{res}_f \text{cosupp}_S(f^!N) \subseteq \text{cosupp}_R(N) = \text{res}_{gf} \text{cosupp}_T((gf)^!N),$$

so \mathfrak{p} lies in image of res_{gf} . Let $\mathcal{Q}_{gf}(\mathfrak{p})$ be a Quillen lift of \mathfrak{p} along gf in $\text{Spec}^h(\pi_*T)$. We claim that

$$\mathcal{Q}_f(\mathfrak{p}) = \text{res}_g(\mathcal{Q}_{gf}(\mathfrak{p}))$$

is then a Quillen lift for \mathfrak{p} along f . It is clear that $\text{res}_g(\mathcal{Q}_{gf}(\mathfrak{p})) \in \text{res}_f^{-1}(\mathfrak{p})$. To prove that $\text{res}_g(\mathcal{Q}_{gf}(\mathfrak{p})) \in \text{supp}_S(f^*M) \cap \text{cosupp}_S(f^!M)$, we will first show that

$$\text{res}_g \text{supp}_T((gf)^*M) \subseteq \text{supp}_S(f^*M) \quad \text{and} \quad \text{res}_g \text{cosupp}_T((gf)^!N) \subseteq \text{cosupp}_S(f^!N).$$

Indeed, by Proposition 3.14 in [\[BCHV17\]](#) applied to g and using that Mod_S is stratified, there are equalities

$$\text{supp}_T((gf)^*M) = \text{supp}_T(g^*(f^*M)) = \text{res}_g^{-1} \text{supp}_S(f^*M),$$

so we get

$$\text{res}_g \text{supp}_T((gf)^*M) = \text{res}_g \text{res}_g^{-1} \text{supp}_S(f^*M) \subseteq \text{supp}_S(f^*M).$$

The inclusion for cosupport and coinduction is proven similarly. It follows that

$$\operatorname{res}_g(\mathcal{Q}_{gf}(\mathfrak{p})) \in \operatorname{res}_g \operatorname{supp}_T((gf)^*M) \cap \operatorname{res}_g \operatorname{cosupp}_T((gf)^!M) \subseteq \operatorname{supp}_S(f^*M) \cap \operatorname{cosupp}_S(f^!N),$$

hence $\operatorname{res}_g(\mathcal{Q}_{gf}(\mathfrak{p}))$ is a Quillen lift for \mathfrak{p} along f . \square

Corollary 2.8. *If f admits an R -module retract and Mod_S is stratified, then Mod_R is stratified.*

Proof. In the previous proposition, let g be the section to f ; the claim follows. \square

We also have a partial converse to this result:

Proposition 2.9. *Suppose $R = \prod_{i \in I} R_i$ is a finite product of Noetherian commutative ring spectra, so that $\operatorname{Mod}_R \simeq \prod_{i \in I} \operatorname{Mod}_{R_i}$ as symmetric monoidal categories. Then Mod_R is stratified if and only if the categories Mod_{R_i} are stratified for all $i \in I$.*

Proof. Since $R = \prod_{i \in I} R_i$, there is a corresponding decomposition $\pi_*R \cong \prod_{i \in I} \pi_*R_i$. Moreover, the prime ideals of π_*R are of the form $\prod_{i \in I} \mathfrak{p}_i$ where for some k the ideal \mathfrak{p}_k is a prime ideal of π_*R_k , and for $i \neq k$ we have $\mathfrak{p}_i = \pi_*R_i$. By the construction of $\Gamma_{\mathfrak{p}}$, given a prime ideal $\mathfrak{p} \in \operatorname{Spec}^h(\pi_*R)$ as above, the localizing subcategory $\Gamma_{\mathfrak{p}} \operatorname{Mod}_R$ is minimal if and only if the localizing subcategory $\Gamma_{\mathfrak{p}_k} \operatorname{Mod}_{R_k}$ is minimal. It follows that if Mod_{R_i} is stratified for each $i \in I$, then so is Mod_R . The *only if* direction is a consequence of [Corollary 2.8](#). \square

2.3. Simple Quillen lifting. As we have seen in [Section 2.1](#), we can descend stratification along a morphism $f: R \rightarrow S$ of commutative ring spectra provided two conditions are satisfied:

- f is biconservative.
- f satisfies Condition (*) or, equivalently, Quillen lifting.

While we discussed the first condition in [Section 1](#), we now turn to the study of the second condition. In particular, we will introduce a variant called simple Quillen lifting that is satisfied in many examples and implies Quillen lifting, and establish several of its basic properties.

Definition 2.10. A morphism $f: R \rightarrow S$ of commutative ring spectra is said to satisfy simple Quillen lifting if for any $M \in \operatorname{Mod}_R$:

$$\operatorname{res}_f^{-1} \operatorname{res}_f \operatorname{supp}_S(f^*M) = \operatorname{supp}_S(f^*M) \quad \text{and} \quad \operatorname{res}_f^{-1} \operatorname{res}_f \operatorname{cosupp}_S(f^!M) = \operatorname{cosupp}_R(f^!M).$$

Note that the inclusions \supseteq hold unconditionally.

Lemma 2.11. *Simple Quillen lifting of a morphism $f: R \rightarrow S$ implies Quillen lifting.*

Proof. Let \mathfrak{p} be in $\operatorname{res}_f(\operatorname{supp}_S(f^*M)) \cap \operatorname{res}_f(\operatorname{cosupp}_S(f^!N))$, so that $\operatorname{res}_f^{-1}(\mathfrak{p})$ is non-empty. It then follows from simple Quillen lifting that we have inclusions

$$\begin{aligned} \operatorname{res}_f^{-1}(\mathfrak{p}) &\subseteq \operatorname{res}_f^{-1} \operatorname{res}_f(\operatorname{supp}_S(f^*M)) \cap \operatorname{res}_f^{-1} \operatorname{res}_f(\operatorname{cosupp}_S(f^!N)) \\ &= \operatorname{supp}_S(f^*M) \cap \operatorname{cosupp}_S(f^!N), \end{aligned}$$

so that f satisfies Quillen lifting. \square

The following observation is immediate from the definition of simple Quillen lifting.

Lemma 2.12. *If the restriction map $\operatorname{res}_f: \operatorname{Spec}^h(\pi_*S) \rightarrow \operatorname{Spec}^h(\pi_*R)$ is injective, then $R \rightarrow S$ satisfies simple Quillen lifting.*

Proof. If res_f is injective, then $\operatorname{res}_f^{-1} \operatorname{res}_f(\mathcal{U}) = \mathcal{U}$ for any subset $\mathcal{U} \subseteq \operatorname{Spec}^h(\pi_*S)$. \square

Definition 2.13. We say that a map of (discrete) rings $g: A \rightarrow B$ is an \mathcal{N} -monomorphism if the kernel of g is nilpotent, and that g is a \mathcal{N} -epimorphism if for every $b \in B$ there exists an ℓ such that $b^\ell \in \operatorname{Im} g$. A morphism of ring spectra $f: R \rightarrow S$ is an \mathcal{N} -monomorphism (resp. \mathcal{N} -epimorphism) if $f_*: \pi_*R \rightarrow \pi_*S$ is.

Lemma 2.14. *If $f: R \rightarrow S$ is an \mathcal{N} -epimorphism, then $\text{res}_f: \text{Spec}^h(\pi_* S) \rightarrow \text{Spec}^h(\pi_* R)$ is injective.*

Proof. The argument given in [MNN19, Proposition 3.24] can easily be adjusted to prove this. Namely, factor f_* as the composite $\pi_* R \rightarrow \pi_* R / \ker(f_*) \rightarrow \pi_* S$. The first map is surjective and so induces an injection on Spec^h . The fact that $\text{Spec}^h(\pi_* S) \rightarrow \text{Spec}^h(\pi_* R / \ker(f_*))$ is injective is shown in [MNN19, Proposition 3.24], and we are done. \square

From Lemmas 2.12 and 2.14 we deduce the following.

Corollary 2.15. *If $f: R \rightarrow S$ is an \mathcal{N} -epimorphism, then f satisfies simple Quillen lifting.*

The next result in particular finishes the proof of Theorem 2.4.

Lemma 2.16. *If Mod_R is stratified, then f satisfies simple Quillen lifting and hence Quillen lifting.*

Proof. Let $M \in \text{Mod}_R$. We have

$$\text{res}_f^{-1} \text{res}_f(\text{supp}_S(f^* M)) \subseteq \text{res}_f^{-1} \text{supp}_R(M) = \text{supp}_S(f^* M),$$

where the second equality uses Proposition 3.14 from [BCHV17], which applies because Mod_R is stratified by assumption. The reverse inclusion holds unconditionally. Similarly, we deduce that

$$\text{res}_f^{-1} \text{res}_f(\text{cosupp}_S(f^! M)) = \text{cosupp}_S(f^! M).$$

This shows that f satisfies simple Quillen lifting and hence also Quillen lifting by Lemma 2.11. \square

Remark 2.17. It is not the case that a morphism $f: R \rightarrow S$ which satisfies Quillen lifting and such that Mod_R and Mod_S are stratified has to be conservative. Indeed, choose a surjective morphism of discrete Noetherian commutative rings $A \rightarrow B$ which gives a morphism of commutative Noetherian ring spectra $f: HA \rightarrow HB$. Since $\text{Mod}_{HA} \simeq \mathcal{D}(A)$, the unbounded derived category of A , and $\text{Mod}_{HB} \simeq \mathcal{D}(B)$ (see [Lur17, Remark 7.1.1.16]), these are both stratified by Neeman's result [Nee92]. Moreover, because $A \rightarrow B$ is surjective, simple Quillen lifting is satisfied by Lemma 2.12. However, by Proposition 1.19 induction along f is conservative if and only if the induced map $\text{res}_f: \text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective, which does not hold in general.

Finally, we establish the following partial transitivity properties.

Proposition 2.18. *Let $R \xrightarrow{f} S \xrightarrow{g} T$ be morphisms of commutative ring spectra and assume that g is biconservative. If both f and g satisfy simple Quillen lifting, then so does gf .*

Proof. We will first verify the claim for induced modules. To this end, let $M \in \text{Mod}_R$; we then have equalities:

$$\begin{aligned} \text{res}_{gf}^{-1} \text{res}_{gf} \text{supp}_T((gf)^* M) &= \text{res}_{gf}^{-1} \text{res}_f \text{res}_g \text{supp}_T(g^* f^* M) \\ &= \text{res}_g^{-1} \text{res}_f^{-1} \text{res}_f \text{supp}_S(f^* M) && \text{Theorem 1.12(1)} \\ &= \text{res}_g^{-1} \text{supp}_S(f^* M) && \text{simple Quillen lifting for } f \\ &= \text{res}_g^{-1} \text{res}_g \text{supp}_T(g^* f^* M) && \text{Theorem 1.12(1)} \\ &= \text{supp}_T((gf)^* M) && \text{simple Quillen lifting for } g, \end{aligned}$$

where we used that g is conservative in the penultimate equality. The argument is similar for coinduction, using Theorem 1.12(3). \square

We can also prove the following converse to the last result.

Proposition 2.19. *Let $R \xrightarrow{f} S \xrightarrow{g} T$ be morphisms of commutative ring spectra such that g is biconservative. If gf satisfies simple Quillen lifting, then so does f .*

Proof. Simple Quillen lifting for gf and the same argument as in [Proposition 2.18](#) gives

$$\begin{aligned} \operatorname{res}_g^{-1} \operatorname{res}_f^{-1} \operatorname{res}_f \operatorname{supp}_S(f^*M) &= \operatorname{res}_{gf}^{-1} \operatorname{res}_{gf} \operatorname{supp}_T((gf)^*M) \\ &= \operatorname{supp}_T((gf)^*M) \\ &\subseteq \operatorname{res}_g^{-1} \operatorname{res}_g \operatorname{supp}_T(g^*f^*M) \\ &= \operatorname{res}_g^{-1} \operatorname{supp}_S(f^*M). \end{aligned}$$

By [Proposition 1.19](#), res_g is surjective, so applying this map to the inclusion above yields an inclusion

$$\operatorname{res}_f^{-1} \operatorname{res}_f \operatorname{supp}_S(f^*M) \subseteq \operatorname{supp}_S(f^*M),$$

while the reverse inclusion always holds. The same argument works for coinduction. \square

2.4. Examples. We finish this section with some examples of ring spectra for which we can check that Mod_R is stratified using the criteria established above. In these cases, it is often easier to check directly that simple Quillen lifting is satisfied when compared to Quillen lifting.

Example 2.20. Consider the complexification morphism $f: KO \rightarrow KU$ from real K -theory to complex K -theory. On homotopy the induced map

$$\pi_*KO \cong \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}] \longrightarrow \mathbb{Z}[u^{\pm 1}] \cong \pi_*KU$$

is given by $\eta \mapsto 0, \alpha \mapsto 2u^2$ and $\beta \mapsto u^4$. It follows that this map is an \mathcal{N} -isomorphism, so simple Quillen lifting is satisfied by [Corollary 2.15](#).

The cofiber sequence of KO -modules

$$\Sigma^1 KO \xrightarrow{\eta} KO \xrightarrow{f} KU$$

shows that f is a finite morphism, and nilpotence of η shows that induction along f is conservative, see [\[Rog08, Proposition 5.3.1\]](#). By [Remark 2.5](#) coinduction along f is also conservative. Moreover, the homotopy groups of KU are concentrated in even degrees, and satisfy the sufficient conditions for stratification given in [\[DS16, Theorem 1.3\]](#). It follows that Mod_{KU} is stratified. By the preceding discussion and [Theorem 2.4](#) we deduce that Mod_{KO} is stratified.

We note that the same argument works in the 2-complete setting, so $\operatorname{Mod}_{KO_2}$ is stratified.

For the next result we recall [\[Lur17, Definition 7.2.4.21\]](#) that a module M over a connective ring spectrum R is said to have finite Tor-amplitude if there exists an integer n such that for any discrete R -module N (i.e., $\pi_i N = 0$ for $i \neq 0$), we have $\pi_i(N \otimes_R M) = 0$ for $i \geq n$.

Example 2.21. Suppose R is a connective (graded) Noetherian commutative ring spectrum with the property that $\pi_i R = 0$ for $i \gg 0$. Consider the morphism $f: R \rightarrow \tau_{\leq 0} R \simeq H\pi_0 R$. In the language of [\[Mat16\]](#) this is a descendable morphism, see [\[Mat16, Proposition 3.34\]](#). Such morphisms have the property that induction is conservative, see [\[Mat16, Proposition 3.19\]](#). Moreover, the map f_* is surjective, so $\operatorname{res}_f: \operatorname{Spec}^h(\pi_* R) \rightarrow \operatorname{Spec}^h(\pi_0 R)$ is injective, and hence simple Quillen lifting is satisfied. Note that since $\pi_* R$ is assumed to be graded Noetherian, $\pi_0 R$ is Noetherian, and each $\pi_i R$ is finitely-generated as a $\pi_0 R$ -module [\[BG09, Lemma 4.12\]](#). By Neeman's result [\[Nee92, Theorem 2.8\]](#) and the equivalence $\operatorname{Mod}_{H\pi_0 R} \simeq \mathcal{D}(H\pi_0 R)$, we thus see that $\operatorname{Mod}_{H\pi_0 R}$ is stratified. It follows that whenever coinduction along f is conservative, then Mod_R is also stratified by [Theorem 2.4](#).

This will be the case if f is a finite morphism, i.e., if $H\pi_0 R$ is a compact R -module. We claim that f is finite if $H\pi_0 R$ has finite Tor-amplitude over R (for example, if $\pi_* R$ is regular).

Indeed, it then follows that $H\pi_0 R$ is compact as an R -module by [Lur17, Proposition 7.2.4.23(4)]. Therefore, Mod_R is stratified.

3. COCHAINS ON CLASSIFYING SPACES

In this section, we specialize the work on stratification in the previous sections to the case of cochains on a space X , in particular, the case where X is the classifying space of a topological group. Using work of Broto and Kitchloo [BK02] we prove Theorem A of the introduction, namely that $\text{Mod}_{C^*(BK)}$ is stratified for K a Kac–Moody group.

3.1. Cochains on topological spaces. Let R be a ring spectrum R and let X be a topological space. We write $C^*(X; R)$ for the ring spectrum of R -valued cochains on X , i.e., the function spectrum $\text{Hom}_{\text{Sp}}(\Sigma_+^\infty X, R)$. In particular, there is an isomorphism $\pi_* C^*(X; R) \cong R^{-*}(X)$, the R -cohomology of X . The multiplication of two elements $x, y \in C^*(X; R)$ is given by the composite

$$\Sigma_+^\infty X \xrightarrow{\Delta} \Sigma_+^\infty X \otimes \Sigma_+^\infty X \xrightarrow{x \otimes y} R \otimes R \xrightarrow{m} R,$$

where the first map is induced by the diagonal of X and the last map m is the multiplication on the ring spectrum R . If R is commutative, then $C^*(X; R)$ inherits the structure of an augmented commutative R -algebra.

In this paper, we are mostly interested in the case when $R = Hk$ is the Eilenberg–Mac Lane spectrum of a field k of characteristic $p > 0$ for a fixed prime p and that X is a connected space with $H^*(X; k)$ Noetherian. If X is additionally assumed to be p -good, then the canonical map $X \rightarrow X_p^\wedge$ induces an equivalence $C^*(X_p^\wedge; k) \simeq C^*(X; k)$ of commutative ring spectra, see for example [BCHV17, Lemma 4.8]. When $k = \mathbb{F}_p$, we will usually omit it from the notation.

3.2. Chouinard’s theorem for classifying spaces. We recall that in Section 2 we isolated two conditions to descend stratification along a morphism $f: R \rightarrow S$ of ring spectra, namely that induction and coinduction along f are conservative, and that f satisfies simple Quillen lifting. Suppose now that $R = C^*(BG)$ for a topological group G . In order to construct a suitable morphism $f: C^*(BG) \rightarrow S$ as above, we associate to G a category, first considered by Quillen [Qui71].

Definition 3.1. Let G be a topological group. The category $\mathcal{A}_p(G)$ has objects the elementary abelian p -subgroups of G , and morphisms are group homomorphisms induced by subconjugation in G .

Suppose that $\mathcal{A}_p(G)$ is equivalent to a finite category. This is true, for example, when G is a compact Lie group [Qui71, Lemma 6.3]. Let $\mathcal{E}_p(G)$ denote a set of representatives of conjugacy classes of elementary abelian p -subgroups of G , then the inclusions $E \leq G$ define a morphism

$$q_G: C^*(BG) \longrightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE).$$

We would like to know when induction and coinduction along q_G are conservative.

Remark 3.2. When G is a finite group, this is essentially Chouinard’s theorem [Cho76] about projective representations of groups. Indeed, let Mod_{kG} be the abelian category of k -linear G -representations. Chouinard proves that $M \in \text{Mod}_{kG}$ is projective if and only if the restrictions of M to all elementary abelian subgroups of G are projective. Equivalently, an object M in the stable module category StMod_{kG} of G is trivial if and only if it becomes trivial in StMod_{kE} for $E \in \mathcal{E}_p(G)$. In this form, Chouinard’s theorem can be reinterpreted homotopically, at least

for finite p -groups: StMod_{kG} embeds fully faithfully into the homotopy category $K(\text{Inj}_{kG})$ of unbounded injective kG -modules, which in turn is equivalent to $\text{Mod}_{C^*(BG)}$:

$$\text{StMod}_{kG} \hookrightarrow K(\text{Inj}_{kG}) \xrightarrow{\sim} \text{Mod}_{C^*(BG)}.$$

Benson, Iyengar, and Krause [BIK11c, Proposition 9.6] then prove that Chouinard's theorem extends to the statement that the functor

$$q_G^*: \text{Mod}_{C^*(BG)} \longrightarrow \prod_{E \in \mathcal{E}_p(G)} \text{Mod}_{C^*(BE)}$$

is conservative, where q_G^* denotes induction along the map $q_G: C^*(BG) \rightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE)$ induced by the subgroup inclusions $E \rightarrow G$.

Example 3.3. When G is a compact Lie group, Benson and Greenlees show in [BG14, Theorem 3.1(i)] that $C^*(BG)$ is in the thick subcategory generated by the $C^*(BG)$ -modules $C^*(BE)$ for $E \leq G$, hence induction and coinduction along q_G are conservative by Lemma 1.22.

Remark 3.4. Example 3.3 shows that induction and coinduction along a morphism $f: R \rightarrow S$ can be satisfied even when S is not a compact R -module. Indeed, if $\pi_0 G$ is not a finite p -group and $E \leq G$ is some elementary abelian, then $C^*(BE)$ is not necessarily finite as a $C^*(BG)$ -module. An explicit example is given by $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq A_4$.

Example 3.5. Let G be a compact Lie group, X a finite G -CW complex. As a generalization of the previous example, consider the category $\mathcal{A}_G(X)$ whose objects are homotopy classes of G -equivariant maps $\iota: G/E \rightarrow X$, where E is an elementary abelian p -subgroup of G , and morphisms from $\iota: G/E \rightarrow X$ to $\iota': G/E' \rightarrow X$ are G -equivariant maps $f: G/E \rightarrow G/E'$ such that $\iota' \circ f \simeq \iota$ (see [Qui71, Section 8]). For any $(\iota: G/E \rightarrow X) \in \mathcal{A}_G(X)$, there is an induced map $C^*(X_{hG}) \rightarrow C^*(BE)$ obtained by applying cochains to the Borel construction on ι . These maps assemble together to give a map

$$q_{G,X}: C^*(X_{hG}) \longrightarrow \prod_{E \in \mathcal{E}_G(X)} C^*(BE),$$

where $\mathcal{E}_G(X)$ denotes a set of representatives of isomorphism classes of objects in $\mathcal{A}_G(X)$. Using the techniques of Benson and Greenlees [BG14, Theorem 3.1(i)], Cameron [Cam18, Proposition 5.4.8] has shown that $q_{G,X}$ is biconservative. In particular, [Cam18, Lemma 5.4.11] shows that $C^*(X_{hG})$ is in the thick subcategory of $\text{Mod}_{C^*(X_{hG})}$ generated by the set of $C^*(X_{hG})$ -modules $\{C^*(EG \times_G G/E): (G/E \rightarrow X) \in \mathcal{A}_G(X)\}$, so that Lemma 1.22 applies. Note that if X is contractible, then one recovers Example 3.3. One can also show that $q_{G,X}$ satisfies Quillen lifting (this is essentially [Cam18, Proposition 5.4.13]), and so deduce that $\text{Mod}_{C^*(X_{hG})}$ is stratified by $H_G^*(X)$, see [Cam18, Theorem 5.4.1].

Taking $X = G$, considered as a G -space with conjugation action, the Borel construction G_{hG} is homotopy equivalent to $\Lambda(BG)$, the free loop space on BG [Smi81].

Example 3.6. Suppose \mathcal{F} is a saturated fusion system on a finite p -group S (see [BLO03, Section 1]). Broto, Levi and Oliver introduced the notion of a classifying space for a fusion system, and Chermak [Che13] showed that it always exists and is unique up to homotopy equivalence. Then we can associate a classifying space $B\mathcal{F}$ which comes equipped with a canonical map $\theta: BS \rightarrow B\mathcal{F}$, inducing $\theta^*: C^*(B\mathcal{F}) \rightarrow C^*(BS)$ which is split as a map of $C^*(B\mathcal{F})$ -modules [Rag06, Rag08].

Let $\mathcal{E}_p(\mathcal{F})$ (resp. $\mathcal{E}_p(S)$) be a set of representatives of \mathcal{F} -conjugacy (resp. S -conjugacy) classes of elementary abelian p -subgroups in S . We obtain a morphism

$$q_{\mathcal{F}}: C^*(B\mathcal{F}) \longrightarrow \prod_{E \in \mathcal{E}_p(\mathcal{F})} C^*(BE)$$

induced by the inclusions $\iota_E: E \leq S$. Those representatives can be chosen in a way that the map $\pi: \mathcal{E}_p(S) \rightarrow \mathcal{E}_p(\mathcal{F})$, which assigns a representative in the \mathcal{F} -conjugacy class, is a surjection. Consider the resulting decomposition $\mathcal{E}_p(S) = \coprod_{E \in \mathcal{E}_p(\mathcal{F})} \pi^{-1}(E)$. For each $V \in \pi^{-1}(E)$ there is an isomorphism $f: V \rightarrow E$ in \mathcal{F} . We can assemble them in a morphism

$$q_E: C^*(BE) \longrightarrow \prod_{V \in \pi^{-1}(E)} C^*(BV).$$

Since $f \in \mathcal{F}$, we have $f^* \circ \iota_E^* \circ \theta^* \simeq \iota_V^* \circ \theta^*$, so it follows that the following diagram commutes:

$$\begin{array}{ccccc} C^*(B\mathcal{F}) & \xrightarrow{\theta^*} & C^*(BS) & \xrightarrow{q_S} & \prod_{V \in \mathcal{E}_p(S)} C^*(BV) \\ & \searrow^{q_{\mathcal{F}}} & & & \downarrow \simeq \\ & & \prod_{E \in \mathcal{E}_p(\mathcal{F})} C^*(BE) & \xrightarrow{\prod q_E} & \prod_{E \in \mathcal{E}_p(\mathcal{F})} \prod_{V \in \pi^{-1}(E)} C^*(BV). \end{array}$$

Since θ^* is split, induction and coinduction along it are conservative by [Lemma 1.22](#). Moreover, q_S is biconservative, because S is a finite p -group (see [Example 3.3](#)). By transitivity, the composite is therefore biconservative as well. But by [Lemma 1.21](#) this implies that $q_{\mathcal{F}}$ is also biconservative.

Remark 3.7. More generally, one can consider saturated fusion systems \mathcal{F} on a discrete p -toral group S , i.e., a group that fits in an extension

$$1 \longrightarrow (\mathbb{Z}/p^\infty)^r \longrightarrow S \longrightarrow \pi \longrightarrow 1$$

where $r \geq 0$ is finite, and π is a finite p -group. As in the case $r = 0$ considered in [Example 3.6](#), there is an associated classifying space $B\mathcal{F}$ [[BLO07](#), [LL15](#)] and a canonical map $\theta: BS \rightarrow B\mathcal{F}$. The induced map $\theta^*: C^*(B\mathcal{F}) \rightarrow C^*(BS)$ is split as a map of $C^*(B\mathcal{F})$ -modules by [[BCHV17](#), Proposition 4.24]. The composite

$$C^*(B\mathcal{F}) \xrightarrow{\theta^*} C^*(BS) \xrightarrow{q_S} \prod_{V \in \mathcal{E}_p(S)} C^*(BV)$$

is known to be biconservative when \mathcal{F} models a compact Lie group, a connected p -compact group, or when S is a finite p -group as in [Example 3.6](#), see [[BCHV17](#), Corollary 4.20 and Theorem 4.25]. We also have the map $q_{\mathcal{F}}$ as in [Example 3.6](#), and a similar argument shows that $q_{\mathcal{F}}$ is biconservative if and only if q_S is biconservative.

3.3. A hierarchy of spaces. In [[KM98](#)] Kropholler and Mislin introduced a class of hierarchically decomposable groups. Given a class \mathcal{X} of groups, they define $H_1\mathcal{X}$ to be the class of groups G which admit a finite dimensional G -CW-complex X with cell stabilizers in \mathcal{X} . Then $H\mathcal{X}$ is defined as the smallest class \mathcal{J} containing \mathcal{X} such that $H_1\mathcal{J} = \mathcal{J}$. Of special interest is the class \mathcal{X} of finite groups.

Inspired by this, Broto and Kitchloo [[BK02](#)] introduced the following class of groups. Let \mathcal{X} be a class of compactly generated Hausdorff topological groups and let p be a fixed prime. Define $\mathcal{K}_1\mathcal{X}$ to the class of compactly generated Hausdorff topological groups G for which there exists a finite G -CW-complex X such that:

- (1) the isotropy subgroups of X belongs to \mathcal{X} , and
- (2) for each finite p -subgroup $\pi < G$, the fixed point space X^π is p -acyclic.

Remark 3.8. The second condition in the definition applied to the trivial subgroup implies that X is mod p acyclic and then $C^*(X_{hG}) \simeq C^*(BG)$. Moreover, from the proof of [[Cam18](#), Lemma 5.4.10] we see that $C^*(BG) \in \text{Thick}(C^*(BH) | H \in \text{Iso}_G(X))$, where $\text{Iso}_G(X)$ denotes a set of representatives of G -conjugacy classes of isotropy subgroups of the G -action on X (which are compactly generated Hausdorff topological groups by the definition of $\mathcal{K}_1\mathcal{X}$).

In what follows \mathcal{X} is the class of compact Lie groups. Examples of groups in $\mathcal{K}_1\mathcal{X}$ are given by Kac–Moody groups and infinite Coxeter groups, see [Example 3.15](#). For any $G \in \mathcal{K}_1\mathcal{X}$ we let $\mathcal{A}_p(G)$ denote the Quillen category associated to G , and $\mathcal{E}_p(G)$ a set of representatives for conjugacy classes of elementary abelian p -subgroups. As with the case of a compact Lie group, we can construct a morphism

$$q_G: C^*(BG) \longrightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE).$$

Proposition 3.9. *Let $G \in \mathcal{K}_1\mathcal{X}$, then induction and coinduction along q_G are conservative.*

Proof. Since $G \in \mathcal{K}_1\mathcal{X}$, by definition, there exists a finite G -CW-complex X whose isotropy subgroups are compact Lie groups, and such that $C^*(X_{hG}) \simeq C^*(BG)$. Let $E \leq G$ be an elementary abelian p -subgroup, then X^E is mod p acyclic by hypothesis, and in particular is non-empty. Then $E \leq G$ is a subgroup of an isotropy subgroup of G , hence E is subconjugated to some $H \in \text{Iso}_G(X)$, where the latter is a set of representatives of G -conjugacy classes of isotropy subgroups of X . Let $\mathcal{E}_p(H)$ be a set of representatives of H -conjugacy classes of elementary abelian p -subgroups in H and write $\mathcal{E}_\mathcal{X}(G) = \{(H, E) | H \in \text{Iso}_G(X), E \in \mathcal{E}_p(H)\}$. Therefore, one can choose representatives in a way so that the natural map $\mathcal{E}_\mathcal{X}(G) \rightarrow \mathcal{E}_p(G)$ is surjective.

The composite

$$q: C^*(BG) \xrightarrow{q_{\text{Iso}}} \prod_{H \in \text{Iso}_G(X)} C^*(BH) \xrightarrow{\prod q_H} \prod_{H \in \text{Iso}_G(X)} \prod_{E \in \mathcal{E}_p(H)} C^*(BE),$$

is biconservative: q_{Iso} is so by [Remark 3.8](#) and q_H by [Example 3.3](#) since $H \leq G$ are compact Lie groups. Now the map q factors through $q_G: C^*(BG) \rightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE)$ by the same argument as in [Example 3.6](#) using conjugation by elements in G instead of morphisms in the fusion systems. We conclude that q_G is also biconservative. \square

Let \mathcal{V}_G denote the homogeneous prime ideal spectrum of $H^*(BG)$. For an elementary abelian subgroup $E \leq G$, let $\mathcal{V}_E^+ = \mathcal{V}_E \setminus \bigcup_{E' < E} \text{res}_E^{E'} \mathcal{V}_{E'}$, where $\text{res}_E^{E'}: \mathcal{V}_{E'} \rightarrow \mathcal{V}_E$ is the restriction map. Finally, we write $\mathcal{V}_{G,E}^+ = \text{res}_G^E \mathcal{V}_E^+$, to denote the restriction of \mathcal{V}_E^+ induced by the map $H^*(BG) \rightarrow H^*(BE)$.

Theorem 3.10. *Let \mathcal{X} be the class of compact Lie groups and $G \in \mathcal{K}_1\mathcal{X}$.*

(1) (Broto–Kitchloo) $H^*(BG)$ is Noetherian and the Quillen map

$$q_G: H^*(BG) \longrightarrow \varprojlim_{E \in \mathcal{A}_p(G)} H^*(BE)$$

is an F -isomorphism.

(2) (Rector) The variety \mathcal{V}_G admits a decomposition

$$\mathcal{V}_G = \coprod_{E \in \mathcal{E}_p(G)} \mathcal{V}_{G,E}^+$$

where $\mathcal{E}_p(G)$ denotes a set of representatives for conjugacy classes of elementary abelian p -subgroups of G .

Proof. The first part is proved by Broto and Kitchloo. The fact that $H^*(BG)$ is Noetherian is shown in [[BK02](#), Theorem 4.8], while the F -isomorphism is proved in [[BK02](#), Theorem 4.1].

For the second part, we can use the methods of Rector [[Rec84](#)] to deduce the decomposition of the variety \mathcal{V}_G from the F -isomorphism theorem as in [[BCHV17](#), Theorem 5.6]. In particular, we claim that the Conditions (1) – (5) of [[Rec84](#), Proposition 2.3] are satisfied for the pair $(\mathcal{A}_p(G), H^*(B-))$, with proofs analogous to that given in the proof of Theorem 5.6 of [[BCHV17](#), Theorem 5.6].

It follows from [Rec84, Proposition 2.5 and Theorem 2.6] that $\Lambda = \varprojlim_{E \in \mathcal{A}_p(G)} H^*(BE)$ is a reduced, Noetherian unstable algebra over the Steenrod algebra. The proof of [Rec84, Theorem 2.6] shows moreover that $\mathcal{V}_\Lambda \cong \prod_{E \in \mathcal{E}_p(G)} \mathcal{V}_{G,E}^+$. Since an F -isomorphism induces an isomorphism on varieties, we deduce from this and the first part of the theorem that

$$\mathcal{V}_G \cong \mathcal{V}_\Lambda \cong \prod_{E \in \mathcal{E}_p(G)} \mathcal{V}_{G,E}^+,$$

as required. \square

Corollary 3.11. *Let $G \in \mathcal{K}_1\mathcal{X}$, then*

$$q: H^*(BG) \longrightarrow \prod_{E \in \mathcal{E}_p(G)} H^*(BE)$$

satisfies Quillen lifting.

Proof. This is the same argument as given in [BCHV17, Theorem 5.10]. \square

Remark 3.12. In fact, inspection of the proof shows that q actually satisfies simple Quillen lifting, by an argument similar to [Proposition 4.3](#).

Finally we need the next result—the stratification result follows from [BIK11c], as in [BG14, Theorem 4.2(i)], while the Avrunin–Scott identities are then a consequence of [Corollary 2.6](#).

Theorem 3.13 (Benson–Iyengar–Krause). *For any elementary abelian p -subgroup E , the category $\text{Mod}_{C^*(BE)}$ is stratified by $H^*(BE)$. Moreover, for any morphism $E' \rightarrow E$ of elementary abelian p -subgroups with induced morphism $f: C^*(BE) \rightarrow C^*(BE')$, the Avrunin–Scott identities are satisfied, i.e.,*

$$\begin{aligned} \text{supp}_{C^*(BE')} (f^* M) &= \text{res}_f^{-1} \text{supp}_{C^*(BE)} (M) \\ \text{cosupp}_{C^*(BE')} (f^! M) &= \text{res}_f^{-1} \text{cosupp}_{C^*(BE)} (M) \end{aligned}$$

for all $M \in \text{Mod}_{C^*(BE)}$.

Combining the previous results we obtain the following.

Theorem 3.14. *Let \mathcal{X} denote the class of compact Lie groups. Then, for any $G \in \mathcal{K}_1\mathcal{X}$, the category $\text{Mod}_{C^*(BG)}$ is stratified by $H^*(BG)$.*

Proof. We apply [Theorem 2.4](#) to the morphism

$$q_G: C^*(BG) \longrightarrow \prod_{E \in \mathcal{E}_p(G)} C^*(BE).$$

Note that $\mathcal{E}_p(G)$ is finite, see the discussion at the bottom of page 630 of [BK02]. By [Corollary 3.11](#) and [Proposition 3.9](#) stratification of $\text{Mod}_{C^*(BG)}$ will follow if $\prod_{E \in \mathcal{E}_p(G)} \text{Mod}_{C^*(BE)}$ is stratified, but this follows from [Proposition 2.9](#) and [Theorem 3.13](#). \square

By [Theorem 1.10](#), we deduce that the telescope conjecture holds for $\text{Mod}_{C^*(BG)}$ with $G \in \mathcal{K}_1\mathcal{X}$.

Example 3.15 (Stratification for Kac–Moody groups). Unitary forms of Kac–Moody groups can be understood as an extension of the class of compact Lie groups with whom they share many properties. For example, they have a maximal torus of finite rank and a notion of Weyl group which is generated by reflections, see [Kum02]. The Weyl group of a Kac–Moody group is a Coxeter group which is, in general, infinite.

The study of the topology of Kac–Moody groups and their classifying spaces was initiated by Kitchloo in his thesis and followed by many authors, see for example [Kit98, BK02, AR03, Kit14].

In fact, the interest in the groups $G \in \mathcal{K}_1\mathcal{X}$ studied in the previous section arose because Kac–Moody groups give examples of groups in $\mathcal{K}_1\mathcal{X}$, see [BK02, Section 5].

Theorem 3.16 (Broto–Kitchloo). *Let \mathcal{X} denote the class of compact Lie groups, then any Kac–Moody group K is in $\mathcal{K}_1\mathcal{X}$. Moreover, the Weyl group of K belongs to the class $\mathcal{K}_1\mathcal{X}$.*

When combined with Theorem 3.14, we deduce the following.

Theorem 3.17. *Let K be a Kac–Moody group and W the Weyl group of K . Then $\text{Mod}_{C^*(BK)}$ and $\text{Mod}_{C^*(BW)}$ are stratified.*

4. HOPF SPACES WITH NOETHERIAN MOD p COHOMOLOGY

The structure of H -spaces with Noetherian mod p cohomology is quite well understood. Using Lannes’ T -functor and Bousfield nullification functors, those spaces fit into a fibration where the base and the fibre are finite and Eilenberg–MacLane H -spaces, respectively, see [BCS01] and [CCS07]. In this section, we use this decomposition and the work in the previous sections to show that for a connected H -space X with Noetherian mod p cohomology ring, the category of modules over $C^*(X)$ is stratified.

4.1. Unstable algebras over the Steenrod algebra. The mod p cohomology of any space X is equipped with an action of the mod p Steenrod algebra \mathcal{A}_p . In fact, it is a graded \mathcal{A}_p -module, and the action of \mathcal{A}_p satisfies additional properties, such that $H^*(X; \mathbb{F}_p)$ is an object in the abelian category of unstable modules over the Steenrod algebra \mathcal{U} , see [Sch94] for example. We let \mathcal{K} denote the category of unstable algebras over the Steenrod algebra; that is, an object $K \in \mathcal{K}$ is an unstable module together with maps $\rho: K \otimes K \rightarrow K$, $\eta: \mathbb{F}_p \rightarrow K$ satisfying some additional axioms, again see [Sch94]. The diagonal of X equips $H^*(X; \mathbb{F}_p)$ with a multiplication, which satisfies certain compatibility relations with the Steenrod algebra, so that the mod p cohomology of any space is in fact an object of \mathcal{K} .

We recall that a morphism $f: R \rightarrow S$ of \mathbb{F}_p -algebras is said to be an F -monomorphism if every element in $\ker(f)$ is nilpotent, and an F -epimorphism if for every $s \in S$, there exists a natural number n such that $s^{p^n} \in \text{Im}(f)$. We say that f is an F -isomorphism if it is both an F -monomorphism and an F -epimorphism. If the number n can be chosen independently of s , then we say that it is a uniform F -isomorphism. Lannes’ machinery gives a criterion for when a morphism of unstable algebras is an F -monomorphism or F -epimorphism.

Theorem 4.1. [HLS93, Corollary II.1.4] *Let $\phi: K \rightarrow K'$ be a morphism of unstable algebras. Then the morphism of \mathbb{F}_p -algebras underlying ϕ is an F -monomorphism (resp. F -epimorphism) if and only if*

$$\text{Hom}_{\mathcal{K}}(K', H^*(BV)) \longrightarrow \text{Hom}_{\mathcal{K}}(K, H^*(BV))$$

is surjective (resp. injective) for all elementary abelian p -groups V .

Given a topological space X , taking cohomology induces a morphism

$$[BV, X] \longrightarrow \text{Hom}_{\mathcal{K}}(H^*(X), H^*(BV))$$

and Lannes’ theory [Lan92] establishes conditions under which it is a bijection. Recall that we write X_p^\wedge for the Bousfield–Kan p -completion of a space X .

Theorem 4.2 (Lannes). *Let X be a topological space such that $H^*(X; \mathbb{F}_p)$ is of finite type, then taking mod p cohomology induces a bijection*

$$[BV, X_p^\wedge] \xrightarrow{\cong} \text{Hom}_{\mathcal{K}}(H^*(X), H^*(BV)).$$

Moreover, if $\pi_1 X$ is a finite p -group, then $[BV, X] \cong [BV, X_p^\wedge]$.

Proof. The first part of the statement is [Lan92, Theorem 3.1.1]. For the second, by the obstruction theoretic argument of [DZ87, proof of Theorem 3.1], the natural map $X \rightarrow X_p^\wedge$ induces an isomorphism $[BV, X] \cong [BV, X_p^\wedge]$. \square

One can combine [Theorems 4.1](#) and [4.2](#) into the following.

Proposition 4.3. *Let X and Y be topological spaces such that $H^*(X)$ and $H^*(Y)$ are of finite type. Let $f: X \rightarrow Y$ be a map such that*

$$[BV, X_p^\wedge] \longrightarrow [BV, Y_p^\wedge]$$

is injective (resp. surjective) for all elementary abelian p -groups V , then $f^: H^*(Y) \rightarrow H^*(X)$ is an F -monomorphism (resp. F -epimorphism).*

Finally, we state a theorem proven by Miller [Mil84, Theorem 1.5] that allows us to remove p -completions when mapping from classifying spaces of elementary abelian p -groups.

Theorem 4.4 (Miller). *Let X be a connected nilpotent space, then the canonical map $X \rightarrow X_p^\wedge$ induces an isomorphism*

$$\text{map}_*(BV, X) \xrightarrow{\cong} \text{map}_*(BV, X_p^\wedge).$$

Corollary 4.5. *Let X be a connected nilpotent space with finite fundamental group, then the canonical map $X \rightarrow X_p^\wedge$ induces an equivalence*

$$[BV, X] \xrightarrow{\cong} [BV, X_p^\wedge].$$

Proof. We recall that if Y is a path-connected space, the forgetful map $[BV, Y]_* \rightarrow [BV, Y]$ is surjective and two elements have the same image if and only if they are in the same orbit by the action of $\pi_1(Y)$. Now consider the following commutative diagram

$$\begin{array}{ccccccc} \pi_1(X) & \longrightarrow & [BV, X]_* & \longrightarrow & [BV, X] & \longrightarrow & \pi_0(X) = * \\ \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ \pi_1(X_p^\wedge) & \longrightarrow & [BV, X_p^\wedge]_* & \longrightarrow & [BV, X_p^\wedge] & \longrightarrow & \pi_0(X_p^\wedge) = *. \end{array}$$

The diagram shows that $[BV, X] \rightarrow [BV, X_p^\wedge]$ is surjective. To prove injectivity, one needs to prove that given two pointed maps $f, g: BV \rightarrow X$ such that $f_p^\wedge, g_p^\wedge \in [BV, X_p^\wedge]$ differ by the action of $\pi_1(X_p^\wedge)$, then f and g differ by the action of $\pi_1(X)$. This is the case if $\pi_1(X) \rightarrow \pi_1(X_p^\wedge)$ is surjective, for example, when $\pi_1(X)$ is finite. \square

4.2. $B\mathbb{Z}/p$ -nullification and nilpotent fibrations. We collect some material about $B\mathbb{Z}/p$ -nullification and nilpotent fibrations that we will use later in this section. Our main references for the former are [Far96, Bou94], while we refer to [BK72] for the latter.

Let A be a space. Dror-Farjoun [Far96] and Bousfield [Bou94] construct a localization functor $P_A: \text{Top} \rightarrow \text{Top}$ that universally inverts the map $A \rightarrow *$, the so-called A -nullification. In more detail, a space X is A -null if the canonical map $A \rightarrow *$ induces a weak equivalence

$$X \simeq \text{map}(*, X) \xrightarrow{\sim} \text{map}(A, X).$$

If A and X are connected, this is equivalent to a weak equivalence $* \simeq \text{map}_*(A, X)$ on pointed mapping spaces. The functor P_A is coaugmented and takes values in A -null spaces. Moreover, if X is a space and Y is an A -null space, then the coaugmentation of P_A induces a weak equivalence

$$\text{map}(P_A X, Y) \xrightarrow{\sim} \text{map}(X, Y),$$

i.e., P_A is left adjoint to the inclusion of the category of A -null spaces into Top . It follows that P_A preserves finite products and hence H -spaces; in fact, if X is an H -space, then $X \rightarrow P_A X$ is an H -map of H -spaces. In [Bou94, Proposition 2.9], Bousfield proves that P_A preserves the property of being simply connected.

Remark 4.6. It is an open problem (see [Far96, Question 9.F.7]) whether P_A also preserves nilpotency.

We now specialize to the case $A = B\mathbb{Z}/p$. Miller shows that $\text{map}_*(B\mathbb{Z}/p, X)$ is weakly contractible for any nilpotent space X with bounded mod p cohomology, see [Mil84, Theorem C], so we get:

Theorem 4.7. *A connected nilpotent space with bounded mod p cohomology is $B\mathbb{Z}/p$ -null.*

Moreover, by Theorem 4.4, if X is connected and of finite type, then there are weak equivalences

$$* \simeq \text{map}_*(B\mathbb{Z}/p, P_{B\mathbb{Z}/p} X) \simeq \text{map}_*(B\mathbb{Z}/p, (P_{B\mathbb{Z}/p} X)_p^\wedge),$$

so $(P_{B\mathbb{Z}/p} X)_p^\wedge$ is $B\mathbb{Z}/p$ -null as well.

Recall that a fibration $f: E \rightarrow B$ of connected spaces is called nilpotent if the fiber of f is connected and the canonical action of $\pi_1 E$ on $\pi_i F$ is nilpotent for all $i \geq 1$. In this case, we will also refer to

$$F \longrightarrow E \longrightarrow B$$

as a nilpotent fiber sequence. Note that a space X is nilpotent if and only if $X \rightarrow *$ is a nilpotent fibration. Furthermore, the following two-for-three property holds for nilpotent fibrations: If

$$X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

are fibrations with connected fibers, then if two of the three maps $f_1, f_2, f_1 f_2$ are nilpotent fibrations, then so is the third, see [BK72, Proposition II.4.4]. In particular, if $f: E \rightarrow B$ is a fibration with E and B both nilpotent, then f is a nilpotent fibration.

In [BK72, Section II.4.8], it is shown that the p -completion of a nilpotent fiber sequence is again a fiber sequence, i.e., if $f: E \rightarrow B$ is a nilpotent fibration, then $f_p^\wedge: E_p^\wedge \rightarrow B_p^\wedge$ is a fibration with fiber F_p^\wedge . In particular, if X is nilpotent, then so is X_p^\wedge .

We collect these results in a lemma for later use.

Lemma 4.8. *Let $f: E \rightarrow B$ be a nilpotent fibration of connected spaces with fiber F such that E is nilpotent and B is $B\mathbb{Z}/p$ -null and of finite type, then p -completion gives a nilpotent fiber sequence*

$$F_p^\wedge \longrightarrow E_p^\wedge \longrightarrow B_p^\wedge$$

with $B\mathbb{Z}/p$ -null base B_p^\wedge .

Proof. It follows from the assumptions on f and E that the base space B is nilpotent, so B_p^\wedge is also $B\mathbb{Z}/p$ -null by Theorem 4.4. Furthermore, p -completion yields the indicated fiber sequence, which is nilpotent because both E_p^\wedge and B_p^\wedge are. \square

4.3. Stratification along fiber sequences. Let $F \rightarrow E \rightarrow B$ be a fiber sequence where E and B are connected. The universal property of the pushout in commutative ring spectra provides a canonical map

$$\iota: C^*(E) \otimes_{C^*(B)} \mathbb{F}_p \longrightarrow C^*(F).$$

As explained in more detail in [Sha09], we can refine this map as follows: Without loss of generality, we can assume that ΩB is a topological group and that F is a ΩB -space (by using Kan's loop group functor [Kan58, GJ09], for example). Consider the ring spectrum $R = C_*(\Omega B)$

and let $R \rightarrow \mathbb{F}_p$ be the canonical map, which allows us to view \mathbb{F}_p as an R -module. Note that the Rothenberg–Steenrod construction shows that $\text{End}_R(\mathbb{F}_p) \simeq C^*(B)$, see [DGI06, Section 4.22]. Note that as in the proof of [Sha09, Lemma 5.5] $C^*(F)$ is an R -module. By [DW09, Lemma 2.10] there is an equivalence of R -modules $C^*(E) \simeq \text{Hom}_R(\mathbb{F}_p, C^*(F))$ which fits into a map of R -modules

$$C^*(E) \otimes_{C^*(B)} \mathbb{F}_p \simeq \text{Hom}_R(\mathbb{F}_p, C^*(F)) \otimes_{\text{End}_R(\mathbb{F}_p)} \mathbb{F}_p \xrightarrow{\iota'} C^*(F), \quad (4.9)$$

where ι' is adjoint to the identity map on $\text{Hom}_R(\mathbb{F}_p, C^*(F))$ via

$$\text{Hom}_R(\text{Hom}_R(\mathbb{F}_p, C^*(F)) \otimes_{\text{End}_R(\mathbb{F}_p)} \mathbb{F}_p, C^*(F)) \simeq \text{Hom}_{\text{End}_R(\mathbb{F}_p)}(\text{Hom}_R(\mathbb{F}_p, C^*(F)), \text{Hom}_R(\mathbb{F}_p, C^*(F))).$$

The composite constructed in (4.9) then coincides with ι as defined above.

Definition 4.10. We say that the fiber sequence $F \rightarrow E \rightarrow B$ with E and B connected is of Eilenberg–Moore type (at the prime p) if the map ι is an equivalence.

- Example 4.11.**
- (1) Every trivial fibration $F \rightarrow F \times B \rightarrow B$ is of Eilenberg–Moore type in light of the equivalence $C^*(F \times B) \simeq C^*(F) \otimes_{\mathbb{F}_p} C^*(B)$.
 - (2) Recall that a fibration $F \rightarrow E \rightarrow B$ is simple if the action of $\pi_1(B)$ on the homotopy groups of F is trivial and $\pi_1(B)$ is abelian. If $F \rightarrow E \rightarrow B$ satisfies an Eilenberg–Moore convergence theorem, such as when $\pi_1(B)$ acts nilpotently on $H_*(F)$ [Dwy74], then it is of Eilenberg–Moore type. In particular, this is satisfied if the fibration is simple, e.g., a fibration of H -spaces and H -maps.
 - (3) In [Sha09, Lemma 5.6], the author shows that the morphism of R -modules ι is a \mathbb{F}_p -cellular approximation of $C^*(F)$ if R is proxy-small, see also [DGI06, Proposition 4.10]. Moreover, in the proof of [Sha09, Proposition 5.8] it is shown that $C^*(F)$ is \mathbb{F}_p -cellular if $\pi_1(B)$ acts nilpotently on $H^n(F)$ for every n . Both of these conditions are satisfied if $\pi_1(B)$ is a finite p -group and F of finite type, so ι is an equivalence in this case.

Lemma 4.12. *Suppose $F \rightarrow E \rightarrow B$ is a fibration of Eilenberg–Moore type with $H^*(B)$ finite. If $g: C^*(E) \rightarrow C^*(F)$ is the induced map on mod p cochains, then both g^* and $g^!$ are conservative.*

Proof. Since $H^*(B)$ is finite, the induced map $f: C^*(B) \rightarrow \mathbb{F}_p$ is cosmall by [DGI06, Proposition 3.16], i.e., $C^*(B) \in \text{Thick}_{C^*(B)}(k)$. In particular, f^* and $f^!$ are conservative by Lemma 1.22. Moreover, because the fibration is of Eilenberg–Moore type, there is a pushout square of commutative ring spectra

$$\begin{array}{ccc} C^*(B) & \longrightarrow & C^*(E) \\ f \downarrow & & \downarrow g \\ k & \longrightarrow & C^*(F). \end{array}$$

It follows that we are in the situation of Lemma 1.23, so g^* and $g^!$ are conservative. \square

Recall that we write $P_{B\mathbb{Z}/p}$ for the $B\mathbb{Z}/p$ -nullification functor; see Section 4.2 for the basic properties of $B\mathbb{Z}/p$ -nullification.

Lemma 4.13. *Let $F \xrightarrow{i} E \rightarrow B$ be a nilpotent fiber sequence of connected spaces with mod p cohomology of finite type and assume that E is connected nilpotent and B is connected and $B\mathbb{Z}/p$ -null. Then the induced map $H^*(i): H^*(E) \rightarrow H^*(F)$ is an F -monomorphism. If the fibration is simple, then $H^*(i)$ is an F -isomorphism.*

Proof. The conditions of Lemma 4.8 are satisfied, so the p -completion of $F \rightarrow E \rightarrow B$ results in a nilpotent fiber sequence

$$F_p^\wedge \longrightarrow E_p^\wedge \longrightarrow B_p^\wedge \quad (4.14)$$

with $B\mathbb{Z}/p$ -null base. The evaluation map then gives an equivalence $\text{map}(B\mathbb{Z}/p, B_p^\wedge) \simeq B_p^\wedge$ since B is connected. Now applying $\pi_* \text{map}(BV, -)$ to the fiber sequence of (4.14) gives an exact sequence

$$\pi_1(B_p^\wedge) \longrightarrow [BV, F_p^\wedge] \xrightarrow{\phi_V} [BV, E_p^\wedge] \longrightarrow \pi_0(B_p^\wedge) = *.$$

We see that ϕ_V is surjective and it identifies $[BV, E_p^\wedge]$ with the orbit space by the action of $\pi_1(B_p^\wedge)$ on $[BV, F_p^\wedge]$. If the fibration is simple, then ϕ_V is a bijection. Then the following natural composite is a monomorphism

$$\text{Hom}_{\mathcal{K}}(H^*(F), H^*(BV)) \simeq [BV, F_p^\wedge] \longrightarrow [BV, E_p^\wedge] \simeq \text{Hom}_{\mathcal{K}}(H^*(E), H^*(BV)),$$

and even an isomorphism when the fibration is simple, as a consequence of Lannes' Theorem, see [Theorem 4.2](#). Since this holds for all elementary abelian p -groups V , we deduce from [Theorem 4.1](#) the conclusion for $H^*(E) \rightarrow H^*(F)$. \square

Proposition 4.15. *Let $F \rightarrow E \rightarrow B$ be a nilpotent fiber sequence of Eilenberg–Moore type with E nilpotent. Assume $H^*(F)$ and $H^*(E)$ are Noetherian and $H^*(B)$ finite, then the induced map $i: C^*(E) \rightarrow C^*(F)$ descends stratification, that is, if $\text{Mod}_{C^*(F)}$ is stratified, then so is $\text{Mod}_{C^*(E)}$.*

Proof. The goal is to apply [Theorem 2.4](#) to the induced morphism of ring spectra

$$i: C^*(E) \longrightarrow C^*(F),$$

which allows us to descend the stratification from $\text{Mod}_{C^*(F)}$ to $\text{Mod}_{C^*(E)}$. For this we need to check that i satisfies Quillen lifting and that induction and coinduction along i are conservative.

Since B is a connected nilpotent space with finite mod p cohomology, it is $B\mathbb{Z}/p$ -null by [Theorem 4.7](#). Therefore, [Lemma 4.13](#) implies that i induces an F -isomorphism $H^*(E) \rightarrow H^*(F)$, so i satisfies simple Quillen lifting by [Corollary 2.15](#), hence Quillen lifting by [Lemma 2.11](#). Moreover, [Lemma 4.12](#) applies to show that induction and coinduction along i are conservative. \square

Remark 4.16. Consider a trivial fibration $F \rightarrow F \times B \rightarrow B$ with $H^*(F)$ Noetherian, $H^*(B)$ finite, and suppose that $\text{Mod}_{C^*(F)}$ stratified. Note that $i: C^*(F \times B) \rightarrow C^*(F)$ is surjective on π_* , and in particular, an F -epimorphism. Using this instead of [Lemma 4.13](#) in the proof of [Proposition 4.15](#) yields the stratification of $\text{Mod}_{C^*(F \times B)}$.

4.4. Noetherian Hopf spaces. Since H -spaces are simple, any fibration of H -spaces and H -maps is simple, and therefore of Eilenberg–Moore type. Moreover, simple spaces are in particular nilpotent, and hence p -good [[BK72](#), Proposition V.3.4]. It follows that $C^*(X) \simeq C^*(X_p^\wedge)$, see [Section 3.1](#).

For the following, we recall that a discrete p -toral group P is a group with a normal subgroup $P_0 \cong (\mathbb{Z}/p^\infty)^r$ such that $\pi = P/P_0$ is a finite p -group. If the discrete p -toral group is abelian, then P splits and it is isomorphic to a product $P \cong (\mathbb{Z}/p^\infty)^r \times \pi$. In particular, the classifying space of any abelian p -discrete toral group is \mathbb{F}_p -equivalent to a compact abelian Lie group since $BP_p^\wedge \cong ((BS^1)^r \times B\pi)_p^\wedge$.

Theorem 4.17. *If X is a connected H -space with Noetherian mod p cohomology, then $\text{Mod}_{C^*(X)}$ is stratified.*

Proof. By [[BCS01](#), Theorems 1.2,1.3] (see also [[CCS07](#)]), the homotopy fiber of the H -map $X_p^\wedge \rightarrow (P_{B\mathbb{Z}/p}X)_p^\wedge$ is equivalent to $K(P, 1)_p^\wedge$, where P is an abelian discrete p -toral group. Moreover, the base space $(P_{B\mathbb{Z}/p}X)_p^\wedge$ is $B\mathbb{Z}/p$ -null and has finite mod p cohomology. Note that since X is an H -space, then $(P_{B\mathbb{Z}/p}X)_p^\wedge$ inherits the structure of an H -space and the nullification

map is an H -map, since both nullification and completion are homotopy functors with commute with finite products. The corresponding fiber sequence is of Eilenberg–Moore type, so the map

$$\iota: C^*(X) \otimes_{C^*(P_{B\mathbb{Z}/p}X)} \mathbb{F}_p \longrightarrow C^*(K(P, 1))$$

is an equivalence. Since $C^*(K(P, 1)) \simeq C^*(BA)$, where A is a compact abelian Lie group, $\text{Mod}_{C^*(K(P, 1))}$ is stratified by [BG14] or [BCHV17], so we can apply Proposition 4.15 to deduce the claimed result. \square

As a consequence of Theorem 1.10, the telescope conjecture holds for $\text{Mod}_{C^*(X)}$ for any connected H -space X with Noetherian mod p cohomology.

4.5. An explicit example. We conclude this section with an explicit example given by $X = S^3\langle 3 \rangle$, the 3-connected cover of S^3 . This space fits in a principal fibration $BS^1 \xrightarrow{i} S^3\langle 3 \rangle \xrightarrow{j} S^3$ induced by the fundamental class $S^3 \rightarrow K(\mathbb{Z}, 3)$. The mod p cohomology of $S^3\langle 3 \rangle$ is

$$H^*(S^3\langle 3 \rangle) \cong \mathbb{F}_p[x_{2p}] \otimes \Lambda(y_{2p+1}).$$

In cohomology, $H^*(Bi): H^*(S^3\langle 3 \rangle) \rightarrow H^*(BS^1)$ sends x_{2p} to x^p , so this is an F -isomorphism. Then $C^*(Bi): C^*(S^3\langle 3 \rangle) \rightarrow C^*(BS^1)$ satisfies simple Quillen lifting by Corollary 2.15. Since S^3 is simply connected and has finite cohomology, Lemma 4.12 implies that $C^*(i): C^*(S^3\langle 3 \rangle) \rightarrow C^*(BS^1)$ is biconservative. As a consequence of Theorem 2.4, $\text{Mod}_{C^*(S^3\langle 3 \rangle)}$ is stratified.

Note that the morphism $C^*(i): C^*(S^3\langle 3 \rangle) \rightarrow C^*(BS^1)$ is neither finite nor split. Indeed, because all spaces involved are simply connected, the fiber sequence $\Omega S^3 \rightarrow BS^1 \rightarrow S^3\langle 3 \rangle$ is of Eilenberg–Moore type, so we have a pushout square of commutative ring spectra

$$\begin{array}{ccc} C^*(S^3\langle 3 \rangle) & \xrightarrow{C^*(i)} & C^*(BS^1) \\ \downarrow & & \downarrow \\ k & \xrightarrow{C^*(j)} & C^*(\Omega S^3). \end{array}$$

Since $H^*(\Omega S^3)$ is infinite-dimensional, the morphism $C^*(j)$ is not finite, and thus neither is $C^*(i)$. Further, the ring $H^*(BS^1; \mathbb{F}_p)$ does not contain nilpotent elements, so the previous cohomology calculation implies that $C^*(i)$ cannot admit a retraction even on homotopy groups.

This means that none of the known techniques for establishing costratification (for example, [BCHV17, Proposition 3.20 or Theorem 3.27]) apply to the morphism $C^*(i): C^*(S^3\langle 3 \rangle) \rightarrow C^*(BS^1)$, so we currently do not know whether $\text{Mod}_{C^*(S^3\langle 3 \rangle)}$ is costratified.

5. CHOUINARD'S CONDITION AND STRATIFICATION

In order to use the descent results of Section 2 to study stratifications for $\text{Mod}_{C^*(X)}$ for general topological spaces X with Noetherian mod p cohomology, we need to first find a candidate morphism $\rho_X: C^*(X) \rightarrow S$ with Mod_S stratified. The goal of this section is to construct such a morphism ρ_X using Lannes' theory and to then deduce from Rector's generalization of Quillen's stratification theorem that ρ_X satisfies Quillen lifting.

5.1. Rector's category. Given the distinguished role played by the cohomology of elementary abelian p -groups V for detecting when a morphism in \mathcal{K} is an F -isomorphism, it is convenient to organize \mathcal{A}_p -algebra morphisms from a fixed $K \in \mathcal{K}$ to $H^*(BV)$ in a category $\mathcal{R}(K)$. This category was introduced by Rector [Rec84], generalizing previous work of Quillen [Qui71] on the cohomology of compact Lie groups.

Definition 5.1. Let K be a Noetherian unstable \mathcal{A}_p -algebra. The Rector category $\mathcal{R}(K)$ of K has as objects pairs (V, φ) , where V is an elementary abelian p -group and $\varphi: K \rightarrow H^*(BV)$ is a morphism of unstable algebras which is finite, i.e., $H^*(BV)$ is a finitely generated K -module via φ . A morphism $f: (V, \varphi) \rightarrow (V', \varphi')$ between objects in $\mathcal{R}(K)$ is given by a monomorphism $f: V \rightarrow V'$ such that $H^*(Bf) \circ \varphi' = \varphi$.

Remark 5.2. This category is in fact the opposite of the so-called fundamental category of K as defined by Rector, cf. Definition 1.2 in [Rec84]; by now, it is more conventional to work with $\mathcal{R}(K)$ instead of Rector's original category $\mathcal{R}(K)^{\text{op}}$.

Example 5.3. Let G be a compact Lie group. Then Lannes' theory shows that $\mathcal{R}(H^*(BG))$ is equivalent to the Quillen category $\mathcal{A}_p(G)$ of G introduced in Section 3, see the discussion in [HLS95, Section 5.3].

If $K \in \mathcal{K}$ is Noetherian, Rector showed that $\mathcal{R}(K)$ is equivalent to a finite category, since isomorphism classes of objects are determined by \mathcal{A}_p -invariant prime ideals and K has finitely many prime ideals. Moreover, there is only a finite set of morphisms since the set $\text{Hom}(V, V')$ of abelian group homomorphisms is finite for two elementary abelian p -groups V, V' . The functor from $\mathcal{R}(K)$ to \mathcal{K} which assigns $H^*(BV; \mathbb{F}_p)$ to (V, φ) induces a natural homomorphism

$$r_K: K \longrightarrow \varprojlim_{\mathcal{R}(K)} H^*(BV).$$

Rector [Rec84] and Broto–Zarati [BZ88] prove r_K provides a close approximation to K :

Theorem 5.4 (Rector, Broto–Zarati). *Let K be a Noetherian unstable \mathcal{A}_p -algebra, then r_K is an F -isomorphism.*

In his work on equivariant cohomology Quillen [Qui71, Section 10 and 11] shows how this gives a decomposition of the prime ideal spectrum of $H^*(BG)$. Rector proves a similar result [Rec84, Theorem 1.7]. Note that Rector states the result in terms of the maximal ideal spectrum instead of the prime ideal spectrum - as explained after Proposition 11.2 of [Qui71], the results also hold for the prime ideal spectrum, with the exception that the ‘Weyl group’ (denoted by W_ϕ below) no longer acts freely, but rather only transitively. In order to state these results, we need to introduce some notation, similar to that used in Section 3.3.

For an elementary abelian group E , we recall that $\mathcal{V}_E = \text{Spec}^h(H^*(BE))$ and that $\mathcal{V}_E^+ = \mathcal{V}_E \setminus \bigcup_{E' < E} \text{res}_E^{E'} \mathcal{V}_{E'}$, where $\text{res}_E^{E'}: \mathcal{V}_{E'} \rightarrow \mathcal{V}_E$ is the restriction map. For $K \in \mathcal{K}$, we will write $\mathcal{V}_K = \text{Spec}^h(K)$. If $\phi: K \rightarrow H^*(BE)$ is a morphism in $\mathcal{R}(K)$, then res_ϕ denotes the corresponding restriction map on varieties. We let

$$\mathcal{V}_{\phi, E} = \text{res}_\phi(\mathcal{V}_E) \quad \text{and} \quad \mathcal{V}_{\phi, E}^+ = \text{res}_\phi(\mathcal{V}_E^+).$$

Finally, we let W_ϕ denote the group of automorphisms of (E, ϕ) in $\mathcal{R}(K)$, and let $\mathcal{E}(K)$ denote a set of representatives for the isomorphism classes of objects in $\mathcal{R}(K)$.

Theorem 5.5 (Rector). *Let K be a Noetherian unstable \mathcal{A}_p -algebra, then:*

(1) *There is a decomposition of varieties*

$$\mathcal{V}_K = \coprod_{(E, \phi) \in \mathcal{E}(K)} \mathcal{V}_{\phi, E}^+.$$

(2) *W_ϕ acts transitively on \mathcal{V}_E^+ , and $\mathcal{V}_{\phi, E}^+ \cong \mathcal{V}_E^+/W_\phi$.*

(3) *$\mathcal{V}_{\phi, E} \subseteq \mathcal{V}_{\phi', E}$ if and only if there is a morphism $(E', \phi') \rightarrow (E, \phi)$ in $\mathcal{R}(K)$.*

We would like to discuss the situation when K is the mod p cohomology of a topological space X . For suitable X , the objects and morphisms in $\mathcal{R}(K) = \mathcal{R}(H^*(X))$ can be realized by

maps between spaces. Indeed, let X be a p -good connected topological space such that $H^*(X)$ is Noetherian. Consider Rector's category $\mathcal{R}(H^*(X))$ associated to the Noetherian unstable \mathcal{A}_p -algebra $H^*(X)$; for simplicity, we write this simply as $\mathcal{R}(X)$. By [Theorem 4.2](#) any map of unstable \mathcal{A}_p -algebras $\phi: H^*(X) \rightarrow H^*(BV)$ with V an elementary abelian p -group is realized by a unique map up to homotopy $\tilde{\phi}: BV \rightarrow X_p^\wedge$, i.e., $H^*(\tilde{\phi}) = \phi$. Since X is p -good, this provides a map

$$\rho_X^\phi: C^*(X) \simeq C^*(X_p^\wedge) \xrightarrow{C^*(\tilde{\phi})} C^*(BV)$$

with $\pi_*\rho_X^\phi = \phi$. We record this result here.

Lemma 5.6. *Let X be a p -good connected topological space X , then for any pair $(V, \phi) \in \mathcal{R}(X)$ there exists $\rho_X^\phi: C^*(X) \rightarrow C^*(BV)$ as above with $\pi_*\rho_X^\phi = \phi$. Moreover, for any morphism $f: (V, \phi) \rightarrow (V', \phi')$ in \mathcal{E} , we obtain a commutative diagram as follows*

$$\begin{array}{ccc} C^*(X) & \xrightarrow{\rho_X^\phi} & C^*(BV) \\ \rho_X^{\phi'} \downarrow & \nearrow C^*(Bf) & \\ C^*(BV') & & \end{array}$$

which realizes the commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\phi} & H^*(V) \\ \phi' \downarrow & \nearrow H^*(Bf) & \\ H^*(V') & & \end{array}$$

on homotopy.

Proof. Everything but the claim that we can realize the commutative diagram on homotopy by maps of cochains has been shown. This follows because the maps produced by applying [Theorem 4.2](#) are unique up to homotopy; since $\phi \cong H^*(Bf) \circ \phi'$, there is a unique map $BV \rightarrow X_p^\wedge$ realizing $\phi \cong H^*(Bf) \circ \phi'$ on cohomology which factors through $Bf: BV \rightarrow BV'$. Taking cochains, we deduce that $C^*(Bf) \circ \rho_X^{\phi'} \simeq \rho_X^\phi$. \square

Let $\mathcal{E}(X)$ be a set of representatives of isomorphism classes of objects in $\mathcal{R}(X)$, i.e., $\mathcal{E}(X) = \mathcal{E}(H^*(X))$. We thus obtain a map of commutative ring spectra

$$\rho_X := \prod_{(E, \phi) \in \mathcal{E}(X)} \rho_X^\phi: C^*(X) \longrightarrow \prod_{(E, \phi) \in \mathcal{E}(X)} C^*(BE) \quad (5.7)$$

for any p -good connected topological space X . This will be our candidate for descending stratification.

5.2. Quillen lifting for cochains of spaces. We now show that the morphism ρ_X constructed in the previous section always satisfies Quillen lifting. We continue to assume that X is a p -good connected topological space with Noetherian mod p cohomology.

Proposition 5.8. *The morphism $\rho_X: C^*(X) \rightarrow \prod_{(E, \phi) \in \mathcal{E}(X)} C^*(BE)$ defined in (5.7) satisfies simple Quillen lifting.*

Proof. In order to prepare for the proof, we start by introducing some notation. Let $\mathcal{V}_X = \text{Spec}^h(H^*(X))$ and $\mathcal{V}_{\mathcal{E}(X)} = \text{Spec}^h(\prod_{E \in \mathcal{E}(X)} H^*(BE))$, and write $\rho = \rho_X$. By [Theorem 5.5](#)

there is a commutative diagram

$$\begin{array}{ccc} \mathcal{V}_{\mathcal{E}(X)} & \xrightarrow{\text{res}_{\pi_*\rho}} & \mathcal{V}_X \\ \cong \uparrow & & \uparrow \mathfrak{S} = \\ \coprod_{(E,\phi) \in \mathcal{E}(X)} \mathcal{V}_E & \longrightarrow & \coprod_{(E,\phi) \in \mathcal{E}(X)} \mathcal{V}_{\phi,E}^+ \end{array}$$

Let $\mathfrak{p} \in \mathcal{V}_X$ be a prime ideal. Since \mathfrak{S} is a bijection, we may choose a pair $(E, \phi) \in \mathcal{E}(X)$ in which \mathfrak{p} originates: this means that there exists a (not necessarily unique) prime ideal $\mathcal{O}(\mathfrak{p}) \in \mathcal{V}_E^+ \subseteq \mathcal{V}_E$ with $\mathfrak{S}([\mathcal{O}(\mathfrak{p})]) = \mathfrak{p}$, where $[\mathcal{O}(\mathfrak{p})]$ denotes the image of $\mathcal{O}(\mathfrak{p})$ under the quotient map $\mathcal{V}_E^+ \rightarrow \mathcal{V}_{\phi,E}^+$ of [Theorem 5.5\(2\)](#).

To simplify the notation further, we write $\text{supp}_{\mathcal{E}(X)}(N)$ for the support of a $\prod_{E \in \mathcal{E}(X)} C^*(BE)$ -module N . As observed in [[BCHV17](#), Section 5.2], the support of ρ^*M for $M \in \text{Mod}_{C^*(X)}$ decomposes as

$$\text{supp}_{\mathcal{E}(X)}(\rho^*M) \cong \coprod_{(E,\phi) \in \mathcal{E}(X)} \text{supp}_{C^*(BE)}((\rho_X^\phi)^*M),$$

where we use [Lemma 5.6](#) to realize morphisms $\phi: H^*(X) \rightarrow H^*(BE)$ as $\rho_X^\phi: C^*(X) \rightarrow C^*(BE)$. This decomposition will be used implicitly throughout the proof. We need to show that, for any module $M \in \text{Mod}_{C^*(X)}$, there is an equality

$$\text{res}_\rho^{-1} \text{res}_\rho \text{supp}_{\mathcal{E}(X)}(\rho^*M) = \text{supp}_{\mathcal{E}(X)}(\rho^*M), \quad (5.9)$$

leaving the modifications for the dual claim about the cosupport of the coinduction of M to the interested reader. From now on, let us fix such a module M .

Since the inclusion “ \supseteq ” in (5.9) holds automatically, it remains to prove the reverse inclusion. To this end, consider a prime ideal $\mathfrak{r} \in \mathcal{V}_{\mathcal{E}(X)}$ with the property that $\mathfrak{p} := \text{res}_\rho(\mathfrak{r}) \in \text{res}_\rho \text{supp}_{\mathcal{E}(X)}(\rho^*M)$; in particular, $\text{supp}_{\mathcal{E}(X)}(\rho^*M)$ is non-empty. Our goal is to show that $\mathfrak{r} \in \text{supp}_{\mathcal{E}(X)}(\rho^*M)$. We first claim that

(♣) any prime ideal $\mathcal{O}(\text{res}_\rho(\mathfrak{r}))$ in which $\text{res}_\rho(\mathfrak{r})$ originates is contained in $\text{supp}_{\mathcal{E}(X)}(\rho^*M)$.

To this end, choose $\mathfrak{q} \in \text{supp}_{\mathcal{E}(X)}(\rho^*M)$ such that $\text{res}_\rho(\mathfrak{r}) = \text{res}_\rho(\mathfrak{q})$. In particular, there exists $(E, \phi) \in \mathcal{E}(X)$ such that $\mathfrak{q} \in \mathcal{V}_E$. Without loss of generality, we may assume that \mathfrak{p} in fact originates in (E, ϕ) with $\mathcal{O}(\mathfrak{p}) = \mathfrak{q}$: Indeed, if not there exists a (E', ϕ') with E' of smaller p -rank than E and a monomorphism $f: E' \rightarrow E$ such that \mathfrak{q} is in the image of the associated restriction map $\text{res}_f: \mathcal{V}_{E'} \rightarrow \mathcal{V}_E$. The map f fits into a commutative diagram of unstable algebras over the Steenrod algebra, as displayed on the left,

$$\begin{array}{ccc} H^*(X) & \xrightarrow{H^*(\phi)} & H^*(BE) \\ & \searrow H^*(\phi') & \downarrow H^*(f) \\ & & H^*(BE') \end{array} \quad \begin{array}{ccc} C^*(X) & \xrightarrow{C^*(\phi)} & C^*(BE) \\ & \searrow C^*(\phi') & \downarrow C^*(f) \\ & & C^*(BE') \end{array},$$

which by virtue of [Lemma 5.6](#) lifts to the commutative diagram on the right. Using the Avrunin–Scott identities ([Theorem 3.13\(2\)](#)) applied to $C^*(f)$, we see that any prime ideal $\mathfrak{q}' \in \mathcal{V}_{E'}$ which maps to \mathfrak{q} under res_f must be in $\text{res}_\rho^{-1}(\mathfrak{p}) \cap \text{supp}_{\mathcal{E}(X)}(\rho^*M)$, so we can replace \mathfrak{q} by \mathfrak{q}' and (E, ϕ) by (E', ϕ') if necessary. Since E has finite p -rank, we see that this process must stop at a finite stage, so we can assume \mathfrak{p} in fact originates in (E, ϕ) .

By [Theorem 5.5\(2\)](#) all prime ideals in which \mathfrak{p} originates are related by an isomorphism in $\mathcal{R}(X)$. Since [Lemma 5.6](#) shows that every such isomorphism can be realized by a commutative

diagram of cochains as in the previous paragraph, we deduce that $\mathcal{O}(\mathfrak{p})$ is in $\text{supp}_S(\rho^*M)$ for any choice of $\mathcal{O}(\mathfrak{p})$, finishing the proof of (\clubsuit) .

In order to finish the proof of the proposition, it remains to prove that \mathfrak{r} itself is contained in $\text{supp}_{\mathcal{E}(X)}(\rho^*M)$. Let $(E, \phi) \in \mathcal{E}(X)$ be such that $\mathfrak{r} \in \mathcal{V}_E \subseteq \mathcal{V}_{\mathcal{E}(X)}$. As before, we can find a monomorphism $f: (E', \phi') \rightarrow (E, \phi)$ in $\mathcal{R}(X)$ and a prime ideal $\mathfrak{r}' \in \mathcal{V}_{E'}^+$ such that $\text{res}_f(\mathfrak{r}') = \mathfrak{r}$. In particular, we see that

$$\text{res}_{\phi'}(\mathfrak{r}') = \text{res}_{\phi} \text{res}_f(\mathfrak{r}') = \text{res}_{\phi}(\mathfrak{r}) = \text{res}_{\rho}(\mathfrak{r}),$$

hence $\mathfrak{r}' = \mathcal{O}(\text{res}_{\rho}(\mathfrak{r}))$ is a prime ideal in which $\text{res}_{\rho}(\mathfrak{r})$ originates. From (\clubsuit) we know that $\mathfrak{r}' \in \text{supp}_{\mathcal{E}(X)}(\rho^*M)$, so [Theorem 1.12\(1\)](#) implies $\mathfrak{r} \in \text{supp}_{\mathcal{E}(X)}(\rho^*M)$ as desired. \square

5.3. Chouinard's condition for cochains on spaces.

Definition 5.10. A p -good connected topological space X with Noetherian mod p cohomology is said to satisfy Chouinard's condition if the map ρ_X of [\(5.7\)](#) is biconservative, i.e., if induction and coinduction along ρ_X are conservative.

Remark 5.11. [Proposition 1.19](#) exhibits a necessary condition for ρ_X to be conservative: res_{ρ_X} needs to be surjective. This is indeed the case and follows from the decomposition of the variety $\mathcal{V}_{H^*(X)}$ in [Theorem 5.5\(1\)](#) and [\(2\)](#).

Theorem 5.12. *Let X be a p -good connected space with Noetherian mod p cohomology, then $\text{Mod}_{C^*(X)}$ is stratified if and only if X satisfies Chouinard's condition.*

Proof. Suppose that ρ_X is biconservative. [Proposition 2.9](#) and [Theorem 3.13](#) show that the category $\text{Mod}_{\prod_{(E, \phi) \in \mathcal{R}(X)} C^*(BE)} \simeq \prod_{(E, \phi) \in \mathcal{R}(X)} \text{Mod}_{C^*(BE)}$ is stratified. Since Quillen lifting for ρ_X has been proven in [Proposition 5.8](#), we can descend stratification along ρ_X by [Theorem 2.4](#), so $\text{Mod}_{C^*(X)}$ is stratified.

The *only if* direction follows from [Proposition 1.19](#) and Rector's strong form of Quillen stratification given in [Theorem 5.5](#). \square

Example 5.13. Let X be a connected finite complex. Then Rector's category consists only of the trivial subgroup: indeed, for any elementary abelian p -group V and any map $f: BV \rightarrow X$, Lannes' theory says that $H^*(f): H^*(X) \rightarrow H^*(BV)$ is trivial. This also follows from Miller's solution of the Sullivan conjecture [\[Mil84\]](#), since any map $BV \rightarrow X$ is nullhomotopic. The map $H^*(f)$ is thus finite if and only if $V = \{e\}$, the trivial group. Moreover, the canonical map $C^*(X) \rightarrow \mathbb{F}_p$ is biconservative. This follows from the same argument in [Lemma 4.12](#): because $H^*(X)$ is finite, we have $C^*(X) \in \text{Thick}_{C^*(X)}(\mathbb{F}_p)$, and so [Lemma 1.22](#) applies. We deduce that $C^*(X)$ is stratified.

Remark 5.14. More generally, suppose X is a connected finite CW-complex, and R a Noetherian commutative ring spectrum so that $\pi_* C^*(X; R)$ is Noetherian. There is a splitting

$$R \longrightarrow C^*(X; R) \xrightarrow{s} R$$

of R -modules, where the morphism s is given by evaluating at a basepoint $* \in X$. We have that $C^*(X; R) \in \text{Thick}_{C^*(X; R)}(R)$ by induction on the number of cells of X . It follows that s is biconservative. Moreover, the splitting implies that $\pi_* s: \pi_* C^*(X; R) \rightarrow \pi_* R$ is surjective, and hence simple Quillen lifting is satisfied by [Lemma 2.11](#). It follows from [Theorem 2.4](#) that $\text{Mod}_{C^*(X; R)}$ is stratified.

5.4. Chouinard’s condition for H -spaces with Noetherian mod p cohomology. Finally, we verify that H -spaces with Noetherian mod p cohomology satisfy Chouinard’s condition. In particular, this demonstrates that the stratification of H -spaces with Noetherian mod p cohomology ([Theorem 4.17](#)) fits into the more general framework of this section.

Theorem 5.15. *Let X be an H -space with Noetherian mod p cohomology, then X satisfies Chouinard’s condition, i.e., the map ρ_X constructed in [\(5.7\)](#) is biconservative.*

Proof. As in the proof of [Theorem 4.17](#), [\[BCS01\]](#) provides a nilpotent fiber sequence of Eilenberg–Moore type

$$K(P, 1)_p^\wedge \xrightarrow{i} X_p^\wedge \xrightarrow{j} (P_{B\mathbb{Z}/p}X)_p^\wedge,$$

in which P is an abelian discrete p -toral group and $(P_{B\mathbb{Z}/p}X)_p^\wedge$ is a $B\mathbb{Z}/p$ -null nilpotent space with finite mod p cohomology. We recall that $C^*(K(P, 1)_p^\wedge) \simeq C^*(BA)$ for A an abelian compact Lie group. We may thus replace $K(P, 1)$ by BA . The proof of [Lemma 4.13](#) shows that the induced map $H^*(i): H^*(X) \rightarrow H^*(BA)$ is an F -isomorphism. Because $H^*(X)$ and $H^*(BA)$ are both finitely generated \mathbb{F}_p -algebras, $H^*(i)$ is even a uniform F -isomorphism, see the remark after A.2 of [\[Hen98\]](#), and hence $H^*(i)$ induces an isomorphism of Rector categories $\mathcal{R}(BA) \cong \mathcal{R}(X)$, by [\[Rec84, Proposition 1.5\]](#). In particular, passage to isomorphism classes gives a bijection $\mathcal{E}(BA) \cong \mathcal{E}(X)$ induced by precomposing with $H^*(i)$.

Given $(E, \phi) \in \mathcal{E}(BA)$, by [Lemma 5.6](#), there is a map $\rho_{BA}^\phi: C^*(BA) \rightarrow C^*(BE)$ such that $\pi_*(\rho_{BA}^\phi) = \phi$. Then, for the corresponding object $(E, \phi \circ H^*(i)) \in \mathcal{E}(X)$, the map $\rho_{BA}^\phi \circ C^*(i)$ is a model for $\rho_{BA}^{\phi \circ H^*(i)}$, since $\pi_*(\rho_{BA}^\phi \circ C^*(i)) = \phi \circ H^*(i)$. Therefore, the map ρ_X defined in [Equation \(5.7\)](#) can be taken to be $\rho_{BA} \circ C^*(i)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} C^*(X) & \xrightarrow{\rho_X} & \prod_{(E, \phi \circ H^*(i)) \in \mathcal{E}(X)} C^*(BE) \\ C^*(i) \downarrow & & \parallel \\ C^*(BA) & \xrightarrow{\rho_{BA}} & \prod_{(E, \phi) \in \mathcal{E}(BA)} C^*(BE). \end{array}$$

[Lemma 4.12](#) applies to show that $C^*(i)$ is biconservative, while ρ_{BA} is biconservative by [\[BG14, Theorem 3.1\(i\)\]](#), as observed in [Example 3.3](#). It follows from [Lemma 1.21](#) that ρ_X is biconservative as well. \square

Alternative proof of [Theorem 4.17](#). Any H -space is simple and hence p -good, so by [Theorem 5.12](#) it suffices to show that X satisfies Chouinard’s condition. This is the content of [Theorem 5.15](#); the claim follows. \square

Example 5.16. We return to the example of $X = S^3\langle 3 \rangle$ considered in [Section 4.5](#). We recall that $S^3\langle 3 \rangle$ fits into a principal fibration $BS^1 \xrightarrow{i} S^3\langle 3 \rangle \xrightarrow{j} S^3$. Rector’s category $\mathcal{R}(S^3\langle 3 \rangle)$ consists of a single non-trivial object, given by the inclusion $\iota: \mathbb{Z}/p \rightarrow S^1$. Let $f: C^*(S^3\langle 3 \rangle) \rightarrow C^*(B\mathbb{Z}/p)$ denote the composite $C^*(B\iota) \circ C^*(i)$. This map is an F -isomorphism, and so f satisfies simple Quillen lifting by [Lemma 2.11](#). As we have already seen, $C^*(i)$ is biconservative, and so is $C^*(B\iota)$ by [Example 3.3](#). It follows that f and hence $\rho_{S^3\langle 3 \rangle}$ are biconservative, and we again deduce stratification of $C^*(S^3\langle 3 \rangle)$.

REFERENCES

- [AR03] Jaume Aguadé and Albert Ruiz. Maps between classifying spaces of Kac-Moody groups. *Adv. Math.*, 178(1):66–98, 2003. (cit. on p. 21).
- [AS82] George S. Avrunin and Leonard L. Scott. Quillen stratification for modules. *Invent. Math.*, 66(2):277–286, 1982. (cit. on p. 13).

- [Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005. (cit. on p. 2).
- [Bar19] Tobias Barthel. A short introduction to the telescope and chromatic splitting conjectures. *arXiv:1902.05046 [math]*, February 2019. (cit. on p. 3).
- [BCHV17] Tobias Barthel, Natalia Castellana, Drew Heard, and Gabriel Valenzuela. Stratification and duality for homotopical groups. 2017. arXiv:1711.03491. (cit. on pp. 2, 3, 6, 7, 8, 9, 11, 12, 13, 15, 17, 19, 20, 21, 27, 30).
- [BCR96] D. J. Benson, Jon F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. II. *Math. Proc. Cambridge Philos. Soc.*, 120(4):597–615, 1996. (cit. on p. 13).
- [BCS01] Carles Broto, Juan A. Crespo, and Laia Saumell. Non-simply connected H -spaces with finiteness conditions. *Math. Proc. Cambridge Philos. Soc.*, 130(3):475–488, 2001. (cit. on pp. 2, 4, 22, 26, 32).
- [BDS16] Paul Balmer, Ivo Dell’Ambrogio, and Beren Sanders. Grothendieck–Neeman duality and the Wirthmüller isomorphism. *Compos. Math.*, 152(8):1740–1776, 2016. (cit. on p. 8).
- [BG09] Winfried Bruns and Joseph Gubeladze. *Polytopes, rings, and K -theory*. Springer Monographs in Mathematics. Springer, Dordrecht, 2009. (cit. on p. 16).
- [BG14] David Benson and John Greenlees. Stratifying the derived category of cochains on BG for G a compact Lie group. *J. Pure Appl. Algebra*, 218(4):642–650, 2014. (cit. on pp. 2, 3, 18, 21, 27, 32).
- [BHV18] Tobias Barthel, Drew Heard, and Gabriel Valenzuela. Local duality for structured ring spectra. *J. Pure Appl. Algebra*, 222(2):433–463, 2018. (cit. on p. 6).
- [BIK08] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008. (cit. on pp. 1, 5).
- [BIK11a] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Localising subcategories for cochains on the classifying space of a finite group. *C. R. Math. Acad. Sci. Paris*, 349(17–18):953–956, 2011. (cit. on p. 2).
- [BIK11b] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying triangulated categories. *J. Topol.*, 4(3):641–666, 2011. (cit. on pp. 1, 6, 7, 9, 11).
- [BIK11c] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying modular representations of finite groups. *Ann. of Math. (2)*, 174(3):1643–1684, 2011. (cit. on pp. 1, 3, 7, 12, 13, 18, 21).
- [BIK12] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Colocalizing subcategories and cosupport. *Journal für die Reine und Angewandte Mathematik. [Crelle’s Journal]*, 673:161–207, 2012. (cit. on pp. 5, 6, 9, 12).
- [BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972. (cit. on pp. 5, 23, 24, 26).
- [BK02] Carles Broto and Nitu Kitchloo. Classifying spaces of Kac-Moody groups. *Math. Z.*, 240(3):621–649, 2002. (cit. on pp. 2, 4, 17, 19, 20, 21, 22).
- [BLO03] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16(4):779–856, 2003. (cit. on p. 18).
- [BLO07] Carles Broto, Ran Levi, and Bob Oliver. Discrete models for the p -local homotopy theory of compact Lie groups and p -compact groups. *Geom. Topol.*, 11:315–427, 2007. (cit. on p. 19).
- [Bou94] A. K. Bousfield. Localization and periodicity in unstable homotopy theory. *J. Amer. Math. Soc.*, 7(4):831–873, 1994. (cit. on pp. 23, 24).
- [BZ88] C. Broto and S. Zarati. Nil-localization of unstable algebras over the Steenrod algebra. *Math. Z.*, 199(4):525–537, 1988. (cit. on p. 28).
- [Cam18] James Cameron. *On the Duflot filtration for equivariant cohomology rings*. PhD thesis, 2018. <https://digital.lib.washington.edu/researchworks/handle/1773/42460>. (cit. on pp. 18, 19).
- [CCS07] Natàlia Castellana, Juan A. Crespo, and Jérôme Scherer. Deconstructing Hopf spaces. *Invent. Math.*, 167(1):1–18, 2007. (cit. on pp. 2, 4, 22, 26).
- [Che13] Andrew Chermak. Fusion systems and localities. *Acta Math.*, 211(1):47–139, 2013. (cit. on p. 18).
- [Cho76] Leo G. Chouinard. Projectivity and relative projectivity over group rings. *J. Pure Appl. Algebra*, 7(3):287–302, 1976. (cit. on pp. 2, 7, 17).
- [DGI06] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006. (cit. on p. 25).
- [DS16] Ivo Dell’Ambrogio and Donald Stanley. Affine weakly regular tensor triangulated categories. *Pacific J. Math.*, 285(1):93–109, 2016. (cit. on p. 16).
- [DW09] W. G. Dwyer and C. W. Wilkerson. The fundamental group of a p -compact group. *Bull. Lond. Math. Soc.*, 41(3):385–395, 2009. (cit. on p. 25).
- [Dwy74] W. G. Dwyer. Strong convergence of the Eilenberg-Moore spectral sequence. *Topology*, 13:255–265, 1974. (cit. on p. 25).

- [DZ87] W. Dwyer and A. Zabrodsky. Maps between classifying spaces. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 106–119. Springer, Berlin, 1987. (cit. on p. 23).
- [Far96] Emmanuel Dror Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996. (cit. on pp. 23, 24).
- [GJ09] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition. (cit. on p. 24).
- [Hen98] Hans-Werner Henn. Unstable modules over the Steenrod algebra and cohomology of groups. In *Group representations: cohomology, group actions and topology (Seattle, WA, 1996)*, volume 63 of *Proc. Sympos. Pure Math.*, pages 277–300. Amer. Math. Soc., Providence, RI, 1998. (cit. on p. 32).
- [HLS93] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects. *Amer. J. Math.*, 115(5):1053–1106, 1993. (cit. on pp. 4, 22).
- [HLS95] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. Localizations of unstable A -modules and equivariant mod p cohomology. *Math. Ann.*, 301(1):23–68, 1995. (cit. on p. 28).
- [HP01] Mark Hovey and John H. Palmieri. Stably thick subcategories of modules over Hopf algebras. *Math. Proc. Cambridge Philos. Soc.*, 130(3):441–474, 2001. (cit. on p. 13).
- [HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997. (cit. on p. 1).
- [Kan58] Daniel M. Kan. A combinatorial definition of homotopy groups. *Annals of Mathematics. Second Series*, 67:282–312, 1958. (cit. on p. 24).
- [Kit98] Nitya R. Kitchloo. *Topology of Kac-Moody groups*. ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)—Massachusetts Institute of Technology. (cit. on p. 21).
- [Kit14] Nitu Kitchloo. On the topology of Kac-Moody groups. *Math. Z.*, 276(3-4):727–756, 2014. (cit. on p. 21).
- [KM98] Peter H. Kropholler and Guido Mislin. Groups acting on finite-dimensional spaces with finite stabilizers. *Comment. Math. Helv.*, 73(1):122–136, 1998. (cit. on pp. 4, 19).
- [Kum02] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002. (cit. on p. 21).
- [Lan87] J. Lannes. Sur la cohomologie modulo p des p -groupes abéliens élémentaires. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 97–116. Cambridge Univ. Press, Cambridge, 1987. (cit. on p. 4).
- [Lan92] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d’un p -groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.*, (75):135–244, 1992. With an appendix by Michel Zisman. (cit. on pp. 2, 4, 22, 23).
- [LL15] Ran Levi and Assaf Libman. Existence and uniqueness of classifying spaces for fusion systems over discrete p -toral groups. *J. Lond. Math. Soc. (2)*, 91(1):47–70, 2015. (cit. on p. 19).
- [Lur17] Jacob Lurie. *Higher Algebra*. 2017. Draft available from author’s website as <http://www.math.harvard.edu/~lurie/papers/HA.pdf>. (cit. on pp. 15, 16, 17).
- [Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Adv. Math.*, 291:403–541, 2016. (cit. on p. 16).
- [Mil84] Haynes Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984. (cit. on pp. 4, 23, 24, 31).
- [MNN19] Akhil Mathew, Niko Naumann, and Justin Noel. Derived induction and restriction theory. *Geom. Topol.*, 23(2):541–636, 2019. (cit. on p. 15).
- [Nee92] Amnon Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt. (cit. on pp. 1, 15, 16).
- [Qui71] Daniel Quillen. The spectrum of an equivariant cohomology ring. I, II. *Ann. of Math. (2)*, 94:549–572; *ibid.* (2) 94 (1971), 573–602, 1971. (cit. on pp. 17, 18, 27, 28).
- [Rag06] Kári Ragnarsson. Classifying spectra of saturated fusion systems. *Algebr. Geom. Topol.*, 6:195–252, 2006. (cit. on p. 18).
- [Rag08] Kári Ragnarsson. Retractive transfers and p -local finite groups. *Proc. Edinb. Math. Soc. (2)*, 51(2):465–487, 2008. (cit. on p. 18).
- [Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. (cit. on p. 3).
- [Rec84] D. L. Rector. Noetherian cohomology rings and finite loop spaces with torsion. *J. Pure Appl. Algebra*, 32(2):191–217, 1984. (cit. on pp. 4, 20, 21, 27, 28, 32).
- [Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008. (cit. on p. 16).

- [Sch94] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994. (cit. on p. 22).
- [Sha09] Shoham Shamir. Cellular approximations and the Eilenberg-Moore spectral sequence. *Algebr. Geom. Topol.*, 9(3):1309–1340, 2009. (cit. on pp. 24, 25).
- [Sha12] Shoham Shamir. Stratifying derived categories of cochains on certain spaces. *Math. Z.*, 272(3-4):839–868, 2012. (cit. on p. 2).
- [Smi81] Larry Smith. On the characteristic zero cohomology of the free loop space. *Amer. J. Math.*, 103(5):887–910, 1981. (cit. on p. 18).

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